ECE 317
Chapter 1
Homework Solutions

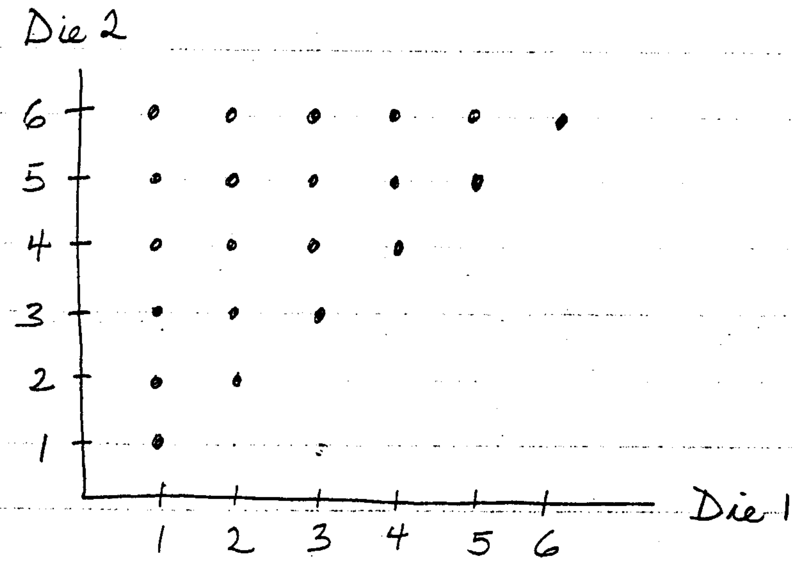
1.2.2) Sketch a sample space to represent possible outcomes of roll of two <u>indistinguishable</u> dice.

From ex 1.2.1, p.6,

$$S_2 = \{11, 12, \ldots, 16, 22, \ldots, 26, 33, \ldots, 66\}$$

(Don't have a 21 because it is indistinguishable from 12)

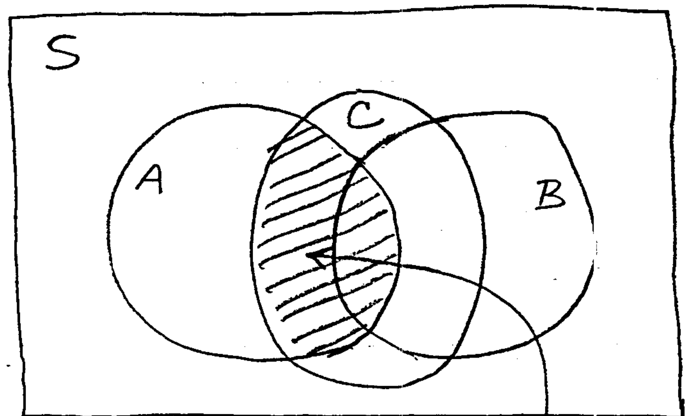
(S_2 contains $6+5+4+3+2+1 = 21$ elements



1.2.5) Given only that $(A \cap B) \subset C$,
fill in the blanks : (one letter per blank)

$$A \cap (B \cup C) = \_\_\_ \cap \_\_\_$$

Venn Diagram :



$A \cap (B \cup C)$

$= A \cap C$

From the diagram,

$$A \cap (B \cup C) = \boxed{A \cap C}$$

1.2.6) Prove Problem 1.2.5

Given $(A \cap B) \subset C$

Must prove that $A \cap (B \cup C) = A \cap C$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

from distributive property.

Given $(A \cap B) \subset C$ ← Take intersection of both sides with A.

$\Rightarrow \underbrace{A \cap (A \cap B)}_{= A \cap B} \subset A \cap C$

$\Rightarrow A \cap B \subset A \cap C$

$\Rightarrow (A \cap B) \cup (A \cap C) = A \cap C$

Substituting:

