ECE 317
Chapter 7
Homework Solutions

7.2.1) Given random process

$$X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t),$$

ω_0 is constant; A and B are random variables with $E(A) = E(B) = 0$, $Var(A) = Var(B) = \sigma^2$, A and B uncorrelated

Need to determine whether $X(t)$ is wide sense stationary.

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$= E\{[A\cos(\omega_0 t_1) + B\sin(\omega_0 t_1)][A\cos(\omega_0 t_2) + B\sin(\omega_0 t_2)]$$

$$= E[A^2\cos(\omega_0 t_1)\cos(\omega_0 t_2) + AB\cos(\omega_0 t_1)\sin(\omega_0 t_2$$
$$+ AB\sin(\omega_0 t_1)\cos(\omega_0 t_2) + B^2\sin(\omega_0 t_1)\sin(\omega_0 t_2)$$

$$= E[A^2\cos(\omega_0 t_1)\cos(\omega_0 t_2)]$$
$$+ E[AB\cos(\omega_0 t_1)\sin(\omega_0 t_2)]$$
$$+ E[AB\sin(\omega_0 t_1)\cos(\omega_0 t_2)]$$
$$+ E[B^2\sin(\omega_0 t_1)\sin(\omega_0 t_2)]$$

$$= \cos(\omega_0 t_1)\cos(\omega_0 t_2)\,E(A^2)$$
$$+ \cos(\omega_0 t_1)\sin(\omega_0 t_2)\,E(AB)$$
$$+ \sin(\omega_0 t_1)\cos(\omega_0 t_2)\,E(AB)$$
$$+ \sin(\omega_0 t_1)\sin(\omega_0 t_2)\,E(B^2)$$

7.2.1) Cont.) A and B are uncorrelated

$$\Rightarrow E\{[A-E(A)][B-E(B)]\} = 0$$

and $E(A) = E(B) = 0$

$$\Rightarrow E(AB) = 0$$

Also, $\mathrm{Var}(A) = \sigma^2$

$$= E(A^2) - [\overset{0}{\cancel{E(A)}}]^2 = E(A^2)$$

and $\mathrm{Var}(B) = \sigma^2$

$$= E(B^2) - [\overset{0}{\cancel{E(B)}}]^2 = E(B^2)$$

Substituting into the expression on p.1 :

$$R_x(t_1, t_2) = \sigma^2 \cos(\omega_0 t_1)\cos(\omega_0 t_2) + 0 + 0 + \sigma^2 \sin(\omega_0 t_1)\sin(\omega_0 t_2)$$

$$= \sigma^2 [\cos(\omega_0 t_1)\cos(\omega_0 t_2) + \sin(\omega_0 t_1)\sin(\omega_0 t_2)]$$

$$= \sigma^2 \cos(\omega_0 t_1 - \omega_0 t_2)$$

OR $\sigma^2 \cos(\omega_0 t_2 - \omega_0 t_1)$

(note: cosine is an even function)

$$\Rightarrow R_x(t_1, t_2) = \sigma^2 \cos[\omega_0(t_2 - t_1)]$$

$$= \sigma^2 \cos(\omega_0 \tau), \text{ where } \tau = t_2 - t_1$$

$$= R_x(\tau)$$

$\Rightarrow$ $\boxed{X(t) \text{ is wide sense stationary.}}$

7.2.2) Given $X(t) = Y\cos(2\pi t)$,
where $f_Y(y) = \frac{1}{2}$, $-1 \le y \le 1$

Need to determine $E[X(t)]$ and $E[X^2(t)]$ and determine whether $X(t)$ is strict sense stationary or wide sense stationary.

$$E[X(t)] = E[Y\cos(2\pi t)] = \cos(2\pi t)\,E(Y)$$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_{-1}^{1} y \cdot \frac{1}{2}\,dy$$

$$= \left.\frac{y^2}{4}\right|_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$$

$$\Rightarrow \boxed{E[X(t)] = 0}$$

$$E[X^2(t)] = E[Y^2\cos^2(2\pi t)] = \cos^2(2\pi t)\,E(Y^2)$$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \int_{-1}^{1} y^2 \cdot \frac{1}{2}\,dy$$

$$= \left.\frac{y^3}{6}\right|_{-1}^{1} = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$\Rightarrow \boxed{E[X^2(t)] = \frac{1}{3}\cos^2(2\pi t) = \frac{1}{6}(1+\cos(4\pi t))}$$

