Name: Solutions

ECE 3170
Test 2
Session 4

INSTRUCTIONS:

This is a closed-book, closed-notes test. You may use only the following items:

- **Two 3-by-5-inch cards of handwritten notes.**
- **A scientific calculator.**
- **A pencil or pen.**

The following items may not be used: additional note cards, additional notes, phones, computers, portable music players, other electronic devices, and other resources of any kind.

All forms of collaboration are prohibited during this test. You may communicate only with the person administering the test, and you may not receive or give aid of any kind.

Write your name in the space provided at the top of this page and on the answer sheet (next page).

Do not separate or remove any of the pages of this test. All pages must be returned to the instructor at the conclusion of the test.

Select the best answer for each problem by drawing a circle around the letter of the correct choice on the answer sheet. Circle only one letter for each problem, and do not make any additional marks on the answer sheet.

Each problem is worth five points. Credit for a problem will be awarded only if the correct letter is circled on the answer sheet. No credit will be awarded if more than one answer is selected.

You may work each problem in the space following the problem statement, but no credit will be given for this work. There should be adequate room for all work on the front of each page, but you also may write on the back or on extra sheets provided by the instructor.

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Answer Sheet

1)	a	b	c	d	e
2)	a	b	c	d	e
3)	a	b	c	d	e
4)	a	b	c	d	e
5)	a	b	c	d	e
6)	a	b	c	d	e
7)	a	b	c	d	e
8)	a	b	c	d	e
9)	a	b	c	d	e
10)	a	b	c	d	e
11)	a	b	c	d	e
12)	a	b	c	d	e
13)	a	b	c	d	e
14)	a	b	c	d	e
15)	a	b	c	d	e
16)	a	b	c	d	e
17)	a	b	c	d	e
18)	a	b	c	d	e
19)	a	b	c	d	e
20)	a	b	c	d	e

1) A fair coin is tossed until the first heads appears, and the random variable X is defined to be the total number of tosses made. Which of the following types of random variables best describes X?

a: binomial
b: Poisson
c: exponential
d: Bernoulli
e: None of the above

X = number of trials required until first "success" in a series of Bernoulli trials.

$\Rightarrow X$ is a geometric R.V.

2) The number of hits on a certain website during any interval of time is a Poisson random variable. On average, 1 hit occurs every minute. Which of the following is closest to the probability that **exactly** 3 hits will be made during an interval of 3 minutes?

a: 0
b: 0.2
c: 0.4
d: 0.6
e: 0.8

Let X = # hits in 3 minutes

Ave. # hits in 3 min. $= \frac{1 \text{ hit}}{\text{min.}} \times 3 \text{ min.} = 3$

$$\Rightarrow P(X=i) = \frac{a^i e^{-a}}{i!}, \text{ where } a = 3$$

$$\Rightarrow P(X=3) = \frac{3^3 e^{-3}}{3!} = \frac{27 e^{-3}}{3 \cdot 2} \cong 0.224$$

3) A fair coin is tossed eight times. Which of the following is closest to the probability that **exactly one** heads is obtained?

a: 0.01
b: 0.02
c: 0.03
d: 0.04
e: 0.05

Let X = # heads in 8 tosses

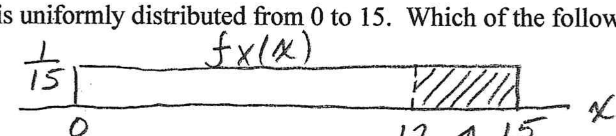
$\Rightarrow X$ is a binomial R.V. with $N = 8$ and $p = \frac{1}{2} = P(\text{heads})$

$$\Rightarrow P(X=i) = C_i^N p^i (1-p)^{N-i}$$

$$\Rightarrow P(X=1) = \cancel{C_1^8}^{\,8} \left(\tfrac{1}{2}\right)^1 \left(\tfrac{1}{2}\right)^7 = 8\left(\tfrac{1}{8}\right)^8 = 0.03125$$

4) The random variable X is uniformly distributed from 0 to 15. Which of the following is closest to $P(X > 12)$?

a: 0.8
b: 0.6
c: 0.4
d: 0.2
e: 0



$$P(X>12) = \int_{12}^{15} f_X(x)\,dx = \left(\tfrac{1}{15}\right)(3) = 0.2$$

5) A fair, six-sided die is rolled five times, and the random variable X is defined to be the number of sixes obtained. Which of the following types of random variables best describes X?

a: geometric
b: Rayleigh
c: Poisson
d: binomial
e: discrete uniform

$X =$ number of "successes" in a set of N $(=5)$ Bernoulli trials.

$\Rightarrow$ binomial

6) Statistically independent Bernoulli random variables X and Y have identical statistics, with $P(X=1) = P(Y=1) = 0.8$. Which of the following is closest to the value of the joint pmf of X and Y at $x = 0$ and $y = 1$?

a: 0.05
b: 0.10
c: 0.15
d: 0.20
e: 0.25

X and Y are S. I.

