Psychological Review
1999, Vol. 106, No. 1, 180–209

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0033-295X/99/$3.00

MINERVA-DM: A Memory Processes Model for Judgments of Likelihood

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This article describes an integration of most of the disparate likelihood judgment phenomena in behavioral decision making using a mathematical memory model. A new theory of likelihood judgments based on D. L. Hintzman's (1984, 1988) MINERVA2 memory model is described. The model, MINERVA-DM (DM = decision making), accounts for a wide range of likelihood judgment phenomena including frequency judgments, conditional likelihood judgments, conservatism, the availability and representativeness heuristics, base-rate neglect, the conjunction error, the validity effect, the simulation heuristic, and the hindsight bias. In addition, the authors extend the model to expert probability judgment and show how MINERVA-DM can account for both good and poor calibration (overconfidence) as a function of varying degrees of expertise. The authors' work is presented as a case study of the advantages of applying memory theory to study decision making.

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One of the most important topics in behavioral decision research has been how people make judgments of likelihood or certainty. In fact, this topic has motivated 4 decades of research spanning several different disciplines, including psychology, economics, meteorology, and auditing. Despite this effort, few comprehensive theories of how people make judgments of likelihood have emerged. Instead, many of the likelihood estimation phenomena have remained a grab bag of heuristics and biases, with no quantitative psychological theory describing the underlying processes.

Throughout the '70s and '80s, scientists studying decision making saw a proliferation of research on judgmental heuristics and their associated biases (see Hogarth, 1987). However, despite the heuristics and biases program's success in motivating a considerable body of research, students of this program have shown an increasing disenchantment with the approach for several reasons (e.g., Gigerenzer, 1996; Gigerenzer, Hoffrage, & Kleinbolting, 1991; Weber, Goldstein, & Barlas, 1995; Weber, Goldstein, & Busemeyer, 1991; but see Kahneman & Tversky, 1996, for a contrasting viewpoint). First, whereas it is sometimes possible to identify which heuristic participants use a posteriori, it is much more difficult to predict which heuristic will be used a priori. As Gigerenzer (1996) pointed out, "one of them [heuristics] can be fitted to almost any experimental result" (p. 592) on a post hoc basis. Second, many of the heuristics and biases have remained only vaguely specified, with no quantitative theory to describe the underlying psychological processes (Gigerenzer, 1996). In fact, many of the heuristics and biases remain nothing more than informally worded verbal descriptions of psychological processes. Finally, the vast majority of the heuristics and biases have remained disconnected, with no unifying theory to link them together. The implied promise of an integrating and coherent theory has not yet been realized and probably will not be realized unless fundamental theoretical changes in approach are made (Wallsten, 1983).[1]

Several recent researchers have attempted to integrate various decision phenomena into a coherent theory. We believe these efforts mark the start of a new paradigm. For example, Gigerenzer et al.'s (1991) probabilistic mental models theory (PMM theory) addresses the overconfidence effect, the hard–easy effect, and the confidence–frequency effect (see also Gigerenzer & Goldstein, 1996; Gigerenzer & Hoffrage, 1995). Likewise, Fiedler's (1996) Brunswikian induction algorithm for social cognition (BIAS) model accounts for several social psychological phenomena, including the mere-thinking effect, group polarization, illusory correlation, and the range–frequency effect, among other things. Although PMM and BIAS postulate some cognitive mechanisms, their foci are on modeling decision making as a function of the environment. Relatively less attention has been devoted to specifying the cognitive processes involved with decision making.

The purpose of this article is to present a comprehensive theory

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We thank Scott Gronlund and Frank Durso for suggesting that we look at MINERVA2 as a starting point for our model. We are deeply indebted to Hal Arkes, David Budescu, Jerry Busemeyer, Frank Durso, Klaus Fiedler, Scott Gronlund, Rob Hamm, Reid Hastie, Douglas Hintzman, Jim Shanteau, Robert Terry, Tom Wallsten, and Elke Weber for their many valuable suggestions and insights during the preparation of this article. Finally, Charles F. Gettys would like to thank Mike Kolodziej for the invaluable services that he rendered during the preparation of this article.

Michael R. P. Dougherty and Eve E. Ogden are indebted to Charles F. Gettys, who passed away in May 1998, for his careful guidance throughout their graduate careers.

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[1] We do not share all of Gigerenzer's criticisms of the heuristics and biases approach. On the contrary, we believe that the heuristics and biases program made significant progress in moving behavioral decision theory from a focus on optimal and normative models to a focus on psychological processes. We also do not dispute (or dismiss) the many empirical findings characterizing judgment as error prone. Indeed, we believe these findings are important and that they should be explained by any theory of judgment and decision making.

of how people make judgments of likelihood. Our theory, MINERVA-DM (MDM; DM stands for decision making), is based on Hintzman's (1984, 1988) MINERVA2 memory model, which has been applied extensively to study memory retrieval phenomena. MDM extends MINERVA2's capability and enables it to account for several likelihood estimation phenomena, including frequency judgments, conditional likelihood judgments, conservatism, the availability and representativeness heuristics, base-rate neglect, the conjunction error, the validity effect, the simulation heuristic, the hindsight bias, and both good and poor calibration (overconfidence).

Our presentation of the theory is directed at achieving two related goals. Our primary goal is to describe an overarching theoretical framework from which many of the heuristics and biases can be understood. Past research, whether intentionally or unintentionally, regarded the heuristics and biases as relatively distinct from one another, each treated as arising from a somewhat different underlying psychological process. In contrast, our view is that many of the existing heuristics and biases can be understood as arising from a single *common* process—the process of memory retrieval. MDM is able to account for a variety of memory retrieval phenomena and heuristics and biases, as well as make several new predictions regarding the factors that affect likelihood judgments. Table 1 presents many of the heuristics and biases. MDM can account for most of the likelihood estimation phenomena.

Our secondary goal is to illustrate the utility of using mathematical memory models to explain judgment and decision phenomena. As noted above, one criticism of the heuristics and biases program was that it failed to specify the precise cognitive processes underlying the heuristics. In contrast, because MDM is a mathematical memory model, it makes explicit both the representational assumptions and the memory retrieval processes that underlie judgments of likelihood. Thus, our model is both empirically testable and falsifiable. We believe that the application of memory models to decision theory will prove useful both for describing the cognitive processes underlying judgment and decision making and for motivating future theory development.

To date, there have been a few attempts to use existing memory models to study decision making (e.g., Hastie, 1988; Kashima & Kerekes, 1994; Smith, 1991; Smith & Zatare, 1992; Weber et al.,

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Table 1
Several of the Heuristics and Biases From the Decision-Making Literature

Heuristic or bias	Description	Literature
	Likelihood estimation phenomena explained by MINERVA-DM	
Conservatism	People were shown to underestimate extreme probabilities in respect to Bayes's theorem.	Edwards (1968)
Overconfidence	Calibration measures showed that people overestimated probability judgments with respect to their proportion correct.	Einhorn & Hogarth (1978); Lichtenstein, Fischhoff, & Phillips (1982)
Hindsight	The tendency to overestimate the predictability of an event after it has already occurred.	Fischhoff (1975)
Availability	Probability judgments are based on the ease with which instances can be brought to mind.	Tversky & Kahneman (1973)
Representativeness	Probability is based on the similarity between an instance and a category in memory.	Kahneman & Tversky (1973)
Conjunction fallacy	The probability of a conjunction is judged more likely than either of its constituents.	Tversky & Kahneman (1983)
Base-rate neglect	The prior probabilities of an event are ignored in probabilistic inferences.	Bar-Hillel (1980)
Validity effect	Statements are seen as more valid if they are familiar.	Hasher, Goldstein, & Toppino (1977)
Illusory correlation	The perception that two unrelated variables are correlated (simulated by Smith, 1991, using MINERVA2).	Chapman & Chapman (1969); Smith (1991)
Simulation	Probability judgments are made by assessing how easily a causal scenario can be constructed and simulated in memory (commented on but not simulated).	Kahneman & Tversky (1982)
Hard–easy effect	Overconfidence increases for difficult questions and decreases for easy questions (not discussed in this article).	Lichtenstein & Fischhoff (1977)
	Phenomena potentially explained by enhancing MINERVA-DM	
Best guess	Best guesses are made when passing from one inference stage to the next, producing overconfidence.	Gettys, Kelly, & Peterson (1973)
	Heuristics or biases where likelihood estimation is not the primary focus	
Anchoring and adjustment	Responses are made by adjusting from a known point suggested by the problem.	Tversky & Kahneman (1974)
Framing	Logically equivalent representations of a decision problem can result in marked changes in decisions.	Tversky & Kahneman (1981)
Sunk–cost effect	The tendency to base future economic decisions on past economic decisions.	Arkes & Blumer (1985)
Gambler's fallacy	The belief in streaks of good and bad luck.	Tversky & Kahneman (1971)
"Hot hands"	The belief that athletes get "hot" and can't miss or get "cold" and miss more than usual.	Gilovich, Vallone, & Tversky (1985)
Belief in the law of small numbers	Overconfidence in inferences made from small samples.	Tversky & Kahneman (1971)
Inertia effect	Inferences made during probability revision tasks change revisions.	Pitz, Downing, & Reinhold (1967)

Note. DM = decision making.

1991). However, most of these model applications have been quite limited in scope, with no real attempt to develop an overarching theory of decision making. In the next section, we present a new model of how people make likelihood judgments. In contrast to the previous modeling attempts, our model is able to integrate a wide array of likelihood phenomena into a coherent whole and provide new predictions. Next, we describe the model in detail and follow with the results of several simulation studies.

The MDM Model

MDM[2] is a modified version of Hintzman's (1984, 1988) MINERVA2 memory model, which has been used to model frequency judgments and recognition memory. The revised model not only retains all of MINERVA2's original capabilities as a memory model but also accounts for several likelihood estimation phenomena by incorporating a two-stage similarity comparison process. This modification enables MDM to account for both simple and conditional likelihood and frequency judgments as well as a variety of likelihood estimation phenomena. We discuss the psychological details for a two-stage conditional likelihood process at length in the section entitled *A Two-Part Conditional Likelihood Estimation Process.* First, we describe MDM's nonconditional processes.

For the sake of clarity, it is useful to first define a few terms. We use the name MINERVA2 to refer to Hintzman's (1984, 1988) original memory model and the abbreviation MDM (MDM corresponds to MINERVA–decision making) to refer to our modified model. The term MINERVA is used generically when describing isomorphic properties of the two models.

Memory Representation

MINERVA assumes that memory consists of a database of past experiences representing the decision maker's environment. However, because of various factors that affect encoding, such as perceptual acuity or the lack of attention to the stimulus event, the items stored in memory are assumed to be degraded copies of the experienced events. Events stored in memory are called memory *traces,* and one can imagine that memory consists of thousands of traces representing all of the events we have experienced over a lifetime.

One of the strengths of MINERVA is that traces and probes can represent anything of interest. Often, theorizing in decision making is concerned with hypotheses and data. For example, physicians formulate disease hypotheses from symptoms that are the data for that judgment. MINERVA has a vector representation; we assume that each trace consists of a series of concatenated minivectors that contain the features necessary to model a variety of inference processes. At present, we model decision making by using up to three concatenated minivectors: a hypothesis vector (**H**), a data vector (**D**), and an environmental context vector (**E**).[3] These concatenated minivectors form the basis of MDM inference processes and enable it to account for both simple (e.g., P[*H*]) and conditional (e.g., P[*H*|*D*]) likelihood and frequency estimates with or without environmental context.

Minivectors may be *filled* or *null.* A null minivector is ignored and does not enter into the calculations, as it does not exist. Each filled minivector consists of N cells, where values of $+1$, 0, or -1 are randomly assigned to each cell with equal probability. For example, a single minivector might contain {0, 1, -1, -1, 0, 0, 1, 1, -1}. Thus, each trace is represented by a vector of up to $3N$ cells, with each cell representing a feature of the stimulus event. A value of 0 corresponds to a feature that is either unknown or irrelevant to the present judgment task. Values of $+1$ and -1 correspond to features that are excitatory or inhibitory, respectively (Hintzman, 1988). For the present purposes, and for all of the simulations presented in this article, $N = 9$ for each minivector.[4]

Encoding in MINERVA consists of creating secondary memory (i.e., long-term memory) traces by copying an event vector. Traces are assumed to be degraded copies of the event vector, and the encoding, or learning rate parameter, L, determines the degree to which traces are degraded. Each feature in the event vector is copied into the secondary memory trace with probability L, where $0 \leq L \leq 1.0$. Degradation is modeled by converting each feature in the event vector to 0, with probability $1 - L$; otherwise, the veridical value of the corresponding feature in the event vector is retained and copied into the secondary memory trace (Hintzman, 1988). Thus, these secondary traces are similar, but degraded, copies of the experienced event. If L is close to 1.0, the trace will be highly similar to the event vector, but if $L = .5$, approximately half the -1, $+1$ entries will have been converted to 0, and the similarity between the secondary trace and the event vector will be much lower.[5]

Retrieval of Information From Memory

Memory in MINERVA is accessed by probing memory with an event vector called a *memory probe.* For example, suppose one was asked to estimate the relative frequency, or likelihood, of Democrats to Republicans. To make this judgment, the participant must create two probes, a "Democrat" probe and a "Republican" probe, and use these probes to interrogate memory. Imagine the Democrat probe is used first. In this case, the similarity between the Democrat probe and each trace is evaluated, and the output of the model, for that probe, is the sum of the similarities over all the traces stored in memory.[6] (In fact, the similarity value is cubed for each trace. We return to this shortly.) The frequency of Republi-

[2] MDM is a 1,154-line Turbo Pascal program written primarily by Charles F. Gettys. A copy of the source code is attainable from Michael R. P. Dougherty.

[3] Hintzman has also used vector concatenation, but for different purposes. By vector concatenation, we mean appending the three minivectors into one larger vector (e.g., the vector **ABC** concatenated with **DEF** concatenated with **GHI** yields **ABCDEFGHI**).

[4] Our choice of $N = 9$ cells for each minivector (or the 27 cells in the concatenated vector) was somewhat arbitrary; we wanted to allow for many possible values in each minivector, and a nine-element vector allows 3^9 possible hypotheses, data, or contexts to be encoded or represented in combination.

[5] Neither MINERVA2 nor MDM allows events to be misencoded (i.e., for a $+1$ to be replaced with a -1 rather than a 0). However, nothing would prohibit such a change to the models, if useful.

[6] This process is referred to as a global matching process because all traces in memory are assumed to be matched simultaneously. Thus, the resulting familiarity value is called a global familiarity signal.

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cans can be estimated by using the Republican probe in the same way. Finally, the Democrat and the Republican sums can be used to make estimates of the relative frequency or likelihood. At this juncture, MDM departs slightly from MINERVA2. The MDM response process assigns numbers proportional to the frequency estimates of Democrats and Republicans.[7] If the task calls for a probability judgment, the response numbers are assumed to be normalized so they sum to 1.

One can imagine the probe as producing an echo. If the similarity between the probe and a trace is high, a very loud echo will be contributed by that trace. If the similarity is low, there will be a faint echo. However, all traces contribute to the sum to some extent, even if the similarity between the probe and the trace is low.

Retrieval proceeds by probing secondary memory with a probe vector. The probe vector is assumed to activate all secondary memory traces simultaneously. This activation gives rise to a single *echo,* which is the sum of the activations of all traces stored in secondary memory. The echo is akin to a familiarity signal and depends on the similarity between the probe and all traces stored in memory. Following Hintzman (1988), the similarity between a single trace, *i,* and the probe is

$$S_i = \frac{\sum_{j=1}^{N} \mathbf{P}_j \mathbf{T}_{i,j}}{N_i}, \tag{1}$$

where $\mathbf{P}_j$ corresponds to feature j in the probe, $\mathbf{T}_{i,j}$ corresponds to feature j in trace i, and N_i is the number of corresponding nonzero features in both the probe and trace i (i.e., if either [or both] $\mathbf{P}_j$ or $\mathbf{T}_{i,j}$ is nonzero, then N_i is incremented). Thus, if either $\mathbf{P}_j$ or $\mathbf{T}_{i,j}$ is zero, the product will be zero, and nothing will be added to the numerator of Equation 1. However, if either $\mathbf{P}_j$ or $\mathbf{T}_{i,j}$ is nonzero, the denominator of Equation 1, N_i, will be incremented. Therefore, zeros in the vectors tend to reduce similarity. In contrast, if an MDM probe or trace vector contains a null minivector, that part of the vector is skipped in the similarity calculation and has no effect on similarity. Note that S_i can be either positive or negative.

The activation of a single memory trace is a positively accelerated function of the similarity between that trace and the probe. This is given by

$$A_i = S_i^3. \tag{2}$$

The cubing function allows those items in memory that are most similar to the test probe to dominate the overall echo from secondary memory, while preserving the sign.

The echo gives rise to two properties: echo intensity and echo content. Echo intensity and content are analogous to characteristics of real echoes. The echo intensity is the sum of all activation produced by the probe. This is given by

$$I = \sum_{i=1}^{\mathbf{M}} A_i, \tag{3}$$

where $\mathbf{M}$ is the number of traces assessed for similarity. The echo intensity is dominated by those traces that match the probe most closely, as the cubing function gives traces that are highly similar to the probe more weight than traces that are only somewhat similar. Thus, I will be close to 0 if there are only a few traces in memory that match the probe and will increase as the frequency of similar traces stored in memory increases. We assume that echo intensity is proportional to judged likelihood or frequency.

The echo content is a vector that corresponds to the sum of all traces in memory weighted by their activation, A_i. Echo content is given by

$$\mathbf{C}_j = \sum_{i=1}^{\mathbf{M}} A_i \mathbf{T}_{i,j}. \tag{4}$$

All terms are as previously defined. The echo content, $\mathbf{C}_j$, is a composite vector consisting of information from all traces stored in memory; however, it is affected most by traces that have a high degree of similarity with the probe. One can think of echo content as an attempt to recall the characteristics of the event vector by combining the multiple secondary traces of that event stored in memory (Hintzman, 1987, 1988). We do not use echo content in the simulations in this article.

A Two-Part Conditional Likelihood Estimation Process

MDM is more flexible than MINERVA2 because it incorporates a conditional process, which enables the model to account for conditional likelihood judgments.[8] The necessity for a conditional process can be understood by changing the Democrat–Republican example to a conditional likelihood problem. Imagine that you are now asked to estimate the frequency of Democrats who are wealthy. In this task, a condition of wealth is imposed, and in probability theory we think of the likelihood of a Democrat given wealth, L(Democrat|wealth). In fact, we now are interested in the frequency of Democrats in the subset of wealthy people.

How would a judgment of L(Democrat|wealth) be made using MDM? First, a decision must be made to determine whether a particular trace is that of a "wealthy" person. A "wealth" probe is created, and each trace is examined to determine whether its wealth component is similar enough to the wealth probe. The wealth assessment is confined to a comparison between the wealth probe and the wealth component of the trace. At this stage, it does not matter if the "political party" component of the trace contains

[7] MINERVA2 assigns numbers in the response process by comparing the echo intensity with fixed response criteria. If the echo intensity is above the lower criterion but below the upper, then the integer associated with that category is assigned, and so forth. MDM responds with a number proportional to the echo intensity. If an integer response is required, the number is rounded up or down appropriately. If probability responses are called for, we assume that they are produced by normalizing echo intensity, so that the sum of the responses is 1.0.

