Name: Solutions

ECE 3170
Test 2
Session 2

INSTRUCTIONS:

This is a closed-book, closed-notes test. You may use only the following items:

- **Two 3-by-5-inch cards of handwritten notes.**
- **A scientific calculator.**
- **A pencil or pen.**

The following items may not be used: additional note cards, additional notes, phones, computers, portable music players, other electronic devices, and other resources of any kind.

All forms of collaboration are prohibited during this test. You may communicate only with the person administering the test, and you may not receive or give aid of any kind.

Write your name in the space provided at the top of this page and on the answer sheet (next page).

Do not separate or remove any of the pages of this test. All pages must be returned to the instructor at the conclusion of the test.

Select the best answer for each problem by drawing a circle around the letter of the correct choice on the answer sheet. Circle only one letter for each problem, and do not make any additional marks on the answer sheet.

Each problem is worth five points. Credit for a problem will be awarded only if the correct letter is circled on the answer sheet. No credit will be awarded if more than one answer is selected.

You may work each problem in the space following the problem statement, but no credit will be given for this work. There should be adequate room for all work on the front of each page, but you also may write on the back or on extra sheets provided by the instructor.

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Answer Sheet

1)	a	b	c	d	e
2)	a	b	c	d	e
3)	a	b	c	d	e
4)	a	b	c	d	e
5)	a	b	c	d	e
6)	a	b	c	d	e
7)	a	b	c	d	e
8)	a	b	c	d	e
9)	a	b	c	d	e
10)	a	b	c	d	e
11)	a	b	c	d	e
12)	a	b	c	d	e
13)	a	b	c	d	e
14)	a	b	c	d	e
15)	a	b	c	d	e
16)	a	b	c	d	e
17)	a	b	c	d	e
18)	a	b	c	d	e
19)	a	b	c	d	e
20)	a	b	c	d	e

1) A 64-bit word in a computer's memory contains a random binary integer. All bits have the same probability of being equal to 1, and each bit is statistically independent of all other bits. The random variable X is defined to be the total number of bits that are equal to 1. Which of the following types of random variables best describes X?

a: Poisson
b: Gaussian
c: geometric
d: Bernoulli
e: None of the above

X = number of "successes" in a set of N $(=64)$ independent, identical Bernoulli trials.

$\Rightarrow X$ is a binomial R.V.

2) A fair, six-sided die is rolled six times. Which of the following is closest to the probability that **exactly one** "2" is obtained?

a: 0.4
b: 0.3
c: 0.2
d: 0.1
e: 0

Let $X = \#$ "2"s in 6 tosses

$\Rightarrow X$ is binomial with $N=6$ and $p = 1/6$

$$\Rightarrow P(X=i) = C_i^N p^i (1-p)^{N-i}$$

$$\Rightarrow P(X=1) = C_1^6 \left(\tfrac{1}{6}\right)^1 \left(\tfrac{5}{6}\right)^5 \cong 0.4019$$

3) Interrupts to a certain microprocessor occur at random times. The length of time between successive interrupts is an exponential random variable whose average is 10 ms. Which of the following is closest to the probability that the time between two successive interrupts will be less than or equal to 5 ms?

a: 0
b: 0.1
c: 0.2
d: 0.3
e: 0.4

Let X = time between interrupts (ms)

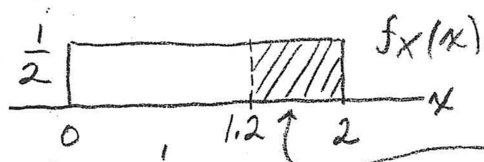
$$\Rightarrow F_X(x) = 1 - e^{-ax} \text{ for } x \geq 0$$

$$a = \frac{\text{ave\# interrupts}}{\text{ms}} = \frac{1}{10}$$

$$\Rightarrow P(X \leq 5) = F_X(5) = 1 - e^{-\left(\frac{1}{10}\right)(5)} = 1 - e^{-0.5} \cong 0.393$$