$A \cap (B \cup C) = A \cap C$

$\therefore$

1.2.7) Given only that $B \subset C$,
fill in the blanks: (one letter per blank)

$$(A \cup B) \cap C = \_\_ \cup (\_\_ \cap \_\_)$$

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

from distributive property

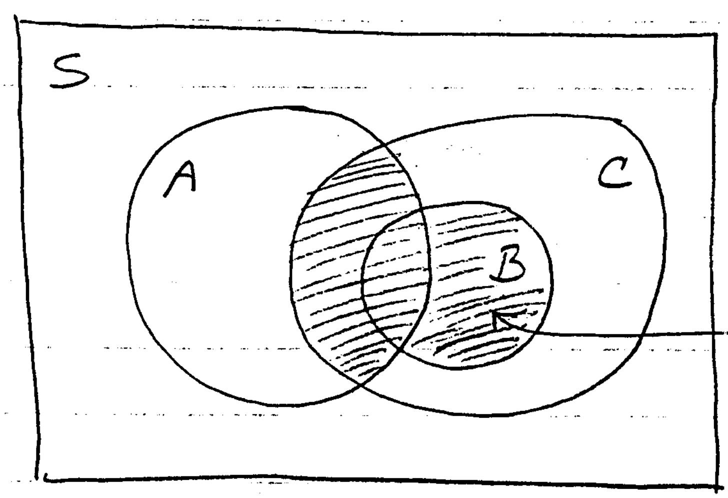
$B \cap C = B$, because $B \subset C$

$$\Rightarrow (A \cup B) \cap C = (A \cap C) \cup B$$

$$\Rightarrow \boxed{(A \cup B) \cap C = B \cup (A \cap C)}$$

from commutative property

Venn Diagram:



$(A \cup B) \cap C$
$= B \cup (A \cap C)$

1.2.11) Given $S = \{i : 1 \le i \le 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{4, 5, 6, 7, 8\}$, $C = \{3, 4, 7, 9, 10\}$

- Determine $(A \cap C) \cup B$

$$A \cap C = \{3, 4\}$$

$$\Rightarrow \boxed{(A \cap C) \cup B = \{3, 4, 5, 6, 7, 8\}}$$

- Determine $(A \cup C)^c \cap B$

$$A \cup C = \{1, 2, 3, 4, 5, 7, 9, 10\}$$

$$(A \cup C)^c = \{6, 8\}$$

$$\Rightarrow \boxed{(A \cup C)^c \cap B = \{6, 8\}}$$

Hubbard

1.2.12

1.2.12) How many subsets of $S = \{1, 2, 3\}$ exist? List them.

S contains 3 mutually exclusive elements.

$\Rightarrow$ $\boxed{\text{There are } 2^3 = 8 \text{ subsets of } S.}$

They are

S,
ϕ,
$\{1\}$,
$\{2\}$,
$\{3\}$,
$\{1, 2\}$,
$\{1, 3\}$,
and $\{2, 3\}$

1.3.2) Given $S = \{1,2,3,4,5\}$, $A = \{1,2,3\} \in \mathcal{A}$,
$B = \{3,4,5\} \in \mathcal{A}$
where $\mathcal{A}$ is an algebra.

Determine $\mathcal{A}$:

$\mathcal{A}$ must include S, $S^c = \emptyset$,
$A = \{1,2,3\}$, $B = \{3,4,5\}$,
$A^c = \{4,5\}$, $B^c = \{1,2\}$,
and all unions and intersections of these :
$A \cup B = S$ (already included)
$A \cap B = \{3\}$
$A \cup B^c = A$ (already included)
$A \cap B^c = B^c$ "
$A^c \cup B = B$ "
$A^c \cap B = A^c$ "
$A^c \cup B^c = \{1,2,4,5\}$
$A^c \cap B^c = \emptyset$ (already included)

$$\Rightarrow \boxed{\mathcal{A} = \{S, \emptyset, \{1,2,3\}, \{3,4,5\}, \{4,5\}, \{1,2\}, \{3\}, \{1,2,4,5\}\}}$$

1.3.5) Given $P(i) = \frac{1}{5}$ in problem 1.3.2, determine the probability assignment for $\mathcal{A}$.