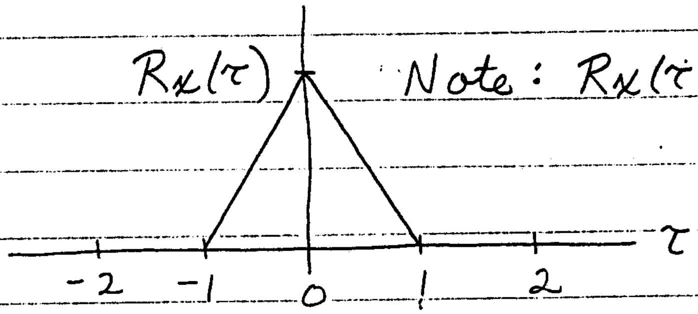
Note: $E[X^2(t)]$ is a function of t.

$\Rightarrow R_X(t,t) = E[X^2(t)]$ is a function of t.

$\Rightarrow$ $\boxed{X(t) \text{ is not wide sense stationary.}}$

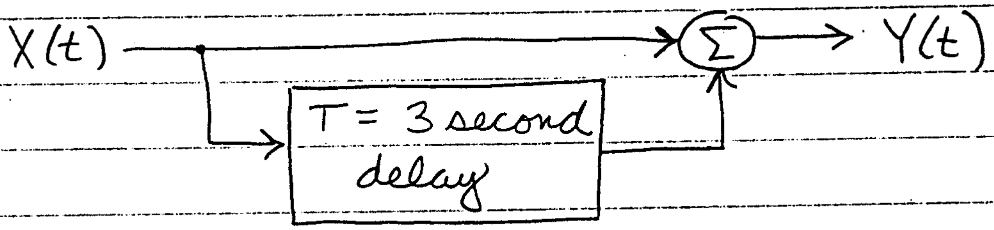
$\Rightarrow$ $\boxed{X(t) \text{ is not strict sense stationary.}}$

7.2.5) Given $X(t)$ with $R_X(\tau)$:



Note: $R_X(\tau) = \begin{cases} 1-\tau, & -1 \le \tau \le 1 \\ 0, & \text{otherwise} \end{cases}$

and the following system:



Need to determine $R_Y(\tau)$

$$Y(t) = X(t) + X(t-3)$$

$$R_Y(\tau) = E\left[Y(t)\,Y(t+\tau)\right]$$

$$= E\left\{\left[X(t) + X(t-3)\right]\left[X(t+\tau) + X(t+\tau-3)\right]\right\}$$

$$= E\left[X(t)X(t+\tau) + X(t)X(t+\tau-3) + X(t-3)X(t+\tau) + X(t-3)X(t+\tau-3)\right]$$

$$= E\left[X(t)X(t+\tau)\right] + E\left[X(t)X(t+\tau-3)\right] + E\left[X(t-3)X(t+\tau)\right] + E\left[X(t-3)X(t+\tau-3)\right]$$

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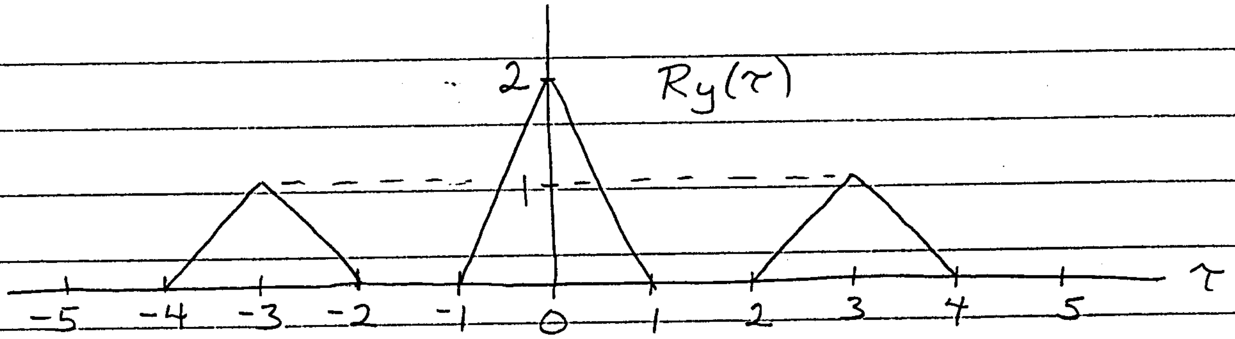
7.2.5
p.2