4) The random variable X is uniformly distributed from 0 to 2. Which of the following is closest to $P(X > 1.2)$?

a: 0
b: 0.1
c: 0.2
d: 0.3
e: 0.4



$$P(X>1.2) = \int_{1.2}^{2} f_X(x)\,dx = \tfrac{1}{2}(0.8) = 0.4$$

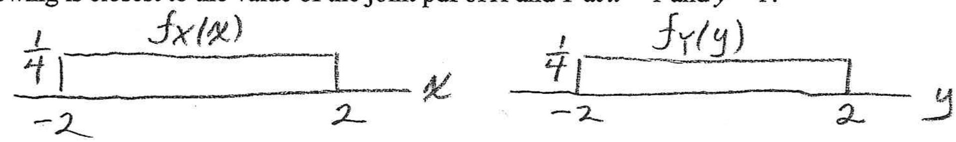
5) An integer from 1 to 10 is chosen at random, and all possible values are equally likely. The random variable X is defined to be the chosen integer. Which of the following types of random variables best describes X?

a: discrete uniform
b: Rayleigh
c: Poisson
d: Bernoulli
e: binomial

Possible values : 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
(equally spaced, discrete, equal probabilities)

6) Statistically independent random variables X and Y are uniformly distributed from –2 to +2. Which of the following is closest to the value of the joint pdf of X and Y at $x = 1$ and $y = 1$?

a: 0.5
b: 0.4
c: 0.3
d: 0.2
e: 0.1



X, Y are S.I. $\Rightarrow f_{XY}(x,y) = f_X(x) f_Y(y)$

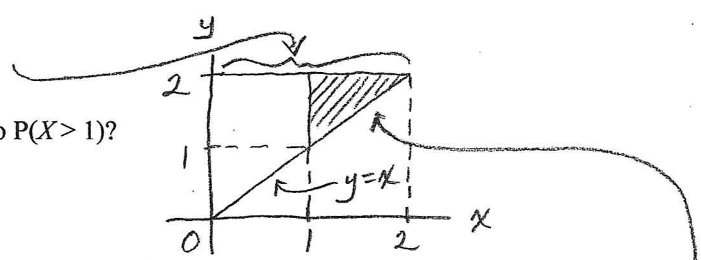
$\Rightarrow f_{XY}(1,1) = f_X(1) f_Y(1) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16} = 0.0625$

7) The joint probability density function of random variables X and Y is

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2}, & 0 \le x < y \le 2 \\ 0, & \text{otherwise} \end{cases}.$$

Which of the following is closest to P(X> 1)?

a: 0.8
b: 0.6
c: 0.4
d: 0.2
e: 0



$P(X>1) = \iint f_{XY}(x,y)\,dx\,dy$ over triangle

$= \left(\frac{1}{2}\right)(\text{area}) = \left(\frac{1}{2}\right)\left(\frac{1}{2} \cdot 1 \cdot 1\right) = \frac{1}{4} = 0.25$

8) For the variables in problem 7, which of the following is closest to the value of the marginal probability distribution function of Y evaluated at $y = 2$?

a: 1.0
b: 0.8
c: 0.6
d: 0.4
e: 0.2

$F_Y(y) = P(Y \le y)$

$\Rightarrow F_Y(2) = P(Y \le 2) = 1$

(Note $f_{XY}(x,y) = 0$ for $y > 2$)

9) The joint probability distribution function of random variables X and Y is
$F_{XY}(x,y) = \left(1-e^{-x^2/2}\right)\left(1-e^{-y}\right)u(x)u(y)$.

Which of the following is closest to $P(X \le 1, Y \le 2)$?

a: 0
b: 0.1
c: 0.2
d: 0.3
e: 0.4

$$P(X \le 1, Y \le 2) = F_{XY}(1,2)$$
$$= \left(1-e^{-(1)^2/2}\right)\left(1-e^{-2}\right) \cong 0.3402$$

10) If $f_{XY}(5, 6) = 4$ and $f_Y(6) = 2$, which of the following is equal to $f_{X|Y}(5 \mid 6)$?

a: 1
b: 2
c: 6
d: 8
e: None of the above

$$f_{XY}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$\Rightarrow f_{XY}(5|6) = \frac{f_{XY}(5,6)}{f_Y(6)} = \frac{4}{2} = 2$$

11) X and Y are standard Gaussian random variables whose correlation coefficient (ρ) is equal to 1. If the value of X has been found to be 2, which of the following is true?