Note: $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$ are mutually exclusive events.

$$\Rightarrow P(\{1,2,3\}) = P(1) + P(2) + P(3) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = \frac{3}{5}$$

Similarly, $P(\{3,4,5\}) = \frac{3}{5}$, $P\{4,5\} = \frac{2}{5}$,

$P(\{1,2\}) = \frac{2}{5}$, $P(\{3\}) = \frac{1}{5}$,

$P(\{1,2,4,5\}) = \frac{4}{5}$

Recall from 1.3.2,

$\mathcal{A} = \{S, \phi, \{1,2,3\}, \{3,4,5\}, \{4,5\}, \{1,2\}, \{3\}, \{1,2,4,5\}\}$

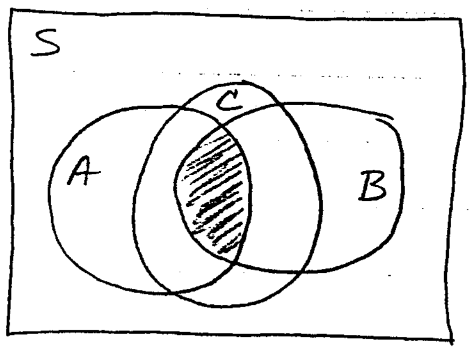
$$\Rightarrow \boxed{P = \left\{1, 0, \frac{3}{5}, \frac{3}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}, \frac{4}{5}\right\}}$$

Note: my order is different from book's.

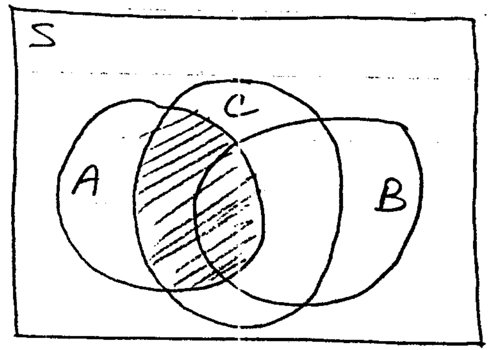
1.3.8) Given $(A \cap B) \subset C$

Determine and prove whether
$P(A \cap B) \leq, =, \text{or} \geq P(A \cap C)$

Venn diagrams are helpful:



$A \cap B$



$A \cap C$

We can see from diagrams that $(A \cap B) \subset (A \cap C)$, but this does not constitute a proof.

To prove: Given $(A \cap B) \subset C$

$$\Rightarrow \underbrace{A \cap (A \cap B)} \subset A \cap C \quad \text{(from ex. 1.2.5)}$$

$$= (A \cap A) \cap B \quad \text{(associative property)}$$

$$= A \cap B \quad (A \cap A = A)$$

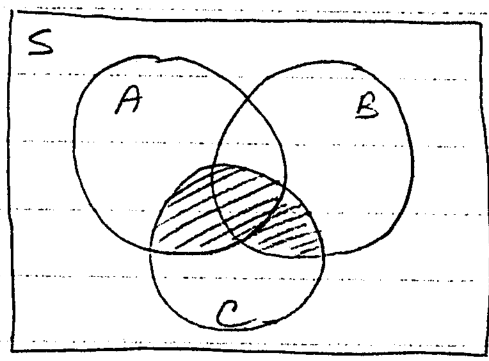
$$\Rightarrow (A \cap B) \subset (A \cap C)$$

$$\Rightarrow \boxed{P(A \cap B) \leq P(A \cap C)}$$

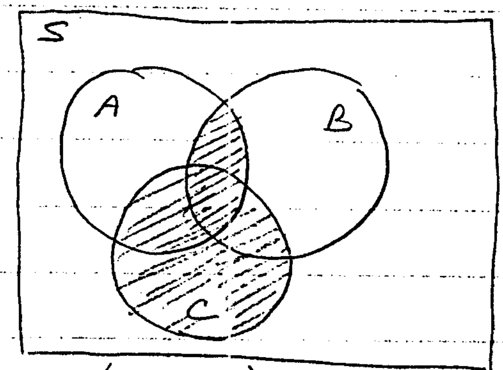
1.3.10) Fill in the blank and prove

$P[(A \cup B) \cap C]$ $\underline{\leq, =, \geq}$ $P[(A \cap B) \cup C]$

Venn diagrams:



$(A \cup B) \cap C$



$(A \cap B) \cup C$

Note: $(A \cup B) \cap C \subset (A \cap B) \cup C$

Proof:

$(A \cup B) \cap C \subset C$ (intersection property, p. 9)

$C \subset (A \cap B) \cup C$ (union property, p. 8)

