ECE 317
Chapter 4
Homework Solutions

4.1.2) Given $f_X(x) = \frac{(a+b+1)!}{a!\,b!} x^a (1-x)^b$,

for $0 \le x \le 1$ and integers $a, b \ge 0$

Need to determine $E(X)$, $E(X^2)$, $Var(X)$

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_0^1 \underbrace{x \frac{(a+b+1)!}{a!\,b!} x^a}_{x \cdot x^a = x^{a+1}} (1-x)^b\, dx$$

$$= \int_0^1 \frac{(a+b+1)!}{a!\,b!} x^{a+1} (1-x)^b\, dx$$

Let $c = a+1$

$$\Rightarrow E(X) = \int_0^1 \frac{(a+b+1)!}{a!\,b!} \cdot \overbrace{\left(\frac{(c+b+1)!\; c!\, b!}{(c+b+1)!\; c!\, b!} \right)}^{=1} x^c (1-x)^b\, dx$$

$$= \frac{(a+b+1)!\; c!\, \cancel{b!}}{(c+b+1)!\; a!\, \cancel{b!}} \int_0^1 \underbrace{\frac{(c+b+1)!}{c!\, b!} x^c (1-x)^b}_{f_X(x) \text{ for integers } c,\, b}\, dx$$

$$= \frac{(a+b+1)!\; c!}{(c+b+1)!\; a!} \underbrace{\int_0^1 f_X(x)\, dx}_{=1}$$

$$= \frac{(a+b+1)!\,(a+1)!}{(a+b+2)!\; a!} = \boxed{\frac{a+1}{a+b+2}}$$

4.1.2) Cont.) $E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$

$$= \int_0^1 x^2 \frac{(a+b+1)!}{a!\,b!} x^a (1-x)^b\,dx$$

$x^2 x^a = x^{a+2}$

$$= \int_0^1 \frac{(a+b+1)!}{a!\,b!} x^{a+2} (1-x)^b\,dx$$

Let $d = a+2$

$$\Rightarrow E(X^2) = \int_0^1 \frac{(a+b+1)!}{a!\,b!} \left(\frac{(d+b+1)!\,d!\,b!}{(d+b+1)!\,d!\,b!} \right) x^d (1-x)^b\,dx$$

$$= \frac{(a+b+1)!\,d!\,\cancel{b!}}{(d+b+1)!\,a!\,\cancel{b!}} \int_0^1 \underbrace{\frac{(d+b+1)!}{d!\,b!} x^d (1-x)^b\,dx}_{f_X(x) \text{ for integers } d,\,b}$$

$$= \frac{(a+b+1)!\,d!}{(d+b+1)!\,a!} \underbrace{\int_0^1 f_X(x)\,dx}_{=1}$$

$$= \frac{(a+b+1)!\,(a+2)!}{(a+b+3)!\,a!} \boxed{= \frac{(a+2)(a+1)}{(a+b+3)(a+b+2)}}$$

4.1.2) Cont.) $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$$= \frac{(a+2)(a+1)}{(a+b+3)(a+b+2)} - \left[\frac{a+1}{a+b+2}\right]^2$$

$$= \frac{(a+2)(a+1)(a+b+2) - (a+1)^2(a+b+3)}{(a+b+3)(a+b+2)^2}$$

$$= \frac{(a+1)\left[(a+2)(a+b+2) - (a+1)(a+b+3)\right]}{(a+b+3)(a+b+2)^2}$$

$$= \frac{(a+1)\left[\cancel{a^2} + \cancel{ab} + \cancel{2a} + \cancel{2a} + 2b + 4 - \cancel{a^2} - \cancel{ab} - \cancel{3a} - \cancel{a} - b - 3\right]}{(a+b+3)(a+b+2)^2}$$

$$= \boxed{\frac{(a+1)(b+1)}{(a+b+3)(a+b+2)^2}}$$

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4.1.4) Given $P(X=K) = p(1-p)^{K-1}$, $K = 1, 2, 3, \ldots$, and $0 < p < 1$

