1) Random variables X and Y are statistically independent and uniform on [0,1]. Random variable Z is equal to the maximum of X and Y. Which of the following is equal to $F_Z(0.5)$?

a: 0
b: 0.25
c: 0.5
d: 1.0
e: None of the above

$F_X(x) = x$, $F_Y(y) = y$ for $x, y \in [0,1]$

$Z = \max(X,Y) \Rightarrow F_Z(z) = F_X(z) F_Y(z)$
$= z \cdot z = z^2$

$\Rightarrow F_Z(0.5) = (0.5)^2 = 0.25$

2) A system is composed of 10 identical components connected in series. The time to failure of each component (in hours) is a random variable having the following probability distribution function:

$$F_X(x) = \begin{cases} 1 - e^{-ax}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}, \quad \text{where } a = 10^{-5}.$$

The failure times of the components are statistically independent. Which of the following is closest to the probability that the system will fail within 5,000 hours of operation?

a: 0
b: 0.2
c: 0.4
d: 0.6
e: 0.8

Let $X_1, \ldots, X_{10}$ = component failure times
and Z = system failure time

$Z = \min(X_1, \ldots, X_{10}) \Rightarrow F_Z(z) = 1 - [1 - F_X(z)]^{10}$

$\Rightarrow F_Z(z) = 1 - [\cancel{1} - (\cancel{1} - e^{-az})]^{10} = 1 - e^{-10az}$, $a = 10^{-5}$

$P(Z \leq 5{,}000) = F_Z(5{,}000) = 1 - e^{-10(10^{-5})(5{,}000)} \cong 0.393$

3) The random variable X has the following probability density function:

$$f_X(x) = \begin{cases} 3x^2, & -1 \leq x \leq 0 \\ 0, & \text{otherwise} \end{cases},$$

Which of the following is equal to the mean of X?

a: −1
b: −3/4
c: −1/2
d: −1/4
e: None of the above

$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-1}^{0} x \cdot 3x^2\,dx$

$= \frac{3}{4} x^4 \Big|_{-1}^{0} = \frac{3}{4}(0 - 1) = -\frac{3}{4}$

4) The random variable X has a mean of −5 and a mean square of 35. Which of the following is equal to the variance of X?

a: 5
b: 10
c: 30
d: 40
e: None of the above

$Var(X) = E(X^2) - [E(X)]^2$

$= 35 - (-5)^2$

$= 35 - 25 = 10$

5) The random variable X is uniform on [0,1]. Which of the following is equal to the mean square of X?

a: 0
b: 1/4
c: 1/2
d: 1
e: None of the above

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$= \int_0^1 x^2 (1)\, dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

6) The random variable X is the number obtained on a single toss of a fair, six-sided die. Which of the following is closest to the value of $E(X^2)$?

a: 0
b: 5
c: 10
d: 15
e: 20

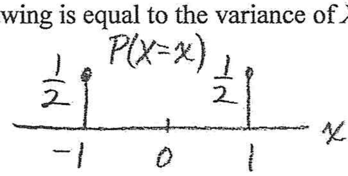
$$E(X^2) = \sum_{K=-\infty}^{\infty} x_K^2 P(X=x_K) = \sum_{K=1}^{6} K^2\left(\tfrac{1}{6}\right)$$

$$= \frac{1}{6}\left[1^2+2^2+3^2+4^2+5^2+6^2\right]$$

$$= \frac{1}{6}\left[1+4+9+16+25+36\right] = \frac{91}{6} \cong 15.17$$

7) The random variable X may equal the values +1 and –1 with equal likelihood, and no other values are possible. Which of the following is equal to the variance of X?

a: 0
b: 1/4
c: 1/2
d: 1
e: None of the above



← $E(X)=0$ by inspection

$$E(X^2) = \left(\tfrac{1}{2}\right)(-1)^2 + \left(\tfrac{1}{2}\right)(1)^2 = \tfrac{1}{2} + \tfrac{1}{2} = 1$$

$$Var(X) = E(X^2) - [E(X)]^2 = 1 - 0 = 1$$

8) The random variable X has the following characteristic function:

$\varphi_X(\omega) = e^{j\omega}$

Which of the following is closest to the mean of X?

a: 1
b: 2
c: 3
d: 4
e: 5

$$\frac{d\Phi_X(\omega)}{d\omega} = j e^{j\omega}$$

$$E(X) = (-j)^1 \frac{d\Phi_X(\omega)}{d\omega}\Big|_{\omega=0} = (-j)(j e^{j(0)})$$

$$= 1$$

9) The characteristic function of random variable X is $\varphi_X(\omega)$.

If $\left.\dfrac{d\varphi_X(\omega)}{d\omega}\right|_{\omega=0} = j5$, Which of the following is equal to the mean of X?

a: 0
b: 5
c: $j5$
d: 1
e: None of the above

$$E(X) = (-j)^1 \left.\frac{d\phi_X(\omega)}{d\omega}\right|_{\omega=0}$$

$$= (-j)(j5) = 5$$

10) The uncorrelated random variables X and Y have a correlation of 6. The mean value of X is 2. Which of the following is equal to the mean value of Y?

a: 1
b: 2
c: 3
d: 4
e: None of the above.

