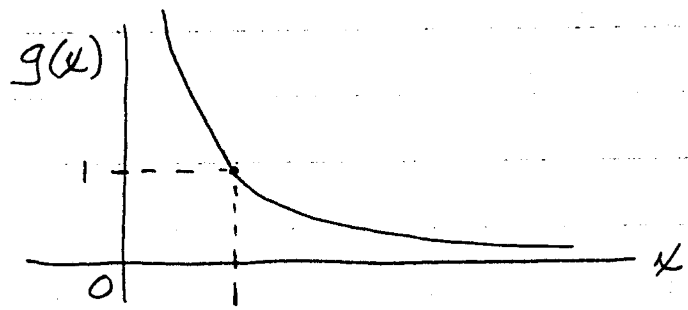
ECE 317
Chapter 3
Homework Solutions

3.1.1) Given $y = 1/x$ and $f_X(x) = 2/x^3$, $1 \le x < \infty$

Need to determine $f_Y(y)$.

$g(x) = 1/x$



Note $g(x)$ is monotonically decreasing for $1 \le x < \infty$.

$\Rightarrow$ Inverse $h(y)$ exists, $x = h(y) = 1/y$

Note: for $1 \le x < \infty$, $0 < y \le 1$

$$f_Y(y) = f_X(h(y))\left|\frac{dh(y)}{dy}\right|$$

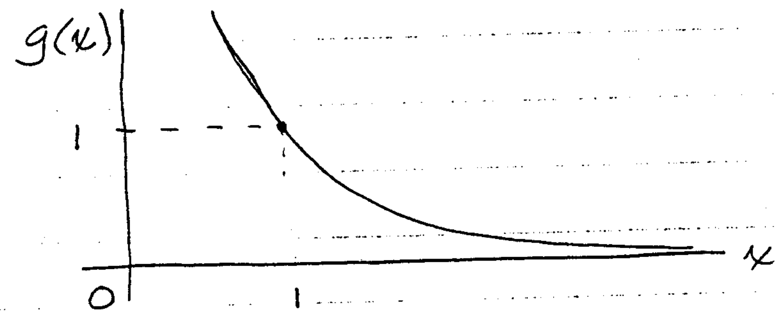
$$= 2(h(y))^{-3} \underbrace{\left|-y^{-2}\right|}_{= y^{-2} \text{ for } y > 0} = 2y^3 \cdot y^{-2}$$

$$\Rightarrow \boxed{f_Y(y) = 2y, \quad 0 < y \le 1}$$

Hubbard

3.1.2) Given $y = 1/x^2$ and $f_X(x) = 1/x^2$, $1 \le x < \infty$
Need to determine $f_Y(y)$.

$$g(x) = \frac{1}{x^2}$$



Note $g(x)$ is monotonically decreasing for $1 \le x < \infty$.

$\Rightarrow$ Inverse $h(y)$ exists, $x = h(y) = \frac{1}{\sqrt{y}} = y^{-\frac{1}{2}}$

Note: for $1 \le x < \infty$, $0 < y \le 1$

$$f_Y(y) = f_X(h(y)) \left| \frac{dh(y)}{dy} \right|$$

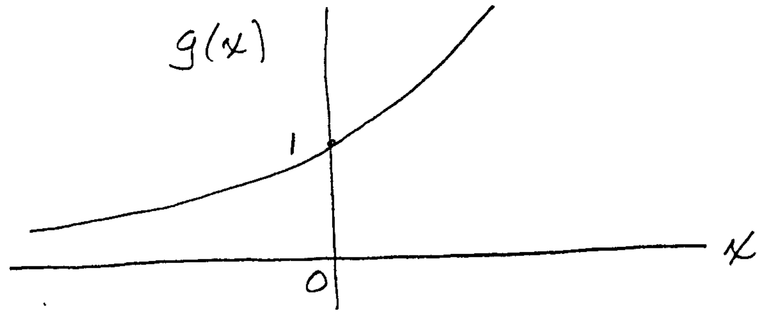
$$= \frac{1}{(h(y))^2} \underbrace{\left| -\frac{1}{2} y^{-\frac{3}{2}} \right|}_{= \frac{1}{2} y^{-\frac{3}{2}} \text{ for } y > 0}$$

$$= \frac{1}{\left(y^{-\frac{1}{2}}\right)^2} \cdot \frac{1}{2} y^{-\frac{3}{2}} = \frac{1}{2} y^{-\frac{1}{2}}$$

$$\Rightarrow \boxed{f_Y(y) = \frac{1}{2} y^{-\frac{1}{2}}, \quad 0 < y \le 1}$$

3.1.4) Given $y = e^{2x}$ and $f_X(x) = \frac{1}{2} e^{-|x|}$, $-\infty < x < \infty$
Need to determine $f_Y(y)$

$$g(x) = e^{2x}$$



Note $g(x)$ is monotonically increasing for $-\infty < x < \infty$

$\Rightarrow$ Inverse $h(y)$ exists:

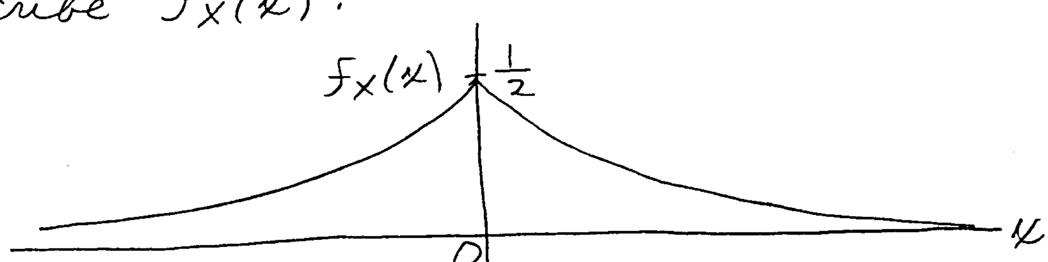
$$y = e^{2x} \Rightarrow \ln y = 2x \Rightarrow x = \frac{\ln y}{2}$$

$$\Rightarrow h(y) = \frac{\ln y}{2} \Rightarrow \frac{dh(y)}{dy} = \frac{1}{2y}$$

Note: for $-\infty < x < \infty$, $0 < y < \infty$

$$f_Y(y) = f_X(h(y)) \left| \frac{dh(y)}{dy} \right|$$

This must be computed over two intervals because of the two functions that describe $f_X(x)$:



$$\left.\begin{aligned} &\text{For } x \geq 0, \quad f_X(x) = \tfrac{1}{2} e^{-x} \\ &\text{For } x \leq 0, \quad f_X(x) = \tfrac{1}{2} e^{x} \end{aligned}\right\} = \frac{1}{2} e^{-|x|}$$

for all x

3.1.4) For $-\infty < x < 0$: $(0 < y < 1)$
Cont.)