Need to determine $E(X)$, $E(X^2)$, $Var(X)$

For a discrete random variable,

$$E(X) = \sum_{K=-\infty}^{\infty} x_K \, P(X = x_K)$$

$$= \sum_{K=1}^{\infty} K p (1-p)^{K-1} = -p \sum_{K=1}^{\infty} -K(1-p)^{K-1}$$

$$= -p \sum_{K=1}^{\infty} \frac{d}{dp}\left\{(1-p)^K\right\}$$

$$= -p \cdot \frac{d}{dp}\left\{\underbrace{\sum_{K=1}^{\infty}(1-p)^K}\right\}$$

$$= \frac{1}{1-(1-p)} - 1 = \frac{1-p}{p}$$

$$= \frac{1}{p} - 1$$

$$\Rightarrow E(X) = -p \frac{d}{dp}\left\{\frac{1}{p} - 1\right\}$$

$$= -p\left(-p^{-2}\right) = p^{-1} = \boxed{\frac{1}{p}}$$

4.1.4) For discrete random variables that
Cont.) take on only integer values,

$$E(X^2) = E[X(X-1)] + E(X)$$

$$E[X(X-1)] = \sum_{K=-\infty}^{\infty} K(K-1)\,P(X=K)$$

$$= \sum_{K=1}^{\infty} K(K-1)\,p(1-p)^{K-1}$$

$$= p(1-p) \sum_{K=1}^{\infty} K(K-1)(1-p)^{K-2}$$

$$= p(1-p) \sum_{K=1}^{\infty} \frac{d^2}{dp^2} \{(1-p)^K\}$$

$$= p(1-p) \frac{d^2}{dp^2} \left\{ \sum_{K=1}^{\infty} (1-p)^K \right\}$$

$$= p(1-p) \frac{d^2}{dp^2} \left\{ \frac{1}{p} - 1 \right\}$$

$$= p(1-p)(2p^{-3}) = \frac{2(1-p)}{p^2}$$

$$E(X^2) = E[X(X-1)] + E(X)$$

$$= \frac{2(1-p)}{p^2} + \frac{1}{p} = \frac{2-2p+p}{p^2}$$

$$= \boxed{\frac{2-p}{p^2}}$$

4.1.4)
Cont.)

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{2-p}{p^2} - \left[\frac{1}{p}\right]^2$$

$$= \frac{2-p-1}{p^2} = \boxed{\frac{1-p}{p^2}}$$

4.1.8) Given X with $E(X) = 1$ and $Var(X) = 4$
and $Y = aX + b$,
with $E(Y) = 9$ and $Var(Y) = 16$

Need to determine a and b

$$Y = aX + b \Rightarrow E(Y) = a\,E(X) + b$$

$$\Rightarrow 9 = a(1) + b \Rightarrow \boxed{a + b = 9}$$

$$Var(Y) = a^2\,Var(X)$$

$$\Rightarrow 16 = a^2(4) \Rightarrow a^2 = 4$$

$$\Rightarrow \boxed{a = \pm 2}$$

$$\Rightarrow \boxed{\begin{array}{c} a = 2, \quad b = 7 \\ \text{OR} \\ a = -2, \quad b = 11 \end{array}}$$

4.1.10) Show that $E[(X-c)^2] \geq Var(X)$

$$Var(X) = E[(X-\bar{x})^2]$$

$$E[(X-c)^2] = E[(X-\bar{x}+\bar{x}-c)^2]$$

$$= E[((X-\bar{x})+(\bar{x}-c))^2]$$

$$= E[(X-\bar{x})^2 + 2(\bar{x}-c)(X-\bar{x}) + (\bar{x}-c)^2]$$

$$= E[(X-\bar{x})^2] + 2(\bar{x}-c)E\cancelto{0}{(X-\bar{x})} + E[(\bar{x}-c)^2]$$

$$\downarrow$$

$$= Var(X) + 0 + (\bar{x}-c)^2$$

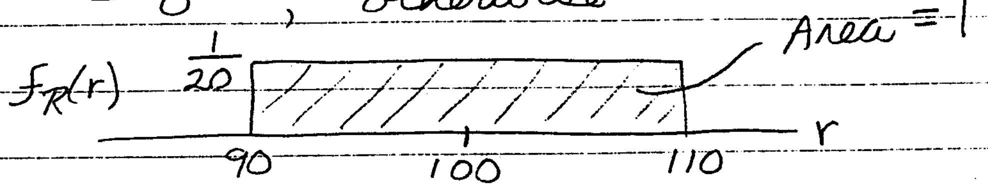
Note that $(\bar{x}-c)^2 \geq 0$

$$\Rightarrow E[(X-c)^2] \geq Var(X)$$

$\therefore$

4.1.11) Given a 100Ω 10% resistor whose resistance is uniformly distributed.
(Let R = the resistance)

$$\Rightarrow f_R(r) = \frac{1}{20} \quad , \quad 90 \leq r \leq 110$$
$$= 0 \quad , \quad \text{otherwise}$$



$$\boxed{E(R)} = \int_{-\infty}^{\infty} r f_R(r)\,dr = \int_{90}^{110} r\left(\frac{1}{20}\right) dr$$

$$= \frac{r^2}{40}\Bigg|_{90}^{110} = \frac{1}{40}(110^2 - 90^2) = \boxed{100}$$

(as expected)