[8] One could, in principle, avoid creating a conditional likelihood estimation process by exploiting the definition of conditional probability: P(**H**|**D**) = P(**H** ∩ **D**)/P(**D**) and using MINERVA2 to estimate the right-hand ingredients. We have investigated this possibility extensively and have found poor fits for empirical data and the lack of desirable theoretical properties.

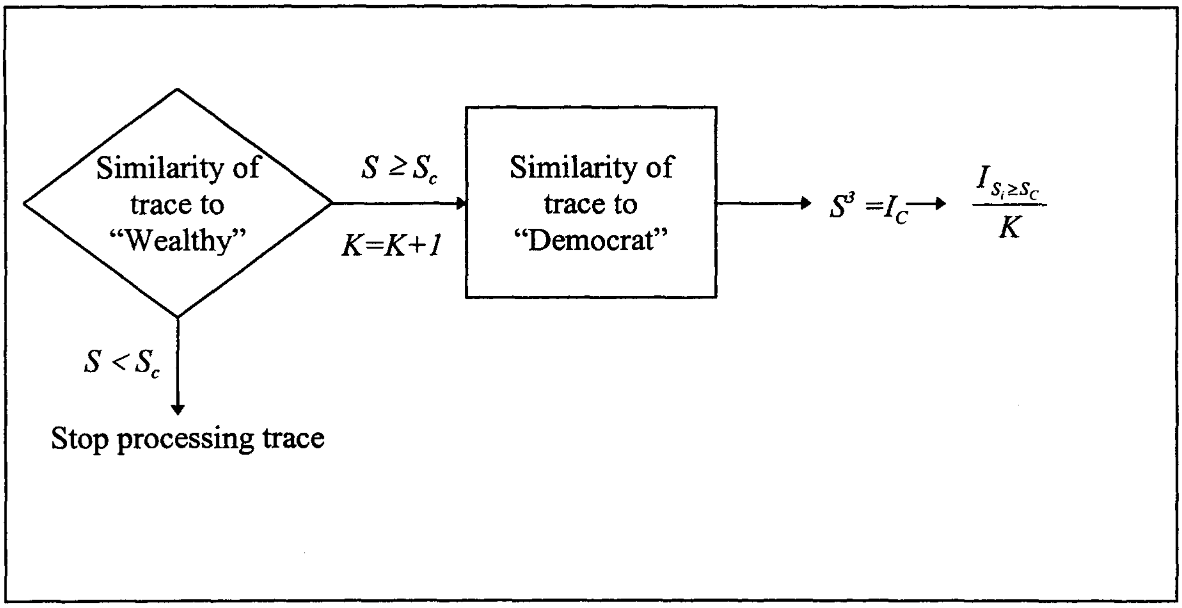

Figure 1. Schematic of the conditional likelihood process for the judgment L(Democrat|wealthy). The conditional likelihood process proceeds by first matching the condition part of the probe (e.g., wealthy) to traces stored in memory. If the similarity value for the condition portion of a trace meets or exceeds S_c, then the conditional portion (e.g., Democrat) is processed. I_c is given by the sum of the activations of all traces that pass the S_c criterion value. S = similarity between the trace and probe; S_c = similarity threshold criterion; I_c = conditional echo intensity; K = number of traces for which $S_i \geq S_c$.

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Democrat, Republican, or Libertarian because processing focuses on wealth. If the "wealth" similarity of that trace exceeds a criterion value, called S_c, the trace is judged to be a wealthy trace and the "political party" component of that trace is evaluated by a second process.

If the trace is judged to be wealthy, it is probed a second time to assess the similarity of the Democrat probe to the memory trace. The echo intensity resulting from the Democrat probe is the sum of the activations of all traces in the wealthy subset. The final step is to divide the sum by the number of wealthy traces, as the size of the sum will vary markedly with the criterion for wealthy. Taking the mean returns a typical value for the similarity to a Democrat, which is what is needed for the frequency estimate. From this point, only a routine response conversion is required; we assume that the conditional frequency or likelihood judgment is proportional to the mean echo intensity. Figure 1 illustrates the conditional likelihood process.[9]

Figure 2 presents a computational example of an L(**H**|**D**) conditional likelihood judgment (assume S_c = .50). MDM starts its search of memory by first identifying which instances in memory have a highly similar **D** minivector. If the criterion for subset membership is met or exceeded (Lines 1–7 in Figure 2), then a secondary analysis is performed to assess the conditional component of the memory trace, **H** (Lines 8–12). For example, suppose the task is to assess L(**H**|**D**). First, subset membership is assessed by creating a subset probe vector that has a null minivector for **H**, and a filled minivector for **D** (Line 4). (The context minivector may be filled or null, depending on the relevance of context to the task.) This condition probe is then analyzed by Equation 1 (Lines 5–7). The result, S_i, is compared with S_c (Line 8), the similarity criterion for deciding whether the condition has been met. The condition is met if

$$S_i \geq S_c, \tag{5}$$

where S_c is a criterion similarity, and S_i is the similarity between the **D** portion of trace *i* and the probe.[10] If, on the other hand, the condition is not met, processing of that trace stops. Thus, the first part of the conditional likelihood estimate is a similarity inference. Processing of the conditional minivector continues if and only if the similarity between the condition probe and the trace is greater than or equal to the similarity criterion value.

[9] Although Figure 1 involves counting, summing, and averaging operations, it should be emphasized that we assume these operations are performed automatically by the "wetware" in the brain. In fact, these processes are known to occur at the neuronal level (Kandel, Schwartz, & Jessell, 1991). One can infer from animal behavior that animals display behavior that amounts to accurate calculations of the mean (K. Cheng, Spetch, & Miceli, 1996). Thus, controlled processes (Shiffrin & Schneider, 1977) are probably unnecessary and not used.

[10] In a review of a draft of this article, D. L. Hintzman suggested that the criterion value be based on activation, *A*, which is the cube of the similarity, *S*. Basing the threshold on *A*, rather than *S*, is a logically equivalent decision rule, and the reader can use either *S* or *A*, whichever seems to be the most intuitive. We prefer *S*.

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Mini-vector:	Hypothesis	Data	Context
1) Memory Event	-1 +1 0 +1 0 -1 +1 -1 0	0 0 +1 -1 0 +1 +1 -1 -1	+1 -1 -1 -1 0 0 +1 +1 0
2) Memory Trace	-1 0 0 +1 0 0 +1 0 0	0 0 +1 0 0 +1 0 -1 -1	0 -1 0 -1 0 0 0 +1 0
3) L(H|D) Calc.			
4) Condition Probe	N U L L	0 0 +1 -1 0 +1 +1 -1 -1	N U L L
5) P_jT_{ij}		0 0 +1 0 0 +1 0 1 +1	$\Sigma P_jT_{i\flat j} = 4$
6) N_i		0 0 1 1 0 1 1 1 1	$\Sigma N_i = 6$
7) Similarity calculation			$S_i = 4/6 = .67$
8) $S_i \geq S_c$, proceed with second similarity calculation			
9) Conditional Probe	-1 +1 0 +1 0 -1 +1 -1 0	N U L L	N U L L
10) $P_jT_{i\flat j}$	+1 0 0 +1 0 0 +1 0 0		$\Sigma P_jT_{i\flat j} = 3$
11) N_i	1 1 0 1 0 1 1 1 0		$\Sigma N_i = 6$
12) Conditional similarity calculation			$S_i = 3/6 = .5$; $I_c = S_i^3 / K = .125$

Figure 2. Example calculation of conditional likelihood in MINERVA–DM (DM = decision making) for a single trace. Assume $S_c = .50$. S_i = similarity between probe and trace; S_c = similarity threshold criterion; Calc. = calculation.

Assuming $S_i \geq S_c$, the conditional component is processed (Lines 9–12). For the present example, the conditional probe for the **H** minivector is filled, but probes for the **D** minivectors and context minivectors are null (Line 9). Processing of the conditional minivectors continues by analyzing the similarity of the conditional probe and the trace using Equation 1 (Lines 10–12). The resulting value of S_i is then processed further by Equations 2 and 3 as before. A final step is to divide the echo intensity by the number of cases that exceeded the similarity criterion. Intensity is transformed to a likelihood by

$$I_c = \frac{I_{S_i \geq S_c}}{K}, \tag{6}$$

where I_c is the conditional intensity and K is the number of traces for which $S_i \geq S_c$ (i.e., the number of traces that pass the criterion value). The sum intensity (I) is a function of the number of cases processed, whereas I_c depends entirely on the similarity process in the second stage, regardless of the number of instances that are processed. The mean seems appropriate because the value of S_c arbitrarily determines the size of I unless a mean is taken. Calculation of the mean in Equation 6 converts I to a typical value, I_c, for each **H** and **D** combination. In cases where participants evaluate two or more hypotheses, we assume that the resulting I_cs are normalized to sum to 1.0 using the following equation:

$$\mathrm{L}(\mathbf{H}_i|\mathbf{D}_j) = \frac{I_c(\mathbf{H}_i \cap \mathbf{D}_j)}{\sum_i I_c(\mathbf{H}_i \cap \mathbf{D}_j)}, \tag{7}$$

where $\mathrm{L}(\mathbf{H}_i|\mathbf{D}_j)$ is the likelihood of hypothesis i given data j, and $I_c(\mathbf{H}_i \cap \mathbf{D}_j)$ is the conditional intensity for the conjunction of the ith hypothesis and jth data minivectors. The numerator of Equation 7 is the conditional intensity for the conjunction $\mathbf{H}_i \cap \mathbf{D}_j$ and the denominator is the sum of the conditional intensities of all $\mathbf{H}_i \cap \mathbf{D}_j$ conjunctions summed across all i hypotheses.

Several things can be discerned from the calculations in Figure 2. First, the echo intensity is increased solely by the similarity between the conditional probe (**H** probe) and the trace stored in memory. Second, the similarity inference is fallible because it is based on the similarity of the subset probe to the memory trace. It is possible for the model to either miss relevant traces that should be processed further, or to "false alarm" by classifying a trace as relevant when it is not. In general, the false-alarm rate will increase as S_c decreases. Of course, it is possible for an irrelevant memory trace to be highly similar to the probe vector by chance (because the traces are generated randomly) and therefore increase I_c. However, in general, false alarms that are included in the conditional calculation tend to reduce echo intensity. Conversely, relevant traces can also be missed, and this depends on L and S_c (the probability of a miss will increase as L decreases and as S_c increases). Increasing the miss rate exaggerates the natural variability of the model.

Psychological Factors That Determine the Choice of the Conditional Variable

In MDM, conditional likelihood judgments are assumed to be made by activating a subset of instances (cf. Gavanski & Hui, 1992; Rothbart, 1981; Sherman, McMullen, & Gavanski, 1992). In

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this case, the condition delineates the appropriate subset of instances in memory to be activated (i.e., the subset of instances that should be used to make the conditional probability judgment). Figure 3 illustrates this process for the two types of conditional likelihood judgments possible: P(hypothesis|data) and P(data|hypothesis). The numbers in the 2×2 table correspond to the number of instances of that type (e.g., there are 4 lawyers who enjoy mathematics and 12 engineers who enjoy mathematics).

The first type of conditional judgment that could be made is P(hypothesis|data). For example, imagine that participants are asked to judge P(engineer|enjoys mathematics). "Enjoys mathematics" defines the appropriate sample space in memory that should be searched in the conditional memory search; therefore, the first row of Figure 3 is the subset of instances that should be activated in memory. After the initial subset of instances is activated, participants are assumed to probe further with the category label, in this case "engineer." The probability is then given by the mean echo intensity produced by the engineer memory probe, as calculated by using Equations 1, 2, 3, and 6. The normative Bayesian calculations are given in the bottom half of Figure 3 for both P(engineer|enjoys mathematics) and P(lawyer|enjoys mathematics). Notice that both of these calculations take into account the number of engineers and lawyers who enjoy mathematics.

The second type of judgment that could be made is P(data|hypothesis). In the example in Figure 3, participants should search memory by conditioning on the category engineer (e.g., P[enjoys mathematics|engineer]). If this is the case, then the process is reversed, and the subset of instances activated in memory corresponds to the columns in Figure 3. The Bayesian calculations for P(enjoys mathematics|engineer) and P(enjoys politics|engineer) are presented at the bottom of Figure 3. Notice that these calculations do not take into account the number of lawyers who enjoy mathematics or politics, respectively.

The calculations in Figure 3 assume people are perfect judges, that they can accurately estimate the veridical frequencies in the

	Lawyer	Engineer	
Enjoys mathematics	4	12	16
Enjoys politics	6	8	14
	10	20	

P(category | instance description)

P(Engineer | Enjoys mathematics) = 12 / 16 = .75

P(Lawyer | Enjoys mathematics) = 4 / 16 = .25

P(instance description | category)

P(Enjoys mathematics | Engineer) = 12 / 20 = .60

P(Enjoys politics | Engineer) = 8 / 20 = .40

Figure 3. Illustration of the conditional memory search process. MINERVA–DM (DM = decision making) can conditionalize on either the rows or the columns depending on the judgment asked for.

2×2 table, and that they can define the appropriate sample space in memory. However, people are not perfect judges for several reasons. First, retrieval processes are often error prone; therefore, people may not be able to make accurate frequency estimates. It is possible for the conditional memory search to false alarm on irrelevant traces (i.e., activate instances from the inappropriate subset) or miss relevant traces (i.e., fail to activate instances from the appropriate subset). Both of these factors are modeled by the parameters L and S_c and will be elaborated on in our simulations that follow. As we demonstrate, faulty retrieval processes may, in part, contribute to people's proclivity to be suboptimal.

Although we assume that retrieval processes are faulty, if retrieval is perfect, then judgments will be Bayesian. In fact, MDM reduces to Bayes's theorem when both L and S_c are 1.0. When $L = 1.0$, the traces are exact copies of the parent event, and when $S_c = 1.0$, only exact copies can pass the criterion value. In this case, retrieval is infallible, and the model will give veridical probability estimates. If $L = 1.0$ and $S_c < 1.0$, then MDM reduces to modified Bayes's theorem (Gettys & Willke, 1969). A formal proof of these properties is presented in Appendix A.

A second factor that may lead to inaccurate judgments is that people may sometimes activate an inappropriate subset of instances in memory. The activation of the subset of instances depends on which variable is used as the condition in the conditional memory search. Activating the appropriate subset is crucial for making accurate conditional likelihood estimates (Gavanski & Hui, 1992; Sherman et al., 1992).

Accessing the Appropriate Subset of Instances

We assume that people activate the appropriate subset of instances when the judgment task is well defined, when the stimulus is carefully worded and the condition variable obvious, and when the participant engages in effortful processing. However, in most cases, the judgment task is not well defined and the stimuli are not carefully constructed. In fact, as Fiedler (1988) has shown, judgment errors can arise from subtle nuances in the linguistic structure of the stimuli used (see also Macchi, 1995). Furthermore, conditional likelihood judgments are probably made unconsciously, with little effortful processing involved (Malt, Ross, & Murphy, 1995). Consequently, people may not search memory in a manner befitting the particular judgment being made.

In the absence of careful analyses of the decision problem, we assume people search whichever subset of instances is most easily accessible. There are several factors that may make a particular subset of instances more accessible: natural categories, category cuing, individuating information, causality, learning order, and confusion of the inverse. We discuss these next.

Natural categories. Our knowledge of the world is partitioned into natural categories, and people organize environmental stimuli into these categories (Rosch, Mervis, Gray, Johnson, & Boyes-Braem, 1976). Given that we are accustomed to thinking in terms of natural categories, it is likely that we exploit this structure when making conditional probability judgments. In particular, we argue that people will activate the subset of instances corresponding to the most natural, or well-defined, category in memory (cf. Gavanski & Hui, 1992; McMullen, Fazio, & Gavanski, 1997; Sherman et al., 1992).

Several studies support the idea that people conditionalize on

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natural categories. Sherman et al. (1992) varied the naturalness of the categories used in the conditional probability task. Some judgments required participants to estimate the probability of a natural category conditional on a less natural category (e.g., P[male|extrovert]). Other judgments required participants to judge the probability of the less natural category conditional on a more natural category (e.g., P[extrovert|male]). Regardless of which likelihood question was posed, participants showed a tendency to condition on the most natural category. For example, when asked to estimate P(gender|extrovert), participants tended to condition on gender instead of extrovert, presumably because gender is a more natural cognitive category, and relatively better defined, than the ad hoc (Barsalou, 1983) extrovert category. Overall, Sherman et al. found that participants were more likely to activate the wrong subset of instances when the conditional event was a more natural category (e.g., gender categories) than the conditioning event (e.g., introverts versus extroverts; see also Gavanski & Hui, 1992).

McMullen et al. (1997) found that how information is categorized in memory affects which variable is used as the condition. In their experiment, participants were instructed to categorize a population of cartoon faces along one of two dimensions, either teeth or hair. When asked to make conditional probability judgments such as P(teeth|hair), participants conditionalized on whichever variable they used to categorize the stimuli initially. For example, if participants learned to categorize the faces by teeth, then they tended to conditionalize on the teeth dimension, even when asked to conditionalize on hair. Thus, depending on how information was categorized in memory, people conditioned on different subsets of instances; participants showed a preference to condition on the variable that was used to categorize the stimuli.

Category cuing. A second factor, related to the idea of natural categories, is category cuing or priming (Barsalou & Ross, 1986; Bruce, Hockley, & Craik, 1991; Nelson, LaLomia, & Canas, 1991). We propose that cuing a subset of instances in memory will lead people to conditionalize on that subset because it is already active in memory and more accessible.

One study that explicitly investigated the effect of category cuing on judgment was done by Arkes and Rothbart (1985; see also Hanita, Gavanski, & Fazio, 1997). Arkes and Rothbart primed a subset of instances in memory by giving participants a cued-recall task and then had them make relative likelihood judgments. In the context of the example given in Figure 3, participants were either cued to recall a row (e.g., enjoys mathematics or enjoys politics) or a column (e.g., engineer or lawyer) of the 2 × 2 table. Interestingly, when the rows were primed, participants' relative likelihood judgments took into account base rates. However, when the columns were primed, base rates were ignored. Thus, when making conditional likelihood judgments, priming a subset of instances in memory leads people to restrict their memory search to that subset. Presumably, this is because the primed subset is more accessible than the unprimed subset of instances.

Individuating information. A third factor that may affect the ease of accessing instances in memory is the amount of individuating information presented in the decision problem (Gavanski & Hui, 1992). Imagine you are asked to judge the likelihood that Alice belongs to the category "engineer" given that she enjoys both mathematics and building models and that she carries a pocket calculator (i.e., P[engineer|enjoys mathematics ∩ enjoys building models ∩ carries pocket calculator]). The additional individuating information will make it more difficult to access similar instances in memory because increasing the number of personality traits necessarily restricts the sample space. The process is further complicated by the fact that increasing the detail may make the appropriate subset of instances obscure and less easily defined. Consequently, people may not even know what subset of instances to activate in memory to make the judgment (Gavanski & Hui, 1992).

