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Monomial Ideals and Their Decompositions

February 28, 2017

Springer

*February 28, 2017

Dedicated to the memory of Diana Taylor,
with gratitude and respect.

Contents

Introduction

The Fundamental Theorem of Arithmetic states that every integer $n \geqslant 2$ factors into a product of prime numbers in an essentially unique way. In algebra class, one learns a similar factorization result for polynomials in one variable with real number coefficients: every non-constant polynomial factors into a product of linear polynomials and irreducible quadratic polynomials in an essentially unique way. These examples share some obvious common ideas.

First, in each case we have a set of objects (in the first example, the set of integers; in the second example, the set of polynomials with real number coefficients) that can be added, subtracted, and multiplied in pairs so that the resulting sums, differences, and products are in the same set. (We say that the sets are "closed" under these operations.) Furthermore, addition and multiplication satisfy certain rules (or axioms) that make them "nice": they are commutative and associative, they have identities and additive inverses, and they interact coherently together via the Distributive Law. In other words, each of these sets is a *commutative ring with identity*. Note that we do not consider division in this setting because, e.g., the quotient of two non-zero integers need not be an integer. Commutative rings with identity arise in many areas of mathematics, like combinatorics, geometry, graph theory, and number theory.

Second, each example deals with factorization of certain elements into finite products of "irreducible" elements, that is, elements that cannot themselves be factored in a nontrivial manner. In general, given a commutative ring R with identity, the fact that elements can be multiplied implies that elements can be factored, even if only trivially. One way to study R is to investigate how well its factorizations behave. For instance, one can ask whether the elements of R can be factored into a finite product of irreducible elements. (There are non-trivial examples where this fails.) Assuming that the elements of R can be factored into a finite product of irreducible elements, one can ask whether the factorizations are unique. The first example one might see where this fails is the ring $\mathbb{Z}[\sqrt{-5}]$ consisting of all complex numbers $a+b\sqrt{-5}$ such that a and b are integers. In this ring, we have $(2)(3)=6=(1+\sqrt{-5})(1-\sqrt{-5})$, and these two factorizations of 6 are fundamentally different.

In the 1800's, Ernst Kummer and Richard Dedekind recognized that this problem can be remedied, essentially by factoring elements into products of sets. More specifically, one replaces the element r to be factored with the set rR of all multiples of that element, and one factors this set into a product $rR = I_1 I_2 \cdots I_n$ of similar sets. (We are being intentionally vague here. For some technical details, see Appendix A.) The "similar sets" are called *ideals* because they are, in a sense, idealized versions of numbers.

In the 1900's Emanuel Lasker and Emmy Noether recognized that it is better in some ways to consider intersections instead of products. The idea is the same, except that factorizations are replaced by decompositions into finite intersections of irreducible ideals, i.e., ideals that cannot be written as a non-trivial intersection of two ideals. In some cases (e.g., in $\mathbb{Z}[\sqrt{-5}]$) these are the same, but it can be shown that decompositions exist in many rings where nice factorizations do not.

Overview

This book is meant to introduce readers to decompositions of ideals in the polynomial ring $A[X_1, \ldots, X_d]$ with coefficients in a commutative ring A and variables $X_1, \ldots, X_d$. Specifically, we focus on monomial ideals, that is, ideals that are generated by monomials $X_1^{n_1} \cdots X_d^{n_d}$. We show that every monomial ideal in R can be written as an intersection of "m-irreducible" monomial ideals, that is, monomial ideals that cannot themselves be written as a non-trivial intersection of two monomial ideals. We call such an intersection an "m-irreducible decomposition"; this is our version of a prime or irreducible factorization in this context. Proving the existence of such decompositions is the first major goal of this text, achieved in Part I.

We have two main reasons for focusing on monomial ideals.

First, monomial ideals have incredible connections to other areas of mathematics. For instance, one can use monomial ideals to study certain objects in combinatorics, geometry, graph theory, and topology. Reciprocally, one can study a monomial ideal using ideas from combinatorics, geometry, graph theory, and/or topology. Part II of the text is devoted entirely to the second major goal of this text: presenting some of the connections between monomial ideals and other areas of mathematics (and electrical engineering).

Second, monomial ideals are the simplest ideals, in a sense, since the generators have only one term each. Accordingly, this makes monomial ideals optimal objects of study for students with little background in abstract algebra. Indeed, we begin studying polynomials in middle school, and we see polynomials in several variables in calculus, so these are familiar objects that are not as intimidating as arbitrary elements of an arbitrary commutative ring. In other words, one should feel more comfortable with the process of formally manipulating polynomials because they are more concrete. When one restricts to ideals generated by monomials, the ideas become even more concrete.

Moreover, the relative simplicity of these ideals allows for concrete descriptions of their decompositions. The third major goal of this text is to provide both general descriptions for arbitrary monomial ideals and specific descriptions for special classes of monomial ideals. The presentation of these begins at the end of Part I, and is the main point of Part III.

On the flip side, the concreteness of these ideals should make it easier for readers to grasp the more general concept of arbitrary ideals. In addition, the material is not terribly difficult, but readers should learn a non-trivial amount of mathematics in the process of working through it.

See the "Summary of Contents" below for more details on the topics covered.

Audience

This book is written for mathematics students (in the broadest sense) who have taken an undergraduate course in abstract algebra. It is appropriate for courses for undergraduate and graduate students. We have used preliminary versions of this text for traditional courses, for individual reading courses, and as a starting point for research projects, each with undergraduate and graduate students.

There are several excellent texts available for readers who are interested in learning about monomial ideals, for instance, Jürgen Herzog and Takayuki Hibi [37], Hibi [38], Ezra Miller and Bernd Sturmfels [58], and Richard Stanley [74]. The topic also receives significant attention in the books of Winfried Bruns and Jürgen Herzog [9], and Rafael Villarreal [77]. However, each of these books is geared toward advanced graduate students (and higher). We think of our book as a gentle introduction to the subject that provides partial preparation for readers interested in these other texts but without the necessary background. For perspective, the level of this book is similar to that of the text by David Cox, John Little, and Donal O'Shea [13]. However, the material covered in that text is very different from ours.

Summary of Contents

The book is divided topically into four parts. Part I sets the stage with a general treatment of monomial ideals divided into three chapters.

Chapter 1 deals with the fundamental properties of monomial ideals in the polynomial ring $R = A[X_1, \ldots, X_d]$ for use in the rest of the text. It is worth noting that we do not require A to be a field for most of the text, unlike many treatments of monomial ideals. We can to do this because, we focus almost exclusively on the decompositions of these ideals, not on properties of their quotient rings.

Chapter 2 addresses ways of modifying monomial ideals to create new monomial ideals. For instance, we show that the intersection of monomial ideals is a monomial ideal, a fundamental fact for proving that every monomial ideal can be decomposed as an intersection of other monomial ideals. Other constructions of this chapter are also used more or less significantly in the subsequent chapters. For instance, the monomial radical of Section 2.3, based on the radical of Section A.7, is used somewhat extensively, throughout Part II (via the square-free monomial ideals of Section 4.1) and throughout Chapter 6, as well as in Section 7.1. Similarly, the colon ideal of Section 2.5, originally from Section A.6, is used throughout Chapter 6, as well as in Section 7.4, and the exploration Sections 2.7, 2.9, and 7.7. On the other hand, bracket powers and generalized bracket powers, from Sections 2.6 and 2.8 are mostly contained to sections where their names appear.

Chapter 3 gets to the issue of decompositions of monomial ideals. Here we achieve the first major goal of the text, proving in Theorem 3.3.3 that every monomial ideal in R has an m-irreducible decomposition. Also, we explicitly characterize the m-irreducible monomial ideals as those ideals generated by a list of "pure powers" $X_i^{m_i}$ of some of the variables. In spite of the explicit characterization of these ideals, the proof of the existence of m-irreducible decompositions is not constructive. Some may view this as a defect, but we feel that it is an important demonstration of the power of abstraction. On the other hand, Section 3.5 describes some algorithms for actually computing decompositions; even though it is an exploration, we strongly recommend that readers work through it.

Part II of the text consists of two chapters that describe some of the connections between the realm of monomial ideals and other areas of mathematics and even other disciplines. This is the second major goal of the text.

Chapter 4 is devoted to certain connections with combinatorics. It treats a class of ideals where m-irreducible decompositions can be described explicitly using combinatorial data. These are the square-free monomial ideals, that is, the ideals generated by monomials of the form $X_{i_1}\cdots X_{i_m}$ with strictly increasing subscripts. Many algebraic properties of these ideals are determined by the combinatorial properties of an associated simplicial complex, and we show how the simplicial complex provides the m-irreducible decomposition of the ideal, via the face ideal construction of Melvin Hochster and Gerald Reisner, also known as the Stanley-Reisner ideal. The chapter begins with the special case of square-free monomial ideals generated by monomials of the form X_iX_j. The decompositions here can be described in terms of "vertex covers" of an associated (finite simple) graph, by the edge ideal construction of Villarreal. Note that we do not assume that the reader is familiar with simplicial complexes or graphs.

Chapter 5 treats some interactions with other areas. For instance, we describe connections with electrical engineering, via the PMU Placement Problem. This uses graphs and edge ideals to obtain information about electrical power systems. We also discuss some connections to topology (via Stanley's proof of the Upper Bound Theorem for simplicial spheres), non-monomial ideals (via initial ideals and Hilbert polynomials), and homological algebra (via the Taylor resolution). It is worth noting that this chapter is quite colloquial in nature. In contrast to the other chapters, this one omits proofs of some results because they are outside the scope of this text. The idea here is to give a taste of these areas as they connect with monomial ideals. We include many references for readers looking for more information on these topics.

Part III of the text contains two chapters dealing with the problem of computing m-irreducible decompositions explicitly. This is the third main goal of the text. The chapters of this part are relatively independent, with topics from Chapter 6 only appearing in some of the exercises of Chapter 7.

Chapter 6 deals with another case of ideals where m-irreducible decompositions can be described explicitly. These are the ideals that contain a power of each one of the variables. This condition means that there are only finitely many monomials that are not in the ideal, and there is an algorithm using these excluded monomials to find the decompositions. In this case, we call the decompositions "parametric decompositions", following Heinzer, Ratliff, Shah [36].

Chapter 7 deals with some cases where we can describe the behavior of the m-irreducible decompositions when one modifies the ideals using the operations from Chapter 2. As a consequence, it contains several algorithms for computing m-irreducible decompositions in general.

Part IV of the text consists of two appendices.

Appendix A serves as a review of (or introduction to) the fundamentals of commutative algebra. Much of the material therein may have been covered in a course on abstract algebra. Accordingly, much of this material may be skipped, though it cannot be ignored entirely as it contains many of the definitions and notations used in the rest of the text. Also, this appendix includes treatments of some ideal constructions that often don't appear in a standard abstract algebra course but arise in the main body of the text (some more centrally than others).

Appendix B is an introduction to the computer algebra system Macaulay2 [28] which is available from the website

`http://www.math.uiuc.edu/Macaulay2/`

for free download. While it is not essential for a reader to use Macaulay2 to get a lot from the book, a certain amount of insight can be gained by working on these ideas with a computer algebra system. For instance, one can perform many computations in a short time period allowing one to formulate conjectures based on empirical data. Readers not familiar with Macaulay2 will want to work through much of this appendix if they hope to work through the computer portions of the main text found at the ends of the sections. On the other hand, some readers may prefer to use other programs like CoCoA [10] or Singular [29]. However, such readers will necessarily have to translate our code to their chosen system.

Most of the computer subsections in the text have two parts. The first part is a tutorial that introduces relevant Macaulay2 commands and concepts for the section, discussing how to format the input and how to interpret the output. The second part contains exercises to work through to practice the ideas from the tutorial, including coding exercises with minor programming assignments, laboratory exercises encouraging readers to run experiments to aid with the challenge exercises in the text, and documentation exercises where readers are instructed to use the existing Macaulay2 documentation to teach themselves. We do not include instructions for installing Macaulay2, referring the interested reader to the website

`http://www.math.uiuc.edu/Macaulay2/Downloads/`

for instructions. Note that the website

`http://www.math.uiuc.edu/Macaulay2/GettingStarted/`

contains tutorials, so the interested reader can get started there if he or she plans to skip most of Appendix B.

Most chapters of the text contain an "exploration" or two. These sections consist of exercise sets, with little or no lecture material, where one investigates a particular aspect of the ideas from the section. The philosophy behind the explorations is that students often learn best by doing instead of reading or listening to lecture. See the "Notes for the Instructor/Independent Reader" below for more about these.

Also, most chapters contain a few sections labeled as "optional." These sections contain fundamental results and ideas that are motivated by the work on monomial ideals, but are not necessary if one is exclusively focused on monomial ideals. For instance, in Section 1.1 one encounters the Cancellation Law for monomials: If f, g, and h are monomials such that $fg = fh$, then $g = h$. However, the Cancellation Law does not hold in most rings, e.g., in $\mathbb{Z}_6$. On the other hand, the Cancellation Law does hold in integral domains, so we devote Section 1.2 to them. Readers who work through these sections should understand that certain properties of monomial ideals do not hold for arbitrary (non-monomial) ideals. We hope that these sections will compel readers to delve more deeply into the subject of abstract algebra.

Each chapter ends with "Concluding Notes" where we discuss some of the history and literature relevant to the topics of the chapter. We have not attempted to give a complete history, nor an exhaustive bibliography. We hope that knowledgeable readers will forgive us for our historical and bibliographical omissions, as well as topical ones.

This text contains over 600 exercises, most of which have multiple parts. A number of these exercises are used explicitly in later sections; these are identified with an asterisk (see, e.g., Exercise A.1.12) and contain references to places where they are used. Several exercises are designated as "challenge exercises", in some cases due to difficulty, in other cases because one is not only tasked with proving something, but also with formulating the statement of the result to be proved. As we note above, the challenge exercises regularly are paired with Macaulay2 laboratory exercises to help with the formulation portion of the exercise.

Notes for the Instructor/Independent Reader

As we discuss above, much of the material in Appendix A should be review. For some readers, this includes the notion of generators of ideals. However, this notion is absolutely crucial for the main text, and we find that students struggle with and resist using it effectively. This includes, in particular, the very important Proposition A.3.6. For this reason, we recommend working through Section A.3, including some exercises for practice.

Some constructions in Appendix A are introduced for their own sake. For instance, products and powers from Section A.5 appear periodically in the text, but aren't central to it, unless one is aiming for the specific decomposition results of Section 6.6 or 7.9.

On the other hand, other constructions from this appendix come up regularly. For instance, sums of ideals from Section A.4 come up periodically, especially Exercise A.4.6 which treats the relation between generators and sums. Similarly, the radical of an ideal is needed, e.g., to discuss the monomial radical of a monomial ideal in Section 2.3, which is central to the treatment of square-free monomial ideals underlying Chapter 4. Colon ideals from Section A.6 are used throughout Chapter 6, in addition to its obvious need in Section 2.5, and in the explorations 2.7 and 2.9. Quotient rings are needed for Chapter 5 as well as for the Macaulay2 work in Section 6.4. Of course, these notions are also key to the material devoted to their decompositions: Sections 7.1, 7.3, 7.4 and 7.7. Partial orders come up implicitly and explicitly in several sections, especially for graphing ideals in Section 1.1, for the material on initial ideals in Section 5.4, for the proofs of Proposition 2.3.2, Theorem 5.3.17, and Theorem 6.3.1, and for a number of the Macaulay2 subsections.

Appendix B on Macaulay2 is presented mostly in parallel with Appendix A. This way, one can readily identify which Macaulay2 sections are needed for later, based on the corresponding commutative algebra sections.

The non-optional sections of Part I of the text form the crucial foundation for Parts II and III. In our experience teaching this foundational material, we have found the need to stress Theorems 1.1.4, 1.1.9, and 2.1.5 somewhat strenuously, for reasons similar to the ones in our discussion of Section A.3 above.

Parts II and III are relatively (though not entirely) independent. So one can use the text for a course that includes some of the topics from each part, or focuses on one part over the other, once one covers the necessary foundational material. See the section "Possible Course Outlines" below for a few course options.

Within Part II, Chapter 5 depends heavily on Chapter 4. Specifically, Section 5.2 on PMU placement uses graphs and edge ideals from Sections 4.1–4.3. Section 5.3 on the Upper Bound Theorem uses simplicial complexes and face ideals from Sections 4.1, 4.4, and 4.3. In addition, it is worth noting that most of Chapter 5 depends on Section 5.1 on Krull dimension (which relies on Exercise A.5.10 dealing with prime ideals), and Section 5.4 on initial ideals uses monomial orders from Section A.9. In addition, Krull dimension, along with Cohen-Macaulayness and depth from Section 5.3 are key for a number of Macaulay2 exercises in Chapter 7.

The nature of Chapter 7 makes it dependent on preceding material. For instance, Section 7.1 on decomposing monomial radicals depends on Section 2.3 which contains the foundational material on monomial radicals. The introduction to each section in this chapter specifies the primary sections it builds on.

We have had some success using the exploration sections as extended writing projects. We have also devoted time in class to using these for inquiry-based learning: brainstorming with students about how to approach these exercises, to give them further insight into the relevant concepts and the process of problem solving around these concepts. One can use the challenge exercises in the text similarly.

The non-exploration sections are mostly independent of the explorations, and of each other. Significant exceptions to this are the following: Sections 1.6 and 5.4 depend on 1.5, Section 6.5 depends on 3.5, Section 6.6 depends on 1.6, Section 7.7 depends on 2.7, and 7.8 depends on 2.8. Furthermore, the Exploration Section A.10 comes equipped with follow-up exercises in many sections. And, as we noted above, Section 3.5 provides many good examples for later use.

Some of the optional sections depend on each other, as well. For instance, Section 1.2 on integral domains is used crucially in Section 2.2 and less crucially (along with Section 2.4) in a couple of exercises. Section 1.4 is essential for Section 3.4. Section 3.2 is used in Section 3.4. In addition, Macaulay2 notions from Section 3.2 are used in 5.2 and 5.4, and one command `basis` from 1.5 is used in 6.2.

In addition, a number of interesting challenge exercises in Chapter 7 use concepts from Chapters 5 and 6, namely, Krull dimension, depth, Cohen-Macaulayness, and corner elements.

Possible Course Outlines

One can use this text in several different ways, as we have done in didactic courses and otherwise. Obviously, one can start at the beginning and work through as much material as one has time for. Another approach is to ignore the optional sections, leaving them as possible reading exercises for motivated students. Similar comments hold for the exploration sections (though we do encourage discussions of Section 3.5, and 6.5 if one gets that far) and for the sections that are not as central and do not feed much into the other sections; for example, Sections 2.6 and 7.2.

Another option is to pick a particular goal and work toward it, covering only the material needed for that goal. A couple of outlines for such course are as follows. Note that some of these are long, and others are short, depending on how much time one spends on particular topics; any of the outlines can be trimmed or combined, depending on ones situation. See also the Supplements following the outlines.

Core material. Appendix A, as much as is needed (see the notes about this appendix in the previous subsection), then Sections 1.1, 1.3, 2.1, 3.1, 3.3, and 3.5 (recommended). Lemma 1.1.10 is used in numerous places in the text, so instructors should assign its proof as an exercise or present it.

Outline 1. Decompositions of edge ideals, face ideals and/or facet ideals. Core material plus Sections 2.3 and 4.1, then the relevant sections of Chapter 4, depending on the desired topics:

(a) Edge ideals: Sections 4.2–4.3;
(b) Face ideals: Sections 4.4–4.5, and possibly 4.7;
(c) Facet ideals: The part of Section 4.4 on simplicial complexes and 4.6.

Notes: The sections on edge ideals and face ideals are largely independent, except for the material on independence complexes, so one can cover most of the material on face ideals without the material on edge ideals. On the other hand, a course based on Sections 4.2–4.5, i.e., items (a)–(b) above, is really tied together by the material on independence complexes. See also the exercises on order complexes of posets.

Similarly, our treatment of facet ideals is mostly independent of the material on face ideals. (On the other hand, facet ideals naturally generalize edge ideals from graphs to simplicial complexes.) Exploration Section 7.6 also ties into this material; note that it depends on Theorem 7.5.9, which in turn uses some other isolated items from Chapter 7 and terminology from Chapter 6.

Also, another perspective on these decompositions is contained in Exploration Section 7.6. This uses Theorem 7.5.9, which in turn uses parametric decompositions; see the notes for Outline 7 below.

Outline 2. PMU placement. Core material plus Sections 2.3, 4.1–4.3 and 5.1–5.2.

Notes: Make sure to include Exercises A.5.10, A.7.14, and 1.1.23 dealing with prime ideals (for Krull dimension). Simplicial complexes and face ideals are mentioned in a few spots in Section 5.1, but they is not crucial for Section 5.2.

Outline 3. Cohen-Macaulayness and the Upper Bound Theorem. Core material, plus Sections 2.3 and 4.1 then 4.4–4.5 and 5.1 and 5.3.

Notes: The note from Outline 2 regarding prime ideals applies here. Graphs and edge ideals are mentioned in a few spots in Section 5.1, but they are not crucial.

Outline 4. Hilbert functions and initial ideals. Core material, Sections 5.1 and 5.4.

Notes: The note from Outline 2 regarding prime ideals applies here. Various concepts from Chapter 4 and Sections 5.2–5.3 are mentioned in Sections 5.1 and 5.4; none of it is indispensable, though one may wish to cover some of Chapter 4 for perspective. Also, this outline is a bit short, so one may wish to augment it with other material.

Outline 5. Resolutions of monomial ideals. Core material, then Section 5.5.

Notes: Sections A.7 and A.8 are not needed for the primary material of Section 5.5. On the other hand, some of the Macaulay2 exercises in 5.5 tie this topic together nicely with material from Sections 4.7–5.4. As with the previous outline, this one is probably worth augmenting.

Outline 6. Parametric decompositions. Core material, plus Sections 2.3 and 2.5, and then Chapter 6.

Notes: The $*$-dual of Exploration Section 4.7 is mentioned for perspective in this chapter, but it is not crucial. Similarly, Exercise 1.6.3 is used once in an exercise. Section 7.4 on decompositions of colon ideals provides a natural conclusion for this material, if there is time for it. If one augments this outline with Macaulay2 material, one should also include Section 2.8, hence probably also Section 2.6, and Section 5.4. See also Supplement 2 below.

Outline 7. Computing general m-irreducible decompositions. Core material plus Sections 2.3, 7.3, and 7.5.

Notes: Theorem 7.5.9 uses parametric decompositions, which are the subject of Chapter 6; see Outline 6 above. To understand the result, one only needs the basic definitions of Section 6.1; but to use the result, one will want some understanding of how the corner elements of Section 6.2 determine these decompositions and how to compute these elements; see Sections 6.3 and 6.4. On the other hand, some nice applications of Theorem 7.5.9 to edge/face/facet ideals are contained in Exploration Section 7.6; see the notes for Outline 1 above.

Outline 8. Decompositions of combinations of monomial ideals. Core material plus the following relevant sections depending on the topic(s) chosen:

(a) Monomial radicals: Sections 2.3 and 7.1
(b) Bracket powers: Sections 2.6 and 7.2
(c) Sums: Section 7.3
(d) Colon ideals: Sections 2.5 and 7.4
(e) Saturations: Sections 2.5, 2.7, 7.4, and 7.7
(f) Generalized bracket powers: Sections 2.6, 2.8, 7.2, and 7.8
(g) Products: Sections 7.3 and 7.9.

Notes: In item (d) here, Section 7.4 contains material about parametric decompositions and corner elements; you don't need it for the main result Theorem 7.4.4, but it's natural material here. Similar comments apply for the exercises of the other sections of Chapter 7. Items (e)–(g) are explorations, so these are particularly good if one is interested in employing inquiry-based learning techniques. The material in item (g) here can be augmented naturally with Section 6.6 on certain powers of ideals; see Outline 6 above. Many of these items have interesting exercises pertaining to the material from Outline 1 and Supplement 1; from the Macaulay2 standpoint, this extends to material from Outlines 3 and 6. If one works on Macaulay2 material in item g, one should include the recursion material from Section 7.5.

As we discussed in the previous subsection, one can augment any of the above outlines with optional sections, explorations, or challenge exercises. In addition, we have included a lot of material for the following.

Supplement 1. Macaulay2. Any of the preceding outlines can be augmented to make an interesting Macaulay2 course. One should include appropriate material from Appendix B; see the Notes for the Instructor/Independent Reader above.

Supplement 2. Algebraic geometry. Include Section A.10 and corresponding exercises within the sections covered; make sure to include Exercises 1.3.15, 3.1.13, and 3.3.11. Our treatment of this topic is necessarily a bit thin, given the nature of monomial ideals. On the other hand, practice with this material in this context can prepare readers for higher-powered treatments in the same way that the main material of this text can prepare readers for more advanced abstract algebra.

Acknowledgments

We would like to express our thanks to the Mathematics Department at the University of Nebraska-Lincoln for its hospitality and generosity during the 2006 IMMERSE program (Intensive Mathematics: a Mentoring, Education and Research Summer Experience) when we taught a course that formed the foundation for this text. Specific thanks are due to the program's staff Allan Donsig and Marilyn Johnson, and to the other graduate student mentors Jen Everson, Martha Gregg, Kris Lund, Lori McCune, Jeremy Parrott, Emily Price, and Mark Stigge.

We are grateful to the students who have worked so enthusiastically through the various versions of these notes: Hannah Altmann, Zech Andersen, Ben Anderson, Katie Ansaldi, Pye Aung, Debbie Berg, Stefan Bock, Sam Cash, Jackie Drewell, Sarah Gilles, Chad Haugen, Colleen Hughes Karvetski, Dan Ingebretson, Erik Insko, Cheryl Jaeger Balm, Nick Johnson, Hannah Kimbrell, Samuel Kolins, Bethany Kubik, Maxx Kureczko, Jeff Langford, David Lovit, Chelsey Morrow, Benjamin Noteboom, Chelsey Paulsen, Trevor Potter, Becky Price, Rebecca Ramos, Tyler Seacrest, Kelly Smith, Sarah Tekansik McMillan, Jon Totushek, Rich Wicklein, and Alexis Wolf.

We based our IMMERSE course and subsequently this text on the paper of William Heinzer, Louis J. Ratliff, Jr., and Kishor Shah [36]. We are indebted to them for their work. We are also grateful to Daniel Grayson for help with our M2 workflow,[1] Rajesh Kavasseri for introducing us to the topic of PMU placement, to Jung-Chen Liu [51] whose lectures form the basis for Section 7.5, to Irena Swanson for initial help with Macaulay2, to Jonathan Totushek for producing our simplicial complexes with TikZ, and to Carolyn Yackel for valuable comments on an earlier draft. Also, we gratefully acknowledge the invaluable Macaulay2 documentation at

`http://www.math.uiuc.edu/Macaulay2/Documentation/`

which we frequently consulted during the writing of this text.

We are also pleased to acknowledge the financial support of the National Science Foundation (NSF grant NSF 0354281), North Dakota EPSCoR (NSF grant EPS-0814442), and the National Security Agency.

[1] See `https://groups.google.com/forum/#!topic/macaulay2/9Li0pXPdSo8`.

Part I
Monomial Ideals

In this part of the text, we explore the algebra of monomial ideals, that is, ideals in a polynomial ring that are generated by sets of monomials. In particular, the results of this part open the door to the connections to combinatorics and other areas described in Part II.

Chapter 1 sets the stage with important basic properties of monomial ideals. Section 1.1 discusses the relation between a monomial ideal I and the set $[\![I]\!]$ of monomials in I. This includes a simple but important criterion for checking ideal membership, when a given monomial is in I. Some of the basic properties of monomials lead naturally to a discussion of integral domains, which is the subject of the optional Section 1.2. Section 1.3 is about generators of monomial ideals, and includes the fact that every monomial ideal is finitely generated. This is closely related to the noetherian property, which we treat in the optional Section 1.4. This chapter ends with two explorations: Section 1.5 shows how binomial coefficients arise in problems of counting monomials, and Section 1.6 introduces the related problem of determining numbers of generators of monomial ideals.

On the Macaulay2 front, this chapter focuses primarily on these basic properties. However, we also begin to develop the syntax (e.g., for functions) needed for more advanced programming in later chapters.

Chapter 2 looks at the behavior of monomial ideals under certain operations. We start with intersections in Section 2.1, which form the framework for our decomposition results. Since we use least common multiples (LCMs) to describe generating sets of intersections, we follow this with the optional Section 2.2 on unique factorization domains (UFDs). Next comes Section 2.3, dealing with monomial radicals (a monomial version of the radical of an ideal), which are very important for the study of square-free monomial ideals in Chapter 4. In general, the radical and monomial radical only agree when we work over "reduced rings", the topic of the exploration Section 2.4. Section 2.5 is all about colon ideals, which are crucial for our treatment of parametric decompositions in Chapter 6. Bracket powers, related to the Frobenius endomorphism in prime characteristic are the next topic, in Section 2.6, and a souped-up version of this notion is treated in the exploration Section 2.8. The exploration Section 2.5 treats saturations, a combination of powers and colons. This chapter ends with Section 2.9, which explores the connections between ordinary and bracket powers; these constructions are generally different, but deeply related.

Most of the basic Macaulay2 constructions of this ilk are introduced in Appendix B, so we only recall them briefly in Chapter 2. Constructions that are specific to monomial ideals are obvious exceptions here: monomial radicals and (generalized) bracket powers. In addition to this, we start looking under the hood at how Macaulay2 represents algebraic data like polynomials, and we work on getting used to some handy shortcuts.

This part culminates in Chapter 3, which is an existential chapter about decompositions of monomial ideals. This chapter begins with Section 3.1 on m-irreducible monomial ideals. These are the analogs of prime numbers in our decomposition results, and we characterize them explicitly here. When working over a field, the m-irreducible monomial ideals are also irreducible, as we prove in the optional Section 3.2. The main point of this part of the text, the first main goal of the text, is the existence and uniqueness of m-irreducible decompositions, which we establish in Section 3.3. The related (but optional) topic of irreducible decompositions is treated in Section 3.4. While most of this chapter is existential in nature, the exploration Section 3.5 gives a first taste of how to actually compute these decompositions; this subject is continued in Chapter 4 and Part III.

The Macaulay2 material in this chapter is a bit deeper than preceding chapters. In Section 3.1 we explain how to build "methods" which are functions that are more powerful and more elaborate than the basic functions from preceding sections. As the chapter progresses, the methods for our various topics and constructions become more involved. In addition, in Section 3.2 we discuss hash tables, structures for representing computational data.

Chapter 1
Fundamental Properties of Monomial Ideals

In this chapter, we develop the basic tools for working with monomial ideals. Section 1.1 introduces the main players. Motivated by some properties from this section, Section 1.2 contains a brief discussion of integral domains. Section 1.3 deals with some aspects of generating sets for monomial ideals. The optional Section 1.4 on noetherian rings is a natural follow-up. The chapter concludes with explorations of some numerical aspects of monomials and monomial ideals in Sections 1.5 and 1.6. The parallel Macaulay2 material here explores these notions computationally and lays the groundwork for more elaborate programming down the road.

1.1 Monomial Ideals

In this section, A is a non-zero commutative ring with identity.

Without further ado, we introduce the main objects of study in this text. For an introductory treatment of rings, polynomial rings, ideals, and generators, see Sections A.1–A.3.

Definition 1.1.1. Set $R = A[X_1, \ldots, X_d]$. A *monomial ideal* in R is an ideal of R that can be generated by monomials in $X_1, \ldots, X_d$; see Definition A.1.5.

For example, consider the polynomial ring $R = A[X,Y]$. The ideal $I = (X^2, Y^3)R$ is a monomial ideal. Note that I contains the polynomial $X^2 - Y^3$, so monomial ideals may contain more than monomials; in fact, every non-zero monomial ideal contains non-monomials. The ideal $J = (Y^2 - X^3, X^3)R$ is a monomial ideal because $J = (Y^2, X^3)R$. The trivial ideals 0 and R are monomial ideals since $0 = (\emptyset)R$ and $R = 1_R R = X_1^0 \cdots X_d^0 R$.

The following notation is non-standard but quite useful.

Definition 1.1.2. Set $R = A[X_1, \ldots, X_d]$. For each monomial ideal $I \subseteq R$, let $[\![I]\!]$ denote the set of all monomials contained in I. In the notation of Definition A.9.5, this translates to $[\![I]\!] = I \cap [\![R]\!]$.

It is important to note that for each non-zero monomial ideal $I \subseteq R$, the set $[\![I]\!] \subset R$ is an infinite set (assuming $d \geqslant 1$) that is not an ideal. However, the next lemma shows that this set is a natural generating set for I.

Lemma 1.1.3. *For each monomial ideal I of $R = A[X_1, \ldots, X_d]$, we have $I = ([\![I]\!])R$.*

Proof. Let S be a set of monomials generating I. It follows that $S \subseteq [\![I]\!] \subseteq I$, so $I = (S)R \subseteq ([\![I]\!])R \subseteq I$. This implies the desired equality. □

The next criterion for containment and equality of monomial ideals is straightforward to prove, but it is incredibly useful.

Theorem 1.1.4 *Set $R = A[X_1, \ldots, X_d]$. Let I and J be monomial ideals of R.*

(a) *We have $I \subseteq J$ if and only if $[\![I]\!] \subseteq [\![J]\!]$.*
(b) *We have $I = J$ if and only if $[\![I]\!] = [\![J]\!]$.*

Proof. (a) If $I \subseteq J$, then $[\![I]\!] = I \cap [\![R]\!] \subseteq J \cap [\![R]\!] = [\![J]\!]$. Conversely, if $[\![I]\!] \subseteq [\![J]\!]$, then Lemma 1.1.3 implies that $I = ([\![I]\!])R \subseteq ([\![J]\!])R = J$.

Part (b) follows directly from (a). □

Definition 1.1.5. Set $R = A[X_1, \ldots, X_d]$.

(a) Let f and g be monomials in R. Then f is a *monomial multiple* of g if there is a monomial $h \in R$ such that $f = gh$.
(b) For a monomial $f = \underline{X}^{\underline{n}} \in R$, the d-tuple $\underline{n} \in \mathbb{N}^d$ is the of f.

Because the monomials in $R = A[X_1, \ldots, X_d]$ are linearly independent over A, the exponent vector of each monomial $f \in R$ is well-defined; see Corollary A.2.5(c). In particular, for $\underline{m}, \underline{n} \in \mathbb{N}^d$, we have $\underline{X}^{\underline{m}} = \underline{X}^{\underline{n}}$ if and only if $\underline{m} = \underline{n}$.

In words, the next result says that a the product of a monomial and a non-monomial is not a monomial. While this may be intuitively clear, we include a proof for completeness. The main point is the linear independence of the monomials in $A[X_1, \ldots, X_d]$. Note that the relation $\succcurlyeq$ is from Definition A.9.3.

Lemma 1.1.6. *Set $R = A[X_1, \ldots, X_d]$. Let $f = \underline{X}^{\underline{m}}$ and $g = \underline{X}^{\underline{n}}$ be monomials in R. If h is a polynomial in R such that $f = gh$, then $\underline{m} \succcurlyeq \underline{n}$ and h is the monomial $h = \underline{X}^{\underline{p}}$ where $p_i = m_i - n_i$.*

Proof. Assume that $f = gh$ where $h = \sum_{\underline{p} \in \Lambda} a_{\underline{p}} \underline{X}^{\underline{p}} \in R$ and $\Lambda \subset \mathbb{N}^d$ is a finite subset such that $a_{\underline{p}} \neq 0_A$ for each $\underline{p} \in \Lambda$. The equation $f = gh$ then reads

$$\underline{X}^{\underline{m}} = \underline{X}^{\underline{n}} \sum_{\underline{p} \in \Lambda} a_{\underline{p}} \underline{X}^{\underline{p}} = \sum_{\underline{p} \in \Lambda} a_{\underline{p}} \underline{X}^{\underline{n}+\underline{p}}.$$

The linear independence of the monomials in R implies that

$$a_{\underline{p}} = \begin{cases} 0_A & \text{when } \underline{n}+\underline{p} \neq \underline{m} \\ 1_A & \text{when } \underline{n}+\underline{p} = \underline{m}. \end{cases}$$

Since we have assumed that each of the coefficients $a_{\underline{p}}$ is non-zero, this implies that the only possible non-zero term in h is $a_{\underline{p}}\underline{X}^{\underline{p}}$ when $\underline{n}+\underline{p}=\underline{m}$; in other words, the set Λ consists of a single element $\Lambda=\{\underline{p}\}$. The equality $a_{\underline{p}}=1_A$ above then implies that $h=\underline{X}^{\underline{p}}$. From the condition $\underline{n}+\underline{p}=\underline{m}$ we have $0\leqslant p_i=m_i-n_i$ for each i, so $\underline{m}\succcurlyeq\underline{n}$ by definition. $\square$

The next lemma builds from the previous one. For our purposes, the main point is the equivalence of conditions (i) and (iv) which gives a numerical/combinatorial ideal membership criterion for a monomial f to be in the ideal generated by another monomial g. This is significantly enhanced in Theorem 1.1.9 and Exercise 1.1.14. See Definition A.9.4 for an explanation of the notation $\langle\underline{n}\rangle$.

Lemma 1.1.7. *Set $R=A[X_1,\ldots,X_d]$. Let $f=\underline{X}^{\underline{m}}$ and $g=\underline{X}^{\underline{n}}$ be monomials in R. The following conditions are equivalent:*

(i) *we have $f\in gR$;*
(ii) *the element f is a multiple of g;*
(iii) *the element f is a monomial multiple of g;*
(iv) *we have $\underline{m}\succcurlyeq\underline{n}$; and*
(v) *we have $\underline{m}\in\langle\underline{n}\rangle$.*

Proof. The equivalence (i) $\Longleftrightarrow$ (ii) is in Proposition A.3.6(d). We have (iv) $\Longleftrightarrow$ (v) and (iii) $\Longrightarrow$ (ii) by definition, and the implications (iv) $\Longleftarrow$ (ii) $\Longrightarrow$ (iii) follow from Lemma 1.1.6.

(iv) $\Longrightarrow$ (iii): Assume that $\underline{m}\succcurlyeq\underline{n}$. By definition, this implies that $m_i-n_i\geqslant 0$ for each i. Setting $p_i=m_i-n_i$, we have $\underline{p}\in\mathbb{N}^d$, and it follows readily that $f=gh$ where $h=\underline{X}^{\underline{p}}$. This says that f is a monomial multiple of g, as desired. $\square$

Remark 1.1.8. Note that the previous result shows that the "divisibility order" on the monomial set $[\![R]\!]$ is a partial order. Indeed, the order $\succcurlyeq$ on $\mathbb{N}^d$ is a partial order by Exercise A.9.9. Thus, the desired conclusion follows from Lemma 1.1.7 since (a) the monomials in $[\![R]\!]$ are in bijection with the elements of $\mathbb{N}^d$, and (b) Lemma 1.1.7 implies that $\underline{X}^{\underline{m}}\succcurlyeq\underline{X}^{\underline{n}}$ if and only if $\underline{m}\succcurlyeq\underline{n}$.

The next result provides a solution to the "Ideal Membership Problem" for monomial ideals; see also Exercise 1.1.10. Note that it assumes that the ideal is generated by a finite set of monomials. We show in Dickson's Lemma 1.3.1 that every monomial ideal is generated by a finite list of monomials, so this result applies to every monomial ideal. See also Exercise 1.1.14. It is worth noting that the non-trivial implication of Theorem 1.1.9 can fail if the f_i are not monomials; see Exercise A.3.16(a).

Theorem 1.1.9 *Set $R=A[X_1,\ldots,X_d]$. Let $f,f_1,\ldots,f_m$ be monomials in R. Then $f\in(f_1,\ldots,f_m)R$ if and only if $f\in f_iR$ for some i.*

Proof. $\Longleftarrow$: Since $f_i\in\{f_1,\ldots,f_m\}$, we have $f_iR=(f_i)R\subseteq(f_1,\ldots,f_m)R$.

$\Longrightarrow$: Assume that $f\in(f_1,\ldots,f_m)R$ and write $f=\sum_{i=1}^m f_ig_i$ with elements $g_i\in R$. By assumption, we have $f=\underline{X}^{\underline{n}}$ and $f_i=\underline{X}^{\underline{n}_i}$ for some $\underline{n},\underline{n}_1,\ldots,\underline{n}_m\in\mathbb{N}^d$. By definition, we can write each $g_i=\sum_{\underline{p}\in\mathbb{N}^d}^{\text{finite}}a_{i,\underline{p}}\underline{X}^{\underline{p}}$ where each $a_{i,\underline{p}}\in A$, so

$$\underline{X}^{\underline{n}} = f = \sum_{i=1}^{m} f_i g_i = \sum_{i=1}^{m} \underline{X}^{\underline{n}_i} \left(\sum_{\underline{p}\in\mathbb{N}^d}^{\text{finite}} a_{i,\underline{p}} \underline{X}^{\underline{p}} \right) = \sum_{i=1}^{m} \sum_{\underline{p}\in\mathbb{N}^d}^{\text{finite}} a_{i,\underline{p}} \underline{X}^{\underline{n}_i+\underline{p}}.$$

Since the monomials in R are linearly independent over A, the monomial $\underline{X}^{\underline{n}}$ must occur in the right-most sum in this display. (Note that we are not suggesting that the monomials occurring in the right-most sum are distinct; see Definition A.2.6. Hence, the monomial $\underline{X}^{\underline{n}}$ it may occur more than once in this expression.) In other words, we have $\underline{X}^{\underline{n}} = \underline{X}^{\underline{n}_i+\underline{p}}$ for some i and some $\underline{p}$. It follows that

$$f = \underline{X}^{\underline{n}} = \underline{X}^{\underline{n}_i+\underline{p}} = \underline{X}^{\underline{n}_i}\underline{X}^{\underline{p}} = f_i\underline{X}^{\underline{p}} \in f_iR$$

for some i, as desired. □

The next result is used in several key results later in the text, so we recommend that every reader work through its proof.

Lemma 1.1.10. *Set $R = A[X_1,\ldots,X_d]$. Let I be an ideal of R. Then the following conditions are equivalent.*

(i) I is a monomial ideal; and
(ii) for each $f \in I$ each monomial occurring in f is in I; see Definition A.2.6.

Proof. Exercise. □

The next construction allows us to visualize monomial ideals, at least in two variables, which is very useful for building intuition.

Definition 1.1.11. Set $R = A[X_1,\ldots,X_d]$. The *graph* of a monomial ideal I is

$$\Gamma(I) = \{\underline{n} \in \mathbb{N}^d \mid \underline{X}^{\underline{n}} \in I\}.$$

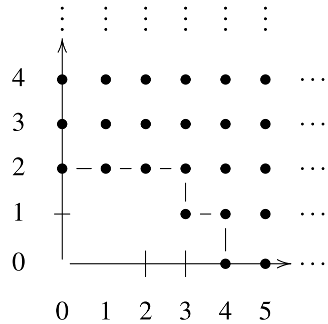
The next result explains the connection between generators of monomial ideals and some basic subsets of $\mathbb{N}^d$. It simplifies the work involved in identifying $\Gamma(I)$.

Theorem 1.1.12 *Set $R = A[X_1,\ldots,X_d]$. If $I = (\underline{X}^{\underline{n}_1},\ldots,\underline{X}^{\underline{n}_m})R$, then one has $\Gamma(I) = \langle \underline{n}_1\rangle \cup \cdots \cup \langle \underline{n}_m\rangle$.*

Proof. $\supseteq$: Let $\underline{m} \in \langle \underline{n}_1\rangle \cup \cdots \cup \langle \underline{n}_m\rangle$. Then we have $\underline{m} \in \langle \underline{n}_i\rangle$ for some i, and so $\underline{m} \succcurlyeq \underline{n}_i$. Lemma 1.1.7 implies that $\underline{X}^{\underline{m}}$ is in $\underline{X}^{\underline{n}_i}R \subseteq (\underline{X}^{\underline{n}_1},\ldots,\underline{X}^{\underline{n}_m})R = I$ and it follows by definition that $\underline{m} \in \Gamma(I)$.

$\subseteq$: Assume that $\underline{p} \in \Gamma(I)$. Then $\underline{X}^{\underline{p}} \in I = (\underline{X}^{\underline{n}_1},\ldots,\underline{X}^{\underline{n}_m})R$, so Theorem 1.1.9 implies that $\underline{X}^{\underline{p}} \in \underline{X}^{\underline{n}_j}R$ for some j. From Lemma 1.1.7 we conclude that $\underline{p} \in \langle \underline{n}_j\rangle \subseteq \langle \underline{n}_1\rangle \cup \cdots \cup \langle \underline{n}_m\rangle$, as desired. □

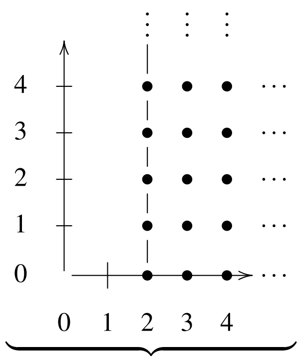
Example 1.1.13. Consider a few examples in $R = A[X,Y]$. For $I = (X^4, X^3Y, Y^2)R$, we have $\Gamma(I) = \langle(4,0)\rangle \cup \langle(3,1)\rangle \cup \langle(0,2)\rangle \subseteq \mathbb{N}^2$, represented in the next diagram.



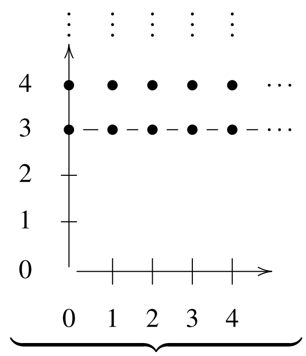
Next, consider the ideals $J = (X^2)R$ and $K = (Y^3)R$. Then $J + K = (X^2, Y^3)R$; see Exercise A.4.6(a). Theorem 1.1.12 shows that $\Gamma(J) = \langle(2,0)\rangle$, $\Gamma(K) = \langle(0,3)\rangle$ and

$$\Gamma(J+K) = \langle(2,0)\rangle \cup \langle(0,3)\rangle = \Gamma(J) \cup \Gamma(K).$$

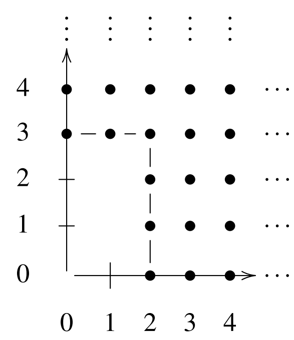
Graphically, we have the following:



$\Gamma(J)$



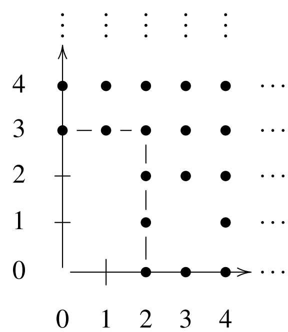
$\Gamma(K)$



$\Gamma(J+K) = \Gamma(J) \cup \Gamma(K)$

More generally, from Exercise 1.3.13(b) we will see that if $I_1,\dots,I_n$ are monomial ideals in $A[X_1,\dots,X_d]$, then $\Gamma(\sum_{j=1}^n I_j) = \bigcup_{j=1}^n \Gamma(I_j)$.

It is straightforward to identify the subsets of $\mathbb{N}^d$ that occur as graphs of monomial ideals: a nonempty subset $\Gamma \subseteq \mathbb{N}^d$ is of the form $\gamma = \Gamma(I)$ for some monomial ideal $I \subseteq A[X_1,\dots,X_d]$ if and only if for each $\underline{m} \in \Gamma$ and each $\underline{n} \in \mathbb{N}^d$ one has $\underline{m}+\underline{n} \in \Gamma$. In two variables, this means that Γ must be closed under "moving to the north and east". For instance, the following is not of the form $\Gamma(I)$



because, e.g., it contains the point $(3,0)$ but not $(3,1)$.

Exercises

Exercise 1.1.14. Set $R = A[X_1,\dots,X_d]$. Let f be a monomial in R, and let S be a set of monomials in R. Prove that $f \in (S)R$ if and only if $f \in sR$ for some $s \in S$.

Exercise 1.1.15. Set $R = A[X_1,\dots,X_d]$. Set $\mathfrak{X} = (X_1,\dots,X_d)R$ and let I be a monomial ideal. Prove that $I \neq R$ if and only if $I \subseteq \mathfrak{X}$.

Exercise 1.1.16. Set $R = A[X_1,\dots,X_d]$. Let f and g be monomials in R.

(a) Prove that if $f \in (g)R$, then $\deg(f) \geqslant \deg(g)$; see Definition A.2.4(c).
(b) Prove or disprove: If $\deg(f) \geqslant \deg(g)$, then $f \in (g)R$. Justify your answer.
(c) Prove that if $\deg(f) = \deg(g)$ and $g \in (f)R$, then $g = f$.

Exercise 1.1.17. Set $R = A[X_1,\dots,X_d]$. Let f be a monomial in R and let n be an integer such that $n \geqslant 1$. Prove that $\deg(f) < n$ if and only if there exists a monomial g of degree $n-1$ such that $g \in (f)R$.

Exercise 1.1.18. Set $R = A[X_1,\dots,X_d]$. Prove that for monomials $f,g \in [\![R]\!]$, one has $(f)R = (g)R$ if and only if $f = g$.

Exercise 1.1.19. Set $R = A[X_1,\dots,X_d]$. Prove that if I is a monomial ideal in R such that $I \neq R$, then $I \cap A = 0$.

Exercise 1.1.20. Set $R = A[X_1, \dots, X_d]$. Let f, g, h be monomials in R.

(a) (Cancellation) Prove that if $fh = gh$, then $f = g$.
(b) Prove that if $fX_i = gX_j$ for some $i \neq j$, then $f \in (X_j)R$ and $g \in (X_i)R$.
(c) Prove that $\deg(fg) = \deg(f) + \deg(g)$.

Exercise 1.1.21. Set $R = A[X,Y]$. Sketch $\Gamma(I)$ for the following ideals and justify your answers:

(a) $I = (X^5, Y^4)R$.
(b) $I = (X^5, XY^2, Y^4)R$.
(c) $I = (X^5Y, X^2Y^2, XY^4)R$.

Exercise 1.1.22. Prove Lemma 1.1.10.

**Exercise 1.1.23.* Prove that if A is a field, then in every polynomial ring $R = A[X_1, \dots, X_d]$, each ideal $I = (X_{i_1}, \dots, X_{i_n})R$ generated by variables (including the 0-ideal) is prime (see Exercise A.5.10). (Hint: Let $f, g \in R \smallsetminus I$. Write $f = f_1 + f_2$ where $f_1 \in I$ and no monomial occurring in f_2 is in I, and similarly for g. Use Lemma 1.1.10 to conclude that $f_2 g_2 \notin I$ and hence $fg \notin I$.) (This is used in the proof of Theorem 5.1.2.)

Monomials in Macaulay2

In this tutorial, we explore some basic operations on monomials in Macaulay2; see Appendix B for an introduction to this computer language. In what follows, lines that begin with `i`n for some n indicate input lines, and lines that begin with `o`n indicate output lines.

First, we define the polynomial ring where we will work.

```
i1 : R = ZZ/101[x, y, z]
o1 = R
o1 : PolynomialRing
```

Next, we will ask Macaulay2 to give us the exponent vector of a monomial.

```
i2 : exponents(x^3*y)
o2 = {{3, 1, 0}}
o2 : List
```

There is a set of redundant braces here because `exponents` actually works on polynomials in general, not just monomials, so `exponents` returns one exponent vector for each term in the polynomial.

We can define functions easily in Macaulay2, for example, the squaring function. The syntax below indicates that i is the input, that we are sending the input i to its square i^2, and that we have given this function the name `f`. Note that comments can be included because Macaulay2 ignores anything on a line after `--`. One can include multi-line comments by surrounding the comment with `{*` and `*}`.

```
i3 : f = i -> i^2 -- the squaring function, for example
o3 = f
o3 : FunctionClosure

i4 : f(3)
o4 = 9
```

As in the earlier part of this section, we wish to compare exponent vectors of monomials. For this, we first need a function to test positivity of vectors.

```
i5 : g = i -> (i > 0)
o5 = g
o5 : FunctionClosure

i6 : g(-1)
o6 = false
```

To compare two exponent vectors, we compute their difference and check if the result has positive entries, as follows.

```
i7 : exponents(x^3*y) - exponents(x^2*y^2*z^3)
o7 = {{1, -1, -3}}
o7 : List

i8 : flatten oo -- remove the external set of braces
o8 = {1, -1, -3}
o8 : List

i9 : all(oo, g) -- test to see if all entries are positive
o9 = false
```

Note that in this code, `oo` refers to the most recent output value, so the commands must be issued in this order for them to work. Putting all these commands together in one line we have the following.

```
i10 : all(flatten (exponents(x^3*y) - exponents(x^2*y^2*z^3)), g)
o10 = false

i11 : exit
```

Exercises

Exercise 1.1.24. Set $R = \mathbb{Z}_{101}[X,Y]$.

(a) Use Macaulay2 to verify that $X^2 - Y^3 \in (X^2, Y^3)R$.
(b) Use Macaulay2 to verify that $(Y^2 - X^3, X^3)R = (Y^2, X^3)R$.

Exercise 1.1.25. Set $R = \mathbb{Z}_{101}[X,Y]$.

(a) Use Macaulay2 to verify that $X^2Y \nmid XY^7$ and $X^2Y \big| X^2Y^7$.
(b) Use Macaulay2 to verify that $XY^7 \notin (X^2Y)R$ and $X^2Y^7 \in (X^2Y)R$.

Exercise 1.1.26. Set $R = \mathbb{Z}_{101}[X,Y]$ and $J = (Y^2 - X^3, X^3)R = (Y^2, X^3)R$.

(a) Use Macaulay2 to verify that $Y^2 \in J$ and $Y^2 - X^3 \nmid Y^2$ and $X^3 \nmid Y^2$.
(b) Use Macaulay2 to verify that $X^2 \notin J$ and $Y^2 \nmid X^2$ and $X^3 \nmid X^2$.

Exercise 1.1.27. Use Macaulay2 to verify that $\deg(fg) = \deg(f) + \deg(g)$ for some monomials $f, g \in \mathbb{Z}_{101}[X,Y]$. Make sure to use `degree` and `==` in your verification.

1.2 Integral Domains (optional)

Given polynomials $f, g \in A[X]$ one may have $\deg(fg) = \deg(f) + \deg(g)$ or maybe $\deg(fg) < \deg(f) + \deg(g)$. For monomials, we have seen in the previous section that equality always holds. This is, in many regards, the nicest situation. This section focuses on rings for which this equality always holds.

Definition 1.2.1. An *integral domain* is a commutative ring R with identity such that $1_R \neq 0_R$ and such that for all $r, s \in R$ if $r, s \neq 0_R$, then $rs \neq 0_R$.

For example, the rings $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{Z}$ are integral domains. The ring $\mathbb{Z}_n$ is an integral domain if and only if n is prime. For instance, the ring $\mathbb{Z}_6$ is not an integral domain because $2, 3 \neq 0$ in $\mathbb{Z}_6$, but $2 \cdot 3 = 0$ in $\mathbb{Z}_6$.

The next proposition shows that every field is an integral domain; for a treatment of the converse, see Exercise 1.2.8.

Proposition 1.2.2 *If R is a field, then R is an integral domain.*

Proof. Assume that R is a field, and let $r, s \in R$ such that $rs = 0_R$. We need to show that $r = 0_R$ or $s = 0_R$. Assume that $r \neq 0_R$. Since R is a field, we have

$$s = 1_R s = (r^{-1}r)s = r^{-1}(rs) = r^{-1}0_R = 0_R$$

as desired. $\square$

The next result contains one of the most important properties of integral domains. Compare it with Exercise 1.1.20(a). See also Exercise 1.2.11 for the converse.

Proposition 1.2.3 (Cancellation) *Let R be an integral domain. If $r, s, t \in R$ such that $r \neq 0_R$ and $rs = rt$, then $s = t$.*

Proof. Given that $rs = rt$, we have $r(s - t) = 0_R$. Since R is an integral domain and $r \neq 0_R$, it follows that $s - t = 0_R$, that is, that $s = t$. $\square$

Next we present the result described in the introduction to this section. See Exercise 1.2.9 for a counterexample when A is not an integral domain. Exercises 1.2.10 and 1.2.12 contain related results.

Proposition 1.2.4 *Let A be an integral domain.*

(a) *Given non-zero polynomials* $f,g \in A[X]$*, one has* $\deg(fg) = \deg(f) + \deg(g)$*, and the leading coefficient of the product* fg *is the product of the leading coefficients of* f *and* g*.*
(b) *The ring* $A[X]$ *is an integral domain.*

Proof. (a) Write $f = \sum_{i=0}^{m} a_i X^i$ and $g = \sum_{i=0}^{n} b_i X^i$ with $m = \deg(f)$ and $n = \deg(g)$. It follows that we have

$$fg = a_0b_0 + (a_1b_0 + a_0b_1)X + \cdots + (a_mb_{n-1} + a_{m-1}b_n)X^{m+n-1} + a_mb_nX^{m+n}.$$

Since R is an integral domain and $a_m, b_n \neq 0_R$, it follows that $a_mb_n \neq 0_R$, and hence the desired properties.

(b) Part (a) shows that the product of two non-zero polynomials in $A[X]$ is non-zero. Since $0_{A[X]} = 0_A \neq 1_A = 1_{A[X]}$, it follows that $A[X]$ is an integral domain. □

Corollary 1.2.5 *If* A *is an integral domain, then so is* $A[X_1,\dots,X_d]$.

Proof. Induct on d, using Proposition 1.2.4(b) and Definition A.2.4(a). □

Exercises

Exercise 1.2.6. Prove that $\mathbb{Z}_n$ is an integral domain if and only if n is prime.

Exercise 1.2.7. Prove that every finite integral domain is a field.

Exercise 1.2.8. Find an example of an integral domain that is not a field. Justify your answer.

Exercise 1.2.9. Find two non-zero polynomials f and g in $\mathbb{Z}_4[X]$ such that $\deg(fg) \neq \deg(f) + \deg(g)$. Justify your answer.

Exercise 1.2.10. Let A be a commutative ring with identity.