Conductance $G = \frac{1}{R}$

$$\boxed{E(G)} = \int_{-\infty}^{\infty} \frac{1}{r} f_R(r)\,dr = \int_{90}^{110} \frac{1}{r}\left(\frac{1}{20}\right) dr$$

$$= \frac{1}{20} \ln(r)\Bigg|_{90}^{110}$$

$$= \frac{1}{20}\left[\ln(110) - \ln(90)\right]$$

$$\boxed{\cong 0.01003}$$

$$\left(\text{Note } E(G) \neq \frac{1}{E(R)}\right)$$

4.2.5) Given $P(X=K) = C_K^N p^K (1-p)^{N-K}$
for $K = 0, 1, \ldots, N$, and $0 < p < 1$

Need to determine $\phi_X(\omega)$

For discrete random variables,

$$\phi_X(\omega) = \sum_{K=-\infty}^{\infty} e^{j\omega x_K} P(X = x_K)$$

$$\Rightarrow \phi_X(\omega) = \sum_{K=0}^{N} e^{j\omega K} C_K^N p^K (1-p)^{N-K}$$

Recall the binomial theorem:

$$(a+b)^N = \sum_{K=0}^{N} C_K^N a^K b^{N-K}$$

$$\phi_X(\omega) = \sum_{K=0}^{N} C_K^N (pe^{j\omega})^K (1-p)^{N-K}$$

$$\Rightarrow \boxed{\phi_X(\omega) = (pe^{j\omega} + 1 - p)^N}$$

4.2.9) Given $\phi_X(\omega) = (1-j\omega b)^{-1}$, $b > 0$

Need to determine $E(X)$, $E(X^2)$, $Var(X)$

Recall $E(X^n) = (-j)^n \dfrac{d^n\phi_X(\omega)}{d\omega^n}\Big|_{\omega=0}$

$$\frac{d\phi_X(\omega)}{d\omega} = (-1)(1-j\omega b)^{-2}(-jb) = jb(1-j\omega b)^{-2}$$

$$\boxed{E(X)} = (-j)^1 \frac{d\phi_X(\omega)}{d\omega}\Big|_{\omega=0} = (-j)(jb)(1-0)^{-2}$$
$$= \boxed{b}$$

$$\frac{d^2\phi_X(\omega)}{d\omega^2} = jb(-2)(1-j\omega b)^{-3}(-jb)$$
$$= -2b^2(1-j\omega b)^{-3}$$

$$\boxed{E(X^2)} = (-j)^2 \frac{d^2\phi_X(\omega)}{d\omega^2}\Big|_{\omega=0}$$
$$= (-1)(-2b^2)(1-0)^{-3} = \boxed{2b^2}$$

$$\boxed{Var(X)} = E(X^2) - [E(X)]^2$$
$$= 2b^2 - (b)^2 = \boxed{b^2}$$

4.2.12) Given $\phi_X(\omega) = e^{a(e^{j\omega}-1)}$, $a > 0$

Need to determine $E(X)$, $E(X^2)$, $Var(X)$

Recall $E(X^n) = (-j)^n \frac{d^n\phi_X(\omega)}{d\omega^n}\Big|_{\omega=0}$

$$\frac{d\phi_X(\omega)}{d\omega} = \left(a e^{a(e^{j\omega}-1)}\right)\left(j e^{j\omega}\right)$$

$$= j a e^{j\omega} e^{a(e^{j\omega}-1)}$$

$$E(X) = (-j)^1 \frac{d\phi_X(\omega)}{d\omega}\Big|_{\omega=0} = (-j)(ja)(e^0)(e^0)$$

$$= \boxed{a}$$

$$\frac{d^2\phi_X(\omega)}{d\omega} = ja\Big[(e^{j\omega})\left(a e^{a(e^{j\omega}-1)}\right)(j e^{j\omega})$$