Given that increasing detail makes accessing the appropriate subset of instances in memory more difficult (and maybe impossible), people may default to an easier strategy, one in which they search the most accessible subset of instances in memory. Thus, when judging P(engineer|enjoys mathematics ∩ enjoys building models ∩ carries pocket calculator), people may inappropriately activate instances of the category "engineer" and then assess how many engineers "enjoy mathematics and enjoy building models and carry a pocket calculator." Although this strategy is easier, it is an inappropriate judgment that ignores category base rates.

Causality. Causality is a fourth factor that may affect how people search memory, and it may operate at several levels. First, people may encode cause–effect relations in a directional manner (Einhorn & Hogarth, 1986) such that causes are encoded before the effects. This may lead people to treat the cause as a superordinate category and the effects as instances of this superordinate (P. W. Cheng & Lien, 1995). As Gavanski and Hui (1992) pointed out, this type of memory structure can lead people to base judgments on the wrong sample space (but see Hanita, 1995). Second, causality may affect people's retrieval processes. Given that the cause always precedes the effect, the cause may prime a subset of instances in memory. Thus, if people are asked to make the judgment P(cause|effect), they may search the subset of instances corresponding to the cause because it is active in memory.

One study suggesting that causal direction influences conditional likelihood judgments was done by Steiger and Gettys (1973). Steiger and Gettys compared participants' conditional likelihood estimates of *height|weight ∩ sex* with *weight|height ∩ sex* and found that participants were quite accurate on the weight|height judgments but much less accurate on the height|weight judgments. Apparently, it was difficult to think of people being too short for their weight, but quite easy to think of a person as being too heavy for their height! This result suggests that participants were responding to the causal direction of the judgment; they saw height as a contributing cause to weight but did not see weight as a contributing cause to height, accounting for participants' difficulties in height|weight judgments.

Learning order. How one has learned to make a conditional judgment may affect which variable is used as the conditional. Waldmann (1996) investigated the effect of learning order on judgments of probability in a causal decision problem. Participants in his experiment learned disease–symptom combinations in one of two orders. In the *predictive* condition, participants were presented the disease of a patient and had to predict the symptoms. In the *diagnostic* condition, participants received the symptoms and had to diagnose the patient as having one of two diseases.

These two tasks presumably involve different learning strategies. Predictive tasks require participants to learn the symptoms conditional on the diseases, whereas diagnostic tasks require participants to learn the diseases conditional on the symptoms (Waldmann, 1996). If this is true, then it should be more natural for

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participants in the predictive group to conditionalize on the disease and for participants in the diagnostic group to conditionalize on symptoms. Therefore, participants in the predictive group should be more likely to invert the conditional probability judgment of P(disease|symptom) to P(symptom|disease), because it is more natural for them to conditionalize on the disease.

This is exactly what Waldmann (1996) found. The accuracy of participants' probability judgments varied depending on which learning strategy was used at encoding. Participants in the diagnostic condition incorporated the disease base rates into their probability judgments, but participants in the predictive condition ignored base rates completely. This asymmetry suggests participants in the diagnostic group correctly conditionalized on the symptoms, but participants in the predictive group incorrectly conditionalized on the disease (Waldmann, 1996). Thus, participants conditioned on whichever variable was more natural for them to use, and, in this case, it was more natural for participants to condition on the variable that was treated as the condition in the initial learning task.

Confusion of the inverse. The sixth factor that may affect which subset is activated in the conditional memory search is confusion of the inverse. This is an explicit misunderstanding of probability theory where people equate the posterior probability, P(H|D), with the diagnosticity of the data or the likelihood, P(D|H). For example, Hamm (1993) gave participants base-rate information, P(H), and reliability information, P(D|H), and asked them to assess the posterior probabilities, P(H|D). He found that participants frequently responded with P(D|H) rather than computing and responding with P(H|D). Eddy (1982) reported similar results with experienced physicians. Both of these studies suggest that at least some people do not understand the difference between the two conditional probabilities. If this logic is extended to how people search memory, it suggests that people may activate the incorrect subset of instances, thereby leading to poor probability judgments.

We have proposed that the nature of the conditional memory search (i.e., which subset of instances is activated by the condition memory probe) depends both on the structure of memory (e.g., natural categories, causality, and learning order) and on how information becomes accessible in memory (e.g., category cuing, causality, and confusion of the inverse). Underlying these factors is the idea that people are cognitive misers and that memory search is done in the easiest and most efficient manner. This notion has been expressed in various forms by several theorists, most recently by Anderson (1989), who proposed that memory is a rational system: "Human memory behaves as an optimal solution to the information-retrieval problems facing humans" (p. 195). In MDM, we assume that efficiency (optimality) is achieved by accessing the subset that comes to mind most easily.

Each of the above mechanisms has been investigated in isolation, and each has received at least some empirical support. There has been no research to our knowledge, however, that has investigated how people choose the condition variable when two or more of these mechanisms are in competition (e.g., when a decision problem has both a natural category and a causal structure). More research is needed to disentangle the antecedent conditions that determine when a particular variable is triggered as the condition in the conditional memory search.

In the next section, we present several Monte Carlo simulations that demonstrate MDM's capacity to account for a variety of judgmental phenomena including raw frequency estimates, conditional likelihood estimates, conservatism, two forms of the availability heuristic, the representativeness heuristic, base-rate neglect, the conjunction error, the validity effect, and the hindsight bias. We then extend MDM to expert probability judgment and show how the model can account for varying degrees of overconfidence as a function of experience and encoding. Although we did not use any formal means to estimate the parameter values, the values used in the simulations provided reasonably good fits when we modeled actual data.[11]

Simulation Studies of Frequency Estimation

Frequency Estimation: Greene's (1988) Experiment 6

In this section, we model a standard frequency estimation task using MDM and then apply the model to more complicated conditional likelihood judgments. Our approach to modeling frequency judgments is somewhat different than Hintzman's (1988). Hintzman assumed participants set multiple criterion values for different frequencies. Frequency judgments are made by comparing the echo intensity to all criterion values. For example, for frequencies of 1, 2, 3, and 4, Hintzman used four criterion values (see Footnote 7). Instead of setting criterion values to model frequency judgments, we treat the raw echo intensity output of the model as proportional to the frequency estimate. In situations where unconditional likelihoods (e.g., L[**H**]) are estimated, MDM is used to calculate an echo intensity for each category of the variable. If conditional judgments (e.g., L[**H**|**D**] and L[**D**|**H**]) are being made, echo intensity is calculated for each **D** and **H** combination.

The first simulation models a standard frequency estimation task. Hintzman (1988) presented several illustrations of MINERVA2's ability to model frequency estimation tasks. We present another as a convenience for the reader. For this simulation, we used only the **H** minivector to estimate frequency. The **D** and **E** minivectors were set to null because they are irrelevant to the judgment task. Thus, for standard frequency estimation tasks, MDM is identical to a nine-element version of Hintzman's MINERVA2 with multiple categories.

Greene (1988) investigated the generation effect in frequency judgments. The generation effect occurs when memory for information that is self-produced (i.e., generated) is more easily remembered than information externally presented (i.e., read; Slamecka & Graf, 1978). One account of the generation effect is that instances that are generated receive more elaborate encoding than instances that are merely read (Clark, 1995). In MDM, this translates into a higher encoding rate (L increases) for the generated words.

In Greene's (1988) experiment, participants were presented with a list of words, with each word appearing either 1, 2, 3, or 4 times.

[11] It is unrealistic to expect any mathematical model to perfectly predict experimental data. There are several sources of variance in most experimental tasks, and most mathematical models are not developed to account for all of these sources of variance. Thus, it is typically the case that mathematical models are used to model the functional form of the data, rather than give precise quantitative fits (Hintzman, 1990).

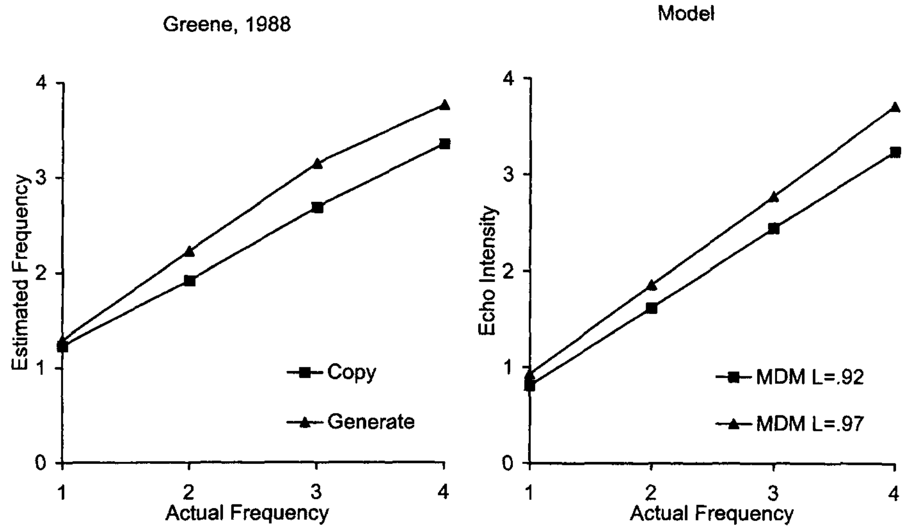

Figure 4. Frequency judgments from Greene (1988; left panel) and MINERVA-DM (DM = decision making; MDM) estimates (right panel). L = likelihood.

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For half of the words, participants were required to copy the word on a sheet of paper (the *copy* condition); for the other half, participants were presented with anagram stimuli and had to generate the words (the *generate* condition). After viewing the list, participants estimated the number of times each word appeared on the list.

The results of Greene (1988) are presented in the left panel of Figure 4. The upper curve illustrates the mean frequency judgments for words that were generated. The lower curve illustrates frequency judgments for words that were copied. There was a clear effect of generation on frequency judgments; generating the words led to higher frequency judgments.

Simulation method. The generation effect is presumably the result of an encoding process. That is, generating a word leads to better encoding than merely copying a word (Greene, 1988). Hence, we simulated Greene's results by varying L. For each simulated participant, we stored four different words in memory with frequencies of 1, 2, 3, or 4. Thus, each simulated participant had a total of 10 instances in memory. The simulation was run twice; the generate condition was simulated using $L = .97$, and the copy condition was simulated using $L = .92$. A total of 1,000 participants were simulated for each condition to give stable results. As stated earlier, N (the number of elements in each vector) = 9. Note that L is higher for the generate condition. This is because generating a word should result in better encoding than copying a word.

Simulation results. The results of the simulation are presented in the right panel of Figure 4. The top curve corresponds to the generate condition ($L = .97$) and the bottom curve to the copy condition ($L = .92$). Two characteristics of these curves are of interest. First, as the frequency with which each word was stored in memory increases, the model prediction increases, showing that the model is sensitive to frequency information. This is the result of the summation process in Equation 3 and the fact that instances most similar to the probe tend to dominate the echo intensity (the cubing function in Equation 2). Thus, increasing the number of similar instances in memory results in an increase in the overall echo intensity. A second noteworthy characteristic is that model estimates are higher for larger values of L. The larger value of L used to simulate the generate condition reflects the notion that generated words received better encoding than copied words.

In terms of the model, decreasing L leads to more zeros in the memory traces. This has the effect of increasing N_i in Equation 1, without producing a corresponding increase in the numerator of Equation 1. This, in turn, results in a lower overall echo intensity. Overall, MDM's predictions are very similar to the mean frequency judgments obtained by Greene (1988).

Conditional Likelihood Tasks: Gettys's (1969) Experiment

Early research on conditional probability judgments used the book bag and poker chip paradigm. In the typical experiment, participants are asked to imagine two book bags (Bags X and Y), each containing 100 red poker chips and blue poker chips. Each bag contains a different ratio of red chips to blue chips (e.g., Bag X might contain 70 red and 30 blue chips, and Bag Y might contain 30 red and 70 blue chips). A book bag is chosen at random, and poker chips are sampled from it. Participants are then asked to estimate the likelihood that the chip came from each bag—an L(Bag X|blue chip) conditional likelihood judgment.

In an unpublished experiment,[12] Gettys (1969) examined the generality of the book bag and poker chip paradigm. Most studies showed participants a table of book bag compositions before the chips were sampled. Other studies used tasks in which participants had prior experiences with the underlying likelihoods, but the history of the acquisition of their knowledge was unknown (e.g., the estimation of the likelihood of sex given height; DuCharme, 1970). In his study, Gettys used a training approach to the book bag and poker chip paradigm, where participants were taught the underlying conditional frequencies. This approach allowed control

[12] This research was conducted in 1969 while Charles F. Gettys was a National Science Foundation postdoctoral fellow at the University of Michigan in Ward Edwards's laboratory.

over participants' experiences and enabled the experimenter to calculate the actual Bayesian P(D|H) and P(H|D) values. We simulated two studies from Gettys's experiment in which participants made P(D|H) and P(H|D) judgments when the veridical frequencies were known.

Participants in Gettys's (1969) study were trained on the contingent relationships (e.g., P[Chip A|blue bag]) and their underlying distributions, as shown in Table 2. The participants were asked to pretend they had three book bags (red, green, and blue); each bag was filled with a different proportion of chips marked by the letters A, B, and C. Participants pushed a button on an apparatus that randomly drew a chip from one of the three bags and then pushed a different button to indicate which chip (A, B, or C) was drawn (the chip was then returned to its bag). After 36 training trials, participants were asked to estimate the proportion of each type of chip in each of the bags, a P(D|H) estimate (e.g., P[Chip A|blue bag]). The entire procedure then was repeated, drawing bags instead of chips, and participants were asked to make P(H|D) (e.g., P[blue bag|Chip A]) estimates. The 36 training trials enabled participants to learn the probabilities of each bag–chip-color combination.

Simulation method. We assumed that participants encoded each training trial in memory as two concatenated minivectors, one corresponding to the letter on the chip (A, B, or C), and one corresponding to the color of the bag (blue, green, or red), resulting in a total of 18 elements in the vector. In this simulation, the chips correspond to the **D** minivector, and the bag color corresponds to the **H** minivector (the **E** minivector was set to null). We modeled Gettys's (1969) results by using the veridical frequencies as input to the model (i.e., the number of instances stored in memory corresponded to the actual number of chips labeled A, B, or C in each of the three bags). For each simulated participant, there were 36 instances stored in memory, with 7 red-A, 3 blue-A, 2 green-A, 3 red-B, 7 blue-B, 2 green-B, 2 red-C, 2 blue-C, and 8 green-C instances. As in the actual experiment, our simulations used the conditional likelihood processes, L(**D**|**H**) and L(**H**|**D**). The simulation was run twice, once to simulate L(**D**|**H**) and once to simulate L(**H**|**D**). Both simulations used 1,000 simulated participants and the parameter values L = .75 and S_c = .75. Table 2 presents the MDM input frequencies (number of traces of each type) and the veridical probabilities for the two simulations. The two-part conditional likelihood estimation process of MDM was used.

Simulation results. The results of the L(**D**|**H**) simulations and the results of Gettys's (1969) experiment are presented in Figure 5 (the simulations of L[**H**|**D**] fit the data equally well and therefore are not presented). As can be seen, the model estimates (dashed lines) almost exactly predict the participants' estimates (solid lines).

The conditional likelihood process is assumed to activate a subset of instances in memory. For the L(**D**|**H**) simulation, participants are assumed to search all instances corresponding to the particular **H** minivector used to probe memory. For example, if participants are asked to judge P(Chip A|green), they are assumed to first activate all instances in memory with a "green" minivector. The subset of instances activated is different depending on which **H** minivector is used to probe memory. Thus, a different subset of instances will be activated for P(Chip A|green) than for P(Chip A|red), resulting in two different likelihood judgments depending on the conditioning event (e.g., red or green).

The above simulations illustrate MDM's ability to simulate both simple and conditional likelihood judgments. In the next section, we show how the model accounts for several of the heuristics and biases.

Table 2
Matrix of Frequencies and Probabilities (in Parentheses) Used in the Gettys (1969) Task and in the MINERVA-DM Simulations

Chip letter	Red bag (H_1)	Blue bag (H_2)	Green bag (H_3)
A Chips (D_1)	7 (.583)	3 (.250)	2 (.167)
B Chips (D_2)	3 (.250)	7 (.583)	2 (.167)
C Chips (D_3)	2 (.167)	2 (.167)	8 (.667)
Total traces	36		

Note. Parameters were L = .75 and S_c = .75, and 1,000 simulated participants were used. DM = decision making; H; = hypothesis; D = data.

Heuristics and Biases Explained by MDM

In this section, we model conservatism, availability, representativeness, base-rate neglect, the conjunction error, the validity effect, and hindsight by using MDM. We model overconfidence in the section on expertise.

Conservatism: DuCharme's (1970) Experiment 1

Conservatism is the tendency to underestimate objective probabilities as determined by Bayes's theorem. Historically, three explanations have been posited to account for conservatism: misaggregation of the true probabilities, misperception of probabilities, and response bias (see DuCharme, 1970, for a discussion of these). However, as Erev, Wallsten, and Budescu (1994) pointed out, the source of the conservatism bias was never really determined. More recently, Erev et al. have argued that conservatism is accounted for by a model that assumes true judgments are accompanied by random error variance. This random error leads to conservatism when probability judgments are compared with Bayes's theorem and overconfidence when compared with proportion correct. Thus, it is possible for judgments to be simultaneously conservative with respect to Bayes's theorem and overconfident with respect to proportion correct. MDM offers an alternative theoretical account of both conservatism and overconfidence and predicts simultaneous overconfidence and conservatism. We discuss MDM's account of conservatism after presenting DuCharme's (1970) Experiment 1 and the simulation study. We return to overconfidence in the section on expertise where we also simulate simultaneous over- and underconfidence.

DuCharme (1970) examined the conservatism effect by using a task in which participants made several conditional likelihood judgments for the heights of men and women. Participants were presented with the height of an individual and then judged the likelihood of either male or female given that height (e.g., L{male|70 in. [177.8 cm]}). The results from DuCharme's Experiment 1 are presented in Figure 6 as a solid line. If the judgments were perfect, they would fall on the identity line. This is approximately true for judgments between posterior odds of 10:1 and 1:10 (±1.0 log odds); however, notice that judgments diverge from the identity line for the more extreme judgments. Participants' estimates were less extreme than those calculated by Bayes's theorem. This S-shaped function is characteristic of conservative behavior.

