Name: Solutions

ECE 3170
Test 3
Session 3

INSTRUCTIONS:

This is a closed-book, closed-notes test. You may use only the following items:

- **Three 3-by-5-inch cards of handwritten notes.**
- **A scientific calculator.**
- **A pencil or pen.**

The following items may not be used: additional note cards, additional notes, phones, computers, portable music players, other electronic devices, and other resources of any kind.

All forms of collaboration are prohibited during this test. You may communicate only with the person administering the test, and you may not receive or give aid of any kind.

Write your name in the space provided at the top of this page and on the answer sheet (next page).

Do not separate or remove any of the pages of this test. All pages must be returned to the instructor at the conclusion of the test.

Select the best answer for each problem by drawing a circle around the letter of the correct choice on the answer sheet. Circle only one letter for each problem, and do not make any additional marks on the answer sheet.

Each problem is worth five points. Credit for a problem will be awarded only if the correct letter is circled on the answer sheet. No credit will be awarded if more than one answer is selected.

You may work each problem in the space following the problem statement, but no credit will be given for this work. There should be adequate room for all work on the front of each page, but you also may write on the back or on extra sheets provided by the instructor.

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Answer Sheet

1)	a	b	c	d	e
2)	a	b	c	d	e
3)	a	b	c	d	e
4)	a	b	c	d	e
5)	a	b	c	d	e
6)	a	b	c	d	e
7)	a	b	c	d	e
8)	a	b	c	d	e
9)	a	b	c	d	e
10)	a	b	c	d	e
11)	a	b	c	d	e
12)	a	b	c	d	e
13)	a	b	c	d	e
14)	a	b	c	d	e
15)	a	b	c	d	e
16)	a	b	c	d	e
17)	a	b	c	d	e
18)	a	b	c	d	e
19)	a	b	c	d	e
20)	a	b	c	d	e

1) Random variables X and Y are statistically independent and uniform on [0,1]. Random variable Z is equal to the minimum of X and Y. Which of the following is equal to $F_Z(0.5)$?

a: 0.75
b: 0.50
c: 0.25
d: 0
e: None of the above

$F_X(x) = x,\ F_Y(y) = y$ for $x, y \in [0,1]$

$Z = \min(X,Y)$

$\Rightarrow F_Z(z) = F_X(z) + F_Y(z) - F_X(z)F_Y(z)$

$= z + z - (z)(z)$

$= 2z - z^2$

$\Rightarrow F_Z(0.5) = 2(0.5) - (0.5)^2 = 1 - 0.25 = 0.75$

2) A system is composed of 10 identical components connected in series. The time to failure of each component (in hours) is a random variable having the following probability distribution function:

$$F_X(x) = \begin{cases} 1-e^{-ax}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases},\quad \text{where } a = 10^{-5}.$$

The failure times of the components are statistically independent. Which of the following is closest to the probability that the system will fail within 5,000 hours of operation?

a: 0
b: 0.1
c: 0.2
d: 0.3
e: 0.4

Let $X_1, \ldots, X_{10}$ = component failure times and Z = system failure time

$Z = \min(X_1, \ldots, X_{10}) \Rightarrow F_Z(z) = 1 - [1 - F_X(z)]^{10}$

$\Rightarrow F_Z(z) = 1 - [\cancel{1}-(\cancel{1}-e^{-az})]^{10} = 1 - e^{-10az},\ a = 10^{-5}$

$P(Z \leq 5{,}000) = F_Z(5{,}000) = 1 - e^{-10(10^{-5})(5{,}000)} \cong 0.393$

3) The random variable X has the following probability density function:

$$f_X(x) = \begin{cases} 5x^4, & -1 \leq x \leq 0 \\ 0, & \text{otherwise} \end{cases},$$

Which of the following is equal to the mean of X?

a: −6/5
b: −1
c: −4/5
d: −3/4
e: None of the above

$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-1}^{0} x \cdot 5x^4\,dx$

$= \frac{5}{6} x^6 \Big|_{-1}^{0} = \frac{5}{6}(0-1) = -\frac{5}{6}$

4) The random variable X has a mean of −18 and a mean square of 500. Which of the following is equal to the variance of X?

a: 518
b: 482
c: 176
d: 50
e: None of the above

$Var(X) = E(X^2) - [E(X)]^2$

$= 500 - (-18)^2$

$= 500 - 324 = 176$

5) The random variable X is uniform on [0,1]. Which of the following is equal to the mean square of X?