Uncorrelated $\Rightarrow Cov(X,Y)=0$

$$Cov(X,Y) = \underbrace{E(XY)}_{=6} - \underbrace{E(X)}_{=2}E(Y) = 0$$

$$\Rightarrow 6 - 2E(Y) = 0 \Rightarrow E(Y) = \frac{6}{2} = 3$$

11) The random variables X and Y are uncorrelated. Which of the following must be true?

a: X and Y are orthogonal.
b: The correlation of X and Y is equal to zero.
c: The covariance of X and Y is equal to zero.
d: X and Y are statistically independent.
e: None of the above

Uncorrelated $\Leftrightarrow Cov(X,Y)=0$

12) The random variables X and Y are related by the equation $Y + 4 = 3X$. The variance of X is 5. Which of the following is equal to the variance of Y?

a: 11
b: 15
c: 19
d: 45
e: None of the above

$$Y = 3X - 4$$

$$\Rightarrow Var(Y) = (3)^2 Var(X)$$

$$= (9)(5) = 45$$

13) The random variables X and Y are orthogonal. Which of the following must be true?

a: The covariance of X and Y is equal to zero.
b: X and Y are uncorrelated.
c: X and Y are statistically independent.
d: The correlation coefficient of X and Y is equal to zero.
e: None of the above

Orthogonal $\Longleftrightarrow E(XY)=0$

14) The random variables X and Y have a covariance of 2. The variance of X is 2, and the variance of Y is 8. Which of the following is the value of the correlation coefficient of X and Y?

a: 0
b: 1/8
c: 1/4
d: 1/2
e: None of the above

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{2}{\sqrt{2\cdot 8}} = \frac{2}{4} = \frac{1}{2}$$

15) The random variables X_1 and X_2 are jointly Gaussian with the following mean vector and covariance matrix:

$$\mu_X = \begin{bmatrix} 2 \\ -7 \end{bmatrix} \qquad \Sigma_X = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

The random variables Y_1 and Y_2 are formed from X_1 and X_2 as follows:
$Y_1 = 2X_1 - 4X_2 - 3$, and $Y_2 = 4X_1 + 5X_2$.
Which of the following is equal to the variance of Y_2?

a: 36
b: 66
c: 32
d: –32
e: None of the above

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & -4 \\ 4 & 5 \end{bmatrix}}_{A} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \underbrace{\begin{bmatrix} -3 \\ 0 \end{bmatrix}}_{B}$$

$$\Sigma_Y = A\Sigma_X A^T = \begin{bmatrix} 2 & -4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -8 \\ 4 & 10 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -4 & 5 \end{bmatrix} = \begin{bmatrix} 36 & -32 \\ -32 & 66 \end{bmatrix}$$

$\leftarrow Var(Y_2)$

16) The random variables X_1 and X_2 are jointly Gaussian with the following mean vector and covariance matrix:

$$\mu_X = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad \Sigma_X = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$$

The random variables Y_1 and Y_2 are formed from X_1 and X_2 as follows:
$Y_1 = X_1 + 2X_2 + 3$, and $Y_2 = 4X_1 - X_2$.
Which of the following is equal to the covariance of Y_1 and Y_2?

a: –14
b: 14
c: –19
d: 19
e: None of the above

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}}_{A} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 3 \\ 0 \end{bmatrix}}_{B}$$

$$\Sigma_Y = A\Sigma_X A^T = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 10 \\ 14 & -14 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 19 & -14 \\ -14 & 70 \end{bmatrix}$$

$\leftarrow Cov(Y_1, Y_2)$

17) The random process $X(t)$ is wide sense stationary and has autocorrelation function $R_x(t_1, t_2)$. If $R_x(0, 2) = -2$, which of the following is equal to the value of $R_x(-1, 1)$?

a: -1

b: 1

c: -2

d: 2

e: None of the above

$R_X(t_1, t_2)$ depends only on $\tau = t_2 - t_1$

$$\Rightarrow \underbrace{R_X(-1, 1)}_{\tau = 2} = \underbrace{R_X(0, 2)}_{\tau=2} = -2$$

18) The ergodic random process $X(t)$ has the following autocorrelation function:

$$R_x(\tau) = 10e^{-2|\tau|} + 25$$

Which of the following is equal to the variance of $X(t)$?

a: 10

b: 25

c: 35

d: 100

e: None of the above

$$E[X^2(t)] = R_X(0) = 10\,e^{-2(0)} + 25 = 35$$

$$\{E[X(t)]\}^2 = \lim_{|\tau| \to \infty} R_X(\tau) = 10\,e^{-2(\infty)} + 25 = 25$$

$$\text{Var}[X(t)] = E[X^2(t)] - \{E[X(t)]\}^2 = 35 - 25 = 10$$

19) The random process $X(t)$ is wide sense stationary and has autocorrelation function $R_x(t_1, t_2)$. Which of the following must be true? Choose the best answer.

a: The mean of $X(t)$ is zero.

b: The mean of $X(t)$ is constant.

c: $R_x(t_1, t_2)$ depends only on $t_1 - t_2$ and not on the individual values of t_1 and t_2.

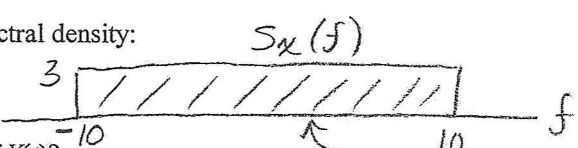
d: Both b and c

e: None of the above

WSS $\Leftrightarrow$ 1) $E[X(t)]$ = constant and 2) $R_X(t_1, t_2)$ depends only on $\tau = t_2 - t_1$.

20) The random process $X(t)$ has the following power spectral density:

$$S_x(f) = \begin{cases} 3, & |f| \le 10 \\ 0, & \text{otherwise} \end{cases}$$



Which of the following is equal to the mean square of $X(t)$?

a: 60

b: 30

c: 10

d: 3

e: None of the above

$$E[X^2(t)] = \int_{-\infty}^{\infty} S_X(f)\,df = \text{area}$$

$$= (20)(3) = 60$$