(Product Rule) $\quad + \left(e^{a(e^{j\omega}-1)}\right)(j e^{j\omega})$

$$= (jae^{j\omega})^2\left(a e^{a(e^{j\omega}-1)}\right)$$

$$+ \cancelto{-1}{j^2} a e^{j\omega}\left(e^{a(e^{j\omega}-1)}\right)$$

$$= \left(-(ae^{j\omega})^2 - ae^{j\omega}\right)\left(e^{a(e^{j\omega}-1)}\right)$$

$$E(X^2) = \cancelto{-1}{(-j)^2} \frac{d^2\phi_X(\omega)}{d\omega^2}\Big|_{\omega=0}$$

$$= (-1)(-a^2 - a)(e^0) = \boxed{a^2 + a}$$

$$Var(X) = E(X^2) - [E(X)]^2 = a^2 + a - a^2 = \boxed{a}$$

4.3.1) Given X_1 with $Var(X_1) = 1$, $E(X_1) = 0$
X_2 with $Var(X_2) = 5$, $E(X_2) = 0$
and $Cov(X_1, X_2) = 2$
and $Y = 3X_1 + 2X_2$

Need to determine $Var(Y)$

$$Y = 3X_1 + 2X_2 \Rightarrow E(Y) = 3E(X_1) + 2E(X_2)$$
$$= 0 + 0 = 0$$

$$Var(Y) = E(Y^2) - [\overset{0}{\cancel{E(Y)}}]^2 = E(Y^2)$$

$$= E[(3X_1 + 2X_2)^2]$$

$$= E[9X_1^2 + 12X_1X_2 + 4X_2^2]$$

$$= 9E(X_1^2) + 12E(X_1X_2) + 4E(X_2^2)$$

Note: $E(X_1) = 0 \Rightarrow E(X_1^2) = Var(X_1)$
$E(X_2) = 0 \Rightarrow E(X_2^2) = Var(X_2)$

$$\Rightarrow Var(Y) = 9Var(X_1) + 12Cov(X_1, X_2) + 4Var(X_2)$$

$$= 9(1) + 12(2) + 4(5)$$

$$= \boxed{53}$$

4.3.5) Given $f_{XY}(x,y) = xy$, $0 \le x \le 1$,
$0 \le y \le 2$

Need to determine $\mathrm{Cov}(X,Y)$, ρ, and whether X and Y are uncorrelated.

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^2 xy\,dy$$

$$= x \cdot \frac{y^2}{2}\Big|_{y=0}^{y=2} = 2x, \quad 0 \le x \le 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^1 xy\,dx$$

$$= y \cdot \frac{x^2}{2}\Big|_{x=0}^{x=1} = \frac{y}{2}, \quad 0 \le y \le 2$$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x(2x)\,dx$$

$$= \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3}$$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^2 y\left(\frac{y}{2}\right)dy$$

$$= \frac{y^3}{6}\Big|_0^2 = \frac{4}{3}$$

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p. 2

4.3.5) Cont.) $E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dx\, dy$

$$= \int_0^2 \int_0^1 xy(xy)\, dx\, dy$$

$$= \int_0^2 y^2 \int_0^1 x^2\, dx\, dy = \int_0^2 y^2 \left[\frac{x^3}{3}\right]\Bigg|_{x=0}^{x=1} dy$$

$$= \int_0^2 y^2\left(\frac{1}{3}\right) dy = \frac{1}{3}\cdot\frac{y^3}{3}\Bigg|_{y=0}^{y=2}$$

$$= \frac{1}{3}\cdot\frac{8}{3} = \frac{8}{9}$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= \frac{8}{9} - \left(\frac{2}{3}\right)\left(\frac{4}{3}\right) = \boxed{0}$$

$\Rightarrow$ $\boxed{\text{X and Y are uncorrelated.}}$

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}} = \frac{0}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}$$

$$= \boxed{0}$$

4.3.10) Given X_1 and X_2,

$Y_1 = X_1$ and $Y_2 = aX_1 + bX_2$,

$E(X_1) = E(X_2) = E(Y_1) = E(Y_2) = 0$

$Var(X_1) = Var(X_2) = Var(Y_1) = Var(Y_2) = 1$,

$Cov(X_1, X_2) = 0.5$, $Cov(Y_1, Y_2) = 0$,

Need to determine a and b

Note: $Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)\cancelto{0}{E(Y_2)}$