$\Rightarrow (A \cup B) \cap C \subset (A \cap B) \cup C$

$\Rightarrow \boxed{P[(A \cup B) \cap C] \leq P[(A \cap B) \cup C]}$

1.4.2) Given three tossings of a die,

$A_i = \{\text{outcome} \le 2 \text{ on ith toss}\}$, $i = 1, 2, 3$

either 1 or 2

$$\Rightarrow P(A_1) = P(A_2) = P(A_3) = \frac{2}{6} = \frac{1}{3}$$

out of 6 possible

$$\boxed{P(A_1 \cup A_2)} = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$
$$= P(A_1) + P(A_2) - \underbrace{P(A_1)P(A_2)}$$

because A_1 and A_2 are statistically independent.

$$= \frac{1}{3} + \frac{1}{3} - \frac{1}{9} = \boxed{\frac{5}{9}} = \boxed{0.556}$$

$$\boxed{P(A_1 \cup A_2 \cup A_3)} = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$
$$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} - \frac{1}{9} - \frac{1}{9} - \frac{1}{9} + \frac{1}{27}$$
$$= \boxed{\frac{19}{27}} = \boxed{0.704}$$

$$\boxed{P(\text{outcomes} > 2 \text{ on all three tosses})}$$
$$= 1 - P(A_1 \cup A_2 \cup A_3)$$
$$= 1 - \frac{19}{27} = \boxed{\frac{8}{27}} = \boxed{0.296}$$

1.4.4) A die is tossed until two successive tosses yield the same outcome. Need to determine P_i = probability of stopping on ith toss.

$$\{\text{stopping on ith toss}\} = \{\underbrace{(\text{toss } 2 \neq \text{toss } 1) \cap (\text{toss } 3 \neq \text{toss } 2) \cap \ldots \cap (\text{toss } i-1 \neq \text{toss } i-2) \cap (\text{toss } i = \text{toss } i-1)}_{\text{these events are all statistically independent.}}\}$$

$$\Rightarrow P_i = P(\text{toss } 2 \neq \text{toss } 1)\, P(\text{toss } 3 \neq \text{toss } 2) \cdots$$
$$= \cdots P(\text{toss } i-1 \neq \text{toss } i-2)\, P(\text{toss } i = \text{toss } i-1)$$

Note: $\left.\begin{array}{l} P(\text{toss } j = \text{toss } K) = \frac{1}{6} \\ \text{and } P(\text{toss } j \neq \text{toss } K) = \frac{5}{6} \end{array}\right\}$ for $j \neq K$

$$\Rightarrow P_i = \underbrace{\left(\tfrac{5}{6}\right)\left(\tfrac{5}{6}\right)\cdots\left(\tfrac{5}{6}\right)}_{i-2 \text{ terms}}\left(\tfrac{1}{6}\right)$$

$$\boxed{P_i = \left(\tfrac{5}{6}\right)^{i-2}\left(\tfrac{1}{6}\right) \quad \text{for } i = 2, 3, 4, \ldots}$$

1.4.7) Given Box A contains 8 white, 2 green balls
Box B " 5 white, 5 green balls
Box C " 6 white, 4 green balls
and $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$, $P(C) = \frac{1}{4}$

- Need $P(C|W)$ (may use Bayes' theorem)

Note: $P(W|A) = \frac{8}{10}$, $P(G|A) = \frac{2}{10}$
$P(W|B) = \frac{5}{10}$, $P(G|B) = \frac{5}{10}$
$P(W|C) = \frac{6}{10}$, $P(G|C) = \frac{4}{10}$

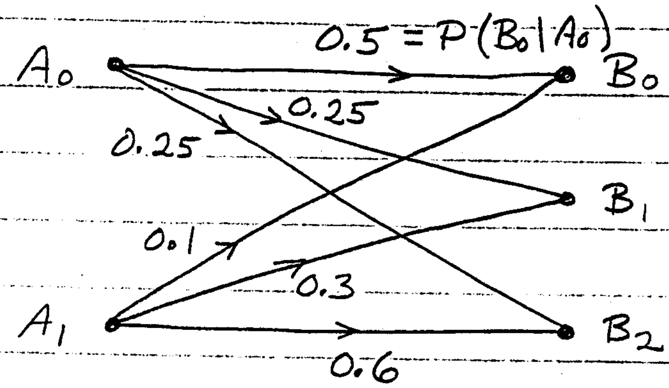
$$P(W) = P(W|A)\,P(A) + P(W|B)\,P(B) + P(W|C)\,P(C)$$
$$= \left(\frac{8}{10}\right)\left(\frac{1}{4}\right) + \left(\frac{5}{10}\right)\left(\frac{1}{2}\right) + \left(\frac{6}{10}\right)\left(\frac{1}{4}\right) = \frac{3}{5}$$

$$\Rightarrow \text{(Bayes')}: \boxed{P(C|W)} = \frac{P(W|C)\,P(C)}{P(W)} = \frac{\left(\frac{6}{10}\right)\left(\frac{1}{4}\right)}{\left(\frac{3}{5}\right)} = \boxed{\frac{1}{4}}$$
$$\boxed{= 0.25}$$