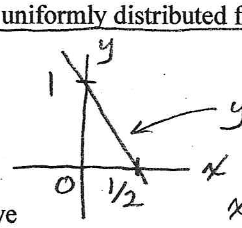
a: The value of Y must be 1.
b: The value of Y must be 2.
c: The value of Y may be any real number, but it probably is greater than zero.
d: The value of Y may be either 2 or –2, but no other values are possible.
e: None of the above

$\rho = 1 \Rightarrow X$ and Y are perfectly correlated
X and Y are standard $\Rightarrow \mu = 0, \sigma = 1$
$\Rightarrow X = Y$ so $X = 2 \Rightarrow Y = 2$

12) The random variable X is uniformly distributed from 0 to 1. If $Y = -2X + 1$, which of the following is equal to $f_Y(0)$?

$\Rightarrow f_X(x) = 1, \; 0 \le x \le 1$

a: –1/2
b: –1/4
c: 1/4
d: 1
e: None of the above



$y = -2x + 1 = g(x)$
(monotonically decreasing)

$$x = \frac{y-1}{-2} = \frac{1-y}{2} = h(y) \Rightarrow \frac{dh(y)}{dy} = -\frac{1}{2}$$

$$f_Y(y) = f_X[h(y)]\left|\frac{dh(y)}{dy}\right| = (1)\left|-\frac{1}{2}\right| = \frac{1}{2}, \quad -1 \le y \le 1$$

$$\Rightarrow f_Y(0) = \frac{1}{2}$$

13) The random variable X is uniformly distributed from 0 to 1. If $Y = 3X + 1$, which of the following is equal to $F_Y(2.5)$?

$\Rightarrow F_X(x) = x, \; 0 \le x \le 1$

a: 0

b: 1/3

c: 1/2

d: 2/3

e: None of the above

$y = 3x + 1 = g(x) \leftarrow$ *Monotonically increasing*

$x = \frac{y-1}{3} = h(y)$

$F_Y(y) = F_X[h(y)] = x|_{x=h(y)} = h(y) = \frac{y-1}{3}$

$\Rightarrow F_Y(2.5) = \frac{2.5-1}{3} = \frac{1.5}{3} = \frac{3/2}{3} = \frac{1}{2}$

14) X is a **discrete** uniform random variable that may equal any value in the set $\underbrace{\{1, 2, 3, 4, 5\}}$. If $Y = 3X + 1$, which of the following is equal to $P(Y = 4)$?

Each has probability $= \frac{1}{5}$

a: 1/5

b: 1/4

c: 1/3

d: 1/2

e: None of the above

$y = 3x + 1 = g(x) \leftarrow$ 1-to-1

$\Rightarrow P(Y=y) = P[X = h(y)]$

$x = \frac{y-1}{3} = h(y)$

$\Rightarrow P(Y=4) = P(X = \frac{4-1}{3}) = P(X=1) = \frac{1}{5}$

15) Random variables Y_1 and Y_2 are derived from random variables X_1 and X_2 as follows:
$Y_1 = X_1 + X_2, \quad Y_2 = X_1 - X_2$
Which of the following is equal to $\mathbf{J}_{h_1h_2}(y_1, y_2)$?