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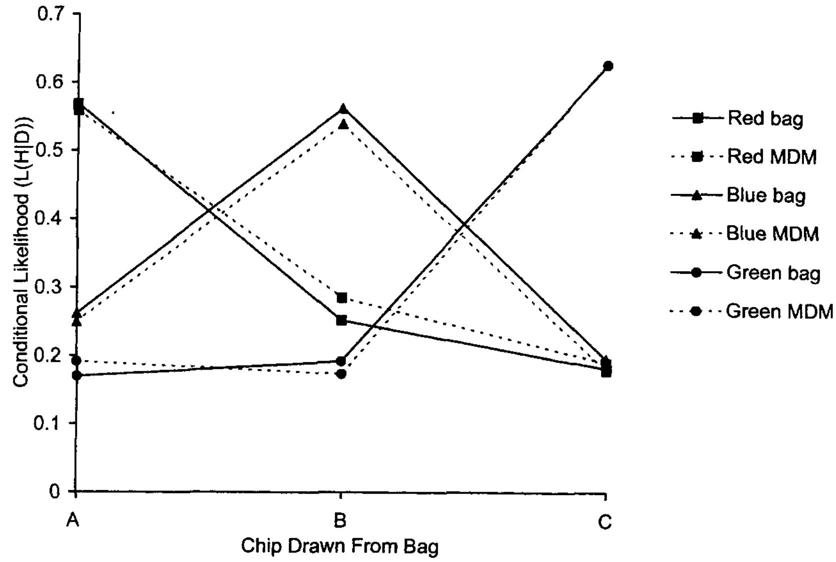

Figure 5. Conditional likelihood judgment from Gettys (1969; solid lines) and conditional echo intensity from MINERVA-DM (DM = decision making; MDM; dashed lines). Conditional echo intensity transformed using Equation 7. L = likelihood.

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Simulation method. To simulate DuCharme's (1970) Experiment 1, we assumed participants' memories consisted of instances corresponding to the heights of females and males. These memory traces were represented in MDM as two concatenated minivectors, with the **H** minivector corresponding to gender and the **D** minivector corresponding to height (resulting in a total of 18 elements for each memory vector). The participants in DuCharme's study probably experienced tens of thousands of males and females with varying heights over their lifetime, and it is likely that many of these heights were stored in memory. It is therefore reasonable to assume that participants' memory reflected the real-life frequencies of the heights of males and females. Because it is impossible to know for certain how many instances to store in MDM, we replicated the veridical likelihood ratios presented in DuCharme's Figure 1.

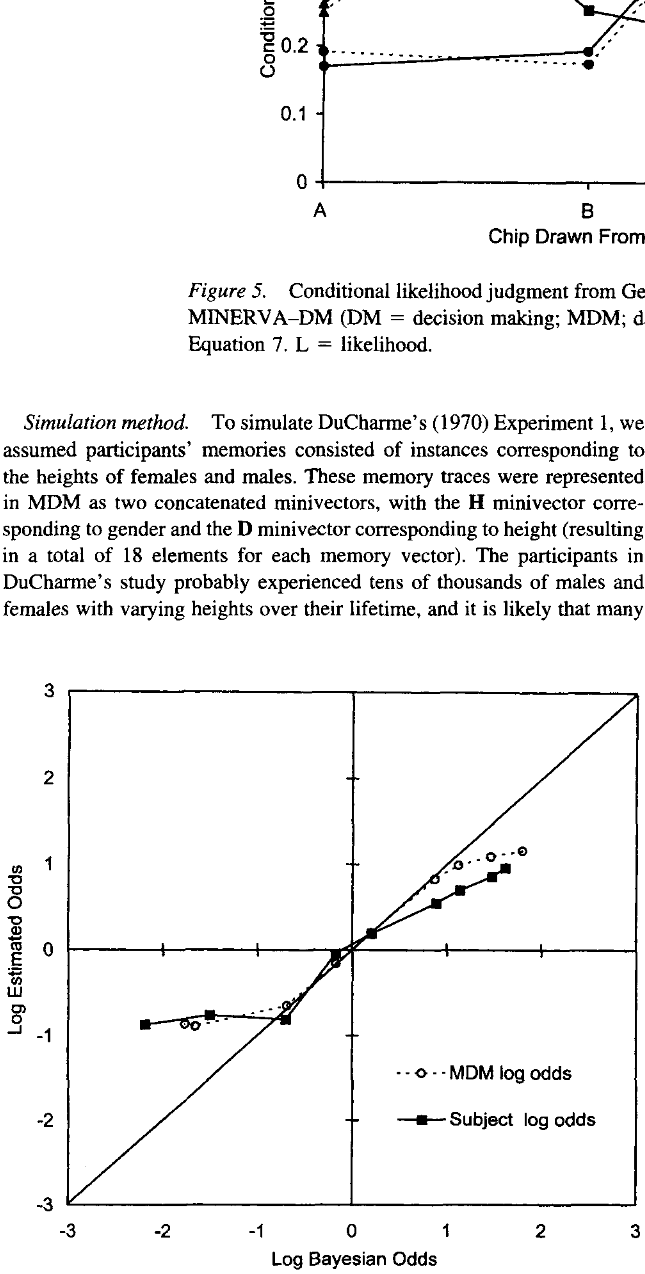

Figure 6. Participants' log estimated odds, log[L($\mathbf{H}_1|\mathbf{D}_1$)/L($\mathbf{H}_2|\mathbf{D}_1$)], from DuCharme (1970; solid line) and MINERVA-DM (DM = decision making; MDM) log estimated odds (dashed line) as a function of the log Bayesian odds.

We reproduced DuCharme's (1970) likelihood ratios by assuming that the heights of males and females were distributed normally with means of 67 in. (170.2 cm) and 63 in. (160.0 cm), respectively, and a standard deviation of 2.64 in. (6.7 cm; as reported by DuCharme). This resulted in a total of 26,901 instances stored in memory for each simulated participant corresponding to each of the following gender–height combinations: 1,127 males = 71.2 in. (180.8 cm); 18 females = 71.2 in. (180.8 cm); 1,276 males = 71 in. (180.3 cm); 44 females = 71 in. (180.3 cm); 2,565 males = 69.5 in. (176.5 cm); 194 females = 69.5 in. (176.5 cm); 3,410 males = 68.5 in. (174.0 cm); 459 females = 68.5 in. (174.0 cm); 3,605 males = 65.8 in. (167.1 cm); 2,275 females = 65.8 in. (167.1 cm); 2,371 males = 64.3 in. (163.3 cm); 3,538 females = 64.3 in. (163.3 cm); 775 males = 62.2 in. (158.0 cm); 3,814 females = 62.2 in. (158.0 cm); 18 males = 58.3 in. (148.1 cm); 818 females = 58.3 in. (148.1 cm); 10 males = 57.8 in. (146.8 cm); 584 females = 57.8 in. (146.8 cm).[13] Each gender–height combination was then used to probe memory in a conditional memory search (L[gender|height]). One hundred participants were simulated to obtain stable results, and L = .60 and S_c = .85 were found to provide a good fit.

Simulation results. The results of the MDM simulations are presented in Figure 6 as the dashed line. The model's predictions have the same S-shaped function as participants' estimates. This pattern indicates that MDM is conservative for extreme posterior odds but is nearly optimal for estimates in the 10:1 to 1:10 range.

Why does the model predict conservatism? The primary factor that produces conservatism in MDM is the criterion value S_c. S_c

[13] A total of 26,901 instances are needed if the lowest frequency is set to 10 and all other frequencies are set to numbers proportional to the posterior likelihoods.

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sets the similarity criterion for whether a trace is relevant or irrelevant. When S_c is high, few irrelevant memory traces pass the criterion value, and the model shows relatively little conservatism. However, as S_c decreases, more irrelevant traces pass the criterion (i.e., the false-alarm rate increases). Because the conditional echo intensity is a mean (Equation 6), and because irrelevant memory traces will be, on average, less similar to the probe than relevant traces, irrelevant traces will decrease the conditional echo intensity. This simulation offers an alternative account of conservatism, namely, that conservatism is the result of fallible retrieval processes—the inability to discriminate relevant from irrelevant traces. Moreover, this simulation suggests that manipulations that entice participants to increase their criterion values may result in less conservatism bias.

The Availability Heuristic

Tversky and Kahneman (1973) argued that people judge the frequency of events on the basis of the ease (or speed) with which relevant instances of those events are recalled or brought to mind. In one experiment, Tversky and Kahneman showed participants a list of 19 famous and 20 nonfamous names. Approximately 81% of the participants indicated that the famous names were more frequent, despite the fact they occurred slightly less frequently on the lists. This result was taken as evidence that the judged frequency of the "famous names" category is related to how easily these names could be brought to mind. A recall task showed a positive correlation between the number of names recalled and the judged frequency.

In a related study, Lewandowsky and Smith (1983) found that increasing the memorability of instances resulted in greater estimates of category size. Using the famous-name paradigm, Lewandowsky and Smith varied both the saliency and the frequency of presentation of famous and nonfamous names; both manipulations increased judgments of category size. These results suggest that increasing the memorability of the instances increased participants' likelihood estimates.

There are two mechanisms in MDM that can produce availability biases. Both mechanisms can be regarded as encoding biases, but the underlying memory processes are quite different. Each of these mechanisms alone is sufficient to account for Tversky and Kahneman's (1973) results.

The first availability mechanism involves context and probably arises because the participant partially confounds intraexperimental memory traces (those created by the experimental procedures) and extraexperimental traces (those naturally present in the participant's memory; cf. Johnson & Raye, 1981). For example, consider a judgment task involving the famous name, "John Wayne." Many individuals have a number of "built-in" memory traces associated with John Wayne, such as "John Wayne, the Duke," "John Wayne and Maureen O'Hara," and so forth. When memory is probed for instances of John Wayne, it is likely that these *extraexperimental* traces are activated, as well as the relevant intraexperimental traces. Thus, people's frequency estimates are biased by their own past experiences. If the participant could completely exclude irrelevant, extraexperimental traces, any availability bias due to the irrelevant traces should disappear.

MDM uses context as a mechanism to discriminate between intra- and extraexperimental memory traces by adding a *context* minivector to all traces (cf. Tulving & Thomson, 1973). In the following simulations, the context minivector (**E**) for intraexperimental traces was fixed to a given value, whereas the context minivectors for each extraexperimental memory trace were randomly generated. The probes contain the fixed intraexperimental context minivector. Consequently, memory traces containing that fixed intraexperimental minivector will be evaluated as considerably more similar by Equation 1 than extraexperimental memory traces having a randomly generated context minivector. Therefore, context gives MDM the ability to partially discount irrelevant traces, but extraexperimental traces will still inflate echo intensity.

The second availability mechanism in MDM is closer to the biased encoding explanation that most readers have in mind when they think about availability. This effect results from either the media's predilection to report the sensational, its audience's interest in the sensational, or both (Lichtenstein, Slovic, Fischhoff, Layman, & Combs, 1978; but see Shanteau, 1978). Thus, sensational press coverage of air crashes and the perfunctory reporting of traffic deaths gives rise to an encoding bias, because the frequency information itself is biased by the information source. Furthermore, our interest in the sensational compounds the problem; we are more likely to encode the details of the latest air disaster than a traffic fatality. This availability mechanism can be modeled by varying the learning rate parameter L. The latest air disaster might have a value of $L = .9$, whereas a driving death might have an $L = .3$. Below, we simulate both the extraexperimental memory traces explanation and the biased encoding explanation of availability.

Availability Effects Produced by Varying the Number of Extraexperimental Traces

We generated four lists of famous and less famous male and female actors. These names were then used as search cues in a news database containing citations to these actors in major newspapers such as the *New York Times*, the *Chicago Tribune*, the *Washington Post*, and so on. The citation frequency for famous male actors was 46 times higher than less famous male actors, and the corresponding ratio for female actors was 26:1. These values were entered into MDM as the number of extraexperimental memory traces to be produced for every famous experimental trace.

Simulation method. Three simulations were conducted. Each simulation used 39 memory traces corresponding to the 19 famous names and the 20 less famous names. Either 0, 26, or 46 extraexperimental traces were created for every famous intraexperimental trace, whereas 0, 1, or 1 extraexperimental traces were used for every less famous trace. These numbers capture the 26:1 and 46:1 ratios discussed earlier. The first simulation consisted of only the 19 famous and 20 nonfamous names, for a total of 39 traces stored in memory. To simulate female actors' names, 26 extraexperimental traces were stored in memory for each of the famous names and 1 extraexperimental trace for each nonfamous name, resulting in 439 total memory traces [$(19 + 19 \times 20) + (20 + 20 \times 1) = 439$] for each simulated participant. To simulate male actors, 46 extraexperimental traces were stored in memory for each famous actor and 1 for each nonfamous actor. This resulted in 1,012 instances stored in memory for each participant [$(46 + 46 \times 19) + (46 + 46 \times 1) = 1{,}012$]. The learning rate parameter, L, was fixed at .75, and the simulation consisted of 1,000 simulated participants.

Simulation results. The first simulation used no extraexperimental traces and is presented as the left-hand pair of bars in

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Figure 7. As can be seen, the zero-extraexperimental traces simulation showed a slight increase in MDM likelihood for the 20 less famous names over the 19 famous names, reflecting the small difference between frequencies (20 vs. 19). Thus, in the absence of extraexperimental memory traces, the model shows a slightly higher echo intensity for the nonfamous names, as it should.

The next four bars in Figure 7 show the expected availability effect when 26 extraexperimental traces are added for every famous female actor and 46 extraexperimental traces are added for every famous male actor. For both simulations, the model predicts that the famous actors will be incorrectly judged more numerous than the less famous actors. Thus, the more extraexperimental memory traces that are stored in memory, the greater the magnitude of the availability effect.

Luce (1959) proposed that choice probabilities were proportional to model output. Applying Luce's choice axiom to the output of MDM yields choice probabilities similar to those obtained by Tversky and Kahneman (1973). Tversky and Kahneman reported that 81% of their participants incorrectly chose the famous names as more frequent. As indicated in Figure 7, MDM's predictions approximate these choice probabilities.

Availability Effects Produced by L, the Learning Rate Parameter

Tversky and Kahneman (1973) reported recall data from one group of participants who were asked to recall the names of the actors. An average of 12.3 of the 19 famous actors' names were recalled, and 8.4 of the 20 less famous actors' names were recalled. These data can be used to make crude estimates of L. If we assume that recall is all or none (contrary to the spirit of the multidimensional memory traces in MDM), L for the famous actors is estimated to be .65 (i.e., 12.3/19 = .65), and L for the less famous actors is .42.[14]

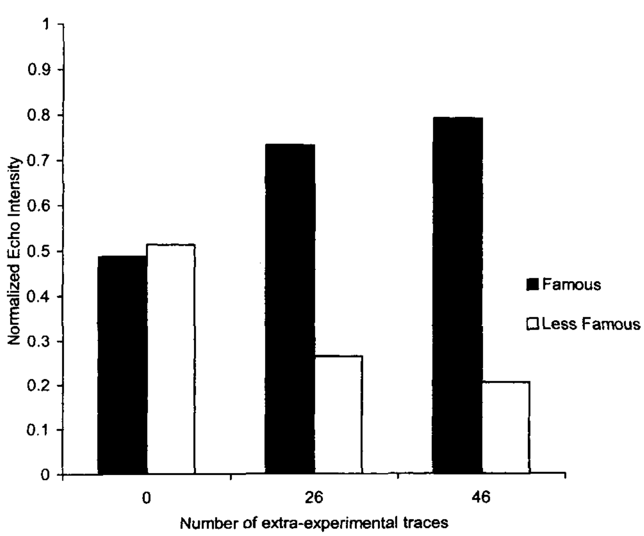

Figure 7. Simulation of famous-names study using extraexperimental memory traces.

Simulation method. The next simulation examined the effect of L on frequency judgments. To simulate the famous-names study using an encoding mechanism, we set L = .65 for the famous names and L = .42 to simulate the less famous names (assuming that people better encode the more famous actors in memory). A total of 39 instances were stored in memory for each simulated participant: 19 instances of famous actors and 20 instances of less famous actors. Again, 1,000 participants were simulated.

Simulation results. The results of this simulation revealed a clear effect of the encoding parameter on frequency judgments, with mean relative likelihoods of .74 (L = .65) and .26 (L = .42) for the 19 famous and 20 less famous names, respectively. Applying Luce's (1959) choice axiom to these likelihoods shows that MDM predicts choice probabilities similar to those obtained by Tversky and Kahneman (1973). In contrast, when L was the same for both conditions, the model predicted a slightly higher likelihood for the less famous (but more frequent) names.

These two simulations illustrate two possible mechanisms for availability effects. Although the accounts represent two very different mechanisms, both produce effects similar in size to those obtained by Tversky and Kahneman (1973). In the extraexperimental traces simulations, an availability bias was produced because extraexperimental memory traces partially influenced the echo intensity: The model was unable to completely discount the extraexperimental memory traces. In the biased encoding simulation, we found that higher values of L for the famous (but less frequent) names relative to the less famous (but more frequent) names were sufficient to produce an availability bias. This latter result is a direct consequence of increasing the number of zeros in the memory traces; as the number of zeros in the memory traces increases, echo intensity decreases.

Theoretically, it is possible for both the biased-encoding and the extraexperimental trace mechanisms to operate together in some availability experiments. Simulations of the two mechanisms together show huge availability effects. Most research on availability has tried to explain availability in terms of metamemorial processes or in terms of recall mechanisms (e.g., the number of instances recalled in a short period of time, or speed of recall). Although multiple factors undoubtedly contribute to the availability effect, we have shown that availability effects can be accounted for by a global familiarity signal[15] without assuming recall is taking place. Our simulations, and recent research by Ogden, Gettys, and Dougherty (1998), indicate that either mechanism is sufficient to produce biased judgments.

The Representativeness Heuristic

Kahneman and Tversky (1973; Tversky & Kahneman, 1982) make a cogent argument that some likelihood judgments are made by judging the representativeness of an instance, *X*, in reference to class, *M*, or a mental model (Gentner & Stevens, 1983) by judging

[14] We recognize that retrieval is not all-or-none; however, we found this to be the most plausible method for estimating the encoding parameter from Tversky and Kahneman's original results.

[15] This is called a *global familiarity signal* because all traces stored in memory contribute to the output of the model.

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the similarity of X to M. Their demonstration has three parts: (a) Likelihood judgments are related to judged similarity, not Bayesian probabilities; (b) participants largely ignore prior probabilities or base rates; and (c) participants commit a conjunction error when the conjunction of two events is more similar to a stored prototype than one of its constituents. In the next section, we show how MDM is compatible with the representativeness heuristic by showing that it is consistent with all three aspects of Kahneman and Tversky's argument.

Similarity

The first part of Kahneman and Tversky's (1973) three-part demonstration is that judgments of probability are made by judging the degree of similarity between the target event and a prototype stored in memory. As the reader surely has realized by now, similarity is the basis of MDM (through Equation 1). Thus, MDM is consistent with this part of Kahneman and Tversky's three-part demonstration, although similarity in their model is assessed with respect to a prototype rather than instances stored in memory.

Base-Rate Neglect

The second part of Kahneman and Tversky's (1973) demonstration is that people show base-rate neglect when using the representativeness heuristic. MDM displays base-rate neglect with the type of problems used by Kahneman and Tversky in their demonstration. Consider the Tom W. problem. Kahneman and Tversky presented participants with a personality sketch of Tom W. as a bright, compulsive, and mechanical thinker. The instructions invited participants to make base-rate, similarity, or likelihood judgments for nine graduate specialties. It was found that 95% of the participants said Tom W. was a computer science major, despite the fact that they stated that the other majors listed had many more students. Their results suggested that likelihood judgments were based primarily on similarity and that base-rate information was largely ignored.