a: 0
b: 1/3
c: 1/2
d: 1
e: None of the above

$$f_X(x) = 1, \quad 0 \le x \le 1$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$= \int_0^1 x^2(1)\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

6) The random variable X is the number obtained on a single toss of a fair, six-sided die. Which of the following is closest to the value of $E(X^2)$?

a: 15
b: 14
c: 13
d: 12
e: 11

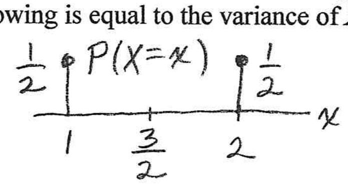
$$E(X^2) = \sum_{K=-\infty}^{\infty} x_K^2\, P(X = x_K) = \sum_{K=1}^{6} K^2 \left(\frac{1}{6}\right)$$

$$= \frac{1}{6}\left[1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\right]$$

$$= \frac{1}{6}\left[1 + 4 + 9 + 16 + 25 + 36\right] = \frac{91}{6} \cong 15.17$$

7) The random variable X may equal the values 1 and 2 with equal likelihood, and no other values are possible. Which of the following is equal to the variance of X?

a: 5/2
b: 3/2
c: 1
d: 1/2
e: None of the above



$\leftarrow E(X) = \frac{3}{2}$ by inspection

$$E(X^2) = \left(\tfrac{1}{2}\right)(1^2) + \left(\tfrac{1}{2}\right)(2^2) = \tfrac{1}{2} + 2 = \tfrac{5}{2}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{5}{2} - \left(\frac{3}{2}\right)^2 = \frac{10}{4} - \frac{9}{4} = \frac{1}{4}$$

8) The random variable X has the following characteristic function:

$$\varphi_X(\omega) = \frac{1}{2}e^{j\omega} + \frac{1}{2}$$

Which of the following is closest to the mean of X?

a: 1/8
b: 1/4
c: 3/8
d: 1/2
e: 5/8

$$\frac{d\phi_X(\omega)}{d\omega} = \frac{j}{2}e^{j\omega}$$

$$E(X) = (-j)^1 \frac{d\phi_X(\omega)}{d\omega}\Big|_{\omega=0} = (-\cancel{j})\left(\frac{\cancel{j}}{2}\cancel{e^{j0}}\right)$$

$$= \frac{1}{2}$$

9) The characteristic function of random variable X is $\varphi_X(\omega)$.

If $\left.\frac{d\varphi_X(\omega)}{d\omega}\right|_{\omega=0} = j$, which of the following is equal to the mean of X?

a: 0
b: j
c: 2
d: $j2$
e: None of the above

$$E(X) = (-j)^1 \left.\frac{d\phi_X(\omega)}{d\omega}\right|_{\omega=0}$$
$$= (-j)(j) = 1$$

10) The uncorrelated random variables X and Y have a correlation of 6. The mean value of X is 2. Which of the following is equal to the mean value of Y?

a: 0
b: 1
c: 2
d: 3
e: None of the above

Uncorrelated $\Rightarrow Cov(X,Y) = 0$

$$Cov(X,Y) = \underbrace{E(XY)}_{6} - \underbrace{E(X)}_{2}E(Y) = 0$$

$$\Rightarrow 6 - 2E(Y) = 0 \Rightarrow E(Y) = \frac{6}{2} = 3$$

11) The random variables X and Y are statistically independent. Which of the following must be true?

a: The correlation of X and Y is equal to zero.
b: X and Y are uncorrelated.
c: The correlation of X and Y is not equal to zero.
d: X and Y are orthogonal.
e: None of the above

Statistically independent $\Rightarrow E(XY) = E(X)E(Y)$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 0$$

$\Rightarrow$ Uncorrelated

12) The random variables X and Y are related by the equation $Y + 4 = 5X$. The variance of X is 3. Which of the following is equal to the variance of Y?

a: 75
b: 45
c: 19
d: 15
e: None of the above

$$Y = 5X - 4$$
$$\Rightarrow Var(Y) = (5)^2 Var(X)$$
$$= (25)(3) = 75$$

13) The random variables X and Y are orthogonal. Which of the following must be true?

a: The covariance of X and Y is equal to zero.
b: X and Y are uncorrelated.
c: X and Y are statistically independent.
d: The correlation coefficient of X and Y is equal to zero.
e: None of the above

Orthogonal $\Longleftrightarrow E(XY) = 0$

14) The random variables X and Y have a covariance of 2. The variance of X is 2, and the variance of Y is 8. Which of the following is the value of the correlation coefficient of X and Y?

