ECE 317
Chapter 6
Homework Solutions

6.2.1) Given mutually independent $X_1, \ldots, X_N$ and $f_{X_i}(x_i;\theta) \equiv \frac{1}{\theta} e^{-x_i/\theta}$, $\theta > 0$

and $x_i \geq 0$, $i = 1, \ldots, N$

Need to determine maximum likelihood estimator $\hat{\theta}$

Likelihood function:

$$f_{\underline{X}}(\underline{x};\theta) = \prod_{i=1}^{N} \left(\frac{1}{\theta} e^{-x_i/\theta} \right) = \frac{1}{\theta^N} e^{-\left(\sum_{i=1}^{N} \frac{x_i}{\theta}\right)}$$

$$\ln f_{\underline{X}}(\underline{x};\theta) = \ln\left(\frac{1}{\theta^N}\right) + \ln\left(e^{-\sum_{i=1}^{N} \frac{x_i}{\theta}}\right)$$

$$= -N \ln \theta - \sum_{i=1}^{N} \frac{x_i}{\theta}$$

$$= -N \ln \theta - \frac{1}{\theta} \sum_{i=1}^{N} x_i$$

$$\frac{\partial \ln f_{\underline{X}}(\underline{x};\theta)}{\partial \theta} = -\frac{N}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{N} x_i$$

The maximum likelihood estimate causes this derivative to equal 0:

$$\left[-\frac{N}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{N} x_i \right] \Bigg|_{\theta = \hat{\theta}} = 0$$

6.2.1) Cont.) $\Rightarrow \frac{1}{\hat{\theta}^2} \sum_{i=1}^{N} x_i = \frac{N}{\hat{\theta}}$

(Multiply both sides by $\frac{\hat{\theta}^2}{N}$):

$$\hat{\theta} = \frac{1}{N} \sum_{i=1}^{N} x_i \Rightarrow \boxed{\hat{\theta} = \frac{1}{N} \sum_{i=1}^{N} X_i}$$

Is $\hat{\theta}$ unbiased?

$$E(\hat{\theta}) = E\left(\frac{1}{N} \sum_{i=1}^{N} X_i\right) = \frac{1}{N} \sum_{i=1}^{N} E(X_i)$$

$$= \frac{1}{\cancel{N}} \cdot \cancel{N} E(X_i) = E(X_i)$$

$$E(X_i) = \int_{-\infty}^{\infty} x_i f_{X_i}(x_i;\theta)\, dx_i$$

$$= \int_{-\infty}^{\infty} x_i \cdot \frac{1}{\theta} e^{-x_i/\theta}\, dx_i$$

$$= \frac{1}{\theta} \int_{0}^{\infty} x_i e^{-x_i/\theta}\, dx_i$$

Note hint in problem statement:

$$\int_0^{\infty} y^m e^{-cy}\, dy = m!/c^{m+1}$$

$$\Rightarrow E(X_i) = \frac{1}{\theta} \cdot \frac{1!}{(1/\theta)^{1+1}} = \theta$$

$$\Rightarrow E(\hat{\theta}) = \theta \Rightarrow \boxed{\hat{\theta} \text{ is unbiased.}}$$

6.2.1) Cont.) Determine the Cramer-Rao bound for $\hat{\theta}$:

$$\frac{\partial^2 \ln f_{\underline{X}}(\underline{x};\theta)}{\partial \theta^2} = \frac{N}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{N} x_i$$

$$\Rightarrow E\left[\frac{\partial^2 \ln f_{\underline{X}}(\underline{x};\theta)}{\partial \theta^2}\right] = E\left[\frac{N}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{N} x_i\right]$$

$$= \frac{N}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{N} E(X_i)$$

$$= \frac{N}{\theta^2} - \frac{2}{\theta^3} N \cancel{E(X_i)}^{\theta} = \frac{N}{\theta^2} - \frac{2N}{\theta^2} = -\frac{N}{\theta^2}$$

$$\text{CR bound} = -\left\{E\left[\frac{\partial^2 \ln(f_{\underline{X}}(\underline{x};\theta))}{\partial \theta^2}\right]\right\}^{-1}$$

$$= \boxed{\frac{\theta^2}{N}}$$

Is $\hat{\theta}$ efficient?