$$f_Y(y) = \frac{1}{2} e^{h(y)} \underbrace{\left|\frac{1}{2y}\right|}_{=\frac{1}{2y} \text{ for } y>0} = \left(\frac{1}{2} e^{\frac{\ln y}{2}}\right)\left(\frac{1}{2y}\right)$$

$$= \frac{1}{2} e^{\ln(y^{\frac{1}{2}})} \cdot \frac{1}{2y} = \frac{1}{2} y^{\frac{1}{2}} \cdot \frac{1}{2y}$$

$$= \boxed{\frac{1}{4} y^{-\frac{1}{2}}}$$

For $0 \le x < \infty$: $(1 \le y < \infty)$

$$f_Y(y) = \frac{1}{2} e^{-h(y)} \underbrace{\left|\frac{1}{2y}\right|}_{=\frac{1}{2y}} = \left(\frac{1}{2} e^{-\frac{\ln y}{2}}\right)\left(\frac{1}{2y}\right)$$

$$= \frac{1}{2} e^{\ln(y^{-\frac{1}{2}})} \cdot \frac{1}{2y} = \frac{1}{2} y^{-\frac{1}{2}} \cdot \frac{1}{2y}$$

$$= \boxed{\frac{1}{4} y^{-\frac{3}{2}}}$$

$$\Rightarrow \boxed{f_Y(y) = \begin{cases} \frac{1}{4} y^{-\frac{1}{2}}, & 0 < y < 1 \\ \frac{1}{4} y^{-\frac{3}{2}}, & 1 \le y < \infty \end{cases}}$$

3.1.7) Given U with $f_U(u) = 1,\ 0 \le u \le 1$

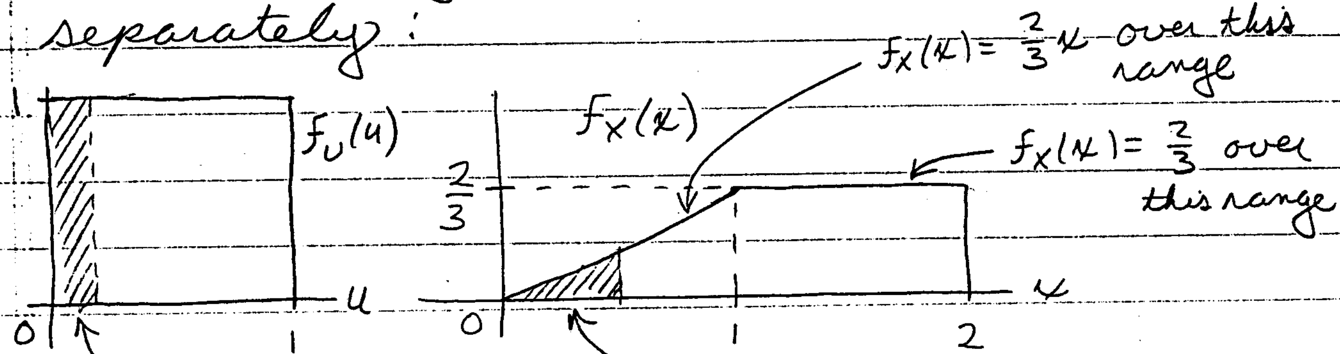
Need to determine the transformation

$$x = g(u)$$

such that $f_X(x) = \begin{cases} \frac{2}{3}x, & 0 \le x \le 1 \\ \frac{2}{3}, & 1 \le x \le 2 \end{cases}$

$$F_U(u) = F_X(x) = F_X(g(u))$$

The two ranges of x must be evaluated separately:



$$F_U(u) = \int_0^u (1)\,dt = u$$

For $0 \le x < 1$, $F_X(x) = \int_0^x \frac{2}{3}t\,dt$

$$= \frac{2}{3} \cdot \frac{t^2}{2} \Big|_0^x = \frac{1}{3}(x^2 - 0) = \frac{1}{3}x^2$$

$$\Rightarrow u = \frac{1}{3}x^2 \Rightarrow \boxed{x = \sqrt{3u}}$$

Note: for $0 \le x < 1$, $\boxed{0 \le u < \frac{1}{3}}$

3.1.7) Cont.) For $1 \le x \le 2$, $F_X(x) = (\text{triangle area}) + \int_1^x \left(\frac{2}{3}\right) dt$

$\left(\begin{array}{l}\text{triangle area} \\ = \frac{1}{2}(1)\left(\frac{2}{3}\right) = \frac{1}{3}\end{array}\right)$

$$\Rightarrow F_X(x) = \frac{1}{3} + \frac{2}{3} t \Big|_1^x$$

$$= \frac{1}{3} + \frac{2}{3}(x-1) = \frac{2}{3}x - \frac{1}{3}$$

$$\Rightarrow u = \frac{2}{3}x - \frac{1}{3} \Rightarrow u + \frac{1}{3} = \frac{2}{3}x$$

$$\Rightarrow \boxed{x = \frac{3}{2}u + \frac{1}{2}}$$

Note: for $1 \le x \le 2$, $\boxed{\frac{1}{3} \le u \le 1}$

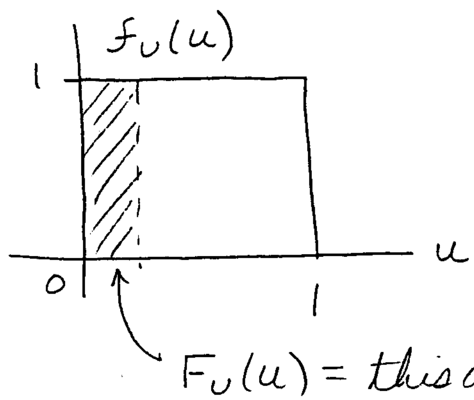
$$\Rightarrow \boxed{g(u) = \begin{cases} \sqrt{3u}, & 0 \le u < \frac{1}{3} \\ \frac{3}{2}u + \frac{1}{2}, & \frac{1}{3} \le u \le 1 \end{cases}}$$

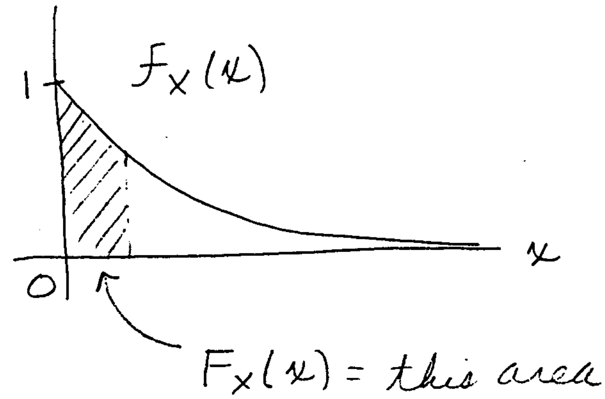
3.1.9) Given U with $f_U(u) = 1, \quad 0 \le u \le 1$

Need to find transformation $x = g(u)$

such that $f_X(x) = \dfrac{1}{(x+1)^2}, \quad 0 \le x < \infty$

$$F_U(u) = F_X(x) = F_X(g(u))$$





$$F_U(u) = \int_0^u (1)\,dt = u$$

$$F_X(x) = \int_0^x \frac{1}{(t+1)^2}\,dt = -\frac{1}{(t+1)}\Bigg|_0^x$$

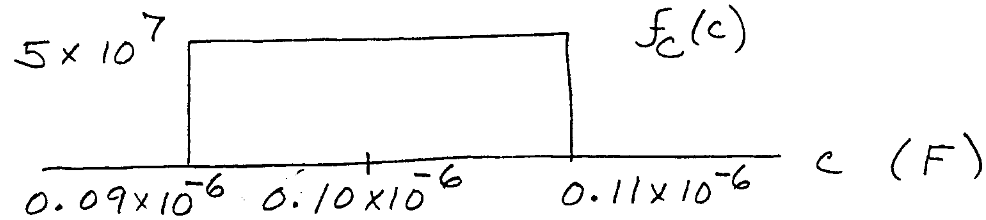
$$= -\frac{1}{x+1} - (-1) = 1 - \frac{1}{x+1}$$

$$u = 1 - \frac{1}{x+1} \Rightarrow \frac{1}{x+1} = 1 - u$$

$$\Rightarrow x + 1 = \frac{1}{1-u} \Rightarrow x = \frac{1}{1-u} - 1 = \frac{u}{1-u}$$

$$\Rightarrow \boxed{g(u) = \frac{u}{1-u}, \quad 0 \le u \le 1}$$

3.1.12) Given a capacitor with random capacitance, C, uniformly distributed about $0.1\,\mu F$ by $\pm 10\%$