(a) Prove that if A is a subring of an integral domain, then A is an integral domain; see Definition A.1.8.
(b) Prove that if the polynomial ring $A[X_1,\dots,X_d]$ in d variables is an integral domain for some $d \geqslant 1$, then A is an integral domain.

Exercise 1.2.11. Let R be a non-zero commutative ring with identity. Assume that for all $r,s,t \in R$ if $r \neq 0_R$ and $rs = rt$, then $s = t$. Prove that R is an integral domain.

Exercise 1.2.12. Let A be a commutative ring with identity such that $1_A \neq 0_A$. Assume that for all non-zero polynomials $f,g \in A[X]$ in one variable, we have $\deg(fg) = \deg(f) + \deg(g)$. Prove that A is an integral domain.

Exercise 1.2.13. Prove or disprove the following: If A is an integral domain, then $A[X_1,X_2,\dots]$ is an integral domain. Does the converse hold? Justify your answers.

**Exercise 1.2.14.* Let A be a non-zero commutative ring with identity, and recall the prime ideals of Exercise A.5.10.

(a) Prove that 0 is prime if and only if A is an integral domain.
(b) Prove that an ideal J of A is prime if and only if A/J is an integral domain.
(c) Prove that the following conditions are equivalent:

(i) A is an integral domain;
(ii) in every polynomial ring $R = k[X_1,\ldots,X_d]$, each ideal $(X_{i_1},\ldots,X_{i_n})R$ generated by variables is prime; and
(iii) there exists a polynomial ring $R = k[X_1,\ldots,X_d]$ with a prime ideal of the form $(X_{i_1},\ldots,X_{i_n})R$.

(This is used optionally in Exercise 4.1.12.)

**Exercise 1.2.15.* In the ring $\mathrm{C}(\mathbb{R})$ of continuous functions, show that for each $r \in \mathbb{R}$, the ideal $I_r = \{f \in \mathrm{C}(\mathbb{R}) \mid f(r) = 0\}$ is prime. (This is used in Exercise 3.2.15.)

Integral Domains in Macaulay2, Exercises

In the next exercises, use the commands `degree`, `==`, and `leadCoefficient`.

Exercise 1.2.16. Set $A = \mathbb{Z}_{101}[Y,Z]$. Choose two non-zero polynomials $f, g \in A[X]$ and use Macaulay2 to verify that $fg \neq 0$ and $\deg(fg) = \deg(f) + \deg(g)$. Use Macaulay2 to show that the leading coefficient of fg is the product of the leading coefficients of f and g.

Exercise 1.2.17. Set $A = \mathbb{Z}_{101}[Y]/(Y^2)\mathbb{Z}_{101}[Y]$. Find two polynomials $f, g \in A[X]$ such that $f, g \neq 0$ but $fg = 0$. Use Macaulay2 to verify your example. Repeat this with $A = \mathbb{Z}_{101}[Y,Z]/(YZ)\mathbb{Z}_{101}[Y,Z]$.

1.3 Generators of Monomial Ideals

In this section, A is a non-zero commutative ring with identity.

Theorem 1.1.12 is only stated for a monomial ideal I that is generated by a finite number of monomials. The next result shows that this condition is automatic for all monomial ideals. It is named after Leonard Dickson. It compares directly to the Hilbert Basis Theorem 1.4.5 below, with one key difference: it does not require A to be "noetherian".

Theorem 1.3.1 (Dickson's Lemma) *Set $R = A[X_1,\ldots,X_d]$. Then every monomial ideal of R is finitely generated; moreover, it is generated by a finite set of monomials.*

Proof. Let $I \subseteq R$ be a monomial ideal, and assume without loss of generality that $I \neq 0$. We proceed by induction on the number of variables d.

Base case: $d = 1$. For this case, we write $R = A[X]$. Let

$$r = \min\{e \geqslant 0 \mid X^e \in I\}.$$

Then $X^r \in I$ and so $X^r R \subseteq I$. We will be done with this case once we show that $X^r R \supseteq I$. Since I is generated by its monomials, it suffices to show that $X^r R \supseteq [\![I]\!]$. For this, note that if $X^s \in I$, then $s \geqslant r$ and so $X^s \in X^r R$ by Lemma 1.1.7.

Induction step: Assume that $d \geqslant 2$ and that every monomial ideal of the ring $R' = A[X_1, \ldots, X_{d-1}]$ is finitely generated. Given a monomial ideal I of R, we set

$$S = \{\text{monomials } z \in R' \mid zX_d^a \in I \text{ for some } a \geqslant 0\}$$

and $J = (S)R'$. By definition J is a monomial ideal in R', so our induction hypothesis implies that it is finitely generated, say $J = (z_1, \ldots, z_n)R'$ where $z_1, \ldots, z_n \in S$; see Proposition A.3.6(f). For $i = 1, \ldots, n$ there exists an integer $e_i \geqslant 0$ such that $z_i X_d^{e_i} \in I$. With $e = \max\{e_1, \ldots, e_n\}$, it follows that $z_i X_d^e \in I$ for $i = 1, \ldots, n$.

For $m = 0, \ldots, e-1$ we set

$$S_m = \{\text{monomials } w \in R' \mid wX_d^m \in I\}$$

and $J_m = (S_m)R'$. (In the case $e = 0$, there are no S_m's nor J_m's to consider.) By definition J_m is a monomial ideal in R', so our induction hypothesis implies that it is finitely generated, say $J_m = (w_{m,1}, \ldots, w_{m,n_m})R'$ where $w_{m,1}, \ldots, w_{m,n_m} \in S_m$.

Let I' be the ideal of R generated by the $z_i X_d^e$ and the $w_{m,i} X_d^m$:

$$I' = (\{z_i X_d^e \mid i = 1, \ldots, n\} \cup \{w_{m,i} X_d^m \mid m = 0, \ldots, e-1; i = 1, \ldots, n_m\})R.$$

By construction I' is a finitely generated monomial ideal of R. By definition, each $z_i X_d^e, w_{m,i} X_d^m \in I$, so we have $I' \subseteq I$.

Claim: $I' = I$. (Once the claim is established, we conclude that I is generated by a finite set of its monomials, completing the proof.) It suffices to show that $I' \supseteq I$. Since I is generated by its monomials, it suffices to show that $I' \supseteq [\![I]\!]$, so let $\underline{X}^{\underline{p}} = X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} X_d^{p_d} \in [\![I]\!]$.

Case 1: $p_d \geqslant e$. We have $X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} \in S \subseteq J = (z_1, \ldots, z_n)R'$. Thus, we have $X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} \in z_i R'$ for some i, by Theorem 1.1.9. Writing $X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} = z_i z$ for some $z \in R'$, we have

$$\underline{X}^{\underline{p}} = X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} X_d^{p_d} = z_i z X_d^e X_d^{p_d - e} = (z_i X_d^e)(z X_d^{p_d - e}) \in (z_i X^e)R \subseteq I'$$

as desired.

Case 2: $p_d < e$. In this case, the monomial $X_1^{p_1} \cdots X_{d-1}^{p_{d-1}}$ is in the set $S_{p_d} \subseteq J_{p_d} = (w_{p_d,1}, \ldots, w_{p_d,n_{p_d}})R'$. Theorem 1.1.9 implies that $X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} \in w_{p_d,i} R'$ for some i. Writing $X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} = w_{p_d,i} w$ for some $w \in R'$, we have

$$\underline{X}^{\underline{p}} = X_1^{p_1} \cdots X_{d-1}^{p_{d-1}} X_d^{p_d} = w_{p_d,i} w X_d^{p_d} = (w_{p_d,i} X_d^{p_d})(w) \in (w_{p_d,i} X_d^{p_d})R \subseteq I'$$

as desired. □

Corollary 1.3.2 *Set $R = A[X_1, \ldots, X_d]$. Let $S \subseteq [\![R]\!]$ and set $I = (S)R$. Then there is a finite sequence $s_1, \ldots, s_n \in S$ such that $I = (s_1, \ldots, s_n)R$.*

Proof. Dickson's Lemma 1.3.1 implies that I is finitely generated, so the desired conclusion follows from Proposition A.3.6(f). □

Part (a) of the next result says that the polynomial ring $R = A[X_1, \ldots, X_d]$ satisfies the *ascending chain condition for monomial ideals*. Part (b) says that every nonempty set Σ of monomial ideals in R has a *maximal element*, and that every element of Σ is contained in a maximal element of Σ. While this result may seem esoteric, it is quite useful. For instance, it is the key to the main result of this part of the text in Section 3.3.

Theorem 1.3.3 *Set $R = A[X_1, \ldots, X_d]$.*

(a) *(ACC) Given a chain $I_1 \subseteq I_2 \subseteq \cdots$ of monomial ideals in R, there is an integer $N \geqslant 1$ such that $I_N = I_{N+1} = \cdots$.*
(b) *Given a nonempty set Σ of monomial ideals in R, there is an ideal $I \in \Sigma$ such that for all $J \in \Sigma$, if $I \subseteq J$, then $I = J$. Moreover, for each $K \in \Sigma$, there is an ideal $I \in \Sigma$ such that $K \subseteq I$ and such that for all $J \in \Sigma$, if $I \subseteq J$, then $I = J$.*

Proof. (a) Consider a chain $I_1 \subseteq I_2 \subseteq \cdots$ of monomial ideals in R. Then the ideal $J = \sum_{j=1}^{\infty} I_j = \bigcup_{j=1}^{\infty} I_j$ is a monomial ideal in R; see Exercises A.4.4(c) and A.4.6(c). Dickson's Lemma 1.3.1 implies that J is generated by a finite list of monomials $f_1, \ldots, f_m \in [\![J]\!]$. Since $J = \bigcup_{j=1}^{\infty} I_j$, for $i = 1, \ldots, m$ there is an index j_i such that $f_i \in I_{j_i}$. The condition $I_1 \subseteq I_2 \subseteq \cdots$ implies that there is an index N such that $f_i \in I_N$ for $i = 1, \ldots, m$. It follows that

$$J = (f_1, \ldots, f_m)R \subseteq I_N \subseteq I_{N+1} \subseteq I_{N+2} \subseteq \cdots \subseteq J.$$

Thus, we have equality at each step, as desired.

(b) Let Σ be a nonempty set of monomial ideals in R, and let $K \in \Sigma$. Suppose by way of contradiction that K is not contained in a maximal element of Σ. In particular, K is not a maximal element of Σ, so there is an element $I_1 \in \Sigma$ such that $K \subsetneq I_1$. Since K is not contained in a maximal element of Σ, it follows that I_1 is not a maximal element of Σ. Thus, there is an element $I_2 \in \Sigma$ such that $K \subsetneq I_1 \subsetneq I_2$. Continue this process inductively to construct a chain $K \subsetneq I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ of elements of Σ. The existence of this chain contradicts part (a). □

Definition 1.3.4. Let I be a monomial ideal of $R = A[X_1, \ldots, X_d]$. Let $z_1, \ldots, z_m \in [\![I]\!]$ such that $I = (z_1, \ldots, z_m)R$. The list $z_1, \ldots, z_m$ is an *irredundant monomial generating sequence* for I if for $i = 1, \ldots, m$ we have $(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)R \neq I$, that is, we have $(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)R \subsetneq I$. The list is a *redundant monomial generating sequence* for I if it is not irredundant, that is, if there exists an index i such that $I = (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)R$.

For an example, we work in $R = A[X,Y]$. The sequence X^3, X^2Y, X^2Y^2, Y^5 is a redundant generating sequence for $(X^3, X^2Y, X^2Y^2, Y^5)R$: since $X^2Y \mid X^2Y^2$, we have $(X^3, X^2Y, X^2Y^2, Y^5)R = (X^3, X^2Y, Y^5)R$.

On the other hand, the sequence X^3, X^2Y, XY^2, Y^3 is an irredundant monomial generating sequence for $(X^3, X^2Y, XY^2, Y^3)R$, as follows. First, we have

$$(X^2Y, XY^2, Y^3)R \subsetneq (X^3, X^2Y, XY^2, Y^3)R$$

as $X^3 \notin (X^2Y, XY^2, Y^3)R$; verify this using exponent vectors with Lemma 1.1.7 and Theorem 1.1.9. Similarly, we have

$$\begin{aligned}(X^3, XY^2, Y^3)R &\subsetneq (X^3, X^2Y, XY^2, Y^3)R\\ (X^3, X^2Y, Y^3)R &\subsetneq (X^3, X^2Y, XY^2, Y^3)R\\ (X^3, X^2Y, XY^2)R &\subsetneq (X^3, X^2Y, XY^2, Y^3)R\end{aligned}$$

so the generating sequence is irredundant.

Our next result contains a practical criterion (condition (i)) for checking if a given monomial generating sequence is irredundant.

Proposition 1.3.5 *Set $R = A[X_1, \ldots, X_d]$. Let I be a monomial ideal of R, and let $z_1, \ldots, z_m \in [\![I]\!]$ such that $I = (z_1, \ldots, z_m)R$. The following conditions are equivalent:*

(i) *z_i is not a monomial multiple of z_j whenever $i \neq j$.*
(ii) *for $i = 1, \ldots, m$ we have $z_i \notin (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)R$; and*
(iii) *the generating sequence $z_1, \ldots, z_m$ is irredundant.*

Proof. (i) $\Longrightarrow$ (ii): Assume that z_i is not a monomial multiple of z_j whenever $i \neq j$. If $z_i \in (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)R$, then Theorem 1.1.9 implies that $z_i \in z_jR$ for some $j \neq i$; Lemma 1.1.7 then implies that z_i is a monomial multiple of z_j, contradicting our assumption.

(ii) $\Longrightarrow$ (iii): Fix an index i. If $z_i \notin (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)R$, then by assumption $z_i \in I \smallsetminus (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)R$. It follows that $(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)R \subsetneq I$, so the generating sequence $z_1, \ldots, z_m$ is irredundant.

(iii) $\Longrightarrow$ (i): Assume that the generating sequence $z_1, \ldots, z_m$ is irredundant. Suppose that there are indices i, j such that $i \neq j$ and z_i is a monomial multiple of z_j. Then $z_i \in z_jR \subseteq (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)R$. It follows that

$$I = (z_1, \ldots, z_{i-1}, z_i, z_{i+1}, \ldots, z_m)R \subseteq (z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)R \subseteq I$$

so we have $(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)R = I$, which contradicts our assumption. $\square$

The previous result does not assume (or conclude) that the ideal I has an irredundant monomial generating sequence. This is dealt with in the next result.

Theorem 1.3.6 *Set $R = A[X_1, \ldots, X_d]$. Let I be a monomial ideal of R.*

(a) *Every monomial generating set S for I contains an irredundant monomial generating sequence for I.*

(b) *The ideal I has an irredundant monomial generating sequence.*
(c) *Irredundant monomial generating sequences are unique up to re-ordering.*

Proof. (a) Assume without loss of generality that $S \neq \emptyset$. Corollary 1.3.2 implies that I can be generated by a finite list of monomials $z_1, \ldots, z_m \in S$. If the list is redundant, then Proposition 1.3.5 provides an index i such that $(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m)R = I$; hence, the shorter list $z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_m$ of monomials generates I. Repeat this process with the new list, removing elements from the list until the remaining elements form an irredundant monomial generating sequence for I. The process will terminate in finitely many steps because the original list is finite.

(b) The ideal I has a monomial generating set by definition, so the desired conclusion follows from part (a).

(c) Assume that $z_1, \ldots, z_m$ and $w_1, \ldots, w_n$ are two irredundant monomial generating sequences. We show that $m = n$ and that there is a permutation $\sigma \in S_n$ such that $z_i = w_{\sigma(i)}$ for $i = 1, \ldots, n$. (Here we use the notation S_n for the *symmetric group* on n letters.)

Fix an index i. Since z_i is in $I = (w_1, \ldots, w_n)R$, Theorem 1.1.9 implies that z_i is a monomial multiple of w_j for some index j. Similarly, there is an index k such that w_j is a monomial multiple of z_k. The transitivity of the divisibility order on the monomial set $[\![R]\!]$ implies that z_i is a multiple of z_k; that is, z_i is a monomial multiple of z_k. Since the generating sequence $z_1, \ldots, z_m$ is irredundant, Proposition 1.3.5 says that $i = k$. It follows that $z_i \big| w_j$ and $w_j \big| z_i$. The fact that the divisibility order on $[\![R]\!]$ is antisymmetric implies $z_i = w_j$.

In summary, we see that for each index $i = 1, \ldots, m$ there exists an index $j = \sigma(i)$ such that $z_i = w_j = w_{\sigma(i)}$. Since the z_i are distinct and the w_j are distinct, we conclude that the function $\sigma \colon \{1, \ldots, m\} \to \{1, \ldots, n\}$ is injective. By symmetry, there is an injective function $\delta \colon \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that $w_i = z_{\delta(i)}$ for $i = 1, \ldots, n$. It follows that $m \leqslant n \leqslant m$ and so $m = n$. Furthermore, since σ is injective and $m = n$, the Pigeonhole Principle implies that σ is also surjective. This is the desired conclusion. $\square$

Here is an algorithm for finding an irredundant monomial generating sequence.

Algorithm 1.3.7. Set $R = A[X_1, \ldots, X_d]$. Fix monomials $z_1, \ldots, z_m \in [\![R]\!]$ and set $J = (z_1, \ldots, z_m)R$. We assume that $m \geqslant 1$.

Step 1. Check whether the generating sequence $z_1, \ldots, z_m$ is irredundant using Proposition 1.3.5.

Step 1a. If, for all indices i and j such that $i \neq j$, we have $z_j \notin (z_i)R$, then the generating sequence is irredundant; in this case, the algorithm terminates.

Step 1b. If there exist indices i and j such that $i \neq j$ and $z_j \in (z_i)R$, then the generating sequence is redundant; in this case, continue to Step 2.

Step 2. Remove a generator that causes a redundancy in the generating sequence. By assumption, there exist indices i and j such that $i \neq j$ and $z_j \in (z_i)R$. Re-order the indices to assume without loss of generality that $j = m$. Thus, we have $i < m$ and $z_m \in (z_i)R$. It follows that $J = (z_1, \ldots, z_m)R = (z_1, \ldots, z_{m-1})R$. Now apply Step 1 to the new list of monomials $z_1, \ldots, z_{m-1}$.

The algorithm will terminate in at most $m-1$ steps because one can remove at most $m-1$ monomials from the list and still form a non-zero ideal.

Example 1.3.8. Set $R = A[X,Y]$. Using Algorithm 1.3.7, one finds that the sequence X^3, X^2Y, Y^5 is an irredundant monomial generating sequence for the ideal $(X^3, X^2Y, X^2Y^2, Y^5)R$.

Here is a proposition that shows how to find an irredundant monomial generating sequence in one shot.

Proposition 1.3.9 *Set $R = A[X_1,\ldots,X_d]$. Fix a non-empty set of monomials $S \subseteq [\![R]\!]$ and set $J = (S)R$. For each $z \in S$, write $z = \underline{X}^{\underline{n}_z}$ with $\underline{n}_z \in \mathbb{N}^d$. Set $\Delta = \{\underline{n}_z \mid z \in S\} \subseteq \mathbb{N}^d$ and consider the order $\succcurlyeq$ on $\mathbb{N}^d$ from Definition A.9.3. Let Δ' denote the set of minimal elements of Δ under this order.*

(a) *The set $S' = \{z \mid \underline{n}_z \in \Delta'\}$ is an irredundant monomial generating set for J.*
(b) *The set Δ' is finite.*

Proof. Note that the set Δ has minimal elements by the Well-Ordering Principle; see Section A.9.

(a) The minimality of the elements of Δ' implies that for each $\underline{n}_z \in \Delta$, there is an element $\underline{n}_w \in \Delta'$ such that $\underline{n}_z \succcurlyeq \underline{n}_w$, hence, $z \in (w)R$. From this, we conclude that

$$J = (S)R \subseteq (\{w \in S \mid \underline{n}_w \in \Delta'\})R \subseteq (S)R = J$$

so $J = (\{w \in S \mid \underline{n}_w \in \Delta'\})R = (S')R$.

For distinct elements $w,z \in S'$ we have $\underline{n}_w \not\succcurlyeq \underline{n}_z$ since $\underline{n}_w$ and $\underline{n}_z$ are both minimal among the elements of Δ and they are distinct. It follows that $w \notin (z)R$.

Theorem 1.3.6(a) implies that S' contains an irredundant monomial generating sequence $s_1,\ldots,s_n \in S'$ for J. We claim that $S' = \{s_1,\ldots,s_n\}$. (From this, it follows that S' is an irredundant monomial generating set for J.) We know that $\{s_1,\ldots,s_n\} \subseteq S'$, so suppose by way of contradiction that $\{s_1,\ldots,s_n\} \subsetneq S'$, and let $s \in S' \smallsetminus \{s_1,\ldots,s_n\}$. Then $s \in J = (s_1,\ldots,s_n)R$ so $s \in (s_i)R$ for some i by Theorem 1.1.9. The previous paragraph implies that $s = s_i \in \{s_1,\ldots,s_n\}$, a contradiction.

(b) The set Δ' is in bijection with S' which is finite. $\square$

The following result explains how an irredundant monomial generating sequence in two variables fits together with respect to the lexicographical order from Definition A.9.8(a).

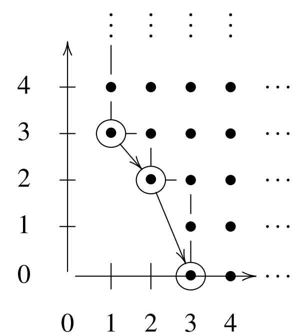
Lemma 1.3.10. *Let J be a non-zero monomial ideal in R and let $f_1,\ldots,f_n \in [\![J]\!]$ be an irredundant monomial generating sequence for J, and assume that $n \geqslant 2$. For $i = 1,\ldots,n$ we write $f_i = X^{a_i}Y^{b_i}$. If $f_i <_{lex} f_j$, then $a_i < a_j$ and $b_i > b_j$.*

Proof. By definition, the inequality $f_i <_{\text{lex}} f_j$ translates to either $(a_i < a_j)$ or $(a_i = a_j$ and $b_i < b_j)$.

Suppose first that $a_i \geqslant a_j$. The previous paragraph shows that this implies that $a_i = a_j$ and $b_i < b_j$. In other words, in the order on $\mathbb{N}^2$ we have $(a_j,b_j) \succcurlyeq (a_i,b_i)$. It follows that f_j is a monomial multiple of f_i, contradicting the irredundancy of the generating sequence.

Suppose next that $b_i \leqslant b_j$. The condition $a_i < a_j$ that we have already established then implies $(a_j, b_j) \succcurlyeq (a_i, b_i)$, and this again contradicts the irredundancy of the generating sequence. $\square$

Graphically, the preceding lemma says that, when the monomials of an irredundant monomial generating sequence for J are arranged in lexicographical order, they move strictly to the southeast.



Exercises

Exercise 1.3.11. Set $R = A[X_1, \ldots, X_d]$. Find irredundant monomial generating sequences for the following monomial ideals and justify your answers.

(a) $I = (X_1^5, X_1X_2, X_1^2X_2^3, X_2^3)R$
(b) $J = (X_1X_2^2X_3^3, X_1X_3, X_2X_4, X_1^3X_2^2X_4X_5)R$
(c) $I + J$, IJ, and J^3

**Exercise 1.3.12.* Set $R = A[X_1, \ldots, X_d]$, and let $I_1, \ldots, I_n, I$ be monomial ideals in R.

(a) Prove that the product $I_1 \cdots I_n$ is a monomial ideal.
(b) Prove that $[\![I_1 \cdots I_n]\!] = \{z_1 \cdots z_n \mid z_1 \in [\![I_1]\!], \ldots, z_n \in [\![I_n]\!]\}$.
(c) Prove that the power I^m is a monomial ideal for each $m \geqslant 0$, and describe $[\![I^m]\!]$. Justify your answer.

(This is used in Section 7.9.)

**Exercise 1.3.13.* Set $R = A[X_1, \ldots, X_d]$, and let $I_1, \ldots, I_n$ be monomial ideals in R.

(a) Prove that the sum $I_1 + \cdots + I_n$ is a monomial ideal.
(b) Prove that $[\![I_1 + \cdots + I_n]\!] = [\![I_1]\!] \cup \cdots \cup [\![I_n]\!]$.

(This exercise is used in the proofs of Proposition 2.3.4 and Theorem 2.5.4, and throughout Section 7.3.)

**Exercise 1.3.14.* Set $R = A[X_1,\dots,X_d]$, and let $I,J,I_1,\dots,I_m$ be monomial ideals in R. Let n be a positive integer.

(a) Prove that $(I+J)^n = \sum_{p=0}^{n} I^p J^{n-p} = \sum_{p+q=n} I^p J^q$.
(b) Prove that $(\sum_{j=1}^{m} I_j)^n = \sum_{p_1+\cdots+p_m=n} I_1^{p_1}\cdots I_m^{p_m}$.

(This exercise is used in Exercise 2.7.6.)

Exercise 1.3.15. This exercise involves the construction $\mathrm{V}(I)$ from Section A.10. Let A be a field, and let I be a monomial ideal of $R = A[X_1,\dots,X_d]$. Prove that there are finitely many monomials $f_1,\dots,f_n \in [\![I]\!]$ such that $\mathrm{V}(I) = \mathrm{V}(f_1,\dots,f_n)$.

Generators of Monomial Ideals in Macaulay2

In this tutorial, we will explore some commands useful for computing generating sets of monomial ideals. First, we define the ring and ideal that we wish to consider.

```
i1 : R=ZZ/101[x,y,z]
o1 = R
o1 : PolynomialRing

i2 : I=ideal(x,y,z,x^2*y,y^2*z)
                          2    2
o2 = ideal (x, y, z, x y, y z)
o2 : Ideal of R
```

Note that this ideal has redundant generators. If we ask Macaulay2 for the number of generators, we get the number that we used to define the ideal.

```
i3 : numgens I
o3 = 5
```

However, one often wishes to work with an irredundant set of generators of I. One can obtain this in a number of ways, and we provide two examples. Firstly, the command `trim` computes an irredundant generating set of the ideal I, and returns this ideal to the user.

```
i4 : trim I
o4 = ideal (z, y, x)
o4 : Ideal of R

i5 : I_*
                    2    2
o5 = {x, y, z, x y, y z}
o5 : List

i6 : (trim I)_*
o6 = {z, y, x}
o6 : List
```

Note that the order of the generators is not preserved when trim is called. We have also illustrated the command `I_*` which returns the generators of the ideal I. This is just shorthand for the following command.

```
i7 : flatten entries gens I
                2    2
o7 = {x, y, z, x y, y z}
o7 : List
```

Therefore, if one wants the minimal number of generators of an ideal, one uses the next command.

```
i8 : numgens trim I
o8 = 3
```

Alternatively, one can obtain a row matrix with an entry for each irredundant generator using the `mingens` command.

```
i9 : mingens I
o9 = | z y x |
             1       3
o9 : Matrix R  <--- R
```

Depending on what one wishes to do with the irredundant generating set, each method has its merits.

If one wishes to obtain the irredundant generators as a list starting from the output of `mingens`, one first calls `entries` on the return value of `mingens`. The command `entries` returns the entries of a matrix as a doubly nested list of the entries of the matrix. If one wishes to remove one level of nesting in a list of lists, one uses the command `flatten`. The following command achieves all of these tasks.

```
i10 : flatten entries mingens I
o10 = {z, y, x}
o10 : List
```

In this tutorial we have used the command `ideal` to generate our monomial ideals. This produces an `Ideal` object generated by monomials, so such an object satisfies our definition of being a monomial ideal. For many purposes, this is good enough for Macaulay2. Sometimes, though, Macaulay2 requires a `MonomialIdeal` object, generated, e.g., with the command `monomialIdeal`. (See Section 3.1 and beyond for examples of such situations.) To keep things relatively simple, we will stick with `ideal` until we need to switch.

```
i11 : exit
```

Exercises

Exercise 1.3.16. Set $R = \mathbb{Z}_{101}[X_1, \ldots, X_5]$, and use Macaulay2 to find irredundant generating sequences for the ideals in Example 1.3.8 and Exercise 1.3.11.

Exercise 1.3.17. Set $R = \mathbb{Z}_{101}[X_1,\ldots,X_5]$ and $I = (X_1^5, X_1X_2, X_1^2X_2^3, X_2^3)R$ and $J = (X_1X_2^2X_3^3, X_1X_3, X_2X_4, X_1^3X_2^2X_4X_5)R$.

(a) Use Macaulay2 to find irredundant generating sequences for $I+J$, IJ and I^3.
(b) Use the command == to verify that $(I+J)^3 = \sum_{p=0}^{3} I^p J^{3-p}$.
(c) Use Macaulay2 to show that $I \cap J$, $(I :_R J)$, $(J :_R I)$, and $\operatorname{rad}(I)$ are monomial ideals and to find irredundant monomial generating sequences for these ideals. See Sections B.3, B.6, and B.7 for the relevant commands.

1.4 Noetherian Rings (optional)

In this section, R is a non-zero commutative ring with identity.

In the previous section, we saw that every monomial ideal in the polynomial ring $A[X_1,\ldots,X_d]$ is finitely generated. This condition can fail for more general ideals. Specifically, there exist rings with ideals that are not finitely generated; see Example 1.4.4(d). This motivates the study of noetherian rings, which are defined after Theorem 1.4.2. We begin with a definition that echoes Theorem 1.3.3(a).

Definition 1.4.1. We say that an ascending chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ of ideals in R *stabilizes* if there is a positive integer n such that $I_n = I_{n+1} = I_{n+2} = \cdots$. The ring R satisfies the *ascending chain condition* (sometimes abbreviated ACC) for ideals if every ascending chain of ideals in R stabilizes.

Compare the next result with Dickson's Lemma 1.3.1 and Theorem 1.3.3.

Theorem 1.4.2 *The following conditions are equivalent:*

(i) *every ideal of R is finitely generated;*
(ii) *every non-empty set of ideals of R contains a maximal element; and*
(iii) *R satisfies the ascending chain condition (ACC) for ideals.*

Proof. (iii) $\Longrightarrow$ (ii): We argue by contradiction. Assume that R satisfies the ascending chain condition for ideals, and suppose that R has a non-empty set Σ of ideals with no maximal element. Let $I_1 \in \Sigma$. Since Σ has no maximal element, I_1 is not maximal, so there is an element $I_2 \in \Sigma$ such that $I_1 \subsetneq I_2$. Similarly, there is an element $I_3 \in \Sigma$ such that $I_2 \subsetneq I_3$. Inductively, this process yields an ascending chain $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ that does not stabilize; this contradicts the assumption that R satisfies the ascending chain condition for ideals.

(ii) $\Longrightarrow$ (i): Assume that every non-empty set of ideals of R contains a maximal element. Let I be an ideal of R, and set

$$\Sigma = \{\text{finitely generated ideals } J \subseteq R \mid J \subseteq I\}.$$

Note that $\Sigma \neq \emptyset$ since $0 = (0_R)R \in \Sigma$. By assumption, the set Σ has a maximal element J. Then $J \subseteq I$ and J is finitely generated.

We claim that $J = I$. By way of contradiction, suppose that $J \subsetneq I$ and let $a \in I \smallsetminus J$. Since J is finitely generated, so is $J + aR$. Furthermore, since $a \in I$, we have $aR \subseteq I$, so the assumption $J \subseteq I$ implies that $J + aR \subseteq I$. Furthermore, we have $a \in aR \subseteq J + aR$, so the condition $a \notin J$ implies that $J + aR$ is an ideal in Σ that properly contains J. This contradicts the maximality of J. Hence, we have $J = I$, so I is finitely generated.

(i) $\Longrightarrow$ (iii): Assume that every ideal of R is finitely generated. Consider an ascending chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ of ideals in R, and set $I = \bigcup_{j=1}^{\infty} I_j$. Fact A.3.3(c) implies that I is an ideal of R. Our assumption implies that $I = (r_1, \ldots, r_k)R$ for some elements $r_i \in R$. For $i = 1, \ldots, k$ we have $r_i \in I = \bigcup_{j=1}^{\infty} I_j$, and so there exists a positive integer n_i such that $r_i \in I_{n_i}$. With $n = \max\{n_1, \ldots, n_k\}$ we have $r_i \in I_{n_i} \subseteq I_n$ and so $I = (r_1, \ldots, r_k)R \subseteq I_n$. Hence, for each $m \geqslant 0$, we have

$$I_{n+m} \subseteq I \subseteq I_n \subseteq I_{n+m}$$

and so $I_{n+m} = I_n$. That is, the chain stabilizes, as desired. $\square$

Definition 1.4.3. The ring R is *noetherian* if it satisfies the equivalent conditions of Theorem 1.4.2.

Example 1.4.4.

(a) The ring $\mathbb{Z}$ is noetherian because every ideal is principal; see Exercise A.3.13.
(b) Every finite ring is noetherian. In particular, the ring $\mathbb{Z}_n$ is noetherian.
(c) Every field k is noetherian since its only ideals are $0 = 0_k k$ and $k = 1_k k$. In particular, the rings $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are noetherian.
(d) Given a non-zero commutative ring A with identity, the polynomial ring $R = A[X_1, X_2, X_3, \ldots]$ in infinitely many variables is not noetherian, since the ideal $(X_1, X_2, X_3, \ldots)R$ is not finitely generated. Also, the chain of ideals $(X_1)R \subsetneq (X_1, X_2)R \subsetneq (X_1, X_2, X_3)R \subsetneq \cdots$ never stabilizes.
(e) The ring $R = \mathrm{C}(\mathbb{R})$ of continuous functions is not noetherian. Indeed, for each $n \in \mathbb{N}$ define

$$I_n = \{f \in R \mid f(k) = 0 \text{ for all } k \in \{n, n+1, n+2, \ldots\}\}.$$

Then the chain of ideals $I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots$ never stabilizes.

Here is the Hilbert Basis Theorem. Compare the statement and proof with those of Dickson's Lemma 1.3.1. In the language of the 1890's, the conclusion of this result says that every ideal of $A[X]$ has a "finite basis", hence the name.

Theorem 1.4.5 (Hilbert Basis Theorem) *Let A be a commutative ring with identity. If A is noetherian, then so is $A[X]$.*

Proof. Let I be an ideal of $A[X]$. We need to show that I is finitely generated. Assume without loss of generality that $I \neq 0$. Let

$$J = \{a \in A \mid a \text{ is the leading coefficient of some } f \in I\} \cup \{0_A\}.$$

The set J is an ideal of A. The fact that A is noetherian implies that there exist $a_1,\dots,a_n \in J$ such that $J = (a_1,\dots,a_n)A$. Assume without loss of generality that each a_i is non-zero. By definition, for each $i = 1,\dots,n$ there is a polynomial $f_i \in I$ such that a_i is the leading coefficient of f_i. By multiplying each polynomial f_i by a power of X, we may assume that the polynomials $f_1,\dots,f_n$ have the same degree; let N denote this common degree.

For $d = 0,1,\dots,N-1$ let

$$J_d = \{a \in A \mid a \text{ is the leading coefficient of some } f \in I \text{ of degree } d\} \cup \{0_A\}.$$

The set J_d is an ideal of A. The fact that A is noetherian implies that there exist $b_{d,1},\dots,b_{d,n_d} \in J_d$ such that $J_d = (b_{d,1},\dots,b_{d,n_d})A$. For each d such that $J_d \neq 0$, we assume without loss of generality that each $b_{d,i}$ is non-zero. By definition, for each d such that $J_d \neq 0$ and for each $i = 1,\dots,n_d$ there is a polynomial $g_{d,i} \in I$ such that $b_{d,i}$ is the leading coefficient of $g_{d,i}$. For each d such that $J_d = 0$, we set $n_d = 1$ and $b_{d,1} = 0 = g_{d,1}$.

Let I' be the ideal of $A[X]$ generated by the polynomials f_i and $g_{d,i}$:

$$I' = \big(\{f_i \mid i = 1,\dots,n\} \cup \{g_{d,i} \mid d = 0,\dots,N-1; i = 1,\dots,n_d\}\big)A[X].$$

By construction I' is a finitely generated ideal of $A[X]$. As $f_i, g_{d,i} \in I$, we have $I' \subseteq I$.

Claim: $I' = I$. (Once the claim is established, we conclude that I is finitely generated, completing the proof.) Suppose by way of contradiction that $I' \subsetneq I$. Let f be a polynomial of minimal degree in the complement $I \smallsetminus I'$. Let e denote the degree of f, and let a be the leading coefficient of f.

Suppose that $e \geqslant N$. Since $f \in I$, we have $a \in J$, and so there exist elements $r_1,\dots,r_n \in A$ such that $a = \sum_{i=1}^n r_i a_i$. For $i = 1,\dots,n$ the polynomial $r_i X^{e-N} f_i$ is an element of I' with degree at most e. Also, the coefficient of X^e in the polynomial $r_i X^{e-N} f_i$ is $r_i a_i$. Since each polynomial $r_i X^{e-N} f_i$ is in I', we have $\sum_{i=1}^n r_i X^{e-N} f_i \in I'$. Exercise A.2.8 implies that $\sum_{i=1}^n r_i X^{e-N} f_i$ has degree e and leading coefficient $\sum_{i=1}^n r_i a_i$. Fact A.3.2 implies that $f - \sum_{i=1}^n r_i X^{e-N} f_i$ is in $I \smallsetminus I'$. Furthermore, the polynomial $f - \sum_{i=1}^n r_i X^{e-N} f_i$ has degree strictly less than e. This contradicts the minimality of e.

Hence, we have $e < N$. It follows that $a \in J_e$ and so there exist elements $s_1,\dots,s_{n_e}$ such that $a = \sum_{i=1}^{n_e} s_i b_{e,i}$. The polynomial $f - \sum_{i=1}^{n_e} s_i g_{e,i}$ is in $I \smallsetminus I'$ and has degree strictly less than e. Again, this contradicts the minimality of e. It follows that $I' = I$, as claimed. $\square$

Corollary 1.4.6 *Let A be a commutative ring with identity. If A is noetherian, then so is $A[X_1,\dots,X_d]$. In particular, if A is a field, then $A[X_1,\dots,X_d]$ is noetherian.*

Proof. Induct on d using Theorem 1.4.5 and Example 1.4.4(c). $\square$

Exercises

Exercise 1.4.7. Let I be an ideal of R, generated by a set $S \subset R$. Prove that if R is noetherian, then I is generated by a finite subset of elements of S.

Exercise 1.4.8. Let I and J be ideals of R.

(a) Prove that for $j = 1, 2, \ldots$ we have $(J :_R I^j) \subseteq (J :_R I^{j+1})$.
(b) Prove that if R is noetherian, then there is an integer $n \geqslant 1$ such that for all $j \geqslant n$ we have $(J :_R I^n) = (J :_R I^j)$. The ideal $(J :_R I^n)$ is the *saturation* of J with respect to I, denoted $(J :_R I^\infty)$. See Section 2.7 for more about this construction.

Exercise 1.4.9. In the proof of the Hilbert Basis Theorem 1.4.5, prove that J and J_d are ideals of $A[X]$.

Exercise 1.4.10. Let A be a commutative ring with identity. Prove that the following conditions are equivalent:

(i) the ring A is noetherian;
(ii) for each integer $d \geqslant 1$ the polynomial ring $A[X_1, \ldots, X_d]$ in d variables is noetherian; and
(iii) there is an integer $d \geqslant 1$ such that the polynomial ring $A[X_1, \ldots, X_d]$ in d variables is noetherian.

Exercise 1.4.11. Set $R = \{a_0 + Xf(X,Y) \in \mathbb{Q}[X,Y] \mid f(X,Y) \in \mathbb{Q}[X,Y]\}$ and $I = \{Xf(X,Y) \in R \mid f(X,Y) \in \mathbb{Q}[X,Y]\}$.

(a) Prove that R is a commutative ring with identity under the usual polynomial addition and multiplication. (That is, prove that R is a subring of $\mathbb{Q}[X,Y]$.)
(b) Prove that I is an ideal of R.
(c) Prove that R is not noetherian by showing that the ideal I is not finitely generated.

Exercise 1.4.12. This exercise involves the construction $\mathrm{V}(I)$ from Exploration Section A.10. Let A be a field, and let I be an ideal of $R = A[X_1, \ldots, X_d]$. Prove that there are finitely many polynomials $f_1, \ldots, f_n \in I$ such that $\mathrm{V}(I) = \mathrm{V}(f_1, \ldots, f_n)$.

1.5 Exploration: Counting Monomials

In this section, A is a non-zero commutative ring with identity.

Here we outline some fundamental combinatorial aspects of monomials in polynomial rings, beginning with the univariate case. The point is to link our algebraic objects (monomials) to combinatorial ones (binomial coefficients, from various counting problems and the Binomial Theorem).

Be sure to justify your answers to these exercises.

Exercise 1.5.1. Set $R = A[X]$.

(a) For each integer $n \geqslant 0$, how many monomials of degree n are in R?
(b) For each integer $n \geqslant 0$, how many monomials of degree at most n are in R?

Next we explore the cases $d = 2,3$. As you work through the next two exercises, be on the lookout for a pattern to help make a good guess about what happens for all $d \geqslant 1$.

Exercise 1.5.2. Set $R = A[X,Y]$.

(a) For each integer $n \geqslant 0$, how many monomials of degree n are in R? (Hint: Write down how many there are for $n = 0,\ldots,3$. Make a conjecture (that is, an educated guess) about the formula for arbitrary n, then prove it.)
(b) Compare the answer from part (a) with the answer to Exercise 1.5.1(b). Explain the similarity.
(c) For each integer $n \geqslant 0$, how many monomials of degree at most n are in R? Interpret your answer as a binomial coefficient.

Exercise 1.5.3. Repeat Exercise 1.5.2 for the ring $R = A[X,Y,Z]$, interpreting each of your answers as a binomial coefficient.

**Exercise 1.5.4.* Given your answers to Exercises 1.5.1–1.5.3, make a conjecture about the number of monomials of degree n in the ring $R = A[X_1,\ldots,X_d]$. Make a conjecture about the number of monomials of degree at most n in R. Prove your conjectures by induction on d. (This exercise is used in Exercise 1.6.3.)

Here is another way to prove one of the formulas from Exercise 1.5.4.

**Exercise 1.5.5.* Set $R = A[X_1,\ldots,X_d]$. Each monomial $X_1^{a_1}\cdots X_d^{a_d} \in R$ of degree k corresponds to a binary number with exactly k ones and $d-1$ zeroes:

$$\underbrace{111\ldots1}_{a_1\text{ ones}}0\underbrace{111\ldots1}_{a_2\text{ ones}}0\cdots0\underbrace{111\ldots1}_{a_d\text{ ones}}.$$

(a) How many digits does each of these binary numbers have?
(b) Each of these binary numbers is determined by the placement of the $d-1$ zeroes. Use this to count the total number of these binary numbers.
(c) Use part (b) to count the number of monomials of degree n in R.
(d) Can you find a similar argument for counting the number of monomials of degree at most n in R?

(This exercise is used in Exercise 1.6.3.)

Counting Monomials in Macaulay2

In this tutorial, we show how to use the `basis` command to find the numbers of monomials of a certain degree, or range of degrees. We first define our polynomial ring, as usual.

```
i1 : R = ZZ/41[x, y, z]
o1 = R
o1 : PolynomialRing
```

To obtain the complete set of monomials of degree 2 in three variables, we use the command `basis` as follows.

```
i2 : basis(2, R) -- Gives degree 2 monomials, in matrix form
o2 = | x2 xy xz y2 yz z2 |
             1       6
o2 : Matrix R  <--- R
```

The reason for this terminology is because the set of monomials of degree n provides a basis of the vector space of all homogeneous polynomials of degree n; see Definition A.2.6.

The `basis` command returns the monomials as a row matrix; to determine the number of monomials returned by basis, we use the combination of commands `numgens` and `source`.

```
i3 : numgens source basis(2,R)
o3 = 6
```

Alternatively, one can use the command `numColumns`.

```
i4 : numColumns basis(2,R)
o4 = 6
```

The following command provides a basis of R in degrees 1–3 inclusive.

```
i5 : basis(1, 3, R)
o5 = | x x2 x3 x2y x2z xy xy2 xyz xz xz2 y y2 y3 y2z yz yz2 z z2 z3 |
             1       19
o5 : Matrix R  <--- R
```

To illustrate the above commands together, we check a basic equality.

```
i6 : numgens source basis(1, R) +
numgens source basis(2, R) +
numgens source basis(3, R) ==
numgens source basis(1, 3, R)
o6 = true
```

Note that in this code we have entered a multi-line command. This is done at the Macaulay2 command prompt using 'shift-return' (marked as 'enter' on some keyboards) at the first three line breaks, then using 'return' at the end of the fourth line. Macaulay2 knows that the the first three lines do not represent our completed command, and it does not execute the command until we use the 'return' key to indicate that our command is complete. This can be useful for lengthy inputs such as large sums, polynomials or function definitions.

One can also pass the optional argument `Variables` to the basis command in order to return monomials that only use the variables provided.

```
i7 : basis(2,R,Variables=>{x,y})
o7 = | x2 xy y2 |
```

```
                1       3
o7 : Matrix R  <--- R

i8 : exit
```

Exercises

Exercise 1.5.6. With $A = \mathbb{Z}_{41}$, use Macaulay2 to test your formulas from the exercises above for a few values of n. Be sure to use the `binomial` command as well as the equality test `==`.

1.6 Exploration: Numbers of Generators

In this section, A is a non-zero commutative ring with identity.

This section is a tour of some of the numerical properties of irredundant generating sequences, especially as they pertain to powers of monomial ideals.

Be sure to justify your answers to the exercises.

Set $R = A[X_1,\ldots,X_d]$. For each monomial ideal $I \subseteq R$, let $\nu_R(J)$ denote the number of elements in an irredundant monomial generating sequence for I.

Example 1.6.1. Set $R = A[X,Y]$. Example 1.3.8 shows that X^3, X^2Y, Y^5 is an irredundant monomial generating sequence for the ideal $I = (X^3, X^2Y, X^2Y^2, Y^5)R$, so we have $\nu_R(I) = 3$.

Exercise 1.6.2. Compute $\nu_R(I)$ for the monomial ideals from Exercise 1.3.11.

The next three examples are pretty special. But we'll see at the end of the section that they can give us information about the general situation.

**Exercise 1.6.3.* Set $R = A[X_1,\ldots,X_d]$ and $\mathfrak{X} = (X_1,\ldots,X_d)R$. Compute $\nu_R(\mathfrak{X}^n)$ for each integer $n \geqslant 0$. (Hint: Exercise 1.5.4 or 1.5.5 may be helpful.) (This is used in Exercise 6.6.1.)

Exercise 1.6.4. Set $R = A[X_1,\ldots,X_d]$ and consider $I = (X_{i_1},\ldots,X_{i_t})R$ where $1 \leqslant t \leqslant d$ and $1 \leqslant i_1 < \cdots < i_t \leqslant d$. Compute $\nu_R(I^n)$ for each integer $n \geqslant 0$.

Exercise 1.6.5. Set $R = A[X_1,\ldots,X_d]$ and $J = (X_{i_1}^{e_1},\ldots,X_{i_t}^{e_t})R$ where $1 \leqslant t \leqslant d$ and $1 \leqslant i_1 < \cdots < i_t \leqslant d$ and $e_1,\ldots,e_t \geqslant 1$. Compute $\nu_R(J^n)$ for each integer $n \geqslant 0$.

See Exercise 1.6.9 below for a discussion of the sharpness of the bound in the next example.

Exercise 1.6.6. Set $R = A[X_1,\ldots,X_d]$. Fix monomials $f_1,\ldots,f_t \in [\![R]\!]$, and consider the ideal $K = (f_1,\ldots,f_t)R$. Compute an upper bound for $\nu_R(K^n)$ in terms of n and t. (Hint: The computation from Exercise 1.6.4 may be helpful.)

Number of Generators in Macaulay2, Exercises

The commands relevant to the following exercises appear in the Macaulay2 tutorials in Sections 1.3 and 1.5.

Exercise 1.6.7. Set $A = \mathbb{Z}_{101}$, and use Macaulay2 to compute $\nu_R(I)$ for the ideals in Example 1.6.1 and Exercise 1.3.11.

Exercise 1.6.8. Set $A = \mathbb{Z}_{101}$, and use Macaulay2 to verify your answer to Exercise 1.6.3 in the following cases:

(a) $d = 2$ and $n = 1,\dots,5$, and
(b) $d = 3$ and $n = 1,\dots,5$.

Exercise 1.6.9. Set $R = \mathbb{Z}_{101}[X_1,\dots,X_d]$ and $K = \mathfrak{X}^2$ where $\mathfrak{X} = (X_1,\dots,X_d)R$. Use Macaulay2 to calculate $\nu_R(I^n)$ in the following cases:

(a) $d = 2$ and $n = 1,\dots,5$, and
(b) $d = 3$ and $n = 1,\dots,5$.

How close are these values to the upper bound you found in Exercise 1.6.6?

Concluding Notes

The problem of finite generation of ideals goes back to the time before the term "ideal" was coined.[1] Paul Gordan [27] dealt with a preliminary version of this, using the constructive techniques of the time. Later, David Hilbert [39] proved what we know now as the Hilbert Basis Theorem 1.4.5, even though the notion of a noetherian ring was not formalized at that point. Note that the use of the term "basis" in this result, in contrast with its modern usage in linear algebra, is synonymous with "generating sequence".

Like the proof we give for this result (and the proof of Dickson's Lemma 1.3.1), Hilbert's proof is non-constructive, unlike Gordan's. This allegedly caused Gordan to proclaim, "Das ist nicht Mathematik. Das ist Theologie." ("This is not mathematics. This is theology.") Regardless of this, Hilbert is one of the most celebrated mathematicians of all time. For instance, his famous list of twenty-three problems had a profound influence on the direction of mathematics research in the 20th century. See Constance Reid's biography [68] for more about Hilbert.

The use of the ascending chain condition was pioneered by Emmy Noether [64], after whom the noetherian rings are named. It is worth noting that Gordan was Noether's dissertation advisor. Noether was a true giant of mathematics who created great works in spite of the difficulties of her time. See the biography by Auguste Dick [15] for more about Noether's life and research.

[1] See the Concluding Notes of Appendix A.

With the two exploration sections from this chapter in mind, it is perhaps worth noting that counting problems in this area yield fascinating connections with other areas of mathematics. For instance, if $f(n)$ is the number of distinct monomial ideals in $A[X,Y]$ generated by monomials of degree at most n, then one can use lattice paths to show that $f(n)$ is the $(n+2)$nd Catalan number C_{n+2}.

Chapter 2
Operations on Monomial Ideals

In this chapter we apply some of the operations of Appendix A to monomial ideals. We have already seen this theme for sums and products in Exercises 1.3.12 and 1.3.13. In Sections 2.1 and 2.5 we show, for instance, that intersections and colons of monomial ideals are monomial ideals. Since we are interested in decomposing monomial ideals into intersections, it is important to know that the set of monomial ideals in a fixed polynomial ring is closed under intersections. We show that generating sequences for intersections of monomial ideals are described using least common multiples (LCMs). This motivates the optional Section 2.2 on unique factorization domains (UFDs), which are rings where least common multiples are guaranteed to exist in general.

On the other hand, the radical of a monomial ideal need not be a monomial ideal. We remedy this by introducing the monomial radical in Section 2.3. The rings where these two constructions agree are the "reduced rings", treated briefly in the exploration Section 2.4. Other constructions we consider are bracket powers, saturations, and generalized bracket powers of monomial ideals, in Section 2.6 and the exploration Sections 2.7, 2.8 and 2.9.

Much of the Macaulay2 groundwork for these operations is laid in Appendix B. In the present chapter, we use these to understand the non-computer results, and to introduce certain Macaulay2 syntaxes in these contexts. In addition, we describe implementations of the monomial-only constructions from this chapter.

2.1 Intersections of Monomial Ideals

In this section, A is a non-zero commutative ring with identity.

Given ideals in a commutative ring R, Fact A.3.3(a) shows that their intersection is also an ideal of R. In other words, the set of ideals of R is closed under intersections. In the next result, we show that the same is true of the set of monomial ideals in a polynomial ring over A. Following this, we show how generators of the ideals being intersected yield generators of the intersection.

Theorem 2.1.1 *Set $R = A[X_1,\ldots,X_d]$. If $I_1,\ldots,I_n$ are monomial ideals of R, then the intersection $I_1 \cap \cdots \cap I_n$ is generated by the set of monomials in $I_1 \cap \cdots \cap I_n$. In particular, the ideal $I_1 \cap \cdots \cap I_n$ is a monomial ideal of R and $[\![I_1 \cap \cdots \cap I_n]\!] = [\![I_1]\!] \cap \cdots \cap [\![I_n]\!]$.*

Proof. Let S denote the set of monomials $\bigcap_{j=1}^n [\![I_j]\!]$ and set $J = (S)R$. By construction J is a monomial ideal such that $J \subseteq \bigcap_{j=1}^n I_j$, since $S \subseteq \bigcap_{j=1}^n I_j$. We claim that $J = \bigcap_{j=1}^n I_j$. To show this, fix an element $f \in \bigcap_{j=1}^n I_j$ and write $f = \sum_{\underline{n}\in\mathbb{N}^d}^{\text{finite}} a_{\underline{n}} \underline{X}^{\underline{n}}$. For $j = 1,\ldots,n$ we have $f \in I_j$. Hence, Lemma 1.1.10 implies that if $a_{\underline{n}} \neq 0$, then $\underline{X}^{\underline{n}} \in [\![I_j]\!]$ for each j, that is, if $a_{\underline{n}} \neq 0$, then $\underline{X}^{\underline{n}} \in \bigcap_{j=1}^n [\![I_j]\!] = S$. Hence, we have $f \in (S)R = J$, as desired.

The previous paragraph shows that $I_1 \cap \cdots \cap I_n$ is a monomial ideal of R and is generated by the monomial set $\bigcap_{j=1}^n [\![I_j]\!]$. We complete the proof with the computation $\left[\!\!\left[\bigcap_{j=1}^n I_j\right]\!\!\right] = \left(\bigcap_{j=1}^n I_j\right) \cap [\![R]\!] = \bigcap_{j=1}^n (I_j \cap [\![R]\!]) = \bigcap_{j=1}^n [\![I_j]\!]$. □

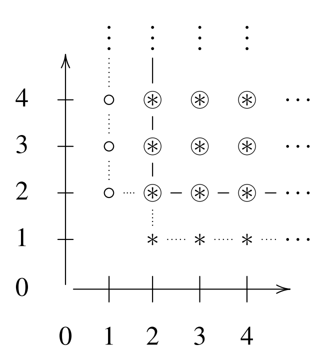
Remark 2.1.2. Set $R = A[X_1,\ldots,X_d]$. Let $I_1,\ldots,I_n$ be monomial ideals of R. The fact that the intersection $I_1 \cap \cdots \cap I_n$ is a monomial ideal such that $[\![I_1 \cap \cdots \cap I_n]\!] = [\![I_1]\!] \cap \cdots \cap [\![I_n]\!]$ implies $\Gamma(I_1 \cap \cdots \cap I_n) = \Gamma(I_1) \cap \cdots \cap \Gamma(I_n)$.

Next we show how to find a monomial generating sequence for an intersection of monomial ideals.

Definition 2.1.3. Set $R = A[X_1,\ldots,X_d]$. Let $f = \underline{X}^{\underline{m}}$ and $g = \underline{X}^{\underline{n}}$ for some $\underline{m},\underline{n} \in \mathbb{N}^d$. For $i = 1,\ldots,d$ set $p_i = \max\{m_i,n_i\}$. Define the *least common multiple* or *LCM* of f and g to be the monomial $\operatorname{lcm}(f,g) = \underline{X}^{\underline{p}}$.

For example, in $R = A[X,Y,Z]$ we compute the least common multiple of $f = XY^4Z^8$ and $g = X^3Z^5$. In the notation of Definition 2.1.3, we have $\underline{m} = (1,4,8)$ and $\underline{n} = (3,0,5)$; thus $\underline{p} = (3,4,8)$ and $\operatorname{lcm}(XY^4Z^8, X^3Z^5) = X^3Y^4Z^8$.

Next, we work in $R = A[X,Y]$ to motivate the connection between intersections of monomial ideals and least common multiples. We compute $(XY^2)R \cap (X^2Y)R$. Because of Theorem 2.1.1 and Remark 2.1.2, we need to compute the intersection $\Gamma((XY^2)R \cap (X^2Y)R) = \Gamma((XY^2)R) \cap \Gamma((X^2Y)R)$.



KEY
$\Gamma((XY^2)R)$: $\circ$
$\Gamma((X^2Y)R)$: $*$
$\Gamma((XY^2)R \cap (X^2Y)R)$: $\circledast$

From this, we see that $(XY^2)R \cap (X^2Y)R = (X^2Y^2)R = (\operatorname{lcm}(XY^2, X^2Y))R$. In words, the intersection of the principal ideals generated by XY^2 and X^2Y is principal and is generated by $\operatorname{lcm}(XY^2, X^2Y)$. The next result shows that this is true for any two principal monomial ideals. The subsequent proposition deals with the non-principal case.

Lemma 2.1.4. *Set $R = A[X_1, \ldots, X_d]$. For monomials $f, g \in [\![R]\!]$, there is an equality $(f)R \cap (g)R = (\operatorname{lcm}(f,g))R$.*

Proof. Exercise. Hint: Apply Lemma 1.1.7 repeatedly. □

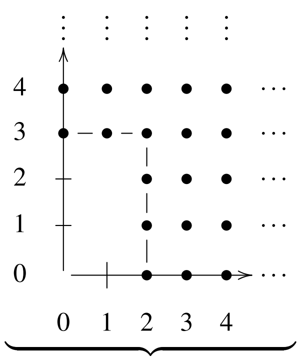
Theorem 2.1.5 *Set $R = A[X_1, \ldots, X_d]$. Suppose I is generated by the set of monomials $\{f_1, \ldots, f_m\}$ and J is generated by the set of monomials $\{g_1, \ldots, g_n\}$. Then $I \cap J$ is generated by the set of monomials $\{\operatorname{lcm}(f_i, g_j) \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$.*

Proof. We begin by setting $K = (\{\operatorname{lcm}(f_i, g_j) \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\})R$. This is a monomial ideal in R since each element $\operatorname{lcm}(f_i, g_j)$ is a monomial in R.