$$\frac{\partial \ln[f_{\underline{X}}(\underline{x};\theta)]}{\partial \theta} = -\frac{N}{\theta} + \frac{1}{\theta^2}\underbrace{\sum_{i=1}^{N} x_i}$$

$$= -\frac{N}{\theta} + \frac{1}{\theta^2}\left(N\hat{\theta}\right) = \frac{N}{\theta^2}\left(\hat{\theta} - \theta\right)$$

$$= (\hat{\theta} - \theta)\, c(\theta), \text{ where } c(\theta) = \frac{N}{\theta^2}$$

$\Rightarrow$ $\boxed{\hat{\theta} \text{ is efficient.}}$

6.2.1) Note: can also verify that $\hat{\theta}$ is
Cont.) efficient by computing $Var(\hat{\theta})$:

$$Var(\hat{\theta}) = Var\left(\frac{1}{N}\sum_{i=1}^{N} X_i\right)$$

$$= \frac{1}{N^2}\sum_{i=1}^{N} Var(X_i)$$ (because X_i are statistically independent)

$$E(X_i^2) = \int_{-\infty}^{\infty} x_i^2 f_{X_i}(x_i;\theta)\, dx_i$$

$$= \int_{-\infty}^{\infty} x_i^2 \cdot \frac{1}{\theta} e^{-x_i/\theta}\, dx_i$$

$$= \frac{1}{\theta}\int_{0}^{\infty} x_i^2 e^{-x_i/\theta}\, dx_i = \frac{1}{\theta} \cdot \frac{2!}{(1/\theta)^{2+1}}$$

$$= 2\theta^2$$

$$Var(X_i) = E(X_i^2) - [E(X_i)]^2 = 2\theta^2 - \theta^2$$

$$= \theta^2$$

$$\Rightarrow Var(\hat{\theta}) = \frac{1}{N^2}\sum_{i=1}^{N}(\theta^2) = \frac{1}{N^2}(N\theta^2)$$

$$= \frac{\theta^2}{N}$$

Note: $Var(\hat{\theta})$ = the CR bound.

$\Rightarrow$ $\boxed{\hat{\theta} \text{ is efficient.}}$

6.2.3) Given mutually independent $X_1, \ldots, X_N$ and

$$f_{X_i}(x_i; b) = \frac{1}{\sqrt{b}} e^{-x_i/\sqrt{b}}, \quad b > 0$$

$$\text{and } x_i \geq 0, \quad i = 1, \ldots, N$$

Need to determine maximum likelihood estimator $\hat{b}$

Likelihood function:

$$f_{\underline{X}}(\underline{x}; b) = \prod_{i=1}^{N} \left(\frac{1}{\sqrt{b}} e^{-x_i/\sqrt{b}} \right) = \frac{1}{b^{N/2}} e^{-\left(\sum_{i=1}^{N} \frac{x_i}{\sqrt{b}}\right)}$$

$$\ln f_{\underline{X}}(\underline{x}; b) = \ln\left(\frac{1}{b^{N/2}}\right) + \ln\left(e^{-\sum_{i=1}^{N} \frac{x_i}{\sqrt{b}}}\right)$$

$$= -\frac{N}{2} \ln b - \sum_{i=1}^{N} \frac{x_i}{\sqrt{b}}$$

$$= -\frac{N}{2} \ln b - \frac{1}{\sqrt{b}} \sum_{i=1}^{N} x_i$$

$$\frac{\partial \ln f_{\underline{X}}(\underline{x}; b)}{\partial b} = -\frac{N}{2b} + \frac{1}{2 b^{3/2}} \sum_{i=1}^{N} x_i$$

The maximum likelihood estimate causes this derivative to equal 0:

$$\left[-\frac{N}{2b} + \frac{1}{2 b^{3/2}} \sum_{i=1}^{N} x_i \right] \Bigg|_{b = \hat{b}} = 0$$

6.2.3) Cont.) $\Rightarrow \frac{1}{\cancel{\beta}\,\hat{b}^{3/2}} \sum_{i=1}^{N} x_i = \frac{N}{\cancel{\beta}\,\hat{b}}$

(Multiply both sides by $\frac{\hat{b}^{3/2}}{N}$):

$$\hat{b}^{1/2} = \frac{1}{N} \sum_{i=1}^{N} x_i \Rightarrow \boxed{\hat{b} = \left(\frac{1}{N} \sum_{i=1}^{N} x_i\right)^2}$$

Is $\hat{b}$ unbiased?