$\Rightarrow P(X=x, Y=y) = P(X=x)\,P(Y=y)$

$\Rightarrow P(X=0, Y=1) = P(X=0)\,P(Y=1)$

$= (0.2)(0.8) = 0.16$

7) The joint probability distribution function of random variables X and Y is

$F_{XY}(x,y) = \left(1-e^{-x^2/2}\right)\left(1-e^{-y}\right)u(x)u(y)$.

Which of the following is closest to $P(X \le 1, Y \le 2)$?

a: 0.3
b: 0.4
c: 0.5
d: 0.6
e: 0.7

$P(X \le 1, Y \le 2) = F_{XY}(1,2)$

$= \left(1-e^{-(1)^2/2}\right)\left(1-e^{-2}\right) \cong 0.3402$

8) For the variables in problem 7, which of the following is closest to the value of the marginal probability distribution function of X evaluated at $x = 2$?

a: 1.2
b: 0.9
c: 0.6
d: 0.3
e: 0

$F_X(x) = F_{XY}(x,\infty) = \left(1-e^{-x^2/2}\right)\left(1-\cancel{e^{-\infty}}^{\,0}\right)u(x)\,\cancel{u(\infty)}^{\,1}$

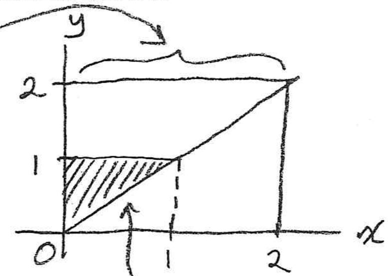
$= \left(1-e^{-x^2/2}\right)u(x)$

$\Rightarrow F_X(2) = 1 - e^{-(2)^2/2}\,\cancel{u(2)}^{\,1} = 1 - e^{-2}$

$\cong 0.8647$

9) The joint probability density function of random variables X and Y is

$$f_{XY}(x,y) = \begin{cases} \frac{1}{2}, & 0 \le x < y \le 2 \\ 0, & \text{otherwise} \end{cases}.$$



Which of the following is closest to P(Y < 1)?

a: 0.2
b: 0.4
c: 0.6
d: 0.8
e: 1.0

$P(Y<1) = \iint f_{XY}(x,y)\,dx\,dy$ over triangle

$= (\frac{1}{2})(\text{area}) = (\frac{1}{2})(\frac{1}{2}\cdot 1\cdot 1) = \frac{1}{4} = 0.25$

10) If $f_{X|Y}(5 \mid 6) = 3$ and $f_Y(6) = 4$, which of the following is equal to $f_{XY}(5, 6)$?

a: 3
b: 4
c: 12
d: 7
e: None of the above

$f_{XY}(x,y) = f_{X|Y}(x|y)\, f_Y(y)$

$\Rightarrow f_{XY}(5,6) = f_{X|Y}(5|6)\, f_Y(6)$

$= (3)(4) = 12$

11) X and Y are standard Gaussian random variables whose correlation coefficient (ρ) is **not** equal to 1. If the value of X can be determined exactly from the value of Y, which of the following is the value of their correlation coefficient (ρ)?

a: −0.5
b: 0
c: −1
d: 0.5
e: None of the above

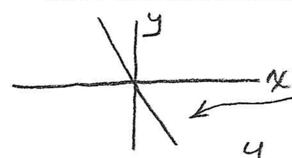
If X can be determined exactly from Y, then X and Y must be perfectly correlated ($\rho = 1$) or perfectly anticorrelated ($\rho = -1$)

$\rho \ne 1 \Rightarrow \rho = -1$

12) The random variable X is uniformly distributed from 0 to 1. If $Y = -2X$, which of the following is equal to $f_Y(-1)$?

a: 1
b: 1/4
c: −1/2
d: 1/2
e: None of the above

$\Rightarrow f_X(x) = 1,\ 0 \le x \le 1$



$y = -2x = g(x)$ (monotonically decreasing)

$x = -\frac{y}{2} = h(y) \Rightarrow \frac{dh(y)}{dy} = -\frac{1}{2}$

$f_Y(y) = f_X[h(y)]\left|\frac{dh(y)}{dy}\right| = (1)\left|-\frac{1}{2}\right| = \frac{1}{2},\ -2 \le y \le 0$

$\Rightarrow f_Y(-1) = \frac{1}{2}$

13) The random variable X is uniformly distributed from 0 to 1. If $Y = 5X - 2$, which of the following is equal to $F_Y(2)$?

$\Rightarrow F_X(x) = x, \quad 0 \le x \le 1$

a: 0

b: 1/3

c: 4/5

d: 2/3

e: None of the above

$y = 5x - 2 = g(x)$ is monotonically increasing

$x = \frac{y+2}{5} = h(y)$

$F_Y(y) = F_X[h(y)] = x|_{x=h(y)} = h(y) = \frac{y+2}{5}$

$\Rightarrow F_Y(2) = \frac{2+2}{5} = \frac{4}{5}$

14) X is a **discrete** uniform random variable that may equal any value in the set $\{1, 2, 3, 4, 5\}$. If $Y = 3X$, which of the following is equal to the value of the pmf of Y at $y = 6$?