Need the density function of the <u>magnitude</u> of the capacitor's impedance (wording of problem in book is misleading)

i.e., we want $|z| = \left|\frac{1}{j\omega c}\right| = \frac{1}{\omega c}$

where $\omega = 5\times10^6$ rad/s

$$\Rightarrow |z| = g(c) = \frac{1}{5\times10^6 c} = 2\times10^{-7}c^{-1}$$

$$\Rightarrow c = h(|z|) = \frac{1}{5\times10^6 |z|} = 2\times10^{-7}|z|^{-1}$$

Note: for $0.09\times10^{-6} \le c \le 0.11\times10^{-6}$,
$1.818 \le |z| \le 2.222$

$$f_{|z|}(|z|) = f_C(h(|z|))\left|\frac{dh(|z|)}{d|z|}\right|$$

$$\frac{dh(|z|)}{d|z|} = -2\times10^{-7}|z|^{-2} \Rightarrow \left|\frac{dh(|z|)}{d|z|}\right| = 2\times10^{-7}|z|^{-2}$$

3.1.12) Cont.)

$$\Rightarrow f_{|z|}(|z|) = \underbrace{\left(5\times10^{7}\right)}_{f_C(h(|z|))}\underbrace{\left(2\times10^{-7}|z|^{-2}\right)}_{\left|\frac{dh(|z|)}{d|z|}\right|}$$

$$= 10\,|z|^{-2}$$

$$\Rightarrow \boxed{\begin{aligned} f_{|z|}(|z|) &= 10\,|z|^{-2}, \quad 1.818 \le |z| \le 2.222 \\ &= 0, \quad \text{otherwise} \end{aligned}}$$

(Units are Ω)

3.2.1) Given $f_{X_1X_2}(x_1, x_2) = e^{-x_1 - x_2}$

for $0 \le x_1 < \infty$, $0 \le x_2 < \infty$

and $y_1 = \dfrac{x_1}{x_1 + x_2}$, $y_2 = x_1 + x_2$

Need to determine $f_{Y_1Y_2}(y_1, y_2)$

$$g_1(x_1, x_2) = \frac{x_1}{\underbrace{x_1 + x_2}} = y_1 \qquad \Rightarrow \frac{x_1}{y_2} = y_1$$

$$g_2(x_1, x_2) = x_1 + x_2 = y_2$$

$$\Rightarrow x_1 = \underbrace{y_1 y_2}$$

$$\Rightarrow y_1 y_2 + x_2 = y_2 \qquad \Rightarrow \qquad x_2 = y_2 - y_1 y_2$$

$$\Rightarrow h_1(y_1, y_2) = y_1 y_2$$
$$h_2(y_1, y_2) = y_2 - y_1 y_2$$

$$J_{h_1 h_2}(y_1, y_2) = \begin{vmatrix} \dfrac{\partial h_1}{\partial y_1} & \dfrac{\partial h_1}{\partial y_2} \\ \dfrac{\partial h_2}{\partial y_1} & \dfrac{\partial h_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} y_2 & y_1 \\ -y_2 & (1 - y_1) \end{vmatrix}$$

$$= y_2(1 - y_1) - (-y_1 y_2)$$
$$= y_2 - y_1 y_2 + y_1 y_2$$
$$= y_2$$

$$f_{Y_1Y_2}(y_1, y_2) = f_{X_1X_2}(h_1(y_1, y_2), h_2(y_1, y_2)) \left| J_{h_1 h_2}(y_1, y_2) \right|$$

3.2.1) Cont.) $\Rightarrow f_{Y_1Y_2}(y_1, y_2) = e^{-(y_1y_2)-(y_2-y_1y_2)} \cdot |y_2|$

$= |y_2|\, e^{-\cancel{y_1y_2}-y_2+\cancel{y_1y_2}}$

$= |y_2|\, e^{-y_2}$

Note: $0 \le x_1 < \infty$ and $0 \le x_2 < \infty$

and $y_1 = \dfrac{x_1}{x_1+x_2}$, $y_2 = x_1 + x_2$

$\Rightarrow \boxed{0 \le y_1 \le 1, \quad 0 \le y_2 < \infty}$

$\Rightarrow |y_2| = y_2$

$\Rightarrow \boxed{f_{Y_1Y_2}(y_1, y_2) = y_2\, e^{-y_2}}$

3.2.2) Given $f_{X_1 X_2}(x_1, x_2) = \frac{1}{6\pi} e^{-(5x_1^2 - 2x_1x_2 + 2x_2^2)/18}$

for $-\infty < x_1 < \infty$, $-\infty < x_2 < \infty$

and $y_1 = \frac{x_1 + x_2}{3}$, $y_2 = \frac{x_2 - 2x_1}{3}$

Need to determine $f_{Y_1 Y_2}(y_1, y_2)$

$g_1(x_1, x_2) = \frac{x_1 + x_2}{3} = y_1 \implies x_1 + x_2 = 3y_1$

$g_2(x_1, x_2) = \frac{x_2 - 2x_1}{3} = y_2 \implies -2x_1 + x_2 = 3y_2$

Subtract eqns: $3x_1 = 3y_1 - 3y_2$

$\implies x_1 = y_1 - y_2 = h_1(y_1, y_2)$

Substitute for x_1 in first equation:

$y_1 - y_2 + x_2 = 3y_1$

$\implies x_2 = 2y_1 + y_2 = h_2(y_1, y_2)$

$$J_{h_1 h_2}(y_1, y_2) = \begin{vmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$= 1 - (-2) = 3$

$f_{Y_1 Y_2}(y_1, y_2) = f_{X_1 X_2}(h_1(y_1, y_2), h_2(y_1, y_2)) \left| J_{h_1 h_2}(y_1, y_2\right.$

3.2.2) Cont.) $\Rightarrow f_{Y_1Y_2} = \left(\frac{1}{6\pi} e^{-\frac{\left[5(y_1-y_2)^2 - 2(y_1-y_2)(2y_1+y_2) + 2(2y_1+y_2)^2\right]}{18}} \cdot (3)\right)$

$$= \frac{1}{2\pi} e^{-\left(5y_1^2 - \cancel{10y_1y_2} + 5y_2^2 - 4y_1^2 + \cancel{2y_1y_2} + 2y_2^2 + 8y_1^2 + \cancel{8y_1y_2} + 2y_2^2\right)/18}$$

$$= \frac{1}{2\pi} e^{-\left(9y_1^2 + 9y_2^2\right)/18}$$

$$\Rightarrow \boxed{f_{Y_1Y_2}(y_1, y_2) = \frac{1}{2\pi} e^{-\frac{(y_1^2+y_2^2)}{2}}}$$