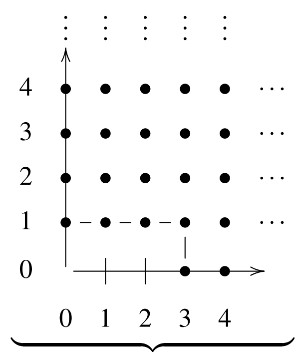
For the containment $I \cap J \subseteq K$, it suffices to show that every monomial $z \in [\![I \cap J]\!]$ is in K. The element z is a monomial in $I = (f_1, \ldots, f_m)R$ so Theorem 1.1.9 implies that $z \in (f_i)R$ for some index i. Similarly, the condition $z \in J = (g_1, \ldots, g_n)R$ implies that $z \in (g_j)R$ for some index j. Hence, Lemma 2.1.4 yields $z \in (f_i)R \cap (g_j)R = (\operatorname{lcm}(f_i, g_j))R \subseteq K$ as desired.

For the containment $I \cap J \supseteq K$, it suffices to show that each monomial generator $\operatorname{lcm}(f_i, g_j)$ of K is in $I \cap J$; see Theorem 2.1.1. Theorem 1.1.9 implies the equality in the next sequence $\operatorname{lcm}(f_i, g_j) \in (\operatorname{lcm}(f_i, g_j))R = (f_i)R \cap (g_j)R \subseteq I \cap J$ while the rest of the steps are standard. This gives the desired conclusion. □

Example 2.1.6. Set $R = A[X, Y]$. We compute a generating sequence for the ideal $I = (X^2, Y^3)R \cap (X^3, Y)R$:

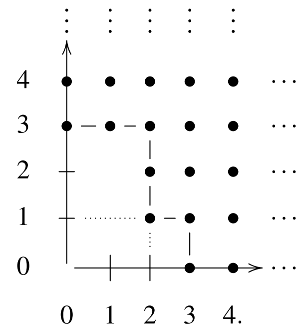


$\Gamma((X^2,Y^3)R)$



$\Gamma((X^3,Y)R.)$

Theorem 2.1.1 implies that $[\![I]\!] = [\![(X^2, Y^3)R]\!] \cap [\![(X^3, Y)R]\!]$, so the graph of I is



Using Proposition 1.3.9 and a visual inspection of the graph, we conclude that an irredundant monomial generating sequence for I is Y^3, X^2Y, X^3.

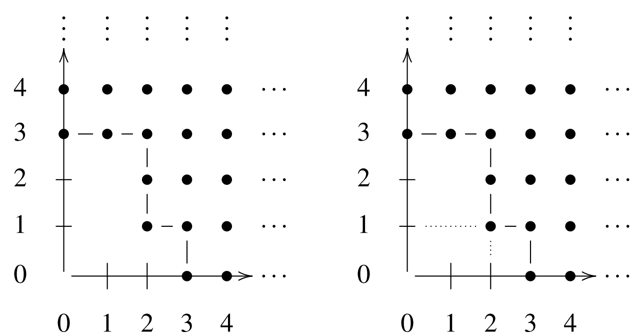
We now use Theorem 2.1.5 and Algorithm 1.3.7 to find an irredundant monomial generating sequence for I. In the notation of Theorem 2.1.5, we have $f_1 = X^2$, $f_2 = Y^3$, $g_1 = X^3$ and $g_2 = Y$. The relevant LCM's are $\operatorname{lcm}(f_1, g_1) = X^3$ and $\operatorname{lcm}(f_2, g_1) = X^3Y^3$ and $\operatorname{lcm}(f_1, g_2) = X^2Y$ and $\operatorname{lcm}(f_2, g_2) = Y^3$. Theorem 2.1.5 implies that the sequence X^3, X^3Y^3, X^2Y, Y^3 generates I.

Now we use Algorithm 1.3.7 to find an irredundant monomial generating sequence for this ideal. The list of z_i's to consider is X^3, X^3Y^3, X^2Y, Y^3.

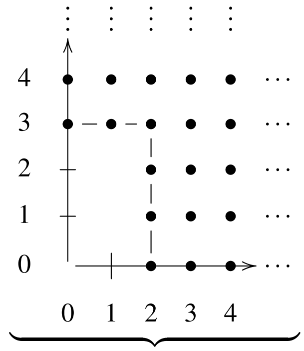
The monomial X^3Y^3 is a multiple of X^3, so we remove X^3Y^3 from the list. The new list of z_i's to consider is X^3, X^2Y, Y^3. No monomial in this list is a multiple of another since the exponent vectors $(3,0)$, $(2,1)$, and $(0,3)$ are incomparable. Hence, the list X^3, X^2Y, Y^3 is an irredundant monomial generating sequence for J.

One goal of this text is the following: given a monomial ideal I, to find simpler monomial ideals $I_1, \ldots, I_n \subseteq R$ such that $I = I_1 \cap \cdots \cap I_n$. A hint as to how this might be done is found in the previous example, as we discuss next.

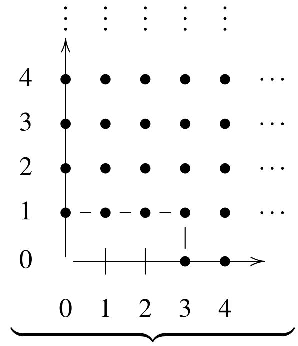
Example 2.1.7. Set $R = A[X,Y]$ and $I = (X^3, X^2Y, Y^3)R$. The graph $\Gamma(I)$ has the following form.



The two corners of the form $\urcorner$ break the "negative space" into two pieces and suggest the decomposition $I = (X^2, Y^3)R \cap (X^3, Y)R$.



$\Gamma((X^2,Y^3)R)$



$\Gamma((X^3,Y)R)$

We conclude this section by noting that the methods from Theorem 2.1.5 and Example 2.1.6 can be extended to intersections of three or more monomial ideals inductively, simply by repeating the process. For instance, to find a monomial generating sequence for $I \cap J \cap K$, first find a monomial generating sequence for $I \cap J$, then find one for $(I \cap J) \cap K$. See also Exercise 2.1.15.

Exercises

Exercise 2.1.8. Set $R = A[X,Y]$. Find irredundant generating sequences for the ideals $I = (X,Y^5)R \cap (X^4,Y)R$ and $J = (X^4,X^3Y^2,Y^3)R \cap (X^3,Y^5)R$.

Exercise 2.1.9. Set $R = A[X_1,\ldots,X_d]$. Prove that if $\{I_\lambda\}_{\lambda\in\Lambda}$ is a (possibly infinite) set of monomial ideals in R, then the intersection $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a monomial ideal with $[\![\bigcap_{\lambda\in\Lambda} I_\lambda]\!] = \bigcap_{\lambda\in\Lambda} [\![I_\lambda]\!]$.

Exercise 2.1.10. Prove Lemma 2.1.4.

Exercise 2.1.11. Set $R = A[X_1,\ldots,X_d]$. Assume that I is generated by the set of monomials S and that J is generated by the set of monomials T. Prove or disprove the following: The intersection ideal $I \cap J$ is generated by the set of monomials $L = \{\mathrm{lcm}(f,g) \mid f \in S \text{ and } g \in T\}$. Justify your answer.

**Exercise 2.1.12.* Set $R = A[X_1,\ldots,X_d]$.

(a) Let $I_1, \ldots, I_k$, and J be monomial ideals in R. Prove that $(I_1 + \cdots + I_k) \cap J = (I_1 \cap J) + \cdots + (I_k \cap J)$.
(b) Give an example (where $d = 2$) to show that this is not true without the assumption that each of the ideals I_1, I_2, and J are monomial ideals; justify your answer.
(c) Prove or disprove the following: Given monomial ideals $I_1, \ldots, I_k, J$ of R, one has $(I_1 \cap \cdots \cap I_k) + J = (I_1 + J) \cap \cdots \cap (I_k + J)$.

(This exercise is used in the proof of Lemma 7.5.1.)

Exercise 2.1.13. Set $R = A[X_1,\ldots,X_d]$. Let $f = \underline{X}^{\underline{m}}$, $g = \underline{X}^{\underline{n}}$, and $w = \underline{X}^{\underline{p}}$ be monomials in R. Prove that the following conditions are equivalent.

(i) $w = \operatorname{lcm}(f,g)$;
(ii) w is a common multiple of f and g, and if $h \in R$ is a common multiple of f and g, then h is a multiple of w (note that we have not assumed that h is a monomial); and
(iii) we have $\underline{m} \leqslant \underline{p}$ and $\underline{n} \leqslant \underline{p}$, and if $\underline{e} \in \mathbb{N}^d$ satisfies the inequalities $\underline{m} \leqslant \underline{e}$ and $\underline{n} \leqslant \underline{e}$, then $\underline{p} \leqslant \underline{e}$.

**Exercise 2.1.14.* Set $R = A[X_1,\ldots,X_d]$. Let $f = \underline{X}^{\underline{m}}$ and $g = \underline{X}^{\underline{n}}$ for some $\underline{m},\underline{n} \in \mathbb{N}^d$. For $i = 1,\ldots,d$ set $q_i = \min\{m_i,n_i\}$. Define the *greatest common divisor* or *GCD* of f and g to be the monomial $\gcd(f,g) = \underline{X}^{\underline{q}}$. Prove that $\operatorname{lcm}(f,g)\gcd(f,g) = fg$. (This exercise is used in Section 5.5.)

**Exercise 2.1.15.* Set $R = A[X_1,\ldots,X_d]$. Let $f = \underline{X}^{\underline{m}}$ and $g = \underline{X}^{\underline{n}}$ and $h = \underline{X}^{\underline{p}}$ for some $\underline{m},\underline{n},\underline{p} \in \mathbb{N}^d$. For $i = 1,\ldots,d$ set $q_i = \max\{m_i,n_i,p_i\}$. Define the *least common multiple* of f, g, and h to be the monomial $\operatorname{lcm}(f,g,h) = \underline{X}^{\underline{q}}$.

(a) Prove that $\operatorname{lcm}(\operatorname{lcm}(f,g),h) = \operatorname{lcm}(f,g,h) = \operatorname{lcm}(f,\operatorname{lcm}(g,h))$.
(b) Prove that $(f)R \cap (g)R \cap (h)R = (\operatorname{lcm}(f,g,h))R$.
(c) Assume that I is generated by the set of monomials $\{f_1,\ldots,f_m\}$ and that J is generated by the set of monomials $\{g_1,\ldots,g_n\}$ and K is generated by the set of monomials $\{h_1,\ldots,h_p\}$. Prove that $I \cap J \cap K$ is generated by the set of monomials $L = \{\operatorname{lcm}(f_i,g_j,h_k) \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n, 1 \leqslant k \leqslant p\}$.
(d) For a sequence $z_1,\ldots,z_a \in [\![R]\!]$, propose a definition of the term "least common multiple of $z_1,\ldots,z_a$. State and prove the versions of (a)–(c) for your definition.

(This exercise is used in Section 5.5 and Exercise 6.6.3.)

Intersections of Monomial Ideals in Macaulay2

In this tutorial we use Macaulay2 to show that the algorithm presented in Theorem 2.1.5 produces a generating set for the intersection of two monomial ideals. To begin, let's create a polynomial ring and some monomial ideals.

```
i1 : R = QQ[x,y]

o1 = R

o1 : PolynomialRing

i2 : I = ideal {x^2,y^3}

                 2   3
o2 = ideal (x , y )

o2 : Ideal of R
```

```
i3 : J = ideal {x^3,x*y^2,y^4}
               3     2   4
o3 = ideal (x , x*y , y )
o3 : Ideal of R
```

Next, we use the commands `apply` and `lcm` to find the least common multiples of the generators of I and J. Note that the `apply` command applies a function to a list; in programming circles, this is sometimes called "mapping a function over a list".

```
i4 : Kgens = apply(I_*, f -> apply(J_*, g -> lcm(f,g)))
         3   2 2   2 4     3 3     3   4
o4 = {{x , x y , x y }, {x y , x*y , y }}
o4 : List
```

To be clear, for every generator f of I (obtained using the command `I_*`), we create a new list containing the least common multiples of f and every generator g of J. To get the ideal generated by all these elements, we first `flatten` this list of lists to a single list, and then take the ideal generated by this set.

```
i5 : K = ideal flatten Kgens
               3   2 2   2 4   3 3     3   4
o5 = ideal (x , x y , x y , x y , x*y , y )
o5 : Ideal of R
```

Notice that this is not a minimal generating set for $I \cap J$ because the generators x^2y^4 and x^3y^3 are redundant.

Next, we verify that we have indeed computed the intersection. Note that even though the ideals have different generating sets, the equality test checks equality of ideals, not the data that defines them[1].

```
i6 : intIJ = intersect(I,J)
               3   4     3   2 2
o6 = ideal (x , y , x*y , x y )
o6 : Ideal of R

i7 : intIJ == K
o7 = true

i8 : exit
```

Exercises

Exercise 2.1.16. Set $R = \mathbb{Z}_{101}[X,Y]$, and use Macaulay2 to show that $(X^2Y^2)R = (XY^2)R \cap (X^2Y)R$ and $(X^3,X^2Y,Y^3)R = (X^2,Y^3)R \cap (X^3,Y)R$.

[1] To test not just equality of ideals but also of the underlying Macaulay2 object, use the *strict* comparison operator `===`.

Exercise 2.1.17. Set $R = \mathbb{Z}_{101}[X,Y]$. Use Macaulay2 to compute irredundant generating sequences for the intersection ideals $I = (X,Y^5)R \cap (X^4,Y)R$ and $J = (X^4,X^3Y^2,Y^3)R \cap (X^3,Y^5)R$. Use Macaulay2 to verify the conclusion of Theorem 2.1.5 for these ideals, as in the tutorial above.

Exercise 2.1.18. Set $R = \mathbb{Z}_{101}[X,Y]$, and consider the ideals $I = (X^2,Y^5)R$, $J = (X^4,Y)R$, and $K = (X^3,XY,Y^5)R$.

(a) Use Macaulay2 to verify that $(I+J)\cap K = (I\cap K)+(J\cap K)$.
(b) Use Macaulay2 to verify your answer to Exercise 2.1.12(b).

Exercise 2.1.19. Set $R = \mathbb{Z}_{101}[X,Y,Z]$. Choose two monomials $f,g \in [\![R]\!]$, and use Macaulay2 to verify the equality $fg = \operatorname{lcm}(f,g)\gcd(f,g)$ from Exercise 2.1.14. The command `gcd` works just like `lcm`.

2.2 Unique Factorization Domains (optional)

In this section, R is an integral domain; see Section 1.2 for background.

In a polynomial ring $A[X_1,\ldots,X_d]$, each monomial $\underline{X}^{\underline{n}}$ has a factorization as a product of powers of the variables X_i. This is similar to the Fundamental Theorem of Arithmetic which states that every positive integer has a unique factorization as a product of powers of prime numbers. This section deals with classes of rings with similar properties. The definition is in 2.2.4; it needs the following prerequisites.

Definition 2.2.1. Two elements $r,s \in R$ are *associates* if $(r)R = (s)R$.

A non-zero non-unit $t \in R$ is *irreducible* if it admits no non-trivial factorization, i.e., if for every $a,b \in R$ such that $t = ab$ either a is a unit or b is a unit.

A non-zero non-unit $p \in R$ is *prime* if for every $a,b \in R$ such that $p\big|ab$ one has either $p\big|a$ or $p\big|b$.

For example, a non-zero positive integer is irreducible in $\mathbb{Z}$ if and only if it is a prime number, i.e., if and only if it is a prime element in the terminology of Definition 2.2.1. Also, two integers m and n are associates in $\mathbb{Z}$ if and only if $m = \pm n$. This corresponds to the fact that for elements $r,s \in R$ the next conditions are equivalent:

(i) r and s are associates;
(ii) $r\big|s$ and $s\big|r$;
(iii) $r \in (s)R$ and $s \in (r)R$; and
(iv) there is a unit $u \in R$ such that $r = us$.

Many other familiar factorization facts from $\mathbb{Z}$ and $A[X]$ hold in arbitrary integral domains. Some of these are described in the following facts, whose proofs are left as exercises.

Fact 2.2.2. Let $r,r',s,u \in R$.

(a) If $u \in R$ is a unit and $r \in R$, then $r \big| u$ if and only if r is a unit.
(b) If r and r' are associates, then $r \big| s$ if and only if $r' \big| s$.

Fact 2.2.3. Let p and q be non-zero non-units in R, and let $a_1, \ldots, a_n \in R$.

(a) If p is prime and $p \big| a_1 \cdots a_n$, then there is an index i such that $p \big| a_i$.
(b) If p is prime, then p is irreducible.
(c) The converse to part (b) can fail in general. See however Lemma 2.2.6.
(d) If p and q are prime and $p \big| q$, then p and q are associates.

Now we are in position to give the main definition of this section, based on the standard factorization properties of $\mathbb{Z}$.

Definition 2.2.4. The integral domain R is a *unique factorization domain* provided that every non-zero non-unit of R can be factored as a product of irreducible elements in an essentially unique way, that is:

(1) for every non-zero non-unit $r \in R$ there are (not necessarily distinct) irreducible elements $s_1, \ldots, s_m \in R$ such that $r = s_1 \cdots s_m$; and
(2) for all irreducible elements $s_1, \ldots, s_m, t_1, \ldots, t_n$ if $s_1 \cdots s_m = t_1 \cdots t_n$, then $m = n$ and there is a permutation σ of the integers $1, \ldots, m$ such that s_i and $t_{\sigma(i)}$ are associates for $i = 1, \ldots, m$.

The term "unique factorization domain" is frequently abbreviated as "UFD".

The Fundamental Theorem of Arithmetic states that the ring $\mathbb{Z}$ is a unique factorization domain. Every field is (vacuously) a unique factorization domain. The ring of Gaussian integers

$$\mathbb{Z}[i] = \{a + bi \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$$

is a unique factorization domain. The ring

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \in \mathbb{R} \mid a, b \in \mathbb{Z}\}$$

is not a unique factorization domain; see [44, Sec. III.3].

Fact 2.2.5. If A is a unique factorization domain, then so is the polynomial ring $A[X_1, \ldots, X_d]$ in d variables for each d. In particular, if k is a field, then $k[X_1, \ldots, X_d]$ is a unique factorization domain. See, e.g., [44, Thm. III.6.12].

In the next result, the unique factorization assumption is essential. For instance, it explains why we do not distinguish between prime and irreducible elements in $\mathbb{Z}$.

Lemma 2.2.6. *Let R be a unique factorization domain. An element $p \in R$ is irreducible if and only if it is prime.*

Proof. One implication is in Fact 2.2.3(b).

For the converse, assume that p is irreducible in R. To show that p is prime, let $a, b \in R$ and assume that $p \big| ab$. We need to prove that $p \big| a$ or $p \big| b$. There is an element $c \in R$ such that $ab = pc$. If $a = 0$, then $p \big| a$ because $a = 0 = 0 \cdot p$, and we are done.

If a is a unit, then $b = a^{-1}pc$, so we have $p\big|b$ and we are done. Similarly, if $b = 0$ or if b is a unit, then we are done. So, we may assume that a and b are non-zero non-units. In particular, the equation $pc = ab$ implies that $c \neq 0$.

Assume that c is not a unit. (The case where c is a unit is handled similarly.) Since R is a unique factorization domain, there are irreducible elements $a_1,\ldots,a_k,b_1,\ldots,b_m,c_1,\ldots c_n \in R$ such that $a = a_1\cdots a_k$ and $b = b_1\cdots b_m$ and $c = c_1\cdots c_n$. The equation $pc = ab$ then reads

$$pc_1\cdots c_n = a_1\cdots a_k b_1\cdots b_m.$$

The uniqueness of factorizations in R implies that p is associate to one of the factors on the right-hand side. (Here is where we use the fact that p is irreducible.) If p and a_i are associates, then $p\big|a_1$, so p divides $a_1\cdots a_k = a$, as desired. Similarly, if p and b_j are associates, then $p\big|b$. Thus p is prime. $\square$

The next two lemmas treat useful bookkeeping notions for factorizations over UFDs that should be familiar over $\mathbb{Z}$.

Lemma 2.2.7. *Let R be a unique factorization domain, and let $r \in R$ be a non-zero non-unit. There exist irreducible elements $p_1,\ldots,p_m \in R$, a unit $u \in R$, and integers $e_1,\ldots,e_m \geqslant 1$ such that*

(1) *$r = up_1^{e_1}\cdots p_m^{e_m}$, and*
(2) *for all $i,j \in \{1,\ldots,m\}$ such that $i \neq j$, the elements p_i and p_j are not associates.*

Proof. By definition there are irreducible elements $s_1,\ldots,s_k \in R$, not necessarily distinct, such that $r = s_1\cdots s_k$. Re-order the s_i so that they are grouped by associates. That is, re-order the s_i to assume that there are integers $i_0 = 0 < 1 = i_1 < i_2 < \cdots < i_m < i_{m+1} = k+1$ such that

(1) for $j = 1,\ldots,m$ and $l = i_j,\ldots,i_{j+1}-1$ the elements s_l, s_{i_j} are associates, and
(2) for $j = 1,\ldots,m$ if $1 \leqslant l < i_j \leqslant h$ the elements s_l, s_h are not associates.

For $j = 1,\ldots,n$ set $p_j = s_{i_j}$ and $e_j = i_j - i_{j-1} \geqslant 1$. For $l = i_j,\ldots,i_{j+1}-1$ fix units $u_l \in R$ such that $s_l = u_l s_{i_j} = u_l p_j$. Note that $u_{i_j} = 1_R$ for $j = 1,\ldots,n$.

For $j = 1,\ldots,n$ set $v_j = u_{i_j}\cdots u_{i_{j+1}-1}$. This yields

$$s_{i_j}s_{i_j+1}\cdots s_{i_{j+1}-1} = p_j(u_{i_j+1}p_j)\cdots(u_{i_{j+1}-1}p_j) = v_j p_j^{e_j}.$$

With $u = \prod_{j=1}^n v_j = \prod_{l=1}^k u_l$, it follows that we have

$$\begin{aligned} r &= s_1\cdots s_k \\ &= [s_{i_1}\cdots s_{i_2-1}][s_{i_2}\cdots s_{i_3-1}]\cdots[s_{i_m}\cdots s_{i_{m+1}-1}] \\ &= [v_1p_1^{e_1}][v_2p_2^{e_2}]\cdots[v_np_n^{e_n}] \\ &= up_1^{e_1}p_2^{e_2}\cdots p_m^{e_m}. \end{aligned}$$

This is the desired factorization. $\square$

Lemma 2.2.8. *Let R be a unique factorization domain, and let $r,s \in R$ be non-zero non-units. There exist irreducible elements $p_1,\ldots,p_n \in R$, units $u,v \in R$, and integers $e_1,\ldots,e_n,f_1,\ldots,f_n \geqslant 0$ such that*

(1) *$r = up_1^{e_1}\cdots p_n^{e_n}$ and $s = vp_1^{f_1}\cdots p_n^{f_n}$, and*
(2) *for all $i,j \in \{1,\ldots,n\}$ such that $i \neq j$, the elements p_i and p_j are not associates.*

Proof. Lemma 2.2.7 yields irreducible elements $p_1,\ldots,p_m,p'_1,\ldots,p'_{m'} \in R$, integers $e_1,\ldots,e_m,e'_1,\ldots,e'_{m'} \geqslant 1$, and units $u,u' \in R$ such that

(1) $r = up_1^{e_1}\cdots p_m^{e_m}$ and $s = u'(p'_1)^{e'_1}\cdots(p'_{m'})^{e'_m}$,
(2) for all distinct $i,j \in \{1,\ldots,m\}$, the elements p_i and p_j are not associates, and
(3) for all distinct $i,j \in \{1,\ldots,m'\}$, the elements p'_i and p'_j are not associates.

Re-order the p_i,p'_i if necessary to assume that there is an integer $\mu \geqslant 1$ such that

(4) for $1 \leqslant i < \mu$ the elements p_i and p'_i are associates, and
(5) for $\mu \leqslant i \leqslant m$ and $\mu \leqslant i' \leqslant m'$ the elements p_i and $p'_{i'}$ are not associates.

Set $n = m + (m' - \mu + 1)$. For $i = m+1,\ldots,n$ set $p_i = p'_{\mu-m-1+i}$ and $e_i = 0$, then

$$r = up_1^{e_1}\cdots p_m^{e_m} = up_1^{e_1}\cdots p_m^{e_m}p_{m+1}^0\cdots p_n^0 = up_1^{e_1}\cdots p_n^{e_n}.$$

For $i = 1,\ldots,\mu-1$ set $f_i = e'_i$ and fix a unit x_i such that $p'_i = x_ip_i$. For $i = \mu,\ldots,m$ set $f_i = 0$. For $i = m+1,\ldots,n$ set $f_i = e'_{\mu-m-1+i}$. Then one has

$$\begin{aligned}
s &= u'(p'_1)^{e'_1}\cdots(p'_{m'})^{e'_m}\\
&= u'(x_1p_1)^{e'_1}\cdots(x_\mu p_\mu)^{e'_\mu}p_{\mu+1}^0\cdots p_m^0(x_{m+1}p_{m+1})^{e'_{\mu+1}}\cdots(x_np_n)^{e'_n}\\
&= vp_1^{f_1}\cdots p_n^{f_n}
\end{aligned}$$

where $v = u'x_1^{e'_1}\cdots x_{m'}^{e'_m}$. $\square$

Again, the next few lemmas should be familiar in the context of the integers. We use them below in our treatment of greatest common divisors and least common multiples in unique factorization domains. The first one characterizes divisibility by a prime element.

Lemma 2.2.9. *Let R be a unique factorization domain, and let $r \in R$ be a non-zero non-unit. Fix irreducible elements $p_1,\ldots,p_n \in R$, integers $e_1,\ldots,e_n \geqslant 0$, and a unit $u \in R$ such that $r = up_1^{e_1}\cdots p_n^{e_n}$, and for all $i,j \in \{1,\ldots,n\}$ such that $i \neq j$, the elements p_i and p_j are not associates. Given a prime element $p \in R$, one has $p \big| r$ if and only if there is an index i such that p and p_i are associates and $e_i \geqslant 1$.*

Proof. First, assume that there is an index i such that $e_i \geqslant 1$ and the elements p and p_i are associates. Then $p \big| p_i$, and hence $p \big| up_1^{e_1}\cdots p_i^{e_i}\cdots p_n^{e_n} = r$.

Conversely, assume that $p \big| r = up_1^{e_1}\cdots p_n^{e_n}$. Fact 2.2.3(a) implies that either $p \big| u$ or $p \big| p_i^{e_i}$ for some index i. Since p is not a unit and u is a unit, Fact 2.2.2(a) implies that $p \nmid u$. Thus, we have $p \big| p_i^{e_i}$ for some index i. If $e_i = 0$ then $p \big| p_i^0 = 1$, which is impossible, again because p is not a unit. It follows that $e_i \geqslant 1$ and $p \big| p_i \cdots p_i$. Another application of Fact 2.2.3(a) shows that $p \big| p_i$. We conclude from Fact 2.2.3(d) that p and p_i are associates. □

The next two results characterize divisibility in a UFD in terms of prime factorizations. Lemma 1.1.7 is a similar result for monomials. This similarity is one of our main motivations for discussing UFD's.

Lemma 2.2.10. *Let R be a unique factorization domain, and let $r, s \in R$ be non-zero non-units. Given irreducible elements $p_1, \ldots, p_n \in R$, units $u, v \in R$, and integers $e_1, \ldots, e_n, f_1, \ldots, f_n \geqslant 0$ as in Lemma 2.2.8, one has $r \big| s$ if and only if $e_i \leqslant f_i$ for $i = 1, \ldots, n$.*

Proof. Assume first that $e_i \leqslant f_i$ for $i = 1, \ldots, n$ and consider the element $x = vu^{-1} p_1^{f_1 - e_1} \cdots p_n^{f_n - e_n}$. It is straightforward to show that $rx = s$, so $r \big| s$.

Assume now that $r \big| s$, and let $t \in R$ such that $s = rt$. Our assumptions imply that

$$v p_1^{f_1} \cdots p_n^{f_n} = u p_1^{e_1} \cdots p_n^{e_n} t. \tag{2.2.10.1}$$

We prove that $f_i \geqslant e_i$ by induction on $e = e_1 + \cdots + e_n$.

Base case: $e = 0$. In this case, each $e_i = 0$, so we have $f_i \geqslant 0 = e_i$ for each i.

Induction step: Assume that $e \geqslant 1$ and that the result holds for elements of the form $r' = u p_1^{e'_1} \cdots p_n^{e'_n}$ where $e'_1 + \cdots + e'_n = e - 1$. Since $e \geqslant 1$, we have $e_i \geqslant 1$ for some index i. It follows that $p_i \big| u p_1^{e_1} \cdots p_n^{e_n} t = v p_1^{f_1} \cdots p_n^{f_n}$. Lemma 2.2.9 implies that there is an index j such that p_i and p_j are associates and $f_j \geqslant 1$. It follows that $i = j$, so we have $f_i \geqslant 1$. Equation (2.2.10.1) now reads as

$$p_i (v p_1^{f_1} \cdots p_i^{f_i - 1} \cdots p_n^{f_n}) = p_i (u p_1^{e_1} \cdots p_i^{e_i - 1} \cdots p_n^{e_n} t).$$

Since R is an integral domain, the cancellation property 1.2.3 implies that

$$v p_1^{f_1} \cdots p_i^{f_i - 1} \cdots p_n^{f_n} = u p_1^{e_1} \cdots p_i^{e_i - 1} \cdots p_n^{e_n} t.$$

The sum of the exponents on the right-hand side of this equation is $e - 1$, so the induction hypothesis implies that $f_j \geqslant e_j$ for each $j \neq i$, and $f_i - 1 \geqslant e_i - 1$ so $f_i \geqslant e_i$, as desired. □

Lemma 2.2.11. *Let R be a unique factorization domain, and let $r \in R$ be a non-zero non-unit. Fix irreducible elements $p_1, \ldots, p_n \in R$, integers $e_1, \ldots, e_n \geqslant 0$, and a unit $u \in R$ such that $r = u p_1^{e_1} \cdots p_n^{e_n}$, and for all $i, j \in \{1, \ldots, n\}$ such that $i \neq j$, the elements p_i and p_j are not associates. Given a non-zero element $t \in R$, one has $t \big| r$ if and only if there exist integers $l_1, \ldots, l_n$, and a unit $w \in R$ such that $t = w p_1^{l_1} \cdots p_n^{l_n}$ and $0 \leqslant l_i \leqslant e_i$ for $i = 1, \ldots, n$.*

Proof. One implication follows from Lemma 2.2.10.

For the converse, assume that $t \mid r$. If t is a unit, then the integers $l_1 = \cdots = l_n = 0$ and the unit $w = t$ satisfy the desired conclusions. Assume that t is not a unit. Since t is also non-zero, it has a prime factor, say p. Since $t \mid r$, we have $p \mid r$, so Lemma 2.2.9 provides an index i such that p and p_i are associates and $e_i \geqslant 1$. Thus, Fact 2.2.2(b) implies that p_i is a prime factor of t. In other words, in a prime factorization of t, the element p can be replaced with p_i. This implies that there exist integers $l_1, \ldots, l_n \geqslant 0$, and a unit $w \in R$ such that $t = wp_1^{l_1} \cdots p_n^{l_n}$. Since $t \mid r$, Lemma 2.2.10 implies that $l_i \leqslant e_i$ for $i = 1, \ldots, n$. □

Other familiar notions from the integers are GCD's and LCM's. Again, these notions extend to general UFD's and compare directly to the notions we have introduced for monomial ideals; see Exercises 2.1.13 and 2.1.14.

Definition 2.2.12. Let $r, s \in R$.

(a) An element $g \in R$ is a *greatest common divisor* for r and s if

  (1) one has $g \mid r$ and $g \mid s$; and
  (2) for all $h \in R$ such that $h \mid r$ and $h \mid s$, one has $h \mid g$.

(b) An element $l \in R$ is a *least common multiple* for r and s if

  (1) one has $r \mid l$ and $s \mid l$; and
  (2) for all $m \in R$ such that $r \mid m$ and $s \mid m$, one has $l \mid m$.

Given elements $r, s \in R$, one can show that GCD's and LCM's are "unique up to associates". That is, if $g \in R$ is a greatest common divisor for r and s, then $g' \in R$ is a greatest common divisor for r and s if and only if g and g' are associates. Also, if $l \in R$ is a least common multiple for r and s, then $l' \in R$ is a least common multiple for r and s if and only if l and l' are associates.

The next result shows how to compute GCD's in terms of prime factorizations, just like in $\mathbb{Z}$. Compare it to the corresponding fact for monomial GCD's in Exercise 2.1.13.

Theorem 2.2.13 *Let R be a unique factorization domain, and let $r, s \in R$ be non-zero non-units. Then R contains a greatest common divisor and a least common multiple for r and s. Specifically, fix irreducible elements $p_1, \ldots, p_n \in R$, integers $e_1, \ldots, e_n, f_1, \ldots, f_n \geqslant 0$, and units $u, v \in R$ as in Lemma 2.2.8. For $i = 1, \ldots, n$ let $m_i = \min\{e_i, f_i\}$ and $M_i = \max\{e_i, f_i\}$. Then $g = p_1^{m_1} \cdots p_n^{m_n}$ is a greatest common divisor for r and s, and $l = p_1^{M_1} \cdots p_n^{M_n}$ is a least common multiple for r and s.*

Proof. We prove that g is a greatest common divisor for r and s. The proof that l is a least common multiple for r and s is left as an exercise.

Lemma 2.2.10 shows that $g \mid r$ and $g \mid s$, since $m_i \leqslant e_i$ and $m_i \leqslant f_i$. Now, assume that $h \in R$ such that $h \mid r$ and $h \mid s$; we need to show that $h \mid g$. Since $h \mid r$, Lemma 2.2.11 provides integers $l_1, \ldots, l_n$, and a unit $w \in R$ such that $h = wp_1^{l_1} \cdots p_n^{l_n}$ and $0 \leqslant l_i \leqslant e_i$ for $i = 1, \ldots, n$. Since $h \mid s$, Lemma 2.2.10 implies that $l_i \leqslant f_i$ for $i = 1, \ldots, n$ so we have $l_i \leqslant \min\{e_i, f_i\} = m_i$ for $i = 1, \ldots, n$. Another application of Lemma 2.2.10 implies that $h \mid g$, as desired. □

Exercises

Exercise 2.2.14. Prove Facts 2.2.2 and 2.2.3.

Exercise 2.2.15. Show by example that the unit u in Lemma 2.2.7 is necessary, and similarly for Lemma 2.2.8. Justify your answers.

Exercise 2.2.16. Let R be a unique factorization domain, and let $r,s \in R$.

(a) Prove that $(r)R \cap (s)R = (\mathrm{lcm}(r,s))R$.
(b) Prove or disprove the following: $(r)R + (s)R = (\gcd(r,s))R$. (Hint: Feel free to use Fact 2.2.5.)
(c) Prove or disprove the following: $(r,s)R \cap (u)R = (\mathrm{lcm}(r,u), \mathrm{lcm}(r,v))R$ for all $u \in R$. (Hint: $(X, X+Y)R = (X,Y)R$.)

Justify your answers.

Exercise 2.2.17. Finish the proof of Theorem 2.2.13 by showing that $l = p_1^{M_1} \cdots p_n^{M_n}$ is a least common multiple for r and s.

Exercise 2.2.18. Let R be a unique factorization domain, and let $r,s \in R$ be non-zero non-units. Let $g \in R$ be a greatest common divisor for r and s, and let $l \in R$ be a least common multiple for r and s. Prove that there is a unit $w \in R$ such that $gl = wrs$. (Compare this with the corresponding result for monomials in Exercise 2.1.14.)

Exercise 2.2.19. Let A be a unique factorization domain, e.g., a field or $\mathbb{Z}$. Use Fact 2.2.5 to prove that $R = A[X_1, \ldots, X_d]$ has infinitely many pair-wise non-associate prime elements. (Hint: Model your proof on Euclid's proof that there are infinitely many prime numbers.)

Exercise 2.2.20. Prove that $t \in R$ is prime if and only if the ideal $(t)R$ is prime.

Exercise 2.2.21. Let A be a non-zero commutative ring with identity. Use Exercise 2.2.19 to prove that $R = A[X_1, \ldots, X_d]$ has infinitely many distinct prime ideals. (Hint: Use the fact that A has a *maximal ideal*: an ideal $\mathfrak{m}$ such that $A/\mathfrak{m}$ is a field.)

Factorization in Macaulay2

In this tutorial we demonstrate Macaulay2's factoring ability. If A is any field or $\mathbb{Z}$, then by Fact 2.2.5, the ring $R = A[X_1, \ldots, X_d]$ is a unique factorization domain. We can ask Macaulay2 to factor any element of such a UFD into a product of irreducibles. Indeed, consider the following.

```
i1 : R = ZZ[a,b]
o1 = R
o1 : PolynomialRing
```

```
i2 : f = 36*a^6+164*a^5*b+252*a^4*b^2+192*a^3*b^3+112*a^2*b^4

           6        5         4 2        3 3        2 4
o2 = 36a  + 164a b + 252a b  + 192a b  + 112a b

o2 : R
```

We can ask for a factorization of this element using the command `factor`.

```
i3 : factor f

        2        2   2             2
o3 = (a) (a + 2b) (9a  + 5a*b + 7b )(4)

o3 : Expression of class Product
```

Note that when Macaulay2 factors a polynomial, if a constant factor is remaining after the irreducible factors have been found, then it is the last factor listed. Also, this factor may or may not be an invertible element of the base ring, depending on whether or not the base ring of the polynomial ring is $\mathbb{Z}$ or a field.

The return value of `factor` is an `Expression` object. This object stores unevaluated expressions for presentation to the user. In this case, the factorization is presented just as in the conclusion of Lemma 2.2.8. One can evaluate an `Expression` using the `value` method. The following example verifies that the `value` of the `Expression` returned from `factor` is equal to its input in our example.

```
i4 : value factor f == f

o4 = true
```

If one wants to use the factors or powers appearing in the factorization in another calculation, one must do a bit more to obtain them; the following command accomplishes this:

```
i5 : facPairs = (toList factor f) / toList

                                   2            2
o5 = {{a, 2}, {a + 2b, 2}, {9a  + 5a*b + 7b , 1}, {4, 1}}

o5 : List

i6 : exit
```

This command converts the return value of `factor` to a `List` object, each of whose entries is a `Power` object. A `Power` object is really just a `List` object[2] whose first entry is the base a in the expression a^b, and whose second entry is the exponent b. Therefore each `Power` object is converted to the underlying `List` object using `toList`, and the `/` function is an application operator like `apply` which we explore further in Section 2.3.

[2] Technically, it is a `BasicList` object.

Exercises

Exercise 2.2.22. Set $R = \mathbb{Z}_{101}[X,Y]$. Use Macaulay2 to factor your two favorite polynomials in R, to compute their `lcm` and `gcd`, and to verify the conclusion of Exercise 2.2.18.

Exercise 2.2.23. Use Macaulay2 to verify your examples for Exercise 2.2.16.

2.3 Monomial Radicals

In this section, A is a non-zero commutative ring with identity.

This section focuses on the following version of the radical for monomial ideals. (See Section A.7 for an introduction to radicals.) To motivate it, note that the radical of a monomial ideal need not be a monomial ideal. Indeed, in the polynomial ring $R = \mathbb{Z}_4[X]$ in one variable, the ideal $J = (X)R$ is a monomial ideal, but the ideal $\operatorname{rad}(J) = (2,X)R$ is not a monomial ideal. See Section 2.4 for more details about this phenomenon.

Definition 2.3.1. Set $R = A[X_1,\ldots,X_d]$. Let J be a monomial ideal in R. The *monomial radical* of J is the monomial ideal $\text{m-rad}(J) = (S)R$ where

$$S = \operatorname{rad}(J) \cap [\![R]\!] = \{z \in [\![R]\!] \mid z^n \in J \text{ for some } n \geqslant 1\}.$$

For instance, in the ring $R = A[X,Y]$, we have $\text{m-rad}\big((X^3,Y^2)R\big) = (X,Y)R$ and $\text{m-rad}\big((X^3Y^2)R\big) = (XY)R$. (This can be verified directly, or using Theorem 2.3.7.) The example preceding Definition 2.3.1 shows that one has $\text{m-rad}(J) \neq \operatorname{rad}(J)$ in general. The next result gives more information about the relation between $\operatorname{rad}(J)$ and $\text{m-rad}(J)$.

Proposition 2.3.2 *Set $R = A[X_1,\ldots,X_d]$. Let J be a monomial ideal in R.*

(a) *One has* $\text{m-rad}(J) \subseteq \operatorname{rad}(J)$.
(b) *One has* $\text{m-rad}(J) = \operatorname{rad}(J)$ *if and only if* $\operatorname{rad}(J)$ *is a monomial ideal.*
(c) *If A is a field, then* $\text{m-rad}(J) = \operatorname{rad}(J)$.

Proof. (a) The ideal $\text{m-rad}(J)$ is generated by the set $S = \operatorname{rad}(J) \cap [\![R]\!] \subseteq \operatorname{rad}(J)$, so we have $\text{m-rad}(J) \subseteq \operatorname{rad}(J)$.

(b) If $\operatorname{rad}(J)$ is a monomial ideal, then

$$\operatorname{rad}(J) = ([\![\operatorname{rad}(J)]\!])R = (S)R = \text{m-rad}(J).$$

Conversely, if $\operatorname{rad}(J) = \text{m-rad}(J)$, then the fact that $\text{m-rad}(J)$ is a monomial ideal implies that $\operatorname{rad}(J)$ is a monomial ideal.

(c) Assume that R is a field. By part (b), it suffices to show that $\operatorname{rad}(J)$ is a monomial ideal. For this, we employ the lexicographical order from Definition A.9.8(a).[3] Let $f \in \operatorname{rad}(J)$. By Lemma 1.1.10 it suffices to show that every monomial occurring in f is in $\operatorname{rad}(J)$. Let $w_1, \ldots, w_n$ be the monomials occurring in f. Then each w_i is a monomial in R, and there are non-zero coefficients $a_1, \ldots, a_n \in A$ such that $f = \sum_{i=1}^n a_i w_i$. Re-order the w_i if necessary to assume that $w_1 <_{\text{lex}} w_2 <_{\text{lex}} \cdots <_{\text{lex}} w_n$. (For this, recall that $<_{\text{lex}}$ is a total order on $[\![R]\!]$ by Exercise A.9.13.) In particular, w_n is the largest monomial occurring in f with respect to the lexicographical ordering.

Claim: for all $t \geqslant 1$, the largest monomial (with respect to $<_{\text{lex}}$) occurring in f^t is w_n^t. If $t = 1$, then this is true by assumption; so assume that $t \geqslant 2$. The power f^t is the sum of all $a_{i_1} \cdots a_{i_t} w_{i_1} \cdots w_{i_t}$ with $1 \leqslant i_1 \leqslant \cdots \leqslant i_t \leqslant n$. To establish the claim, we first show that if $i_1 < n$, then $w_{i_1} \cdots w_{i_t} <_{\text{lex}} w_n^t$. For this, note that $i_j \leqslant n$ for $j = 2, \ldots, t$ so we have $w_{i_j} \leqslant_{\text{lex}} w_n$ by construction. Since $<_{\text{lex}}$ is a monomial order, it respects monomial multiplication. Thus, an induction argument on t implies that we have $w_{i_2} \cdots w_{i_t} \leqslant_{\text{lex}} w_n^{t-1}$. The condition $i_1 < n$ implies that $w_{i_1} <_{\text{lex}} w_n$; with the previous inequality, this shows that

$$w_{i_1} \cdots w_{i_t} \leqslant_{\text{lex}} w_{i_1} w_n^{t-1} <_{\text{lex}} w_n^t$$

so $w_{i_1} \cdots w_{i_t} <_{\text{lex}} w_n^t$. From this, we see that the coefficient of w_n^t in f^t is a_n^t; this coefficient is non-zero since a_n is a non-zero element of the field A. It follows that w_n^t is a monomial occurring in f^t, and the inequality $w_{i_1} \cdots w_{i_t} <_{\text{lex}} w_n^t$ for $i_1 < n$ implies that w_n^t is the largest monomial occurring in f^t, as claimed.[4]

Now, we argue by induction on n to show that $w_1, \ldots, w_n \in \operatorname{rad}(J)$.

For the base case $n = 1$, we have $f = a_1 w_1 = a_1 \underline{X}^{\underline{m}}$ where $w = \underline{X}^{\underline{m}}$ and $0 \neq a_1 \in A$. The assumption $f \in \operatorname{rad}(J)$ implies that there is a positive integer t such that $f^t \in J$, that is, $a_1^t \underline{x}^{t\underline{m}} \in J$. The fact that A is a field with $0 \neq a_1 \in A$ implies that $a_1^t \neq 0$. So a_1^t is a unit in A, thus the condition $a_1^t \underline{x}^{t\underline{m}} \in J$ implies $w_1^t = \underline{x}^{t\underline{m}} \in J$. By definition, it follows that $w_1 \in \operatorname{rad}(J)$, concluding the base case.

For the induction step, assume that $n \geqslant 2$ and that the result holds for polynomials with $n-1$ monomials occurring. Again, by assumption there is a positive integer t such that $f^t \in J$. Since J is a monomial ideal, Lemma 1.1.10 implies that every monomial occurring in f^t is in J. By the claim, the largest monomial occurring in f^t is w_n^t, so in particular we have $w_n^t \in J$. In particular, this says that $w_n \in \operatorname{rad}(J)$. As $\operatorname{rad}(J)$ is an ideal containing f and w_n, we have $\sum_{i=1}^{n-1} a_i w_i = f - a_n w_n \in \operatorname{rad}(J)$. By our induction hypothesis, we conclude that the remaining monomials $w_1, \ldots, w_{n-1}$ are in $\operatorname{rad}(J)$, as desired. $\square$

The next result contains some fundamental properties of the monomial radical. It compares to Proposition A.7.2.

---

[3] In fact, any monomial order will work here.

[4] A similar proof shows that w_1^t is the smallest monomial occurring in f^t. Other than this, though, it is not obvious which of the monomials $w_{i_1} \cdots w_{i_t}$ actually occur in f^t with non-zero coefficient, because of the cancellation that can occur when one collects like terms in the expansion of f^t. This is the main reason for our use of a monomial order here.

Proposition 2.3.3 *Set $R = A[X_1, \dots, X_d]$. Let J be a monomial ideal in R.*

(a) *There is a containment* $J \subseteq \operatorname{m-rad}(J)$.
(b) *One has* $[\![\operatorname{m-rad}(J)]\!] = \operatorname{rad}(J) \cap [\![R]\!]$.
(c) *If $I \subseteq J$, then* $\operatorname{m-rad}(I) \subseteq \operatorname{m-rad}(J)$.
(d) *There is an equality* $\operatorname{m-rad}(J) = \operatorname{m-rad}(\operatorname{m-rad}(J))$.
(e) *One has* $\operatorname{m-rad}(J) = R$ *if and only if* $J = R$.
(f) *One has* $\operatorname{m-rad}(J) = 0$ *if and only if* $J = 0$.
(g) *For each integer $n \geqslant 1$, one has* $\operatorname{m-rad}(J) = \operatorname{m-rad}(J^n)$.

Proof. (a) The set $S = \operatorname{rad}(J) \cap [\![R]\!] \supseteq J \cap [\![R]\!] = [\![J]\!]$ generates $\operatorname{m-rad}(J)$, so we have $\operatorname{m-rad}(J) = (S)R \supseteq ([\![J]\!])R = J$.

(b) Since $S = \operatorname{rad}(J) \cap [\![R]\!]$ is a monomial generating set for $\operatorname{m-rad}(J)$, we have $[\![\operatorname{m-rad}(J)]\!] \supseteq \operatorname{rad}(J) \cap [\![R]\!]$. For the reverse containment, the condition $\operatorname{m-rad}(J) \subseteq \operatorname{rad}(J)$ implies that $[\![\operatorname{m-rad}(J)]\!] = \operatorname{m-rad}(J) \cap [\![R]\!] \subseteq \operatorname{rad}(J) \cap [\![R]\!]$.

(c) Assume that $I \subseteq J$. The containment $\operatorname{rad}(I) \subseteq \operatorname{rad}(J)$ is from Proposition A.7.2(c), so we have $\operatorname{rad}(I) \cap [\![R]\!] \subseteq \operatorname{rad}(J) \cap [\![R]\!]$ and

$$\operatorname{m-rad}(I) = (\operatorname{rad}(I) \cap [\![R]\!])R \subseteq (\operatorname{rad}(J) \cap [\![R]\!])R = \operatorname{m-rad}(J).$$

(d) The containment $\operatorname{m-rad}(J) \subseteq \operatorname{m-rad}(\operatorname{m-rad}(J))$ follows from part (a). For the reverse containment, is suffices to show that $[\![\operatorname{m-rad}(J)]\!] \supseteq [\![\operatorname{m-rad}(\operatorname{m-rad}(J))]\!]$. Let $f \in [\![\operatorname{m-rad}(\operatorname{m-rad}(J))]\!] = \operatorname{rad}(\operatorname{m-rad}(J)) \cap [\![R]\!]$, and fix an exponent $n \geqslant 1$ such that $f^n \in \operatorname{m-rad}(J)$. Then

$$f^n \in \operatorname{m-rad}(J) \cap [\![R]\!] = [\![\operatorname{m-rad}(J)]\!] = \operatorname{rad}(J) \cap [\![R]\!]$$

so there is an exponenet $m \geqslant 1$ such that $f^{mn} = (f^n)^m \in J$. This shows that $f \in \operatorname{rad}(J) \cap [\![R]\!] = [\![\operatorname{m-rad}(J)]\!]$, as desired.

The proofs of parts (e)–(g) are left as exercises. $\square$

The next result describes some of the behavior between monomial radicals and other operations on ideals. It compares to Proposition A.7.3. However, the properties in parts (c) and (d) show that the monomial radical is somewhat better behaved than the regular radical; see Example A.7.4.

Proposition 2.3.4 *Set $R = A[X_1, \dots, X_d]$. Let $I, J, I_1, I_2, \dots, I_n$ be monomial ideals of R with $n \geqslant 1$.*

(a) *There are equalities* $\operatorname{m-rad}(IJ) = \operatorname{m-rad}(I \cap J) = \operatorname{m-rad}(I) \cap \operatorname{m-rad}(J)$.
(b) *There are equalities*

$$\operatorname{m-rad}(I_1 I_2 \cdots I_n) = \operatorname{m-rad}\left(\bigcap_{j=1}^{n} I_j\right) = \bigcap_{j=1}^{n} \operatorname{m-rad}(I_j).$$

(c) *There is an equality* $\operatorname{m-rad}(I + J) = \operatorname{m-rad}(I) + \operatorname{m-rad}(J)$.
(d) *There is an equality* $\operatorname{m-rad}\left(\sum_{j=1}^{n} I_j\right) = \sum_{j=1}^{n} \operatorname{m-rad}(I_j)$.

Proof. (a) As in the proof of Proposition A.7.3(a), we have

$$\text{m-rad}\,(IJ) \subseteq \text{m-rad}\,(I \cap J) \subseteq \text{m-rad}\,(I) \cap \text{m-rad}\,(J)$$

by Proposition 2.3.3(c). For the containment $\text{m-rad}\,(I) \cap \text{m-rad}\,(J) \subseteq \text{m-rad}\,(IJ)$, it suffices to show that $[\![\text{m-rad}\,(IJ)]\!] = [\![\text{m-rad}\,(I) \cap \text{m-rad}\,(J)]\!]$. We compute:

$$\begin{aligned}
[\![\text{m-rad}\,(I) \cap \text{m-rad}\,(J)]\!] &= [\![\text{m-rad}\,(I)]\!] \cap [\![\text{m-rad}\,(J)]\!] \\
&= (\text{rad}\,(I) \cap [\![R]\!]) \cap (\text{rad}\,(J) \cap [\![R]\!]) \\
&= (\text{rad}\,(I) \cap \text{rad}\,(J)) \cap [\![R]\!] \\
&= \text{rad}\,(IJ) \cap [\![R]\!] \\
&= [\![\text{m-rad}\,(IJ)]\!].
\end{aligned}$$

The first step in this sequence is from Theorem 2.1.1. The second and fifth steps are from Proposition 2.3.3(b). The third step is routine, and the fourth step is from Proposition A.7.3(a).

(c) We first show that

$$\text{rad}\,(I+J) \cap [\![R]\!] = (\text{rad}\,(I) \cap [\![R]\!]) \cup (\text{rad}\,(J) \cap [\![R]\!]). \tag{2.3.4.1}$$

For the containment "$\subseteq$", let $f \in \text{rad}\,(I+J) \cap [\![R]\!]$ and fix an integer $n \geqslant 1$ such that $f^n \in I+J$. Since f^n is a monomial, Exercise 1.3.13(b) implies that $f^n \in I \cup J$. If $f^n \in I$, then $f \in \text{rad}\,(I) \cap [\![R]\!]$. If $f^n \in J$, then $f \in \text{rad}\,(J) \cap [\![R]\!]$. So, we conclude that $f \in (\text{rad}\,(I) \cap [\![R]\!]) \cup (\text{rad}\,(J) \cap [\![R]\!])$. For the reverse containment, we compute:

$$\begin{aligned}
(\text{rad}\,(I) \cap [\![R]\!]) \cup (\text{rad}\,(J) \cap [\![R]\!]) &= (\text{rad}\,(I) \cup \text{rad}\,(J)) \cap [\![R]\!] \\
&\subseteq (\text{rad}\,(I) + \text{rad}\,(J)) \cap [\![R]\!] \\
&\subseteq \text{rad}\,(\text{rad}\,(I) + \text{rad}\,(J)) \cap [\![R]\!] \\
&= \text{rad}\,(I+J) \cap [\![R]\!].
\end{aligned}$$

The first step is routine, the second step is from the containment $\text{rad}\,(I) \cup \text{rad}\,(J) \subseteq \text{rad}\,(I) + \text{rad}\,(J)$, and the other steps are from Propositions A.7.2(b) and A.7.3(c).

In the next sequence, the second step is from equation (2.3.4.1):

$$\begin{aligned}
[\![\text{m-rad}\,(I+J)]\!] &= \text{rad}\,(I+J) \cap [\![R]\!] \\
&= (\text{rad}\,(I) \cap [\![R]\!]) \cup (\text{rad}\,(J) \cap [\![R]\!]) \\
&= [\![\text{m-rad}\,(I)]\!] \cup [\![\text{m-rad}\,(J)]\!] \\
&= [\![\text{m-rad}\,(I) + \text{m-rad}\,(J)]\!].
\end{aligned}$$

The first and third steps are from Proposition 2.3.3(b), and the fourth step is from Exercise 1.3.13(b).

The remaining statements follow by induction and are left as exercises. $\square$

Our next goal is to find monomial generating sequences for m-rad(J) in terms of the generators of J. This is accomplished in Theorem 2.3.7, which is based on the following constructions.

Definition 2.3.5. Set $R = A[X_1, \dots, X_d]$. The *support* of a monomial $f = \underline{X}^{\underline{n}} \in [\![R]\!]$ is

$$\mathrm{Supp}(f) = \{i \in \mathbb{N} \mid 1 \leqslant i \leqslant d \text{ and } n_i \neq 0\}.$$

The *reduction* of f is the monomial

$$\mathrm{red}(f) = \prod_{i \in \mathrm{Supp}(f)} X_i.$$

In words, the support of a monomial $f \in [\![R]\!]$ is the set of indices i such that $X_i \big| f$. The reduction of f is the product of the variables dividing f:

$$\mathrm{red}(f) = \prod_{X_i \big| f} X_i.$$

For instance, in the ring $R = A[X_1, X_2, X_3]$, we have $\mathrm{Supp}(X_1^2 X_3^5) = \{1, 3\}$ and $\mathrm{red}(X_1^2 X_3^5) = X_1 X_3$.

In general in $R = A[X_1, \dots, X_d]$, for each integer $n \geqslant 1$ and each monomial $f \in [\![R]\!]$, we have $\mathrm{Supp}(f^n) = \mathrm{Supp}(f) = \mathrm{Supp}(\mathrm{red}(f))$ and $\mathrm{red}(f^n) = \mathrm{red}(f) \big| f$. For monomials $f, g \in [\![R]\!]$, one has $\mathrm{Supp}(f) \subseteq \mathrm{Supp}(g)$ if and only if $\mathrm{red}(f) \big| \mathrm{red}(g)$. Also, if $f \big| g$, then $\mathrm{Supp}(f) \subseteq \mathrm{Supp}(g)$ and $\mathrm{red}(f) \big| \mathrm{red}(g)$. The next lemma contains similar properties that are used in our description of the generators of m-rad(J) in Theorem 2.3.7.

Lemma 2.3.6. *Set $R = A[X_1, \dots, X_d]$. Let J be a monomial ideal in R and $f \in [\![R]\!]$.*

(a) *There is an integer $n \geqslant 1$ such that* $\mathrm{red}(f)^n \in (f)R$.
(b) *If $f \in J$, then* $\mathrm{red}(f) \in$ m-rad(J).

Proof. (a) Write $f = \underline{X}^{\underline{m}}$ and let $n = \max\{m_1, \dots, m_d\}$. Let $\mathrm{Supp}(f) = \{i_1, \dots, i_k\}$ with $1 \leqslant i_1 < \cdots < i_k \leqslant d$. It follows that $\mathrm{red}(f) = X_{i_1} \cdots X_{i_k}$. Then $f = X_{i_1}^{m_{i_1}} \cdots X_{i_k}^{m_{i_k}}$. Since $n \geqslant m_i$ for $i = 1, \dots, d$ we have

$$f = X_{i_1}^{m_{i_1}} \cdots X_{i_k}^{m_{i_k}} \big| X_{i_1}^n \cdots X_{i_k}^n = \mathrm{red}(f)^n$$

so $\mathrm{red}(f)^n \in (f)R$, as desired.

(b) Assume that $f \in J$. Part (a) yields an integer $n \geqslant 1$ such that $\mathrm{red}(f)^n \in (f)R \subseteq J$. Hence $\mathrm{red}(f) \in$ m-rad(J). $\square$

Now we are in a position to describe the generators of m-rad(J) in terms of the reduced versions of the generators of J.