$= E(Y_1, Y_2)$

$= E[X_1(aX_1 + bX_2)]$

$= E[aX_1^2 + bX_1X_2]$

$= a\,E(X_1^2) + b\,E(X_1X_2)$

Note: $E(X_1) = 0 \Rightarrow E(X_1^2) = Var(X_1)$

and $Cov(X_1, X_2) = E(X_1X_2) - E(X_1)\cancelto{0}{E(X_2)}$

$= E(X_1X_2)$

$\Rightarrow Cov(Y_1, Y_2) = a\,Var(X_1) + b\,Cov(X_1, X_2)$

$= a(1) + b(0.5) = 0$

$\Rightarrow b = -2a$

$Var(Y_2) = E(Y_2^2) - [\cancelto{0}{E(Y_2)}]^2 = E(Y_2^2)$

$= E[(aX_1 + bX_2)^2] = E[a^2X_1^2 + 2abX_1X_2 + b^2X_2^2]$

$= a^2E(X_1^2) + 2ab\,E(X_1X_2) + b^2E(X_2^2)$

4.3.10) Cont.) $= a^2 \mathrm{Var}(X_1) + 2ab\,\mathrm{Cov}(X_1, X_2) + b^2 \mathrm{Var}(X_2)$

$$= a^2(1) + \cancel{2}ab(\cancel{0.5}) + b^2(1) \quad = 1$$

Substitute for $b = -2a$:

$$a^2 + a(-2a) + (-2a)^2 \quad = 1$$

$$\Rightarrow 3a^2 = 1 \quad \Rightarrow a = \pm 0.577$$

$$\Rightarrow \boxed{\begin{array}{c} a = 0.577, \quad b = -1.154 \\ \text{OR} \\ a = -0.577, \quad b = 1.154 \end{array}}$$

4.3.16) Given 3 photomultiplier tubes, number of photons counted in each tube is Poisson distributed with mean 3. Need to determine probability that the total number of photons counted in all 3 tubes (combined) is 7.

Let X_1, X_2, X_3 be the numbers of photons counted in tubes 1, 2, and 3, respectively.

X_1, X_2, and X_3 are Poisson:

$$P(X_i = K) = \frac{e^{-a} a^K}{K!}, \quad K = 0, 1, 2, \ldots$$

where a = the mean value of X_i

Given $a = 3$

$$\Rightarrow P(X_i = K) = \frac{e^{-3} 3^K}{K!}, \quad K = 0, 1, 2, \ldots$$

From Appendix C, for a Poisson random variable with parameter a:

$$\phi_X(\omega) = \exp[a(\exp(j\omega) - 1)]$$

$$\Rightarrow \phi_{X_i}(\omega) = e^{[3(e^{j\omega} - 1)]}, \quad i = 1, 2, 3$$

Let $Y = X_1 + X_2 + X_3$

(we assume X_1, X_2, X_3 are statistically independent)

4.3.16) Cont.) $\Rightarrow \phi_Y(\omega) = \phi_{X_1}(\omega)\, \phi_{X_2}(\omega)\, \phi_{X_3}(\omega)$

$$= \left(e^{[3(e^{j\omega}-1)]}\right)^3 = e^{3[3(e^{j\omega}-1)]}$$

$$= e^{[9(e^{j\omega}-1)]}$$

This is the characteristic function of a Poisson random variable with parameter 9.

$$\Rightarrow P(Y=K) = \frac{e^{-9}\, 9^K}{K!}, \quad K = 0, 1, 2, \ldots$$

$$\Rightarrow \boxed{P(Y=7)} = \frac{e^{-9}\, 9^7}{7!} \cong \boxed{0.117}$$

4.4.2) Given the jointly Gaussian variables X_1, X_2, and X_3, with

$$\underset{\sim}{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad \text{and } \underset{\sim}{\mu}_x = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

$$\text{and } \Sigma_x = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 5 & -2 \\ 2 & -2 & 4 \end{bmatrix}$$

Given $Y_1 = 3X_1 - X_2 + 2X_3 + 4$, and $Y_2 = X_1 + 2X_2 - 4X_3 + 5$, i.e.,

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & -4 \end{bmatrix}}_{} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\text{or, } \underset{\sim}{Y} = \underset{\sim}{A}\,\underset{\sim}{X} + \underset{\sim}{B}$$

Need $\underset{\sim}{\mu}_y$ and Σ_y:

$$\boxed{\underset{\sim}{\mu}_y} = \underset{\sim}{A}\,\underset{\sim}{\mu}_x + \underset{\sim}{B}$$

$$= \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & -4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} (9 - 4 + 2 + 4) \\ (3 + 8 - 4 + 5) \end{bmatrix} = \boxed{\begin{bmatrix} 11 \\ 12 \end{bmatrix}}$$