- Are C and W statistically independent?
$\boxed{\text{Yes}}$, because $P(C) = P(C|W) = \frac{1}{4}$

- Need $P(G)$
$$\boxed{P(G)} = 1 - P(W) = 1 - \frac{3}{5} = \boxed{\frac{2}{5}} = \boxed{0.4}$$

1.4.9)



Given:
$P(A_0) = 0.6$
$P(A_1) = 0.4$

- Determine best choice for A given B_0, B_1, B_2:

Total probability of B_0:

$$P(B_0) = P(B_0|A_0)P(A_0) + P(B_0|A_1)P(A_1)$$
$$= (0.5)(0.6) + (0.1)(0.4) = 0.34$$

$$P(A_0|B_0) = \frac{P(B_0|A_0)P(A_0)}{P(B_0)} = \frac{(0.5)(0.6)}{(0.34)} = 0.882$$

$\Rightarrow P(A_1|B_0) = 1 - P(A_0|B_0) = 1 - 0.882 = 0.118$

$\Rightarrow$ $\boxed{\text{Best choice given } B_0 \text{ is } A_0.}$

Total probability of B_1:

$$P(B_1) = P(B_1|A_0)P(A_0) + P(B_1|A_1)P(A_1)$$
$$= (0.25)(0.6) + (0.3)(0.4) = 0.27$$

$$P(A_0|B_1) = \frac{P(B_1|A_0)P(A_0)}{P(B_1)} = \frac{(0.25)(0.6)}{(0.27)} = 0.556$$

$\Rightarrow P(A_1|B_1) = 1 - P(A_0|B_1) = 1 - 0.556 = 0.444$

$\Rightarrow$ $\boxed{\text{Best choice given } B_1 \text{ is } A_0.}$

1.4.9) Cont.) Total probability of B_2:

$$P(B_2) = 1 - P(B_0) - P(B_1)$$
$$= 1 - 0.34 - 0.27 = 0.39$$

$$P(A_0|B_2) = \frac{P(B_2|A_0)P(A_0)}{P(B_2)} = \frac{(0.25)(0.6)}{(0.39)} = 0.385$$

$$\Rightarrow P(A_1|B_2) = 1 - P(A_0|B_2) = 1 - 0.385 = 0.615$$

$\Rightarrow$ $\boxed{\text{Best choice given } B_2 \text{ is } A_1.}$

– Determine the probability of error, $P(e)$:

$P(e) = 1 - P(C)$, where $P(C)$ = probability of being correct.

$$P(C) = P(C|B_0)P(B_0) + P(C|B_1)P(B_1) + P(C|B_2)P(B_2)$$

Note: if we use the above decision rules, then

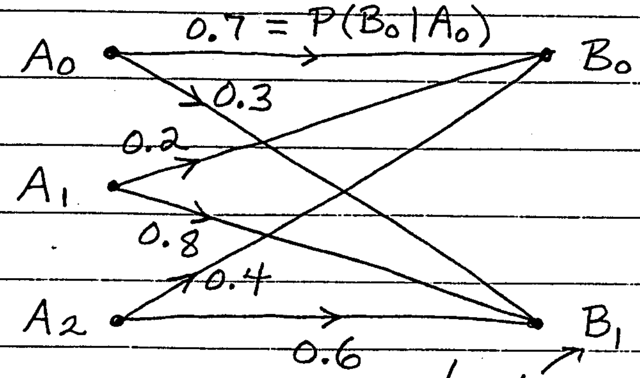
$$P(C|B_0) = P(A_0|B_0)$$
$$P(C|B_1) = P(A_0|B_1)$$

and $P(C|B_2) = P(A_1|B_1)$

$$\Rightarrow P(C) = P(A_0|B_0)P(B_0) + P(A_0|B_1)P(B_1) + P(A_1|B_2)P(B_2)$$
$$= (0.882)(0.34) + (0.556)(0.27) + (0.615)(0.39)$$
$$= 0.300 + 0.150 + 0.240 = 0.69$$

$$\Rightarrow \boxed{P(e)} = 1 - P(C) = 1 - 0.69 = \boxed{0.31}$$

1.4.10)