Theorem 2.3.7 *Set $R = A[X_1, \dots, X_d]$. Let $S \subseteq [\![R]\!]$ and set $J = (S)R$. Then one has* m-rad$(J) = (\{\mathrm{red}(f) \mid f \in S\})R$.

Proof. Set $T = \{\mathrm{red}(f) \mid f \in S\}$ and $K = (T)R$.

For each $f \in S \subseteq [\![J]\!]$, we have $\mathrm{red}(f) \in \text{m-rad}\,(J)$ by Lemma 2.3.6(b). This explains the containment $\text{m-rad}\,(J) \supseteq T$, hence $\text{m-rad}\,(J) \supseteq K$.

For the reverse containment, let $g \in [\![\text{m-rad}\,(J)]\!] = \mathrm{rad}\,(J) \cap [\![R]\!]$. Then there is an integer $n \geqslant 1$ such that $g^n \in [\![J]\!]$. Thus, there is a monomial $f \in S$ such that $f \big| g^n$, so $\mathrm{red}(f) \big| \mathrm{red}(g^n) = \mathrm{red}(g) \big| g$. Hence, we have $g \in (\mathrm{red}(f))R \subseteq (T)R = K$. It follows that $\text{m-rad}\,(J) \subseteq K$. $\square$

In the notation of Theorem 2.3.7, if the generating sequence $f_1, \ldots, f_n$ is irredundant, then the generating sequence $\mathrm{red}(f_1), \ldots, \mathrm{red}(f_n)$ of $\text{m-rad}\,(J)$ may be redundant. Indeed, in the ring $R = A[X,Y]$, the sequence X^2Y, XY^2 is an irredundant monomial generating sequence for the ideal $J = (X^2Y, XY^2)R$. However, we have $\mathrm{red}(X^2Y) = XY = \mathrm{red}(XY^2)$, so the generating sequence XY, XY of $\text{m-rad}\,(J)$ from Theorem 2.3.7 is redundant.

We end this section with an important special case of Theorem 2.3.7.

Corollary 2.3.8 *Set $R = A[X_1, \ldots, X_d]$. Let $k, t_1, \ldots, t_k, e_1, \ldots, e_k \geqslant 1$, and set $J = (X_{t_1}^{e_1}, \ldots, X_{t_k}^{e_k})R$. Then* $\text{m-rad}\,(J) = (X_{t_1}, \ldots, X_{t_k})R$ *In particular, if $\mathfrak{X} = (X_1, \ldots, X_d)R$, then* $\text{m-rad}\,(\mathfrak{X}) = \mathfrak{X}$.

Proof. For $i = 1, \ldots, k$ we have $\mathrm{red}(X_{t_i}^{e_i}) = X_{t_i}$, so the desired conclusion follows from Theorem 2.3.7. $\square$

Exercises

Exercise 2.3.9. Prove parts (e)–(g) of Proposition 2.3.3.

Exercise 2.3.10. Prove parts (b) and (d) of Proposition 2.3.4.

Exercise 2.3.11. Set $R = A[X,Y,Z]$, and consider the ideals $I = (X^3, Y^2Z^4)R$ and $J = (X^4Y, Y^3, X^2Y^3Z^2, Z^9)R$. Compute irredundant monomial generating sequences for $\text{m-rad}\,(I)$ and $\text{m-rad}\,(J)$. Justify your answers.

**Exercise 2.3.12.* Set $R = A[X,Y]$. Prove that there are exactly five monomial ideals in R of the form $\text{m-rad}\,(I)$. Specifically, prove the following.

(a) $\text{m-rad}\,(0) = 0$ and $\text{m-rad}\,(R) = R$.
(b) If $I \neq 0, R$ and I has generators of the form X^a and Y^b, then $\text{m-rad}\,(I) = (X,Y)R$.
(c) If $I \neq 0, R$ and I has a generator of the form X^a but no generator of the form Y^b, then $\text{m-rad}\,(I) = (X)R$.
(d) If $I \neq 0, R$ and I has a generator of the form Y^b but no generator of the form X^a, then $\text{m-rad}\,(I) = (Y)R$.
(e) If $I \neq 0, R$ and I has no generator of the form X^a and no generator of the form Y^b, then $\text{m-rad}\,(I) = (XY)R$.

(This exercise is used in Laboratory Exercise 2.7.7.)

Exercise 2.3.13. Set $R = A[X_1,\ldots,X_d]$. Let $f_1,\ldots,f_s,g_1,\ldots,g_t \in [\![R]\!]$, and set $I = (f_1,\ldots,f_s)R$ and $J = (g_1,\ldots,g_t)R$.

(a) Prove that $\text{m-rad}\,(I) \subseteq \text{m-rad}\,(J)$ if and only if for each $i = 1,2,\ldots,s$ there exists a positive integer n_i such that $f_i^{n_i} \in J$.
(b) Prove that $\text{m-rad}\,(I) = \text{m-rad}\,(J)$ if and only if for each $i = 1,2,\ldots,s$ there exists a positive integer n_i such that $f_i^{n_i} \in J$, and for each $j = 1,2,\ldots,t$ there exists a positive integer m_j such that $g_j^{m_j} \in I$.
(c) Assume that $I \subseteq J$. Prove that $\text{m-rad}\,(I) = \text{m-rad}\,(J)$ if and only if for each $j = 1,2,\ldots,t$ there exists an integer m_j such that $g_j^{m_j} \in I$.

Compare these properties with Exercise A.7.11.

Exercise 2.3.14. Set $R = A[X_1,\ldots,X_d]$, and let I and J be monomial ideals in R. Prove that the following conditions are equivalent.

(i) $J \subseteq \text{m-rad}\,(I)$.
(ii) $\text{m-rad}\,(J) \subseteq \text{m-rad}\,(I)$.
(iii) there is an integer $N \geqslant 1$ such that $J^N \subseteq I$.
(iv) there is an integer $N \geqslant 1$ such that $J^n \subseteq I$ for all $n \geqslant N$.
(v) for every integer $j \geqslant 1$, there is an integer $i \geqslant 1$ such that $J^i \subseteq I^j$. (In fancier language, this condition says that the "J-adic topology" on R is finer than the I-adic topology.)

Exercise 2.3.15. Set $R = A[X_1,\ldots,X_d]$, and let I be a monomial ideal in R. Prove that there is an integer $N \geqslant 1$ such that $(\text{m-rad}\,(I))^n \subseteq I$ for all $n \geqslant N$.

**Exercise 2.3.16.* Set $R = A[X_1,\ldots,X_d]$, and $\mathfrak{X} = (X_1,\ldots,X_d)R$. Let I be a monomial ideal such that $I \neq R$. Prove that the following conditions are equivalent:

(i) $\text{m-rad}\,(I) = \mathfrak{X}$;
(ii) an irredundant monomial generating sequence for I contains a power of each variable;
(iii) for each $i = 1,\ldots,d$ there exists an integer $n_i > 0$ such that $X_i^{n_i} \in I$; and
(iv) the set $[\![R]\!] \smallsetminus [\![I]\!]$ is finite.

(This exercise is used in the proofs of Proposition 6.1.3, Corollary 6.1.6, and Propositions 6.2.7 and 6.4.1.)

**Exercise 2.3.17.* Set $R = A[X_1,\ldots,X_d]$ and $\mathfrak{X} = (X_1,\ldots,X_d)R$. Consider monomial ideals $I_1,\ldots,I_n$ of R such that $I_j \neq R$ for $j = 1,\ldots,n$. Prove that the following conditions are equivalent:

(i) $\text{m-rad}\,(I_j) = \mathfrak{X}$ for $j = 1,\ldots,n$;
(ii) $\text{m-rad}\,(I_1 \cdots I_n) = \mathfrak{X}$; and
(iii) $\text{m-rad}\,(I_1 \cap \cdots \cap I_n) = \mathfrak{X}$.

(This exercise is used in the proof of Theorem 6.1.5.)

Exercise 2.3.18. Set $R = A[X_1,\ldots,X_d]$. Let $I_1,\ldots,I_n$ be monomial ideals in R and set $\mathfrak{X} = (X_1,\ldots,X_d)R$.

(a) Prove that if m-rad $(I_j) = \mathfrak{X}$ for $j = 1,\dots,n$, then m-rad $(I_1 + \cdots + I_n) = \mathfrak{X}$.
(b) Prove or give a counterexample for the following: if m-rad $(I_1 + \cdots + I_n) = \mathfrak{X}$, then m-rad $(I_j) = \mathfrak{X}$ for $j = 1,\dots,n$. Justify your answer.

**Exercise 2.3.19.* Let A be a field, and let I be a monomial ideal of $R = A[X_1,\dots,X_d]$. Prove that $\mathrm{V}(\text{m-rad}\,(I)) = \mathrm{V}(I)$ where $\mathrm{V}(I)$ is the construction from Exploration Section A.10. (This is used in Exercises 2.6.18, 2.8.14, and 6.1.14.)

Monomial Radicals in Macaulay2

In this tutorial we illustrate two ways to compute the monomial radical of a monomial ideal. As usual, we set the stage.

```
i1 : R = ZZ/41[x,y,z]
o1 = R
o1 : PolynomialRing

i2 : I = ideal {x^3*y^2,y^3*z}
                3 2   3
o2 = ideal (x y , y z)
o2 : Ideal of R
```

First, we use Proposition 2.3.7 to compute the monomial radical of our ideal I. To do this, for each generator f of I, we must compute the product of the variables in the support of f. Then, return the ideal generated by all these elements.

This seems like a good place to use the `apply` command, but we will use an alternative syntax to map over lists: the `/` operator. For example, if we wish to find the supports of all the generators of I as a list, we can use the `support` command.

```
i3 : I_* / support
o3 = {{x, y}, {y, z}}
o3 : List
```

We can compute the product of the elements in each list by using another `/` with the `product` command:

```
i4 : I_* / support / product
o4 = {x*y, y*z}
o4 : List
```

Finally, we create a function which computes the ideal generated by the output of the above command. Here, we use the postfix operator `//` to avoid having to use parenthesis to surround the output of the previous command.

```
i5 : mRadical = I -> I_* / support / product // ideal
o5 = mRadical
o5 : FunctionClosure

i6 : mRadical I
o6 = ideal (x*y, y*z)
```

```
o6 : Ideal of R
```

On the other hand, since $\mathbb{Z}_{41}$ is a field, we know from Proposition 2.3.2(c) that $\text{m-rad}\,(J) = \text{rad}\,(J)$ for each monomial ideal $J \subseteq R$; see also Exercise 2.4.5 below. Thus, we can use the command `radical` from Section B.7 to compute $\text{m-rad}\,(J)$.

```
i7 : radical I
o7 = monomialIdeal (x*y, y*z)
o7 : MonomialIdeal of R

i8 : mRadical I == radical I
o8 = true

i9 : exit
```

Exercises

Exercise 2.3.20. Set $R = \mathbb{Z}_{101}[X,Y,Z]$ and $I = (X^2Y, YZ, Z^5)R$. Use Macaulay2 to do the following.

(a) Verify that $\text{rad}\,(I)$ is a monomial ideal.
(b) Verify that $\text{m-rad}\,(I) = \text{rad}\,(I)$.
(c) Find an irredundant monomial generating sequence for $\text{m-rad}\,(I)$.
(d) Verify the conclusion of Theorem 2.3.7 for I.

Do the same for the ideals from Exercise 2.3.11 with $A = \mathbb{Z}_{101}$.

Exercise 2.3.21. Set $R = \mathbb{Z}_{101}[X,Y,Z]$, and consider the ideals $I = (X^2Y, YZ, Z^5)R$ and $J = (X^3Y^2, Y^3Z)R$. Work with Macaulay2 to verify the conclusions of parts (a) and (c) of Proposition 2.3.4 for I and J.

Exercise 2.3.22. Set $R = \mathbb{Z}_{101}[X,Y]$. Compute some examples to support Exercise 2.3.12. Compute similar examples in three variables; how many different ideals of the form $\text{m-rad}\,(I)$ can you generate here?

2.4 Exploration: Reduced Rings

In this section, A is a non-zero commutative ring with identity.

In the proof of Proposition 2.3.2(c), we used the fact that, if A is a field and $0 \neq a \in A$, then $a^t \neq 0$. In this section, we briefly explore the general class of rings satisfying this property, the "reduced" rings.

An element $a \in A$ is *nilpotent* if there exists an integer $n \geqslant 0$ such that $a^n = 0_A$. The *nilradical* of A is

$$\operatorname{Nil}(A) = \{\text{nilpotent elements of } A\}.$$

The ring A is *reduced* if $\operatorname{Nil}(A) = 0$.

Exercise 2.4.1.

(a) Prove that $\operatorname{Nil}(A) = \operatorname{rad}(0)$; so $\operatorname{Nil}(A)$ is an ideal of A. See Section A.7.
(b) Prove that if $a \in A$ and $n \geqslant 0$ such that $a^n \in \operatorname{Nil}(A)$, then $a \in \operatorname{Nil}(A)$.
(c) Prove that the quotient ring $A/\operatorname{Nil}(A)$ is reduced. (See Section A.8 for background on quotient rings.)

In light of the previous exercise, it is natural to wonder about the general behavior of reduced rings with respect to taking quotients.

Exercise 2.4.2. Let $I \subsetneq A$ be an ideal.

(a) Prove or disprove the following: if A is reduced, then A/I is reduced.
(b) Prove or disprove the following: if A/I is reduced, then A is reduced.

Justify your answers.

For the next exercise, see Section 1.2 for background on integral domains.

Exercise 2.4.3.

(a) Prove that the following conditions on an integer $n \geqslant 2$ are equivalent:

    (i) $\mathbb{Z}_n$ is a field,
    (ii) $\mathbb{Z}_n$ is an integral domain,
    (iii) $\mathbb{Z}_n$ is reduced, and
    (iv) n is prime.

(b) Prove that if A is an integral domain (e.g., if A is a field), then A is reduced.
(c) Prove or disprove the following: if A is reduced, then A is an integral domain.
(d) Prove or disprove the following: any subring of a reduced ring is reduced.
(e) Prove or disprove the following: any ring with a reduced subring is reduced.

Justify your answers.

Given the behavior of the reduced property with respect to subrings from the previous exercise, the next exercise gives some indication of how special the subring $A \subseteq A[X_1,\ldots,X_d]$ really is.

Exercise 2.4.4. Prove that the following conditions are equivalent:

(i) the ring A is reduced;
(ii) every polynomial ring $A[X_1,\ldots,X_d]$ is reduced; and
(iii) there is an integer $d \geqslant 1$ such that the polynomial ring $A[X_1,\ldots,X_d]$ is reduced.

The next two exercises tie back to our initial motivation for considering reduced rings, from our proof of Proposition 2.3.2 dealing with the difference between the radical and monomial radical of a monomial ideal.

For perspective on the first of these, recall that the ideal $0 \subseteq A[X_1,\ldots,X_d]$ is a monomial ideal with $\text{m-rad}\,(0) = 0$. So the condition of A being reduced is equivalent to the condition $\text{m-rad}\,(0) = \text{rad}\,(0)$ by the previous exercise. The next exercise complements this by saying that the equality $\text{m-rad}\,(I) = \text{rad}\,(I)$ for any single proper monomial ideal I implies the same condition for every monomial ideal, in particular, for the ideal 0.

The subsequent exercise takes this a step further by explicitly writing $\text{rad}\,(I)$ in terms of $\text{rad}\,(0)$ and $\text{m-rad}\,(I)$. In particular, one can obtain much of 2.4.5 as a corollary of 2.4.6.

Exercise 2.4.5. Set $R = A[X_1,\ldots,X_d]$. Prove that the next conditions are equivalent:

(i) the ring A is reduced;
(ii) for every monomial ideal $I \subseteq R$, the ideal $\text{rad}\,(I)$ is a monomial ideal;
(iii) for every monomial ideal $I \subseteq R$, one has $\text{m-rad}\,(I) = \text{rad}\,(I)$;
(iv) there exists a monomial ideal $I \subsetneq R$ such that $\text{m-rad}\,(I) = \text{rad}\,(I)$; and
(v) there exists a monomial ideal $I \subsetneq R$ such that $\text{rad}\,(I)$ is a monomial ideal.

Exercise 2.4.6. Let $R = A[X_1,\ldots,X_d]$ and let I be a monomial ideal of R. Prove that $\text{rad}\,(I) = \text{m-rad}\,(I) + \text{Nil}(A)R$, where $\text{Nil}(A)R$ is the ideal of R generated by the elements in the nilradical of A.

Reduced Rings in Macaulay2

In this tutorial, we provide another example showing that $\text{m-rad}\,(I)$ can be properly contained in $\text{rad}\,(I)$ if the base ring A is not reduced. Consider the following.

```
i1 : A = QQ[a]/ideal{a^5}
o1 = A
o1 : QuotientRing

i2 : S = A[b,c,d]
o2 = S
o2 : PolynomialRing
```

Note that A is indeed not reduced since $a \neq 0$ and $a^5 = 0$.

```
i3 : a == 0
o3 = false

i4 : a^5 == 0
o4 = true
```

Thus, we have $\text{Nil}(A) \neq 0$. Let's also reload `mRadical` from Section 2.3.

```
i5 : mRadical = I -> I_* / support / product // ideal
o5 = mRadical
o5 : FunctionClosure
```

Now consider the following ideal.

```
i6 : I = ideal{b^2*c,c^2*d^3}
               2     2 3
o6 = ideal (b c, c d )
o6 : Ideal of S
```

By Proposition 2.3.7, we have that bc and cd are generators of m-rad(I), as we see from the following command.

```
i7 : mRadI = mRadical I
o7 = ideal (b*c, c*d)
o7 : Ideal of S
```

However, if we ask Macaulay2 for the radical of I using the command `radical`, we obtain the following.

```
i8 : radI = radical I
o8 = monomialIdeal (b*c, c*d, a)
o8 : MonomialIdeal of S

i9 : exit
```

As you can see, we have $a \in \text{rad}(I)$ while $a \notin \text{m-rad}(I)$.

Exercises

Exercise 2.4.7. Set $R = \mathbb{Z}_{101}[X,Y]$ and $I = (XY)R$ and $J = (X^2,Y^3)R$.

(a) Use Macaulay2 to compute $\text{rad}(I)$ and $\text{Nil}(R/I)$.
(b) Use Macaulay2 to compute $\text{rad}(J)$ and $\text{Nil}(R/J)$.
(c) [Laboratory/Challenge] Based on parts (a) and (b), make a conjecture about the relation between $\text{rad}(K)$ and $\text{Nil}(R/K)$. Test your conjecture with some more examples. Can you prove your conjecture?

Coding Exercise 2.4.8. Using Exercise 2.4.6, combine the Macaulay2 command `mRadical` defined above and the commands `substitute` and `radical`, write a function called `newRadical` that takes a monomial ideal of a polynomial ring $A[X_, \ldots, X_d]$ and returns its radical. Check that your function works properly on the example in the above tutorial and on an example where the base ring is a field.

2.5 Colons of Monomial Ideals

In this section, A is a non-zero commutative ring with identity.

This section focuses on the colon ideal of two monomial ideals. (See Section A.6 for an introduction to colons.) Similarly to Section 2.1, we begin by showing that the set of monomial ideals is closed under taking colons.

Theorem 2.5.1 *Set $R = A[X_1,\ldots,X_d]$. If I and J are monomial ideals of R, then the colon ideal $(J :_R I)$ is a monomial ideal of R.*

Proof. Case 1: $I = zR$ for some monomial $z = \underline{X}^{\underline{m}} \in R$. To show that the ideal $(J :_R I) = (J :_R z)$ is a monomial ideal, we utilize Lemma 1.1.10. For this let $f = \sum_{\underline{n}\in\mathbb{N}^d}^{\text{finite}} a_{\underline{n}}\underline{X}^{\underline{n}} \in (J :_R z)$; we need to show that every monomial occurring in f is in $(J :_R z)$. By assumption, we have $fz = \sum_{\underline{n}\in\mathbb{N}^d}^{\text{finite}} a_{\underline{n}}\underline{X}^{\underline{n}+\underline{m}} \in J$. Lemma 1.1.10 says that if $a_{\underline{n}} \neq 0$, then $\underline{X}^{\underline{n}+\underline{m}} \in J$, since J is a monomial ideal. So, if $a_{\underline{n}} \neq 0$, then $z\underline{X}^{\underline{n}} = \underline{X}^{\underline{n}+\underline{m}} \in J$. In other words, if $a_{\underline{n}} \neq 0$, then $\underline{X}^{\underline{n}} \in (J :_R zR)$, as desired.

Case 2: The general case. The ideal I is generated by a finite list of monomials $z_1,\ldots,z_n$, by Dickson's Lemma 1.3.1. It follows from Proposition A.6.3(d) that

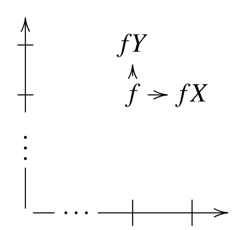
$$(J :_R I) = (J :_R (z_1,\ldots,z_n)R) = \bigcap_{i=1}^{n}(J :_R z_iR).$$

Thus, the ideal $(J :_R I)$ is a finite intersection of monomial ideals. By Theorem 2.1.1, it follows that $(J :_R I)$ is a monomial ideal. $\square$

Given monomial ideals I, J of the polynomial ring $R = A[X_1,\ldots,X_d]$, we are interested in describing the monomial set $[\![(J :_R I)]\!]$ in terms of $[\![I]\!]$ and $[\![J]\!]$. Of course, we have $J \subseteq (J :_R I)$, so $[\![J]\!] \subseteq [\![(J :_R I)]\!]$. Sometimes these containments are proper, and sometimes not. For one solution to this problem, see Exercise 2.5.7.

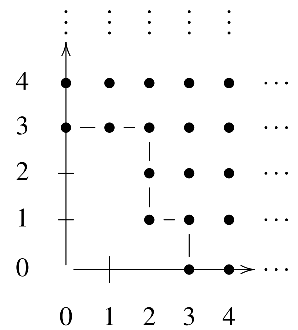
For Chapter 6, we are interested in the special case of this problem where $I = \mathfrak{X} = (X_1,\ldots,X_d)R$. A hint as to how we might find monomials in $(J :_R \mathfrak{X}) \smallsetminus J$ is found in the graph $\Gamma(I)$, as we see in the next examples in $R = A[X,Y]$.

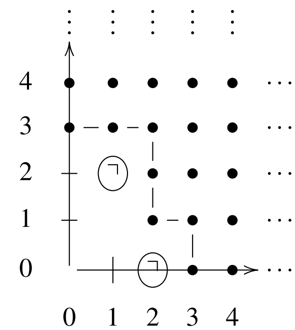
Let I be a monomial ideal in R and set $\mathfrak{X} = (X,Y)R$. A monomial $f \in R$ is in $(I :_R \mathfrak{X})$ if and only if $fX, fY \in I$; see Proposition A.6.2(b). The elements f, fX and fY relate to each graphically as follows.



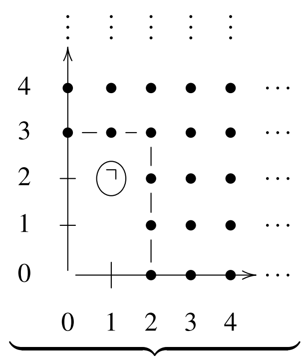
Thus, the point $(a,b) \in \mathbb{N}^2$ represents a point in $(I :_R \mathfrak{X})$ if and only if the ordered pairs $(a+1,b)$ and $(a,b+1)$ are in the graph $\Gamma(I)$.

Example 2.5.2. Consider the ideal $I = (X^3, X^2Y, Y^3)R$ with $R = A[X,Y]$. The graph $\Gamma(I)$ has the following form.

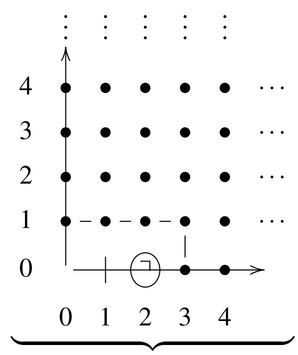




The two corners of the form $\urcorner$ show us where to find elements of $(I :_R \mathfrak{X})$ no in I. It is not difficult to show that the monomials X^2 and XY^2 are precisely the monomials in $(I :_R \mathfrak{X}) \smallsetminus I$; see Exercise 2.5.6. Note that these "corners" correspond to the "corners" in the ideals $(X^2,Y^3)R$ and $(X^3,Y)R$ in the decomposition $I = (X^2,Y^3)R \cap (X^3,Y)R$; see Examples 2.1.6 and 2.1.7.



$\Gamma((X^2,Y^3)R)$



$\Gamma((X^3,Y)R)$

One point of Chapter 6 is that the "corner elements" of certain monomial ideals $J \subseteq R$ give rise to the decomposition of J as an intersection of monomial ideals of the form $(X^a, Y^b)R$.

We end this section with a discussion of how to compute a monomial generating sequence for $(J :_R I)$ in terms of monomial generators of J and I. For this, we use the following non-standard notation. Given sequences $\underline{p}, \underline{q} \in \mathbb{N}^d$ we set $\left(\underline{p} - \underline{q}\right)_i^+ = \max\{p_i - q_i, 0\}$, in other words

$$\left(\underline{p} - \underline{q}\right)_i^+ = \begin{cases} p_i - q_i & \text{if } p_i - q_i \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

By definition, this makes $\left(\underline{p}-\underline{q}\right)^+ \in \mathbb{N}^d$.

Example 2.5.3. If $d=2$, then we have

$$((1,3)-(2,1))^+ = (0,2).$$

Exercise 2.5.8 then says that in $R = A[X,Y]$ we have

$$(XY^3R :_R X^2Y) = Y^2R.$$

The next result shows how to use this to compute monomial generating sequences for colon ideals in general.

Theorem 2.5.4 *Set $R = A[X_1,\ldots,X_d]$, and let I and J be monomial ideals. Let $w_1,\ldots,w_m$ be a monomial generating sequence for J, and let $z_1,\ldots,z_n$ be a monomial generating sequence for I. Then one has*

$$(J :_R I) = \bigcap_{i=1}^{n}\left(\sum_{j=1}^{m}(w_jR :_R z_i)\right).$$

Proof. We first treat the case $n=1$, where we need to prove that

$$(J :_R z_1) = \sum_{j=1}^{m}(w_jR :_R z_1). \tag{2.5.4.1}$$

The containment $\supseteq$ here does not use any properties of monomials: indeed, we have $w_jR \subseteq J$ since w_j is part of a generating sequence for J; so Proposition A.6.2(e) implies that $(w_jR :_R z_1) \subseteq (J :_R z_1)$, thus $\sum_{j=1}^{m}(w_jR :_R z_1) \subseteq (J :_R z_1)$ by Theorem A.4.2(c). For the reverse containment, we use the fact that the ideals $(J :_R z_1)$ and $\sum_{j=1}^{m}(w_jR :_R z_1)$ are monomial ideals; see Exercise 1.3.13(a) and Theorem 2.5.1. Thus, it suffices by Theorem 1.1.4(a) to show that each monomial $f \in (J :_R I)$ is in $\sum_{j=1}^{m}(w_jR :_R z_1)$. Given such a monomial, we have $fz_1 \in J = (w_1,\ldots,w_m)R$, so Theorem 1.1.9 implies that there is an index j such that $fz_1 \in w_jR$; that is, we have $f \in (w_jR :_R z_1) \subseteq \sum_{j=1}^{m}(w_jR :_R z_1)$. Thus, equation (2.5.4.1) is established.

Now, we deal with the general case. By assumption and Exercise A.4.6(a), we have $I = (z_1,\ldots,z_n)R = \sum_{i=1}^{n} z_iR$. This explains the first equality in the next display, and the second equality is from Proposition A.6.3(d):

$$(J :_R I) = \left(J :_R \sum_{i=1}^{n} z_iR\right) = \bigcap_{i=1}^{n}(J :_R z_i) = \bigcap_{i=1}^{n}\left(\sum_{j=1}^{m}(w_jR :_R z_i)\right).$$

The third equality is from equation (2.5.4.1). This completes the proof. $\square$

Example 2.5.5. In $R = A[X,Y]$, we compute $(J :_R I)$ where $J = (X^3, XY^2)R$ and $I = (X^2Y, Y^4)R$. Theorem 2.5.4 explains the first equality in the next display

$$\begin{aligned}(J :_R I) &= [(X^3R :_R X^2Y) + (XY^2R :_R X^2Y)] \cap [(X^3R :_R Y^4) + (XY^2R :_R Y^4)]\\ &= (XR + YR) \cap (X^3R + XR)\\ &= (X,Y)R \cap XR\\ &= XR\end{aligned}$$

and the second equality is from Exercise 2.5.8.

Exercises

Exercise 2.5.6. Set $R = A[X,Y]$. Set $J = (X^3, X^2Y, Y^3)R$ and $\mathfrak{X} = (X,Y)R$. Verify that the monomials in $(J :_R \mathfrak{X}) \smallsetminus J$ are XY^2 and X^2.

Exercise 2.5.7. Let I, J be monomial idelas of $R = A[X_1, \ldots, X_d]$. Prove that one has $[\![(J :_R I)]\!] = \{f \in [\![R]\!] \mid gf \in [\![J]\!] \text{ for all } g \in [\![I]\!]\}$.

Exercise 2.5.8. Set $R = A[X_1, \ldots, X_d]$, and consider vectors $\underline{p}, \underline{q} \in \mathbb{N}^d$. Prove that one has $(\underline{X}^{\underline{p}}R :_R \underline{X}^{\underline{q}}) = \underline{X}^{(\underline{p}-\underline{q})^+}R$.

Exercise 2.5.9. Set $R = A[X_1, \ldots, X_d]$, and let $\{J_\lambda\}_{\lambda \in \Lambda}$ be a set of monomial ideals of R. Prove or disprove the following.

(a) If $z \in [\![R]\!]$, then $(\sum_\lambda J_\lambda :_R z) = \sum_\lambda (J_\lambda :_R z)$.
(b) If I is a monomial ideal of R, then $(\sum_\lambda J_\lambda :_R I) = \sum_\lambda (J_\lambda :_R I)$.

Justify your answers.

Exercise 2.5.10. Set $R = A[X_1, \ldots, X_d]$, and let $S, T \subseteq [\![R]\!]$. Prove or disprove the following: $((T)R :_R (S)R) = \bigcap_{s \in S} (\sum_{t \in T} (tR :_R s))$. Justify your answer.

Colons of Monomial Ideals in Macaulay2

Using the Macaulay2 command from Section B.6, we can compute colons of monomial ideals quickly.

```
i1 : R = QQ[x,y]

o1 = R

o1 : PolynomialRing

i2 : I = ideal {x^2,y^3}

             2   3
o2 = ideal (x , y )

o2 : Ideal of R
```

```
i3 : J = ideal {x^3,x*y^2,y^4}
                3     2   4
o3 = ideal (x , x*y , y )
o3 : Ideal of R

i4 : I : J
o4 = ideal (y, x)
o4 : Ideal of R

i5 : J : I
                   2
o5 = ideal (x, y )
o5 : Ideal of R

i6 : exit
```

Note that each colon ideal here is a monomial ideal, as promised by Theorem 2.5.1.

Exercises

Exercise 2.5.11. Set $R = \mathbb{Z}_{101}[X,Y]$. Use Macaulay2 to verify the computations in Examples 2.5.3 and 2.5.5 and Exercise 2.5.6.

Exercise 2.5.12. Use Macaulay2 to verify your examples for Exercise 2.5.9–2.5.10.

2.6 Bracket Powers of Monomial Ideals

In this section, A is a non-zero commutative ring with identity.

As we mention in Section A.5, the power I^n of an ideal I is not generated by the nth powers of the generators of I. This section investigates the ideal that *is* generated by powers of the generators of I.

Definition 2.6.1. Set $R = A[X_1,\ldots,X_d]$. Let J be a monomial ideal of R. For $k = 1,2,\ldots$ the *kth bracket power* of J is the ideal $J^{[k]} = (T_k)R$ where $T_k = \{f^k \mid f \in [\![J]\!]\}$.

By definition, the ideal $J^{[k]}$ is a monomial ideal for each $k = 1,2,\ldots$. The next lemma helps us find generating sets for $J^{[k]}$; this is made explicit in Propositions 2.6.3 and 2.6.5.

Lemma 2.6.2. *Set $R = A[X_1,\ldots,X_d]$. Consider a set of monomials $S \subseteq [\![R]\!]$ and an integer $k \geqslant 1$. Set $J = (S)R$ and $I = (\{f^k \mid f \in S\})R$. For each monomial $g \in [\![R]\!]$ we have $g \in J$ if and only if $g^k \in I$.*

Proof. For the forward implication, assume that $g \in J$. Since J is a monomial ideal, Dickson's Lemma 1.3.1 provides a finite subset $S' \subseteq S$ such that $J = (S')R$. Theorem 1.1.9 implies that $g \in (f)R$ for some $f \in S'$, and it follows that $g^k \in (f^k)R \subseteq I$.

For the converse, assume that $g^k \in I$. The set $S_k = \{f^k \mid f \in S\}$ is a monomial generating set for I. Hence, there is a finite subset $S'_k \subseteq S_k$ such that $I = (S'_k)R$. Theorem 1.1.9 implies that $g^k \in (f^k)R$ for some $f^k \in S'_k$. Note that $f \in S$ by definition. Write $f = \underline{X}^{\underline{m}}$ and $g = \underline{X}^{\underline{n}}$ with $\underline{m}, \underline{n} \in \mathbb{N}^d$. Then $f^k = \underline{X}^{k\underline{m}}$ and $g^k = \underline{X}^{k\underline{n}}$, so Lemma 1.1.7 implies that $k\underline{m} \succcurlyeq k\underline{n}$. It follows readily that $\underline{m} \succcurlyeq \underline{n}$, so $g = \underline{X}^{\underline{n}} \in (\underline{X}^{\underline{m}})R = (f)R \subseteq J$. $\square$

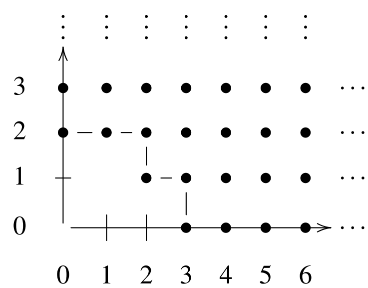
Proposition 2.6.3 *Set $R = A[X_1, \ldots, X_d]$. Let J be a monomial ideal in R.*

(a) *If S is a monomial generating set for J, then the set $S_k = \{f^k \mid f \in S\}$ is a monomial generating set for $J^{[k]}$.*
(b) *If $f_1, \ldots, f_n \in [\![J]\!]$ is a generating sequence for J, then $J^{[k]} = (f_1^k, \ldots, f_n^k)R$.*

Proof. (a) Let $T_k = \{f^k \mid f \in [\![J]\!]\}$. By definition, we have $J^{[k]} = (T_k)R$, so we need to show that $(S_k)R = (T_k)R$. Also, we have $S_k \subseteq T_k$ by construction, so $(S_k)R \subseteq (T_k)R$. To verify the reverse containment, we need to show that $T_k \subseteq (S_k)R$. An arbitrary element of T_k has the form f^k for some $f \in [\![J]\!]$. Lemma 2.6.2 implies that $f^k \in (S_k)R$, so $T_k \subseteq (S_k)R$, as desired.

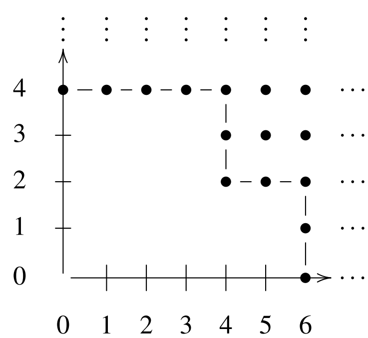
(b) This is the special case of part (a) with $S = \{f_1, \ldots, f_n\}$. $\square$

For example, in $R = A[X,Y]$, the ideal $J = (X^3, X^2Y, Y^2)R$ has the next graph.



(2.6.3.1)

We have $J^{[2]} = (X^6, X^4Y^2, Y^4)R$ which has the following graph.



(2.6.3.2)

Notice that the graph of $J^{[2]}$ is essentially a scale model of the graph of J.

The next result is a useful formulation of Lemma 2.6.2, using $S = [\![J]\!]$. Note that it does *not* imply that $[\![J^{[k]}]\!]$ equals $\{h^k \mid h \in [\![J]\!]\}$; see Exercise 2.6.8.

Lemma 2.6.4. *Set $R = A[X_1, \ldots, X_d]$. Let J be a monomial ideal of R, and let $g \in [\![R]\!]$ be a monomial in R. For $k = 1, 2, \ldots$ we have $g \in J$ if and only if $g^k \in J^{[k]}$.* □

The next result augments Proposition 2.6.3 by showing how to find an *irredundant* monomial generating sequence for bracket powers of monomial ideals.

Proposition 2.6.5 *Set $R = A[X_1, \ldots, X_d]$. Let J be a monomial ideal of R and let $f_1, \ldots, f_n \in [\![J]\!]$ be an irredundant monomial generating sequence for J. For $k = 1, 2, \ldots$ an irredundant monomial generating sequence for $J^{[k]}$ is $f_1^k, \ldots, f_n^k$.*

Proof. By Proposition 2.6.3(b), the sequence $f_1^k, \ldots, f_n^k$ is a monomial generating sequence for $J^{[k]}$, so it suffices to show irredundancy. Suppose that the sequence is redundant. Then there are indices i, j such that $i \neq j$ and $f_i^k \in (f_j^k)R = ((f_j)R)^{[k]}$. Then Lemma 2.6.4 implies that $f_i \in (f_j)R$, contradicting the irredundancy of the original generating sequence. □

The next result provides a useful criterion for checking containment or equality of bracket powers. An application is in the subsequent Proposition 2.6.7.

Lemma 2.6.6. *Set $R = A[X_1, \ldots, X_d]$. Let I and J be monomial ideals in R and fix an integer $k \geqslant 1$.*

(a) *$I \subseteq J$ if and only if $I^{[k]} \subseteq J^{[k]}$.*
(b) *$I = J$ if and only if $I^{[k]} = J^{[k]}$.*

Proof. (a) For the forward implication, we assume that $I \subseteq J$, and we show that $I^{[k]} \subseteq J^{[k]}$. The ideal $I^{[k]}$ is generated by $\{f^k \mid f \in [\![I]\!]\}$. The assumption $I \subseteq J$ implies that each element of this generating set is in $\{g^k \mid g \in [\![J]\!]\} \subseteq J^{[k]}$, so we have $I^{[k]} \subseteq J^{[k]}$.

For the converse, assume that $I^{[k]} \subseteq J^{[k]}$. To show that $I \subseteq J$, we show that $[\![I]\!] \subseteq J$; so let $g \in [\![I]\!]$. Then we have $g^k \in I^{[k]} \subseteq J^{[k]}$, so $g \in J$ by Lemma 2.6.4.

(b) This follows directly from part (a). □

The next result shows that the bracket power operation commutes with intersections. For this, recall that the intersection of monomial ideals is a monomial ideal by Theorem 2.1.1. Hence, the ideal $(\bigcap_{i=1}^n J_i)^{[k]}$ is defined, and it is a monomial ideal. Similarly, the ideal $\bigcap_{i=1}^n J_i{}^{[k]}$ is also a monomial ideal.

Proposition 2.6.7 *Set $R = A[X_1, \ldots, X_d]$. Let $J_1, \ldots, J_n$ be monomial ideals in R. For each integer $k \geqslant 1$, we have $(\bigcap_{i=1}^n J_i)^{[k]} = \bigcap_{i=1}^n J_i{}^{[k]}$.*

Proof. We proceed by induction on n, the number of ideals.

Base case: $n = 2$. Let $f_1, \ldots, f_m \in [\![J_1]\!]$ be a monomial generating sequence for J_1. Let $g_1, \ldots, g_n \in [\![J_2]\!]$ be a monomial generating sequence for J_2.

For the containment $(J_1 \cap J_2)^{[k]} \subseteq J_1{}^{[k]} \cap J_2{}^{[k]}$, observe that $J_1 \cap J_2 \subseteq J_1$, so Lemma 2.6.6(a) implies that $(J_1 \cap J_2)^{[k]} \subseteq J_1{}^{[k]}$. Similarly, we have $(J_1 \cap J_2)^{[k]} \subseteq J_2{}^{[k]}$, and hence $(J_1 \cap J_2)^{[k]} \subseteq J_1{}^{[k]} \cap J_2{}^{[k]}$.

For the containment $(J_1 \cap J_2)^{[k]} \supseteq J_1{}^{[k]} \cap J_2{}^{[k]}$, we need only show that every monomial $z \in [\![J_1{}^{[k]} \cap J_2{}^{[k]}]\!] = [\![J_1{}^{[k]}]\!] \cap [\![J_2{}^{[k]}]\!]$ is in $(J_1 \cap J_2)^{[k]}$. The condition $z \in [\![J_1{}^{[k]}]\!] = [\![(f_1^k, \ldots, f_m^k)R]\!]$ implies that $z \in (f_i^k)R$ for some index i. Similarly, the condition $z \in [\![J_2{}^{[k]}]\!] = [\![(g_1^k, \ldots, g_n^k)R]\!]$ implies that $z \in (g_j^k)R$ for some index j. Write $f_i = \underline{X}^{\underline{m}}$ and $g_j = \underline{X}^{\underline{n}}$, so we have $f_i^k = \underline{X}^{k\underline{m}}$ and $g_j^k = \underline{X}^{k\underline{n}}$. For $l = 1, \ldots, d$ set $p_l = \max\{m_l, n_l\}$. It is straightforward to show that $kp_l = \max\{km_l, kn_l\}$, so Lemma 2.1.4 yields the first and third equalities in the next sequence

$$z \in (f_i^k)R \cap (g_j^k)R = (\underline{X}^{k\underline{p}})R = ((\underline{X}^{\underline{p}})R)^{[k]} = ((f_i)R \cap (g_j)R)^{[k]} \subseteq (J_1 \cap J_2)^{[k]}.$$

The second equality is by definition. The containment at the end of the sequence follows from Lemma 2.6.6(a) since $(f_i)R \cap (g_j)R \subseteq J_1 \cap J_2$.

Induction step: Exercise. □

Exercises

Exercise 2.6.8. Set $R = A[X_1, \ldots, X_d]$. Let J be a non-zero monomial ideal in R. Prove that for each integer $k \geqslant 2$, we have $[\![J^{[k]}]\!] \supsetneq \{h^k \mid h \in [\![J]\!]\}$.

Exercise 2.6.9. Set $R = A[X_1, \ldots, X_d]$. Let J be a monomial ideal in R with monomial generating sequence $f_1, \ldots, f_n$. Fix an integer $k \geqslant 1$, and prove the converse of Proposition 2.6.5: If $f_1^k, \ldots, f_n^k$ is an irredundant monomial generating sequence for $J^{[k]}$, then $f_1, \ldots, f_n$ is an irredundant monomial generating sequence for J.

Exercise 2.6.10. Set $R = A[X_1, \ldots, X_d]$. Let J be a monomial ideal of R. Prove that for $k = 1, 2, \ldots$ we have $J^{[k]} \subseteq J$ and $\text{m-rad}\left(J^{[k]}\right) = \text{m-rad}\,(J)$. See Section 2.3.

Exercise 2.6.11. Complete the induction step of Proposition 2.6.7.

Exercise 2.6.12. Set $R = A[X_1, \ldots, X_d]$, and let J be a monomial ideal in R. Prove that for all integers $k, n \geqslant 1$, one has $(J^{[k]})^{[n]} = J^{[kn]}$.

Exercise 2.6.13. Let p be a prime number and set $R = \mathbb{Z}_p[X_1, \ldots, X_d]$. Let $f_1, \ldots, f_n \in R$ and set $I = (f_1, \ldots, f_n)R$. (Note that the f_i need not be monomials.) For each integer $e \geqslant 1$, set $I^{[p^e]} = (T_{p^e})R$ where $T_{p^e} = \{f^{p^e} \mid f \in I\}$. Prove that $I^{[p^e]} = (f_1^{p^e}, \ldots, f_n^{p^e})R$. Show that the analogous result for $I^{[k]}$ need not hold when k is not a power of p.

Exercise 2.6.14. Let I and J be monomial ideals of $R = A[X_1, \ldots, X_d]$. Prove or disprove the following.

(a) We have $(IJ)^{[n]} = I^{[n]}J^{[n]}$ for each integer $n \geqslant 1$.
(b) We have $(I+J)^{[n]} = I^{[n]} + J^{[n]}$ for each integer $n \geqslant 1$.
(c) We have $(I :_R J)^{[n]} = (I^{[n]} :_R J^{[n]})$ for each integer $n \geqslant 1$. See Section 2.6.

Justify your answers.

Exercise 2.6.15. Let I be a monomial ideal in the ring $R = A[X_1, \ldots, X_d]$. Prove that for all $n,t \geqslant 1$ we have $(I^{[n]})^t = (I^t)^{[n]}$.

Exercise 2.6.16. Let I be a monomial ideal in the ring $R = A[X_1, \ldots, X_d]$, and let a be an integer. Prove that there is an integer n such that $I^{na} \subseteq I^{[a]} \subseteq I^a$. (Hint: Let n be the number of elements in a finite generating sequence for I.) (In fancier language, this implies that the "I-adic topology" on R is equivalent to the topology defined by bracket powers of I.)

Exercise 2.6.17. Let I and J be monomial ideals of $R = A[X_1, \ldots, X_d]$. Prove that the following conditions are equivalent.

(i) There is an integer $n \geqslant 1$ such that $J^n \subseteq I$.
(ii) There is an integer $n \geqslant 1$ such that $J^{[n]} \subseteq I$.
(iii) For every integer $m \geqslant 1$, there is an integer $n \geqslant 1$ such that $J^{[n]} \subseteq I^{[m]}$.
(iv) We have $J \subseteq \text{m-rad}\,(I)$.

(Compare to Exercise 2.3.14.)

Exercise 2.6.18. This exercise involves the construction $\mathrm{V}(I)$ from Exploration Section A.10. Let A be a field, and let I be a monomial ideal of $R = A[X_1, \ldots, X_d]$. Prove that for all integers $n \geqslant 1$, one has $\mathrm{V}(I^{[n]}) = \mathrm{V}(I)$. (See Exercise 2.3.19.)

Bracket Powers of Monomial Ideals in Macaulay2

In this tutorial, we show how to compute bracket powers of monomial ideals. The syntax for this was first introduced in Macaulay2 version 1.8, so make sure you have the most recent version if you wish to follow along.

```
i1 : R=ZZ/101[x,y,z];

i2 : I=ideal(x^2,y^3,z^4,x*y,y*z)
             2   3   4
o2 = ideal (x , y , z , x*y, y*z)
o2 : Ideal of R

i3 : I^[3]
             6   9   12   3 3   3 3
o3 = ideal (x , y , z  , x y , y z )
o3 : Ideal of R

i4 : exit
```

Note our use of the semicolon ; to suppress unnecessary output.

Exercises

Exercise 2.6.19. Set $R = \mathbb{Z}_{101}[X,Y,Z]$, and consider the ideals $J = (XY,Z)R$ and $I = (X^2Y,YZ,Z^5)R$.

(a) Use Macaulay2 to compute the bracket power $I^{[4]}$.
(b) Use Macaulay2 to verify that $Y^2Z \in I$, $(Y^2Z)^5 \in I^{[5]}$, $XY \notin I$, and $(XY)^5 \notin I^{[5]}$.
(c) Use Macaulay2 to verify that $I^{[4]} \subseteq I$ and $\text{m-rad}\left(I^{[4]}\right) = \text{m-rad}\,(I)$.
(d) Use Macaulay2 to verify that $I \subseteq J$, $I^{[3]} \subseteq J^{[3]}$, $J \not\subseteq I$, and $J^{[3]} \not\subseteq I^{[3]}$.
(e) Use Macaulay2 to verify that $\left(I \cap J^{[2]}\right)^{[3]} = I^{[3]} \cap \left(J^{[2]}\right)^{[3]} = I^{[3]} \cap J^{[6]}$.
(f) Use Macaulay2 to check whether or not $(IJ)^{[3]} = I^{[3]}J^{[3]}$.
(g) Use Macaulay2 to check whether or not $(I+J)^{[3]} = I^{[3]} + J^{[3]}$.
(h) Use Macaulay2 to check whether or not $(I :_R J)^{[3]} = (I^{[3]} :_R J^{[3]})$.
(i) Use Macaulay2 to verify that $(I^{[3]})^4 = (I^4)^{[3]}$.

Exercise 2.6.20. Use Macaulay2 to verify your examples for Exercise 2.6.14.

2.7 Exploration: Saturation

In this section, A is a non-zero commutative ring with identity.

Here we explore the concept of the saturation of one monomial ideal with respect to another. This combines colons and powers (and optionally, bracket powers); see Section 2.5 and Exercise 1.3.12 (and Section 2.6). On first inspection, it isn't clear at all how this is a useful construction. (In contrast, this text contains several applications of colon ideals.) The motivation for this concept comes from geometry, actually, where one sees that saturations are the key to understanding the complement of one algebraic variety with respect to another; see Challenge Exercise 7.7.6.

We begin by showing how to define the saturation of one monomial ideal with respect to another.

**Exercise 2.7.1.* Set $R = A[X_1,\ldots,X_d]$, and let I,J be monomial ideals in R.

(a) Prove that we have $J \subseteq (J :_R I) \subseteq (J :_R I^2) \subseteq (J :_R I^3) \subseteq \cdots$.
(b) Prove that the chain of ideals in part (a) stabilizes, that is, there is an integer $N \geqslant 1$ such that $(J :_R I^n) = (J :_R I^N)$ for all $n \geqslant N$. The ideal $(J :_R I^N) = \bigcup_{i=1}^{\infty}(J :_R I^i)$ is the *saturation* of J with respect to I, and is denoted $(J :_R I^\infty)$.
(c) Prove that $(J :_R I^\infty)$ is a monomial ideal.
(d) Let N be a positive integer. Prove that $(J :_R I^{N+1}) = (J :_R I^N)$ if and only if $(J :_R I^n) = (J :_R I^N)$ for all $n \geqslant N$.

(This exercise is used in Exercise 2.8.13.)

Next, we give some perspective on this construction by showing how to compute it for some examples.

**Exercise 2.7.2.* Set $R = A[X_1, \dots, X_d]$, and let I, J be monomial ideals in R.

(a) Prove that

$$(J :_R I^\infty) = \bigcup_{j=1}^{\infty} (J :_R I^j) = \sum_{j=1}^{\infty} (J :_R I^j) = \{f \in R \mid fI^j \subseteq J \text{ for some } j \geqslant 1\}.$$

(b) Prove that if I is generated by $g \in [\![R]\!]$, then $(J :_R I^\infty) = (J :_R (gR)^\infty)$ is the monomial ideal generated by

$$\{f \in [\![R]\!] \mid fg^i \in [\![J]\!] \text{ for some } i \geqslant 1\}.$$

For example, in $R = A[X, Y]$, prove that $(X^2Y^3R :_R (YR)^\infty)$ is generated by

$$\{f \in [\![R]\!] \mid fY^i \in [\![(X^2Y^3)R]\!] \text{ for some } i \geqslant 1\}$$

so $(X^2Y^3R :_R (YR)^\infty) = X^2R$.

(c) Set $R = A[X, Y]$. Argue as in part (b) to show that for $a, b \geqslant 1$ one has

$$(X^aR :_R (X^pY^qR)^\infty) = \begin{cases} X^aR & \text{if } p = 0 \\ R & \text{if } p \geqslant 1 \end{cases}$$

$$(Y^bR :_R (X^pY^qR)^\infty) = \begin{cases} Y^bR & \text{if } q = 0 \\ R & \text{if } q \geqslant 1. \end{cases}$$

$$(X^aY^bR :_R (X^pY^qR)^\infty) = \begin{cases} (X^aY^b)R & \text{if } p = 0 = q \\ Y^bR & \text{if } p \geqslant 1 \text{ and } q = 0 \\ X^aR & \text{if } p = 0 \text{ and } q \geqslant 1 \\ R & \text{if } p, q \geqslant 1 \end{cases}$$

Given that saturations are defined in terms of colon ideals, it is natural to suspect that properties of colons with respect to containments of monomial ideals have counterparts with saturations. The next exercise explores these and other properties.

**Exercise 2.7.3.* Set $R = A[X_1, \dots, X_d]$, and let I, J, and K be monomial ideals in R.

(a) Prove that if $J \subseteq K$, then $(J :_R I^\infty) \subseteq (K :_R I^\infty)$.
(b) Prove or disprove the converse of part (a).
(c) Prove that if $I \subseteq K$, then $(J :_R I^\infty) \supseteq (J :_R K^\infty)$.
(d) Prove or disprove the converse of part (c).
(e) Prove that if $\text{m-rad}\,(I) \subseteq \text{m-rad}\,(K)$, then $(J :_R I^\infty) \supseteq (J :_R K^\infty)$.
(f) Prove or disprove the converse of part (e).
(g) Prove that if $\text{m-rad}\,(I) = \text{m-rad}\,(K)$, then $(J :_R I^\infty) = (J :_R K^\infty)$.
(h) Prove or disprove the converse of part (e).

(i) Prove that $(J :_R I^\infty) = (J :_R \text{m-rad}\,(I)^\infty)$. (This equality is used in Challenge Exercise 7.7.5(b).)
(j) Prove that $((J :_R I^\infty) :_R K^\infty) = (J :_R (IK)^\infty) = ((J :_R K^\infty) :_R I^\infty)$.

Justify your answers.

Sometimes, ideals that remain unchanged when an operation is applied to them are particularly nice. The next exercise explores the monomial ideals that have this property with respect to specific saturations, the "saturated" ideals.

Exercise 2.7.4. Set $R = A[X_1,\ldots,X_d]$, and let I, J be monomial ideals in R.

(a) Prove that $(J :_R I^\infty) = J$ if and only if $(J :_R I) = J$. An ideal J satisfying these equivalent conditions is said to be *saturated* with respect to I.
(b) Prove that the following conditions are equivalent.
   (i) $(J :_R I) = R$
   (ii) $(J :_R I^\infty) = R$
   (iii) there is an integer $N \geqslant 1$ such that $I^N \subseteq J$
   (iv) $I \subseteq \text{m-rad}\,(J)$
(c) Prove that if $\text{m-rad}\,(J) = (X_1,\ldots,X_d)R$ and $I \neq R$, then $(J :_R I^\infty) = R$.
(d) Prove that the saturation of J with respect to I is saturated with respect to I, that is, we have $((J :_R I^\infty) :_R I^\infty) = (J :_R I^\infty)$.

As we mentioned in the introduction to this section, saturations are also related to bracket powers. The point, explored in the next exercise, is that, where the saturation is defined in terms of colons of powers of an ideal, it can also be obtained by colons of bracket powers.

Exercise 2.7.5. Set $R = A[X_1,\ldots,X_d]$, and let I, J be monomial ideals in R.

(a) Prove that we have $J \subseteq (J :_R I) \subseteq (J :_R I^{[2]}) \subseteq (J :_R I^{[3]}) \subseteq \cdots$.
(b) Prove that there is an integer $N \geqslant 1$ such that for all $n \geqslant N$ one has

$$(J :_R I^{[n]}) = (J :_R I^{[N]}) = \bigcup_{j=1}^{\infty} (J :_R I^{[j]}).$$

(c) Prove that $\bigcup_{j=1}^{\infty} (J :_R I^{[j]}) = (J :_R I^\infty)$. In particular, this says that the chains $(J :_R I^j)$ and $(J :_R I^{[j]})$ have the same stable value; that is, if we have $(J :_R I^M) = (J :_R I^{M+1}) = \cdots$ and $(J :_R I^{[N]}) = (J :_R I^{[N+1]}) = \cdots$, then $(J :_R I^M) = (J :_R I^{[N]})$.

The next exercise is much like 2.7.3, exploring how saturation behaves with respect to operations on the input ideals. This time, we treat sums and intersections.

**Exercise 2.7.6.* Let I, J, I_1, I_2, J_1, J_2 be monomial ideals in $R = A[X_1,\ldots,X_d]$.

(a) Prove that $(J_1 \cap J_2 :_R I^\infty) = (J_1 :_R I^\infty) \cap (J_2 :_R I^\infty)$. (Hint: Proposition A.6.3(b).)
(b) Prove that $(J :_R (I_1 + I_2)^\infty) = (J :_R I_1^\infty) \cap (J :_R I_2^\infty)$. (Hint: Exercise 1.3.14 and Proposition A.6.3(d).)

(c) Prove that $(J_1 \cap J_2 :_R (I_1 + I_2)^\infty) = \bigcap_{i=1}^2 \bigcap_{j=1}^2 (J_i :_R I_j^\infty)$.
(d) Repeat parts (a)–(c) for monomial ideals $I_1, \ldots, I_m, J_1, \ldots, J_n$.
(e) Prove that if $f_1, \ldots, f_t \in [\![R]\!]$ and $I = (f_1, \ldots, f_t)R$, then one has $(J :_R I^\infty) = \bigcap_{j=1}^t (J :_R (f_j R)^\infty)$.

(This exercise is used in Exercise 7.7.3.)

Saturation in Macaulay2, Exercises

Commands for this section can be found in Sections 2.6, B.5, and B.6.

Laboratory Exercise 2.7.7. Set $R = \mathbb{Z}_{101}[X,Y]$, with the ideals $I = (X^2Y, XY^3)R$ and $J = (X^{10}Y^2, X^7Y^5, X^3Y^8)R$ and $\mathfrak{X} = (X,Y)R$.

(a) Use Macaulay2 to compute $(I :_R \mathfrak{X}^n)$ for $n = 1,2,3,4$. Sketch the graphs of these ideals. Identify the saturation $(I :_R \mathfrak{X}^\infty)$.
(b) Use Macaulay2 to compute $(J :_R \mathfrak{X}^n)$ for $n = 1,2,\ldots,11$. Sketch the graphs of these ideals. Identify the saturation $(J :_R \mathfrak{X}^\infty)$.
(c) Use Macaulay2 to compute $((X^2Y^3)R :_R \mathfrak{X}^n)$ for $n = 1,2,3$. Sketch the graphs of these ideals. Identify the saturation $((X^2Y^3)R :_R \mathfrak{X}^\infty)$.
(d) [Challenge] Based on parts (a)–(c), make a conjecture about the saturation $(L :_R \mathfrak{X}^\infty)$ where L is any monomial ideal in R. Can you prove your conjecture? (The lexicographical order from Definition A.9.8(a). may be useful here.)
(e) [Challenge] Repeat parts (a)–(d) with the ideal $(X)R$ in place of $\mathfrak{X}$.
(f) [Challenge] Repeat parts (a)–(d) with the ideal $(XY)R$ in place of $\mathfrak{X}$.
(g) [Challenge] Use parts (d)–(f) to give a complete description of $(L :_R K^\infty)$ in terms of the generators of L and K. (Hint: Exercise 2.3.12 and 2.7.3(i).)