Each has probability $= \frac{1}{5}$

a: 1

b: 1/2

c: 1/4

d: 1/8

e: None of the above

$y = 3x = g(x)$ is 1-to-1

$\Rightarrow P(Y=y) = P[X = h(y)]$

$x = \frac{y}{3} = h(y)$

$P(Y=6) = P(X = \frac{6}{3}) = P(X=2) = \frac{1}{5}$

15) Random variables Y_1 and Y_2 are derived from random variables X_1 and X_2 as follows:

$Y_1 = X_1 + X_2, \quad Y_2 = X_1 - X_2$

Which of the following is equal to $\mathbf{J}_{h_1 h_2}(y_1, y_2)$?

$y_1 = x_1 + x_2 = g_1(x_1, x_2)$

$y_2 = x_1 - x_2 = g_2(x_1, x_2)$

a: 1/2

b: 1/4

c: −1/4

d: −1/2

e: None of the above

$\Rightarrow x_1 = \frac{y_1 + y_2}{2} = h_1(y_1, y_2)$

$x_2 = \frac{y_1 - y_2}{2} = h_2(y_1, y_2)$

$$J_{h_1 h_2}(y_1, y_2) = \begin{vmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

16) Random variables X_1 and X_2 are statistically independent and uniform on [0,1]. Random variables Y_1 and Y_2 are derived from random variables X_1 and X_2 as follows:

$Y_1 = X_1 + X_2, \quad Y_2 = X_1 - X_2$

Which of the following is equal to the minimum possible value of Y_2?

a: 2

b: 1

c: 0

d: −1

e: None of the above

$Y_2 = X_1 - X_2$

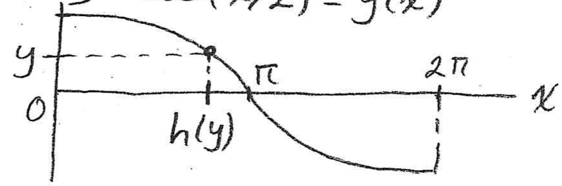
$Y_2(min) = \cancel{X_1(min)}^{\,0} - \cancel{X_2(max)}^{\,1}$

$= -1$

17) The random variable X is an angle (in units of radians) that is uniformly distributed from 0 to 2π. If $Y = \cos(X/2)$, which of the following is equal to the number of different inverse functions $h(y)$ required in the computation of $f_Y(y)$?

$y = \cos(x/2) = g(x)$

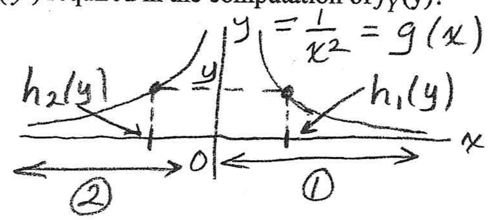
a: 4
b: 3
c: 2
d: 1
e: None of the above



$g(x)$ is 1-to-1 over the entire range of x
$\Rightarrow$ only 1 inverse $h(y)$ exists

18) X is a standard, Gaussian random variable. If $Y = 1/X^2$, which of the following is equal to the number of different inverse functions $h(y)$ required in the computation of $f_Y(y)$?

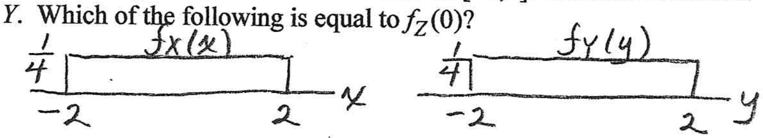
a: 5
b: 4
c: 3
d: 2
e: None of the above



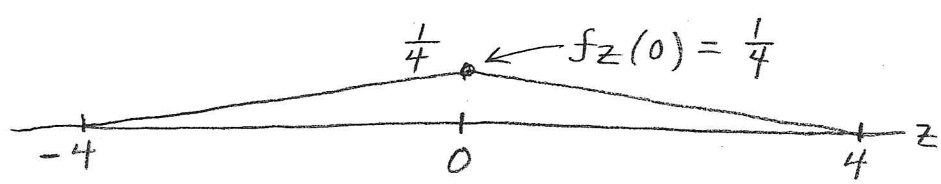
$g(x)$ is 1-to-1 on 2 intervals
$\Rightarrow$ 2 functions $h(y)$

19) Random variables X and Y are statistically independent and uniform on [–2,2]. Random variable Z is equal to the sum of X and Y. Which of the following is equal to $f_Z(0)$?

a: 0
b: 1/8
c: 1/6
d: 1/4
e: None of the above



$f_Z(z) = f_X(x) * f_Y(y)$



20) The random variable Z is equal to the sum of random variables X and Y. Which of the following must be true if $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$ for all z? $\leftarrow f_Z(z) = f_X(x) * f_Y(y)$

a: Both X and Y must be Gaussian random variables.
b: X and Y must be correlated.
c: X and Y must be statistically independent.
d: Both a and b must be true.
e: None of the above

if X and Y are S.I.