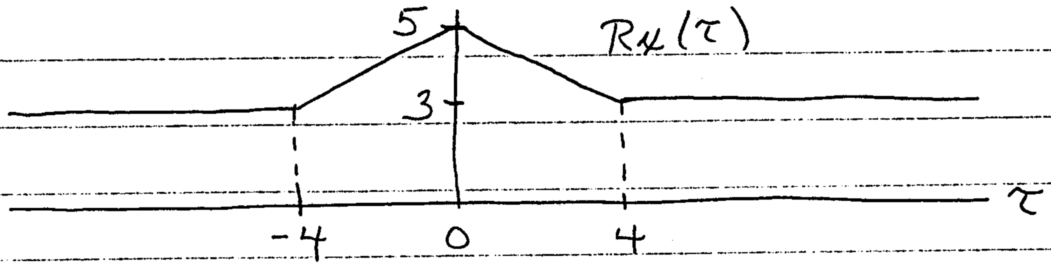
7.2.5) Cont.) $\Rightarrow R_y(\tau) = R_x(\tau) + R_x(\tau-3)$

$+ R_x(\tau+3) + R_x(\tau)$

$$= \boxed{2R_x(\tau) + R_x(\tau-3) + R_x(\tau+3)}$$



7.2.9) Given $R_X(\tau)$ as shown:



Need to determine $E[X(t)]$, $E[X^2(t)]$, and $Var[X(t)]$.

Note: $R_X(\tau) = \begin{cases} 5 - \frac{1}{2}|\tau|, & -4 \le \tau \le 4 \\ 3, & \text{otherwise} \end{cases}$

$$\{E[X(t)]\}^2 = \lim_{\tau \to \infty} R_X(\tau) = 3$$

$$\Rightarrow E[X(t)] = \pm\sqrt{3} \cong \boxed{\pm 1.732}$$

$$E[X^2(t)] = R_X(0) = \boxed{5}$$

$$Var(X) = E[X^2(t)] - \{E[X(t)]\}^2$$

$$= 5 - 3 = \boxed{2}$$

7.3.1) Given $R_x(\tau) = 1 \ , \ -2 \le \tau \le 2$

Need to determine $S_x(f)$

$$S_x(f) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j2\pi f\tau} d\tau$$

$$= \int_{-2}^{2} (1) e^{-j2\pi f \tau} d\tau$$

$$= \frac{1}{-j2\pi f} e^{-j2\pi f\tau} \Big|_{-2}^{2}$$

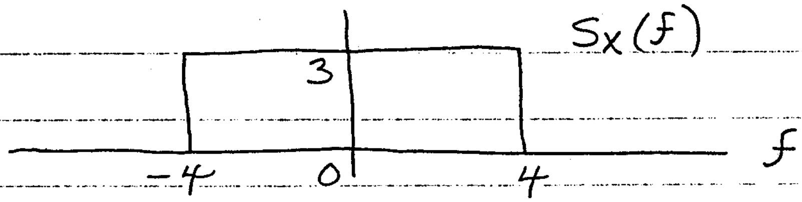
$$= \frac{-1}{j2\pi f} \left(e^{-j4\pi f} - e^{j4\pi f} \right)$$

$$= \frac{1}{\pi f} \left(\frac{e^{j4\pi f} - e^{-j4\pi f}}{2j} \right)$$

$$= \frac{1}{\pi f} \sin(4\pi f)$$

$$= \boxed{4 \frac{\sin(4\pi f)}{4\pi f} \ , \quad -\infty < f < \infty}$$

7.3.5) Given $S_x(f)$ as shown:



Need to determine $E[X(t)]$, $E[X^2(t)]$, and $R_x(\tau)$.

$$\text{Note } S_x(f) = \begin{cases} 3, & -4 \leq f \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(f) e^{j2\pi f\tau} df$$

$$E[X^2(t)] = R_x(0) = \int_{-\infty}^{\infty} S_x(f) \overset{1}{\cancel{e^{0}}} df$$

$$= \int_{-4}^{4} (3) df = 3f \Big|_{-4}^{4} = 3(8) = \boxed{24}$$

$$R_x(\tau) = \int_{-4}^{4} (3) e^{j2\pi f\tau} df$$

$$= \frac{3}{j2\pi\tau} e^{j2\pi f\tau} \Big|_{f=-4}^{f=4}$$

$$= \frac{3}{j2\pi\tau} \left(e^{j8\pi\tau} - e^{-j8\pi\tau} \right)$$

$$= \frac{3}{\pi\tau} \left(\frac{e^{j8\pi\tau} - e^{-j8\pi\tau}}{2j} \right)$$

7.3.5) Cont.) $\Rightarrow R_X(\tau) = \dfrac{3}{\pi\tau}\sin(8\pi\tau)$

$$= \boxed{24\,\frac{\sin(8\pi\tau)}{8\pi\tau}}$$

$$\{E[X(t)]\}^2 = \lim_{\tau\to\infty} R_X(\tau) = 0$$

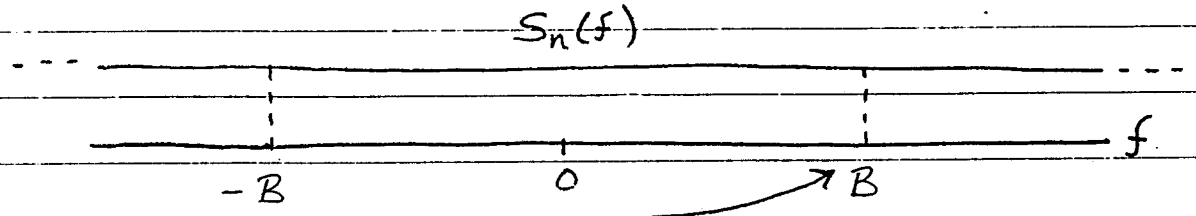
$$\Rightarrow \boxed{E[X(t)] = 0}$$

Note: The fact that $S_X(f)$ does not contain an impulse at $f=0$ (i.e., $\delta(f)$) also indicates that $E[X(t)] = 0$.

7.3.7) Determine the thermal noise voltage in a 1-Ω resistor when measured on an oscilloscope:

From equation 7.3.10, we assume thermal noise is approximately white:

$$S_n(f) = 7.946\times10^{-21}\,\overset{1\Omega}{\cancel{R}}, \quad -6\times10^{11} < f < 6\times10^{11}$$



Let B = bandwidth of oscilloscope $\ll 6\times10^{11}$

From equation 7.3.2, $E[N^2(t)] = \int_{-\infty}^{\infty} S_n(f)\,df$

As measured by the oscilloscope,

$$E[N^2(t)] = \int_{-B}^{B \leftarrow (\text{Limited Bandwidth})} S_n(f)\,df = S_n(f)\cdot 2B$$

$\Rightarrow$ RMS thermal noise voltage $= \sqrt{E[N^2(t)]}$

$$= \sqrt{S_n(f)\cdot 2B} = \sqrt{7.946\times10^{-21}(1)\cdot(2B)} = V$$

a) $B = 10\,\text{MHz} \Rightarrow V = \sqrt{7.946\times10^{-21}(2\cdot10^{7})}$

$$\cong \boxed{0.399\ \mu V}$$

b) $B = 50\,\text{MHz} \Rightarrow V = \sqrt{7.946\times10^{-21}(2\cdot50\cdot10^{6})} \cong \boxed{0.891\,\mu V}$