The above task requires participants to judge P(graduate major|Tom W.'s personality sketch). In making this judgment, participants must first access all instances corresponding to Tom W.'s personality sketch and then judge how many of those instances belong to the various graduate majors. As pointed out earlier in our discussion of the conditional memory search, this may not be a simple process (and perhaps impossible, given the amount of data in the personality sketch). Instances corresponding to Tom W.'s personality sketch are presumably distributed over a number of different categories, and accessing them might be relatively difficult.

It is much easier for participants to activate a subset of instances in memory corresponding to a well-defined category, such as college major, than it is to activate a subset of instances distributed across several disparate categories. Probing memory with the category label "computer science major" would activate instances of computer science majors. The secondary probe, Tom W.'s personality sketch, can then be used to assess the number of instances in the activated subset that correspond to Tom's personality. Therefore, if participants conditionalize on the category, the subset of instances initially activated in memory will be *computer science major* and not instances similar to Tom W.'s personality sketch. This gives rise to an L(Tom W.'s personality sketch|computer science major) (i.e., an L[**D**|**H**] judgment), rather than an L(computer science major|Tom W.'s personality sketch) (i.e., an L[**H**|**D**] judgment).

Simulation method. Two simulation studies were run using the same parameter values and the same number of instances stored in memory. The first simulation conditionalized on **D** and estimated L(**H**|**D**) (e.g., L[computer science major|Tom W.'s personality sketch]). This was done to show MDM is sensitive to base-rate information when it conditionalizes on the appropriate variable. The **D** vector defines the appropriate subset of instances in memory for this judgment. The second simulation conditionalized on **H** and estimated L(**D**|**H**) (e.g., L[Tom W.'s personality sketch|computer science major]). In this case, MDM should show base-rate neglect because it is restricting its memory search only to instances corresponding to the **H** minivector. The **H** vector defines the inappropriate subset of instances in memory for this judgment. In our example, this means the model will ignore people with Tom W.'s personality that belong to different college majors.

The MDM input frequencies are presented in the top right of Figure 8. A total of 300 instances, consisting of both the **H** and **D** minivectors, were stored in memory; of these instances, 40 corresponded to $\mathbf{H}_1$–$\mathbf{D}_1$, 60 corresponded to $\mathbf{H}_1$–$\mathbf{D}_2$, 120 corresponded to $\mathbf{H}_2$–$\mathbf{D}_1$, and 80 corresponded to $\mathbf{H}_2$–$\mathbf{D}_2$. The parameter values for these simulations were L = .75 and S_c = .50, and 1,000 participants were simulated. Notice that the likelihood ratios are either 40:60 or 60:40, but that $\mathbf{H}_2$ is twice as likely as $\mathbf{H}_1$. If the model responds to base rates, the output should indicate that $\mathbf{H}_2$ is more likely than $\mathbf{H}_1$ for both $\mathbf{D}_1$ and $\mathbf{D}_2$.

Simulation results. Figure 8 shows a Bayesian calculation for P(*H*|*D*), an MDM estimate for L(**H**|**D**), and an MDM estimate for L(**D**|**H**). The Bayesian calculation of P(*H*|*D*) and the MDM L(**H**|**D**) simulation, shown in the left and middle parts of the figure, are presented for comparison purposes. Notice that the result of the MDM L(**H**|**D**) simulation is similar to the Bayesian calculation, except that the MDM simulation is slightly conservative (i.e., less extreme) in respect to Bayes's theorem (as is the case with human participants). Also, the differences between the pairs

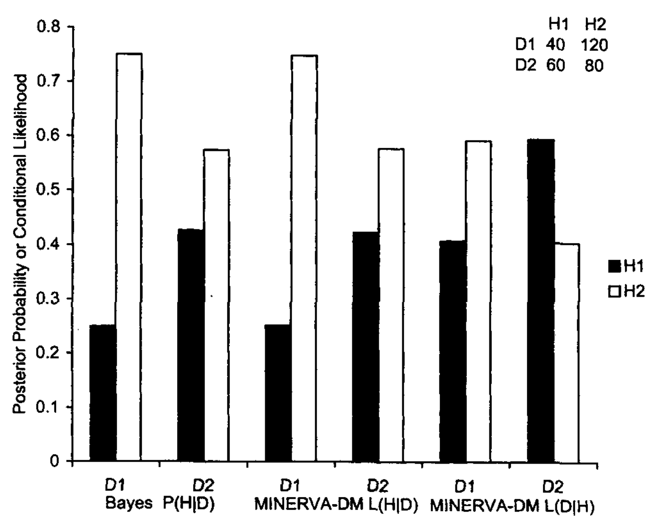

Figure 8. Predicted likelihood for Bayesian P(**H**|**D**) judgment, MINERVA-DM (DM = decision making; MDM) L(**H**|**D**) judgment, and MDM L(**D**|**H**) judgment. Base-rate neglect occurs only in the MDM L(**D**|**H**) judgments. Conditional echo intensity transformed using Equation 7. **H** = hypothesis; **D** = data; L = likelihood.

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of bars for the Bayesian calculation and the L(**H**|**D**) simulation indicate that Bayes and MDM are both responding to the base rate. (In fact, the 2:1 base rate overwhelms the 4:6 likelihood ratio, so that $\mathbf{H}_2$ is the most likely alternative for both $\mathbf{D}_1$ and $\mathbf{D}_2$.)

However, if a conditional likelihood judgment of the form L(**D**|**H**) is made (e.g., L[Tom W.'s personality sketch|graduate major]), base-rate neglect is shown. The right pairs of bars in Figure 8 show the MDM L(**D**|**H**) simulation. Notice that base rates are largely ignored, as the two right-most pairs of bar graphs are close to being the inverse of each other and are roughly in the ratio 60:40 or 40:60. This satisfies the second proposition of Kahneman and Tversky's (1973) three-part demonstration, that participants display base-rate neglect.

The reason MDM shows base-rate neglect for the L(**D**|**H**) simulation is that the inappropriate subset of instances is activated in memory. Recently, several researchers have argued that base-rate neglect arises when people invert the conditional probability judgment and condition on the wrong subset of instances (see Gavanski & Hui, 1992; McMullen et al., 1997; Sherman et al., 1992). MDM is consistent with this account and specifies the underlying memory retrieval mechanisms that take place.[16]

It should be noted that our theory predicts that base rates will be attended to in some situations. For example, in some situations it may be easier to conditionalize on **D** than **H.** In these cases, the L(**H**|**D**) routine is used and base rates are incorporated into the judgment. However, as stated earlier, exactly which variable is used as the condition will depend on which subset of instances is more easily accessed in memory, and this may be determined by one or more of the factors discussed earlier: the naturalness of the categories, category cuing, the amount of individuating information, causality, learning order, and explicit confusion of the inverse.

The Conjunction Error

The third aspect of Kahneman and Tversky's (1973) representativeness heuristic is that its use may lead to a conjunction error (Tversky & Kahneman, 1983). The conjunction error is the tendency to estimate the probability of the conjunction of two events as higher than one or both of its constituent events (e.g., $P[A \cap B] > P[A]$). This is an error because it violates the product rule of probability theory, which states that the probability of two co-occurring events must be less than or equal to the probability of each event in isolation (e.g., $P[A \cap B] \leq P[A]$ and $P[A \cap B] \leq P[B]$; see Gigerenzer, 1991, 1996, for arguments why this is not an error).

In their classic study of the conjunction error, Tversky and Kahneman (1983) had participants read a short vignette, such as the Linda problem below, and then rate the probability of several statements pertaining to the vignette. Three of the eight statements used by Tversky and Kahneman are given after the vignette.

> Linda is 31 years old, single, outspoken, and very bright. In college, she majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.
>
> 1) "Linda is active in the feminist movement."
> 2) "Linda is a bank teller."
> 3) "Linda is a bank teller and is active in the feminist movement."
>
> (Tversky & Kahneman, 1983, p. 297)

Eighty-five percent of the participants in Tversky and Kahneman's (1983) original study judged the conjunction ("bank teller and feminist," Sentence 3) to be more probable than one of its constituents ("bank teller," Sentence 2). This is a violation of the product rule of probability theory.

We assume that the conjunction error can arise from either (or both) of two memory mechanisms. The first mechanism is a biased retrieval mechanism. In their original study, Tversky and Kahneman (1983) constructed vignettes explicitly designed to elicit a conjunction error: The vignettes contained several details relevant to one constituent event but very few details relevant to the other constituent event. We assume that the retrieval probes are a function of the details contained in the vignette. In the Linda problem, almost all of the details are supportive of Linda being a feminist: Linda is female, single, outspoken, has a college education, majored in philosophy, and concerned with issues of discrimination and social justice. There are virtually no details consistent with Linda being a bank teller. People may construct memory probes by extracting details from the vignette relevant to each particular judgment question. When details are missing or are not present in the vignette, the slots in the memory probe may be filled in with zeros because zeros indicate "value not known."

Assume that bank teller and feminist are modeled by two minivectors. If asked to judge the likelihood that "Linda is a feminist," we assume that participants construct a feminist minivector probe that contains details from the vignette relevant to feminist. Because the vignette contains many details relevant to feminist, this probe will have relatively few zeros in it (e.g., +1, 0, −1, 0, +1, −1, 0, +1, −1). Likewise, if asked to judge the likelihood that "Linda is a bank teller," we assume that participants construct a bank teller probe using the details from the vignette. However, because the vignette contains few details relevant to bank teller, the bank teller probe will have only a few details, and the remaining slots will be filled in with zeros (e.g., 0, 0, 0, 0, 0, +1, 0, 0, 0). The "feminist and bank teller" probe is assumed to be a concatenation of the feminist and bank teller probe minivectors (e.g., +1, 0, −1, 0, +1, −1, 0, +1, −1, 0, 0, 0, 0, 0, +1, 0, 0, 0).

The response to lack of detail (i.e., increased number of zeros) in the probe is a reduction in echo intensity. Therefore, probing memory with the feminist probe will return a relatively high echo intensity, whereas probing memory with bank teller will return a relatively low echo intensity. Probing memory with feminist ∩ bank teller will return an echo intensity between that returned by the feminist probe and the bank teller probe.

Notice that this explanation of the conjunction error relies on the biasing properties of the vignette. If the vignette contains details relevant to only one event (e.g., feminist is detailed but bank teller is much less detailed), then the model predicts that feminist will be rated the most likely, feminist ∩ bank teller the second most

[16] It should be noted that when MDM calculates L(**D**|**H**) it does not always demonstrate complete base-rate neglect. Instead, the magnitude of base-rate neglect primarily depends on the value of S_c; base-rate neglect decreases as S_c decreases. This is because more irrelevant traces pass the lower threshold criterion, thus decreasing the overall similarity between the probe and the traces stored in memory. This property enables the model to account for the varying degrees of base-rate neglect often found in the literature (e.g., Ajzen, 1977; Bar-Hillel, 1980).

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likely, and bank teller the least likely. However, if neither event is detailed, the model does not predict a conjunction error. In fact, both of these predictions are supported by past research. Tversky and Kahneman (1983) found that people often judged P(feminist) > P(feminist ∩ bank teller) > P(bank teller) when the Linda vignette contained details relevant to feminists and no details relevant to bank tellers, but judged P(feminist) > P(feminist ∩ bank teller) and P(bank teller) > P(feminist ∩ bank teller) when neither feminist nor bank teller were detailed.

In theory, one should be able to determine a priori which event (A, B, or $A \cap B$) will be judged most likely by increasing or decreasing the details in the vignette relevant to one or both events. For example, adding details to the Linda vignette relevant to bank tellers and eliminating details relevant to feminists should lead people to rate P(bank teller) as most likely, whereas increasing the details relevant to both alternatives should lead people to rate the conjunction P(feminist ∩ bank teller) as the most likely event (a double conjunction error)—a prediction that has not been tested. We do not simulate this mechanism because such artificially created vignettes are of less interest than real-world tasks where vignettes are not present.

The second mechanism for modeling the conjunction error arises from the interaction between the structure of memory and the global matching property of the model and does not rely on the biasing properties of a vignette. In the real world, judgments typically are concerned with stimuli directly experienced (repeatedly) in the environment, rather than stimuli presented in a vignette format. In such cases, we assume that the probes are based on whatever stimuli participants are asked to judge.

Dougherty, Thomas, Gettys, and Ogden (1998) examined the conjunction error in a situation where participants had direct experience with the underlying event frequencies. In their study, Dougherty et al. used an exemplar training paradigm in which participants studied a population of fictitious animals characterized by five independent-binary traits. For example, in one condition, 80% of the animals were classified as being mean (20% were nice), 20% were large (80% were small), and 16% were mean and large (the conjunction of the two traits was determined using the product rule for independent events $.8 \cdot .2 = .16$). At test, participants were asked to judge P(mean), P(large), and P(mean ∩ large). Participants consistently overestimated the conjunction P(mean ∩ large) relative to the objective probabilities and committed a substantial number of conjunction errors (up to 53% in one condition).

In this type of task there is no vignette present from which to construct the memory probes. Therefore, it is assumed that participants probe memory with whatever event they are asked to judge. For example, if asked to judge P(mean) and P(large), participants probe with a minivector representing mean and a minivector representing large, respectively. If asked to judge the conjunction, P(mean ∩ large), we assume that participants probe with the concatenation of the two constituent minivectors (mean + large).

In the simulations that follow, we illustrate how the conjunction error can arise from the structure of memory without using the biased-probing mechanism (such as would be used if we were simulating the Linda problem). We simulated three conditions by varying the likelihood of the constituent events. Yates and Carlson (1986) showed that the number and type of conjunction errors people make depends on the likelihood of the two constituent events (see also Reeves & Lockhart, 1993). When both events were highly likely, participants tended to commit a double conjunction error (i.e., $P[A] < P[A \cap B] > P[B]$). When one event was highly likely and the other unlikely, participants tended to commit a single conjunction error (i.e., $P[A] > P[A \cap B] > P[B]$). Finally, when both events were highly unlikely, participants did not commit a conjunction error of either type (i.e., $P[A] > P[A \cap B]$ and $P[B] > P[A \cap B]$). In our simulations, we operationalized the likelihood of the constituent events by varying the frequency of traces stored in memory and showed that MDM can readily account for all three results without assuming a biased-probing mechanism or a biased-memory representation.

Simulation method. Three simulations were done to simulate each of Yates and Carlson's (1986) likely–likely, likely–unlikely, and unlikely–unlikely conditions. In each simulation, the number of memory traces entered into the model for A intersection B was determined using the product rule for independent events, $P(\mathbf{A} \cap \mathbf{B}) = P(\mathbf{A}) \cdot P(\mathbf{B})$. In the likely–likely condition, $P(\mathbf{A}) = .90$, $P(\mathbf{B}) = .90$, and $P(\mathbf{A} \cap \mathbf{B}) = .90 \cdot .90 = .81$. In the likely–unlikely condition, $P(\mathbf{A}) = .90$, $P(\mathbf{B}) = .10$, and $P(\mathbf{A} \cap \mathbf{B}) = .09$. In the unlikely–unlikely condition, $P(\mathbf{A}) = .10$, $P(\mathbf{B}) = .10$, and $P(\mathbf{A} \cap \mathbf{B}) = .01$. These probabilities were transformed into MDM memory trace input frequencies by multiplying each probability by 100. Thus, in the likely–likely condition each simulated participant had 261 instances (90 + 90 + 81), in the likely–unlikely simulation each had 109 instances (90 + 10 + 9), and in the unlikely–unlikely condition each had 21 instances (10 + 10 + 1). The memory probes for each constituent event (e.g., P[**A**] and P[**B**]) were a 9-element vector, whereas the memory probe for the conjunctive event (e.g., P[**A** ∩ **B**]) was an 18-element vector (the concatenation of the **A** and **B** probes). Each simulation used $L = .70$ and 1,000 simulated participants.

Simulation results. The results for the three conditions are shown in Figure 9. Each graph plots the conjunction, **A** ∩ **B**, as the center bar. Notice that MDM inflates the likelihood of the conjunction so that it is greater than both constituent events in the likely–likely condition (producing a double conjunction error). Notice also that the conjunction is greater than only one of the

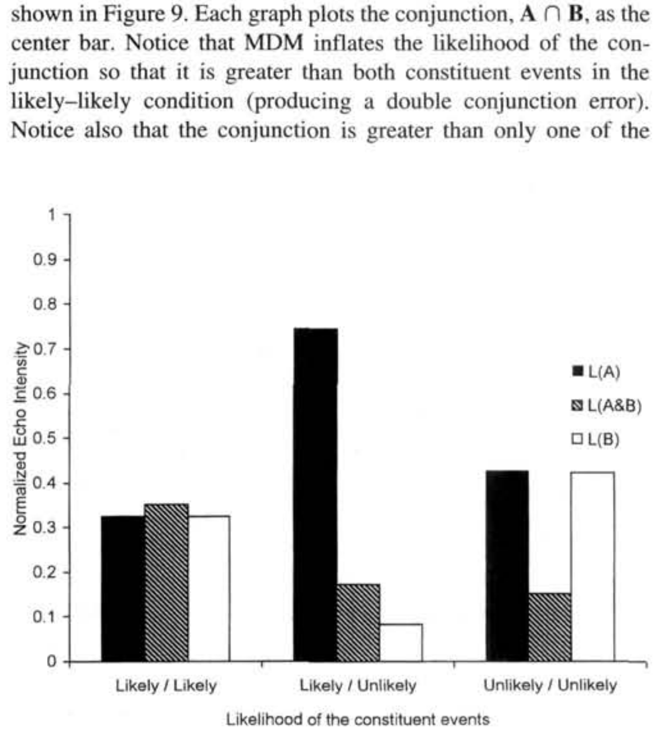

Figure 9. The conjunction effect in MINERVA–DM (DM = decision making). The middle bar shows how the model predicts a conjunction effect for the likely/likely and likely/unlikely conditions but not the unlikely/unlikely condition. L = likelihood.

constituents in the likely–unlikely condition (producing a single conjunction error) and less than both constituents in the unlikely–unlikely condition (producing no conjunction error). This pattern of results is consistent with Yates and Carlson's (1986) and Reeves and Lockhart's (1993) findings.

MDM's conjunction error prediction arises from the global matching property of the model (i.e., the fact that each probe is matched against all traces stored in memory). This property exaggerates the echo intensity when memory is probed with the $\mathbf{A} \cap \mathbf{B}$ memory probe because any trace that has an **A** or a **B** component will return a positive value. For example, in the likely–unlikely simulation, the 9 traces corresponding to $\mathbf{A} \cap \mathbf{B}$ will have relatively high activations and will therefore have a relatively large contribution to the overall echo intensity. However, because echo intensity is computed over all traces in memory, the 90 traces corresponding to **A** and the 10 traces corresponding to **B** will also contribute a positive value. This is because the **A** component of the $\mathbf{A} \cap \mathbf{B}$ probe is similar to all the **A** traces stored in memory, and the **B** component of the $\mathbf{A} \cap \mathbf{B}$ probe is similar to all the **B** traces stored in memory. Thus, both the **A** and the **B** traces will contribute to the overall echo intensity. In contrast, when memory is probed with **A** alone, all 90 traces corresponding to **A** return high activations as do the **A** components of the 9 $\mathbf{A} \cap \mathbf{B}$ traces. The 10 **B** traces will contribute very little, if anything, in response to the **A** probe. The same processes apply to the **B** probe. Thus, the net result of probing memory with a conjunctive event is the overestimation of the conjunction relative to the less frequent constituent event.