a: 1/2
b: 1/4
c: 1/8
d: 0
e: None of the above

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{2}{\sqrt{2\cdot 8}} = \frac{2}{4} = \frac{1}{2}$$

15) The random variables X_1 and X_2 are jointly Gaussian with the following mean vector and covariance matrix:

$$\mu_X = \begin{bmatrix} 2 \\ -7 \end{bmatrix} \qquad \Sigma_X = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

The random variables Y_1 and Y_2 are formed from X_1 and X_2 as follows:
$Y_1 = 2X_1 - 4X_2 - 3$, and $Y_2 = 4X_1 + 5X_2$.
Which of the following is equal to the variance of Y_2?

a: 66
b: 36
c: 32
d: −32
e: None of the above

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 2 & -4 \\ 4 & 5 \end{bmatrix}}^{A}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \overbrace{\begin{bmatrix} -3 \\ 0 \end{bmatrix}}^{B}$$

$$\Sigma_y = A\Sigma_x A^T = \begin{bmatrix} 2 & -4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -8 \\ 4 & 10 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -4 & 5 \end{bmatrix} = \begin{bmatrix} 36 & -32 \\ -32 & 66 \end{bmatrix}$$

16) The random variables X_1 and X_2 are jointly Gaussian with the following mean vector and covariance matrix:

$$\mu_X = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \Sigma_X = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$$

The random variables Y_1 and Y_2 are formed from X_1 and X_2 as follows:
$Y_1 = X_1 + 2X_2 + 3$, and $Y_2 = 4X_1 - X_2$.
Which of the following is equal to the covariance of Y_1 and Y_2?

a: −14
b: 14
c: −19
d: 19
e: None of the above

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}}^{A}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \overbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}^{B}$$

$$\Sigma_y = A\Sigma_x A^T = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 10 \\ 14 & -14 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 19 & -14 \\ -14 & 70 \end{bmatrix}$$

17) The random process $X(t)$ is wide sense stationary and has autocorrelation function $R_x(t_1, t_2)$. If $R_x(0, 2) = -2$, which of the following is equal to the value of $R_x(-2, 0)$?

a: -1
b: -2
c: 1
d: 2
e: None of the above

$R_x(t_1, t_2)$ depends only on $\tau = t_2 - t_1$

$$\Rightarrow R_x\underbrace{(-2,0)}_{\tau=2} = R_x\underbrace{(0,2)}_{\tau=2} = -2$$

18) The ergodic random process $X(t)$ has the following autocorrelation function:

$$R_x(\tau) = 15e^{-2|\tau|} + 10$$

Which of the following is equal to the variance of $X(t)$?

a: 25
b: 15
c: 10
d: 5
e: None of the above

$$E[X^2(t)] = R_x(0) = 15e^{-2(0)} + 10 = 25$$

$$\{E[X(t)]\}^2 = \lim_{|\tau|\to\infty} R_x(\tau) = 15e^{-2(\infty)} + 10 = 10$$

$$\mathrm{Var}[X(t)] = E[X^2(t)] - \{E[X(t)]\}^2 = 25 - 10 = 15$$

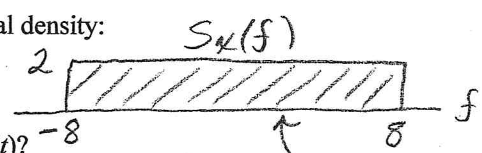
19) The random process $X(t)$ is wide sense stationary and has autocorrelation function $R_x(t_1, t_2)$. Which of the following must be true? Choose the best answer.

a: The mean of $X(t)$ is zero.
b: The mean of $X(t)$ is greater than zero.
c: $R_x(t_1, t_2) > 0$ for all t_1 and t_2.
d: Both b and c
e: None of the above

WSS $\Leftrightarrow$ 1) $E[X(t)]$ = constant and 2) $R_x(t_1, t_2)$ depends only on $\tau = t_2 - t_1$

20) The random process $X(t)$ has the following power spectral density:

$$S_X(f) = \begin{cases} 2, & |f| \le 8 \\ 0, & \text{otherwise} \end{cases}$$



Which of the following is equal to the mean square of $X(t)$?

a: 64
b: 40
c: 32
d: 16
e: None of the above

$$E[X^2(t)] = \int_{-\infty}^{\infty} S_x(f)\,df = \text{area}$$

$$= (16)(2) = 32$$