Documentation Exercise 2.7.8. The Macaulay2 command `saturate` is used for computing saturations. Explore the Macaulay2 documentation for this command. Perform some sample computations with the ideals from Laboratory Exercise 2.7.7, comparing your computer computations with your manual calculations.

Exercise 2.7.9. Use the Macaulay2 command `saturate` to verify any examples you construct for Exercise 2.7.3.

Exercise 2.7.10. Set $R = \mathbb{Z}_{101}[X,Y,Z]$.

(a) Find monomial ideals $I,J,K \subseteq R$ such that $J \subseteq K$ and $0 \neq (J :_R I^\infty) \subseteq (K :_R I^\infty) \neq R$. Use Macaulay2 to verify these properties.
(b) Repeat part (a) for the other positive results in Exercise 2.7.3.

Exercise 2.7.11. Set $R = \mathbb{Z}_{101}[X,Y,Z]$. Find monomial ideals I and J in R such that $I \not\subseteq J$ and $J \not\subseteq I$ and $(J :_R I) = J$. Use Macaulay2 to verify that your ideals satisfy these conditions as well as $(J :_R I^\infty) = J$ and $I \subseteq \text{m-rad}(J)$.

Exercise 2.7.12. Set $R = \mathbb{Z}_{101}[X,Y,Z]$. Find monomial ideals I and J in R such that $I \not\subseteq J$ and $J \not\subseteq I$ and $(J :_R I) \neq J, R$. Use Macaulay2 to verify that your ideals satisfy these conditions and that $(J :_R I^\infty)$ is saturated with respect to I.

Exercise 2.7.13. Set $R = \mathbb{Z}_{101}[X,Y,Z]$. Find monomial ideals I_1, I_2, J_1, J_2 in R such that no ideal in the list is contained in any other ideal of the list. Use Macaulay2 to verify that $(J_1 \cap J_2 :_R (I_1 + I_2)^\infty) = \bigcap_{i=1}^{2} \bigcap_{j=1}^{2} (J_i :_R I_j^\infty)$.

2.8 Exploration: Generalized Bracket Powers

In this section, A is a non-zero commutative ring with identity.

In Section 2.6, we discussed the bracket powers $I^{[n]}$ for a monomial ideal I in $R = A[X_1, \ldots, X_d]$. As seen in (2.6.3.1) and (2.6.3.2), the graph of $I^{[n]}$ is the graph of I scaled by n in each direction. In this section, we explore a definition which allows for the scaling factors in each direction to be independent of one another. To do this, fix a d-tuple $\underline{e} \in \mathbb{N}^d$ such that $e_1, \ldots, e_d \geqslant 1$. Let $f, f_1, \ldots, f_m, g_1, \ldots, g_n$ be monomials in R.

In order to define the generalized bracket power, we need to understand some of the properties of the following operation. For each monomial $z = \underline{X}^{\underline{n}}$, set $z^{\underline{e}} = (\underline{X}^{\underline{n}})^{\underline{e}} = X_1^{n_1 e_1} \cdots X_d^{n_d e_d}$. For instance, we have $(XY^3)^{(2,5)} = X^2 Y^{15}$.

Exercise 2.8.1.

(a) Prove that one has $f^{\underline{e}} \in (f_1^{\underline{e}}, \ldots, f_m^{\underline{e}})R$ if and only if $f \in (f_1, \ldots, f_m)R$.
(b) Prove that we have $(f_1, \ldots, f_m)R \subseteq (g_1, \ldots, g_n)R$ if and only if $(f_1^{\underline{e}}, \ldots, f_m^{\underline{e}})R \subseteq (g_1^{\underline{e}}, \ldots, g_n^{\underline{e}})R$.
(c) Prove that we have $(f_1, \ldots, f_m)R = (g_1, \ldots, g_n)R$ if and only if $(f_1^{\underline{e}}, \ldots, f_m^{\underline{e}})R = (g_1^{\underline{e}}, \ldots, g_n^{\underline{e}})R$.

Definition 2.8.2. If $I = (f_1, \ldots, f_m)R$, define the $\underline{e}$th *generalized bracket power* of I to be the ideal

$$I^{[\underline{e}]} = (f_1^{\underline{e}}, \ldots, f_m^{\underline{e}})R.$$

For instance, we have $(X^2, XY^3, Y^4)R)^{[(2,5)]} = (X^4, X^2Y^{15}, Y^{20})R$. Note that Exercise 2.8.1(c) shows that $I^{[\underline{e}]}$ is independent of the choice of monomial generating sequence for I.

Here is an example to consider for perspective on this construction.

Exercise 2.8.3. Let $R = A[X,Y]$ and set $I = (X^2, XY^2, Y^3)R$ and $\underline{e} = (2,3)$. Write out an irredundant monomial generating sequence for $I^{[\underline{e}]}$. Sketch the graphs $\Gamma(I)$ and $\Gamma(I^{[\underline{e}]})$, indicating the generators in each case.

The term "generalized bracket power" suggests that this construction subsumes the regular bracket powers as a special case. Our next exercise makes this explicit.

Exercise 2.8.4. Let I, be a monomial ideal in the ring $R = A[X_1, \ldots, X_d]$. Prove that for every integer $n \geqslant 1$, the vector $\underline{n} = (n, n, \ldots, n) \in \mathbb{N}^d$ satisfies $I^{[\underline{n}]} = I^{[n]}$.

Notice that the definition of $I^{[\underline{e}]}$ is in terms of monomial generating sequences that are not necessarily irredundant. However, part of the previous example suggests that one can obtain an irredundant monomial generating sequence for $I^{[\underline{e}]}$ from an irredundant monomial generating sequence for I. The next exercise substantiates this in general.

Exercise 2.8.5. Prove that $f_1, \ldots, f_n$ is an irredundant monomial generating sequence for I if and only if $f_1^{\underline{e}}, \ldots, f_n^{\underline{e}}$ is an irredundant monomial generating sequence for $I^{[\underline{e}]}$.

We started this section by indicating that the generalized bracket power is very similar to the regular bracket power from Section 2.6. However, it is important to note that the definitions of these two constructions are fundamentally different: $I^{[\underline{e}]}$ is defined in terms of a generating sequence for I, while $I^{[n]}$ is not. The next exercise asks you to reconcile these differences.

Exercise 2.8.6. Prove or disprove the following: $I^{[\underline{e}]} = (f^{\underline{e}} \mid f \in [\![I]\!])R$.

The next few exercises demonstrate the many similarities between the generalized and regular bracket powers, beginning with an analogue of Exercise 2.6.10 for our new construction.

Exercise 2.8.7. Prove that $I^{[\underline{e}]} \subseteq I$ and $\text{m-rad}\left(I^{[\underline{e}]}\right) = \text{m-rad}\,(I)$. See Section 2.3.

We continue with this theme by exploring how generalized bracket powers behave with respect to intersections, products, sums, colons, and regular powers.

Exercise 2.8.8. Let $J_1, \ldots, J_n$ be monomial ideals in R, and set $J = \bigcap_{i=1}^n J_i$. Prove that $J^{[\underline{e}]} = \bigcap_{i=1}^n J_i{}^{[\underline{e}]}$.

Exercise 2.8.9. Let I and J be monomial ideals in the ring $R = A[X_1, \ldots, X_d]$. Prove or disprove the following.

(a) We have $(IJ)^{[\underline{e}]} = I^{[\underline{e}]} J^{[\underline{e}]}$.
(b) We have $(I+J)^{[\underline{e}]} = I^{[\underline{e}]} + J^{[\underline{e}]}$.
(c) We have $(J :_R I)^{[\underline{e}]} = (J^{[\underline{e}]} :_R I^{[\underline{e}]})$. See Section 2.6.

Justify your answers.

**Exercise 2.8.10.* Let I be a monomial ideal in R. Prove that for all $t \geqslant 1$, we have $(I^{[\underline{e}]})^t = (I^t)^{[\underline{e}]}$. (This exercise is used in Exercise 7.8.5.)

Part of what makes bracket powers useful is that they compare directly with regular powers. One point here is that while the number of generators for a regular power is usually larger than for the original ideal, the number of generators for the bracket power is the same as for the original ideal, even though the regular power and the bracket power are in some ways similar. The next exercises extend this comparison to the generalized bracket power.

Exercise 2.8.11. Let I be a monomial ideal in the ring $R = A[X_1,\ldots,X_d]$. Prove that there is an integer n such that $I^n \subseteq I^{[\underline{e}]}$.

Exercise 2.8.12. Let I, J be monomial ideals in the ring $R = A[X_1,\ldots,X_d]$. Prove that the following conditions are equivalent.

(i) There is an integer n such that $J^n \subseteq I$.
(ii) There is an integer n such that $J^{[n]} \subseteq I$.
(iii) There is a tuple $\underline{e} \in \mathbb{N}^d$ such that $e_1,\ldots,e_d \geqslant 1$ and $J^{[\underline{e}]} \subseteq I$.
(iv) For every tuple $\underline{a} \in \mathbb{N}^d$ with $a_1,\ldots,a_d \geqslant 1$ there is a tuple $\underline{e} \in \mathbb{N}^d$ such that $e_1,\ldots,e_d \geqslant 1$ and $J^{[\underline{e}]} \subseteq I^{[\underline{a}]}$.
(v) We have $J \subseteq \text{m-rad}(I)$.

(Compare to Exercise 2.6.17.)

The next exercise gives a stabilization result for colons of generalized bracket powers like the ones in Exercises 2.7.1(b) and 2.7.5.

Exercise 2.8.13. Let I, J be monomial ideals in the ring $R = A[X_1,\ldots,X_d]$.

(a) Prove that there is an integer $n \geqslant 1$ such that the vector $\underline{n} = (n,n,\ldots,n) \in \mathbb{N}^d$ satisfies the following: for all $\underline{e} \in \mathbb{N}^d$ with $\underline{e} \succcurlyeq \underline{n}$ we have
$$(J :_R I^{[\underline{e}]}) = (J :_R I^{[\underline{n}]}) = \bigcup_{\underline{a}} (J :_R I^{[\underline{a}]})$$
where the union is taken over all $\underline{a} \in \mathbb{N}^d$ with $a_1,\ldots,a_d \geqslant 1$.
(b) Prove that $\bigcup_{\underline{a}}(J :_R I^{[\underline{a}]}) = (J :_R I^{\infty})$ where the union is taken over all $\underline{a} \in \mathbb{N}^d$ with $a_1,\ldots,a_d \geqslant 1$ and where $(J :_R I^{\infty})$ is the saturation of Section 2.7.

Next, we discuss how generalized bracket powers behave with respect to the construction $\mathrm{V}(I)$ from Exploration Section A.10.

**Exercise 2.8.14.* Let A be a field, and let I be a monomial ideal of $R = A[X_1,\ldots,X_d]$. Prove that $\mathrm{V}(I^{[\underline{e}]}) = \mathrm{V}(I)$. (See Exercise 2.3.19. This is used in Exercise 3.1.13.)

Generalized Bracket Powers in Macaulay2

In this tutorial we show how to compute generalized bracket powers in Macaulay2. The command for this is essentially the same as that for ordinary bracket powers from Section 2.6. (As we mentioned there, this was added to Macaulay2 in version 1.8.)

```
i1 : R = ZZ/101[a..d];

i2 : X = ideal vars R
o2 = ideal (a, b, c, d)
o2 : Ideal of R
```

Note that we used the command `vars` to obtain a matrix of the variables of R.

```
i3 : X^[2,3,4,7]

             2   3   4   7
o3 = ideal (a , b , c , d )

o3 : Ideal of R
```

When taking the generalized bracket power of an ideal, be sure that the length of the sequence in brackets is the same as the number of variables in the ring. An error is also generated if any of the entries are negative. This is a handy way to generate m-irreducible ideals, which are the subject of Chapter 3.

For a more elaborate example, consider the following.

```
i4 : I = ideal (a^4,a*b,b^2,b*c^2,c^4,d^5)

             4        2     2   4   5
o4 = ideal (a , a*b, b , b*c , c , d )

o4 : Ideal of R

i5 : I^[1,2,3,4]

             4     2   4   2 6   12   20
o5 = ideal (a , a*b , b , b c , c  , d  )

o5 : Ideal of R
```

As predicted by Exercise 2.8.5, the minimal number of generators of an ideal and any bracket power are the same.

```
i6 : numgens trim I == numgens trim I^[1,2,3,4]

o6 = true
```

Lastly, we perform a reality-check by verifying the fact that bracket powers are just special cases of generalized bracket powers.

```
i7 : I^[2,2,2,2] == I^[2]

o7 = true

i8 : exit
```

Exercises

Exercise 2.8.15. Set $R = \mathbb{Z}_{101}[X,Y,Z]$. Consider the ideals $J = (XY,Z)R$ and $I = (X^2Y, YZ, Z^5)R$, and the vectors $\underline{e} = (3,2,4)$ and $\underline{n} = (2,3,2)$.

(a) Use Macaulay2 to compute the generalized bracket powers $I^{[\underline{e}]}$ and $J^{[\underline{e}]}$.
(b) Use Macaulay2 to verify that $Y^2Z \in I$ and $(Y^2Z)^{\underline{e}} \in I^{[\underline{e}]}$.
(c) Use Macaulay2 to verify that $XY \notin I$ and $(XY)^{\underline{e}} \notin I^{[\underline{e}]}$.
(d) Use Macaulay2 to verify that $I^{[\underline{n}]} \subseteq I$ and $\text{m-rad}\left(I^{[\underline{n}]}\right) = \text{m-rad}\,(I)$.
(e) Use Macaulay2 to verify that $I \subseteq J$ and $I^{[\underline{e}]} \subseteq J^{[\underline{e}]}$ and $J \not\subseteq I$ and $J^{[\underline{e}]} \not\subseteq I^{[\underline{e}]}$.
(f) Use Macaulay2 to check whether or not $(I \cap J^{[\underline{e}]})^{[\underline{n}]} = I^{[\underline{n}]} \cap (J^{[\underline{e}]})^{[\underline{n}]} = I^{[\underline{n}]} \cap J^{[\underline{en}]}$ where $(\underline{en})_i = e_i n_i$.

(g) Use Macaulay2 to check whether or not $(IJ)^{[\underline{e}]} = I^{[\underline{e}]}J^{[\underline{e}]}$.
(h) Use Macaulay2 to check whether or not $(I+J)^{[\underline{e}]} = I^{[\underline{e}]} + J^{[\underline{e}]}$.
(i) Use Macaulay2 to check whether or not $(J :_R I)^{[\underline{e}]} = (J^{[\underline{e}]} :_R I^{[\underline{e}]})$.
(j) Use Macaulay2 to verify that $(I^{[\underline{e}]})^4 = (I^4)^{[\underline{e}]}$.

Laboratory Exercise 2.8.16. Let $\underline{e}$ and $\underline{f}$ be d-tuples in $\mathbb{N}_+^d$, and let I be the ideal in the above tutorial. Investigate the ideal $(I^{[\underline{e}]})^{[\underline{f}]}$. In particular, is it of the form $I^{[\underline{p}]}$? Use the data you collect to make a conjecture about the ideal $(I^{[\underline{e}]})^{[\underline{f}]}$ in general (that is, for any monomial ideal I), and prove your conjecture.

2.9 Exploration: Comparing Bracket Powers and Ordinary Powers

In this section, A is a non-zero commutative ring with identity.

Here, we investigate the difference between bracket powers and ordinary powers, using colon ideals. See Section 2.5, Exercise 1.3.12, and Section 2.6. Our first exercise shows that bracket powers are pinched between two regular powers in a controlled manner. It improves on Exercise 2.6.16 in a special case.

Exercise 2.9.1. Let $R = A[X_1,\ldots,X_d]$, and let $I = (X_1^{a_1},\ldots,X_d^{a_d})$. Prove that we have $I^{nd-d+1} \subseteq I^{[n]} \subseteq I^n$ for all n.

It is straightforward to give examples showing that the equalities from the previous exercise can be strict. The next exercise says, however, that in a special case with $d = 2$ one can control the difference between $I^{[n]}$ and I^n.

Exercise 2.9.2. Let $R = A[X,Y]$ and let $I = (X^a,Y^b)R$.

(a) Use the previous exercise to show that $I^{n-1} \subseteq (I^{[n]} : I^n)$.
(b) Let $X^lY^m \in [\![(I^{[n]} : I^n)]\!]$. Prove that for all $0 \leqslant k \leqslant n$, either $l \geqslant (n-k)a$ or $m \geqslant kb$.
(c) Let $X^lY^m \in [\![(I^{[n]} : I^n)]\!]$. Use the previous part to show that there exists an i such that $l \geqslant (n-i)a$ and $m \geqslant (i-1)b$.
(d) Conclude that $(I^{[n]} : I^n) = I^{n-1}$.

The next exercise asks you to explore the question of just how special the situation of the previous exercise really is.

Challenge Exercise 2.9.3. Let $R = A[X,Y]$. By the previous exercise, ideals of the form $I = (X^a,Y^b)$ have the property that $I^k \subseteq (I^{[n]} : I^n)$ for some $k < n$. Can you find any other classes of monomial ideals of R that have this property? Perform the same analysis for ideals in the polynomial ring $A[X_1,\ldots,X_d]$. Justify your answers. (See Laboratory Exercise 2.9.6.)

Next, we switch from bracket powers to the generalized bracket powers from Section 2.8.

Challenge Exercise 2.9.4. Let $R = A[X_1, \ldots, X_d]$, let $I = (X_1^{a_1}, \ldots, X_d^{a_d})$, and fix a d-tuple $\underline{e} \in \mathbb{N}_+^d$. Prove that there are integers m and n such that $I^m \subseteq I^{[\underline{e}]} \subseteq I^n$. Let $m(\underline{a}, \underline{e})$ be the smallest integer m such that $I^m \subseteq I^{[\underline{e}]}$, and let $n(\underline{a}, \underline{e})$ be the largest integer n such that $I^{[\underline{e}]} \subseteq I^n$. Investigate the behavior of the functions $m(\underline{a}, \underline{e})$ and $n(\underline{a}, \underline{e})$ for some examples. Make conjectures about the values of these functions, and prove your conjectures. (See Laboratory Exercise 2.9.7.)

Comparing Bracket Powers and Ordinary Powers in Macaulay2, Exercises

Commands for this section can be found in Sections 2.6, B.5, and B.6.

Exercise 2.9.5. Set $R = \mathbb{Z}_{101}[X, Y]$. Use Macaulay2 to verify the conclusion of Exercise 2.9.2(d) for the ideals $(X^3, Y^b)R$ for $b = 1, \ldots, 4$ and $n = 2, \ldots, 5$.

Laboratory Exercise 2.9.6. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to verify special cases of any classes you find for Challenge Exercise 2.9.3. If you are having trouble finding any more classes of ideals for 2.9.3, use Macaulay2 to compare I^k and $(I^{[n]} :_R I^n)$ for some other monomial ideals I to try to identify some candidate classes.

Laboratory Exercise 2.9.7. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate $m(\underline{a}, \underline{e})$ and $n(\underline{a}, \underline{e})$ from Challenge Exercise 2.9.4 for some specific $\underline{a}$ and $\underline{e}$, starting with $d = 2, 3$, to help you make your conjecture for 2.9.4.

Concluding Notes

Operations on ideals appear almost certainly in every subject where ideals are used. One instance of this is in algebraic geometry, which we touch on briefly; see the exercises e.g., of Exploration Section A.10, as well as the text of Cox, Little, and O'Shea [13] for more on this.

Given a set of polynomials $S \subseteq R = \mathbb{C}[X_1, \ldots, X_d]$, the vanishing locus of S is the set of all points in $\mathbb{C}^d$ making all the polynomials in S vanish.[5] Reciprocally, given a subset $Z \subseteq \mathbb{C}^d$, the vanishing ideal of Z is the ideal of all polynomials in R vanishing at all points of Z. Under these operations, roughly speaking, intersections of ideals correspond to unions of vanishing loci, sums of ideals correspond to intersections of vanishing loci, and colon ideals (more precisely, saturations) correspond to complements of vanishing loci. Colon ideals are also important for Christian Peskine and Lucien Szpiro's theory of linkage, a.k.a., liaison [66].

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[5] This can all be done in significantly more generality.

On the other hand, the powers, (generalized) bracket powers, and (monomial) radical of an ideal all determine the same vanishing locus. In spite of this, there is significant geometric content here as well. For instance, the powers of an ideal give rise to the Rees algebra of the ideal, a.k.a., the blow-up algebra which can be used to "desingularize" the corresponding vanishing locus. Powers also determine the Hilbert-Samuel[6] multiplicity, which is a measure of the severity of the singularity of the vanishing locus. See, e.g., the advanced texts of David Eisenbud [18], Robin Hartshorne [33], and Jean-Pierre Serre [72] for additional details on these topics, and Section 5.4 for some information about multiplicities.

Bracket powers occur when the field $\mathbb{C}$ is replaced with a field k like $\mathbb{Z}_p$ or its algebraic closure $\overline{\mathbb{Z}_p}$. (Again, there is significant geometric content here.) In this case, the bracket powers $I^{[p^n]}$ are related to the Frobenius endomorphism $k[X_1,\ldots,X_d] \to k[X_1,\ldots,X_d]$ given by $f \mapsto f^p$. For instance, these powers are key for Melvin Hochster and Craig Huneke's theory of tight closure [43] and Paul Monsky's Hilbert-Kunz multiplicity [59], based on ideas of Hilbert and Ernst Kunz [50].

Unique factorization is an important and special property for an integral domain. As we have seen, many of our favorite rings are UFD's, though most rings are not. In fact, the lack of unique factorization in some rings of integers is one of many factors making life hard for many mathematicians, notably including Gabriel Lamé, who were trying to prove Fermat's Last Theorem: in short, if rings of cyclotomic integers were all UFD's (which they aren't) then one could prove Fermat's Last Theorem handily. Coincidentally, Ernst Kummer developed the theory of ideals for rings of algebraic integers in order to advance Lamé's approach to this problem. See the survey articles of Pierre Samuel [71] and Daniel D. Anderson [2] for more about UFD's and their extensions.

[6] Again, we find Hilbert's influence.

Chapter 3
M-Irreducible Ideals and Decompositions

M-irreducible ideals are, in a sense, the simplest monomial ideals, in that they cannot be written as non-trivial intersections of monomial ideals. We study these ideals in Section 3.1. In many texts, these notions are only considered when the ground ring A is a field, where the m-irreducible ideals are actually irreducible; this means that they cannot be written as non-trivial intersections of any ideals. This is the topic of the optional Section 3.2.

One of the main goals of this book is to show that every monomial ideal has an m-irreducible decomposition, that is, it can be written as a finite intersection of m-irreducible monomial ideals. This goal is accomplished in Section 3.3. The case of irreducible decompositions is treated in the optional Section 3.4. The chapter concludes with Section 3.5, which is an exploration of m-irreducible decompositions in two variables. Even though this section is optional, it is extremely useful for computing examples.

The Macaulay2 material of this chapter explores these topics computationally. In addition, we discuss how to write "methods" in Section 3.1. These are functions that are more versatile that the ones in the preceding chapter, and we develop more and more elaborate methods as we move through the chapter. In addition, in Section 3.2 we treat hash tables, structures used by Macaulay2 to represent computational data.

3.1 M-Irreducible Monomial Ideals

In this section, A is a non-zero commutative ring with identity.

We are now ready to introduce the building blocks of our decompositions. Note that our assumption $A \neq 0$ is crucial here because if $A = 0$, then $R = A[X_1, \ldots, X_d] = 0$, and it follows that every ideal $J \subseteq R$ satisfies $J = 0 = R$.

Definition 3.1.1. Set $R = A[X_1, \ldots, X_d]$. A monomial ideal $J \subsetneq R$ is *m-reducible* if there are monomial ideals $J_1, J_2 \neq J$ such that $J = J_1 \cap J_2$. A monomial ideal $J \subsetneq R$ is *m-irreducible* if it is not m-reducible.

By definition, a monomial ideal $J \subseteq R$ is m-irreducible if and only if $J \neq R$ and, given two monomial ideals J_1, J_2 such that $J = J_1 \cap J_2$, either $J_1 = J$ or $J_2 = J$. Inductively, if J is m-irreducible and $J_1, \ldots, J_n$ are monomial ideals (with $n \geqslant 2$) such that $J = \bigcap_{i=1}^n J_i$, then $J = J_i$ for some index i. (This uses Theorem 2.1.1.) According to our definition, the unit ideal R is neither m-irreducible nor m-reducible.

Example 3.1.2. Set $R = A[X,Y]$. The monomial ideal $J = (X^3, X^2Y, Y^3)R$ is m-reducible. Indeed, we have

$$J = (X^2, Y^3)R \cap (X^3, Y)R$$

by Example 2.1.6. Also, we have $X^2 \in (X^2, Y^3)R \smallsetminus J$ so $J \neq (X^2, Y^3)R$. Also $Y \in (X^3, Y)R \smallsetminus J$, so $J \neq (X^3, Y)R$.

On the other hand, the ideals $(X^2, Y^3)R$ and $(X^3, Y)R$ are m-irreducible. This can be verified directly, or by appealing to Theorem 3.1.3.

The next theorem is the main result of this section. It characterizes the non-zero m-irreducible monomial ideals as the ideals generated by "pure powers" of the variables. See Exercise 3.1.5 for information about the zero ideal.

Theorem 3.1.3 *Set $R = A[X_1, \ldots, X_d]$, and let J be a non-zero monomial ideal of R. The ideal J is m-irreducible if and only if there exist positive integers $k, t_1, \ldots, t_k, e_1, \ldots, e_k$ such that $1 \leqslant t_1 < \cdots < t_k \leqslant d$ and $J = (X_{t_1}^{e_1}, \ldots, X_{t_k}^{e_k})R$.*

Proof. Assume that there are integers $k, t_1, \ldots, t_k, e_1, \ldots, e_k \geqslant 1$ such that $t_1 < \cdots < t_k \leqslant d$ and $J = (X_{t_1}^{e_1}, \ldots, X_{t_k}^{e_k})R$. Note that $J \subseteq (X_{t_i}, \ldots, X_{t_k})R \subseteq (X_1, \ldots, X_d)R$, so we have $J \neq R$.

Fix monomial ideals J_1, J_2 in R such that $J = J_1 \cap J_2$. Suppose that $J \subsetneq J_i$ for $i = 1, 2$ and fix a monomial $f_i \in [\![J_i]\!] \smallsetminus [\![J]\!]$. Write $f_1 = \underline{X}^{\underline{m}}$ and $f_2 = \underline{X}^{\underline{n}}$. For $i = 1, \ldots, d$ set $p_i = \max\{m_i, n_i\}$.

For $i = 1, \ldots, k$ we have $m_{t_i} < e_i$: otherwise, we have $m_i \geqslant e_i$ for some i, so a comparison of exponent vectors shows that $f_1 \in (X_{t_i}^{e_i})R \subseteq J$, a contradiction. Similarly, for $i = 1, \ldots, k$ we have $n_i < e_i$, and hence $p_i = \max\{m_i, n_i\} < e_i$. A similar argument shows that $\operatorname{lcm}(f_1, f_2) = \underline{X}^{\underline{p}} \notin J$. However, we have $\operatorname{lcm}(f_1, f_2) \in J_1 \cap J_2 = J$ by Lemma 2.1.4, a contradiction.

For the converse, we argue by contrapositive. Let $f_1, \ldots, f_k$ be an irredundant monomial generating sequence for J. Note that the assumption $J \neq 0$ implies that $k \geqslant 1$. Assume that some f_i is not a power of one of the variables. Re-order the f_j if necessary to assume that f_k is not a power of one of the variables. This means that we can write $f_k = X_{t_i}^{e_i} g$ for some i, where $e_i \geqslant 1$ and $X_{t_i} \nmid g$ and $g \neq 1$. Re-order the variables if necessary to assume that $f_k = X_1^e g$ where $e \geqslant 1$ and $X_1 \nmid g$ and $g \neq 1$. Set $I = (f_1, \ldots, f_{k-1}, X_1^e)R$ and $I' = (f_1, \ldots, f_{k-1}, g)R$.

Claim: $J = I \cap I'$. For this, we use Theorem 2.1.5 to conclude that the following sequence generates $I \cap I'$:

$$f_1, \ldots, f_{n-1}, \underbrace{\operatorname{lcm}(f_1, X_1^e), \operatorname{lcm}(f_1, g)}_{\in (f_1)R}, \ldots, \underbrace{\operatorname{lcm}(f_{n-1}1, X_1^e), \operatorname{lcm}(f_{n-1}, g)}_{\in (f_{n-1})R}, \underbrace{\operatorname{lcm}(X_1^e, g)}_{= f_n}.$$

Removing redundancies from this list (by Algorithm 1.3.7) we see that $I \cap I'$ is generated by $f_1, \ldots, f_{n-1}, f_n$. As this sequence generates J, we have the desired equality.

Claim: $J \subsetneq I$. To show that $J \subseteq I$, it suffices to note that $J = I \cap I' \subseteq I$. To show that $J \neq I$, we need to show that $X_1^e \notin J$. Suppose by way of contradiction that $X_1^e \in J$. Then $f_j \big| X_1^e$ for some index j, by Theorem 1.1.9. Since $X_1^e \big| f_k$, this implies that $f_j \big| f_k$. The sequence $f_1, \ldots, f_k$ is irredundant, so we have $j = k$. Thus, we have $f_k = X_1^e g = f_j \big| X_1^e$. By comparing exponent vectors, we conclude that $g = 1$, a contradiction.

Similarly, we have $J \subsetneq I'$. In short, we have $J = I \cap I'$ and $J \subsetneq I$ and $J \subsetneq I'$. This shows that J is m-reducible, completing the proof. □

The next result is for our work below on decompositions; see Proposition 3.3.5. Note that the conclusion is a souped-up version of the definition of m-irreducible.

Lemma 3.1.4. *Set $R = A[X_1, \ldots, X_d]$. Let $I, J_1, \ldots, J_n$ be monomial ideals in R such that I is m-irreducible. If $\bigcap_{i=1}^n J_i \subseteq I$, then there is an index j such that $J_j \subseteq I$.*

Proof. If $I = 0$, then the condition $\bigcap_{i=1}^n J_i \subseteq I = 0$ implies that $\bigcap_{i=1}^n J_i = 0$; it is straightforward to show that this implies that $J_i = 0 = I$ for some index i. Thus, we assume that $I \neq 0$. Also, the case $n = 1$ is trivial, so we assume that $n \geqslant 2$. Theorem 3.1.3 provides positive integers $k, t_1, \ldots, t_k, e_1, \ldots, e_k$ such that $1 \leqslant t_1 < \cdots < t_k \leqslant d$ such that $I = (X_{t_1}^{e_1}, \ldots, X_{t_k}^{e_k})R$.

We proceed by induction on n.

Base case: $n = 2$. Assume that $J_1 \cap J_2 \subseteq I$. Suppose by way of contradiction that $J_1 \not\subseteq I$ and $J_2 \not\subseteq I$. This implies that $[\![J_1]\!] \not\subseteq [\![I]\!]$ and $[\![J_2]\!] \not\subseteq [\![I]\!]$, by Theorem 1.1.4(a), so there are monomials $f_1 \in J_1 \smallsetminus I$ and $f_2 \in J_2 \smallsetminus I$. Write $f_1 = \underline{X}^{\underline{m}}$ and $f_2 = \underline{X}^{\underline{n}}$. For $i = 1, \ldots, d$ set $p_i = \max\{m_i, n_i\}$ so we have

$$\underline{X}^{\underline{p}} = \operatorname{lcm}(f_1, f_2) \in J_1 \cap J_2 \subseteq I = (X_{t_1}^{e_1}, \ldots, X_{t_k}^{e_k})R.$$

It follows that there is an index j such that $X_{t_j}^{e_j} \big| \underline{X}^{\underline{p}}$. Comparing exponent vectors, we find that $e_j \leqslant p_{t_j} = \max\{m_{t_j}, n_{t_j}\}$. It follows that either $e_j \leqslant m_{t_j}$ or $e_j \leqslant n_{t_j}$. If $e_j \leqslant m_{t_j}$, then another comparison of exponent vectors implies that $X_{t_j}^{e_j} \big| \underline{X}^{\underline{m}} = f_1$, so $f_1 \in (X_{t_j}^{e_j})R \subseteq I$, a contradiction. Similarly, if $e_j \leqslant n_{t_j}$, we conclude that $f_2 \in I$, a contradiction. This concludes the proof of the base case.

Induction step: Exercise. □

Exercises

**Exercise 3.1.5.* Set $R = A[X_1, \ldots, X_d]$. Prove that 0 is m-irreducible. (This exercise is used in the proof of Theorem 3.3.3.)

**Exercise 3.1.6.* Set $R = A[X_1, \ldots, X_d]$. Let J be a monomial ideal in R.

(a) Prove that if J is non-zero and m-irreducible, then there are positive integers $n, t_1, \ldots, t_n$ such that $\operatorname{m-rad}(J) = (X_{t_1}, \ldots, X_{t_n})R$.

(b) Prove that if J is m-irreducible, then $\text{m-rad}(J)$ is m-irreducible.
(c) Prove or disprove the following: if $\text{m-rad}(J)$ is m-irreducible, then J is m-irreducible. Justify your answer.

(This exercise is used in the proof of Proposition 7.1.1.)

Exercise 3.1.7. Set $R = A[X]$. Prove that every monomial ideal in R is m-irreducible.

**Exercise 3.1.8.* Set $R = A[X_1,\dots,X_d]$. Let $k, t_1,\dots,t_k, e_1,\dots,e_k \geqslant 1$ be integers such that $1 \leqslant t_1 < \cdots < t_k \leqslant d$, and set $J = (X_{t_1}^{e_1},\dots,X_{t_k}^{e_k})R$. Prove that the monomial $X_{t_1}^{e_1-1}\cdots X_{t_k}^{e_k-1}$ is not in J. (This exercise is used in the proof of Theorem 3.2.4.)

Exercise 3.1.9. Set $R = A[X_1,\dots,X_d]$. Let J be a non-zero m-irreducible monomial ideal in R. Prove or disprove the following: If $\{I_\lambda\}_{\lambda\in\Lambda}$ is a (possibly infinite) set of monomial ideals in R such that $J = \bigcap_{\lambda\in\Lambda} I_\lambda$, then there is an index $\lambda \in \Lambda$ such that $J = I_\lambda$. Justify your answer.

Exercise 3.1.10. Set $R = A[X_1,\dots,X_d]$. Let J be a monomial ideal in R such that $J \neq R$. Prove that the following conditions are equivalent:

(i) J is m-irreducible;
(ii) for all monomial ideals J_1, J_2 if $J_1 \cap J_2 \subseteq J$, then either $J_1 \subseteq J$ or $J_2 \subseteq J$; and
(iii) for all monomials $f, g \in [\![R]\!]$ if $\operatorname{lcm}(f,g) \in J$, then either $f \in J$ or $g \in J$.

Exercise 3.1.11. Complete the induction step of Lemma 3.1.4.

Exercise 3.1.12. Set $R = A[X_1,\dots,X_d]$, and consider a chain $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ of m-irreducible monomial ideals of R. Prove that $\bigcap_{i=1}^\infty J_i$ is m-irreducible.

**Exercise 3.1.13.* This exercise involves the constructions $\mathrm{V}(I)$ and $\mathrm{I}(Y)$ from Exploration Section A.10. Let A be a field, and let I be an m-irreducible monomial ideal of $R = A[X_1,\dots,X_d]$. Prove that there is an integer $n \in \mathbb{N}$, and there are variables $X_{i_1},\dots,X_{i_n}$ such that $\mathrm{V}(I) = \mathrm{V}(X_{i_1},\dots,X_{i_n})$; in particular, $\mathrm{V}(I)$ is a "linear subspace" of A^d. (See Exercise 2.8.14.) Use this to prove that $\mathrm{I}(\mathrm{V}(I)) = (X_{i_1},\dots,X_{i_n})R = \text{m-rad}(I)$. (This is used in Exercises 3.3.11, 7.1.6, 7.2.6 and 7.8.6.)

M-Irreducible Monomial Ideals in Macaulay2

In this section we implement a "method function" (a "method", for short) that uses Theorem 3.1.3 to determine whether a monomial ideal is m-irreducible. Before doing this, we first explain a bit about methods in Macaulay2, and why they are useful. In Macaulay2, a method knows the type of its input, and can perform different operations on the input, depending on the type of input passed in. (This behavior is called *overloading* in the computer science literature. We have already seen one example of this with the command `/`. Indeed, in Sections B.8 and 2.2–2.3 respectively we show how to use this command to construct quotient rings and to apply functions. Another example is discussed later in this section and in the next section: the command `#`, for computing the size of a set and for accessing values in a hash table.) For our current purposes, one may want different behavior when a `MonomialIdeal` object, rather than an `Ideal` object, is passed as input.

Another useful feature of a method is that it understands inheritance of Macaulay2 types. For an example of this in action, see the Macaulay2 tutorial for Section 5.4. For a detailed overview of how inheritance and methods work in Macaulay2, view the help pages for `inheritance` and `installing methods`.

A simple function does not have any of these capabilities, nor does it ensure for you that the object passed as input is what the programmer expected it to be when she wrote the code. Even when you only want to implement a function on one kind of input, it is good Macaulay2 coding practice to write it as a method because input type safety is ensured, and the code is often more readable.

So with this in mind, let us define our method.

```
i1 : isMIrreducible = method()
o1 = isMIrreducible
o1 : MethodFunction
```

This command (not surprisingly) creates a method called `isMIrreducible`. It doesn't do anything yet, so we want to install a function that takes a single argument of type `MonomialIdeal`. Here is the code for our method. We go through each command below.

```
i2 : isMIrreducible MonomialIdeal := I -> (
Itrim := trim I;
all(Itrim_*, m -> #(support m) == 1)
)
o2 = {*Function[stdio:2:35-4:35]*}
o2 : FunctionClosure
```

Before discussing the code itself, a few comments regarding syntax inside a function are necessary. First, the body of the function should be surrounded by parentheses. Second, inside functions one can have temporary variables that are not visible to the user of the function. These are called local variables, and when one creates them one uses the `:=` operator rather than the `=` operator. If you define a variable using `=` instead, that variable will persist outside the function, which is often not desired. Finally, all the lines inside a function should be terminated by a semicolon `;`, except the last line, whose result is the return value of the function.

Now, back to our specific method. The first thing we do in our code is to ensure that we are working with an irredundant generating sequence of our input ideal. This is accomplished using the `trim` command. Next, we check that all monomials in this irredundant generating sequence only contain a single variable using the commands `#` and `support`. The command `all` returns true if all elements of the first argument (the list of generators) return true when passed into the second argument (the function). Let's test our new method.

```
i3 : R = QQ[a..d];I = monomialIdeal (a^2,b^3,c^4,d^5)
                     2   3   4   5
o4 = monomialIdeal (a , b , c , d )
o4 : MonomialIdeal of R

i5 : isMIrreducible I
o5 = true
```

We would like this method to also accept `Ideal` objects as input. We can do so by installing another method function for that type.

```
i6 : isMIrreducible Ideal := I -> (
if I != monomialIdeal I then error "Expected a monomial ideal as input.";
isMIrreducible monomialIdeal I
)
o6 = {*Function[stdio:8:27-10:30]*}
o6 : FunctionClosure
```

Note that we have included a check, using an `if-then` statement with the `error` command, to ensure that the ideal input is indeed generated by monomials. This is because the command `monomialIdeal` returns the ideal generated by the leading terms of all elements in the ideal. (An ideal is monomial if and only if it is equal to its ideal of "leading terms".) We discuss this ideal in Definition 5.4.5 below.

```
i7 : I = ideal (a^4,a*b,b^2,b*c^2,c^4,d^5)
             4         2     2   4   5
o7 = ideal (a , a*b, b , b*c , c , d )
o7 : Ideal of R

i8 : isMIrreducible I
o8 = false

i9 : exit
```

Exercises

Coding Exercise 3.1.14. Let $R = \mathbb{Q}[X,Y]$.

(a) Determine how to define the ideal R of R in Macaulay2, and verify that the command `isMIrreducible` does indeed return `false` on this input. Examine the code for `isMIrreducible` and explain why the code handles this case correctly.
(b) According to Exercise 3.1.5, the zero ideal is m-irreducible. Verify that the command `isMIrreducible` works correctly on this input as well, and again examine the code in order to explain why this case is also handled correctly.

Exercise 3.1.15. Use Macaulay2 verify the conclusions of Example 3.1.2 over $\mathbb{Q}$.

Exercise 3.1.16. Set $R = \mathbb{Q}[X,Y,Z]$, and use Macaulay2 to verify the conclusion of Exercise 3.1.8 for some ideals J of the form $(X^a,Y^b)R$ and $(X^a,Y^b,Z^c)R$.

3.2 Irreducible Ideals (optional)

In this section, A is a non-zero commutative ring with identity.

The notion of m-irreducibility for monomial ideals is derived from the notion of irreducibility for arbitrary ideals, which is the focus of this section. In words, an ideal is irreducible if it cannot be written as a non-trivial intersection of two ideals.

The main result of this section is Theorem 3.2.4. It shows that, when A is a field, every m-irreducible monomial ideal is also irreducible.

Definition 3.2.1. An ideal $J \subsetneq A$ is *reducible* if there are ideals $J_1, J_2 \neq J$ such that $J = J_1 \cap J_2$. An ideal $J \subsetneq A$ is *irreducible* if it is not reducible.

By definition, an ideal $J \subseteq A$ is irreducible if and only if $J \neq A$ and given two ideals J_1, J_2 with $J = J_1 \cap J_2$, either $J_1 = J$ or $J_2 = J$. Inductively, if J is irreducible and $J_1, \ldots, J_n$ are ideals (with $n \geqslant 2$) with $J = \bigcap_{i=1}^n J_i$, then $J = J_i$ for some i. According to our definition, the unit ideal R is neither irreducible nor reducible.

For example, the ideal $0 \subseteq \mathbb{Z}$ is irreducible. If p is a prime number and n is a positive integer, then the ideal $p^n\mathbb{Z} \subseteq \mathbb{Z}$ is irreducible. These are the only irreducible ideals of $\mathbb{Z}$; for instance, we have $6\mathbb{Z} = 3\mathbb{Z} \cap 2\mathbb{Z}$. In the ring $\mathbb{Z}_6$, the ideal 0 is reducible since $2\mathbb{Z}_6 \cap 3\mathbb{Z}_6 = 0$. In the ring $\mathbb{Z}_4$, the ideal 0 is irreducible since the only ideals in $\mathbb{Z}_4$ are 0, $2\mathbb{Z}_4$ and $\mathbb{Z}_4$.

Let k be a field and let R be the polynomial ring $R = k[X]$ in one variable. For each $a \in k$ and each positive integer n, the ideal $(X-a)^n R \subseteq R$ is irreducible.

The proof of the main result of this section uses the following notion.

Definition 3.2.2. Set $R = A[X_1, \ldots, X_d]$. For each $f = \sum_{\underline{n} \in \mathbb{N}^d}^{\text{finite}} a_{\underline{n}} \underline{X}^{\underline{n}} \in R$, we set $\gamma(f) = \{\underline{n} \in \mathbb{N}^d \mid a_{\underline{n}} \neq 0\}$. This is sometimes called the "support" of f, though we avoid this terminology in order to avoid confusion with the support of a monomial.

For instance, for the polynomial $f = X^2 + XY + X^2Z^3 - XY^2Z^3$ in $A[X,Y,Z]$ we have $\gamma(f) = \{(2,0,0), (1,1,0), (2,0,3), (1,2,3)\} \subseteq \mathbb{N}^3$. In general, $\gamma(f)$ is a finite set such that $f = \sum_{\underline{n} \in \gamma(f)} a_{\underline{n}} \underline{X}^{\underline{n}}$. Furthermore, we have $\gamma(f) = \emptyset$ if and only if $f = 0$.

The following technical lemma is included for the proof of Theorem 3.2.4.

Lemma 3.2.3. *Set $R = A[X_1, \ldots, X_d]$. Fix integers $k, e_1, \ldots, e_k \geqslant 1$, and set $J = (X_1^{e_1}, \ldots, X_k^{e_k})R$. Let I be an ideal of R such that $J \subsetneq I$. Then there is a polynomial $h_k = z\widehat{h}(X_{k+1}, \ldots, X_d)$ in $I \smallsetminus J$ where $z = X_1^{e_1-1} \cdots X_k^{e_k-1}$.*

Proof. Fix a polynomial $h \in I \smallsetminus J$. Then we have

$$h = \sum_{\underline{n} \in \gamma(h)} a_{\underline{n}} \underline{X}^{\underline{n}} = \sum_{\substack{\underline{n} \in \gamma(h) \\ \underline{n} \notin \Gamma(J)}} a_{\underline{n}} \underline{X}^{\underline{n}} + \sum_{\substack{\underline{n} \in \gamma(h) \\ \underline{n} \in \Gamma(J)}} a_{\underline{n}} \underline{X}^{\underline{n}} = f + g$$

where

$$f = \sum_{\substack{\underline{n}\in\gamma(h)\\ \underline{n}\notin\Gamma(J)}} a_{\underline{n}}\underline{X}^{\underline{n}} \qquad \text{and} \qquad g = \sum_{\substack{\underline{n}\in\gamma(h)\\ \underline{n}\in\Gamma(J)}} a_{\underline{n}}\underline{X}^{\underline{n}}.$$

By construction, every monomial occuring in g is in J, so $g \in J$. On the other hand, since $h \notin J$ and $g \in J$, we have $f = h - g \notin J$. In particular, we have $f \neq 0$. Also, we have $h \in I$ by assumption, so the condition $g \in J \subseteq I$ implies that $f = h - g \in I$.

Furthermore, we have

$$\gamma(f) = \{\underline{n} \in \gamma(h) \mid \underline{n} \notin \Gamma(J)\}$$

so for each $\underline{n} \in \gamma(f)$ and for $i = 1,\ldots,k$, we have $n_i < e_i$. Indeed, if $n_i \geqslant e_i$, then $\underline{X}^{\underline{n}} \in (X_i^{e_i})R \subseteq J$, contradicting the condition $\underline{n} \notin \Gamma(J)$.

In a sense, the existence of f is stronger than the existence of h. Not only is f in I and not in J, but also no monomial occurring in f is in J. For this reason, we turn our attention from h to f.

Claim: For $j = 1,\ldots,k$ there exists a polynomial $h_j \in I \smallsetminus J$ such that for each $\underline{m} \in \gamma(h_j)$, we have $m_i = e_i - 1$ when $1 \leqslant i \leqslant j$, and we have $m_i \leqslant e_i - 1$ when $j+1 \leqslant i \leqslant k$. We prove the claim by induction on j.

Base case: $j = 1$. Consider the smallest power of X_1 appearing in the monomials of f, which is

$$r_1 = \min\{n_1 \in \mathbb{N} \mid \underline{n} \in \gamma(f)\}.$$

It follows that $r_1 < e_1$ since, if not, then every monomial occurring in f would be in $(X_1^{e_1})R \subseteq J$; this would imply that $f \in J$, a contradiction. This implies that $e_1 - r_1 > 0$, that is, that $e_1 - r_1 \geqslant 1$.

Write

$$f = \sum_{\substack{\underline{n}\in\gamma(f)\\ n_1=r_1}} a_{\underline{n}}\underline{X}^{\underline{n}} + \sum_{\substack{\underline{n}\in\gamma(f)\\ n_1>r_1}} a_{\underline{n}}\underline{X}^{\underline{n}} = f_1 + g_1$$

where

$$f_1 = \sum_{\substack{\underline{n}\in\gamma(f)\\ n_1=r_1}} a_{\underline{n}}\underline{X}^{\underline{n}} \qquad \text{and} \qquad g = \sum_{\substack{\underline{n}\in\gamma(f)\\ n_1>r_1}} a_{\underline{n}}\underline{X}^{\underline{n}}$$

and note that

$$\gamma(f_1) = \{\underline{n} \in \gamma(f) \mid n_1 = r_1\} \neq \emptyset$$
$$\gamma(g_1) = \{\underline{n} \in \gamma(f) \mid n_1 \geqslant r_1 + 1\}.$$

We set $h_1 = X_1^{e_1-r_1-1} f_1 \neq 0$.

To show that h_1 has the desired properties, we first show that $X_1^{e_1-r_1-1} g_1 \in J \subseteq I$. For each d-tuple $\underline{n} \in \gamma(g_1)$ we have $n_1 \geqslant r_1 + 1$. By construction, we have

$$X_1^{e_1-r_1-1} g_1 = X_1^{e_1-r_1-1} \sum_{\underline{n}\in\gamma(g_1)} a_{\underline{n}}\underline{X}^{\underline{n}} = \sum_{\underline{n}\in\gamma(g_1)} a_{\underline{n}} X_1^{e_1-r_1-1}\underline{X}^{\underline{n}}.$$

It follows that every d-tuple $\underline{m} \in \gamma(X_1^{e_1-r_1-1} g_1)$ satisfies

$$m_1 = (e_1 - r_1 - 1) + n_1 \geqslant (e_1 - r_1 - 1) + r_1 + 1 = e_1$$

so $X_1^{e_1 - r_1 - 1} g_1 \in (X_1^{e_1})R \subseteq J$.

Since $f \in I$, it follows that

$$h_1 = X_1^{e_1 - r_1 - 1} f_1 = X_1^{e_1 - r_1 - 1} f - X_1^{e_1 - r_1 - 1} g_1 \in I.$$

Each monomial occurring in h_1 has the form $\underline{X}^{\underline{m}} = X_1^{e_1 - r_1 - 1} \underline{X}^{\underline{n}}$ for some $\underline{n} \in \gamma(f_1)$. The condition $\underline{n} \in \gamma(f_1)$ implies that $n_1 = r_1$ so

$$m_1 = e_1 - r_1 - 1 + n_1 = e_1 - r_1 + r_1 - 1 = e_1 - 1.$$

Furthermore, for $i \geqslant 2$, the condition $\underline{n} \in \gamma(f_1) \subseteq \gamma(f)$ implies that $m_i = n_i < e_i$, that is, $m_i = n_i \leqslant e_i - 1$.

To complete the proof of the base case, we need to show that $h_1 \notin J$. Since J is a monomial ideal, it suffices to show that no monomial $\underline{X}^{\underline{m}}$ occurring in h_1 is in J; see Lemma 1.1.10. Again write $\underline{X}^{\underline{m}} = X_1^{e_1 - r_1 - 1} \underline{X}^{\underline{n}}$ for some $\underline{n} \in \gamma(f_1)$, and suppose that $\underline{X}^{\underline{m}} \in J = (X_1^{e_1}, \ldots, X_k^{e_k})R$. Theorem 1.1.9 implies $\underline{X}^{\underline{m}} \in (X_i^{e_i})R$ for some $i \leqslant k$, so $m_i \geqslant e_i$ by Lemma 1.1.7. This contradicts the condition $m_i \leqslant e_i - 1$, which has already been shown for each i.

This completes the proof of the base case of our induction. The induction step is left as an exercise. This step is quite similar to the base case. Here are some hints. Assume that $1 \leqslant j \leqslant d - 1$ and that h_j has been constructed. We want to construct h_{j+1}. Set

$$r_{j+1} = \min\{n_{j+1} \in \mathbb{N} \mid \underline{n} \in \gamma(h_j)\}$$

and show that $r_{j+1} \leqslant e_{j+1} - 1$. Then write

$$h_j = \sum_{\substack{\underline{n} \in \gamma(h_n) \\ n_{j+1} = r_{j+1}}} a_{\underline{n}} \underline{X}^{\underline{n}} + \sum_{\substack{\underline{n} \in \gamma(h_n) \\ n_{j+1} > r_{j+1}}} a_{\underline{n}} \underline{X}^{\underline{n}} = f_{j+1} + g_{j+1}.$$

Set $h_{j+1} = X_{j+1}^{e_{j+1} - r_{j+1} - 1} f_{j+1}$ and show that h_{j+1} has the desired properties.

It follows that there is a polynomial $h_k \in I \smallsetminus J$ such that for each $\underline{n} \in \gamma(h_k)$, we have $n_i = e_i - 1$ when $1 \leqslant i \leqslant k$. In other words, every monomial occurring in h_k has the form zw where w is a monomial in $X_{k+1}, \ldots, X_d$. This implies that there is a polynomial $\widehat{h}(X_{k+1}, \ldots, X_d)$ such that $h_k = z\widehat{h}(X_{k+1}, \ldots, X_d)$, as desired. $\square$

The next theorem is the main result of this section. Note that Exercises 3.2.5, 3.2.9, and 3.2.10 show why the field assumption is essential. Also, it is important to observe that not every irreducible ideal in a polynomial ring over a field is a monomial ideal. For instance, let k be a field and let R denote the polynomial ring $R = k[X]$ in one variable. Then the ideal $(X+1)R$ is irreducible; see Exercise 3.2.7.

Theorem 3.2.4 *Let A be a field, and set $R = A[X_1, \ldots, X_d]$. A non-zero monomial ideal $J \subseteq R$ is irreducible if and only if it is m-irreducible.*

Proof. The forward implication is straightforward: if J cannot be written as a non-trivial intersection of any two ideals, it cannot be written as a non-trivial intersection of two monomial ideals.

For the converse, assume that J is m-irreducible. Theorem 3.1.3, yields positive integers $k, t_1, \ldots, t_k, e_1, \ldots, e_k$ such that $1 \leqslant t_1 < \cdots < t_k \leqslant d$ such that $J = (X_{t_1}^{e_1}, \ldots, X_{t_k}^{e_k})R$. By re-ordering the variables if necessary, we may assume without loss of generality that $J = (X_1^{e_1}, \ldots, X_k^{e_k})R$. Set $z = X_1^{e_1-1} \cdots X_n^{e_n-1}$, and note that Exercise 3.1.8 implies that $z \notin J$.

By way of contradiction, suppose that there are ideals $I, K \subseteq R$ such that $J = I \cap K$ and $J \subsetneq I$ and $J \subsetneq K$. Lemma 3.2.3 provides polynomials

$$f_k = z\widehat{f}(X_{k+1}, \ldots, X_d) \in I \smallsetminus J$$
$$g_k = z\widehat{g}(X_{k+1}, \ldots, X_d) \in K \smallsetminus J.$$

Write $\widehat{f} = \widehat{f}(X_{k+1}, \ldots, X_d)$ and $\widehat{g} = \widehat{g}(X_{k+1}, \ldots, X_d)$. Since $f_k \in I$, we have $z\widehat{f}\widehat{g} = f_k\widehat{g} \in I$. Similarly, the condition $g_k \in K$ implies that $z\widehat{f}\widehat{g} \in K$, hence $z\widehat{f}\widehat{g} \in I \cap K = J$.

Because of the conditions $\widehat{f} = \widehat{f}(X_{k+1}, \ldots, X_d)$ and $\widehat{g} = \widehat{g}(X_{k+1}, \ldots, X_d)$, every monomial w occurring in $z\widehat{f}\widehat{g}$ has the form

$$w = zv = X_1^{e_1-1} \cdots X_n^{e_n-1} X_{n+1}^{m_{n+1}} \cdots X_d^{m_d}.$$

As J is a monomial ideal, every monomial occurring in $z\widehat{f}\widehat{g}$ is in J, by Lemma 1.1.10. The condition $w \in J$ implies that there is an index j such that $1 \leqslant j \leqslant k$ and $X_j^{e_j} \big| w$. By comparing exponent vectors, we deduce that $e_j \leqslant e_j - 1$, which is impossible.

We conclude that the polynomial $z\widehat{f}\widehat{g}$ does not have any monomials, that is, we have $z\widehat{f}\widehat{g} = 0$. Since A is a field and z is a monomial, it follows that either $\widehat{f} = 0$ or $\widehat{g} = 0$. If $\widehat{f} = 0$, then $0 = z\widehat{f} = f_k \notin J$, which is impossible. A similar contradiction arises if $\widehat{g} = 0$. Thus, the ideal J is irreducible, as desired. □

Exercises

Exercise 3.2.5. Set $R = A[X_1, \ldots, X_d]$.

(a) Prove that the ideal 0 is irreducible in R if and only if it is irreducible in A.
(b) Prove that if A is a field, then 0 is irreducible in R.

**Exercise 3.2.6.* Prove that the non-zero irreducible ideals in $\mathbb{Z}$ are exactly the ideals of the form $p^n\mathbb{Z}$ where p is prime and $n \geqslant 1$. (This is used in Example 3.4.2.)

**Exercise 3.2.7.* Let A be a field.

(a) Set $R = A[X]$, and let $a \in A$. Use a change of variables with Theorem 3.2.4 to show that for all $n \geqslant 1$ the ideal $(X-a)^nR$ is irreducible.
(b) Set $R = A[X,Y]$. Show that $(X^2, X+Y)R$ and $(X, (X+Y)^2)R$ are irreducible.

(This exercise is used in Examples 3.4.3 and 3.4.5.)

Exercise 3.2.8. Let $I \subseteq A$ be an ideal.

(a) Assume that I has the following property: there exists an element $f \in R$ such that f is not in I, but $f \in J$ for every ideal J of R that properly contains I. Prove that I is irreducible.
(b) Does the converse of part (a) hold? That is, if I is irreducible, must there exist an element $f \in R$ such that f is not in I, but $f \in J$ for every ideal J of R that properly contains I? Justify your answer.

Exercise 3.2.9. Find an example of a commutative ring A with identity such that the ideal $(X_1,\dots,X_d)R$ in the polynomial ring $R = A[X_1,\dots,X_d]$ is reducible.

Exercise 3.2.10. Set $R = A[X_1,\dots,X_d]$. Prove that the next conditions are equivalent:

(i) the ideal 0 is irreducible in A;
(ii) R has an irreducible monomial ideal; and
(iii) every non-zero m-irreducible monomial ideal $J \subsetneq R$ is irreducible.