MDM predicted a conjunction error despite the fact that veridical frequencies were stored in memory (we used the product rule to determine the conjunction frequency)—a direct result of the global matching property of the model and the structure of memory (i.e., the frequency of traces stored in memory). These simulations suggest that the conjunction error occurs even when retrieval is unbiased and when memory conforms to the rules of probability theory.

In summary, MDM provides a coherent account of the representativeness heuristic and is consistent with Kahneman and Tversky's (1982) three-part demonstration. MDM is based on similarity, predicts base-rate neglect when the L(**D**|**H**) search process is triggered, and predicts a conjunction error. More important, MDM's account of the representativeness heuristic specifies the memory retrieval processes and the representational structures necessary for judgment to appear fallible: MDM predicts base-rate neglect only when the inappropriate subset of instances is activated and predicts a conjunction error only in the likely–likely and likely–unlikely conditions.

Validity Effect

In an uncertain environment, people are often faced with having to judge the validity of various propositions (Hasher, Goldstein, & Toppino, 1977). For example, the proposition "Singapore has the seventh-largest gross national product among industrialized nations" is both plausible and verifiable in that it is either valid (true) or invalid (false). Although most of us probably do not have direct access to the knowledge that would allow us to decide the validity of this statement, we can still estimate the degree (i.e., how confident) to which we believe it is valid.

An interesting aspect of validity judgments is that merely repeating a statement can make that statement seem more valid (Arkes, 1993; Arkes, Boehm, & Xu, 1991; Boehm, 1994; Hasher et al., 1977). This tendency is known as the "validity effect" (Einhorn & Hogarth, 1978). Judgments of validity are presumably based on the familiarity of the stimulus being judged (Arkes et al., 1991; Boehm, 1994), with more familiar statements rated as more valid than less familiar statements. The feeling of familiarity is also assumed to be the basis of recognition memory in forced-choice memory tasks and is modeled by echo intensity in MDM. We assume that judged validity is proportional to echo intensity.

We modeled the validity effect by using two different MDM mechanisms and by using the frequency routine with a single nine-element vector. In the first simulation, we simulated experiments where validity has been manipulated through mere repetition of the stimulus (see Hasher et al., 1977). In the second simulation, we modeled a biased-encoding mechanism for the validity effect (see Begg, Armour, & Kerr, 1985). Here, some of the traces stored in memory receive higher encoding than other traces.

Memory Trace Explanation of the Validity Effect

In the typical experiment on the validity effect, participants are presented with a set of validity statements on several successive trials. On each successive trial, a subset of the statements is repeated from a previous trial (often the statements are repeated several times), and the remaining statements are new. The participants are required to rate the validity of the statements on each trial. The typical result, of course, is that repeated statements are rated as more valid.

Simulation method. The first mechanism we used to model the validity effect was simply to add traces to memory corresponding to the number of times a statement was repeated. For example, Hasher et al. (1977) repeated a statement such as "Singapore has the seventh-largest gross national product among industrialized nations" 0, 1, 2, or 3 times. To simulate this effect, we assumed for each time the statement was repeated a new memory trace was added to memory. Thus, we ran four simulations using different numbers of memory traces depending on how many times the target statement was repeated. The memory trace input frequencies are summarized in Table 3. For example, for the first simulation, no traces had been repeated, so there were 20 total traces stored in memory for each simulated participant: 10 corresponding to $\mathbf{H}_1$ (first column) and 10 corresponding to $\mathbf{H}_2$ (second column). For the second simulation, one trace was repeated, so there were 11 traces corresponding to $\mathbf{H}_1$ and 10 corresponding to $\mathbf{H}_2$ (21 total memory traces for each simulated participant). The same procedure was used for the third and fourth simulations, except in the third simulation

Table 3
Cell Frequencies for Simulation of Validity Effect by Adding Traces to Memory

Presentation trial	Repeated statement	Nonrepeated statement
First	10	10
Second	11	10
Third	12	10
Fourth	13	10

Note. Parameter values were $L = .75$ and $S_c = .70$, and 1,000 simulated participants were used.

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there were 2 traces repeated, and for the fourth simulation there were 3 traces repeated. Each memory trace consisted of a nine-element vector and $L = .75$ was used. Each simulation used 1,000 simulated participants. These simulations most closely match Hasher et al.'s experimental design, where participants saw a proposition either 0, 1, 2, or 3 times during the experiment.

Simulation results. Echo intensity increased by about 5%, from zero repetitions to three repetitions. This result is similar in magnitude to those reported by Hasher et al. (1977), who found only a modest increase in judged validity (4% to 5%) from zero to three presentations. The increase in echo intensity is the direct result of increasing the number of instances in memory for $\mathbf{H}_1$. Echo intensity increases as the number of similar instances in memory increases, suggesting that judged validity should increase as the number of similar instances in memory increases.

Encoding Explanation of the Validity Effect

Simulation method. Two simulations were performed; both had a total of 20 instances stored in memory (10 corresponding to $\mathbf{H}_1$ and 10 corresponding to $\mathbf{H}_2$). In the first simulation, all memory traces were encoded into secondary memory with $L = .75$; this represents the unbiased case where none of the instances received higher encoding. For the second simulation, all 10 instances in $\mathbf{H}_1$ were encoded with $L = .75$, but for $\mathbf{H}_2$, half of the 10 instances were encoded with $L = .95$ (the other half with $L = .75$); this situation modeled the encoding explanation of the validity effect. Some of the instances in $\mathbf{H}_2$ received higher encoding. Thus, we held the number of traces entered into the model constant but varied how well some of the traces were encoded. This type of differential encoding may result from attentional mechanisms (Boronat & Logan, 1997).

Simulation results. The echo intensity was close to .50 for both $\mathbf{H}_1$ and $\mathbf{H}_2$ when all instances in $\mathbf{H}_1$ and $\mathbf{H}_2$ were encoded with $L = .75$. However, when half of the instances in $\mathbf{H}_2$ received $L = .95$ and half received $L = .75$, the echo intensity showed a modest increase (6%). This increase reflects the increased encoding for half of the traces in $\mathbf{H}_2$. Thus, MDM predicts that judged validity should increase as the encoding level increases. In fact, these results are consistent with Begg et al. (1985), who found that statements receiving higher encoding were later judged more valid than statements that presumably received lower encoding, even though both sets of statements were presented with equal frequency.

In summary, both sets of simulations showed that the validity effect can be accounted for by simple memory mechanisms. Our first set of simulations demonstrated that MDM's validity judgment increased as the frequency of similar traces stored in memory increased. Our second set of simulations modeled the encoding explanation of the validity effect. The model's judged validity was higher when instances received higher encoding (a higher L was applied to some instances). Both accounts are consistent with previous research on the validity effect (e.g., Begg et al., 1985; Boehm, 1994; Hasher et al., 1977).

Hindsight Bias

The hindsight bias is the tendency to overestimate the predictability of an event after it has already occurred. In his classic study, Fischhoff (1975) presented participants with vignettes describing causal scenarios, such as the war between the British and Gurkas. In the British–Gurka vignette, some participants were given an outcome of the war, and other participants were not. After reading the vignette, participants rated the probability of four possible outcomes: British victory, Gurka victory, stalemate with no peace treaty, and stalemate with peace treaty. Participants tended to rate the outcome that they believed occurred as more likely than the alternatives that did not occur, even though they were told to ignore their hindsight knowledge (Fischhoff, 1975)—the well-known hindsight bias. Since Fischhoff's original experiment, numerous experiments have documented the robustness of the hindsight bias, and several different explanations of hindsight have been proposed (see Hawkins & Hastie, 1990).

Hawkins and Hastie (1990) described four general explanations of the hindsight bias: rejudgment of the outcome, motivated self-presentation, direct recall of the old belief, and anchoring on the current belief and adjustment. Our modeling is somewhat similar to the rejudgment explanation of hindsight, which suggests that the hindsight bias arises because people use their outcome knowledge as the retrieval cue (Slovic & Fischhoff, 1977) and fail to use plausible events that did not occur as retrieval cues (cf. Hoch, 1985; Koriat, Lichtenstein, & Fischhoff, 1980). This biased cuing of memory results in a biased activation of instances, which in turn leads to an inflated level of certainty. Our view of the hindsight process is an extension of the rejudgment explanation.

Our interest is in modeling real-world hindsight biases, rather than participants' responses to artificial vignettes. Unfortunately, many of the experiments on the hindsight bias have used vignettes as stimuli (vignettes are the "limiting cases" of our more general examples) and therefore may not provide an adequate test of our theoretical explanation. In what follows, we describe the hindsight judgment process in terms of probing memory with cues that are either presented by the decision task or generated internally. We focus on describing the hindsight judgment process after the retrieval probes have been brought into working memory. We do not explicitly model the process of generating the probes when they are not provided by the task; however, we assume that the generation process is similar in nature to the scenario-generation process described elsewhere (see Dougherty, Gettys, & Thomas, 1997; Gettys & Fisher, 1979).[17] The net result of this scenario-generation process is the activation of one or a few scenarios that can be used to probe memory (Dougherty et al., 1997), rather than the generation of all possible scenarios. In hindsight judgment tasks, one of these scenarios necessarily is the outcome that "actually happened," whereas the other scenarios that are generated by this process are those that "might have happened but didn't."

Level of Detail in the Probe

Crucial to our explanation of the hindsight bias is the amount of detail in the memory probes. We assume that the probe for what actually happened is specified in considerable detail because the actual outcome scenario is used as the retrieval cue. In contrast, we assume that the probes for what might have happened but didn't

[17] In theory, one could model the scenario-generation process using MDM. We have decided to not model the scenario-generation process because the process is quite complicated and would presumably involve a much more elaborate model. We include our explanation of the hindsight bias in this article because it represents a logical break point in our thinking. However, we realize that there is further work to be done to specify the scenario-generation process in the context of the model.

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are specified in much less detail because these probes rely exclusively on the scenario-generation process, and the details must be generated internally or "filled in" by the participant. The scenario-generation process is assumed to produce alternative scenario probes that are less detailed than the scenario that actually occurred. However, the level of detail in the alternative scenario probes will depend on how much detail is produced in the scenario-generation process, and may be mediated by the experimental task. Instructions to imagine or explain an alternative scenario may entice participants to generate a more detailed scenario than would be generated spontaneously (Koehler, 1991).

The response of MDM to a lack of detail (i.e., replacing +1s and −1s with 0s) is a reduction in echo intensity. As discussed earlier, 0s mean value not known and therefore represent missing detail. Probing memory with the detailed outcome scenario returns a relatively large echo intensity, whereas probing with other, less-detailed scenarios returns a noticeably smaller echo intensity. Our explanation of the hindsight bias does not depend on the failure to generate scenarios, although we allow for this possibility, but rather it depends on the reduced detail in the what-might-have-happened-but-didn't scenarios. Accordingly, hindsight is assumed to arise from the biased probing of memory. The probe for what did happen is highly detailed, with relatively more +1s and −1s (e.g., +1, −1, 0, +1, −1, 0, 0, −1, +1, −1, +1, +1, 0, 0, −1, +1, +1, −1), but the probe for what might have happened but didn't is relatively less detailed, with fewer +1s and −1s (e.g., 0, 0, 0, +1, 0, −1, 0, 0, −1, +1, 0, 0, 0, +1, 0, 0, −1, 0).

The G Parameter

Our simulation of hindsight with MDM involves the introduction of a new parameter, G, which specifies the level of detail in the memory probes (this is in contrast to L, which degrades the traces). If $G = 1.0$, all details in the scenario are retained in the probe; we assume that $G = 1.0$ for the outcome scenario probe. For the what-might-have-happened-but-didn't scenario probes, G has some lesser value, such as .5, for all or part of the scenario. With a value of $G = .5$, half (on average) of the +1s and −1s in the probe minivectors are converted to 0, reducing similarity and, therefore, echo intensity. The range of G is $0 \leq G \leq 1.0$.

Assume there are two possible causes ($\mathbf{C}_1$ and $\mathbf{C}_2$) and two possible outcomes ($\mathbf{O}_1$ and $\mathbf{O}_2$), resulting in four possible scenarios ($\mathbf{C}_1\mathbf{O}_1$, $\mathbf{C}_1\mathbf{O}_2$, $\mathbf{C}_2\mathbf{O}_1$, and $\mathbf{C}_2\mathbf{O}_2$). Assume that what actually happened is $\mathbf{C}_1\mathbf{O}_1$. Because they represent what actually happened, both the $\mathbf{C}_1$ and $\mathbf{O}_1$ probe minivectors are completely specified ($G = 1.0$). However, for the $\mathbf{C}_1\mathbf{O}_2$ scenario, $G = 1.0$ for the $\mathbf{C}_1$ minivector, but $G < 1.0$ for the $\mathbf{O}_2$ minivector. This is because $\mathbf{C}_1$ actually occurred and $\mathbf{O}_2$ did not (therefore $G = 1.0$ for $\mathbf{C}_1$, but $G < 1.0$ for $\mathbf{O}_2$). In the $\mathbf{C}_2\mathbf{O}_2$ probe, both the $\mathbf{C}_2$ and $\mathbf{O}_2$ probe minivectors receive $G < 1.0$ because neither actually occurred. The net result of degrading probes is a reduction in echo intensity (this is a result of increasing the number of 0s in the probe vector). Thus, the G parameter results in a biased retrieval of instances from memory by degrading the memory probe.[18]

As an aside, we find this idea intellectually appealing, as it creates a means to explain the simulation heuristic (Kahneman & Tversky, 1982). One can model the counterfactual "undoing" process by assuming that a value of a scenario is changed, in the same way that we model changes in the outcome scenario in

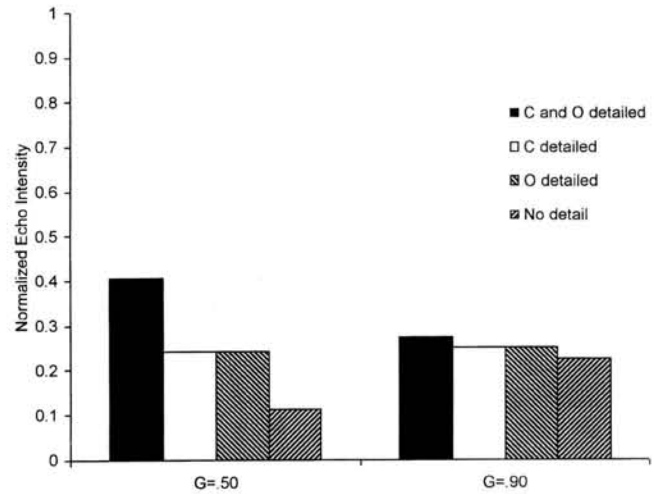

Figure 10. The hindsight bias in MINERVA–DM (DM = decision making). The hindsight bias is reproduced using a detailed probe for what happened (the actual outcome) and less detailed probes for events that did not happen. C = cause; O = outcome; G = parameter.

hindsight. Thus, the undoing process is analogous to the scenario-generation process in which alternative scenarios are generated spontaneously by the decision maker. These alternative scenarios will include details that could have happened but are not represented in the primary focal scenario. Our account of hindsight also suggests that it is intimately related to the simulation heuristic. When a person imagines an alternative scenario that undoes the outcome scenario by changing a value of that scenario, one is simulating a partially different scenario in memory, one that contains less detail. We believe this happens both in the hindsight and the simulation heuristic situations, and that this altered scenario would be judged to be less likely than the outcome scenario that actually happened.

Simulation method. Two simulations were performed. For the first simulation $G = .5$ was used. This value simulates the case where the probe vectors for what did not happen have relatively little detail. The second simulation used $G = .9$. In this case, participants were assumed to have relatively high detail in the probes corresponding to what did not happen. There were 1,000 simulated participants in each simulation, and L was set to .75. There were a total of 100 instances stored in memory for each simulated participant: 25 $\mathbf{C}_1\mathbf{O}_1$ instances, 25 $\mathbf{C}_2\mathbf{O}_1$ instances, 25 $\mathbf{C}_1\mathbf{O}_2$ instances, and 25 $\mathbf{C}_2\mathbf{O}_2$ instances. Thus, an unbiased matrix with 25 in each cell was used in this simulation. In contrast to the conditional memory search used in some of the previous simulations, memory was probed with the concatenation of **H** and **D** simultaneously.

Simulation results. Figure 10 presents the results from the hindsight simulations. In each of the two graphs, the leftmost bar is the likelihood (echo intensity) for the scenario that actually happened. The middle two bars are the likelihoods when one of the two minivectors lacks detail, and in the rightmost bar of each graph, both minivectors lack detail. As can be seen, the likelihoods

[18] We do not model the possibility of completely failing to generate a scenario for reasons of parsimony. To do so would introduce another parameter, and the effect of modeling generation failure can only exaggerate the magnitude of the predicted hindsight effect.

are as predicted by the hindsight bias. The graph where $G = .5$ shows a pronounced hindsight effect because fewer details are present in the changed minivectors; however, when $G = .9$, the magnitude of the bias is reduced considerably (for comparison, if $G = 1.0$ for all probes, then the likelihood for all four scenarios would be equivalent), thereby showing that MDM can explain varying degrees of hindsight bias (Christensen-Szalanski & Willham, 1991).

Our account of the hindsight bias predicts that the effect should be reduced when participants are enticed to specify the details of alternative scenarios. This prediction is supported by a body of literature that indicates the hindsight bias is mitigated when people are asked to think of alternatives (e.g., Hell, Gigerenzer, Gauggel, Mall, & Müller, 1988; Slovic & Fischhoff, 1977) or when the foresight retrieval cues are reinstated (Davies, 1987). This is not to say that generating alternatives is sufficient to reduce or eliminate the hindsight bias. On the contrary, MDM predicts that it is not whether the alternative scenarios are used as memory probes but rather the level of detail contained in the alternative causal scenarios that reduces the degree of hindsight bias. Tasks that entice participants to imagine or generate alternative scenarios (e.g., Gregory, Cialdini, & Carpenter, 1982; Koehler, 1991; Lord, Lepper, & Preston, 1984) most likely also result in more detailed scenarios than what would be generated spontaneously. We are unaware of any research that has explicitly examined the effect of increasing the *details* of the alternative scenarios on the magnitude of the hindsight bias.

Summary of Heuristics and Biases

The simulations presented so far illustrate that MDM is able to account for several of the heuristics and biases. It is important to realize that all of these simulations have used essentially the same memory processes. Thus, MDM has been able to account for a number of judgmental biases as arising from a *single cognitive process* (i.e., memory), instead of several isolated cognitive processes. In the next section, we extend the model to predict new findings in the domain of expert calibration. The model is also used to account for the simultaneous overconfidence–underconfidence revealed by Erev et al. (1994).