Note : from the transformations

$$y_1 = \frac{x_1 + x_2}{3} \quad \text{and} \quad y_2 = \frac{x_2 - 2x_1}{3}$$

If $x_1 \to \infty$ and $|x_2| < \infty$, $y_1 \to \infty$
If $x_1 \to -\infty$ and $|x_2| < \infty$, $y_1 \to -\infty$

If $x_2 \to \infty$ and $|x_1| < \infty$, $y_2 \to \infty$
If $x_2 \to -\infty$ and $|x_1| < \infty$, $y_2 \to -\infty$

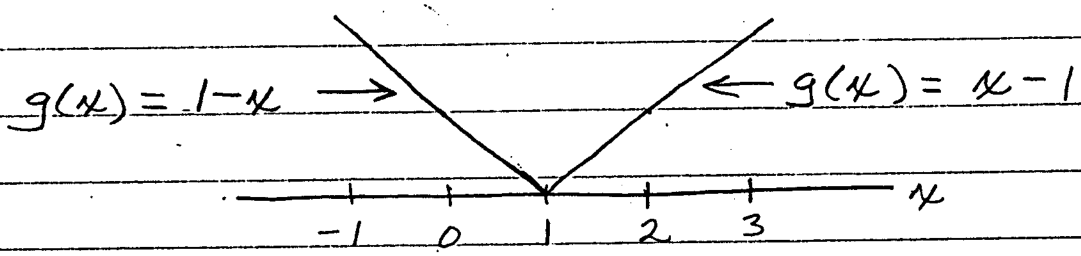
Thus the ranges for y_1 and y_2 are

$$\boxed{\begin{array}{c} -\infty < y_1 < \infty \\ -\infty < y_2 < \infty \end{array}}$$

3.3.2) Given $f_X(x) = \dfrac{1}{\pi(x^2+1)}$, $-\infty < x < \infty$

and $y = |x-1| = g(x)$

Need to determine $f_Y(y)$



Note that $g(x)$ is not one-to-one for all x but it is one-to-one for $-\infty < x < 1$ and also for $1 \le x < \infty$.

For $1 \le x < \infty$, $y = x - 1 \Rightarrow x = y + 1 = h_1(y)$

For $-\infty < x < 1$, $y = 1 - x \Rightarrow x = 1 - y = h_2(y)$

$$f_Y(y) = f_X(h_1(y))\left|\frac{dh_1(y)}{dy}\right| + f_X(h_2(y))\left|\frac{dh_2(y)}{dy}\right|$$

$$= \frac{1}{\pi[(y+1)^2+1]}|1| + \frac{1}{\pi[(1-y)^2+1]}|-1|$$

$$\Rightarrow \boxed{f_Y(y) = \frac{1}{\pi(y^2+2y+2)} + \frac{1}{\pi(y^2-2y+2)}}$$

For $-\infty < x < \infty$, $0 \le |x-1| < \infty$

$$\Rightarrow \boxed{0 \le y < \infty}$$

3.4.3) Given X and Y statistically independent,

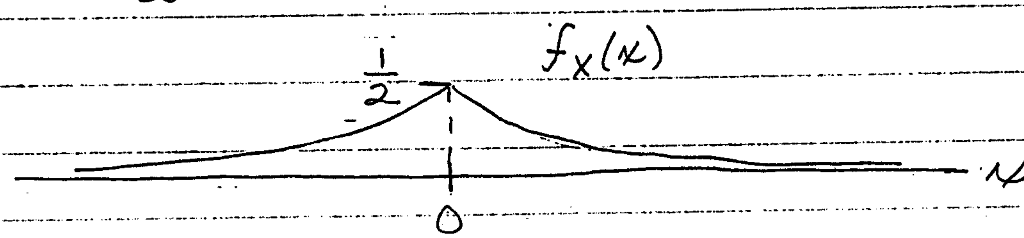
$$f_X(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$$

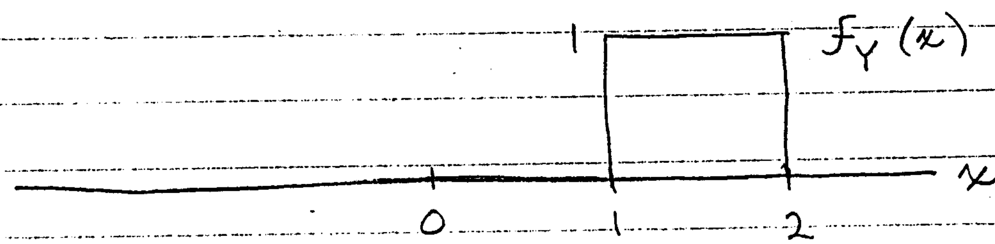
$$f_Y(y) = 1, \quad 1 \le y \le 2$$

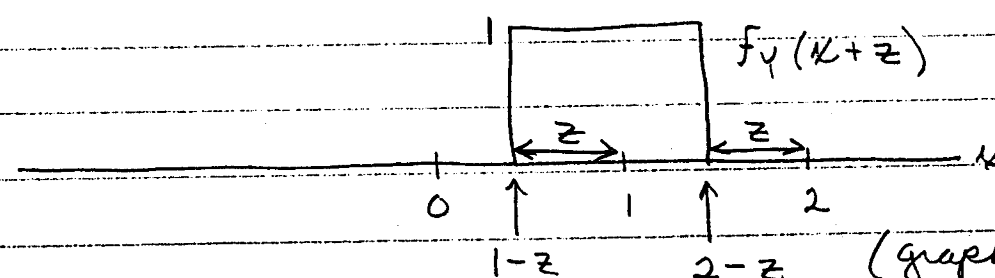
and $z = x + y$

Need to determine $f_Z(z)$

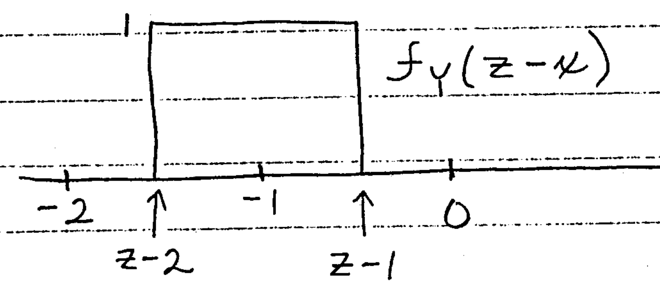
$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$







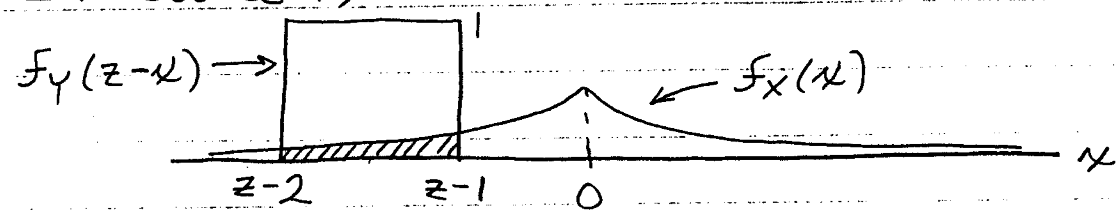
(graph shown for $z > 0$)



Evaluate different conditions of overlap of nonzero values of $f_X(x)$ and $f_Y(z-x)$ for different ranges of z:

3.4.3) Interval 1) $z-1<0 \Rightarrow z<1$

Cont.)