$y_1 = x_1 + x_2 = g_1(x_1, x_2)$

$y_2 = x_1 - x_2 = g_2(x_1, x_2)$

a: −1/2

b: 1/2

c: 1/4

d: −1/4

e: None of the above

$\Rightarrow x_1 = \frac{y_1 + y_2}{2} = h_1(y_1, y_2)$

$x_2 = \frac{y_1 - y_2}{2} = h_2(y_1, y_2)$

$$J_{h_1h_2}(y_1, y_2) = \begin{vmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

16) Random variables X_1 and X_2 are statistically independent and uniform on [0,1]. Random variables Y_1 and Y_2 are derived from random variables X_1 and X_2 as follows:
$Y_1 = X_1 + X_2, \quad Y_2 = 2X_1 + X_2$
Which of the following is equal to the maximum possible value of Y_2?

a: 3

b: 2

c: 1

d: 0

e: None of the above

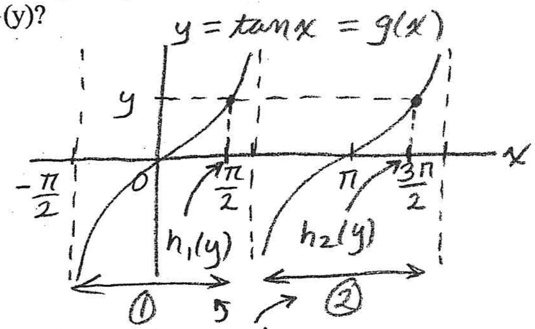
$Y_2 = 2X_1 + X_2$

$Y_2(max) = 2X_1(max) + X_2(max)$

$= 2(1) + 1 = 3$

17) The random variable X is an angle (in units of radians) that is uniformly distributed from $-\pi/2$ to $+3\pi/2$. If $Y = \tan(X)$, which of the following is equal to the number of different inverse functions $h(y)$ required in the computation of $f_Y(y)$?

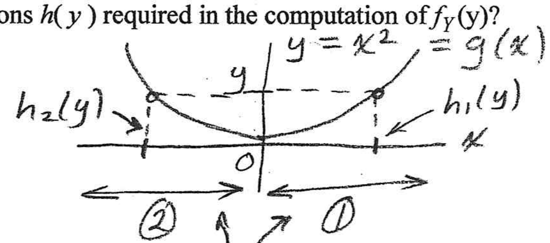
a: 4
b: 3
c: 2
d: 1
e: None of the above



$g(x)$ is 1-to-1 on 2 intervals
$\Rightarrow$ 2 functions $h(y)$

18) X is a standard, Gaussian random variable. If $Y = X^2$, which of the following is equal to the number of different inverse functions $h(y)$ required in the computation of $f_Y(y)$?

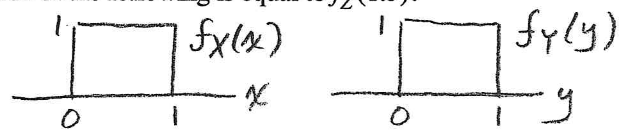
a: 5
b: 4
c: 3
d: 2
e: None of the above



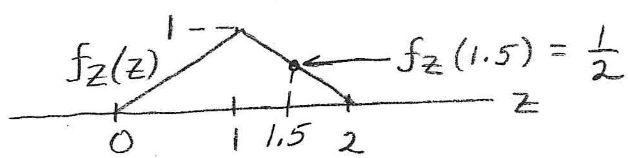
$g(x)$ is 1-to-1 on 2 intervals
$\Rightarrow$ 2 functions $h(y)$

19) Random variables X and Y are statistically independent and uniform on [0,1]. Random variable Z is equal to the sum of X and Y. Which of the following is equal to $f_Z(1.5)$?

a: 3/4
b: 1/2
c: 1/4
d: 1/8
e: None of the above



$f_Z(z) = f_X(x) * f_Y(y)$:



20) Random variables X and Y are statistically independent, and random variable Z is equal to the sum of X and Y. Which of the following is equal to the probability density function of Z?

a: The difference of the pdf of X and the pdf of Y.
b: The sum of the pdf of X and the pdf of Y.
c: The product of the pdf of X and the pdf of Y.
d: The quotient of the pdf of X and the pdf of Y.
e: None of the above

$f_Z(z)$ = the convolution of $f_X(x)$ and $f_Y(y)$.