Exercise 3.2.11. Let J be a non-zero irreducible ideal in A. Prove or disprove the following: If $\{I_\lambda\}_{\lambda\in\Lambda}$ is a (possibly infinite) set of ideals in A such that $J = \bigcap_{\lambda\in\Lambda} I_\lambda$, then there is an index $\lambda \in \Lambda$ such that $J = I_\lambda$. Justify your answer.

Exercise 3.2.12. Set $R = \mathbb{Q}[X,Y,Z]$. Set $J = (X^4,Y^5,Z^3)R$ and

$$h = 2X^2Y + 3X^2YZ^2 + 5X^2Y^2 + 2X^2Y^4Z + X^3Y^2 + 9X^4Y^2 + 6XY^4Z^4 + 11Y^3Z^5 + Y^6$$

and suppose that I is an ideal of R that contains h. Work through the proof of Lemma 3.2.3 in this special case by completing the following steps.

(a) Prove that $h \notin J$.
(b) List the elements in the set $\gamma(h)$.
(c) What is $a_{(4,2,0)}$?
(d) Find f and g.
(e) Find r_1. Is $r_1 \leqslant 3$?
(f) Find f_1 and g_1.
(g) Find h_1. Does h_1 have the property that X appears with exponent 3 in each monomial of h_1, while Y appears with exponent at most 4 and Z appears with exponent at most 2?
(h) Find r_2. Is $r_2 \leqslant 4$?
(i) Find f_2 and g_2.
(j) Find h_2. Does h_2 have the property that X appears with exponent 3 and Y appears with exponent 4 in each monomial, while Z appears with exponent at most 2?
(k) Find r_3. Is $r_3 \leqslant 2$?
(l) Find f_3 and g_3.
(m) Find h_3. Does h_3 have the property that X appears with exponent 3 in each monomial, while Y appears with exponent 4 and Z appears with exponent 2?

(n) What is $\widehat{h}$? What is z? Is it true that $h_3 = z\widehat{h}$?

Exercise 3.2.13. Complete the proof of Lemma 3.2.3 by establishing the induction step of the Claim.

Exercise 3.2.14. Prove that the next conditions on an ideal $J \subseteq A$ are equivalent:

(i) J is prime (see Exercise A.5.10);
(ii) $J \neq A$ and for all ideals I and K in A if $IK \subseteq J$, then either $I \subseteq J$ or $K \subseteq J$; and
(iii) J is irreducible and $\operatorname{rad}(J) = J$.

In the language of algebraic geometry, the equivalence of conditions (i) and (iii) says that an "algebraic set" is "integral" if and only if it is "irreducible and reduced".

(Here is an outline for the proof of (iii) $\Longrightarrow$ (i). Assume that J is irreducible and $\operatorname{rad}(J) = J$, and suppose that J is not prime. Then there exist $f, g \in A \smallsetminus J$ such that $fg \in J$. Set $I = J + fA$ and $K = J + gA$. The containments $IK \subseteq J \subseteq I \cap K$ imply that

$$J = \operatorname{rad}(J) = \operatorname{rad}(I \cap K) = \operatorname{rad}(I) \cap \operatorname{rad}(K).$$

Since J is irreducible, we have $J = \operatorname{rad}(I)$ or $J = \operatorname{rad}(K)$. The fact that $f \in I \subseteq \operatorname{rad}(I)$ and $g \in K \subseteq \operatorname{rad}(K)$ yields a contradiction.)

**Exercise 3.2.15.* In the ring $\mathrm{C}(\mathbb{R})$ of continuous functions, prove that the ideal $I_r = \{f \in \mathrm{C}(\mathbb{R}) \mid f(r) = 0\}$ is irreducible. (Hint: Exercises 3.2.14 and 1.2.15.) (This exercise is used in Example 3.4.4.)

**Exercise 3.2.16.* (For students familiar with integral domains; see Section 1.2 for background.) Prove that the conclusions of following results hold when A is only assumed to be an integral domain: Theorem 3.2.4 and Exercises 3.2.5(b) and 3.2.7. (This exercise is used in Exercise 3.4.15.)

Hash Tables in Macaulay2

In this Macaulay2 subsection, we will explore a data structure that makes keeping track of computational data in Macaulay2 much easier, called a hash table. This is a set of ordered pairs (k, v) where k is called the *key* and v is called the *value*, such that there is at most one pair (k, v) with a given key k. In other words, we can think of the pairs in a hash table as defining a function whose domain is the set of all keys, and whose range is the set of all values. Usually, we think of the key as an attribute of a larger whole, and the value associated to that key as the value of that attribute. Many data types in Macaulay2 have the `HashTable` type as an ancestor, so having a basic understanding of how they are used is helpful for peeking behind the Macaulay2 curtain. See the tutorial in Section 5.4 for an example of this.

Let's create a `HashTable` object, and see how to use it to store information.

```
i1 : ht = hashTable {(1,2),(3,4),(foo, bar),(color,red)}
o1 = HashTable{1 => 2      }
               3 => 4
               color => red
               foo => bar
o1 : HashTable
```

This hash table has keys `1`, `3`, `foo`, and `color`, with associated values `2`, `4`, `bar`, and `red` respectively. We can access this information as follows.

```
i2 : ht#1
o2 = 2

i3 : ht#foo
o3 = bar
o3 : Symbol
```

Oftentimes, we do not know if a key is present or not. If you try to access the value associated to a key that is not present, an error is given.

```
i4 : ht#5
stdio:39:3:(3): error: key not found in hash table
```

To avoid this, we can ask Macaulay2 if a key is present with the next syntax.

```
i4 : ht#?length
o4 = false

i5 : ht#?color
o5 = true
```

We can also ask for the list of the keys, the values, or the underlying list of pairs of a hash table, as follows.

```
i6 : keys ht
o6 = {color, 1, 3, foo}
o6 : List

i7 : values ht
o7 = {red, 2, 4, bar}
o7 : List

i8 : pairs ht
o8 = {(color, red), (1, 2), (3, 4), (foo, bar)}
o8 : List
```

It is worth noting that the order of the pairs that make up the hash table is not preserved; indeed, this is one factor that makes hash tables so efficient to use.

One common use of hash tables is to store the terms of a polynomial, where the keys will be the exponent vectors of the monomials, and the values will be the corresponding coefficients. Let's create a `HashTable` object with this information for a random polynomial. First, we generate a random[1] homogeneous polynomial of degree 3 using the command `random (3,R)`.

```
i9 : R = ZZ[x,y];

i10 : setRandomSeed(12); f = random (3,R)
        3       2     3
o11 = 8x  + 5x*y  + 7y
o11 : R
```

To get the terms of f, we use the command `terms`.

```
i12 : terms f
         3      2    3
o12 = {8x , 5x*y , 7y }
o12 : List
```

Now to each of these elements, we want to generate a pair whose first element is the exponent vector of the monomial, and whose second element is the coefficient. We do this using the `apply` command, coupled with the `exponents` and `leadCoefficient` commands.

```
i13 : coeffHash = hashTable apply(terms f, t ->
(first exponents t, leadCoefficient t))
o13 = HashTable{{0, 3} => 7}
                {1, 2} => 5
                {3, 0} => 8
o13 : HashTable
```

We can now use `#?` to obtain the coefficients from the exponent vectors using an `if-then-else` statement.

```
i14 : if coeffHash#?{1,2} then coeffHash#{1,2} else 0_R
o14 = 5

i15 : if coeffHash#?{2,1} then coeffHash#{2,1} else 0_R
o15 = 0
o15 : R
```

In the first case, the exponent vector $(1,2)$ does occur as a key in the hash table `coeffHash` because the monomial xy^2 does occur in f, so our command returns the coefficient 5. In the second case, the exponent vector $(2,1)$ does not occur as a key in the hash table `coeffHash` because the monomial x^2y does not occur in f, so our command returns the coefficient 0.

On occasion, one may want to create a hash table where the set of key/value pairs can be changed. For this, one must use a `MutableHashTable` object; see the documentation of `MutableHashTable` for details.

---

[1] Since we wish to comment on some of the properties of f below, we can't allow it to be too random, hence our use of the command `setRandomSeed`. This command sets the starting point for Macaulay2's random number generator and is useful for debugging code with random input parameters. See the Macaulay2 documentation for details about this command.

```
i16 : exit
```

Exercises

Exercise 3.2.17. Use Macaulay2 to verify the example you devise for Exercise 3.2.9.

Exercise 3.2.18. Let R and h be as in Exercise 3.2.12. Use the code above to create a `HashTable` object with keys given by the exponent vectors of the monomials in h, and values their corresponding coefficients. Use this object to find the coefficient of X^4Y^2 in h.

3.3 M-Irreducible Decompositions

In this section, A is a non-zero commutative ring with identity.

Section 3.1 characterizes the monomial ideals that cannot be decomposed as non-trivial intersections of two monomial ideals. The next step is to show that every monomial ideal can be decomposed in terms of these ideals. This is the first main goal of this text, which is accomplished in Theorem 3.3.3 below.

Definition 3.3.1. Set $R = A[X_1,\ldots,X_d]$. Let $J \subsetneq R$ be a monomial ideal. An *m-irreducible decomposition* of J is an expression $J = \bigcap_{i=1}^n J_i$ with $n \geqslant 1$, where each J_i is m-irreducible.

For instance, any m-irreducible ideal J in $R = A[X_1,\ldots,X_d]$ has a trivial m-irreducible decomposition $J = \bigcap_{i=1}^1 J_i$ with $J_1 = J$.

Example 3.3.2. Set $R = A[X,Y]$. An m-irreducible decomposition of the monomial ideal $J = (X^3, X^2Y, Y^3)R$ is

$$J = (X^2,Y^3)R \cap (X^3,Y)R.$$

See Example 3.1.2 and Theorem 3.1.3.

The next result accomplishes the first main goal of this text, by showing that every monomial ideal admits an m-irreducible decomposition. The proof is essentially due to Emmy Noether. Afterward, we discuss conditions guaranteeing that such decompositions are unique.

Theorem 3.3.3 *Set $R = A[X_1,\ldots,X_d]$. Every monomial ideal $J \subsetneq R$ has an m-irreducible decomposition.*

Proof. If $J = 0$, then J is m-irreducible by Exercise 3.1.5, so it has a trivial m-irreducible decomposition.

Suppose that there is a non-zero monomial ideal $J \subsetneq R$ that does not have an m-irreducible decomposition. Then the set Σ of all non-zero monomial ideals $J \subsetneq R$ that do not have an m-irreducible decompositions is a non-empty set of monomial ideals of R. Theorem 1.3.3(b) implies that Σ has a maximal element J. In particular J is not m-irreducible, so there exist monomial ideals $I, K \subseteq R$ such that $J = I \cap K$ and $J \subsetneq I, K$. In particular, we have $0 \neq I \neq R$ and $0 \neq K \neq R$. Since J is maximal in Σ, we have $I, K \notin \Sigma$. Hence, these ideals have m-irreducible decompositions $I = \bigcap_{j=1}^{m} I_j$ and $K = \bigcap_{i=1}^{n} K_i$. It follows that

$$J = I \cap K = \left(\bigcap_{j=1}^{m} I_j\right) \cap \left(\bigcap_{i=1}^{n} K_i\right)$$

so J has an m-irreducible decomposition, a contradiction. $\square$

According to our definitions, the converse of this result holds as well: if J is an ideal of $R = A[X_1, \ldots, X_d]$ with an m-irreducible decomposition, then J is a monomial ideal such that $J \subseteq R$. Indeed, assume that $J = \bigcap_{i=1}^{n} J_i$ with $n \geqslant 1$, where each J_i is m-irreducible. Theorem 2.1.1 implies that J is a monomial ideal since each J_i is so. Also, we have $J_i \subsetneq R$ for each i, so the condition $n \geqslant 1$ implies that $J \subsetneq R$.

One can say that the trivial ideal R has a "degenerate" m-irreducible decomposition $R = \bigcap_{i=1}^{n} J_i$ with $n = 0$, since the empty intersection is defined to be R. However, we avoid this degenerate case in general, to keep from having to deal with trivial cases separately in our proofs.

As with monomial generating sequences, we are interested in finding and understanding m-irreducible decompositions that are as efficient as possible. An added benefit of such decompositions is that they are unique, as we show below.

Definition 3.3.4. Set $R = A[X_1, \ldots, X_d]$. Let $J \subsetneq R$ be a monomial ideal. An m-irreducible decomposition $J = \bigcap_{i=1}^{n} J_i$ is *redundant* if there is an index j such that $J = \bigcap_{i \neq j} J_i$, where the intersection is taken over $i = 1, \ldots, n$ such that $i \neq j$. An m-irreducible decomposition $J = \bigcap_{i=1}^{n} J_i$ is *irredundant* if it is not redundant, that is if for all indices j one has $J \neq \bigcap_{i \neq j} J_i$. As we have $J = \bigcap_{i=1}^{n} J_i \subseteq \bigcap_{i \neq j} J_i$ automatically, the given decomposition is irredundant if and only if for all j one has $J \subsetneq \bigcap_{i \neq j} J_i$.

For example, consider the monomial ideal $J = (X^3, X^2Y, Y^3)R$ in $R = A[X, Y]$. The m-irreducible decomposition of J from Example 3.3.2

$$J = (X^2, Y^3)R \cap (X^3, Y)R$$

is irredundant. Indeed, we have $X^2 \in (X^2, Y^3)R \smallsetminus J$, so $J \neq (X^2, Y^3)R$. Also, we have $Y \in (X^3, Y)R \smallsetminus J$, so $J \neq (X^3, Y)R$.

On the other hand, the containment $J \subseteq (X, Y)R$ yields the next decomposition

$$J = (X^2, Y^3)R \cap (X^3, Y)R \cap (X, Y)R$$

which is redundant because, by assumption, we have

$$(X^2,Y^3)R\cap(X^3,Y)R\cap(X,Y)R = J = (X^2,Y^3)R\cap(X^3,Y)R.$$

This shows that m-irreducible decompositions are not unique in general. However, we show below that *irredundant* m-irreducible decompositions are unique. First, we give a handy characterization of redundant m-irreducible decompositions.

Proposition 3.3.5 *Set $R = A[X_1,\ldots,X_d]$. Let J be a monomial ideal in R with m-irreducible decomposition $J = \bigcap_{i=1}^n J_i$. Then the next conditions are equivalent:*

(i) *the decomposition $J = \bigcap_{i=1}^n J_i$ is redundant; and*
(ii) *there are indices $j \neq j'$ such that $J_{j'} \subseteq J_j$.*

Proof. (i) $\Longrightarrow$ (ii) Assume that the decomposition $J = \bigcap_{i=1}^n J_i$ is redundant. (This implies that $n \geqslant 2$.) Then there is an index j such that $\bigcap_{i\neq j} J_i = J = \bigcap_{i=1}^n J_i \subseteq J_j$. Lemma 3.1.4 gives an index j' such that $J_{j'} \subseteq J_j$, as desired.

(ii) $\Longrightarrow$ (i) Assume that there are indices $j \neq j'$ such that $J_{j'} \subseteq J_j$. This explains the second step in the next display.

$$J = \bigcap_{i=1}^n J_i \subseteq \bigcap_{i\neq j'} J_i \subseteq \bigcap_{i=1}^n J_i = J$$

The other steps are by assumption and basic properties of intersections. It follows that we have $J = \bigcap_{i\neq j'} J_i$, so the intersection is redundant. $\square$

Using criterion (ii) from the last result, we next show that every m-irreducible decomposition can be turned into an irredundant one by removing redundancies.

Algorithm 3.3.6. Set $R = A[X_1,\ldots,X_d]$. Let J be a monomial ideal with m-irreducible decomposition $J = \bigcap_{i=1}^n J_i$. Note that $n \geqslant 1$.

Step 1. Check whether the intersection $J = \bigcap_{i=1}^n J_i$ is irredundant, using Propositions 1.3.5 and 3.3.5.

Step 1a. If, for all indices j and j' such that $j \neq j'$, we have $J_j \not\subseteq J_{j'}$, then the intersection is irredundant; in this case, the algorithm terminates.

Step 1b. If there exist indices j and j' such that $j \neq j'$ and $J_j \subseteq J_{j'}$, then the intersection is redundant; in this case, continue to Step 2.

Step 2. Remove an ideal that causes a redundancy in the intersection. By assumption, there exist indices j and j' such that $j \neq j'$ and $J_j \subseteq J_{j'}$. Re-order the indices to assume without loss of generality that $j' = n$. Thus, we have $j < n$ and $J_j \subseteq J_n$. It follows that $J = \bigcap_{i=1}^n J_i = \bigcap_{i=1}^{n-1} J_i$.

Step 3. Apply Step 1 to the new decomposition $J = \bigcap_{i=1}^{n-1} J_i$.

The algorithm will terminate in at most $n-1$ steps because one can remove at most $n-1$ ideals from the list and still form an ideal that is a non-empty intersection of m-irreducible ideals.

Corollary 3.3.7 *Set $R = A[X_1,\ldots,X_d]$. Every monomial ideal $J \subsetneq R$ has an irredundant m-irreducible decomposition.*

Proof. This follows from Theorem 3.3.3 and Algorithm 3.3.6. □

The next result shows that irredundant m-irreducible decompositions are unique up to re-ordering the terms. Given the similarities between m-irreducible decompositions of monomial ideals and prime factorizations of integers, this result compares to the uniqueness theorem for prime factorizations. (See Section 4.7 for a rigorous explanation of this comparison in a special case.)

Theorem 3.3.8 *Set $R = A[X_1,\ldots,X_d]$. Let J be a monomial ideal in R with irredundant m-irreducible decompositions $J = \bigcap_{i=1}^n J_i = \bigcap_{h=1}^m I_h$. Then $m = n$ and there is a permutation $\sigma \in S_n$ such that $J_t = I_{\sigma(t)}$ for $t = 1,\ldots,n$.*

Proof. Claim: For $t = 1,\ldots,n$ there is a unique index u such that $I_u = J_t$. To show this, we compute:

$$\bigcap_{h=1}^m I_h = J = \bigcap_{i=1}^n J_i \subseteq J_t.$$

Lemma 3.1.4 implies that there is an index u such that $I_u \subseteq J_t$. Similarly, we have

$$\bigcap_{i=1}^n J_i = J = \bigcap_{h=1}^m I_h \subseteq I_u$$

so Lemma 3.1.4 implies that there is an index v such that $J_v \subseteq I_u \subseteq J_t$. Since the decomposition $\bigcap_{i=1}^n J_i$ is irredundant, the containment $J_v \subseteq J_t$ implies that $v = t$, so we have $J_t \subseteq I_u \subseteq J_t$, that is $I_u = J_t$. Note that u is unique: if $I_u = J_t = I_{u'}$, then the irredundancy of the intersection $\bigcap_{h=1}^m I_h$ implies that $u = u'$.

Define the function $\sigma\colon \{1,\ldots,n\} \to \{1,\ldots,m\}$ by letting $\sigma(t)$ be the unique index u such that $I_u = J_t$.

Symmetrically, for $u = 1,\ldots,m$ there is a unique index t such that $I_u = J_t$. Define the function $\omega\colon \{1,\ldots,m\} \to \{1,\ldots,n\}$ by letting $\omega(u)$ be the unique index t such that $I_u = J_t$. By construction, the function ω is a two-sided inverse function for σ, hence the desired conclusions. □

Exercises

Exercise 3.3.9. Which of the following m-irreducible decompositions in $R = A[X,Y]$ are irredundant? For each redundant decomposition, use Algorithm 3.3.6 to generate an irredundant m-irreducible decomposition of the same ideal. Justify your answers.

(a) $(X,Y^4)R \cap (X^4,Y^2)R$
(b) $(X^2,Y^5)R \cap (Y)R \cap (X^4,Y^2)R$
(c) $(X,Y^4)R \cap (X^5,Y^3)R \cap (X^4,Y^2)R$

Exercise 3.3.10. Sketch the graphs of the following ideals in $R = A[X,Y]$. Use the graphs as in Example 2.1.7 to find irredundant m-irreducible decompositions of each ideal. Use Theorem 2.1.5 to check your decompositions.

(a) $(X^2, XY, Y^2)R$
(b) $(X^4, X^3Y^2, Y^5)R$
(c) $(X^a, XY, Y^b)R$ where $a, b \geqslant 2$
(d) $(X^4, X^3Y, XY^2, Y^5)R$

The next two exercises involve the constructions $\mathrm{V}(I)$ and $\mathrm{I}(Y)$ from the Exploration Section A.10.

**Exercise 3.3.11.* Let A be a field, and let I be a monomial ideal of $R = A[X_1,\ldots,X_d]$.

(a) Prove that there is an integer $k \in \mathbb{N}$, and there are linear subspaces $V_1,\ldots,V_k \subseteq A^d$ such that $\mathrm{V}(I) = V_1 \cup \cdots \cup V_k$. (See Exercise 3.1.13.)
(b) Use the idea behind Algorithm 3.3.6 to prove that the linear subspaces in part (a) can be chosen such that $V_i \not\subseteq V_j$ whenever $i \neq j$.
(c) Use part (a) to prove that $\mathrm{I}(\mathrm{V}(I)) = \mathrm{I}(V_1) \cap \cdots \cap \mathrm{I}(V_k) = \text{m-rad}\,(I)$.

(This is used in the following Exercises and Challenge Exercises: 3.3.12, 4.1.14, 4.3.19, 4.5.15, 4.5.16, 4.5.17, 4.6.14(e), 4.6.15, 5.1.8, 5.3.33, 6.1.14, 7.3.14, 7.4.22, 7.7.6, 7.9.7.)

Challenge Exercise 3.3.12. Let A be a field, and let I and J be monomial ideals of $R = A[X_1,\ldots,X_d]$. Exercise 3.3.11(b) shows that there is an integer $k \in \mathbb{N}$, and there are linear subspaces $V_1,\ldots,V_k \subseteq A^d$ such that $\mathrm{V}(I) = V_1 \cup \cdots \cup V_k$ and such that $V_i \not\subseteq V_j$ whenever $i \neq j$. This exercise also yields such irredundant subspace decompositions $\mathrm{V}(J) = W_1 \cup \cdots \cup W_l$ and $\mathrm{V}(I \cap J) = U_1 \cup \cdots \cup U_m$. Describe the linear subspaces U_n in terms of the subspaces V_i and W_j. Justify your answer.

M-Irreducible Decompositions in Macaulay2

In this section, let us implement Algorithm 3.3.6, which takes a m-irreducible decomposition of an ideal (in the form of a list) and returns an irredundant m-irreducible decomposition. First, let's set up an example.

```
i1 : R = QQ[x,y,z];

i2 : J1 = ideal {x^3,y^4,z^5};
o2 : Ideal of R

i3 : J2 = ideal {x^4,y^5,z^3};
o3 : Ideal of R

i4 : J3 = ideal {x^5,y^3,z^4};
o4 : Ideal of R
```

```
i5 : J4 = ideal {x^4,y^5,z^5};
o5 : Ideal of R

i6 : J5 = ideal {x^4,y^5,z^5};
o6 : Ideal of R

i7 : mDecomp = {J1,J2,J3,J4,J5};
```

We wish to write a method that takes a list of ideals such as `mIrredDecomp` as input, and returns an irredundant list of ideals as output. We will use the methods `select` and `positions` for this purpose. The method `select` returns the elements of a list that return true when given as input to a supplied boolean function. For example, we list the prime numbers less than or equal to 50.

```
i8 : select(toList(1..50), isPrime)
o8 = {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47}
o8 : List
```

The method `positions` has a similar behavior, except it returns the *positions* of the elements in the list that would have been chosen by `select`.

```
i9 : positions(1..50, isPrime)
o9 = {1, 2, 4, 6, 10, 12, 16, 18, 22, 28, 30, 36, 40, 42, 46}
o9 : List
```

We use these together in our new method.

```
i10 : makeIrredundant = method()
o10 = makeIrredundant
o10 : MethodFunction

i11 : makeIrredundant List := decomp -> (
uniqDecomp := unique(decomp / trim);
select(uniqDecomp, I -> #positions(uniqDecomp, J -> isSubset(J,I)) == 1)
)
o11 = {*Function[stdio:11:32-13:71]*}
o11 : FunctionClosure

i12 : mIrredDecomp = makeIrredundant mDecomp
                3   4   5           4   5   5
o12 = {ideal (y , z , x ), ideal (x , z , y )}
o12 : List

i13 : intersect mDecomp == intersect makeIrredundant mDecomp
o13 = true

i14 : exit
```

In the above code, we first `trim` each ideal to make sure that an irredundant monomial generating set for each ideal is chosen. This allows the `unique` command to remove any duplicates in the list. Then, those elements in the unique list are selected that do not contain any elements in the list other than themselves.

Exercises

Exercise 3.3.13. Set $R = \mathbb{Z}_{101}[X,Y]$, and use Macaulay2 to verify the decomposition in Example 3.3.2.

Exercise 3.3.14. Set $A = \mathbb{Z}_{101}$, and use Macaulay2 to verify your decompositions from Exercise 3.3.10, parts (a), (b), and (d).

Exercise 3.3.15. Set $R = \mathbb{Q}[X,Y]$, and use the above Macaulay2 code to verify your answers for Exercise 3.3.9.

Coding Exercise 3.3.16. In what ways is the code above not optimal? That is, do you see ways in which to improve it? Change the code to reflect your improvements. (Hint: Some of the features of `positions` may be helpful here. Discover these features using the documentation.)

Documentation Exercise 3.3.17. Explore the documentation for the Macaulay2 command `irreducibleDecomposition` which generates m-irreducible decompositions of monomial ideals. Use this command to verify your decompositions from Exercise 3.3.10, parts (a), (b), and (d).

3.4 Irreducible Decompositions (optional)

In this section, A is a non-zero commutative ring with identity.

This section treats decompositions of non-monomial ideals of A in terms of the irreducible ideals of Section 3.2. Such decompositions are guaranteed to exist when A is noetherian; see Section 1.4 and Theorem 3.4.6. We also explore irredundant decompositions in this context, which are not in general unique. This gives yet another indication of how special monomial ideals are.

Definition 3.4.1. Let $J \subsetneq A$ be an ideal. An *irreducible decomposition* of J is an expression $J = \bigcap_{i=1}^{n} J_i$ with $n \geqslant 1$, where each J_i is irreducible.

An irreducible decomposition $J = \bigcap_{i=1}^{n} J_i$ is *redundant* if if there exists an index i' such that $J = \bigcap_{i \neq i'} J_i$. An irreducible decomposition $J = \bigcap_{i=1}^{n} J_i$ is *irredundant* if it is not redundant, that is, if for all indices i' one has $J \neq \bigcap_{i \neq i'} J_i$.

Example 3.4.2. Let n be an integer with $n \geqslant 2$. The Fundamental Theorem of Arithmetic says that n has a factorization $n = p_1^{e_1} \cdots p_m^{e_m}$ where the p_i are distinct prime numbers and each e_i is a positive integer. It follows that $n\mathbb{Z} = p_1^{e_1}\mathbb{Z} \cap \cdots \cap p_m^{e_m}\mathbb{Z}$ is an irredundant irreducible decomposition; see Exercise 3.2.6.

Example 3.4.3. Next, let R be the polynomial ring $R = \mathbb{C}[X]$ in one variable, and let f be a non-constant polynomial in R. The Fundamental Theorem of Algebra says that f has a factorization $f = c(X-a_1)^{e_1} \cdots (X-a_m)^{e_m}$ where the a_i are distinct complex numbers, each e_i is a positive integer, and c is the leading coefficient of f. It follows that $fR = (X-a_1)^{e_1}R \cap \cdots \cap (X-a_m)^{e_m}R$ is an irredundant irreducible decomposition; see Exercise 3.2.7(a).

Example 3.4.4. In the ring $\mathrm{C}(\mathbb{R})$ of continuous functions, the ideal

$$I = \{f \in \mathrm{C}(\mathbb{R}) \mid f(n) = 0 \text{ for all } n \in \mathbb{N}\}$$

does not have an irreducible decomposition. For an indication of why this is true, note that $I = \bigcap_{n\in\mathbb{Z}} I_n$ where

$$I_n = \{f \in \mathrm{C}(\mathbb{R}) \mid f(n) = 0\}.$$

Each ideal I_n is irreducible by Exercise 3.2.15, but there is no finite sequence $n_1,\ldots,n_t$ such that $I = I_{n_1} \cap \cdots \cap I_{n_t}$.

Let k be a field, and set $A = k[X,Y]$. An irredundant irreducible decomposition of the ideal $J = (X^3, X^2Y, Y^3)A$ is

$$J = (X^2,Y^3)A \cap (X^3,Y)A.$$

See Example 3.1.2 and Theorem 3.2.4. On the other hand, the decomposition

$$J = (X^2,Y^3)A \cap (X^3,Y)A \cap (X,Y)A$$

is redundant because $J = (X^2,Y^3)A \cap (X^3,Y)A$. This shows that irreducible decompositions are not unique in general. The situation is even worse, however, as the next example shows that even irredundant irreducible decompositions are not unique in general. Contrast this with the situation of m-irreducible decompositions in Theorem 3.3.8. However, see Exercise 3.4.14 for a weak uniqueness statement.

Example 3.4.5. Let k be a field, and set $A = k[X,Y]$. The following ideals are irreducible and pair-wise distinct:

$$(X,Y^2)A \qquad (X^2,Y)A \qquad (X^2,X+Y)A \qquad (X,(X+Y)^2)A.$$

See Exercise 3.2.7(b). Furthermore, one has

$$(X,Y^2)A \cap (X^2,Y)A = (X^2,XY,Y^2)A = (X^2,X+Y)A \cap (X,(X+Y)^2)A.$$

The next decomposition result is akin to Theorem 3.3.3. Its proof is due to Emmy Noether. Note that Theorem 3.2.4 shows that, over a field, m-irreducible decompositions of monomial ideals are also irreducible decompositions.

Theorem 3.4.6 *If A is noetherian and $J \subsetneq A$ is a proper ideal, then J has an irreducible decomposition.*

Proof. Exercise. Argue as in the proof of Theorem 3.3.3. $\square$

The following procedure shows how to pare down an arbitrary irreducible decomposition to an irredundant one. It compares to Algorithm 3.3.6.

Algorithm 3.4.7. Let J be an ideal of A with irreducible decomposition $J = \bigcap_{i=1}^{n} J_i$. Note that $n \geqslant 1$.

Step 1. Check whether the intersection $J = \bigcap_{i=1}^{n} J_i$ is irredundant.
Step 1a. If, for all indices j, we have $J \neq \bigcap_{i \neq j} J_i$, then the intersection is irredundant; in this case, the algorithm terminates.
Step 1b. If there exists an index j such that $J = \bigcap_{i \neq j} J_i$, then the intersection is redundant; in this case, continue to Step 2.
Step 2. Remove an ideal that causes a redundancy in the intersection. By assumption, there exists an index j such that $J = \bigcap_{i \neq j} J_i$. Re-order the indices to assume without loss of generality that $j = n$. Thus, we have $J = \bigcap_{i=1}^{n-1} J_i$.
Step 3. Apply Step 1 to the new decomposition $J = \bigcap_{i=1}^{n-1} J_i$.

The algorithm will terminate in at most $n-1$ steps because one can remove at most $n-1$ ideals from the intersection and still form an ideal that is a non-empty intersection of irreducible ideals.

Corollary 3.4.8 *If A is noetherian, then every proper ideal in A has an irredundant irreducible decomposition.*

Proof. Theorem 3.4.6 and Algorithm 3.4.7. □

Corollary 3.4.9 *If k is a field, then every proper ideal in $k[X_1, \ldots, X_n]$ has an irredundant irreducible decomposition.*

Proof. Corollaries 1.4.6 and 3.4.8. □

The next result gives a criterion for detecting whether a given irreducible decomposition is redundant. It corresponds to one implication of Theorem 3.3.8.

Proposition 3.4.10 *Let J be an ideal in A with irreducible decomposition $J = \bigcap_{i=1}^{n} J_i$. If there are indices $j \neq j'$ such that $J_j \subseteq J_{j'}$, then the decomposition $J = \bigcap_{i=1}^{n} J_i$ is redundant.*

Proof. If there are indices $j \neq j'$ such that $J_j \subseteq J_{j'}$, then $J = \bigcap_{i \neq j'} J_i$. □

It is worth noting that the converse of the previous result fails in general. (Contrast this with the situation for monomial ideals from Theorem 3.3.8.) For instance, let k be a field, and set $A = k[X,Y]$. We consider the irreducible decompositions

$$(X,Y^2)A \cap (X^2,Y)A = (X^2,XY,Y^2)A = (X^2,X+Y)A \cap (X,(X+Y)^2)A$$

from Example 3.4.5. It follows that the decomposition

$$(X^2,XY,Y^2)A = (X,Y^2)A \cap (X^2,Y)A \cap (X^2,X+Y)A$$

is redundant; however, there are no containment relations between the three ideals in this decomposition. Thus, the converse of Proposition 3.4.10 fails in general.

Also, we have

$$(X,Y^2)A \cap (X^2,Y)A = (X^2,XY,Y^2)A \subseteq (X^2,X+Y)A$$

even though $(X,Y^2)A \not\subseteq (X^2,X+Y)A$ and $(X^2,Y)A \not\subseteq (X^2,X+Y)A$. This shows that the version of Lemma 3.1.4 fails in this setting.

Exercises

Exercise 3.4.11. Let k be an integral domain, e.g., a field, and set $A = k[X,Y]$.

(a) Use a change of variables to verify the equality

$$(X^2,X+Y)A \cap (X,(X+Y)^2)A = (X^2,XY,Y^2)A.$$

(b) Prove that there are no containment relations between the following ideals:

$$(X,Y^2)A \qquad (X^2,Y)A \qquad (X^2,X+Y)A \qquad (X,(X+Y)^2)A.$$

Exercise 3.4.12. Prove Theorem 3.4.6.

Exercise 3.4.13. Let J be an ideal in A with an irredundant irreducible decomposition $J = \bigcap_{i=1}^n J_i$.

(a) Assume that for $i = 1,\ldots,n$ one has $\operatorname{rad}(J_i) = J_i$. Prove that $J = \operatorname{rad}(J)$.
(b) Prove or disprove the following: If $J = \operatorname{rad}(J)$, then $\operatorname{rad}(J_i) = J_i$ for $i = 1,\ldots,n$. Justify your answer.

Exercise 3.4.14. Let J be an ideal in A with irredundant irreducible decompositions $J = \bigcap_{i=1}^n J_i = \bigcap_{h=1}^m I_h$. Then $m = n$ and there is a permutation $\sigma \in S_n$ such that $\operatorname{rad}(J_t) = \operatorname{rad}\left(I_{\sigma(t)}\right)$ for $t = 1,\ldots,n$. (Hint: Show that for $i = 1,\ldots,m$ there is an index j such that $J = J_1 \cap \cdots \cap J_{i-1} \cap I_j \cap J_{i+1} \cap \cdots \cap J_n$.)

Exercise 3.4.15. Let A be an integral domain, e.g., a field, and set $R = A[X_1,\ldots,X_d]$. Use Exercise 3.2.16 and Corollary 3.3.7 to prove that every non-zero monomial ideal in R has an irredundant irreducible decomposition.

Irreducible Decompositions in Macaulay2, Exercises

Some commands relevant to the next exercise appear in Sections 2.1 and B.3.

Exercise 3.4.16. Set $R = \mathbb{Z}_{101}[X,Y]$, and use Macaulay2 to verify the decompositions in Example 3.4.5 and the conclusion of Exercise 3.4.11(b).

3.5 Exploration: Decompositions in Two Variables I

In this section, A is a non-zero commutative ring with identity and $R = A[X,Y]$.

Exercise 3.1.7 shows how to compute m-irreducible decompositions in $A[X]$. Here we focus on the case of two variables, building from Example 3.3.2. The idea is to factor the generators of the form X^mY^n, one at a time, implicitly using the lex monomial order to track the generators; see Definition A.9.6 and Lemma 1.3.10. We begin with the case where the ideal has one such generator.

Exercise 3.5.1. Set $I = (X^a, X^bY^c, Y^d)R$ where $a > b \geqslant 1$ and $d > c \geqslant 1$.

(a) Prove that X^a, X^bY^c, Y^d is an irredundant monomial generating sequence for I.
(b) Prove that $I = (X^b,Y^d)R \cap (X^a,Y^c)R$ is an irredundant m-irreducible decomposition. (Hint: Use Theorem 2.1.5. It may be helpful to think of the decomposition as factoring the generator X^bY^c and removing the redundant generators, as in the following display:

$$(X^a,X^bY^c,Y^d)R = (X^a,X^b,Y^d)R \cap (X^a,Y^c,Y^d)R = (X^b,Y^d)R \cap (X^a,Y^c)R.$$

This perspective is not only useful for proving the result, but also for remembering and applying it.)
(c) Use part (b) to find an m-irreducible decomposition of $(X^6,X^5Y^4,Y^7)R$.

Next, we move on to the case where the ideal has two or three generators that are not pure powers. The point in each case is to reduce to the preceding case by factoring one of the generators. As you work through these, keep this pattern in mind for the subsequent induction exercise.

Exercise 3.5.2. Set $J = (X^a, X^bY^c, X^dY^e, Y^f)R$ where we have $a > b > d \geqslant 1$ and $f > e > c \geqslant 1$.

(a) Prove that X^a, X^bY^c, X^dY^e, Y^f is an irredundant monomial generating sequence for J.
(b) Prove that $J = (X^b, X^dY^e, Y^f)R \cap (X^a,Y^c)R$. As in the previous exercise, it may be helpful to think of this as factoring the generator X^bY^c and removing the redundant generators, as in the following display:

$$\begin{aligned}(X^a,X^bY^c,X^dY^e,Y^f)R &= (X^a,X^b,X^dY^e,Y^f)R \cap (X^a,Y^c,X^dY^e,Y^f)R \\ &= (X^b,X^dY^e,Y^f)R \cap (X^a,Y^c)R\end{aligned}$$

and similarly in the next item.
(c) Prove that the expression $J = (X^d,Y^f)R \cap (X^b,Y^e)R \cap (X^a,Y^c)R$ is an irredundant m-irreducible decomposition.
(d) Use part (c) to find an m-irreducible decomposition of $(X^6,X^5Y^4,X^3Y^7,Y^8)R$.

Exercise 3.5.3. Repeat Exercise 3.5.2(a)–(c) for the ideal

$$K = (X^{a_1}, X^{a_2}Y^{b_2}, X^{a_3}Y^{b_3}, X^{a_4}Y^{b_4}, Y^{b_5})R$$

where $a_1 > a_2 > a_3 > a_4 \geqslant 1$ and $b_5 > b_4 > b_3 > b_2 \geqslant 1$.

Next, we treat general monomial ideals containing powers of X and Y.

**Exercise 3.5.4.* Use induction to repeat Exercise 3.5.3 for the ideal

$$L = (X^{a_1}, X^{a_2}Y^{b_2}, X^{a_3}Y^{b_3}, \ldots, X^{a_{m-1}}Y^{b_{m-1}}, Y^{b_m})R$$

where $a_1 > a_2 > a_3 > \cdots > a_{m-1} \geqslant 1$ and $b_m > b_{m-1} > \cdots > b_3 > b_2 \geqslant 1$. (This exercise is used in Coding Exercise 3.5.10.)

The previous exercise is a general case in a sense: it is the general situation for monomial ideals in R that contain powers of X and Y. Next, we explore the remaining situations of ideals that don't contain such powers.

Challenge Exercise 3.5.5. Formulate similar decompositions for ideals of the form

$$(X^{a_2}Y^{b_2}, X^{a_3}Y^{b_3}, \ldots, X^{a_{m-1}}Y^{b_{m-1}}, Y^{b_m})R$$
$$(X^{a_1}, X^{a_2}Y^{b_2}, X^{a_3}Y^{b_3}, \ldots, X^{a_{m-1}}Y^{b_{m-1}})R$$
$$(X^{a_2}Y^{b_2}, X^{a_3}Y^{b_3}, \ldots, X^{a_{m-1}}Y^{b_{m-1}})R.$$

Justify your answers. (See Laboratory Exercise 3.5.9.)

Section 7.5 contains a version of the decomposition results of this section that works with more variables, though it usually yields redundant decompositions. The idea is to factor each generator step-by-step, removing redundancies along the way. As with the above exercises, the technique uses only results we have already proved (specifically, Algorithm 1.3.7, Theorem 2.1.5, and Algorithm 3.3.6); however, the general formulation is messy, and the proof is facilitated by some of the results from Chapter 7, so we put it of for now, satisfying ourselves for the moment with the next example which should get the point across.

Example 3.5.6. We decompose the ideal $I = (X^2Y^3, Y^2Z, XYZ)R$ as follows:

$$I = (X^2Y^3, Y^2Z, XYZ)R$$

split the last generator

$$= (X^2Y^3, Y^2Z, X)R \cap (X^2Y^3, Y^2Z, Y)R \cap (X^2Y^3, Y^2Z, Z)R$$

remove redundant generators

$$= (Y^2Z, X)R \cap (Y)R \cap (X^2Y^3, Z)R$$

split the first generators where necessary

$$= (Y^2,X)R \cap (Z,X)R \cap (Y)R \cap (X^2,Z)R \cap (Y^3,Z)R$$

remove the redundant ideal $(Z,X)R$

$$= (Y^2,X)R \cap (Y)R \cap (X^2,Z)R \cap (Y^3,Z)R.$$

Exercise 3.5.7. Verify the decomposition from Example 3.5.6 using Theorem 2.1.5.

Decompositions in Two Variables I in Macaulay2

In the formulas from the previous exercises, the systematic listing of the minimal monomial generators is the first step in providing a decomposition of the monomial ideal. The ordering of monomials used here is the lexicographic (or lex) ordering from Definition A.9.6. This is a monomial order as described in Definition A.9.5.

It is beneficial in applications to allow for the use of other monomial orders, and Macaulay2 has the capability to use many standard orders, as well as ones defined by the user. Let us look at an example.

```
i1 : R = QQ[x,y];

i2 : I = ideal (x^5,x^4*y^2,y^6)
                5   4 2   6
o2 = ideal (x , x y , y )
o2 : Ideal of R

i3 : sort I_*
           5   6   4 2
o3 = {x , y , x y }
o3 : List
```

As we discuss in Section B.9, and as you can see here, Macaulay2 used a different monomial order (the graded reverse lexicographic or grevlex order from Definition A.9.8(c)) to sort the monomial generators of I in this example. As in Section B.9, we remedy this by specifying the monomial order when creating the ring.

```
i4 : R = QQ[x,y,MonomialOrder=>Lex]
o4 = R
o4 : PolynomialRing

i5 : I = ideal (x^5,x^4*y^2,y^6)
                5   4 2   6
o5 = ideal (x , x y , y )
o5 : Ideal of R

i6 : sort I_*
           6   4 2   5
```

```
o6 = {y , x y , x }
o6 : List
```

It turns out that the `sort` function also sorts elements in increasing order, and if one would rather have it in decreasing order, `rsort` should be used.

```
i7 : rsort I_*
        5   4 2   6
o7 = {x , x y , y }
o7 : List
```

As a side note, one can also obtain the exponents of the above monomials, and sort the resulting exponent vectors.

```
i8 : rsort (I_* / exponents / first)
o8 = {{5, 0}, {4, 2}, {0, 6}}
o8 : List

i9 : exit
```

Macaulay2 uses a left-to-right ordering when sorting any ordered pair, in that the first item is compared, and to break ties, the second is compared.

Exercises

Exercise 3.5.8. Use the Macaulay2 command `irreducibleDecomposition` to verify the decompositions from Exercises 3.5.1(c) and 3.5.2(d) and Example 3.5.6.

Laboratory Exercise 3.5.9. If you are stuck on any of the decompositions from Challenge Exercise 3.5.5, use the command `irreducibleDecomposition` to investigate the decompositions of such ideals to help you formulate your conjectures.

Coding Exercise 3.5.10. Write a Macaulay2 method that performs the decomposition of a monomial ideal in two variables as in Exercise 3.5.4. Test your method on the ideals in Exercise 3.3.10, parts (a), (b), and (d). Compare run times between your method and the command `irreducibleDecomposition`. (One can obtain the run time of an operation in Macaulay2 by placing `time` before the command.)

Concluding Notes

As we have mentioned elsewhere, this chapter contains one of the main goals of this text: a proof of the existence of m-irreducible decompositions of monomial ideals and the characterization of m-irreducible monomial ideals. In principle, a reader at this point is prepared to start looking at some of the more advanced texts in this area, e.g., [9, 37, 38, 58, 74, 77]. These include much deeper treatments of certain topics, for instance, Cohen-Macaulay and Gorenstein properties and canonical modules of quotients by square-free monomial ideals, as well as other interactions with algebraic geometry, combinatorics, homological algebra, and topology. In many ways, these are *the* reasons for looking at monomial ideals in the first place. It is worth noting that at least two of these texts [9, 58] consider significantly more general settings than monomial ideals in polynomial rings.

On the other hand, the next part of this text treats some of these (and other) topics, in greater or lesser detail, depending on the topic. And the third part contains numerous algorithms for computing m-irreducible decompositions. So, readers should not feel any pressure to put this text away any time soon.

Part II
Monomial Ideals and Other Areas

In this part of the text, we accomplish the second main goal of this text by presenting some connections between monomial ideals and other areas of research. For many mathematicians, these connections are the main motivation for studying monomial ideals.

Chapter 4 describes some of the interactions between monomial ideals and combinatorics. The main players in this area are the square-free monomial ideals, which we introduce in Section 4.1. A special example of such an ideal is the edge ideal of a graph, which we introduce in Section 4.2. Section 4.3 is about the m-irreducible decompositions of these ideals, which are determined by the vertex covers of the graphs. Conversely, one can read the vertex covers directly from the decompositions. This is one of the important themes in this subject: one can get algebraic information about an edge ideal from the combinatorial information encoded in the graph, and one can get combinatorial information about the graph from algebraic information encoded in its edge ideal. Similarly, combinatorial properties of a simplicial complex are reflected in algebraic properties of its face ideal and facet ideal; these are discussed in Sections 4.4–4.6. This chapter ends with an exploration of algebraic Alexander duality for simplicial complexes.

The Macaulay2 material in this chapter continues where the previous chapter left off with the construction of methods. We also see here that Macaulay2 can represent combinatorial objects like graphs and simplicial complexes, in addition to the algebraic objects we have already seen. This uses packages which provide a lot more functionality for our computational work.

The subsequent Chapter 5 looks into some connections between monomial ideals and other areas of mathematics and beyond. It begins with Section 5.1 on Krull dimension, an important measure of the size of a ring. The chapter continues with some ties to electrical engineering in Section 5.2, and continues with links to topology in 5.3, non-monomial ideals in 5.4, and homological algebra in 5.5. This chapter is necessarily colloquial in nature, as the technical details are outside the scope of this text. However, the chapter's Concluding Notes give a number of references to the literature where the interested reader can find many more details.

Macaulay2 is particularly useful in this chapter, since one can use it to verify unproven results for specific examples. Similarly, one can use Macaulay2 like a mathematical laboratory, to perform otherwise laborious computations quickly in the search for patterns, ideally to help one make conjectures that can be proven. In this chapter, we expand our use of Macaulay2 in this capacity significantly.

Chapter 4
Connections with Combinatorics

This chapter investigates three special cases of monomial ideals that are important for graph theory and combinatorics: the edge ideal of a simple graph, and the face and facet ideals of a simplicial complex. Each of these cases is a monomial ideal that is "square free." These ideals are treated in general in Section 4.1. Graphs and their edge ideals are introduced in Section 4.2, and the decompositions of edge ideals are described in Section 4.3. This includes, as a consequence, a method for finding decompositions of quadratic square-free monomial ideals. Simplicial complexes and their face ideals are presented in Section 4.4, and the decompositions of face ideals are described in Section 4.5. This includes, as a consequence, a method for finding decompositions of arbitrary square-free monomial ideals. Section 4.6 treats the facet ideals associated to simplicial complexes, and their m-irreducible decompositions. The chapter ends in Section 4.7 with an exploration of Alexander duality, a process that transforms monomial generating sequences to m-irreducible decompositions, and vice versa.

The computer material in this chapter explores how Macaulay2 can represent combinatorial objects like graphs and simplicial complexes, as well as the associated algebraic objects, using packages. Additionally, we use these constructions to create more methods, building on our work in the preceding chapter.

4.1 Square-Free Monomial Ideals

In this section, A is a non-zero commutative ring with identity.

The following notion of "square-free" monomials compares directly to the same notion for integers. The main point of this section is to characterize the square-free monomial ideals in terms of their monomial radicals; see Section 2.3.

Definition 4.1.1. Set $R = A[X_1, \ldots, X_d]$. A monomial $\underline{X}^{\underline{n}} \in [\![R]\!]$ is *square-free* if, for $i = 1, \ldots, d$ one has $n_i \in \{0, 1\}$. A monomial ideal $J \subseteq R$ is if it has a square-free monomial generating sequence.

For instance, the square-free monomials in $R = A[X,Y,Z]$ are

$$1, X, Y, Z, XY, XZ, YZ, XYZ.$$

The ideal $(XY, YZ)R$ is square-free. The ideal $(X^2Y, YZ^2)R$ is not square-free; see Exercise 4.1.7. The ideal 0 is square-free, since it is generated by the set $\emptyset$ which vacuously consists of square-free monomials.

A monomial $f \in [\![R]\!] = A[X_1,\ldots,X_d]$ is square-free if and only if it has no factor of the form X_i^2, i.e., if and only if $f = \operatorname{red}(f)$. (In particular, the monomial $\operatorname{red}(f)$ is square-free.) Thus, the term "square-free" refers to the fact that the monomial is free of square factors.

Proposition 4.1.2 *Set $R = A[X_1,\ldots,X_d]$. A monomial ideal $J \subseteq R$ is square-free if and only if* $\operatorname{m-rad}(J) = J$. *In particular, the ideal* $\operatorname{m-rad}(J)$ *is square-free.*

Proof. Let $f_1,\ldots,f_n$ be an irredundant monomial generating sequence for J.

Assume first that J is square-free; we show that $\operatorname{m-rad}(J) = J$. The ideal J has a square-free monomial generating sequence, and Theorem 1.3.6 shows that this sequence contains the f_i's. Thus, each monomial f_i is square-free, so $f_i = \operatorname{red}(f_i)$ for $i = 1,\ldots,n$. Thus, Theorem 2.3.7 implies that

$$\operatorname{m-rad}(J) = (\operatorname{red}(f_1),\ldots,\operatorname{red}(f_n))R = (f_1,\ldots,f_n)R = J$$

as desired.

Assume next that $\operatorname{m-rad}(J) = J$. To show that J is square-free, we show that each f_i is square-free, that is, that $f_i = \operatorname{red}(f_i)$ for $i = 1,\ldots,n$. Theorem 2.3.7 implies that

$$J = \operatorname{m-rad}(J) = (\operatorname{red}(f_1),\ldots,\operatorname{red}(f_n))R$$

so an irredundant monomial generating sequence for J is a sequence of the form $\operatorname{red}(f_{i_1}),\ldots,\operatorname{red}(f_{i_k})$. The uniqueness of such generating sequences implies that $k = n$, so an irredundant monomial generating sequence for J is $\operatorname{red}(f_1),\ldots,\operatorname{red}(f_n)$, and further that $\{f_1,\ldots,f_n\} = \{\operatorname{red}(f_1),\ldots,\operatorname{red}(f_n)\}$. Thus, for $i = 1,\ldots,n$ there is an index j_i such that $f_i = \operatorname{red}(f_{j_i})$. It follows that $\operatorname{red}(f_{j_i}) \big| f_{j_i}$, so we have $f_i \big| f_{j_i}$. The irredundancy of the sequence $f_1,\ldots,f_n$ implies that $j_i = i$, so $f_i = \operatorname{red}(f_i)$.

Finally, if J is an arbitrary monomial ideal, it follows from Proposition 2.3.3(d) that $\operatorname{m-rad}(\operatorname{m-rad}(J)) = \operatorname{m-rad}(J)$. So, the previous paragraph shows that $\operatorname{m-rad}(J)$ is square-free. $\square$

Next, we present an extremely useful characterization of the m-irreducible monomial ideals that are square-free.

Proposition 4.1.3 *Set $R = A[X_1,\ldots,X_d]$. A monomial ideal $J \subseteq R$ is square-free and m-irreducible if and only if there exist positive integers $k, t_1,\ldots,t_k$ such that $1 \leqslant t_1 < \cdots < t_k \leqslant d$ and $J = (X_{t_1},\ldots,X_{t_k})R$.*

Proof. For the forward implication, assume that J is square-free and m-irreducible. Let $f_1,\ldots,f_n$ be a square-free monomial generating sequence for J. Theorem 3.1.3

implies that there are positive integers $k, t_1, \ldots, t_k, e_1, \ldots, e_k$ such that $1 \leqslant t_1 < \cdots < t_k \leqslant d$ and $J = (X_{t_1}^{e_1}, \ldots, X_{t_k}^{e_k})R$. The generating sequence $X_{t_1}^{e_1}, \ldots, X_{t_k}^{e_k}$ is irredundant, so Theorem 1.3.6 shows that the sequence of f_i's contains the sequence of $X_{t_j}^{e_j}$'s. Thus, each monomial $X_{t_j}^{e_j}$ is square-free, that is, we have $e_j = 1$ for $j = 1, \ldots, k$. The conclusion $J = (X_{t_1}, \ldots, X_{t_k})R$ follows directly.

Conversely, if we have $J = (X_{t_1}, \ldots, X_{t_k})R$, then J is m-irreducible by Theorem 3.1.3, and it is square-free by definition. □

We end this section with a characterization of square-freeness in terms of m-irreducible decompositions. This uses the following notation for the relevant m-irreducible monomial ideals.

Definition 4.1.4. Let $V = \{v_1, \ldots, v_d\}$, and set $R = A[X_1, \ldots, X_d]$. For each subset $V' \subseteq V$, let $P_{V'} \subseteq R$ be the ideal "generated by the elements of V'":

$$P_{V'} = (\{X_i \mid v_i \in V'\})R.$$

For instance, with $V = \{v_1, \ldots, v_d\}$ and $R = A[X_1, \ldots, X_d]$, we have

$$P_\emptyset = 0 \qquad P_{\{v_1, v_3\}} = (X_1, X_3)R \qquad P_V = (X_1, \ldots, X_d)R.$$

Fact 4.1.5. Let $V = \{v_1, \ldots, v_d\}$ be a finite set, and set $R = A[X_1, \ldots, X_d]$.

(a) Given subsets $V', V'' \subseteq V$, one has $P_{V'} \subseteq P_{V''}$ if and only if $V' \subseteq V''$.
(b) A monomial ideal $J \subseteq R$ is square-free and m-irreducible if and only if there exists a subset $V' \subseteq V$ such that $J = P_{V'}$; see Proposition 4.1.3.

Here is our characterization of square-freeness.

Proposition 4.1.6 *Let $V = \{v_1, \ldots, v_d\}$, and set $R = A[X_1, \ldots, X_d]$. A monomial ideal $J \subsetneq R$ is square-free if and only if there are subsets $V_1, \ldots, V_n \subseteq V$ such that $J = \bigcap_{i=1}^n P_{V_i}$.*

Proof. First, assume that J is square-free. Theorem 3.3.3 implies that J has an irredundant m-irreducible decomposition $J = \bigcap_{i=1}^n J_i$, and Exercise 4.1.9(b) implies that each J_i is square-free. We conclude from Fact 4.1.5(b) that there exist subsets $V_1, \ldots, V_n \subseteq V$ such that for $i = 1, \ldots, n$ we have $J_i = P_{V_i}$, so $J = \bigcap_{i=1}^n P_{V_i}$.

Conversely, if $J = \bigcap_{i=1}^n P_{V_i}$ for some subsets $V_1, \ldots, V_n \subseteq V$. Fact 4.1.5(b) implies that each P_{V_i} is square-free, so J is square-free by Exercise 4.1.9(a). □

Exercises

Exercise 4.1.7. Set $R = A[X_1, \ldots, X_d]$, and let $J \subseteq R$ be a monomial ideal. Prove that J is square-free if and only if its irredundant monomial generating sequences consist of square-free monomials.

Exercise 4.1.8. Verify the statements in Fact 4.1.5.

**Exercise 4.1.9.* Set $R = A[X_1,\dots,X_d]$. Let J be a monomial ideal in R with m-irreducible decomposition $J = \bigcap_{i=1}^n J_i$.

(a) Assume that for $i = 1,\dots,n$ the ideal J_i is square-free. Prove that J is square-free.
(b) Prove that if J is square-free and the decomposition $J = \bigcap_{i=1}^n J_i$ is irredundant, then each ideal J_i is square-free.

(This exercise is used in the proof of Proposition 4.1.6.)

**Exercise 4.1.10.* Set $R = A[X_1,\dots,X_d]$; let I and J be monomial ideals of R.

(a) Prove that if I and J are square-free then so is the sum $I+J$. See Exercise 1.3.13. (This is used in Challenge Exercise 7.3.13.)
(b) Prove or disprove: If $I+J$ is square-free, then either I or J is square-free. Justify your answer.

**Exercise 4.1.11.* Set $R = A[X_1,\dots,X_d]$; let I, J be monomial ideals of R.

(a) Prove that if I is square-free, then so is the colon ideal $(I :_R J)$ and we have $(I :_R J) = (I :_R \text{m-rad}(J))$. See Section 2.6. (This is used in Challenge Exercise 7.4.21.)
(b) Prove or disprove: If J is square-free, then so is $(I :_R J)$.
(c) Prove or disprove: If $(I :_R J)$ is square-free, then so is I.
(d) Prove or disprove: If $(I :_R J)$ is square-free, then so is J.

Justify your answers.

Exercise 4.1.12. Set $R = A[X_1,\dots,X_d]$. We say that a monomial ideal J in R is *m-prime* if it satisfies the following condition: for all monomials $f,g \in [\![R]\!]$ if $fg \in J$, then either $f \in J$ or $g \in J$.

(a) Prove that 0 is m-prime.
(b) Prove that the following conditions are equivalent when $J \neq 0$:

  (i) J is m-prime;
  (ii) for all non-zero monomial ideals I, K, if $IK \subseteq J$, then either $I \subseteq J$ or $K \subseteq J$;
  (iii) J is m-irreducible and square-free; and
  (iv) there are positive integers $k, t_1,\dots,t_k$ such that $J = (X_{t_1},\dots,X_{t_k})R$.