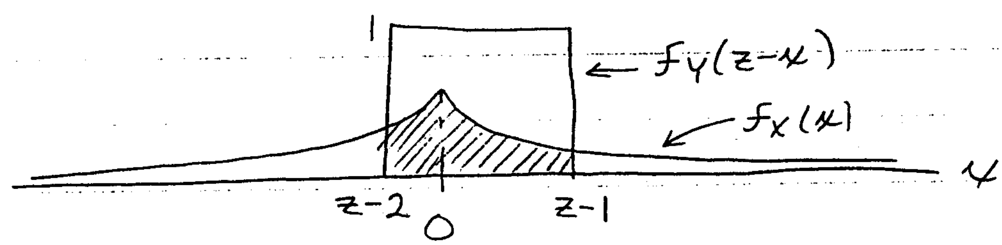
- Overlap from $x = z-2$ to $x = z-1$
- For x in that range, $f_X(x) = \frac{1}{2}e^{-|x|} = \frac{1}{2}e^{x}$ and $f_Y(z-x) = 1$

$$\Rightarrow f_Z(z) = \int_{z-2}^{z-1} \frac{1}{2}e^{x}(1)\,dx = \frac{1}{2}e^{x}\Big|_{z-2}^{z-1}$$

$$= \frac{1}{2}\left(e^{z-1} - e^{z-2}\right)$$

Interval 2) $z-1 \geq 0$ and $z-2 < 0$

$\Rightarrow 1 \leq z < 2$



- Overlap from $x = z-2$ to $x = z-1$
- For x in that range, $f_Y(z-x) = 1$ and,

For $z-2 \leq x \leq 0$, $f_X(x) = \frac{1}{2}e^{-|x|} = \frac{1}{2}e^{x}$

For $0 \leq x \leq z-1$, $f_X(x) = \frac{1}{2}e^{-|x|} = \frac{1}{2}e^{-x}$

(i.e., $f_X(x)$ = different functions over different ranges of x)

$\Rightarrow$ Must integrate different functions over 2 ranges.

3.4.3) Cont.)

$$f_Z(z) = \int_{z-2}^{0} \frac{1}{2} e^{x}(1)\,dx + \int_{0}^{z-1} \frac{1}{2} e^{-x}(1)\,dx$$

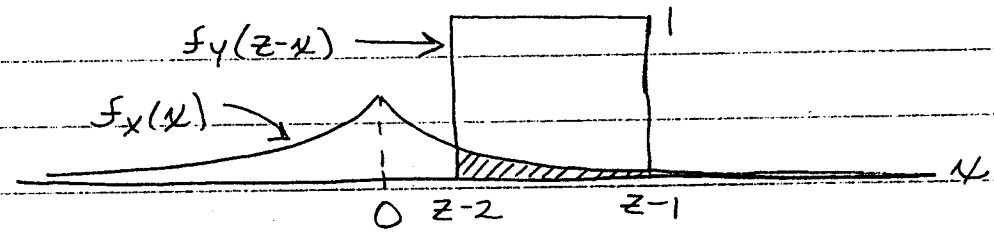
$$= \frac{1}{2} e^{x}\Big|_{z-2}^{0} - \frac{1}{2} e^{-x}\Big|_{0}^{z-1}$$

$$= \frac{1}{2}\left(1 - e^{z-2}\right) - \frac{1}{2}\left(e^{-(z-1)} - 1\right)$$

$$= \frac{1}{2} - \frac{1}{2} e^{z-2} - \frac{1}{2} e^{-(z-1)} + \frac{1}{2}$$

$$= 1 - \frac{1}{2}\left(e^{z-2} + e^{-(z-1)}\right)$$

Interval 3) $z - 2 \geq 0 \Rightarrow z \geq 2$



- Overlap from $x = z-2$ to $x = z-1$
- For x in that range, $f_X(x) = \frac{1}{2} e^{-|x|} = \frac{1}{2} e^{-x}$
  and $f_Y(z-x) = 1$

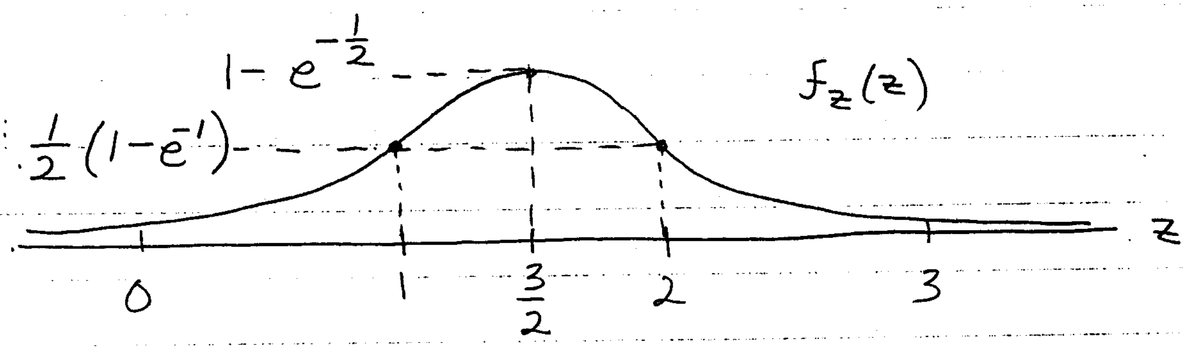
$$\Rightarrow f_Z(z) = \int_{z-2}^{z-1} \frac{1}{2} e^{-x}(1)\,dx = -\frac{1}{2} e^{-x}\Big|_{z-2}^{z-1}$$

$$= -\frac{1}{2}\left(e^{-(z-1)} - e^{-(z-2)}\right)$$

$$= \frac{1}{2}\left(e^{-(z-2)} - e^{-(z-1)}\right)$$

3.4.3) Cont.)

$$\Rightarrow f_Z(z) = \begin{cases} \frac{1}{2}\left(e^{z-1} - e^{z-2}\right), & z < 1 \\ 1 - \frac{1}{2}\left(e^{z-2} + e^{-(z-1)}\right), & 1 \leq z < 2 \\ \frac{1}{2}\left(e^{-(z-2)} - e^{-(z-1)}\right), & z \geq 2 \end{cases}$$