4.4.2) (Cont.) $\boxed{\Sigma_y} = \underset{\sim}{A}\, \Sigma_x\, \underset{\sim}{A}^T$

$$= \begin{bmatrix} 3 & -1 & 2 \\ 1 & 2 & -4 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ -1 & 5 & -2 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -12 & 16 \\ -7 & 17 & -18 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \\ 2 & -4 \end{bmatrix} = \boxed{\begin{bmatrix} 86 & -74 \\ -74 & 99 \end{bmatrix}}$$

Given $\underset{\sim}{Z}^T = (X_2, X_3) \Rightarrow Z_1 = X_2$ and $Z_2 = X_3$

$$\Rightarrow \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$\Rightarrow \underset{\sim}{Z} = \underset{\sim}{A}\,\underset{\sim}{X}$$

Need $\underset{\sim}{\mu}_z$ and Σ_z :

$$\boxed{\underset{\sim}{\mu}_z} = \underset{\sim}{A}\,\underset{\sim}{\mu}_x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}$$

$$\boxed{\Sigma_z} = \underset{\sim}{A}\, \Sigma_x\, \underset{\sim}{A}^T$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ -1 & 5 & -2 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 5 & -2 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 5 & -2 \\ -2 & 4 \end{bmatrix}}$$

4.4.4) Given jointly Gaussian random variables X_1 and X_2 with $\underset{\sim}{\mu}_x^T = (3,-2)$

and $\Sigma_x = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

Need to find $f_{X_1X_2}(x_1, x_2)$

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = \frac{1}{(2\pi)^{N/2}\sqrt{|\Sigma_x|}} \exp\left[\frac{-(\underset{\sim}{x}-\underset{\sim}{\mu}_x)^T \Sigma_x^{-1} (\underset{\sim}{x}-\underset{\sim}{\mu}_x)}{2}\right]$$

(where N = 2)

$$|\Sigma_x| = (1)(5) - (2)(2) = 1$$

$$\Sigma_x^{-1} = \frac{\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}}{|\Sigma_x|} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

$$\underset{\sim}{x} - \underset{\sim}{\mu}_x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} (x_1-3) \\ (x_2+2) \end{bmatrix}$$

$$(\underset{\sim}{x}-\underset{\sim}{\mu}_x)^T \Sigma_x^{-1} (\underset{\sim}{x}-\underset{\sim}{\mu}_x)$$

$$= \begin{bmatrix} (x_1-3) & (x_2+2) \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} (x_1-3) \\ (x_2+2) \end{bmatrix}$$

$$= \begin{bmatrix} (5(x_1-3)-2(x_2+2)) & (-2(x_1-3)+1(x_2+2)) \end{bmatrix} \begin{bmatrix} (x_1-3) \\ (x_2+2) \end{bmatrix}$$

$$= 5(x_1-3)^2 - 2(x_1-3)(x_2+2) - 2(x_1-3)(x_2+2) + 1(x_2+2)^2$$

$$= 5(x_1-3)^2 - 4(x_1-3)(x_2+2) + (x_2+2)^2$$

$$\Rightarrow \boxed{f_{X_1X_2}(x_1,x_2) = \frac{1}{2\pi}\exp\left[-\frac{5(x_1-3)^2-4(x_1-3)(x_2+2)+(x_2+2)^2}{2}\right]}$$

4.4.6) Given

$$f_{X_1 X_2}(x_1,x_2) = \frac{1}{4\pi}\exp\left[-\frac{5(x_1+1)^2+8(x_1+1)x_2+4x_2^2}{8}\right]$$

(i.e, X_1 and X_2 are jointly Gaussian).

Need $\underset{\sim}{\mu}_x$ and Σ_x :

Note terms in exponent are $(x_1-(\underbrace{-1}_{\mu_{x1}}))$ and $(x_2-\underbrace{0}_{\mu_{x2}})$

$$\Rightarrow \boxed{\underset{\sim}{\mu}_x = \begin{bmatrix}-1\\0\end{bmatrix}}$$

$$\frac{1}{4\pi} = \frac{1}{(2\pi)^{N/2}\sqrt{|\Sigma_x|}}, \quad \text{where } N=2$$

$$\Rightarrow 4\cancel{\pi} = 2\cancel{\pi}\sqrt{|\Sigma_x|} \Rightarrow |\Sigma_x| = 4$$

$$\text{Let } \Sigma_x = \begin{bmatrix}a & b\\ b & c\end{bmatrix} \quad \text{(must be symmetric)}$$