7.4.1) Given a linear system with $h(t) = e^{-t}$, $t \geq 0$, and input $X(t)$ a white noise process with

$$S_x(f) = \frac{N_0}{2}, \quad -\infty < f < \infty$$

Need power spectral density of output, $Y(t)$:

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi f t} dt = \int_0^{\infty} e^{-t} e^{-j2\pi f t} dt$$

$$= \int_0^{\infty} e^{-(1+j2\pi f)t} dt = -\frac{1}{1+j2\pi f} e^{-(1+j2\pi f)t} \Big|_{t=0}^{\infty}$$

$$= \cancel{-} \frac{1}{1+j2\pi f} (0 \cancel{-} 1) = \frac{1}{1+j2\pi f}$$

$$S_y(f) = \underset{\downarrow}{S_x(f)} \underset{\downarrow}{|H(f)|^2}$$

$$= \frac{N_0}{2} \left| \frac{1}{1+j2\pi f} \right|^2 = \frac{N_0}{2} \cdot \frac{1}{1^2 + (2\pi f)^2}$$

$$\Longrightarrow \boxed{S_y(f) = \frac{N_0}{2} \cdot \frac{1}{1+(2\pi f)^2}, \quad -\infty < f < \infty}$$

Also need autocorrelation function of output:

$$R_x(\tau) = \mathcal{F}^{-1}\left\{\frac{N_0}{2}\right\} = \frac{N_0}{2} \delta(\tau)$$

$$R_y(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_x(\tau - v + u) h(u) h(v) \, du \, dv$$

7.4.1) Cont.) $$\Rightarrow R_y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{N_0}{2}\,\delta(\underbrace{\tau - v + u}_{})\,h(u)\,h(v)\,du\,dv$$

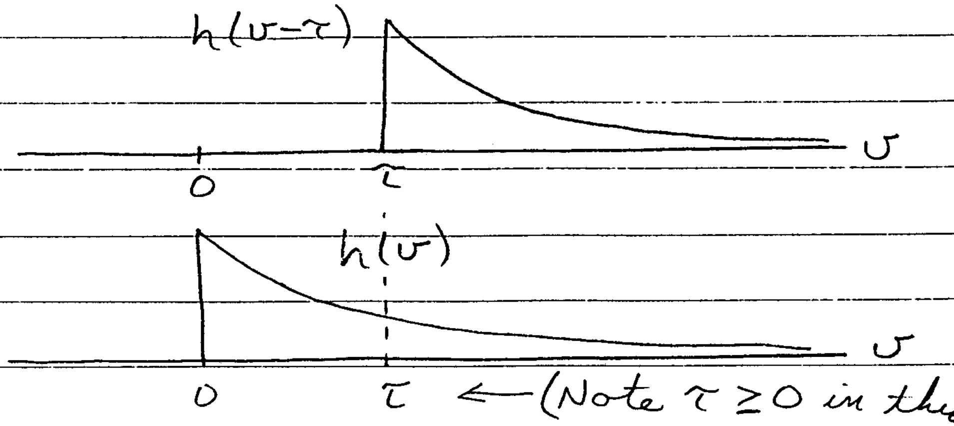
Use sifting property of $\delta()$ to evaluate inner integral

(Argument of $\delta()$) $= 0$ when $u = v - \tau$

$$\Rightarrow R_y(\tau) = \int_{-\infty}^{\infty} \frac{N_0}{2}\,h(v-\tau)\,h(v)\,dv$$

Recall $h(t) = e^{-t}$ for $t \geq 0$

$\Rightarrow h(v-\tau) = e^{-(v-\tau)}$ for $v - \tau \geq 0$
(for $v \geq \tau$)



← (Note $\tau \geq 0$ in this plot).

Note: the product $h(v-\tau)h(v) = 0$ for $v < \boxed{\tau}$
if $\tau \geq 0$ (shown in plot),
but $h(v-\tau)h(v) = 0$ for $v < \boxed{0}$
if $\tau < 0$.