Application of MDM to Expert Probability Judgment

Research on expert probability judgment has revealed conflicting results. On the one hand, considerable research suggests that, in some domains, expert judgment is no better than novice judgment (Lichtenstein & Fischhoff, 1977). For example, overconfidence has been found in domains such as clinical psychology (Oskamp, 1965), medical diagnosis (Christensen-Szalanski & Bushyhead, 1981), and stock forecasting (Yates, McDaniel, & Brown, 1991). On the other hand, experts in domains such as bridge playing (Keren, 1987) and weather forecasting (Murphy & Winkler, 1977) reliably perform much better than novices. Why do experts sometimes outperform novices and other times not? The answer seems to be that good calibration can be achieved only through extensive practice in a domain that lends itself to timely and accurate feedback (Keren, 1991; Shanteau, 1992). Keren (1991) has further argued that for good calibration, the task must comprise essentially similar events. Indeed, most studies have found good calibration in highly repetitive tasks where good feedback was available, such as bridge playing and weather forecasting.

Our account for expert probability judgment relies on the processes of experience and encoding. Accordingly, our simulations rest on two assumptions. First, we assume that judges encode instances in memory and that these instances accumulate. Thus, the experienced judge may have hundreds or even thousands of instances stored in memory. Second, we assume encoding is affected by two factors. The first factor is the judge's level of expertise: Expert judges better know what attributes to attend to, which in turn may lead to better encoding of relevant information (Shanteau, 1988). In general, the degree to which one attends to a stimulus is assumed to affect how well it is encoded (cf. Boronat & Logan, 1997; Logan, 1988), regardless of experience. However, the fact that experts can selectively attend to the relevant attributes may further enhance the level of encoding. The second factor is the quality of feedback. We assume encoding is poor in tasks where feedback is slow, unreliable, or ambiguous, and high in domains where feedback is quick and reliable (Keren, 1991; Shanteau, 1992). Poor encoding can result from a number of different factors. Whether feedback actually affects encoding is an empirical question, one that is not addressed in this article. We have included feedback under the encoding mechanism because it is generally believed to affect judgment quality.

Before we proceed further, a theoretical note about the reality of overconfidence in judgments is warranted. Erev et al. (1994) argued that overconfidence is due, in part, to the error variance associated with participants' responses (cf. Thurstone, 1927). They illustrated in a number of studies that the same data can show both overconfidence and underconfidence, depending on how objective probability is defined. In the same sense, our model has an error component associated with it; however, the error component arises naturally from MDM (i.e., through computations on the memory vectors). Thus, our model captures the functional form of both overconfidence and conservatism and specifies the underlying mechanisms that produce them. We demonstrate through the application of MDM to expert judgment that overconfidence is reduced to the extent judgment variance is reduced (i.e., conditions under which experience and encoding are both high). Conservatism, on the other hand, generally increases as the S_c criterion value decreases (resulting in a higher false-alarm rate).

Calibration as a Function of Experience and Encoding

We reproduced nine different calibration curves to simulate the effect of experience and encoding accuracy on calibration. As in the previous simulations, echo intensity is assumed to be proportional to subjective probability. The present simulations used the same $L(\mathbf{H}|\mathbf{D})$ routine that was used to model the DuCharme (1970) data. However, as in the typical calibration study, the model estimates are compared with the *proportion of correct inferences* (i.e., hit rate), rather than to the *objective probability* as defined by Bayes's theorem.

Simulation method. We varied experience by manipulating the number of instances stored in memory across three levels (80, 200, and 600 traces) by using matrices with two hypotheses ($\mathbf{H}_1$ and $\mathbf{H}_2$) and 10 data ($\mathbf{D}_1$–$\mathbf{D}_{10}$)

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vectors.[19] We examined MDM's prediction for $P(\mathbf{H}_1|\mathbf{D}_j)$. The specific memory trace input matrices used for each simulation are presented in Table 4 in the columns labeled *Low, Medium,* and *High experience.* For example, in the medium experience condition there was a total of 200 instances stored in memory: 19 corresponded to $\mathbf{H}_1\mathbf{D}_1$, 1 to $\mathbf{H}_2\mathbf{D}_1$, 18 to $\mathbf{H}_1\mathbf{D}_2$, 2 to $\mathbf{H}_2\mathbf{D}_2$, and so forth. Notice that the prior odds are always 50:50. The far right column presents the objective probabilities for $\mathbf{H}_1\mathbf{D}_j$. The three encoding conditions were created by varying L across three levels: L = .75, .55, and .35. Each simulation consisted of 1,000 simulated participants, and S_c was held constant at .70.

The hit rate was computed by assuming each participant's decision strategy is to always choose the option with the highest echo intensity. Because of random error resulting from the computations on the vectors, the model is an imperfect predictor and will sometimes fail to choose the normatively correct answer. That is, occasionally the echo intensity will be highest for the lower probability alternative, leading the model to choose the normatively incorrect alternative.

It is easy to imagine an experimental task where this situation might be relevant. For example, the data minivectors may correspond to "symptoms" and the hypothesis minivectors may correspond to "diseases" in a medical diagnosis task. Participants would be presented with the symptoms and have to diagnose patients as having one of two diseases and then state their confidence in their diagnosis.

Simulation results. The nine confidence curves are presented in Panels A, B, and C of Figure 11. Each panel represents a different value of L. The simulations reproduced the signature signs of overconfidence; that is, the models' conditional echo intensities generally exceed the proportion correct. Notice the overconfidence effect is most pronounced under two conditions: when encoding is poor and experience is low, as seen in Panel A. These results are supported empirically by a number of studies (e.g., Christensen-Szalanski & Bushyhead, 1981; Keren, 1987; Lichtenstein & Fischhoff, 1977). As stated earlier, poor encoding may result from several possible factors, such as a judge's inability to selectively attend to relevant attributes and the quality of feedback provided by the task.

There are two other characteristics of the calibration curves that are particularly interesting. First, calibration in these simulations generally improves as a function of experience; this can be clearly seen in Panels A and B. As the number of instances stored in memory increases, the model's estimates come closer to the identity line. Therefore, MDM predicts that overconfidence decreases as experience increases. The decrease in overconfidence is due primarily to the reduction in echo intensity variance associated with each $\mathbf{H}_i\mathbf{D}_j$ probe. This is a direct consequence of increasing the number of similar instances stored in memory; the larger number of similar instances stored in memory naturally decreases the variance (the law of large numbers). As an example, the number of similar instances in memory for each $\mathbf{H}_i\mathbf{D}_j$ combination triples from the medium to the high condition. This results in lower overall error variance in the mean echo intensity (Equation 6) and in better calibration for all three values of L. Variance also decreases as L increases; however, this is because the number of zeros in the minivectors decreases with higher values of L.

The second characteristic is that the model is slightly conservative with respect to *proportion correct* when experience is high and L is high, as is shown in Panel C. The typical conservative result is found when subjective probabilities are compared with *objective probabilities* (e.g., DuCharme, 1970). However, this underconfidence result was obtained using the traditional overconfidence computations; that is, when subjective probabilities are compared with proportion correct. There is at least one study that has shown this type of underconfidence result. Keren's (1987) data showed that expert bridge players were slightly underconfident with respect to their proportion correct, whereas novice bridge players were largely overconfident. Thus, our model is able to explicate an apparently anomalous result. In fact, our model suggests that Keren's result is not anomalous, as it is exactly what MDM predicts!

Table 4
Cell Frequencies for the Three Levels of Experience Used in the Overconfidence Simulations

	Low experience		Medium experience		High experience		Objective probability
Data	H_1	H_2	H_1	H_2	H_1	H_2	$P(H_1\|D)$
D_1	—	—	19	1	57	3	.95
D_2	9	1	18	2	54	6	.90
D_3	8	2	16	4	48	12	.80
D_4	7	3	14	6	42	18	.70
D_5	6	4	12	8	36	24	.60
D_6	4	6	8	12	24	36	.40
D_7	3	7	6	14	18	42	.30
D_8	2	8	4	16	12	48	.20
D_9	1	9	2	18	6	54	.10
D_{10}	—	—	1	19	3	57	.05
Column totals	40	40	100	100	300	300	
Total instances	80		200		600		

Note. For the expertise simulations, we used symmetrical matrices with prior odds of 50:50. The objective probabilities varied across 10 response categories from .05 to .95 (.05, .1, .2, .3, .4, .6, .7, .8, .9, .95). Only 8 response categories were used in the low experience condition because it is impossible to simulate probabilities of .95 and .05 with fewer than 20 instances for each level. These instances are indicated by dashes. The matrices indicate the frequency of instances used to simulate the three different experience conditions. Note that the objective probabilities remain the same but cell frequencies increase. This was done to simulate the effect of experience on judgment. Three different values of L (.35, .55, and .75) were used to simulate different encoding conditions, and S_c was held constant at .70 for all simulations. H = hypothesis; D = data.

Simultaneous Over- and Underconfidence

Erev et al. (1994) showed that participants in several studies were simultaneously overconfident, with respect to proportion correct, and underconfident (conservative), with respect to veridical probabilities. MDM also predicts simultaneous over- and underconfidence. We have already shown underconfidence in the section on conservatism and overconfidence in the expertise sec-

[19] In a review of a draft of this article, Shanteau pointed out that experts may have several thousand relevant memory traces stored in memory. It is perhaps important to point out that the same functions can be obtained using 12,000 instances stored in memory and lower values of L. However, assuming a memory decay function, experts may have only a few hundred relevant memory traces readily accessible in memory. Therefore, we do not believe that our simulations unfairly model the memory representations of expert decision makers.

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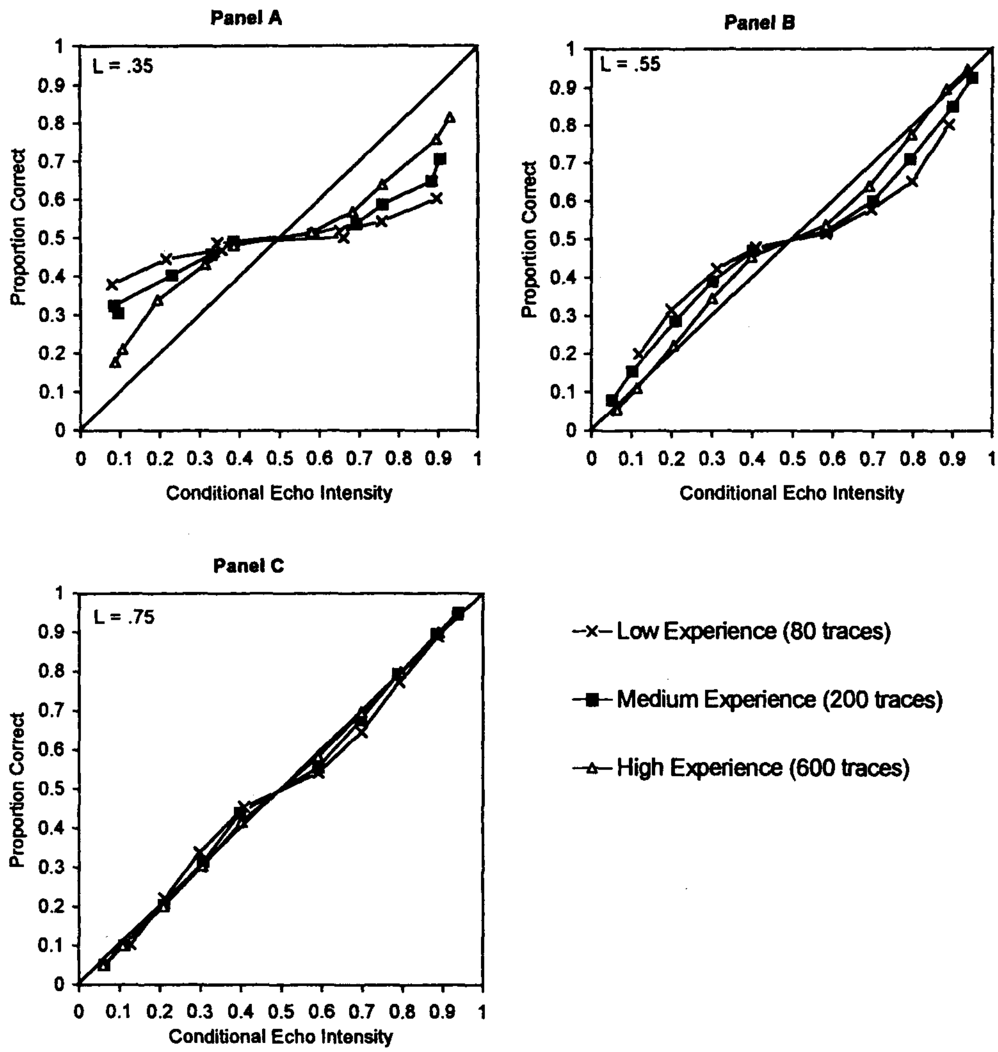

Figure 11. MINERVA–DM (DM = decision making) calibration curves for three different levels of experience (80, 200, or 600 traces stored in memory) across three encoding conditions (Panel A [L = .35], Panel B [L = .55], and Panel C [L = .75]). Conditional echo intensity transformed using Equation 7.

tion. However, it may be useful to show that MDM is simultaneously over- and underconfident using a single data set.

Simulation method. We used the conditional memory search L(**H**|**D**) to simulate conditional probability judgments. Secondary memory for this simulation consisted of the 2 × 10 matrix of instances presented in the middle column of Table 3. This resulted in a total of 200 instances stored in memory for each simulated participant. The parameter values used in this simulation were L = .30 and S_c = .50, and 1,000 participants were simulated.

Simulation results. The results of the simulation are presented in Figure 12. The O_p-S_p line plots the Bayesian probabilities against the model's conditional echo intensity. Notice that this line largely underestimates the Bayesian probabilities; conditional echo intensities are lower than the Bayesian probabilities. This is because S_c is relatively low, and as pointed out earlier, lower values of S_c result in higher false-alarm rates, which in turn lead to a reduction in I_c. This is the typical underconfidence (or conservative) result when veridical probabilities are compared with participants' estimates.

In contrast, the P(c)-S_p line plots proportion correct against the model's conditional echo intensities. In this case, MDM's estimates are substantially larger than the proportion correct, thus showing overconfidence. The overconfidence is the result of the random error produced by the computations on the minivectors. Decreasing L increases the error variance and therefore increases the probability that the model will choose the normatively incorrect alternative.

Summary of Expertise Simulations

The above simulations provide a memory-processes account for the development of expert judgment. Most notably, the model illustrates how task characteristics and experience both contribute to the development of good calibration. We were also able to show that MDM is consistent with simultaneous over- and underconfidence. Erev et al. (1994) argued that overconfidence is a product of the variability associated with people's response processes.

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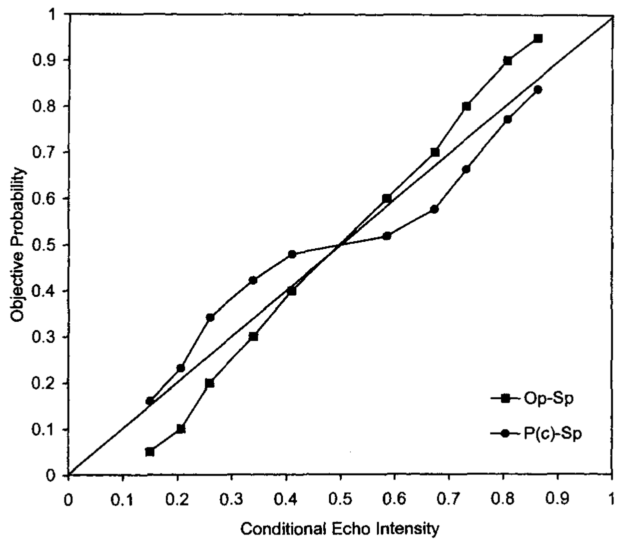

Figure 12. Simultaneous underconfidence–overconfidence. The O_p-S_p line plots Bayesian probability against subjective probability (the underconfidence curve). The P(c)-S_p line plots proportion correct against subjective probability (the overconfidence curve). Conditional echo intensity transformed using Equation 7.

MDM is consistent with the random error account but suggests that the locus of the random error is the process of memory retrieval. Overconfidence is predicted when people lack experience with the judgment domain and when the target stimulus (i.e., the to-be-retrieved stimulus event) has been poorly encoded in memory. Moreover, overconfidence is predicted to decrease both as the number of similar instances stored in memory increases and as the quality of encoding increases. Although several studies on expert probability judgment support these predictions (cf. Dougherty, 1998; Keren, 1987), further empirical tests are needed.

Conclusions

Our primary concern in this article was to present a comprehensive theory of likelihood judgments. In so doing, we have shown how MDM can account for a wide array of decision-making behavior and provide new predictions regarding the factors that affect judgments of likelihood. The model accounts for simple frequency judgments (e.g., Greene, 1988) and conditional likelihood judgments (e.g., DuCharme, 1970; Gettys, 1969) and provides an account of many of the major heuristics and biases including availability (Tversky & Kahneman, 1973), representativeness, base-rate neglect, the conjunction effect (Kahneman & Tversky, 1973; Tversky & Kahneman, 1983), the hindsight bias (Fischhoff, 1975), the simulation heuristic (Kahneman & Tversky, 1982), and the validity effect (Arkes et al., 1991; Hasher et al., 1977). Finally, the model is able to account for the simultaneous over- and underconfidence reported by Erev et al. (1994) and make several new predictions concerning the calibration of probability judgments. Perhaps the most impressive aspect of the theory is that all of the heuristics and biases simulated can be accounted for by a *single cognitive process*—memory—without the need for supplemental and complicating higher level processes.

A secondary goal of this article was to illustrate the utility of using mathematical memory models to describe decision behavior. Clearly, our ability to integrate many of the heuristics and biases using MDM demonstrates the power of the memory models approach; we do not believe this feat could have been accomplished using verbal theories and (or) analytic arguments. More important, our extension of MINERVA to model decision behavior has created a single model that simultaneously accounts for a wide range of memory and decision-making phenomena. The power demonstrated by MDM is not necessarily unique to the MINERVA architecture, as several of the other memory models have similar capabilities (for reviews see Clark & Gronlund, 1996; Raaijmakers & Shiffrin, 1992).

A unique property of MDM is that it is Bayesian in the limit (see Appendix A) and a degraded version of Bayes when $S_c < 1.0$. Interestingly, the first decision research of Edwards (1968) used a degraded version of Bayes's theorem as a model for human judgment. However, Edwards (rather arbitrarily) applied a power function to the likelihood ratio to degrade Bayes. Some 30 years later, we have developed a memory model of likelihoods that also is a degraded version of Bayes's theorem. The only differences are that judgments are assumed to be based on memory retrieval and the degradation is assumed to result from the natural fallibility and imprecision of memory. In addition to affording MDM enormous capability as a decision model, this Bayesian-like property makes it a natural bridge between past research using optimal models and future research that will draw increasingly on cognitive models.