$$\Rightarrow \Sigma_x^{-1} = \frac{\begin{bmatrix}c & -b\\ -b & a\end{bmatrix}}{\cancel{|\Sigma_x|} \rightarrow \frac{1}{4}} = \frac{1}{4}\begin{bmatrix}c & -b\\ -b & a\end{bmatrix}$$

$$\Rightarrow (\underset{\sim}{x}-\underset{\sim}{\mu}_x)^T \Sigma_x^{-1} (\underset{\sim}{x}-\underset{\sim}{\mu}_x)$$

$$= \begin{bmatrix}(x_1-\mu_1) & (x_2-\mu_2)\end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix}c & -b\\ -b & a\end{bmatrix}\begin{bmatrix}(x_1-\mu_1)\\(x_2-\mu_2)\end{bmatrix}$$

4.4.6)
(Cont.)

$$= \frac{1}{4}\Big[\big(c(x_1-\mu_1)-b(x_2-\mu_2)\big)\ \big(-b(x_1-\mu_1)+a(x_2-\mu_2)\big)\Big]\begin{bmatrix}(x_1-\mu_1)\\(x_2-\mu_2)\end{bmatrix}$$

$$= \frac{1}{4}\Big(c(x_1-\mu_1)^2-b(x_1-\mu_1)(x_2-\mu_2)-b(x_1-\mu_1)(x_2-\mu_2)+a(x_2-\mu_2)^2\Big)$$

$$= \boxed{\frac{1}{4}\Big(c(x_1-\mu_1)^2-2b(x_1-\mu_1)(x_2-\mu_2)+a(x_2-\mu_2)^2\Big)}$$

equal

$$-\frac{5(x_1+1)^2+8(x_1+1)x_2+4x_2^2}{8} = -\frac{\boxed{(\underset{\sim}{x}-\underset{\sim}{\mu}_x)^T\,\Sigma_x^{-1}\,(\underset{\sim}{x}-\underset{\sim}{\mu}_x)}}{2}$$

Equate coefficients:

$$c = 5,\quad -2b = 8,\quad a = 4$$

$$\Downarrow$$

$$b = -4$$

$$\Rightarrow \boxed{\Sigma_x = \begin{bmatrix}4 & -4\\ -4 & 5\end{bmatrix}}$$

4.5.1) Need to determine number of samples (N) required to achieve a probability of error $P(e) \leq 0.001$ with no quantization and $\frac{a}{\sigma} = 1$.

$$P(e) = Q\left(\frac{a}{\sigma}\sqrt{N}\right) = Q(\sqrt{N}) \leq 0.001$$

From Table A.1, $Q(3.09) \cong 0.001$

$$\Rightarrow \sqrt{N} > 3.09 \Rightarrow N > (3.09)^2 \cong 9.55$$

$$\Rightarrow \boxed{N = 10}$$

4.5.3) Need to determine $P(e)$ for no quantization and $P_H(e)$ for hard decisions (1-bit quantization), given $N = 3$ and $\frac{a}{\sigma} = 2$.

No quantization : $P(e) = Q\left(\frac{a}{\sigma}\sqrt{N}\right)$

$$= Q(2\sqrt{3}) \cong Q(3.464)$$

From Table A.1 :

$Q(3.46) = 0.000270$

$Q(3.47) = 0.000260$

Using linear interpolation,

$$Q(3.464) \cong \boxed{0.000266}$$

1-bit quantization : $P_H(e) = \sum_{i=2}^{3} C_i^3 P_1^i(e)\left[1-P_1(e)\right]^{3-i}$

where $P_1(e) = Q\left(\frac{a}{\sigma}\right) = Q(2) = 0.02275$

$$\Rightarrow P_H(e) = C_2^3 P_1^2(e)\left[1-P_1(e)\right]^1 + C_3^3 P_1^3(e)\left[1-P_1(e)\right]^0$$

$$= 3(0.02275)^2(0.97725) + (1)(0.02275)^3(1)$$

$$\cong \boxed{0.001529}$$

4.6.1) Given the sum of $N = 500$ Bernoulli random variables $(X_1, X_2, \ldots, X_N)$ with $p = 0.5$ (also, statistically independent) (i.e., $P(X_i = K) = p^K (1-p)^{1-p}$, $K = 0, 1$, and $i = 1, 2, \ldots, N$)

Need a bound on the probability that the sample mean differs from p (the true mean) by more than 20%:

Chebyshev's inequality for the sample mean of N statistically independent, identically distributed random variables: (equation 4.6.6):

$$P\left[\left|\frac{1}{N}\sum_{i=1}^{N} X_i - E(X)\right| \geq \varepsilon\right] \leq \frac{Var(X)}{N\varepsilon^2}$$

Recall for a Bernoulli random variable,
$Var(X) = p(1-p)$ and $E(X) = p$

$$\varepsilon = (20\%)\,p \;= (0.2)(0.5) = 0.1$$

$$\Rightarrow P\left[\left|\frac{1}{500}\sum_{i=1}^{500} X_i - p\right| \geq 0.1\right] \leq \frac{(0.5)(1-0.5)}{(500)(0.1)^2}$$

$$= \boxed{0.050} \Leftarrow \text{Chebyshev inequality}$$

4.6.1) For N statistically independent
Cont.) Bernoulli random variables:

$$P\left[\frac{1}{N}\sum_{i=1}^{N} X_i \ge c\right] \le \left[\left(\frac{p}{c}\right)^{c}\left(\frac{1-p}{1-c}\right)^{1-c}\right]^{N}, \quad p < c \le 1$$

$$P\left[\frac{1}{N}\sum_{i=1}^{N} X_i \le c\right] \le \left[\left(\frac{p}{c}\right)^{c}\left(\frac{1-p}{1-c}\right)^{1-c}\right]^{N}, \quad 0 \le c \le p$$

(equations 4.6.10 a, b)

so,

$$P\left[\left|\frac{1}{N}\sum_{i=1}^{N} X_i - p\right| \ge \varepsilon\right]$$

$$= \underbrace{P\left[\frac{1}{N}\sum_{i=1}^{N} X_i \ge \underset{d_1}{\boxed{p+\varepsilon}}\right]}_{\text{bounded by}\downarrow} + \underbrace{P\left[\frac{1}{N}\sum_{i=1}^{N} X_i \le \underset{d_2}{\boxed{p-\varepsilon}}\right]}_{\text{bounded by}\downarrow}$$

$$\le \left[\left(\frac{p}{d_1}\right)^{d_1}\left(\frac{1-p}{1-d_1}\right)^{1-d_1}\right]^{N} + \left[\left(\frac{p}{d_2}\right)^{d_2}\left(\frac{1-p}{1-d_2}\right)^{1-d_2}\right]^{N}$$

where $d_1 = p+\varepsilon = 0.5+0.1 = 0.6$
and $d_2 = p-\varepsilon = 0.5-0.1 = 0.4$

$$= \left[\left(\frac{0.5}{0.6}\right)^{0.6}\left(\frac{1-0.5}{1-0.6}\right)^{1-0.6}\right]^{500} + \left[\left(\frac{0.5}{0.4}\right)^{0.4}\left(\frac{1-0.5}{1-0.4}\right)^{1-0.4}\right]^{500}$$

$$\cong 4.24\times10^{-5} + 4.24\times10^{-5}$$

$$= \boxed{8.48\times10^{-5}} \Leftarrow \text{Chernoff bound}$$

4.7.1) Determine the Gaussian approximation for problem 4.6.1 (i.e., the sum of $N = 500$ Bernoulli random variables $(X_1, \ldots, X_N)$ with $p = 0.5$, and $X_1, \ldots, X_N$ are statistically independent.)

From equation 4.7.1, (central limit theorem)

$\frac{1}{N}\sum_{i=1}^{N} X_i$ is approximately Gaussian,

with mean $= E(X_i) = p = 0.5$

and variance $= Var(X_i)/N$

$= p(1-p)/N$

$$\Rightarrow \left(\frac{1}{N}\sum_{i=1}^{N} X_i - \overset{= E(X_i)}{p}\right)$$ is approximately Gaussian with <u>zero</u> mean, and variance $p(1-p)/N$

$$\Rightarrow P\left[\left|\frac{1}{N}\sum_{i=1}^{N} X_i - p\right| \geq \varepsilon\right]$$

$$\cong 2Q\left[\frac{\varepsilon - \overset{\text{Mean}}{0}}{\underbrace{\sqrt{p(1-p)/N}}_{\text{standard deviation}}}\right]$$

(from problem 4.6.1, $\varepsilon = (20\%)p = 0.1$)

$$= 2Q\left[\frac{0.1}{\sqrt{(0.5)(0.5)/500}}\right] \cong 2Q(4.472)$$

$$\cong 2(3.876 \times 10^{-6})$$ linear interpolation from Table A.1

$$\cong \boxed{7.75 \times 10^{-6}}$$