  (The outline for Exercise 3.2.14 may be useful here.)
(c) Prove that if A is a field (or more generally an integral domain), then a monomial ideal in R is m-prime if and only if it is prime. (See Exercise A.5.10 (more generally, Exercise 1.2.14).)

Exercise 4.1.13. Assume that A is a field, and let I be a monomial ideal in the ring $R = A[X_1,\dots,X_d]$.

(a) Prove that I is square-free if and only if I is radical.

(b) Prove that the conclusion of part (a) holds if A is an integral domain or, more generally, if A is reduced. See Sections 1.2 and 2.4 for discussion of integral domains and reduced rings.

Exercise 4.1.14. This exercise involves the constructions $\mathrm{V}(I)$ and $\mathrm{I}(Y)$ from Exploration Section A.10. Let A be a field, and let I be a monomial ideal of $R = A[X_1,\ldots,X_d]$. Prove that $\mathrm{I}(\mathrm{V}(I))$ is a square-free monomial ideal of R. (See Exercise 3.3.11(c).)

Square-Free Monomial Ideals in Macaulay2

Here we use the Macaulay2 command `isSquareFree` to determine whether a monomial ideal is square-free. This command only works for monomial ideals, so we must define our ideals using the command `monomialIdeal`, which will create a `MonomialIdeal` object.

```
i1 : R = ZZ/101[x,y,z];

i2 : I = monomialIdeal(x^2,x*y,y*z)
                        2
o2 = monomialIdeal (x , x*y, y*z)
o2 : MonomialIdeal of R

i3 : isSquareFree I
o3 = false

i4 : J = monomialIdeal(x*y, y*z)
o4 = monomialIdeal (x*y, y*z)
o4 : MonomialIdeal of R

i5 : isSquareFree J
o5 = true
```

This function is of course easy enough to implement yourself, and it is also probably easy enough to understand the source code. For functions and methods that have the source code available, one way to do that is via the `code` command. First, we determine what type of object `isSquareFree` is.

```
i6 : isSquareFree
o6 = isSquareFree
o6 : MethodFunction
```

As we can see, the command `isSquareFree` is a `MethodFunction` object, which means that it was defined using the `method` command. To find what types this method accepts as input, we use the command `methods`.

```
i7 : methods isSquareFree
o7 = {(isSquareFree, MonomialIdeal)}
o7 : VerticalList
```

So we can see that the type `MonomialIdeal` is the only allowable type as input to `isSquareFree`. If the function is implemented in the Macaulay2 programming language[1], one can access the code for all instantiations of this method using the command `code`.

```
i8 : code methods isSquareFree
o8 = -- code for method: isSquareFree(MonomialIdeal)
    /Applications/Macaulay2-1.9.2/share/Macaulay2/Core/monideal.m2:270:35
    isSquareFree MonomialIdeal := (I) ->
    all(first entries generators I, m -> all(first exponents m, i -> i < 2))


i8 : exit
```

As you can see[2], this gives both the source code of the function, as well as the location of the file from which the code was loaded. This can be very useful for tracking down good examples of code already written in the Macaulay2 language from which to learn.

Exercises

Exercise 4.1.15. Use Macaulay2 to test any counterexamples you devised for Exercise 4.1.11.

Exercise 4.1.16. Set $R = \mathbb{Q}[X,Y,Z,W]$ and $I = (XY,YZ,ZW)R$. Use Macaulay2 to show that I is square-free, to find an irredundant m-irreducible decomposition of I (using the command `irreducibleDecomposition`), and to verify that each m-irreducible ideal in the decomposition is square-free.

Coding Exercise 4.1.17. Use the functions `all` and `exponents` to write your own method that accepts a `MonomialIdeal` object and returns whether or not it is square-free. If you are unfamiliar with `all`, use the documentation to learn more.

4.2 Graphs and Edge Ideals

In this section, A is a non-zero commutative ring with identity.

[1] Macaulay2 is open source, but not all its functions are written in the Macaulay2 language. Many of the low-level routines are written in C or C++ and therefore the source code for such functions does not come with the standard distribution. To check out the source code, visit the website `https://github.com/Macaulay2/M2`.

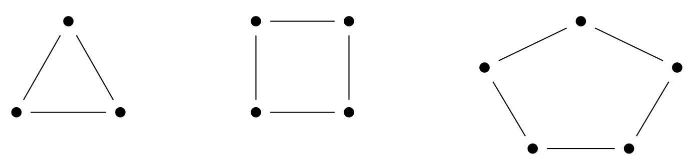
[2] Note that the return value of the `code methods isSquareFree` command was slightly edited to fit on the page.

Geometrically, a graph consists of a set of points (called "vertices") and a set of lines or arcs (called "edges") connecting pairs of vertices. (For us, the term "graph" is short for "finite simple graph".) We will take the more combinatorial approach (as opposed to the geometric approach) to the study of graphs. Our treatment of graph theory is brief, but self-contained. However, the interested reader may wish to consult the text of Reinhard Diestel [16] as a reference.

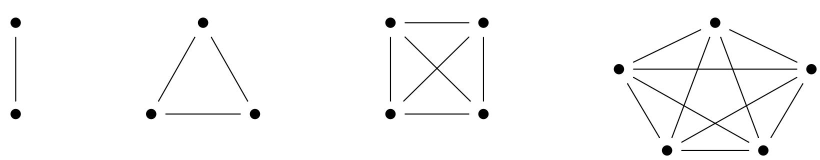
Definition 4.2.1. Let $V = \{v_1,\dots,v_d\}$ be a finite set. A *graph* with *vertex set* V is an ordered pair $G = (V,E)$ where E is a set of un-ordered pairs v_iv_j with $v_i \neq v_j$. (Since the pairs are un-ordered, we have $v_iv_j = v_jv_i$.) An element $v_i \in V$ is a *vertex* of G. (The plural of vertex is "vertices".) The set E is the *edge set* of G. Given an edge $e = v_iv_j$, the *endpoints* of e are the vertices v_i and v_j. Two vertices $v_i, v_j \in V$ are *adjacent* if there is an edge $e \in E$ with endpoints v_i and v_j, that is, if $v_iv_j \in E$; in this case, we also say that the edge v_iv_j is *incident* to its endpoints v_i and v_j.

Our definition implies that our graphs are finite (i.e., have finite vertex sets) and are simple (i.e., have no loops and no multiple edges). Some standard examples of graphs are as follows.

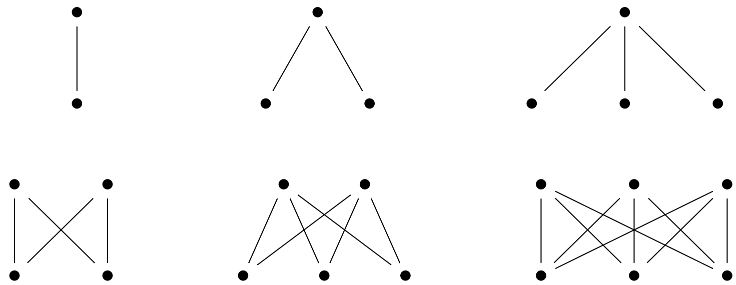
For each integer $d \geqslant 3$, a *d-cycle* is the graph C_d with vertex set $\{v_1,v_2,\dots,v_d\}$ and edge set $\{v_1v_2,v_2v_3,\dots,v_{d-1}v_d,v_dv_1\}$. Geometric versions of C_3, C_4, and C_5 are displayed next.



For each $d \geqslant 2$, the *complete graph on d vertices* is the graph K_d with vertex set $\{v_1,\dots,v_d\}$ and edge set $\{v_iv_j \mid 1 \leqslant i < j \leqslant d\}$. Geometric versions of K_2, K_3, K_4, and K_5 are as follows.



Given $m,n > 1$, the *complete bipartite graph* $K_{m,n}$ is the graph with vertex set $\{u_1,\dots,u_m,v_1,\dots,v_n\}$ and edge set $\{u_iv_j \mid 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$. Geometric versions of $K_{1,1}$, $K_{1,2}$, $K_{1,3}$, $K_{2,2}$, $K_{2,3}$, and $K_{3,3}$ are as follows.



The next definition shows how to use a graph to construct a monomial ideal.

Definition 4.2.2. Let G be a graph with vertex set $V = \{v_1, \dots, v_d\}$. The *edge ideal* of G is the ideal $I_G \subseteq R = A[X_1, \dots, X_d]$ that is "generated by the edges of G":

$$I_G = (\{X_iX_j \mid v_iv_j \text{ is an edge in } G\})R.$$

By definition, the edge ideal I_G is square-free.

For example, the edge ideals associated to C_3, C_4, and C_5 are

$$\begin{aligned}
I_{C_3} &= (X_1X_2, X_2X_3, X_1X_3) \subseteq A[X_1, X_2, X_3] \\
I_{C_4} &= (X_1X_2, X_2X_3, X_3X_4, X_1X_4) \subseteq A[X_1, X_2, X_3, X_4] \\
I_{C_5} &= (X_1X_2, X_2X_3, X_3X_4, X_4X_5, X_1X_5) \subseteq A[X_1, X_2, X_3, X_4, X_5].
\end{aligned}$$

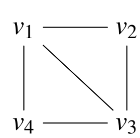
The edge ideals associated to K_2, K_3, and K_4 are

$$\begin{aligned}
I_{K_2} &= (X_1X_2) \subseteq A[X_1, X_2] \\
I_{K_3} &= (X_1X_2, X_1X_3, X_2X_3) \subseteq A[X_1, X_2, X_3] \\
I_{K_4} &= (X_1X_2, X_1X_3, X_1X_4, X_2X_3, X_2X_4, X_3X_4) \subseteq A[X_1, X_2, X_3, X_4].
\end{aligned}$$

The edge ideals associated to some bipartite graphs are

$$\begin{aligned}
I_{K_{1,1}} &= (X_1Y_1) \subseteq A[X_1, Y_1] \\
I_{K_{1,2}} &= (X_1Y_1, X_1Y_2) \subseteq A[X_1, Y_1, Y_2] \\
I_{K_{1,3}} &= (X_1Y_1, X_1Y_2, X_1Y_3) \subseteq A[X_1, Y_1, Y_2, Y_3] \\
I_{K_{2,2}} &= (X_1Y_1, X_1Y_2, X_2Y_1, X_2Y_2) \subseteq A[X_1, X_2, Y_1, Y_2] \\
I_{K_{2,3}} &= (X_1Y_1, X_1Y_2, X_1Y_3, X_2Y_1, X_2Y_2, X_2Y_3) \subseteq A[X_1, X_2, Y_1, Y_2, Y_3].
\end{aligned}$$

The edge ideal of the graph



is $(X_1X_2, X_1X_3, X_1X_4, X_2X_3, X_3X_4) \subseteq A[X_1, X_2, X_3, X_4]$.

Example 4.2.3. Let G be a graph with vertex set $V = \{v_1,\dots,v_d\}$. If G has no edges, i.e., it is a discrete set of vertices, then $I_G = 0$. Moreover, we have $I_G = 0$ if and only if G has no edges.

Exercises

Exercise 4.2.4. Let G be a graph with vertex set $\{v_1,\dots,v_d\}$. Prove that the set $\{X_iX_j \mid v_iv_j \text{ is an edge in } G\}$ is an irredundant generating sequence for I_G.

Exercise 4.2.5. Let G and G' be graphs with vertex sets $\{v_1,\dots,v_d\}$.

(a) Prove that $G \subseteq G'$ if and only if $I_G \subseteq I_{G'}$.
(b) Prove that $I_G \subseteq I_{K_d}$.

Graphs and Edge Ideals in Macaulay2

Macaulay2 has the capability for users to write *packages* that expand its functionality beyond what is available by default. Many such packages come distributed with Macaulay2. An example of this is the package `EdgeIdeals` written by Christopher Francisco, Andrew Hoefel and Adam Van Tuyl [23]. This package is part of the standard distribution of Macaulay2, and is loaded via the following command.

```
i1 : needsPackage "EdgeIdeals"
o1 = EdgeIdeals
o1 : Package
```

In this package, there is no distinction between a vertex of a graph, and the corresponding variable of a polynomial ring. One defines a `Graph` object via the `graph` command, as follows.

```
i2 : R = QQ[w,x,y,z];

i3 : G = graph(R,{{w,x},{x,y},{y,z},{z,w}})
o3 = Graph{edges => {{w, x}, {x, y}, {y, z}, {w, z}}}
           ring => R
           vertices => {w, x, y, z}
o3 : Graph
```

We obtain the edge ideal of this graph with the command `edgeIdeal`.

```
i4 : I = edgeIdeal G
o4 = monomialIdeal (w*x, x*y, w*z, y*z)
o4 : MonomialIdeal of R
```

Let's write a method that creates a complete bipartite graph on any pair of sets of variables in a polynomial ring. To help us, we will introduce the `Set` type. The code is given below.

```
i5 : bipartiteGraph = method()
o5 = bipartiteGraph
o5 : MethodFunction

i6 : bipartiteGraph(Ring,List,List) := (R,L1,L2) -> (
sL1 := set L1;
sL2 := set L2;
if #(sL1 * sL2) > 0 then error "Expected disjoint sets.";
edgeList := toList(sL1 ** sL2) / toList;
graph(R,edgeList)
)
o6 = {*Function[stdio:6:45-11:9]*}
o6 : FunctionClosure
```

This method takes the polynomial ring, and two lists of variables as input. First, the lists are converted to the `Set` type using the command `set`. The reason for this conversion is that the type `Set` has some useful operations defined on it, such as intersection and cartesian product, that will be useful for us. `Set` also ensures that all the elements in the container[3] are unique.

Next, a check is performed to ensure that the two input lists do not have any variables in common. This is done via the hypothetical `#(sL1 * sL2) > 0`; here `sL1 * sL2` is the intersection of the two sets `sL1` and `sL2`, and `#` computes the size of this set.

Next, we use the cartesian product operator `**` to form the cartesian product of `sL1` and `sL2` and convert this set to a list, using the command `toList`. This gives us a list of ordered pairs that are `Sequence` objects. The input to `graph` must be a list of lists of variables corresponding to the (undirected) edges of the graph, so we must convert each sequence to a list using `toList`. Finally, we call the `graph` function on this edge list to return the desired graph.

As an example, a 4-cycle is bipartite, and the example above could also be generated via the following command.

```
i7 : G' = bipartiteGraph(R,{w,y},{x,z})
o7 = Graph{edges => {{w, x}, {y, z}, {x, y}, {w, z}}}
           ring => R
           vertices => {w, x, y, z}
o7 : Graph
```

One can also use the == operator on `Graph` objects to test equality.

```
i8 : G == G'
o8 = true
```

There are many other operations one can perform on the edge ideal of a graph using the `EdgeIdeals` package. To use the commands `viewHelp EdgeIdeals` or `help EdgeIdeals` in the Macaulay2 terminal, you'll need to load the package first.

[3] A *container* is a programming construct that holds objects.

Exercises

**Documentation Exercise 4.2.6.* Set $A = \mathbb{Q}$, and use Macaulay2 to generate the graphs and edge ideals for the examples in the section above. (You may save some time by consulting the documentation for the `EdgeIdeals` package, e.g., because of the `completeGraph` command. This is used in Exercise 4.3.20.)

4.3 Decompositions of Edge Ideals

In this section, A is a non-zero commutative ring with identity.

Our goal for this section is to characterize the m-irreducible decompositions of edge ideals. These are given in terms of the graph's vertex covers, using Proposition 4.1.6. One consequence is a method for finding m-irreducible decompositions of quadratic square-free monomial ideals. We begin with some notions that we use to identify which ideals $P_{V'}$ occur in an (irredundant) m-irreducible decomposition of an edge ideal.

Definition 4.3.1. Let G be a graph with vertex set $V = \{v_1, \ldots, v_d\}$. A *vertex cover* of G is a subset $V' \subseteq V$ such that for each edge $v_i v_j$ in G either $v_i \in V'$ or $v_j \in V'$. A vertex cover V' is *minimal* if it does not properly contain another vertex cover of G.

For instance, the vertex set V is a vertex cover of G. Thus, G has a vertex cover.

Example 4.3.2. Let G be a graph with vertex set $V = \{v_1, \ldots, v_d\}$. If G has no edges, then every subset of V, including $\emptyset$, is a vertex cover of G. Moreover, the graph G has no edges if and only if $\emptyset$ is a vertex cover of G, and this is so if and only if every subset of V is a vertex cover of G.

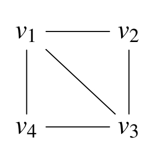
The next fact states that the set of vertex covers of G is closed under supersets, and that every graph has a minimal vertex cover.

Fact 4.3.3. Let G be a graph with vertex set V, and let $V', V'' \subseteq V$.

(a) If V' is a vertex cover of G and $V' \subseteq V''$, then V'' is a vertex cover of G.
(b) Since V is finite, every vertex cover of G contains a minimal vertex cover of G.

The next example shows that a given graph can have several distinct vertex covers. Moreover, it can have minimal vertex covers of differing sizes.

Example 4.3.4. We compute the vertex covers of the following graph G.



In light of Fact 4.3.3, parts (a) and (b), we really only need to find the *minimal* vertex covers of G.

First, we find the minimal vertex covers containing v_1. If $v_1 \in V'$, then the edges v_1v_2, v_1v_3, and v_1v_4 are "covered". This leaves only the edges v_2v_3, and v_3v_4 "uncovered". These edges can be covered either by adding v_3 or by adding v_2, v_4. From this, it is straightforward to show that the minimal vertex covers containing v_1 are $\{v_1, v_3\}$ and $\{v_1, v_2, v_4\}$.

Next, we find the minimal vertex covers that do not contain v_1. If $v_1 \notin V'$, we must have $v_2, v_3, v_4 \in V'$ to cover the edges v_1v_2, v_1v_3, and v_1v_4. It is straightforward to show that the set $\{v_2, v_3, v_4\}$ is a minimal vertex cover of G.

The connection between vertex covers and m-irreducible decompositions begins with the next result.

Lemma 4.3.5. *Let G be a graph with vertex set $V = \{v_1, \ldots, v_d\}$, and let $V' \subseteq V$. Set $R = A[X_1, \ldots, X_d]$. Then $I_G \subseteq P_{V'}$ if and only if V' is a vertex cover of G.*

Proof. Write $V' = \{v_{i_1}, \ldots, v_{i_n}\}$, so that $P_{V'} = (X_{i_1}, \ldots, X_{i_n})R$.

For the forward implication, assume that $I_G \subseteq P_{V'}$. We show that V' is a vertex cover of G. Let v_jv_k be an edge in G. Then we have $X_jX_k \in I_G \subseteq P_{V'} = (X_{i_1}, \ldots, X_{i_n})R$. It follows that $X_jX_k \in (X_{i_m})R$ for some index m. A comparison of exponent vectors shows that either $j = i_m$ or $k = i_m$, that is, either $v_j = v_{i_m} \in V'$ or $v_k = v_{i_m} \in V'$. Thus V' is a vertex cover of G.

For the reverse implication, assume that V' is a vertex cover of G. To show that $I_G \subseteq P_{V'}$, we need to show that each generator of I_G is in $P_{V'}$. To this end, fix a generator $X_iX_j \in I_G$, corresponding to an edge v_iv_j in G. Since V' is a vertex cover of G, either $v_i \in V'$ or $v_j \in V'$. Thus, either $X_i \in P_{V'}$ or $X_j \in P_{V'}$, so $X_iX_j \in P_{V'}$. $\square$

Without further ado, here is the decomposition theorem for edge ideals. It shows explicitly how the combinatorial properties of a graph inform some algebraic properties of its edge ideal. The subsequent discussion explains the reciprocal relation of how the algebraic properties of the edge ideal inform some combinatorial properties of the graph.

Theorem 4.3.6 *Let G be a graph with vertex set $V = \{v_1, \ldots, v_d\}$, and set $R = A[X_1, \ldots, X_d]$. Then the edge ideal $I_G \subseteq R$ has the next m-irreducible decompositions*

$$I_G = \bigcap_{V'} P_{V'} = \bigcap_{V' \text{ min.}} P_{V'}$$

where the first intersection is taken over all vertex covers of G; the second intersection is taken over all minimal vertex covers of G and is irredundant.

Proof. Fact 4.1.5(a) shows that the second intersection is irredundant. The containment $\bigcap_{V'} P_{V'} \subseteq \bigcap_{V' \text{ min.}} P_{V'}$ is straightforward. The reverse containment $\bigcap_{V'} P_{V'} \supseteq \bigcap_{V' \text{ min.}} P_{V'}$ follows from the fact that every vertex cover V' contains a minimal vertex cover V''; see Facts 4.1.5(a) and 4.3.3(b). The containment $I_G \subseteq \bigcap_{V'} P_{V'}$ is from Lemma 4.3.5.

For the final containment $I_G \supseteq \bigcap_{V'} P_{V'}$ recall that I_G is square-free. Hence, Proposition 4.1.6 provides subsets $V_1, \ldots, V_n$ such that $I_G = \bigcap_{j=1}^n P_{V_j}$. For each index j, we then have $I_G \subseteq P_{V_j}$, so Lemma 4.3.5 implies that V_j is a vertex cover of G. It follows that $I_G = \bigcap_{j=1}^n P_{V_j} \supseteq \bigcap_{V'} P_{V'}$, as desired. $\square$

Example 4.3.7. We compute an irredundant m-irreducible decomposition of the ideal I_G where G is the graph from Example 4.3.4. Using Theorem 4.3.6, this can be read from the list of minimal vertex covers that we computed:

$$I_G = (X_1, X_3)R \cap (X_1, X_2, X_4)R \cap (X_2, X_3, X_4)R.$$

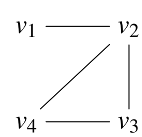
In general, given an irredundant m-irreducible decomposition $I_G = \bigcap_{i=1}^n P_{V_i}$ as in Proposition 4.1.6, one concludes the minimal vertex covers of G are precisely $V_1, \ldots, V_n$. Indeed, Theorem 4.3.6 gives an irredundant m-irreducible decomposition $I_G = \bigcap_{V' \text{ min.}} P_{V'}$, so the uniqueness of such decompositions from Theorem 3.3.8 provides the desired conclusion.

It is straightforward to identify the monomial ideals $J \subseteq R = A[X_1, \ldots, X_d]$ that are of the form I_G for some graph G with vertex set $V = \{v_1, \ldots, v_d\}$: they are precisely the ideals whose irredundant monomial generating sequences contain only elements of the form $X_i X_j$ with $i \neq j$. (In other words, they are precisely the square-free "quadratic" monomial ideals of R.) See Exercise 4.3.18. Thus, we can use the techniques of this section to find m-irreducible decompositions of such ideals, as in the following example.

Example 4.3.8. Set $R = A[X_1, X_2, X_3, X_4]$. We compute an irredundant m-irreducible decomposition of the ideal

$$J = (X_1X_2, X_2X_3, X_2X_4, X_3X_4)R.$$

First, we find a graph G with vertex set $V = \{v_1, v_2, v_3, v_4\}$ such that $J = I_G$ by adding an edge for each generator.



Next, we find the minimal vertex covers for G:

$$\{v_1, v_3, v_4\} \qquad \{v_2, v_3\} \qquad \{v_2, v_4\}.$$

Finally, we read off the decomposition using Theorem 4.3.6:

$$J = I_G = (X_1, X_3, X_4)R \cap (X_2, X_3)R \cap (X_2, X_4)R.$$

Exercises

Exercise 4.3.9. Verify the statements in Fact 4.3.3.

**Exercise 4.3.10.* Set $R = A[X_1, \ldots, X_5]$, and let G be the graph represented by the following sketch:

$$\begin{tikzcd} v_1 \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & v_2 \arrow[d, dash] & \\ v_3 \arrow[r, dash] & v_4 \arrow[r, dash] & v_5. \end{tikzcd}$$

(a) Find an irredundant monomial generating sequence for I_G.
(b) Find all minimal vertex covers of G.
(c) Use Theorem 4.3.6 to find an irredundant m-irreducible decomposition of I_G.
(d) Verify the decomposition $I_G = \bigcap_{V'} P_{V'}$ from part (c) using Theorem 2.1.5.

Justify your answers. (This is used in Exercises 4.4.16 and 4.5.13.)

Exercise 4.3.11. Verify the decomposition

$$I_G = (X_1, X_3)R \cap (X_1, X_2, X_4)R \cap (X_2, X_3, X_4)R$$

from Example 4.3.7 using Theorem 2.1.5.

Exercise 4.3.12. Verify the decomposition

$$I_G = (X_1, X_3, X_4)R \cap (X_2, X_3)R \cap (X_2, X_4)R$$

from Example 4.3.8 using Theorem 2.1.5.

Exercise 4.3.13. Set $R = A[X_1, \ldots, X_5]$ and compute an irredundant m-ir-reducible decomposition of $J = (X_1X_2, X_1X_4, X_1X_5, X_2X_3, X_2X_5, X_3X_4, X_4X_5)R$ as in Example 4.3.8. Check your decomposition using Theorem 2.1.5. Justify your answers.

Exercise 4.3.14. Let G be a graph with vertex set $\{v_1, \ldots, v_d\}$. Set $R = A[X_1, \ldots, X_d]$ and $\mathfrak{X} = (X_1, \ldots, X_d)R$.

(a) Prove that for $i = 1, \ldots, d$ the set $V \smallsetminus \{v_i\} = \{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}$ is a vertex cover of G.
(b) Let $I_G = \bigcap_{i=1}^n J_i$ be an irredundant m-irreducible decomposition of the edge ideal I_G. Prove that for $i = 1, \ldots, n$ we have $J_i \neq \mathfrak{X}$.

Exercise 4.3.15. Let $d \geqslant 3$.

(a) Prove that the minimal vertex covers of the complete graph K_d are exactly the sets $V \smallsetminus \{v_i\} = \{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_d\}$.
(b) Find an irredundant m-irreducible decomposition of I_{K_d}. Justify your answer.

Exercise 4.3.16. Let $m, n \geqslant 1$.

(a) Prove that the minimal vertex covers of the complete bipartite graph $K_{m,n}$ are the sets $\{u_1,\dots,u_m\}$ and $\{v_1,\dots,v_n\}$.
(b) Find an irredundant m-irreducible decomposition of $I_{K_{m,n}}$. Justify your answer.

Exercise 4.3.17. Let $d \geqslant 3$. Find the minimal vertex covers of the d-cycle C_d, and find an irredundant m-irreducible decomposition of I_{C_d}. Justify your answer.

Exercise 4.3.18. Let $V = \{v_1,\dots,v_d\}$ and $R = A[X_1,\dots,X_d]$. Prove that the association $G \mapsto I_G$ describes a bijection between the set of graphs with vertex set V and the set of square-free "quadratic" monomial ideals of R, that is, the ideals whose irredundant monomial generating sequences contain only elements of the form X_iX_j with $i \neq j$. (Hint: Explicitly construct an inverse for this function.)

Challenge Exercise 4.3.19. This exercise involves the construction $\mathrm{V}(I)$ from Exploration Section A.10. Let A be a field, and let G be a graph with vertex set $\{v_1,\dots,v_d\}$. Exercise 3.3.11(b) shows that there is an integer $k \in \mathbb{N}$, and there are linear subspaces $V_1,\dots,V_k \subseteq A^d$ such that $\mathrm{V}(I_G) = V_1 \cup \cdots \cup V_k$ and such that $V_i \not\subseteq V_j$ whenever $i \neq j$. Describe the linear subspaces V_i in terms of the minimal vertex covers of G. Justify your answer.

Decompositions of Edge Ideals in Macaulay2

The `EdgeIdeals` package contains a method to compute the minimal vertex covers of a graph. If we continue with the example from the previous section we can compute the minimal vertex covers of the 4-cycle G using the command `vertexCovers`.

```
i9 : G
o9 = Graph{edges => {{w, x}, {x, y}, {y, z}, {w, z}}}
           ring => R
           vertices => {w, x, y, z}
o9 : Graph

i10 : vertexCovers G
o10 = {w*y, x*z}
o10 : List
```

Note that the return value of this function is a list consisting of square-free monomials whose *supports* give the minimal vertex covers of G.

We use this to write a method that returns an irredundant m-irreducible decomposition of a square-free quadratic monomial ideal. The procedure is straightforward: we first convert the monomial ideal to a graph via the `graph` function (which ensures that its input is quadratic and square-free), then convert each monomial in the return value of `vertexCovers` to the ideal generated by the variables in its support.

```
i11 : mIrreducibleDecomposition = method()
o11 = mIrreducibleDecomposition
o11 : MethodFunction
```

```
i12 : mIrreducibleDecomposition MonomialIdeal := I -> (
G := graph I;
vertCov := vertexCovers G;
apply(vertCov, mon -> ideal support mon)
)
o12 = {*Function[stdio:18:46-21:37]*}
o12 : FunctionClosure

i13 : mIrreds = mIrreducibleDecomposition I
o13 = {ideal (w, y), ideal (x, z)}
o13 : List
```

To verify this, we check that the intersection is indeed I.

```
i14 : intersect mIrreds == I
o14 = true

i15 : exit
```

Exercises

Exercise 4.3.20.

(a) Use the method from the tutorial above (with the command `edgeIdeal`) to compute the minimal vertex covers and the irredundant m-irreducible decompositions of the edge ideals of the graphs from Documentation Exercise 4.2.6, Example 4.3.8, and Exercise 4.3.10.
(b) Use these examples to compare run times between the method from part (a) and the command `irreducibleDecomposition edgeIdeal`.
(c) Verify the conclusion of Exercise 4.3.14(b) for the examples from part (a).
(d) Compare your computations here with your answers for Exercises 4.3.15, 4.3.16, and 4.3.17.

Exercise 4.3.21. Set $A = \mathbb{Q}$, and use Macaulay2 to verify the decomposition from Example 4.3.7.

Exercise 4.3.22. Use the `irreducibleDecomposition` command to verify your answer for Exercise 4.3.13.

4.4 Simplicial Complexes and Face Ideals

In this section, A is a non-zero commutative ring with identity.

The previous section gave a method for computing m-irreducible decompositions for quadratic square-free monomial ideals. The next section introduces some tools to accomplish this for arbitrary square-free monomial ideals. This uses the notion of a simplicial complex, defined next. One often thinks of this as a higher dimensional graph. not only does it have vertices and edges, but it also can have shaded triangles, solid tetrahedra, and so on. Again, we take a purely combinatorial approach here.

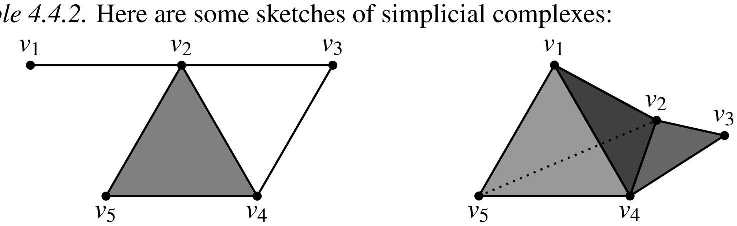
Definition 4.4.1. Let $V = \{v_1, \dots, v_d\}$ be a finite set. A *simplicial complex* on V is a non-empty collection Δ of subsets of V that is closed under subsets, that is, such that for all subsets $F, G \subseteq V$, if $F \subseteq G$ and $G \in \Delta$, then $F \in \Delta$. An element of Δ is called a *face* of Δ. A face of the form $\{v_i\}$ is called a *vertex* of Δ. A face of the form $\{v_j, v_k\}$ with $j \neq k$ is called an *edge* of Δ. A maximal element of Δ with respect to containment is a *facet* of Δ. The $(d-1)$-*simplex* consists of all the subsets of V and is denoted Δ_{d-1}.

By definition, a simplicial complex Δ on $V = \{v_1, \dots, v_d\}$ is a subset of the power set $\mathsf{P}(V) = \Delta_{d-1}$. Note that we do not assume that each singleton $\{v_i\}$ is in Δ. This differs slightly from some definitions. However, this convention allows for some added flexibility, for instance, in Section 4.7.

Since V is finite, every face of Δ is contained in a facet of Δ. In particular, since Δ is non-empty, it has at least one facet. Since Δ is non-empty and closed under subsets, we have $\emptyset \in \Delta$, that is, $\emptyset$ is a face of Δ.

Every graph G with vertex set $V = \{v_1, \dots, v_d\}$ gives rise to a simplicial complex, namely the complex that contains $\emptyset$ along with every singleton $\{v_i\}$ and every pair $\{v_j, v_k\}$ such that $v_j v_k$ is an edge in G. As with graphs, it can be helpful to sketch the "geometric realization" of a simplicial complex. We will not give a technical definition of this here. The idea is the following: every vertex corresponds to a point; every edge corresponds to a line segment between two vertices; every face of the form $\{v_i, v_j, v_k\}$ with i, j, k distinct corresponds to a shaded triangle with vertices v_i, v_j, and v_k; every face $\{v_i, v_j, v_k, v_l\}$ of cardinality 4 corresponds to a solid tetrahedron with vertices v_i, v_j, v_k, and v_l; etc. We demonstrate this in the next example.

Example 4.4.2. Here are some sketches of simplicial complexes:



The first one Δ consists of an edge, a shaded triangle, and an unshaded triangle. This is the simplicial complex with the following faces:

$$\begin{array}{rl}
\text{trivial:} & \emptyset \\
\text{vertices:} & \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\} \\
\text{edges:} & \{v_1, v_2\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_4, v_5\} \\
\text{shaded triangle:} & \{v_2, v_4, v_5\} \\
\text{facets:} & \{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_2, v_4, v_5\}.
\end{array}$$

The second sketched simplicial complex Δ' consists of a solid tetrahedron and a shaded triangle. This is the simplicial complex with the following faces:

$$\begin{array}{rl} \text{trivial:} & \emptyset \\ \text{vertices:} & \{v_1\},\{v_2\},\{v_3\},\{v_4\},\{v_5\} \\ \text{edges:} & \{v_1,v_2\},\{v_1,v_4\},\{v_1,v_5\},\{v_2,v_3\}, \\ & \{v_2,v_4\},\{v_2,v_5\},\{v_3,v_4\},\{v_4,v_5\} \\ \text{shaded triangles:} & \{v_1,v_2,v_4\},\{v_1,v_2,v_5\},\{v_1,v_4,v_5\},\{v_2,v_3,v_4\},\{v_2,v_4,v_5\} \\ \text{solid tetrahedron:} & \{v_1,v_2,v_4,v_5\} \\ \text{facets:} & \{v_2,v_3,v_4\},\{v_1,v_2,v_4,v_5\}. \end{array}$$

The next definition shows how one can use a simplicial complex to construct a very useful monomial ideal. For instance, this construction is the key to Stanley's solution of the Upper Bound Conjecture; see Section 5.3.

Definition 4.4.3. Let Δ be a simplicial complex on $V = \{v_1,\ldots,v_d\}$, and set $R = A[X_1,\ldots,X_d]$. The *face ideal* of R associated to Δ is the ideal "generated by the non-faces of Δ":

$$J_\Delta = (X_{i_1}\cdots X_{i_s} \mid 1 \leqslant i_1 < \cdots < i_s \leqslant d \text{ and } \{v_{i_1},\ldots,v_{i_n}\} \notin \Delta)R.$$

By definition, the face ideal J_Δ is square-free. Next, we compute this ideal for our previous examples.

Example 4.4.4. Consider the simplicial complexes from Example 4.4.2. The "non-faces" of Δ are

$$\begin{gathered} \{v_1,v_3\} \quad \{v_1,v_4\} \quad \{v_1,v_5\} \quad \{v_3,v_5\} \quad \{v_1,v_2,v_3\} \quad \{v_1,v_2,v_4\} \quad \{v_1,v_2,v_5\} \\ \{v_1,v_3,v_4\} \quad \{v_1,v_3,v_5\} \quad \{v_1,v_4,v_5\} \quad \{v_2,v_3,v_4\} \quad \{v_2,v_3,v_5\} \quad \{v_3,v_4,v_5\} \\ \{v_1,v_2,v_3,v_4\} \quad \{v_1,v_2,v_3,v_5\} \quad \{v_1,v_2,v_4,v_5\} \quad \{v_1,v_3,v_4,v_5\} \quad \{v_2,v_3,v_4,v_5\} \\ \{v_1,v_2,v_3,v_4,v_5\}. \end{gathered}$$

It follows that the generators for J_Δ are

$$\begin{gathered} X_1X_3 \quad X_1X_4 \quad X_1X_5 \quad X_3X_5 \quad X_1X_2X_3 \quad X_1X_2X_4 \quad X_1X_2X_5 \quad X_1X_3X_4 \\ X_1X_3X_5 \quad X_1X_4X_5 \quad X_2X_3X_4 \quad X_2X_3X_5 \quad X_3X_4X_5 \quad X_1X_2X_3X_4 \quad X_1X_2X_3X_5 \\ X_1X_2X_4X_5 \quad X_1X_3X_4X_5 \quad X_2X_3X_4X_5 \quad X_1X_2X_3X_4X_5 \end{gathered}$$

Removing redundancies, we have

$$J_\Delta = (X_1X_3, X_1X_4, X_1X_5, X_3X_5, X_2X_3X_4)R.$$

Similarly, for Δ' we have

$$J_{\Delta'} = (X_1X_3, X_3X_5)R = (X_3)R \cap (X_1, X_5)R.$$

Remark 4.4.5. Set $V = \{v_1, \ldots, v_d\}$ and $R = A[X_1, \ldots, X_d]$. It is straightforward to identify the monomial ideals $J \subseteq R$ that are of the form J_Δ for some simplicial complex Δ on V: they are precisely the square-free monomial ideals $I \subsetneq R$; see Exercise 4.4.19. In particular, given a graph G with vertex set V, this implies that the edge ideal I_G of Sections 4.2–4.3 is of the form J_{Δ_G} for some simplicial complex Δ_G on V. We continue this section by identifying and investigating Δ_G.

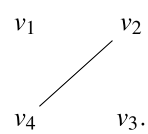
Definition 4.4.6. Let G be a graph with vertex set V. A subset $F \subseteq V$ is *independent in G* if none of the vertices in F are adjacent in G. An independent subset in G is *maximal* if it is maximal with respect to containment. Let Δ_G denote the set of independent subsets of G. This is the *independence complex* of G. (Independence complexes are also constructed as "flag complexes".)

For instance, every singleton $\{v_i\} \subseteq V$ is independent in G, as is the empty set $\emptyset \subset V$. Furthermore, every subset of an independent set in G is also independent in G, so Δ_G is a simplicial complex on V. Here is a concrete example of this construction.

Example 4.4.7. Consider the graph G from Example 4.3.4. The independent subsets in G are exactly the following:

$$\emptyset \qquad \{v_1\} \qquad \{v_2\} \qquad \{v_3\} \qquad \{v_4\} \qquad \{v_2, v_4\}.$$

That is, the geometric realization of Δ_G is as follows:



The maximal independent subsets in G are $\{v_1\}$, $\{v_3\}$, and $\{v_2, v_4\}$.

The next result shows that the faces of Δ_G are in bijection with the vertex covers of G, and the facets of Δ_G are in bijection with the minimal vertex covers of G.

Lemma 4.4.8. *Let G be a graph with vertex set V.*

(a) *A subset $F \subseteq V$ is independent in G if and only if $V \smallsetminus F$ is a vertex cover of G.*
(b) *An independent subset $F \subseteq V$ in G is maximal if and only if the vertex cover $V \smallsetminus F$ of G is minimal.*

Proof. Exercise. $\square$

The next result makes explicit the algebraic connection between a given graph G and the simplicial complex Δ_G.

Theorem 4.4.9 *Let G be a graph with vertex set $V = \{v_1, \ldots, v_d\}$ and edge set E. Set $R = A[X_1, \ldots, X_d]$. Then we have $I_G = J_{\Delta_G}$.*

Proof. For the containment $I_G \subseteq J_{\Delta_G}$, consider an arbitrary generator X_iX_j of I_G, given by the edge $v_iv_j \in E$. It follows that the set $\{v_i, v_j\}$ is not independent in G, so it is a non-face of Δ_G by definition. It follows that we have $X_iX_j \in J_{\Delta_G}$.

For the reverse containment $I_G \supseteq J_{\Delta_G}$, consider a generator $X_{i_1} \cdots X_{i_n}$ of J_{Δ_G}, given by the non-face $\{v_{i_1}, \ldots, v_{i_n}\} \notin \Delta_G$. By definition, this means that the set $\{v_{i_1}, \ldots, v_{i_n}\}$ is not independent in G, so it must contain a pair of adjacent vertices v_{i_k}, v_{i_m}. It follows that $X_{i_k}X_{i_m}$ is a generator of I_G. Thus, we have $X_{i_1} \cdots X_{i_n} \in (X_{i_k}X_{i_m})R \subseteq I_G$, as desired. $\square$

For example, consider the graph G from Example 4.3.4, with Δ_G identified in Example 4.4.7. The non-faces of Δ_G are exactly the following subsets of $V = \{v_1, v_2, v_3, v_4\}$:

$$\begin{array}{ccccc} \{v_1,v_2\} & \{v_1,v_3\} & \{v_1,v_4\} & \{v_2,v_3\} & \{v_3,v_4\} \\ \{v_1,v_2,v_3\} & \{v_1,v_2,v_4\} & \{v_1,v_3,v_4\} & \{v_2,v_3,v_4\} & \{v_1,v_2,v_3,v_4\}. \end{array}$$

Thus, the ideal J_{Δ_G} is generated by the following list of monomials: X_1X_2, X_1X_3, X_1X_4, X_2X_3, X_3X_4, $X_1X_2X_3$, $X_1X_2X_4$, $X_1X_3X_4$, $X_2X_3X_4$, $X_1X_2X_3X_4$. Removing redundancies from this list, we see that

$$J_{\Delta_G} = (X_1X_2, X_1X_3, X_1X_4, X_2X_3, X_3X_4)R = I_G$$

as in Theorem 4.4.9.

The following notions are for use in Chapter 5. In particular, the dimension of a simplicial complex gives a measure of its size.

Definition 4.4.10. Let Δ be a simplicial complex on $V = \{v_1, \ldots, v_d\}$. The *dimension* of a face $F \in \Delta$ is $|F| - 1$. The *dimension* of Δ, denoted $\dim(\Delta)$, is the maximal dimension of a face of Δ. The simplicial complex Δ is *pure* if every facet of Δ has the same dimension.

For $i = -1, 0, \ldots, \dim(\Delta)$, let $f_i(\Delta)$ denote the number of i-dimensional faces of Δ. The *f-vector* of Δ is the vector $f(\Delta) = (f_0(\Delta), f_1(\Delta), \ldots, f_{\dim(\Delta)}(\Delta))$.

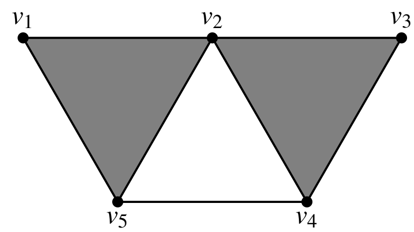
By definition, and as one might expect, vertices have dimension 0, edges have dimension 1, and so on. Thus, the dimension of a graph with at least one edge is 1. Since the facets of Δ are its maximal faces, one can compute $\dim(\Delta)$ as the maximal dimension of a facet of Δ. Also, for the simplicial complexes from Example 4.4.2, we compute

$$\begin{array}{ll} \dim(\Delta) = 2 & f(\Delta) = (5,6,1) \\ \dim(\Delta') = 3 & f(\Delta') = (5,8,5,1). \end{array}$$

Exercises

Exercise 4.4.11. Sketch geometric realizations of all simplicial complexes on d vertices for $d = 1,2,3,4$. (Don't forget to include complexes with non-adjacent vertices.)

Exercise 4.4.12. Set $R = A[X_1,\ldots,X_5]$, and let Δ be the simplicial complex represented by the following sketch:



(a) Find $\dim(\Delta)$ and $f(\Delta)$.
(b) Find an irredundant monomial generating sequence for J_Δ.
(c) Find all facets of Δ.

Justify your answers.

Exercise 4.4.13. Let Δ and Δ' be simplicial complexes on $V = \{v_1,\ldots,v_d\}$, and set $R = A[X_1,\ldots,X_d]$.

(a) Prove that $\Delta \subseteq \Delta'$ if and only if $J_{\Delta'} \subseteq J_\Delta$.
(b) Prove that $\Delta = \Delta'$ if and only if $J_{\Delta'} = J_\Delta$.
(c) Prove that $\Delta = \mathsf{P}(V)$ if and only if $J_\Delta = 0$.
(d) Prove that $\Delta = \{\emptyset\}$ if and only if $J_\Delta = (X_1,\ldots,X_d)R$.

Exercise 4.4.14. Let Δ be a simplicial complex on $V = \{v_1,\ldots,v_d\}$, and set $R = A[X_1,\ldots,X_d]$. Prove that the irredundant generators of J_Δ are in bijection with the minimal non-faces of Δ.

**Exercise 4.4.15.* Let G be a graph with vertex set $V = \{v_1,\ldots,v_d\}$ and edge set E. Identify G with its associated simplicial complex. Set $R = A[X_1,\ldots,X_d]$. Prove that the face ideal J_G is "generated by the non-edges of G and all the closed triangles":

$$J_G = (\{X_iX_j \mid v_iv_j \text{ is not an edge in } G\} \cup \{X_iX_jX_k \mid 1 \leqslant i < j < k \leqslant d\})R.$$

(This exercise is used in Exercise 4.5.12.)

Exercise 4.4.16. Set $R = A[X_1,\ldots,X_5]$, and let G be the graph from Exercise 4.3.10.

(a) Find the independent subsets in G and the maximal independent subsets in G. Sketch the geometric realization of the independence complex Δ_G.
(b) Find $\dim(\Delta_G)$ and $f(\Delta_G)$.

(c) Compute an irredundant monomial generating sequence for J_{Δ_G}, and compare it to the generating sequence from Exercise 4.3.10(a).

Justify your answers.

Exercise 4.4.17. Prove Lemma 4.4.8.

Exercise 4.4.18. Let G be a graph on $V = \{v_1, \ldots, v_d\}$. Describe the dimension and f-vector of the independence complex of G in terms of G. Justify your answers.

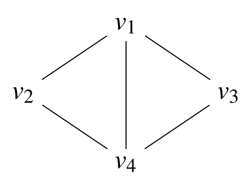
**Exercise 4.4.19.* Let $V = \{v_1, \ldots, v_d\}$ and $R = A[X_1, \ldots, X_d]$. Prove that the association $\Delta \mapsto J_\Delta$ describes a bijection between the set of simplicial complexes on V and the set of square-free monomial ideals of R. (Hint: Show that the following rule describes an inverse for this function. For a square-free monomial ideal I of R, map I to the simplicial complex Δ described by the exponent vectors of the square-free monomials of $[\![R]\!] \smallsetminus [\![I]\!]$.) (This is used in Example 4.5.6 and Challenge Exercises 7.1.7, 7.3.13, and 7.4.21.)

**Exercise 4.4.20.* Fix a partially ordered set (i.e., a poset) $\Pi = (V, \leqslant)$ with $V = \{v_1, \ldots, v_d\}$. The *order complex* associated to Π is the set of all chains in Π:

$$\Delta(\Pi) = \{\{v_{i_1}, \ldots, v_{i_n}\} \subseteq V \mid n \geqslant 0 \text{ and } v_{i_1} \leqslant \cdots \leqslant v_{i_n}\}.$$

Set $R = A[X_1, \ldots, X_d]$.

(a) Prove that the order complex associated to Π is a simplicial complex on V.
(b) Sketch the geometric realization of the order complex $\Delta(\Pi)$ associated to the following partially ordered set:



Here the order is represented vertically by the graph structure. For instance, we have $v_4 < v_2 < v_1$, hence $v_4 < v_1$.
(c) Prove that the face ideal $J_{\Delta(\Pi)}$ is "generated by the products of pairs of incomparable elements":

$$J_{\Delta(\Pi)} = (X_i X_j \mid v_i \not\leqslant v_j \text{ and } v_j \not\leqslant v_i)R.$$

(d) Describe the dimension and f-vector of $\Delta(\Pi)$ in terms of the partial order on Π. Justify your answers.

(This exercise is used in Exercises 4.5.14, 4.5.19, 4.6.14, 4.6.17, and 4.7.24.)

Simplicial Complexes and Face Ideals in Macaulay2

As the package `EdgeIdeals` facilitates the study of edge ideals, the package `SimplicialComplexes` can be used to study simplicial complexes and face ideals; it was written by Sorin Popescu, Gregory Smith, and Michael Stillman. This package also comes with the default distrubution of Macaulay2 and is loaded using the following command.

```
i1 : needsPackage "SimplicialComplexes"
o1 = SimplicialComplexes
o1 : Package
```

One creates a `SimplicialComplex` object using the `simplicialComplex` method, and there are two ways to do so. Either way requires a polynomial ring in which to work, so let us create that first.

```
i2 : R = QQ[x_1..x_5];
```

The first way to create a `SimplicialComplex` object is to provide a list of monomials, with the support of each monomial a facet of the simplicial complex in question. So, to illustrate we would input the simplicial complex Δ' in Example 4.4.2 using the following command.

```
i3 : Delta = simplicialComplex {x_1*x_2*x_4*x_5, x_2*x_3*x_4 }
o3 = | x_1x_2x_4x_5 x_2x_3x_4 |
o3 : SimplicialComplex
```

Alternatively, one can also provide a `MonomialIdeal` object as input that contains the generators of the face ideal. Note that an error is reported should any of the generators not be square-free. Using the computation performed in Example 4.4.4, we input as follows.

```
i4 : Delta2 = simplicialComplex monomialIdeal {x_1*x_3,x_3*x_5}
o4 = | x_1x_2x_4x_5 x_2x_3x_4 |
o4 : SimplicialComplex
```

Note that when a `SimplicialComplex` object is displayed to the user, the facets are listed (and hence computed, if necessary). We can of course check if these determine the same `SimplicialComplex` object.

```
i5 : Delta == Delta2
o5 = true

i6 : Delta === Delta2
o6 = true
```

One nice feature of the `SimplicialComplexes` packages is the `fVector` method, which returns the f-vector of the simplicial complex; see Definition 4.4.10.

```
i7 : fVector Delta
o7 = HashTable{-1 => 1}
               0 => 5
               1 => 8
```

```
                 2 => 5
                 3 => 1
o7 : HashTable
```

One can check that these face counts are the same as those in Example 4.4.2.

Exercises

Exercise 4.4.21. Use the two techniques described in the tutorial above to create the simplicial complex Δ from Example 4.4.2. Verify that the two techniques give the same simplicial complex. Use Macaulay2 to find the f-vector of Δ. Repeat this with the simplicial complex from Exercise 4.4.12.

Documentation Exercise 4.4.22. Use Macaulay2 to find the face ideals and the dimensions of the simplicial complexes from Example 4.4.2. (For help, see the documentation for the `SimplicialComplexes` package, specifically the commands `monomialIdeal` and `dim`.)

Exercise 4.4.23. Use Macaulay2 to verify the conclusion of Exercise 4.4.15 for the graphs in Section 4.2. (Be sure to define the graphs as `SimplicialComplex` objects, not as `Graph` objects.)

**Documentation Exercise 4.4.24.* In this exercise, we consider a graph G and its independence complex Δ_G; see Definition 4.4.6.

(a) Consult the documentation for the command `independenceComplex` in the Macaulay2 `EdgeIdeals` package.
(b) Use Macaulay2 to find the independence complexes of the graphs from Example 4.4.7 and Exercise 4.3.10, and to compute the dimensions and the f-vectors of these complexes. Compare with your answer to Exercise 4.4.18.
(c) Use Macaulay2 to find the face ideals of the independence complexes of the graphs from Example 4.4.7 and Exercise 4.4.16.

(This exercise is used in Coding Exercise 4.5.22.)

**Documentation Exercise 4.4.25.* In this exercise, we consider a poset Π and its order complex $\Delta(\Pi)$; see Exercise 4.4.20.

(a) Consult the documentation for the Macaulay2 package `Posets`, including the command `orderComplex`; see Exercise 4.4.20.
(b) Use Macaulay2 to find the order complex of the poset from Exercise 4.4.20(b), and to compute the dimension and the f-vector of this complex. Compare this with your answer to Exercise 4.4.20(d).
(c) Use Macaulay2 to find the face ideal of the order complex of the poset from Exercise 4.4.20(b), and to verify the conclusion of Exercise 4.4.20(c).

(This exercise is used in Exercise 4.5.19.)

4.5 Decompositions of Face Ideals

In this section, A is a non-zero commutative ring with identity.

This section gives a method for computing m-irreducible decompositions for face ideals of simplicial complexes. As for edge ideals, this decomposition is given in terms of combinatorial information about the simplicial complex, namely its facets. Moreover, we show how to use this to find m-irreducible decompositions for arbitrary square-free monomial ideals. We begin with some notation for the relevant m-irreducible monomial ideals.

Definition 4.5.1. Let $V = \{v_1,\ldots,v_d\}$, and set $R = A[X_1,\ldots,X_d]$. For each subset $F \subseteq V$, let $Q_F \subseteq R$ be the ideal "generated by the non-elements of F":

$$Q_F = (\{X_i \mid v_i \notin F\})R.$$

For instance, in the ring $R = A[X_1,\ldots,X_5]$ with $V = \{v_1,\ldots,v_5\}$, we have

$$Q_\emptyset = (X_1,\ldots,X_5)R \qquad Q_{\{v_1,v_3\}} = (X_2,X_4,X_5)R \qquad Q_V = 0.$$

In general, given a subset $F \subseteq V$, one has $Q_F = P_{V\smallsetminus F}$ by definition. Accordingly, a monomial ideal $J \subseteq R$ is square-free and m-irreducible if and only if there exists a subset $F \subseteq V$ such that $J = Q_F$, by Proposition 4.1.3.

Fact 4.5.2. Let $V = \{v_1,\ldots,v_d\}$ be a finite set, and set $R = A[X_1,\ldots,X_d]$.

(a) Given subsets $F,G \subseteq V$, one has $Q_F \subseteq Q_G$ if and only if $G \subseteq F$.
(b) A monomial ideal $J \subsetneq R$ is square-free if and only if there are subsets $F_1,\ldots,F_n \subseteq V$ such that $J = \bigcap_{i=1}^n Q_{F_i}$; see Proposition 4.1.6.

Like Lemma 4.3.5, the connection between faces and m-irreducible decompositions begins with the next result.

Lemma 4.5.3. *Let Δ be a simplicial complex on $V = \{v_1,\ldots,v_d\}$, and let $F \subseteq V$. Set $R = A[X_1,\ldots,X_d]$. Then $J_\Delta \subseteq Q_F$ if and only if F is a face of Δ.*

Proof. Write $F = \{v_{i_1},\ldots,v_{i_n}\}$ and $V \smallsetminus F = \{v_{j_1},\ldots,v_{j_p}\}$, so that we have $Q_F = (X_{j_1},\ldots,X_{j_p})R$.

For the forward implication, assume that $J_\Delta \subseteq Q_F$. By way of contradiction, suppose that F is not a face of Δ, that is, $F \notin \Delta$. By definition, this implies that $X_{i_1}\cdots X_{i_n} \in J_\Delta \subseteq Q_F$. It follows that there is an index k such that $X_{i_1}\cdots X_{i_n} \in (X_{j_k})R$. An inspection of exponent vectors shows that there is an index l such that $j_k = i_l$. This says that $F \cap (V \smallsetminus F) \neq \emptyset$, a contradiction.

For the reverse implication, assume that $F \in \Delta$. To show that $J_\Delta \subseteq Q_F$, we need to show that each generator of J_Δ is in Q_F. To this end, fix a generator $X_{r_1}\cdots X_{r_q} \in J_\Delta$, corresponding to a "non-face" $V' = \{v_{r_1},\ldots,v_{r_q}\} \notin \Delta$. Since $F \in \Delta$, the defining condition for a simplicial complex shows that $V' \nsubseteq F$. It follows that there is an index s such that $v_{r_s} \in V' \smallsetminus F$, so $X_{r_s} \in Q_F$. We conclude that the generator $X_{r_1}\cdots X_{r_q}$ is in $(X_{r_s})R \subseteq Q_F$, as desired. □

Next, we present the decomposition theorem for face ideals. As with the corresponding result for edge ideals, it shows how the combinatorial properties of a simplicial complex determine algebraic properties of its face ideal. Remark 4.4.5 and Exercise 4.4.19 show how this applies to the study of arbitrary square-free monomial ideals.

Theorem 4.5.4 *Let Δ be a simplicial complex on $V = \{v_1, \ldots, v_d\}$, and set $R = A[X_1, \ldots, X_d]$. The ideal $J_\Delta \subseteq R$ has the following m-irreducible decompositions*

$$J_\Delta = \bigcap_{F\in\Delta} Q_F = \bigcap_{F \text{ facet}} Q_F$$

where the first intersection is taken over all faces of Δ, and the second intersection is taken over all facets of Δ. The second intersection is irredundant.

Proof. Fact 4.5.2(a) shows that the second intersection is irredundant. The containment $\bigcap_{F\in\Delta} Q_F \subseteq \bigcap_{F \text{ facet}} Q_F$ is routine. The reverse containment $\bigcap_{F\in\Delta} Q_F \supseteq \bigcap_{F \text{ facet}} Q_F$ follows from the fact that every face of Δ is contained in a facet, along with Fact 4.5.2(a). The containment $J_\Delta \subseteq \bigcap_{F\in\Delta} Q_F$ is from Lemma 4.5.3.

For the final containment $J_\Delta \supseteq \bigcap_{F\in\Delta} Q_F$ recall that J_Δ is square-free. Hence, Fact 4.5.2(b) provides subsets $F_1, \ldots, F_n$ such that $J_\Delta = \bigcap_{j=1}^n Q_{F_j}$. For each index j, we then have $J_\Delta \subseteq Q_{F_j}$, so Lemma 4.5.3 implies that F_j is a face of Δ. It follows that $J_\Delta = \bigcap_{j=1}^n Q_{F_j} \supseteq \bigcap_{F\in\Delta} Q_F$, as desired. $\square$

Example 4.5.5. We compute an irredundant m-irreducible decomposition of the ideals J_Δ and $J_{\Delta'}$ from Example 4.4.4. Using Theorem 4.5.4, this can be read from the lists of facets that we computed in Example 4.4.2:

$$\begin{aligned} J_\Delta &= (X_2, X_4, X_5)R \cap (X_2, X_3, X_5)R \cap (X_2, X_3, X_4)R \\ &\qquad \cap (X_1, X_2, X_4)R \cap (X_1, X_5)R \\ J_{\Delta'} &= (X_1, X_5)R \cap (X_3)R. \end{aligned}$$

This second computation agrees with the decomposition of $J_{\Delta'}$ from Example 4.4.4.