$\Rightarrow$ Lower limit of integration will
be $v = \tau$ if $\tau \geq 0$
and $v = 0$ if $\tau < 0$.

If $\tau \geq 0$, $R_y(\tau) = \int_{\tau}^{\infty} \frac{N_0}{2}\,e^{-(v-\tau)}\,e^{-v}\,dv$

7.4.1) Cont.)

$$R_y(\tau) = \frac{N_0}{2}\cdot e^{\tau}\int_{\tau}^{\infty} e^{-2\upsilon}\,d\upsilon$$

$$= \frac{N_0}{2} e^{\tau}\cdot\left(-\frac{1}{2}\right) e^{-2\upsilon}\Big|_{\tau}^{\infty}$$

$$= -\frac{N_0}{4} e^{\tau} e^{-2\upsilon}\Big|_{\tau}^{\infty} = \cancel{-}\frac{N_0}{4} e^{\tau}\left(0 \cancel{-} e^{-2\tau}\right)$$

$$= \frac{N_0}{4} e^{-\tau} \quad \text{for } \tau \geq 0$$

If $\tau < 0$, $R_y(\tau) = \int_{0}^{\infty} \frac{N_0}{2} e^{-(\upsilon-\tau)} e^{-\upsilon}\,d\upsilon$

(Same integral as before, but lower limit $= 0$)

$$\Rightarrow R_y(\tau) = -\frac{N_0}{4} e^{\tau} e^{-2\upsilon}\Big|_{0}^{\infty}$$

$$= \cancel{-}\frac{N_0}{4} e^{\tau}\left(0 \cancel{-} 1\right) = \frac{N_0}{4} e^{\tau} \quad \text{for } \tau < 0$$

$$\Rightarrow R_y(\tau) = \begin{cases} \frac{N_0}{4} e^{-\tau}, & \tau \geq 0 \\ \frac{N_0}{4} e^{\tau}, & \tau < 0 \end{cases}$$

OR, $\boxed{R_y(\tau) = \frac{N_0}{4} e^{-|\tau|} \quad \text{for all } \tau}$

Note: The formula for $R_y(\tau)$ for $\tau < 0$ could have been obtained without performing the integration, because $R_y(\tau)$ must be an even function of τ ($\Rightarrow R_y(\tau) = R_y(-\tau)$).

Also note: $R_y(\tau)$ could be obtained instead from

$$R_y(\tau) = \mathcal{F}^{-1}\{S_y(f)\}.$$

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7.4.3

7.4.3) Need cross-correlation of $Z_1(t) = X(t) + Y(t)$ and $Z_2(t) = X(t) - Y(t)$, given $X(t)$ and $Y(t)$ are statistically independent random processes with $E[X(t)] = E[Y(t)] = 0$ and

$$R_x(\tau) = e^{-|\tau|}, \quad R_y(\tau) = 2e^{-|\tau|}$$

$$R_{Z_1Z_2}(t_1, t_2) = E\left[Z_1(t_1)\, Z_2(t_2)\right]$$

$$= E\left[\{X(t_1) + Y(t_1)\}\{X(t_2) - Y(t_2)\}\right]$$

$$= E\left[X(t_1)X(t_2) - X(t_1)Y(t_2) + X(t_2)Y(t_1) - Y(t_1)Y(t_2)\right]$$

$$= E\left[X(t_1)X(t_2)\right] - \cancelto{0}{E\left[X(t_1)Y(t_2)\right]} + \cancelto{0}{E\left[X(t_2)Y(t_1)\right]} - E\left[Y(t_1)Y(t_2)\right]$$

(X(t) and Y(t) are statistically independent)

$$= E\left[X(t_1)X(t_1+\tau)\right] - E\left[Y(t_1)Y(t_1+\tau)\right]$$

where $\tau = t_2 - t_1$

$$= R_x(\tau) - R_y(\tau) = e^{-|\tau|} - 2e^{-|\tau|} = -e^{-|\tau|}$$

$$\Rightarrow \boxed{R_{Z_1Z_2}(\tau) = -e^{-|\tau|} \text{ for all } \tau.}$$