Robustness to Changes in Parameters

One potential criticism of MDM and other mathematical memory models is that they are too powerful and can account for any pattern of results. This criticism implies that slight variations in the parameter values will produce large differences in a model's output, thus enabling the model to fit any anomalous result. This is certainly not the case with our model. The parameters in MDM, L and S_c, are used to fine-tune the fit between the model and the data. In fact, even relatively large adjustments in parameter values do not produce wildly different results that are inconsistent with existing data. Only under unrealistically low values of S_c (e.g., $S_c = 0.0$) and L (e.g., $L = .05$) does the model begin to break down, but even then MDM predicts the data ordinally. Appendix B presents sensitivity analyses of both L and S_c for conditional likelihood judgments. Varying L exaggerates the variability but does not produce large changes in the model output. Varying S_c changes the degree to which the model produces conservatism.

Our selection of model parameter values was constrained to values we believed were psychologically realistic and reflected differences between task characteristics and conditions. For example, we varied L to reflect the different encoding conditions inherent in Greene's (1988) data set. These simulations were performed with relatively high values of L; encoding is likely to be higher under laboratory conditions with relatively simple tasks than under more naturalistic conditions. Under naturalistic conditions, learning typically occurs unintentionally (Hasher & Zacks, 1979), thus resulting in lower values of L, such as was used in the DuCharme (1970) simulation. In a similar vein, we tried to select plausible values of S_c, although changing the value of S_c had little effect on the functional form of the model's output.

It is useful to point out that although the model has three parameters, we limited ourselves to using at most only two of the parameters at one time. For example, we used one parameter (L) to model frequency judgments and two parameters to model both conditional likelihood judgments (L and S_c) and the hindsight bias (L and G). We believe our use of three parameters is justified because decision behavior is often more complex than other basic memory phenomena and therefore calls for a more complex model. Furthermore, we believe that the parameters are justified psychologically. It is well known that memory is imperfect and that people often do not encode all of the details of an event (Ebbinghaus, 1885/1964). The encoding parameter, L, captures this phenomenon. It is also reasonable to assume that people set subjective criterion values to discriminate relevant traces from irrelevant traces (Tiberghien, Cauzinille, & Mathieu, 1979). The criterion parameter, S_c, allowed us to model this. Finally, it is reasonable to assume that memory probes for events that have actually occurred (as in hindsight) are more detailed than those probes that must be generated. We used G to model situations in which participants are asked to generate alternatives that did not occur but could have. Research in our lab (e.g., Dougherty et al., 1997) supports the usefulness of the G parameter. Dougherty et al. found that participants who constructed several causal scenarios judged the focal scenario as less likely. In the context of MDM, the generation of alternative scenarios may lead people to more fully specify the details of the memory probe for alternative causal scenarios, which in turn leads to a decrease in the likelihood (echo intensity) of the focal causal scenario. Similar explanation-based and imagination (e.g., Gregory et al., 1982; Koehler, 1991; Lord et al., 1984) tasks may also cause people to construct well-specified and more detailed probes and therefore reduce their confidence in alternative hypotheses.

Limitations

MDM is a memory model, and our efforts have been to model the memory mechanisms underlying judgments of frequency and probability. Consequently, MDM's theoretical application is limited to explaining how judgments arise from memory (cf. Hastie & Park, 1986), and it cannot account for the many higher order cognitive processes that may contribute to people's decision processes. Indeed, many likelihood estimation phenomena are undoubtly the result of multiple psychological processes. For example, availability biases have been shown to arise from memory processes as well as from metacognitive processes (see Schwarz et al., 1991). Likewise, there are several plausible explanations for the hindsight bias (e.g., rejudgment, motivated self-presentation, direct recall of the old belief, anchoring on the current belief and adjustment; see Hawkins & Hastie, 1990). Our explanation is most similar to only one of these—the rejudgment explanation.

As with many psychological phenomena, there are multiple factors that affect the accuracy of one's probability judgments, many of which are not incorporated into our theoretical framework. For example, it is now widely understood that the accuracy of participants' judgments depends on whether they judge frequencies or probabilities (e.g., Gigerenzer, 1991). We view the differences between probability and frequency estimation tasks as a manifestation of different *higher level* cognitive processes (i.e., those pertaining to how people combine frequentistic vs. probabilistic information), not as differences in memory processes (Gigerenzer & Hoffrage, 1995). Obviously, MDM does not speak to explanations other than those having to do with memory retrieval; they are outside the model's boundary conditions.

Relation to Other Decision Models

There are several similarities between the memory processes approach to decision making and the Brunswikian approach of Fiedler (BIAS; 1996) and Gigerenzer et al. (PMM; 1991; see also Gigerenzer & Goldstein, 1996). First, MDM, BIAS, and PMM all model decision making as a function of memory. Although the underlying assumptions of memory are not the same, all three theories assume that memory roughly corresponds to the environment. Second, all three theories assume that memory representation is frequentistic. Memory is assumed to consist of a database of stored instances, although the properties of these representations may differ in detail. In addition, both the Brunswikian (BIAS and PMM) theories and MDM model information loss. In the context of our model, information loss is due to the fallibility and imprecision of *memory* (as determined by L). However, in BIAS or PMM, information loss corresponds to the quality of the information in the environment (e.g., imperfect cue reliability; see Fiedler, 1996, for further explanation). Finally, all three theories are consistent with the idea of bounded rationality (Simon, 1956). Although judgment can show systematic biases, these biases arise from a cognitive system that often produces good judgments (i.e., following Bayes's theorem) without much cost of thinking.[20]

There are also several dissimilarities between our model and the Brunswikian theories in general and PMM in particular. MDM does not use higher level cognitive processes or algorithms. In contrast, PMM assumes that people have several "fast-and-frugal" cognitive algorithms or heuristics, such as the "take the best," "minimalist," and "take the last" algorithms, that can be used to make inferences or judgments (Gigerenzer & Goldstein, 1996).[21] MDM, on the other hand, assumes that judgments are based on familiarity (i.e., echo intensity) instead of these higher level cognitive algorithms. A second difference between our model and the Brunswikian models is that MDM specifies the memory retrieval processes that give rise to judgments and has a conditional memory search that enables it to account for conditional likelihood judgments. Although BIAS and PMM assume that judgments are based on memory, neither model specifies the memory retrieval processes that give rise to judgments. A final difference between our model and the Brunswikian models is the level of analysis at which the models operate. In terms of Marr's (1982) levels of an information-processing framework, MDM is at the representation–algorithm level, whereas BIAS and PMM are closer to the computational level. Both BIAS and PMM describe how the structure

[20] We thank Klaus Fielder for pointing out some of these similarities in his review.

[21] For the most part, these algorithms involve cue substitution and therefore can be thought of as involving higher level cognitive processes. A special case of the cue substitution process is what Gigerenzer and Goldstein (1996) referred to as the "recognition principle," which assumes that judgments are based on whichever alternative is recognized. However, the precise memory retrieval processes underlying the recognition principle are not specified by their model.

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of the *environment* can lead to good and poor judgments (without necessarily assuming specific memory processes), whereas MDM describes how the structure of *memory* and *memory retrieval processes* lead to both good and poor judgments (without necessarily assuming particular environmental structures). Thus, we do not see our model as an alternative to BIAS and PMM but rather as complementary.

Final Remarks

It is important to realize that our work on MDM does not replace, or render obsolete, the influential work of those who developed the heuristics and biases approach, particularly that of Kahneman, Tversky, and Fischhoff. In fact, their discoveries provided the grist for our theoretical mill, and without these discoveries, we would have had little to explain. Our work on MDM is largely integrative. It provides a coherent theoretical explanation of the cognitive processes underlying many of the heuristics and biases and other likelihood judgments. We also see our work as an attempt to bring traditional behavioral decision theory closer to mainstream cognitive psychology. Although decision making is, without doubt, a cognitive activity, decision theorists have been slow to explore cognitive models as viable models of decision behavior. Our use of MDM shows the utility of using memory models to study decision making. We believe that the widespread application of memory models (and other cognitive theories) to decision making will lead decision researchers to a greater understanding of, and a greater appreciation for, the cognitive processes underlying judgment and decision making.

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(Appendixes follow)

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Appendix A

Proof That MINERVA-DM Is Bayes's Theorem in the Limit

Assume $L = 1.0$, $S_c = 1.0$. The fundamental equation for similarity is

$$S_i = \frac{\sum_{j=1}^{N} \mathbf{p}_j \mathbf{T}_{i,j}}{N_i}$$

(which is Equation 1 in the text), where **p** is a probe vector used to probe memory. Memory traces, **t**, have the same structure as the probe vector, **p**. Secondary memory is a concatenation of traces into a matrix called **T**. **T** is an array of memory trace vectors, $\mathbf{T} = \mathbf{t}_1 \| \mathbf{t}_2 \| \mathbf{t}_3 \| \ldots \| \mathbf{t}_m$. Because $L = 1.0$, $\mathbf{t}_i \in \mathbf{T}$ such that $\mathbf{p} = \mathbf{t}_i$, for all i. Therefore, when we compare **p** and **t** there are only two cases that must be considered.

A: $\mathbf{p} = \mathbf{t}$ Here the probe vector and the trace vector are identical, each and every element in **p** is identical to **t**. This case arises because both the trace and the probe are virtual copies of the same event.

B: $\mathbf{p} <> \mathbf{t}$ Here **p** and **t** are randomly drawn vectors that have zero covariance. In this case the probe and trace are virtual copies of different events.

Case A: If $\mathbf{p} = \mathbf{t}$, then the similarity of **p** and **t** will be 1.0 because two identical events have perfect similarity.

Case B: Here we want to calculate what the similarity will be. First, the equiprobable sets of possible values that **p** and **t** can assume are $\mathbf{p} = \{1, 0, -1\}$; $\mathbf{t} = \{1, 0, -1\}$. The dot product of the two sets: $\mathbf{p} \cdot \mathbf{t} = \{1, 0, -1, 0, 0, 0, -1, 0, 1\}$. Each event in the dot product set has a probability of 1/9. Calculating the expected value:

$$E(S) = \{1/9(1) + 1/9(-1) + 1/9(-1) + 1/9(1)\}$$

$$E(S) = \{1/9(0)\} = 0$$

$$\text{Therefore, } E(S_i) = 0.$$

Thus, we find that in the limit, the similarity equation from MINERVA-DM is a simple event counter. It returns a 1.0 if $\mathbf{p} = \mathbf{t}$, and it returns the expectation $E(S_i) = 0$, if $\mathbf{p} <> \mathbf{t}$. Additional transformations are performed by combining Equations 2 and 3 from the section *Retrieval of Information From Memory*:

$$I = \sum_{i=1}^{M} S^3,$$

which sums the result over all memory traces processed. The cubes of 1 and 0 are 1 and 0, respectively, so I becomes a count of the events processed where $\mathbf{p} = \mathbf{t}$. The equation for conditional likelihoods is

$$I_c = \frac{I}{K_{S_i \geq S_c}}. \tag{A1}$$

Consider the numerator of Equation A1. This will be a count of the events where $\mathbf{p} = \mathbf{t}$. This is the number of traces for which the condition $S_i \geq S_c$ is met and $\mathbf{p} = \mathbf{t}$. The denominator of the equation is $K_{S_i \geq S_c}$, the number of cases where $S_i \geq S_c$. A little further reflection should convince the reader that Equation A1 is a valid conditional probability of the type $P(\mathbf{H}|\mathbf{D})$, that is,

$$\frac{f(S_i \geq S_c \cap \mathbf{p} = \mathbf{t})}{f(S_i \geq S_c)},$$

where f is the frequency of the event.

Proof That MINERVA-DM Is Modified Bayes's Theorem (MBT) When $L = 1.0$ and $S_c < 1.0$

From Gettys and Willke's (1969) Equation 7:

$$P(\mathbf{H}_\alpha|\boldsymbol{\omega}) = P(\mathbf{H}_\alpha) \sum_i \frac{P(\mathbf{D}_i|\boldsymbol{\omega})P(\mathbf{D}_i|\mathbf{H}_\alpha)}{P(\mathbf{D}_i)}.$$

The event ω is a hypothetical event that gives rise to a vector of probabilities, $P(\mathbf{D}_i|\boldsymbol{\omega})$. Bayes's theorem will be seen to be a special case where one of the $P(\mathbf{D}_i|\boldsymbol{\omega})$ is 1.0, and the rest have a probability of zero. In MBT we do not know for sure which $\mathbf{D}_i$ event occurred, but we do have probabilistic information, $P(\mathbf{D}_i|\boldsymbol{\omega})$, that we want to use. The above MBT equation takes a weighted average of the posterior probabilities, which allows a probabilistic input of the form $P(\mathbf{D}_i|\boldsymbol{\omega})$.

$P(\mathbf{H}_\alpha)$ is a constant under the summation $\therefore P(\mathbf{H}_\alpha|\boldsymbol{\omega})$

$$= \sum_i P(\mathbf{D}_i|\boldsymbol{\omega})_i \frac{P(\mathbf{H}_\alpha)P(\mathbf{D}_i|\mathbf{H}_\alpha)}{P(\mathbf{D}_i)}$$

$$P(\mathbf{H}_\alpha)P(\mathbf{D}|\mathbf{H}_\alpha) = P(\mathbf{H}_\alpha,\mathbf{D}_i)$$

$$P(\mathbf{H}_\alpha|\boldsymbol{\omega}) = \sum_i P(\mathbf{D}_\alpha|\boldsymbol{\omega}) \frac{P(\mathbf{H}_\alpha,\mathbf{D}_i)}{P(\mathbf{D}_i)}.$$

If $P(\mathbf{D}_i|\boldsymbol{\omega}) = 1.0$, then MBT becomes Bayes's theorem. However, if $S_c < 1.0$, $P(\mathbf{D}_i|\boldsymbol{\omega}) < 1.0$. In fact, one can think of a particular value of S_c when combined with the rest of the problem as producing a vector of $P(\mathbf{D}_i|\boldsymbol{\omega})$. Thus, we see when $L = 1.0$ and $S_c < 1.0$, MDM becomes MBT, in that it calculates a weighted average of posterior probabilities by weighing each $P(\mathbf{H}_\alpha, \mathbf{D}_i)/P(\mathbf{D}_i)$ posterior by the probability of obtaining that posterior, $P(\mathbf{D}_i|\boldsymbol{\omega})$.

Appendix B

Sensitivity Analysis for Conditional Likelihood Judgments in MINERVA-DM

Simulation Method

We used the conditional memory search $\mathbf{L}(\mathbf{H}|\mathbf{D})$ to simulate the sensitivity of the model to variations in L and S_c. L and S_c were varied independently in 10 simulations. In the first 5 simulations, S_c was held constant at .70, and L was varied across five levels in steps of .2 (L = 1.0, .80, .60, .40, .20). In the second 5 simulations, L was held constant at .70, and S_c was varied across five levels (S_c = 1.0, .80, .60, .40, .20). Each

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simulation consisted of 1,000 simulated participants, and secondary memory for each simulated participant consisted of the 2 × 8 matrix of instances presented in the low experience column of Table 4. For example, there were 9 instances corresponding to $\mathbf{H}_1\mathbf{D}_1$, 1 instance corresponding to $\mathbf{H}_2\mathbf{D}_1$, 8 instances corresponding to $\mathbf{H}_1\mathbf{D}_2$, 2 instances corresponding to $\mathbf{H}_2\mathbf{D}_2$, and so forth, for a total of 80 instances stored in memory.

Simulation Results

The results of the 10 simulations are presented in Figures B1 and B2. In both figures, the x-axis corresponds to the Bayesian posterior probabilities, P(**H**|**D**), and the y-axis corresponds to MDM's predicted conditional likelihood, L(**H**|**D**) (from Equation 7). As can be seen in Figure B1, the net effect of decreasing L (while holding S_c constant) is exaggerated variability in the model output. This is because fewer relevant traces pass S_c as L decreases (i.e., miss rate increases), and as the number of traces that pass the criterion decreases, the variability increases (the inverse of the law of large numbers). Notice that the variability becomes noticeable only under extremely low values of L (e.g., L = .20) and that there is no other systematic effect on the model output in these conditional likelihood tasks.

Figure B2 presents the effect of decreasing S_c while holding L constant. In this case, the model systematically becomes more conservative as S_c decreases. This is because as S_c decreases, more irrelevant traces are processed in the second-stage echo intensity calculations (i.e., false-alarm rate increases). Because irrelevant traces are, on average, less similar to the probe than relevant traces, the net effect is a lower echo intensity (and greater conservatism).

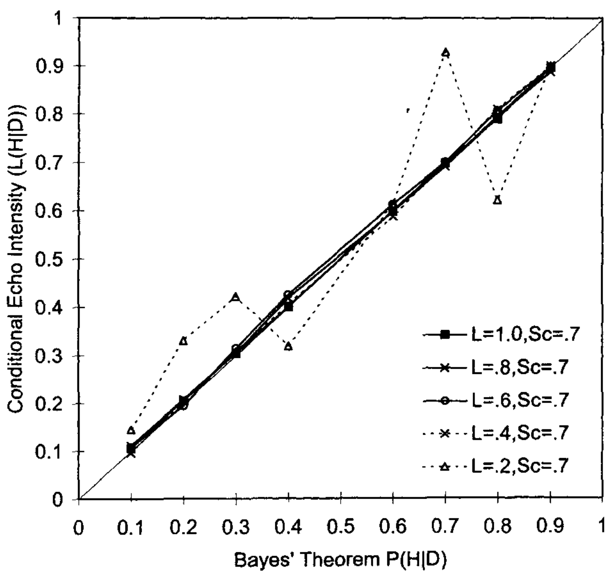

Figure B1. Effect of decreasing L, while holding S_c constant, in a conditional likelihood task. L = encoding parameter; S_c = similarity threshold criterion; H = hypothesis; D = data.

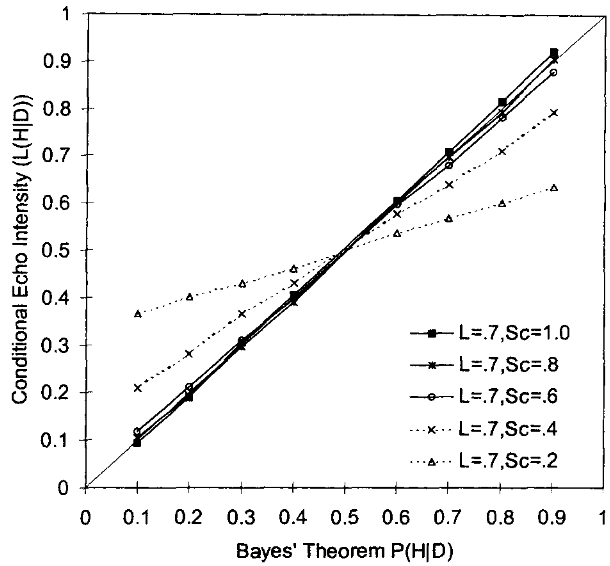

Figure B2. Effect of decreasing S_c, while holding L constant, in a conditional likelihood task. S_c = similarity threshold criterion; L = encoding parameter; H = hypothesis; D = data.

Received November 18, 1996
Revision received July 17, 1998
Accepted July 24, 1998 ■

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