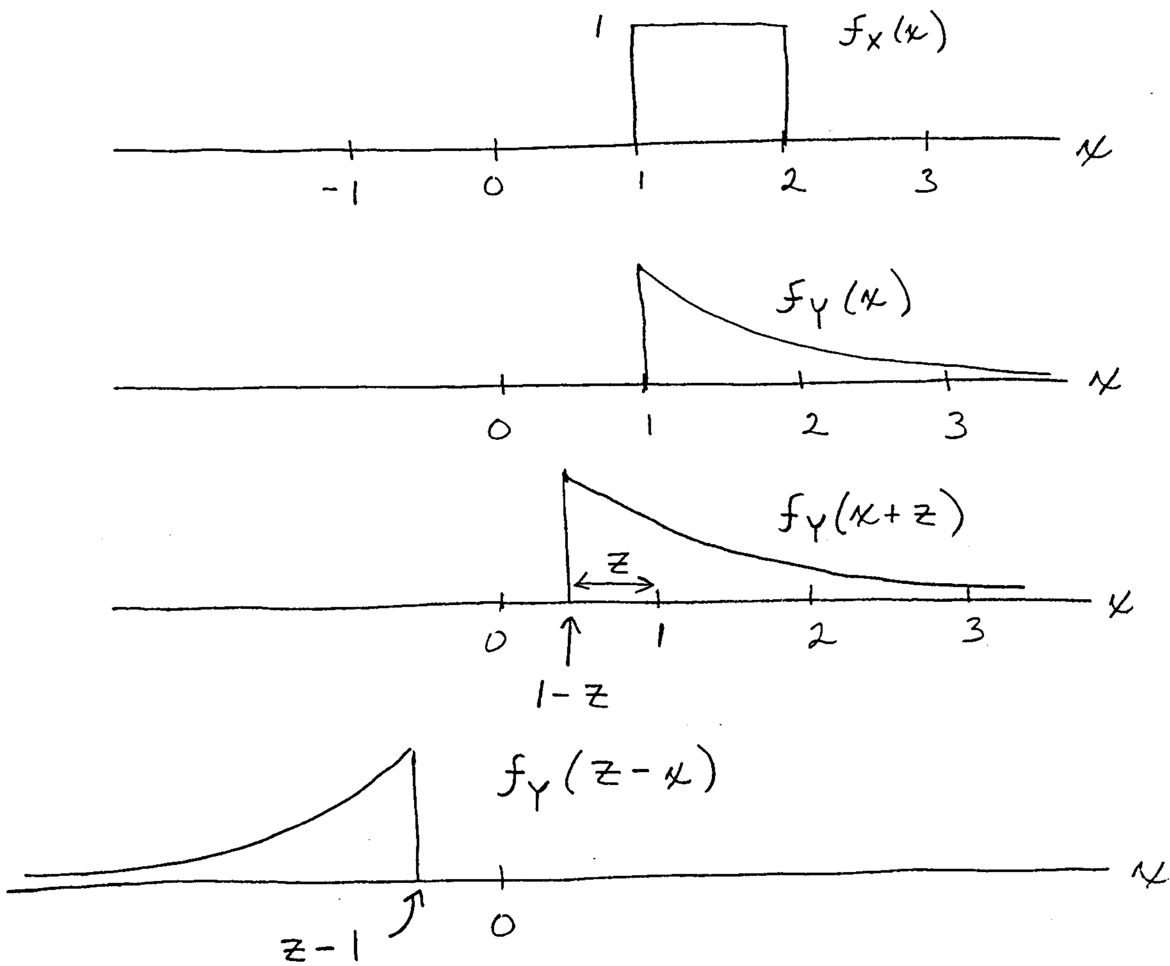
3.4.4) Given X and Y statistically independent,

$f_X(x) = 1, \quad 1 \le x \le 2$

$f_Y(y) = e^{-(y-1)}, \quad 1 \le y < \infty$

and $z = x + y$, need to determine $f_Z(z)$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$



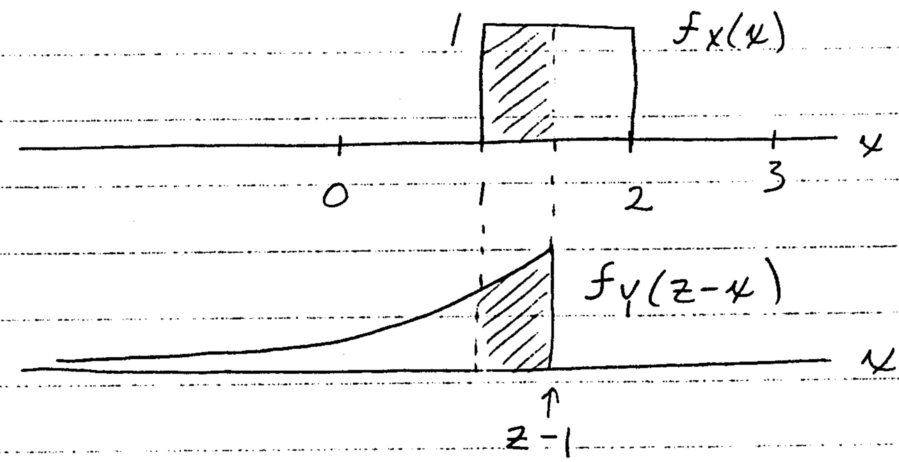
Evaluate different conditions of overlap of nonzero values of $f_X(x)$ and $f_Y(z-x)$ for different ranges of z:

3.4.4) Cont.) Interval 1) $z-1 < 1 \Rightarrow z < 2$:

No overlap $\Rightarrow f_X(x) f_Y(z-x) = 0$

$\Rightarrow f_Z(z) = 0$

Interval 2) $1 \le z-1 < 2 \Rightarrow 2 \le z < 3$:



- Overlap from $x=1$ to $x = z-1$
- For x in that range, $f_X(x) = 1$

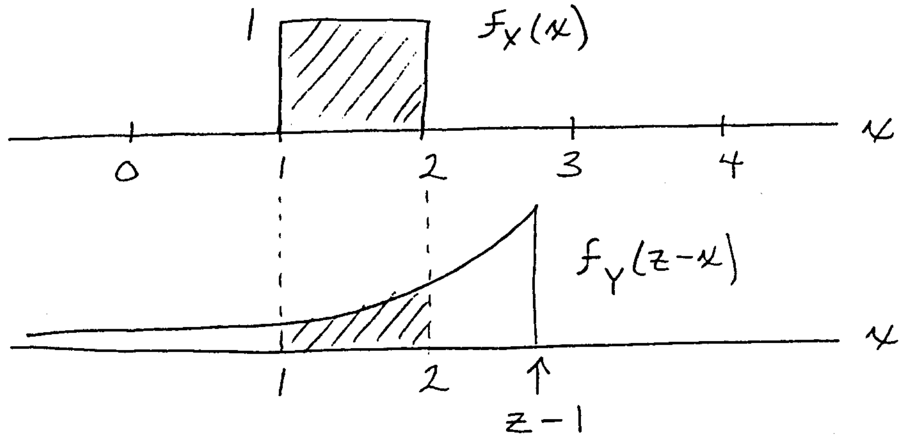
and $f_Y(z-x) = e^{-(z-x-1)}$

$$\Rightarrow f_Z(z) = \int_1^{z-1} e^{-(z-x-1)}\,dx = \int_1^{z-1} e^{-z} e^{x} e^{1}\,dx$$

$$= e^{1-z} e^{x} \Big|_{x=1}^{x=z-1} = e^{1-z}\left(e^{z-1} - e\right)$$

$$= e^{0} - e^{2-z} = 1 - e^{2-z}$$

3.4.4) Interval 3) $z-1 \geq 2 \Rightarrow z \geq 3$:
Cont.)