Given:

$P(A_0) = 0.5$

$P(A_1) = 0.4$

$P(A_2) = 0.1$

(This is "B_2" in Fig. P1.4.10)

Need to determine best choice for A given B_0, B_1.

Total probability of B_0:

$$P(B_0) = P(B_0|A_0)P(A_0) + P(B_0|A_1)P(A_1) + P(B_0|A_2)P(A_2)$$
$$= (0.7)(0.5) + (0.2)(0.4) + (0.4)(0.1) = 0.47$$

$$P(A_0|B_0) = \frac{P(B_0|A_0)P(A_0)}{P(B_0)} = \frac{(0.7)(0.5)}{0.47} \cong 0.745$$

$$P(A_1|B_0) = \frac{P(B_0|A_1)P(A_1)}{P(B_0)} = \frac{(0.2)(0.4)}{0.47} \cong 0.170$$

$$P(A_2|B_0) = \frac{P(B_0|A_2)P(A_2)}{P(B_0)} = \frac{(0.4)(0.1)}{0.47} \cong 0.085$$

Note: could compute $P(A_2|B_0)$ instead in this way:

$$P(A_2|B_0) = 1 - P(A_0|B_0) - P(A_1|B_0)$$
$$= 1 - 0.745 - 0.170 = 0.085$$

$\Rightarrow$ $\boxed{\text{Best choice given } B_0 \text{ is } A_0.}$

Total probability of B_1:

$$P(B_1) = P(B_1|A_0)P(A_0) + P(B_1|A_1)P(A_1) + P(B_1|A_2)P(A_2)$$
$$= (0.3)(0.5) + (0.8)(0.4) + (0.6)(0.1) = 0.53$$

1.4.10 Cont.) Note: it is easier to compute $P(B_1)$ as

$$P(B_1) = 1 - P(B_0) = 1 - 0.47 = 0.53$$

$$P(A_0|B_1) = \frac{P(B_1|A_0)P(A_0)}{P(B_1)} = \frac{(0.3)(0.5)}{0.53} \cong 0.283$$

$$P(A_1|B_1) = \frac{P(B_1|A_1)P(A_1)}{P(B_1)} = \frac{(0.8)(0.4)}{0.53} \cong 0.604$$

$$P(A_2|B_1) = \frac{P(B_1|A_2)P(A_2)}{P(B_1)} = \frac{(0.6)(0.1)}{0.53} = 0.113$$

Or, an easier method:

$$P(A_2|B_1) = 1 - P(A_0|B_1) - P(A_1|B_1)$$
$$= 1 - 0.283 - 0.604 = 0.113$$

$\Rightarrow$ $\boxed{\text{Best choice given } B_1 \text{ is } A_1.}$

Need to determine probability of error:

$P(e) = 1 - P(C)$, where $P(C)$ = probability of being correct.

$$P(C) = P(C|B_0)P(B_0) + P(C|B_1)P(B_1)$$

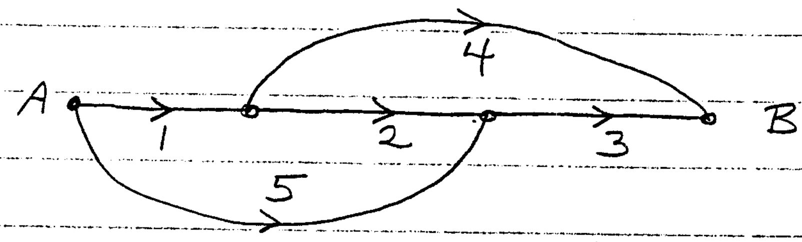
Note: if we use the above decision rules, then $P(C|B_0) = P(A_0|B_0)$ and $P(C|B_1) = P(A_1|B_1)$

$$\Rightarrow P(C) = (0.745)(0.47) + (0.604)(0.53)$$
$$\cong 0.67$$

$$\Rightarrow \boxed{P(e)} = 1 - P(C) = 1 - 0.67 = \boxed{0.33}$$

Hubbard 1.4.15

1.4.15) Determine $P(\text{comm})$ for this network:



Given $P_1 = 0.75$, $P_2 = 0.8$, $P_3 = 0.6$, $P_4 = 0.4$, and $P_5 = 0.5$

(lines are all statistically independent).