$$E(\hat{b}) = E\left[\left(\frac{1}{N} \sum_{i=1}^{N} X_i\right)^2\right] = E\left[\left(\frac{1}{N}\right)^2 \left(\sum_{i=1}^{N} X_i\right)^2\right]$$

$$= \left(\frac{1}{N}\right)^2 E\left[\left(\sum_{i=1}^{N} X_i\right)^2\right]$$

$$= \frac{1}{N^2} E\left[\sum_{i=1}^{N} X_i^2 + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} X_i X_j\right]$$

$$= \frac{1}{N^2} \left[\sum_{i=1}^{N} E(X_i^2) + \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} E(X_i X_j)\right]$$

$$E(X_i) = \int_{-\infty}^{\infty} x_i f_{X_i}(x_i; b)\, dx_i$$

$$= \int_{-\infty}^{\infty} x_i \cdot \frac{1}{\sqrt{b}} e^{-x_i/\sqrt{b}}\, dx_i$$

$$= \frac{1}{\sqrt{b}} \int_{0}^{\infty} x_i e^{-x_i/\sqrt{b}}\, dx_i$$

Note hint in problem 6.2.1:

$$\int_{0}^{\infty} y^m e^{-cy}\, dy = m!/c^{m+1}$$

6.2.3) Cont.) $\Rightarrow E(X_i) = \frac{1}{\sqrt{b}} \cdot \frac{1!}{(1/\sqrt{b})^{1+1}} = \frac{b}{\sqrt{b}} = \sqrt{b}$

$$E(X_i^2) = \int_{-\infty}^{\infty} x_i^2 f_{X_i}(x_i; b)\, dx_i$$

$$= \int_{-\infty}^{\infty} x_i^2 \cdot \frac{1}{\sqrt{b}} e^{-x_i/\sqrt{b}}\, dx_i$$

$$= \frac{1}{\sqrt{b}} \int_{0}^{\infty} x_i^2 e^{-x_i/\sqrt{b}}\, dx_i$$

$$= \frac{1}{\sqrt{b}} \cdot \frac{2!}{(1/\sqrt{b})^{2+1}} = \frac{2 b^{3/2}}{b^{1/2}} = 2b$$

Because X_i and X_j are statistically independent for $i \neq j$,

$$E(X_i X_j) = E(X_i) E(X_j), \quad i \neq j$$

$\Rightarrow$ (from previous page)

$$E(\hat{b}) = \frac{1}{N^2}\left[\sum_{i=1}^{N} 2b + \sum_{i=1}^{N} \sum_{j=1}^{N} (\sqrt{b})(\sqrt{b})\right]$$

$$= \frac{1}{N^2}\left[N(2b) + (N^2 - N)b\right]$$

$$= \frac{1}{N^2}\left[2Nb + N^2 b - Nb\right]$$

$$= \frac{1}{N^2}\left[N^2 + N\right] b = \left(\frac{N+1}{N}\right) b \neq b$$

$\Rightarrow$ $\boxed{\hat{b} \text{ is biased.}}$

6.3.1) For the estimator $\hat{\theta}_N = \frac{1}{2N}\sum_{i=1}^{N} X_i$,

need to determine sequential form:

$$\hat{\theta}_N = \frac{1}{2N}\sum_{i=1}^{N} X_i = \frac{1}{2N}\left(\sum_{i=1}^{N-1} X_i + X_N\right)$$

$$= \frac{1}{2N}\left[\underbrace{2(N-1)}_{2N-2}\hat{\theta}_{N-1} + X_N\right]$$

$$= \frac{2N-2}{2N}\hat{\theta}_{N-1} + \frac{1}{2N}X_N$$

$$= \hat{\theta}_{N-1} - \frac{1}{N}\hat{\theta}_{N-1} + \frac{1}{2N}X_N$$

$$\Longrightarrow \boxed{\hat{\theta}_N = \hat{\theta}_{N-1} + \frac{1}{N}\left(\frac{X_N}{2} - \hat{\theta}_{N-1}\right)}$$

6.3.3) Given $\hat{W}_N = \hat{W}_{N-1} + \frac{1}{N-1}\left[\frac{N-1}{N}\underbrace{(X_N - \hat{\mu}_{N-1})}^2 - \hat{W}_{N-1}\right]$,

$N = 2, 3, \ldots$ and $\hat{W}_0 = \hat{W}_1 = 0$

Need $\hat{W}_N$ in terms of $\hat{W}_{N-1}$ and μ_N, i.e., in form $\hat{W}_N = \hat{W}_{N-1} + a[b(X_N - \hat{\mu}_N)^2 - c]$