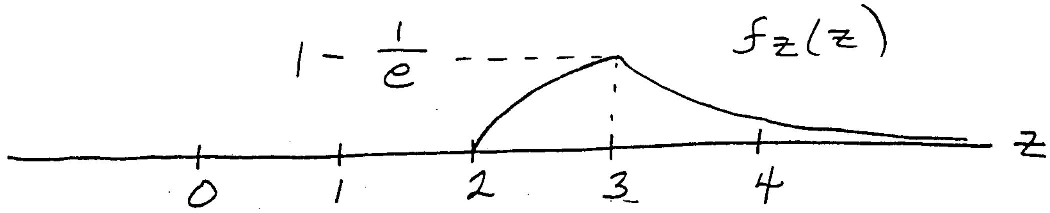


Overlap from $x=1$ to $x=2$
For x in that range, $f_X(x) = 1$
and $f_Y(z-x) = e^{-(z-x-1)}$

$$\Rightarrow f_Z(z) = \int_1^2 e^{-(z-x-1)}\,dx = \int_1^2 e^{-z} e^{x} e^{1}\,dx$$

$$= e^{1-z} e^{x}\Big|_{x=1}^{x=2} = e^{1-z}\left(e^2 - e\right) = e^{3-z} - e^{2-z}$$

$$\Rightarrow \boxed{f_Z(z) = \begin{cases} 0, & z < 2 \\ 1 - e^{2-z}, & 2 \leq z < 3 \\ e^{3-z} - e^{2-z}, & z \geq 3 \end{cases}}$$



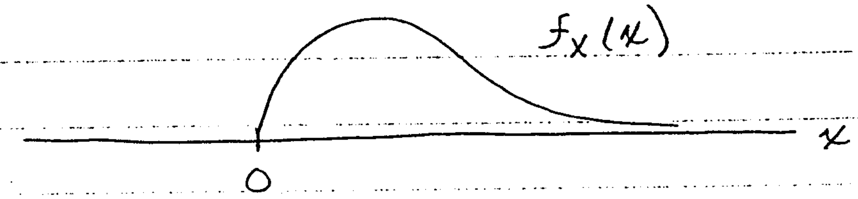
3.4.6) Given X and Y statistically independent,

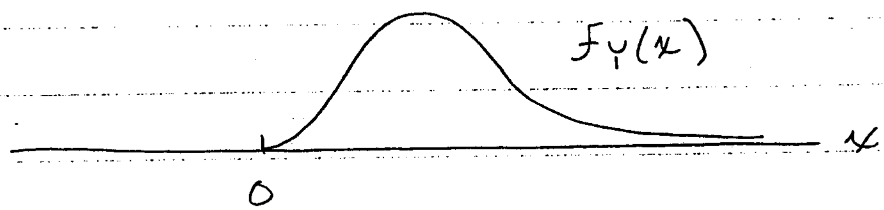
$f_X(x) = x e^{-x}, \quad 0 \le x < \infty$

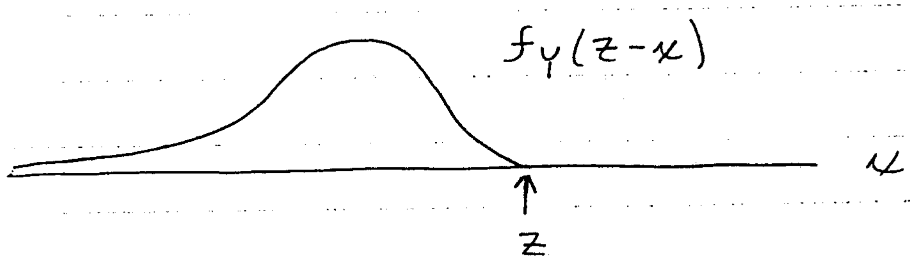
$f_Y(y) = \frac{1}{2} y^2 e^{-y}, \quad 0 \le y < \infty$

and $z = x + y$, need to determine $f_Z(z)$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$







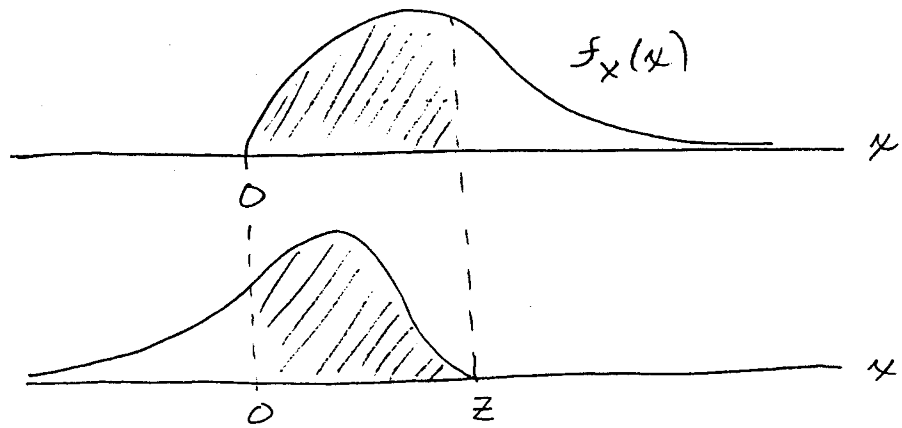
Evaluate different conditions of overlap of nonzero values of $f_X(x)$ and $f_Y(z-x)$ for different ranges of z:

Interval 1) $z < 0$: No overlap

$\Rightarrow f_X(x)\, f_Y(z-x) = 0$

$\Rightarrow f_Z(z) = 0$

3.4.6) Interval 2) $z \geq 0$:
Cont.)



Overlap from $x=0$ to $x=z$
For x in that range, $f_X(x) = x e^{-x}$
and $f_Y(z-x) = \frac{1}{2}(z-x)^2 e^{-(z-x)}$

$$\Rightarrow f_Z(z) = \int_0^z x e^{-x} \cdot \frac{1}{2}(z-x)^2 e^{-(z-x)} \, dx$$

$$= \frac{1}{2} e^{-z} \int_0^z x \cancel{e^{-x}} (z-x)^2 \cancel{e^{x}} \, dx$$

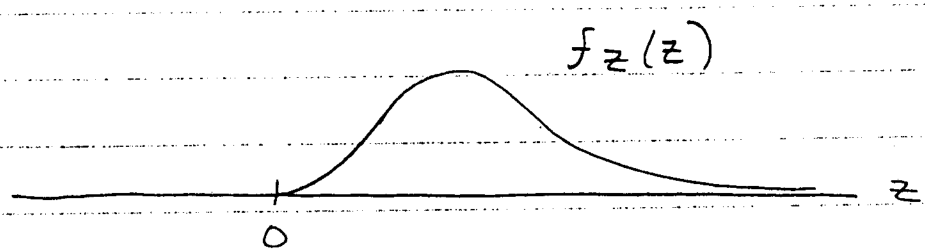
$$= \frac{1}{2} e^{-z} \int_0^z x(z-x)^2 \, dx = \frac{1}{2} e^{-z} \int_0^z x(z^2 - 2zx + x^2) \, dx$$

$$= \frac{1}{2} e^{-z} \left[\frac{z^2 x^2}{2} - \frac{2 z x^3}{3} + \frac{x^4}{4} \right] \Bigg|_{x=0}^{x=z}$$

$$= \frac{1}{2} e^{-z} \left[\frac{z^4}{2} - \frac{2z^4}{3} + \frac{z^4}{4} - 0 \right]$$

$$= \frac{1}{2} e^{-z} \left[\frac{z^4}{12} \right] = \frac{1}{24} z^4 e^{-z}$$

3.4.6)
Cont.) $\Rightarrow$ $\boxed{f_Z(z) = \begin{cases} 0\ , & z < 0 \\ \frac{1}{24} z^4 e^{-z}\ , & z \geq 0 \end{cases}}$