Note: none of the lines are in series or parallel, so we have to analyze the system by considering all possible paths from A to B:

$$P(\text{comm}) = P[(1 \cap 2 \cap 3) \cup (1 \cap 4) \cup (5 \cap 3)]$$

(Also note that these paths are not statistically independent).

$$\Rightarrow \boxed{P(\text{comm})} = P(1 \cap 2 \cap 3) + P(1 \cap 4) + P(5 \cap 3) - P[(1 \cap 2 \cap 3) \cap (1 \cap 4)] - P[(1 \cap 2 \cap 3) \cap (5 \cap 3)] - P[(1 \cap 4) \cap (5 \cap 3)] + P[(1 \cap 2 \cap 3) \cap (1 \cap 4) \cap (5 \cap 3)]$$

$$= P(1 \cap 2 \cap 3) + P(1 \cap 4) + P(5 \cap 3) - P(1 \cap 2 \cap 3 \cap 4) - P(1 \cap 2 \cap 3 \cap 5) - P(1 \cap 3 \cap 4 \cap 5) + P(1 \cap 2 \cap 3 \cap 4 \cap 5)$$

$$= P_1 P_2 P_3 + P_1 P_4 + P_3 P_5 - P_1 P_2 P_3 P_4 - P_1 P_2 P_3 P_5 - P_1 P_3 P_4 P_5 + P_1 P_2 P_3 P_4 P_5$$

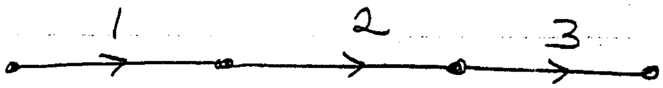
$$= 0.36 + 0.3 + 0.3 - 0.144 - 0.18 - 0.09 + 0.072 = \boxed{0.618}$$

Hubbard

1.4.19

1.4.19) Given 3 statistically independent components with reliabilities $P_1 = P_2 = P_3$ and $P(\text{oper}) = 0.9$ for entire system

For components in series:



For proper system operation, all three components must operate properly:

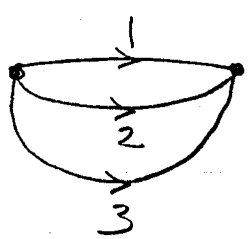
$$\Rightarrow P(\text{oper}) = P(1 \cap 2 \cap 3) = P_1 P_2 P_3$$

$$= P_1^3 \quad \text{because of statistical independence.}$$

$$\Rightarrow 0.9 = P_1^3$$

$$\Rightarrow \boxed{P_1 = P_2 = P_3 = \sqrt[3]{0.9} = 0.965}$$

For components in parallel:



At least one component must operate properly.

$$\Rightarrow P(\text{oper}) = P(1 \cup 2 \cup 3)$$

$$= 1 - P[(1 \cup 2 \cup 3)^c]$$

$$= 1 - P(1^c \cap 2^c \cap 3^c)$$

$$= 1 - P(1^c)P(2^c)P(3^c) \quad \text{because of statistical independence}$$

$$= 1 - (1-P_1)(1-P_2)(1-P_3)$$

$$= 1 - (1-P_1)^3$$

$$0.9 = 1 - (1-P_1)^3 \Rightarrow \boxed{P_1} = 1 - \sqrt[3]{1-0.9} = \boxed{0.536}$$

1.4.29) Given that $P(A|B) \geq P(A)$,
Determine and prove
$$P(A^c|B) \ \underline{\leq, =, \geq} \ P(A^c)$$

Given $P(A|B) \geq P(A)$ (and both are nonnegative),
then $-P(A|B) \leq -P(A)$

Add 1 to both sides:

$$\underbrace{1 - P(A|B)}_{P(A^c|B)} \leq \underbrace{1 - P(A)}_{P(A^c)}$$

$$\Rightarrow \boxed{P(A^c|B) \leq P(A^c)}$$