From Eq. (6.3.2),

$$X_N - \hat{\mu}_N = \frac{N-1}{N}(X_N - \hat{\mu}_{N-1})$$

$$\Rightarrow X_N - \hat{\mu}_{N-1} = \underbrace{\frac{N}{N-1}(X_N - \hat{\mu}_N)}$$

Substitute

$$\Rightarrow \hat{W}_N = \hat{W}_{N-1} + \frac{1}{N-1}\left[\left(\frac{N-1}{N}\right)\left(\frac{N}{N-1}[X_N - \hat{\mu}_N]\right)^2 - \hat{W}_{N-1}\right]$$

$$= \hat{W}_{N-1} + \frac{1}{N-1}\left[\cancel{\left(\frac{N-1}{N}\right)}\frac{N^{\cancel{2}}}{(N-1)^{\cancel{2}}}(X_N - \hat{\mu}_N)^2 - \hat{W}_{N-1}\right]$$

$$\Rightarrow \boxed{\hat{W}_N = \hat{W}_{N-1} + \frac{1}{N-1}\left[\frac{N}{N-1}(X_N - \hat{\mu}_N)^2 - \hat{W}_{N-1}\right], \quad N = 2, 3, \ldots \text{ and } \hat{W}_0 = \hat{W}_1 = 0}$$

6.3.5) Need sequential estimators, $\hat{\mu}_N$ and $\hat{V}_N$ for the following 10 numbers:

0.236, −1.337, −0.724, 0.347, −0.699
0.072, 0.153, −0.800, −0.857, −1.504

Using $\hat{\mu}_N = \hat{\mu}_{N-1} + \frac{1}{N}(X_N - \hat{\mu}_{N-1})$

and $\hat{V}_N = \hat{V}_{N-1} + \frac{1}{N}\left[\frac{N-1}{N}(X_N - \hat{\mu}_{N-1})^2 - \hat{V}_{N-1}\right]$,

$\hat{\mu}_0 = \hat{V}_0 = 0$

$\hat{\mu}_1 = 0 + 1/1(0.236 - 0) = 0.236$
$\hat{V}_1 = 0 + 1/1[0/1(0.236-0)^2 - 0] = 0$

$\hat{\mu}_2 = 0.236 + 1/2(-1.337 - 0.236) = -0.551$
$\hat{V}_2 = 0 + 1/2[1/2(-1.337 - 0.236)^2 - 0] = 0.619$

$\hat{\mu}_3 = -0.551 + 1/3(-0.724 + 0.551) = -0.609$
$\hat{V}_3 = 0.619 + 1/3[2/3(-0.724 + 0.551)^2 - 0.619] = 0.419$

$\hat{\mu}_4 = -0.609 + 1/4(0.347 + 0.609) = -0.370$
$\hat{V}_4 = 0.419 + 1/4[3/4(0.347 + 0.609)^2 - 0.419] = 0.486$

$\hat{\mu}_5 = -0.370 + 1/5(-0.699 + 0.370) = -0.436$
$\hat{V}_5 = 0.486 + 1/5[4/5(-0.699 + 0.370)^2 - 0.486] = 0.406$

$\hat{\mu}_6 = -0.436 + 1/6(0.072 + 0.436) = -0.351$
$\hat{V}_6 = 0.406 + 1/6[5/6(0.072 + 0.436)^2 - 0.406] = 0.374$

6.35) Cont.)

$$\hat{\mu}_7 = -0.351 + 1/7(0.153 + 0.351) = -0.279$$

$$\hat{V}_7 = 0.374 + 1/7\left[6/7(0.153+0.351)^2 - 0.374\right] = 0.352$$

$$\hat{\mu}_8 = -0.279 + 1/8(-0.800 + 0.279) = -0.344$$

$$\hat{V}_8 = 0.352 + 1/8\left[7/8(-0.800+0.279)^2 - 0.352\right] = 0.338$$

$$\hat{\mu}_9 = -0.344 + 1/9(-0.857 + 0.344) = -0.401$$

$$\hat{V}_9 = 0.338 + 1/9\left[8/9(-0.857+0.344)^2 - 0.338\right] = 0.326$$

$$\hat{\mu}_{10} = -0.401 + 1/10(-1.504 + 0.401) = -0.511$$

$$\hat{V}_{10} = 0.326 + 1/10\left[9/10(-1.504+0.401)^2 - 0.326\right] = 0.403$$