3.4.10) Given X and Y statistically independent,

$$P(X=i) = C_i^2 (0.3)^i (0.7)^{2-i} \quad , \quad i = 0, 1, 2$$

and

$$P(Y=j) = C_j^3 (0.3)^j (0.7)^{3-j} \quad , \quad j = 0, 1, 2, 3$$

and $z = x + y$, need to determine $P(Z=K)$

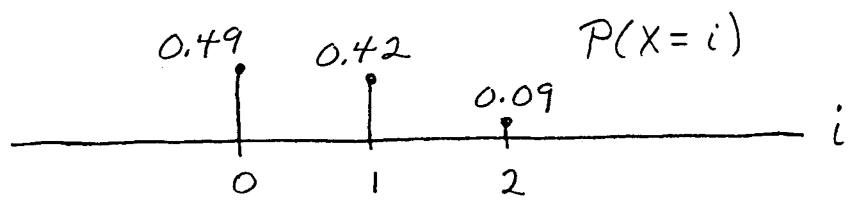
$$P(Z=K) = \sum_{i=-\infty}^{\infty} P(X=i)\,P(Y=K-i)$$

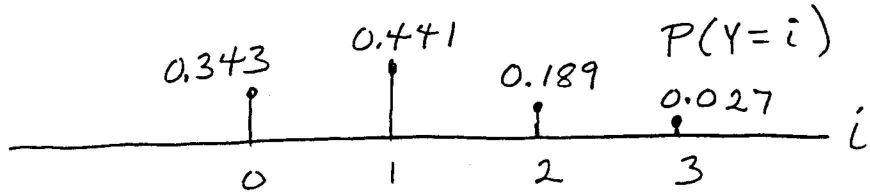
From the above formulas,

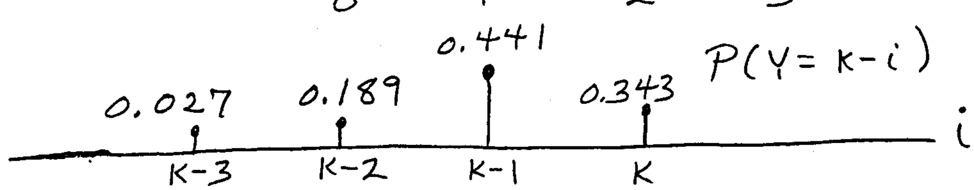
$$P(X=0) = (1)(1)(0.7)^2 = 0.49$$
$$P(X=1) = (2)(0.3)(0.7) = 0.42$$
$$P(X=2) = (1)(0.3)^2(1) = 0.09$$

and

$$P(Y=0) = (1)(1)(0.7)^3 = 0.343$$
$$P(Y=1) = (3)(0.3)(0.7)^2 = 0.441$$
$$P(Y=2) = (3)(0.3)^2(0.7) = 0.189$$
$$P(Y=3) = (1)(0.3)^3(1) = 0.027$$







3.4.10) Cont.) Note from graphs of $P(X=i)$ and $P(Y=k-i)$ that $P(X=i)P(Y=k-i) = 0$
except for $k =$ an integer (otherwise the nonzero values of $P(X=i)$ and $P(Y=k-i)$ do not coincide)

Also, for $k < 0$ there is no overlap
and for $k-3 > 2 \Rightarrow k > 5 \Rightarrow$ no overlap

For $k = 0, 1, 2,$
there is overlap from $i = 0$ to $i = k$

$$\Rightarrow P(Z=k) = \sum_{i=0}^{k} P(X=i)P(Y=k-i)$$

$$\Rightarrow P(Z=0) = P(X=0)P(Y=0) = (0.49)(0.343) = 0.168$$

$$P(Z=1) = P(X=0)P(Y=1) + P(X=1)P(Y=0) = (0.49)(0.441) + (0.42)(0.343) = 0.360$$

$$P(Z=2) = P(X=0)P(Y=2) + P(X=1)P(Y=1) + P(X=2)P(Y=0)$$
$$= (0.49)(0.189) + (0.42)(0.441) + (0.09)(0.343) = 0.309$$

For $k = 3, 4, 5$
there is overlap from $i = k-3$ to $i = 2$

$$\Rightarrow P(Z=k) = \sum_{i=k-3}^{2} P(X=i)P(Y=k-i)$$

3.4.10) Cont.) $\Rightarrow P(Z=3) = P(X=0)\,P(Y=3) + P(X=1)\,P(Y=2) + P(X=2)\,P(Y=1)$

$= (0.49)(0.027) + (0.42)(0.189) + (0.09)(0.441)$

$= 0.132$

$P(Z=4) = P(X=1)\,P(Y=3) + P(X=2)\,P(Y=2)$

$= (0.42)(0.027) + (0.09)(0.189)$

$= 0.028$

$P(Z=5) = P(X=2)\,P(Y=3) = (0.09)(0.027)$

$= 0.002$

$$\Rightarrow \boxed{P(Z=i) = \begin{cases} 0.168, & i=0 \\ 0.360, & i=1 \\ 0.309, & i=2 \\ 0.132, & i=3 \\ 0.028, & i=4 \\ 0.002, & i=5 \\ 0, & \text{otherwise} \end{cases}}$$

Note: this turns out to be

$$P(Z=K) = C_K^5\,(0.3)^K (0.7)^{5-K}, \quad K=0,1,2,3,4,5$$

because X and Y are statistically independent binomial random variables with $p = 0.3$,

so $Z = X + Y$ is also a binomial random variable with $p = 0.3$.

3.5.2) Given 3 statistically independent, uniform random variables (over 0 to 1). Need to determine the probability density function of the minimum, and the probability that the minimum is > 0.25.

$f_{U_i}(u_i) = 1$, $F_{U_i}(u_i) = u_i$, $0 \le u_i \le 1$, $i = 1, 2, 3$

and $z = \min(u_1, u_2, u_3)$

$$F_Z(z) = 1 - [1 - F_{U_i}(z)]^N, \quad \text{where } N = 3$$

$$= 1 - [1 - z]^3, \quad 0 \le z \le 1$$

$$\boxed{f_Z(z)} = \frac{d}{dz}\{F_Z(z)\} = -3(1-z)^2(-1)$$

$$= \boxed{3(1-z)^2}$$

OR

$$\boxed{3(z-1)^2}$$

$$\boxed{P(z > 0.25)} = 1 - F_Z(0.25)$$

$$= \cancel{1} - [\cancel{1} - (1-0.25)^3]$$

$$= (1-0.25)^3 = \boxed{\sim 0.422}$$

3.5.3) Given 3 statistically independent electronic components in series, each with time-to-failure X_i $(i=1,2,3)$ and $f_{X_i}(x_i) = 0.02\, e^{-0.02 x_i}$, $x_i \geq 0$

(These are identically distributed exponential random variables with parameter $a = 0.02$)

$$\Rightarrow F_{X_i}(x_i) = 1 - e^{-0.02 x_i}, \quad x_i \geq 0$$

We want the density function of the time-to-failure (call it z) of the series combination of 3 components.

$$\Rightarrow z = \min(x_1, x_2, x_3)$$

$$\Rightarrow F_Z(z) = 1 - \prod_{i=1}^{3} [1 - F_{X_i}(z)]$$

$$= 1 - [\cancel{1} \cancel{-} (\cancel{1} \cancel{-} e^{-0.02z})]^3 = 1 - (e^{-0.02z})^3$$

$$= 1 - e^{-0.06z}, \quad z \geq 0$$

$$\boxed{f_Z(z)} = \frac{d}{dz}\{F_Z(z)\} = \boxed{0.06\, e^{-0.06z}, \quad z \geq 0}$$

$$\boxed{P(Z \leq 30)} = F_Z(30) = 1 - e^{-0.06(30)}$$

$$\boxed{\cong 0.835}$$