In light of Remark 4.4.5 and Exercise 4.4.19, we can use the techniques of this section to find m-irreducible decompositions of arbitrary square-free monomial ideals, as in the following example.

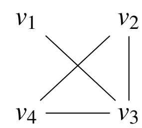
Example 4.5.6. We compute an irredundant m-irreducible decomposition of the square-free monomial ideal

$$J = (X_1X_2, X_2X_3X_4, X_1X_4)R \subseteq R = A[X_1, X_2, X_3, X_4].$$

First, we find a simplicial complex Δ on $V = \{v_1, v_2, v_3, v_4\}$ such that $J = J_\Delta$. To do this, we need to add a face for every square-free monomial that is not in J:

$$\Delta = \{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}\}.$$

The geometric realization of Δ is the following graph:



Next, we list the facets of Δ:

$$\{v_1, v_3\} \qquad \{v_2, v_3\} \qquad \{v_2, v_4\} \qquad \{v_3, v_4\}.$$

Finally, we read off the decomposition using Theorem 4.5.4:

$$J = J_\Delta = (X_2, X_4)R \cap (X_1, X_4)R \cap (X_1, X_3)R \cap (X_1, X_2)R.$$

The next result shows how one can use the independence complex of a graph to find m-irreducible decompositions of the edge ideals of Sections 4.2–4.3.

Theorem 4.5.7 *Let G be a graph with vertex set $V = \{v_1, \ldots, v_d\}$ and edge set E. Set $R = A[X_1, \ldots, X_d]$. Then the ideal $I_G \subseteq R$ has the following m-irreducible decompositions*

$$I_G = \bigcap_{F \text{ indep.}} Q_F = \bigcap_{F \text{ max. indep.}} Q_F$$

where the first intersection is taken over all independent subsets in G, and the second intersection is taken over all maximal independent subsets in G. The second intersection is irredundant.

Proof. By definition, the independent subsets in G are the faces of Δ_G, and the maximal independent subsets in G are the facets of Δ_G. So, the result follows from Theorem 4.5.4. (Alternately, one can apply Theorem 4.5.4 and Lemma 4.4.8.) $\square$

For instance, consider the graph G from Example 4.3.4, with maximal independent sets identified in Example 4.4.7. Theorem 4.5.7 implies that the irredundant m-irreducible decomposition of I_G is

$$I_G = Q_{\{v_1\}} \cap Q_{\{v_3\}} \cap Q_{\{v_2, v_4\}} = (X_2, X_3, X_4)R \cap (X_1, X_2, X_4)R \cap (X_1, X_3)R.$$

Compare to Example 4.3.7.

In general, given an irredundant m-irreducible decomposition $I_G = \bigcap_{i=1}^n P_{V_i}$ as in Proposition 4.1.6, one concludes that the maximal independent subsets in G are precisely $V \smallsetminus V_1, \ldots, V \smallsetminus V_n$. Indeed, Theorem 4.5.7 gives an irredundant m-irreducible decomposition $I_G = \bigcap_{F \text{ max. indep.}} Q_F$, so the uniqueness of such decompositions from Theorem 3.3.8 provides the desired conclusion.

Exercises

Exercise 4.5.8. Verify the statements in Fact 4.5.2.

Exercise 4.5.9. Set $R = A[X_1,\dots,X_5]$, and let Δ be the simplicial complex from Exercise 4.4.12.

(a) Use Theorem 4.5.4 to find an irredundant m-irreducible decomposition of J_Δ.
(b) Verify the decomposition $J_\Delta = \bigcap_F Q_F$ from part (a) using Theorem 2.1.5.

Justify your answers.

Exercise 4.5.10. Verify the following decompositions

$$\begin{aligned} J_\Delta &= (X_2,X_4,X_5)R \cap (X_2,X_3,X_5)R \cap (X_2,X_3,X_4)R \cap (X_1,X_2,X_4)R \cap (X_1,X_5)R \\ J &= (X_2,X_4)R \cap (X_1,X_4)R \cap (X_1,X_3)R \cap (X_1,X_2)R \end{aligned}$$

from Examples 4.5.5 and 4.5.6 using Theorem 2.1.5.

Exercise 4.5.11. Set $R = A[X_1,\dots,X_4]$ and find an irredundant m-irreduci-ble decomposition of the ideal $J = (X_1X_2X_3, X_1X_2X_4, X_1X_3X_4, X_2X_3X_4)R$ as in Example 4.5.6. Verify that your decomposition is correct using Theorem 2.1.5. Justify your answer.

Exercise 4.5.12. Let G be a graph with vertex set $V = \{v_1,\dots,v_d\}$ and edge set E. Identify G with its associated simplicial complex; see Exercise 4.4.15. Set $R = A[X_1,\dots,X_d]$.

(a) Prove that J_G can be decomposed in terms of the edges in G:
$$\begin{aligned} J_G &= \left(\bigcap_{v_i \in V} Q_{\{v_i\}}\right) \cap \left(\bigcap_{v_iv_j \in E} Q_{\{v_i,v_j\}}\right) \\ &= \left(\bigcap_{\substack{v_i \in V \\ \text{isolated}}} Q_{\{v_i\}}\right) \cap \left(\bigcap_{v_iv_j \in E} Q_{\{v_i,v_j\}}\right). \end{aligned}$$
The first intersection is taken over all vertices of G. The second and fourth intersections are taken over all edges v_iv_j in G. The third intersection is taken over all isolated vertices of G. Prove that the second decomposition is irredundant.
(b) Use part (a) to find an irredundant m-irreducible decomposition of the face ideal J_G where G is the graph from Exercise 4.3.10. Justify your answer.
(c) Verify that your decomposition from part (b) is correct using Theorem 2.1.5.

Exercise 4.5.13. Set $R = A[X_1,\dots,X_5]$, and let G be the graph from Exercise 4.3.10. Use Theorem 4.5.7 to find an irredundant m-irreducible decomposition of I_G, and compare it to the decomposition from Exercise 4.3.10(c). Justify your answers.

Exercise 4.5.14. Fix a partially ordered set (i.e., a poset) $\Pi = (V, \leqslant)$ with $V = \{v_1, \dots, v_d\}$. Consider the order complex $\Delta(\Pi)$ from Exercise 4.4.20. Set $R = A[X_1, \dots, X_d]$.

(a) Prove that $J_{\Delta(\Pi)}$ can be decomposed in terms of the chains in Π:

$$J_{\Delta(\Pi)} = \bigcap_{v_{i_1} < \cdots < v_{i_n}} Q_{\{v_{i_1}, \cdots, v_{i_n}\}} = \bigcap_{\substack{v_{i_1} < \cdots < v_{i_n} \\ \text{max.}}} Q_{\{v_{i_1}, \cdots, v_{i_n}\}}.$$

Here the first intersection is taken over all chains $v_{i_1} < \cdots < v_{i_n}$ in Π, and the second intersection is taken over all maximal chains $v_{i_1} < \cdots < v_{i_n}$ in Π. Prove that the second decomposition is irredundant.

(b) Use part (a) to find an irredundant m-irreducible decomposition of $J_{\Delta(\Pi)}$ where Π is the partially ordered set from Exercise 4.4.20(b). Justify your answer.

(c) Verify that your decomposition from part (b) is correct using Theorem 2.1.5.

The next exercises involve the construction $\mathrm{V}(I)$ from Section A.10.

Challenge Exercise 4.5.15. Let A be a field, and let Δ be a simplicial complex on $\{v_1, \dots, v_d\}$. Exercise 3.3.11(b) shows that there is an integer $k \in \mathbb{N}$, and there are linear subspaces $V_1, \dots, V_k \subseteq A^d$ such that $\mathrm{V}(J_\Delta) = V_1 \cup \cdots \cup V_k$ and such that $V_i \not\subseteq V_j$ whenever $i \neq j$. Describe the linear subspaces V_i in terms of the facets of Δ. Justify your answer.

Challenge Exercise 4.5.16. Let A be a field, and let G be a graph with vertex set $V = \{v_1, \dots, v_d\}$ and edge set E. Identify G with its associated simplicial complex; see Exercise 4.5.12. Exercise 3.3.11(b) shows that there is an integer $k \in \mathbb{N}$, and there are linear subspaces $V_1, \dots, V_k \subseteq A^d$ such that $\mathrm{V}(J_\Delta) = V_1 \cup \cdots \cup V_k$ and such that $V_i \not\subseteq V_j$ whenever $i \neq j$. Describe the linear subspaces V_i in terms of the edges and isolated vertices of G. Justify your answer.

Challenge Exercise 4.5.17. Let A be a field, and let $\Pi = (V, \leqslant)$ be a poset with $V = \{v_1, \dots, v_d\}$. Consider the order complex $\Delta(\Pi)$ associated to Π. Exercise 3.3.11(b) shows that there is an integer $k \in \mathbb{N}$, and there are linear subspaces $V_1, \dots, V_k \subseteq A^d$ such that $\mathrm{V}(J_{\Delta(\Pi)}) = V_1 \cup \cdots \cup V_k$ and such that $V_i \not\subseteq V_j$ whenever $i \neq j$. Describe the linear subspaces V_i in terms of the maximal chains of Π. Justify your answer.

Decompositions of Face Ideals in Macaulay2

By Theorem 4.5.4, an irredundant m-irreducible decomposition of the face ideal of a simplicial complex Δ is determined by the facets of Δ. In this tutorial, we show how to assemble these data. We continue the Macaulay2 session from the last section, wherein `Delta` is the simplicial complex Δ' in Example 4.4.2.

```
i8 : Delta
o8 = | x_1x_2x_4x_5 x_2x_3x_4 |
o8 : SimplicialComplex
```

We can obtain the facets of `Delta` via the command `facets`.

```
i9 : facets Delta
o9 = | x_1x_2x_4x_5 x_2x_3x_4 |
             1       2
o9 : Matrix R  <--- R
```

Note that this is a `Matrix` object whose entries are monomials whose supports are the facets of the simplicial complex. We really want to obtain the facets as lists of variables, which we can do using the next command.

```
i10 : facetSupports = (flatten entries facets Delta) / support
o10 = {{x , x , x , x }, {x , x , x }}
         1   2   4   5     2   3   4
o10 : List
```

Recall that the component corresponding to a facet in the m-irreducible decomposition of a face ideal is generated by the variables that *do not* appear in the facet. So, we must take the ideal generated by the complement of each list of variables, and this is the m-irreducible decomposition we seek. To do this, first create a `Set` object that has all the variables of the ring we are working in.

```
i11 : allVars = set gens R
o11 = set {x , x , x , x , x }
            1   2   3   4   5
o11 : Set
```

Now, we find the complement using the `-` operator for set difference.

```
i12 : mIrredDecomp = apply(facetSupports, f ->
monomialIdeal (if #(allVars - f) == 0 then 0_R else toList(allVars - f))
)
o12 = {monomialIdeal x , monomialIdeal (x , x )}
                      3                  1   5
o12 : List
```

Note that we need the conditional in the last line since if the input is the full simplex, then `allVars - f` will be the empty list, and then the command `monomialIdeal` would fail. As usual, we check to ensure that this is indeed an m-irreducible decomposition of the face ideal J_Δ.

```
i13 : intersect mIrredDecomp == monomialIdeal Delta
o13 = true
```

Exercises

Exercise 4.5.18. Use Macaulay2 to verify the conclusions of Exercise 4.5.12 for the graphs from Example 4.4.7 and Exercise 4.3.10. (Make sure to define the graphs as `SimplicialComplex` objects, not as `Graph` objects.)

Exercise 4.5.19. Use Macaulay2 to verify the conclusion of Exercise 4.5.14(a) for the order complex of the poset from Exercise 4.4.20(b); see Documentation Exercise 4.4.25.

Coding Exercise 4.5.20. Write a method that takes a simplicial complex as input and returns an irrendundant m-irreducible decomposition of its face ideal via Theorem 4.5.4. Compare the outputs and run times of your method and the command `irreducibleDecomposition monomialIdeal` with the examples and exercises from this section and the previous section.

Coding Exercise 4.5.21. Write a method that takes a square-free monomial ideal as input and uses a simplicial complex to return an irredundant m-irreducible decomposition. Compare the outputs and run times of your method and the command `irreducibleDecomposition` with the examples and exercises from this section and the previous sections of this chapter.

Coding Exercise 4.5.22. Recall from Documentation Exercise 4.4.24 the command `independenceComplex`.

(a) Write a method that takes a graph as input and uses the Macaulay2 command `independenceComplex` to return an irredundant m-irreducible decomposition of its edge ideal via Theorem 4.4.9 and Coding Exercise 4.5.20.
(b) Compare the outputs and run times of your method from part (a) and the command `irreducibleDecomposition edgeIdeal` with the examples and exercises from this section and the previous sections of this chapter.

4.6 Facet Ideals and Their Decompositions

In this section, A is a non-zero commutative ring with identity.

Here, we investigate a version of the edge ideal for simplicial complexes (instead of just for graphs). It gives another algebraic construction defined in terms of combinatorial data from a simplicial complex, and we show how other combinatorial information provides m-irreducible decompositions.

Definition 4.6.1. Let Δ be a simplicial complex on $V = \{v_1, \dots, v_d\}$, and set $R = A[X_1, \dots, X_d]$. The *facet ideal* of R associated to Δ is the ideal "generated by the facets of Δ":

$$K_\Delta = (X_{i_1} \cdots X_{i_s} \mid 1 \leqslant i_1 < \cdots < i_s \leqslant d \text{ and } \{v_{i_1}, \dots, v_{i_n}\} \text{ is a facet in } \Delta)R.$$

For example, we consider the simplicial complexes from Example 4.4.2. The facets of Δ are $\{v_1,v_2\}$, $\{v_2,v_3\}$, $\{v_3,v_4\}$, $\{v_2,v_4,v_5\}$. It follows that we have $K_\Delta = (X_1X_2, X_2X_3, X_3X_4, X_2X_4X_5)$R. Similarly, for the complex Δ' we have $K_{\Delta'} = (X_2X_3X_4, X_1X_2X_4X_5)R$. Since the empty product is 1, by convention, we have $K_{\{\emptyset\}} = R$ if and only if $\Delta = \{\emptyset\}$. Also, we have $K_\Delta \neq 0$ for all Δ.

If Δ has no edges, that is, if $\Delta = \{\{v_{i_1}\},\dots,\{v_{i_n}\},\emptyset\}$ for some vertices v_{i_j}, then $K_\Delta = (X_{i_1},\dots,X_{i_n})R$. Contrast this with the *edge ideal* of a graph G with no edges in Example 4.2.3.

In general, the facet ideal K_Δ is square-free, by definition. Moreover, since the facets of Δ are incomparable with respect to containment, they generate K_Δ irredundantly, that is, the set

$$\{X_{i_1}\cdots X_{i_s} \mid 1 \leqslant i_1 < \cdots < i_s \leqslant d \text{ and } \{v_{i_1},\dots,v_{i_n}\} \text{ is a facet in } \Delta\}$$

describes an irredundant monomial generating sequence for K_Δ.

The following notions are used to identify which ideals $P_{V'}$ occur in an (irredundant) m-irreducible decomposition of a facet ideal.

Definition 4.6.2. Let Δ be a simplicial complex on V. A *vertex cover* of Δ is a subset $V' \subseteq V$ such that for each facet F in Δ the set $F \cap V'$ is non-empty. A vertex cover of Δ is *minimal* if it does not properly contain another vertex cover of Δ.

If $\Delta \neq \{\emptyset\}$, then V is a vertex cover of Δ; in particular, Δ has a vertex cover. On the other hand, the complex $\{\emptyset\}$ has no vertex covers. If $\Delta \neq \{\emptyset\}$ has no edges, that is, if $\Delta = \{\{v_{i_1}\},\dots,\{v_{i_n}\},\emptyset\}$ for some vertices v_{i_j} with $n \geqslant 1$, then the vertex covers of Δ are precisely the subsets of V containing all the v_{i_j}'s. Contrast this with the vertex covers of the *graph* G with no edges in Example 4.3.2.

The set of vertex covers of Δ is closed under supersets: if $V' \subseteq V$ is a vertex cover of Δ and $V' \subseteq V'' \subseteq V$, then V'' is a vertex cover of Δ. Since V is finite, every vertex cover of Δ contains a minimal vertex cover of Δ.

Example 4.6.3. We compute the minimal vertex covers of the simplicial complexes from Example 4.4.2.

First, we find the minimal vertex covers of Δ containing v_2. If $v_2 \in V'$, then the facets $\{v_1,v_2\}$, $\{v_2,v_3\}$, and $\{v_2,v_4,v_5\}$ are "covered", leaving only the facet v_3v_4 "uncovered". This facet can be covered by adding either v_3 or v_4. It is now routine to show that the minimal vertex covers containing v_2 are $\{v_2,v_3\}$ and $\{v_2,v_4\}$.

Next, we find the minimal vertex covers of Δ that do not contain v_2. If $v_2 \notin V'$, we must have $v_1, v_3 \in V'$ in order to cover the facets $\{v_1,v_2\}$ and $\{v_2,v_3\}$. To cover the facet $\{v_2,v_4,v_5\}$, we must add either v_4 or v_5. It is straightforward to show that the sets $\{v_1,v_3,v_4\}$ and $\{v_1,v_3,v_5\}$ are minimal vertex covers of Δ.

A similar argument shows that the minimal vertex covers of Δ' are $\{v_2\}$, $\{v_4\}$, $\{v_1,v_3\}$, and $\{v_3,v_5\}$.

Similarly to previous constructions in this chapter, the connection between vertex covers and m-irreducible decompositions begins with the next result. It uses the notation from Definition 4.1.4.

Lemma 4.6.4. *Let Δ be a simplicial complex on $V = \{v_1, \dots, v_d\}$, and let $V' \subseteq V$. Set $R = A[X_1, \dots, X_d]$. Then $K_\Delta \subseteq P_{V'}$ if and only if V' is a vertex cover of Δ.*

Proof. Write $V' = \{v_{i_1}, \dots, v_{i_n}\}$, so that $P_{V'} = (X_{i_1}, \dots, X_{i_n})R$.

For the forward implication, assume that $K_\Delta \subseteq P_{V'}$. We show that V' is a vertex cover of Δ. Let $\{v_{j_1}, \dots, v_{j_k}\}$ be a facet of Δ. Then we have $X_{j_1} \cdots X_{j_k} \in K_\Delta \subseteq P_{V'} = (X_{i_1}, \dots, X_{i_n})R$. It follows that $X_{j_1} \cdots X_{j_k} \in (X_{i_m})R$ for some index m. A comparison of exponent vectors shows that $j_l = i_m$ for some l, that is, that $v_{j_l} = v_{i_m} \in V'$. Thus V' is a vertex cover of Δ.

For the reverse implication, assume that V' is a vertex cover of Δ. To show that $K_\Delta \subseteq P_{V'}$, we need to show that each generator of K_Δ is in $P_{V'}$. To this end, fix a generator $X_{j_1} \cdots X_{j_k} \in K_\Delta$, corresponding to a facet $\{v_{j_1}, \dots, v_{j_k}\}$ in Δ. Since V' is a vertex cover of Δ, we have $v_{j_l} \in V'$ for some index l. It follows that $X_{j_l} \in P_{V'}$, so $X_{j_1} \cdots X_{j_k} \in P_{V'}$. □

Here is the decomposition theorem for facet ideals. As in previous results, it shows how combinatorial information from a given simplicial complex informs algebraic properties of its facet ideal. Again, the subsequent remark shows how this applies to arbitrary square-free monomial ideals.

Theorem 4.6.5 *Let Δ be a simplicial complex on $V = \{v_1, \dots, v_d\}$, and set $R = A[X_1, \dots, X_d]$. The facet ideal $K_\Delta \subseteq R$ has the next m-irreducible decompositions*

$$K_\Delta = \bigcap_{V'} P_{V'} = \bigcap_{V' \text{ min.}} P_{V'}.$$

The first intersection is taken over all vertex covers of Δ; the second intersection is taken over all minimal vertex covers of Δ and is irredundant.

Proof. Exercise. Argue as in the proof of Theorem 4.3.6. □

Example 4.6.6. We compute an irredundant m-irreducible decomposition of the ideals K_Δ and $K_{\Delta'}$ where Δ and Δ' are the simplicial complexes from Example 4.4.2. Using Theorem 4.3.6, this can be read from the list of minimal vertex covers that we computed in Example 4.6.3:

$$\begin{aligned} K_\Delta &= (X_2, X_3)R \cap (X_2, X_4)R \cap (X_1, X_3, X_4)R \cap (X_1, X_3, X_5)R \\ K_{\Delta'} &= (X_2)R \cap (X_4)R \cap (X_1, X_3)R \cap (X_3, X_5)R. \end{aligned}$$

In general, given an irredundant m-irreducible decomposition $K_\Delta = \bigcap_{i=1}^n P_{V_i}$ as in Proposition 4.1.6, one concludes the minimal vertex covers of Δ are precisely $V_1, \dots, V_n$. Indeed, Theorem 4.6.5 gives an irredundant m-irreducible decomposition $K_\Delta = \bigcap_{V' \text{ min.}} P_{V'}$, so the uniqueness of such decompositions from Theorem 3.3.8 provides the desired conclusion.

Set $V = \{v_1, \dots, v_d\}$ and $R = A[X_1, \dots, X_d]$. It is straightforward to identify the monomial ideals $J \subseteq R$ that are of the form K_Δ for some simplicial complex Δ on V: they are precisely the non-zero square-free monomial ideals of R; see Exercise 4.6.12. Thus, we can use the techniques of this section to find m-irreducible decompositions of such ideals, as in the following example.

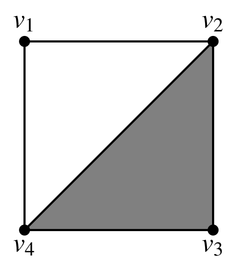
Example 4.6.7. We compute an irredundant m-irreducible decomposition of the square-free monomial ideal

$$J = (X_1X_2, X_2X_3X_4, X_1X_4)R \subseteq R = A[X_1, X_2, X_3, X_4].$$

First, we find a simplicial complex Δ on $V = \{v_1, v_2, v_3, v_4\}$ such that $J = K_\Delta$. To do this, we need to add a facet for every monomial in the irredundant generating sequence of J; the faces of Δ are then the subsets of these facets.

$$\begin{aligned}
\text{facets:}\quad & \{v_1,v_2\},\{v_2,v_3,v_4\},\{v_1,v_4\}\\
\text{trivial face:}\quad & \emptyset\\
\text{vertices:}\quad & \{v_1\},\{v_2\},\{v_3\},\{v_4\}\\
\text{edges:}\quad & \{v_1,v_2\},\{v_2,v_3\},\{v_2,v_4\},\{v_3,v_4\},\{v_1,v_4\}\\
\text{shaded triangles:}\quad & \{v_2,v_3,v_4\}
\end{aligned}$$

The geometric realization of Δ is the following.



Next, we list the minimal vertex covers of Δ:

$$\{v_1, v_2\} \qquad \{v_1, v_3\} \qquad \{v_1, v_4\} \qquad \{v_2, v_4\}.$$

Finally, we read off the decomposition using Theorem 4.5.4:

$$J = K_\Delta = (X_1, X_2)R \cap (X_1, X_3)R \cap (X_1, X_4)R \cap (X_2, X_4)R.$$

Compare this to the decomposition from Example 4.5.6.

Exercises

Exercise 4.6.8. Set $R = A[X_1, \ldots, X_5]$, and let Δ be the simplicial complex from Exercise 4.4.12.

(a) Find an irredundant monomial generating sequence for K_Δ.
(b) Find all minimal vertex covers of Δ.
(c) Use Theorem 4.6.5 to find an irredundant m-irreducible decomposition of K_Δ.
(d) Verify the decomposition $K_\Delta = \bigcap_{V'} P_{V'}$ from part (c) using Theorem 2.1.5.

Justify your answers.

Exercise 4.6.9. Verify the following decompositions

$$\begin{aligned} K_\Delta &= (X_2,X_3)R\cap(X_2,X_4)R\cap(X_1,X_3,X_4)R\cap(X_1,X_3,X_5)R \\ K_{\Delta'} &= (X_2)R\cap(X_4)R\cap(X_1,X_5)R\cap(X_3,X_5)R. \end{aligned}$$

from Example 4.6.6 using Theorem 2.1.5.

Exercise 4.6.10. Prove Theorem 4.6.5.

Exercise 4.6.11. Set $R = A[X_1,\ldots,X_4]$ and find an irredundant m-irreduci-ble decomposition of the ideal $J = (X_1X_2X_3, X_1X_2X_4, X_1X_3X_4, X_2X_3X_4)R$ as in Example 4.6.7. Justify your answer. Verify that your decomposition is correct using Theorem 2.1.5.

**Exercise 4.6.12.* Let $V = \{v_1,\ldots,v_d\}$ and $R = A[X_1,\ldots,X_d]$. Prove that the association $\Delta \mapsto K_\Delta$ describes a bijection between the set of simplicial complexes on V and the set of non-zero square-free monomial ideals of R. (Hint: Show that the following rule describes an inverse for this function. For a non-zero square-free monomial ideal I of R, map I to the simplicial complex Δ whose facets are described by the exponent vectors of the square-free monomial generators of I.) (This is used in Challenge Exercises 7.1.7, 7.3.13, and 7.4.21.)

**Challenge Exercise 4.6.13.* Let $V = \{v_1,\ldots,v_d\}$ and $R = A[X_1,\ldots,X_d]$. Let Δ be a simplicial complex on V. The facet ideal K_Δ is a square-free monomial ideal of R, so there is a simplicial complex $\Lambda(\Delta)$ such that $K_\Delta = J_{\Lambda(\Delta)}$. Describe $\Lambda(\Delta)$ in terms of Δ. Justify your answer. (This is used in Challenge Exercise 5.4.19.) (See also Laboratory Exercise 4.6.18.)

**Challenge Exercise 4.6.14.* Consider the order complex $\Delta(\Pi)$ of a poset Π; see Exercise 4.4.20.

(a) Describe an irredundant monomial generating sequence of the facet ideal $K_{\Delta(\Pi)}$ in terms of chains in Π. (Start by experimenting with the poset from Exercise 4.4.20(b).)
(b) Describe the (minimal) vertex covers of the order complex $\Delta(\Pi)$ in terms of chains in Π.
(c) Use your answer to part (b) to describe an (irredundant) m-irreducible decomposition of the facet ideal $K_{\Delta(\Pi)}$. (This is used in Challenge Exercise 7.6.4(c).)
(d) Use Theorem 2.1.5 to verify your decomposition from part (c) for the poset from Exercise 4.4.20(b).
(e) Consider the construction $\mathrm{V}(I)$ from Section A.10. Exercise 3.3.11(b) shows that there is an integer $k \in \mathbb{N}$, and there are linear subspaces $V_1,\ldots,V_k \subseteq A^d$ such that $\mathrm{V}(K_{\Delta(\Pi)}) = V_1 \cup\cdots\cup V_k$ and such that $V_i \not\subseteq V_j$ whenever $i \neq j$. Describe the linear subspaces V_i in terms of data from Π.

Justify your answers. (See also Laboratory Exercise 4.6.19.)

Challenge Exercise 4.6.15. This exercise involves the construction $\mathrm{V}(I)$ from Exploration Section A.10. Let A be a field, and let Δ be a simplicial complex on $\{v_1,\ldots,v_d\}$. Exercise 3.3.11(b) shows that there is an integer $k \in \mathbb{N}$, and there are linear subspaces $V_1,\ldots,V_k \subseteq A^d$ such that $\mathrm{V}(J_\Delta) = V_1 \cup \cdots \cup V_k$ and such that $V_i \not\subseteq V_j$ whenever $i \neq j$. Describe the linear subspaces V_i in terms of the minimal vertex covers of Δ. Justify your answer.

Facet Ideals and Their Decompositions in Macaulay2

The `EdgeIdeals` package has the capability to compute minimal vertex covers of simplicial complexes as well as graphs. However, in the vocabulary of the package a facet ideal is called the edge ideal of a *hypergraph* and so all functions referring to facet ideals are named as such. The facets of a hypergraph are called *edges* to parallel the language of the edge ideal of a graph.

We continue the Macaulay2 session from the previous two tutorials; recall that `Delta` is the simplicial complex Δ' from Exercise 4.4.2.

```
i14 : Delta
o14 = | x_1x_2x_4x_5 x_2x_3x_4 |
o14 : SimplicialComplex
```

So, we can define `H` to be the hypergraph corresponding to the monomial ideal generated by the facets of `Delta`.

```
i15 : needsPackage "EdgeIdeals"
o15 = EdgeIdeals
o15 : Package

i16 : H = hyperGraph flatten entries facets Delta
o16 = HyperGraph{edges => {{x , x , x }, {x , x , x , x }}}
                             2   3   4     1   2   4   5
                 ring => R
                 vertices => {x , x , x , x , x }
                               1   2   3   4   5
o16 : HyperGraph
```

We also could have used the function `simplicialComplextoHyperGraph` to convert directly from a `SimplicialComplex` object to a `HyperGraph` object.

```
i17 : H == simplicialComplexToHyperGraph Delta
o17 = true
```

To obtain the set of minimal vertex covers of `H`, we use the same function `vertexCovers` as before

```
i18 : vertCovers = vertexCovers H
o18 = {x , x x , x , x x }
        2   1 3   4   3 5
o18 : List
```

and we can obtain an irredundant m-irreducible decomposition of the facet ideal of `Delta` by converting each monomial in the return value from `vertexCovers` to the ideal generated by its support.

```
i19 : mIrredDecomp = apply(vertCovers, c -> monomialIdeal support c);

i20 : netList pack(mIrredDecomp,2)
      +----------------+----------------------+
o20 = |monomialIdeal x |monomialIdeal (x , x )|
      |               2|               1   3 |
      +----------------+----------------------+
      |monomialIdeal x |monomialIdeal (x , x )|
      |               4|               3   5 |
      +----------------+----------------------+
```

Note in the above code, we suppressed the output for `mIrredDecomp`, and printed the result using the commands `netList` and `pack` instead. The `netList` command is a handy way to display a list in a readable way; it converts every element of a list to a `Net` object[4], which is the default way all objects are displayed in Macaulay2. The command `pack` 'packs' a list into a list of lists of the specified length, and is useful to make long lists look nice when using `netList`. Finally, we check our work by intersecting our decomposition and comparing it with the edge ideal of H.

```
i21 : intersect mIrredDecomp == edgeIdeal H
o21 = true
```

Exercises

Exercise 4.6.16. Use Macaulay2 to do the following for some examples from previous sections.

(a) Find the minimal vertex covers and the facet ideals.
(b) Find irredundant m-irreducible decompositions of the facet ideals using the command `irreducibleDecomposition`.
(c) Repeat part (b) using the method from the above tutorial. Compare the outputs and run times for these computations.

Exercise 4.6.17. Consider the order complex $\Delta(\Pi)$ of a poset Π; cf. Exercises 4.4.20 and 4.6.14. Use Macaulay2 to do the following for the poset from Exercise 4.4.20(b).

(a) Find the minimal vertex covers of $\Delta(\Pi)$ and the facet ideal $K_{\Delta(\Pi)}$.
(b) Use the Macaulay2 command `irreducibleDecomposition` to find an irredundant m-irreducible decomposition of $K_{\Delta(\Pi)}$.
(c) Repeat part (b) using the method from the above tutorial. Compare the outputs and run times for these computations.

---

[4] A `Net` is a two dimensional string; get it?

(d) Verify your conclusions for Challenge Exercise 4.6.14(a)–(c).

Laboratory Exercise 4.6.18. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate the ideal K_Λ for some specific complexes Λ, to help with Challenge Exercise 4.6.13.

Laboratory Exercise 4.6.19. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate the order complex $\Lambda(\Pi)$ for some specific posets Π, to help with Challenge Exercise 4.6.14.

4.7 Exploration: Alexander Duality

In this section, A is a non-zero commutative ring with identity.

This section explores the connection between monomial generating sequences and m-irreducible decompositions of square-free monomial ideals. The connection is called Alexander duality. The big-picture idea is this: Given a monomial generating sequence $f_1,\ldots,f_m$ for a square-free monomial ideal I, one creates a new square-free monomial ideal $J = I_1 \cap \cdots \cap I_m$ where each I_j is a square-free m-irreducible monomial ideal determined by the corresponding f_i; see Exercise 4.7.4(b). This process is quite useful in that one can get a lot of useful information about the original ideal I using the new ideal J.

We begin with a useful notation for square-free monomials.

Definition 4.7.1. Set $V = \{v_1,\ldots,v_d\}$ and $R = A[X_1,\ldots,X_d]$. For each $V' \subseteq V$, set $\underline{n}(V') = (n_1,\ldots,n_d)$ where

$$n_i = \begin{cases} 0 & \text{if } v_i \notin V' \\ 1 & \text{if } v_i \in V' \end{cases}$$

and set $\underline{X}^{V'} = \underline{X}^{\underline{n}(V')}$.

For example, consider $V = \{v_1,v_2,v_3,v_4\}$ and $R = A[X_1,X_2,X_3,X_4]$. Then

$$\begin{array}{lll} \underline{n}(\emptyset) = (0,0,0,0) & \underline{n}(\{v_1,v_4\}) = (1,0,0,1) & \underline{n}(V) = (1,1,1,1) \\ \underline{X}^{\emptyset} = 1 & \underline{X}^{\{v_1,v_4\}} = X_1X_4 & \underline{X}^{V} = X_1X_2X_3X_4. \end{array}$$

For arbitrary $d \geqslant 1$, by definition, for each $V' \subseteq V$, the monomial $\underline{X}^{V'}$ is square-free. On the other hand, every square-free monomial f in R is of the form $\underline{X}^{V'}$ for some $V' \subseteq V$, namely for $V' = \operatorname{Supp}(f)$; see Definition 2.3.5.

The first exercise in this section is useful for detecting redundancies in the subsequent construction.

Exercise 4.7.2. Set $V = \{v_1,\ldots,v_d\}$ and $R = A[X_1,\ldots,X_d]$. Prove that, given subsets $V',V'' \subseteq V$ one has $P_{V'} \subseteq P_{V''}$ if and only if $\underline{X}^{V''} \in (\underline{X}^{V'})R$.

Our definition of the $*$-dual of a monomial ideal I is presented next. In words, to construct the $*$-dual of I, we take all the square-free monomials in I, write each one in the form $\underline{X}^{V'}$, and intersect all the corresponding ideals $P_{V'}$.

Definition 4.7.3. Set $V = \{v_1,\dots,v_d\}$ and $R = A[X_1,\dots,X_d]$. Given a square-free monomial ideal $I \subseteq R$ define the $*$-*dual* of I to be the ideal

$$I^* = \bigcap_{\underline{X}^{V'} \in I} P_{V'} = \bigcap_{\underline{X}^{V'} \in I} Q_{V \smallsetminus V'}$$

where each intersection is taken over the set of all subsets $V' \subseteq V$ such that $\underline{X}^{V'} \in I$; define $I^{**} = (I^*)^*$.

For instance, with $V = \{v_1, v_2\}$ and $R = A[X_1, X_2]$, we have the following.

$$(X_1X_2R)^* = P_{\{1,2\}} = (X_1,X_2)R$$
$$(X_1R)^* = P_{\{1\}} \cap P_{\{1,2\}} = (X_1)R \cap (X_1,X_2)R = (X_1)R$$
$$((X_1,X_2)R)^* = (X_1)R \cap (X_2)R \cap (X_1,X_2)R = (X_1)R \cap (X_2)R = (X_1X_2)R$$

The next exercise shows that the construction we described in the introduction for this section is exactly the $*$-dual. In addition, it deals with the issue of redundancies in the intersections describing it.

Exercise 4.7.4. Set $V = \{v_1,\dots,v_d\}$ and $R = A[X_1,\dots,X_d]$. Let I be a square-free monomial ideal of R with monomial generating sequence $\underline{X}^{V_1},\dots,\underline{X}^{V_n}$.

(a) Prove that I^* is square-free. (In particular, the definition of I^{**} makes sense.)
(b) Prove that $I^* = \bigcap_{i=1}^n P_{V_i}$.
(c) Prove that the generating sequence $\underline{X}^{V_1},\dots,\underline{X}^{V_n}$ is irredundant if and only if the intersection $\bigcap_{i=1}^n P_{V_i}$ is irredundant.

One of the goals of this section is to prove that $*$-duality is actually a duality, that is, when you take the $*$-dual of the $*$-dual of a monomial ideal, you get back the ideal you started with. The next exercise starts us on the path toward this property.

Exercise 4.7.5. Set $V = \{v_1,\dots,v_d\}$ and $R = A[X_1,\dots,X_d]$. Prove that $R^* = 0$ and $0^* = R$. For each $V' \subseteq V$, find $(P_{V'})^*$ and prove that $(P_{V'})^{**} = P_{V'}$.

If we are to believe that, given an arbitrary square-free monomial ideal I, we have $I^{**} = I$, then we must also believe the following: if $I^* = J^*$, then $I = J$. The next exercise establishes this implication.

Exercise 4.7.6. Let I, J be square-free monomial ideals in $R = A[X_1,\dots,X_d]$, and set $V = \{v_1,\dots,v_d\}$.

(a) Prove that $I \subseteq J$ if and only if every square-free monomial in I is in J.
(b) Prove that $I \subseteq J$ if and only if for every $V' \subseteq V$ such that $J \subseteq P_{V'}$ we have $I \subseteq P_{V'}$.
(c) Prove that $I \subseteq J$ if and only if $J^* \subseteq I^*$.

(d) Prove that $I = J$ if and only if $J^* = I^*$.

Our claim that $I^{**} = I$ for all square-free monomial ideals I is not unreasonable, given the exercises thus far. However, proving this claim is another matter. The key tool for this is the "Alexander dual" defined next. Essentially, it reverse-engineers the ideal I from its $*$-dual. Later, we will prove that this construction is the same as the $*$-dual, and this will establish our claim.

In words, to construct the Alexander dual of I, we take an irredundant m-irreducible decomposition of I, write each m-irreducible ideal from the decomposition in the form P_{V_i}, then take the ideal generated by the corresponding square-free monomials $\underline{X}^{V_i}$.

Definition 4.7.7. Set $V = \{v_1,\dots,v_d\}$ and $R = A[X_1,\dots,X_d]$. Fix a square-free monomial ideal $I \subsetneq R$ with irredundant m-irreducible decomposition $I = \bigcap_{i=1}^n P_{V_i}$ where $V_1,\dots,V_n$ are subsets of V. (See Proposition 4.1.6.) Define the *Alexander dual* or $\vee$-*dual* of I to be the ideal

$$I^\vee = (\underline{X}^{V_1},\dots,\underline{X}^{V_n})R.$$

When $I \neq 0$, set $I^{\vee\vee} = (I^\vee)^\vee$.

For instance, with $V = \{v_1,v_2\}$ and $R = A[X_1,X_2]$, we have the following.

$$((X_1,X_2)R)^\vee = (\underline{X}^{\{1,2\}})R = (X_1X_2)R$$
$$(X_1R)^\vee = (\underline{X}^{\{1\}})R = (X_1)R$$
$$(X_1X_2R)^\vee = (X_1R \cap X_2)R^\vee = (\underline{X}^{\{1\}},\underline{X}^{\{2\}})R = (X_1,X_2)R$$

Since we claim that the $*$-dual and the Alexander dual are the same, it is reasonable for us to show that they have similar properties. Accordingly, the next exercises are similar to 4.7.4–4.7.6.

Exercise 4.7.8. Set $V = \{v_1,\dots,v_d\}$ and $R = A[X_1,\dots,X_d]$. Let $I \subsetneq R$ be a square-free monomial ideal with irredundant m-irreducible decomposition $I = \bigcap_{i=1}^n P_{V_i}$ where $V_1,\dots,V_n$ are subsets of V.

(a) Prove that $I^\vee$ is square-free.
(b) Prove that, if $I \neq 0$, then $I^\vee \neq R$. (In particular, the definition of $I^{\vee\vee}$ makes sense.)
(c) Prove that the monomial generating sequence $\underline{X}^{V_1},\dots,\underline{X}^{V_n}$ is irredundant.
(d) Prove that if $I = \bigcap_{j=1}^m P_{W_j}$ where $W_1,\dots,W_m$ are subsets of V, then $I^\vee = (\underline{X}^{W_1},\dots,\underline{X}^{W_m})R$.

Exercise 4.7.9. Set $V = \{v_1,\dots,v_d\}$ and $R = A[X_1,\dots,X_d]$. Prove that $0^\vee = R$. For each $V' \subseteq V$, compute $(P_{V'})^\vee$ and prove that $(P_{V'})^{\vee\vee} = P_{V'}$. Justify your answers.

Exercise 4.7.10. Let $I,J \subsetneq R = A[X_1,\dots,X_d]$ be square-free monomial ideals.

(a) Prove that $I \subseteq J$ if and only if $J^\vee \subseteq I^\vee$.
(b) Prove that $I = J$ if and only if $J^\vee = I^\vee$.

Part of the next exercise substantiates our comment above about the Alexander-dual reverse-engineering the $*$-dual.

Exercise 4.7.11. Let I be a square-free monomial ideal of $R = A[X_1, \ldots, X_d]$.

(a) Prove that if $I \neq R$, then $I^{\vee *} = I$.
(b) Prove that if $I \neq 0$, then $I^{* \vee} = I$.

The goal of the next part of this section is to prove that we have $I^{\vee} = I^{*}$ and $I^{\vee\vee} = I$ and $I^{**} = I$, when these make sense. This is done using a corresponding construction for simplicial complexes.

Definition 4.7.12. Let Δ be a simplicial complex on $V = \{v_1, \ldots, v_d\}$. The *Alexander dual* of Δ is the set of all complements of the "non-faces" of Δ:

$$\Delta^{\vee} = \{V \smallsetminus F \mid F \subset V \text{ and } F \notin \Delta\}.$$

When $\Delta \neq \mathsf{P}(V)$, set $\Delta^{\vee\vee} = (\Delta^{\vee})^{\vee}$.

Exercise 4.7.13. Sketch the geometric realizations of the Alexander duals of the simplicial complexes from the examples and exercises from Sections 4.4–4.6.

The next two results parallel some of the properties we have exhibited for the ideals I^{*} and $I^{\vee}$.

Exercise 4.7.14. Let Δ and Δ' be simplicial complexes on $V = \{v_1, \ldots, v_d\}$.

(a) Prove that $\Delta' \subseteq \Delta$ if and only if $\Delta^{\vee} \subseteq \Delta'^{\vee}$.
(b) Prove that $\Delta' = \Delta$ if and only if $\Delta^{\vee} = \Delta'^{\vee}$.

Exercise 4.7.15. Let Δ be a simplicial complex on $V = \{v_1, \ldots, v_d\}$.

(a) Prove that $\mathsf{P}(V)^{\vee} = \emptyset$.
(b) Prove that, if $\Delta \neq \mathsf{P}(V)$, then $\Delta^{\vee}$ is a simplicial complex on V. (In particular, the definition of $\Delta^{\vee\vee}$ makes sense.)
(c) Prove that $\Delta^{\vee} \neq \mathsf{P}(V)$. (In particular, if $\Delta \neq \mathsf{P}(V)$, then $\Delta^{\vee\vee}$ is a simplicial complex on V.)

Now we are finally in a position to prove that $*$-duality and Alexander duality are actually dualities.

Exercise 4.7.16. Set $R = A[X_1, \ldots, X_d]$, and let I be a square-free monomial ideal of R such that $0 \neq I \neq R$. Let Δ be a simplicial complex on $V = \{v_1, \ldots, v_d\}$ such that $I = J_{\Delta}$. See Sections 4.4–4.5.

(a) Prove that $J_{\Delta^{\vee}} = (J_{\Delta})^{*}$.
(b) Prove that $J_{\Delta^{\vee}} = (J_{\Delta})^{\vee}$.
(c) Prove that $I^{\vee} = I^{*}$.
(d) Prove that $I^{\vee\vee} = I$.
(e) Prove that $I^{**} = I$.

The goal of the remainder of this section is to consider a version of Alexander duality for facet ideals; see Section 4.6. We begin with the associated construction for simplicial complexes.

Definition 4.7.17. Let Δ be a simplicial complex on $V = \{v_1, \ldots, v_d\}$. The $\star$-*dual* of Δ is defined to be the set of all subsets of minimal vertex covers of Δ, and is denoted $\Delta^\star$. When $\Delta \neq \{\emptyset\}$, set $\Delta^{\star\star} = (\Delta^\star)^\star$.

Exercise 4.7.18. Sketch the geometric realizations of the $\star$-duals of the simplicial complexes from the examples and exercises from Sections 4.4–4.6.

The next exercise shows that the $\star$-dual does not have the same properties as the $\vee$-dual. This is not unexpected, though, given the differences between face ideals and facet ideals. On the other hand, the subsequent exercise displays some similarities between our new construction and the previous one.

Exercise 4.7.19.

(a) Show by example that the $\star$-dual operation need not respect containments by finding simplicial complexes Δ and Δ' such that $\Delta' \subseteq \Delta$ and $\Delta'^\star \not\subseteq \Delta^\star$.
(b) Show by example that the $\star$-dual operation need not reverse containments by finding simplicial complexes Δ and Δ' such that $\Delta' \subseteq \Delta$ and $\Delta^\star \not\subseteq \Delta'^\star$.

Exercise 4.7.20. Let Δ be a simplicial complex on $V = \{v_1, \ldots, v_d\}$.

(a) Prove that $\{\emptyset\}^\star = \emptyset$.
(b) Prove that, if $\Delta \neq \{\emptyset\}$, then $\Delta^\star$ is a simplicial complex on V. (In particular, the definition of $\Delta^{\star\star}$ makes sense.)
(c) Prove that $\Delta^\star \neq \{\emptyset\}$. (Thus, if $\Delta \neq \{\emptyset\}$, then $\Delta^{\star\star}$ is a simplicial complex on V.)

Next, we have, in particular, a version of Exercise 4.7.16(a).

Exercise 4.7.21. Let Δ, Δ' be simplicial complexes on $V = \{v_1, \ldots, v_d\}$ such that $\Delta \neq \{\emptyset\} \neq \Delta'$.

(a) Prove that $K_{\Delta^\star} = (K_\Delta)^\vee$.
(b) Prove that $\Delta^{\star\star} = \Delta$.
(c) Prove that $\Delta' = \Delta$ if and only if $\Delta^\star = \Delta'^\star$.

Every graph is a simplicial complex, so it is natural to ask whether these operations respect the property of being a graph. The next exercise investigates this.

Exercise 4.7.22. Let G be a graph, considered as a simplicial complex.

(a) Prove or disprove the following: the Alexander dual $G^\vee$ is a graph.
(b) Repeat part (a) for the $\star$-dual.

Justify your answers.

Next, we have two more natural questions about these operations.

Challenge Exercise 4.7.23. Let Δ be a simplicial complex on the vertex set $V = \{v_1,\dots,v_d\}$ such that $\Delta \neq \{\emptyset\}$. The ideal $K_{\Delta^\star}$ is square-free, so it is of the form $J_{\Gamma(\Delta)}$ for some simplicial complex $\Gamma(\Delta)$. Describe $\Gamma(\Delta)$ in terms of Δ. Justify your answer. (Hint: Start with some examples from the previous sections.) (See also Laboratory Exercise 4.7.26.)

Challenge Exercise 4.7.24. Consider the order complex $\Delta(\Pi)$ of a poset Π; see Exercise 4.4.20.

(a) Describe the Alexander dual $\Delta(\Pi)^\vee$ in terms of the partial order on Π. (Start by experimenting on the poset from Exercise 4.4.20(b).)
(b) Repeat part (a) for the $\star$-dual.

Justify your answers. (See also Laboratory Exercise 4.7.27.)

Alexander Duality in Macaulay2

To compute the Alexander dual of a monomial ideal in Macaulay2, one uses the `dual` command. The `dual` command operates both on `MonomialIdeal` objects as well as `SimplicialComplex` objects. Indeed, let us compute the Alexander duals of some of the various examples we have seen thus far. Continuing with the examples from the previous sections in this chapter, recall that we have two objects under discussion at this point: first, we have `Delta` which is the simplicial complex Δ' from Example 4.4.2

```
i22 : Delta
o22 = | x_1x_2x_4x_5 x_2x_3x_4 |
o22 : SimplicialComplex
```

whose underlying ideal is $J_{\Delta'}$.

```
i23 : JDelta = monomialIdeal Delta
o23 = monomialIdeal (x x , x x )
                      1 3   3 5
o23 : MonomialIdeal of R
```

We also have `H`

```
i24 : H
o24 = HyperGraph{edges => {{x , x , x }, {x , x , x , x }}}
                              2   3   4     1   2   4   5
                 ring => R
                 vertices => {x , x , x , x , x }
                               1   2   3   4   5
o24 : HyperGraph
```

which is the corresponding `HyperGraph` object representing the facet ideal $K_{\Delta'}$.

```
i25 : KDelta = edgeIdeal H

o25 = monomialIdeal (x x x , x x x x )
                      2 3 4   1 2 4 5

o25 : MonomialIdeal of R
```

Illustrating Exercise 4.7.16(b), we check that $J_{\Delta'^\vee} = (J_{\Delta'})^\vee$.

```
i26 : monomialIdeal dual Delta

o26 = monomialIdeal (x , x x )
                      3   1 5

o26 : MonomialIdeal of R

i27 : dual monomialIdeal Delta

o27 = monomialIdeal (x , x x )
                      3   1 5

o27 : MonomialIdeal of R
```

One can also verify that Alexander duality really is a duality in this case by checking the following.

```
i28 : Delta == dual dual Delta

o28 = true
```

The $\star$-duality operation on simplicial complexes discussed above is not implemented directly in the `SimplicialComplexes` package, but we can use the existing functionality to add it. Indeed, by Exercise 4.7.21(a), the $\star$-dual of `Delta` is the simplicial complex whose facets are the minimal vertex covers of `Delta`. We implement this operation on `SimplicialComplex` objects, as well as on `HyperGraph` objects for consistency.

```
i29 : starDual = method()

o29 = starDual

o29 : MethodFunction

i30 : starDual SimplicialComplex := S -> (
simplicialComplex vertexCovers simplicialComplexToHyperGraph S
)

o30 = {*Function[stdio:32:33-33:62]*}

o30 : FunctionClosure

i31 : starDual HyperGraph := H -> hyperGraph vertexCovers H

o31 = {*Function[stdio:35:26-35:53]*}

o31 : FunctionClosure
```

Lastly, we verify Exercise 4.7.21(a)–(b) in this case.

```
i32 : dual edgeIdeal H == edgeIdeal starDual H

o32 = true

i33 : H == starDual starDual H

o33 = true

i34 : exit
```

Exercises

Exercise 4.7.25. Use the Macaulay2 techniques described in the tutorial above to verify the conclusions of the exercises in this section for some examples, and to verify any counterexamples from Exercise 4.7.22.

Laboratory Exercise 4.7.26. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate the ideal $K_{\Delta^\star}$ for some specific complexes Δ, to help with Challenge Exercise 4.7.23.

Laboratory Exercise 4.7.27. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate the duals $\Delta(\Pi)^\vee$ and $\Delta(\Pi)^\star$ for some specific posets Π, to help with Challenge Exercise 4.7.24.

Concluding Notes

Edge ideals of graphs were introduced by Villarreal [76]. Face ideals of simplicial complexes are due to Melvin Hochster [42] and Gerald Reisner [69], and facet ideals were defined by Sara Faridi [20]. Facet ideals are sometime studied as edge ideals of clutters or of hypergraphs; see the papers of Sara Faridi [21], Isidoro Gitler, Tài Hà and Adam Van Tuyl [32], Carlos Valencia, and Villarreal [26], for more about this. Excellent surveys of this area are given by Hà and Van Tuyl [31] and Susan Morey and Rafael Villarreal [60]. Each of these contains an extensive bibliography for the subject.

Several variations on the edge ideal have appeared recently. For instance, the ideals generated by the paths of a fixed length in a tree are the path ideals of Aldo Conca and Emanuela De Negri [12]. (Paths of length 1 recover the edge ideal.) Chelsey Paulsen and Sean Sather-Wagstaff [65] study a version of the edge ideal that tracks the weights on the edges of a weighted graph; Bethany Kubik and Sather-Wagstaff [49] soup this up to path ideals for weighted graphs.

In our context, Alexander duality goes back at least to Hochster [42]. Its origins are topological in nature, beginning with work of James Alexander II [1]. Miller [56, 57] introduced a version that applies to all monomial ideals, not just the square-free ones. Also, the $\star$-dual given here is due to Hà, Morey, and Villarreal [30].

Because of Exercise 4.4.19, the number of square-free monomial ideals in d variables equals the number of simplicial complexes on d vertices. This is related to the Dedekind number $M(d)$. It is interesting to note that, while there is an explicit formula for $M(d)$, as of the writing of this text, it has only been calculated for $d \leqslant 8$.

Chapter 5
Connections with Other Areas

This chapter deals with some of the other areas of mathematics and engineering that intersect with the notions from the preceding chapters. We set the stage in Section 5.1 with a measure of the size of a ring called the Krull dimension, for use throughout the chapter. We then discuss applications to electrical engineering, via the PMU Placement Problem, in Section 5.2. We continue in Section 5.3 with an application to topology: Stanley's famous solution of the Upper Bound Conjecture. The remaining sections of this chapter are devoted to connections with commutative algebra and homological algebra.

Be warned that we omit many details in this chapter. Our purpose here is to give some big-picture idea of the significance of these notions. The interested reader should investigate the references given in the Concluding Notes at the end of this chapter for more information on these subjects.

The nature of this chapter allows us to showcase the usefulness of Macaulay2. In each section, we use it to verify conclusions (for some examples) of results where the proofs are outside the scope of the text. In a similar vein, we expand our use of Macaulay2 as a laboratory, giving the reader a number of opportunities to use it to explore topics with the goal of making conjectures.

5.1 Krull Dimension

In algebra, as in many areas of mathematics, is useful to have a notion of the size of an object. For instance, in linear algebra, the dimension (or "rank") of a vector space tells you how large it is. One measure of the size of a ring is its Krull dimension, which we define next.

Definition 5.1.1. Let R be a commutative ring with identity.

An ideal $I \subseteq R$ is *prime* if $I \neq R$ and the complement $R \smallsetminus I$ is closed under multiplication, i.e., for all $a, b \in R$ if $ab \in I$, then either $a \in I$ or $b \in I$. (See Exercise A.5.10 (and optionally A.7.14, 1.2.14, 3.2.14, and 4.1.12) for more about prime ideals.)

Note that we do not assume that I is a monomial ideal, nor is it even necessarily in a polynomial ring.

The *Krull dimension*, denoted $\dim(R)$, is the supremum of the lengths of chains of prime ideals in R. In symbols, we have

$$\dim(R) = \sup\{n \geqslant 0 \mid \text{there is a chain of prime ideals } \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } R\}.$$

The term "Krull dimension" is frequently shortened to "dimension".

For instance, if A is a field, then the dimension of the polynomial ring $R = A[X_1,\ldots,X_d]$ is exactly d. The inequality $\dim(R) \geqslant d$ is straightforward to verify because of the chain of prime ideals

$$0 \subsetneq (X_1)R \subsetneq (X_1,X_2)R \subsetneq \cdots \subsetneq (X_1,\ldots,X_d)R$$

see Exercise 1.1.23. However, the reverse inequality takes more work; consult, e.g. [54, Corollary to Theorem 5.6].

It is worth noting that the Krull dimension of a ring need not be finite, hence the supremum in the definition. However, if there is a bound on the lengths of the chains of prime ideals in the ring, then the supremum is the same as the maximum; such is the case for polynomial rings over fields in finitely many variables, as we discussed in the preceding paragraph.

The dimension of a ring R is an important measure of its size. While the definition is purely algebraic, it has significant geometric content. For instance, the polynomial ring $\mathbb{R}[X]$ in one variable, which has Krull dimension 1, represents the real line (the X-axis, if you like), which is a 1-dimensional geometric object. Similarly, the polynomial ring $\mathbb{R}[X,Y]$ in two variables has Krull dimension 2 and represents the XY-plane, which is a 2-dimensional geometric object. It turns out that the quotient ring $\mathbb{R}[X,Y]/(Y-X^2)$ has Krull dimension 1 and represents the parabola $Y = X^2$, a 1-dimensional geometric object. (See Section A.8 for background on quotient rings, and the end of Section A.10 for discussion of representing geometric objects by quotients.)

For quotients of polynomial rings by monomial ideals, the next result shows how to read the Krull dimension from an m-irreducible decomposition.

Theorem 5.1.2 *Let A be a field, and set $R = A[X_1,\ldots,X_d]$. Let I be a monomial ideal in R with an m-irreducible decomposition $I = \bigcap_{i=1}^m J_i$. Then $\dim(R/I) = d - n$ where n is the smallest number of generators needed for one of the J_i.*

Proof (Sketch of proof). Let k be such that J_k requires n generators. Relabel the variables if necessary to assume that $J_k = (X_1^{e_1},\ldots,X_n^{e_n})R$. Since A is a field, the ideal $(X_1,\ldots,X_n)R \subset R$ is prime by Exercise 1.1.23, and it contains J_k, hence it contains $I = \bigcap_{i=1}^m J_i$. It can be shown that this implies that $(X_1,\ldots,X_n)R$ corresponds to a prime ideal $\mathfrak{p}_0$ in the quotient ring R/I. Similarly, the following chain of prime ideals of length $d-n$

$$(X_1,\ldots,X_n)R \subsetneq (X_1,\ldots,X_n,X_{n+1})R \subsetneq \cdots \subsetneq (X_1,\ldots,X_n,\ldots,X_d)R$$

corresponds to a chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_{d-n}$ of prime ideals in R/I. This explains the inequality $\dim(R/I) \geqslant d-n$.

For the reverse inequality, it can be shown that every chain $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_k$ of prime ideals in R/I corresponds to a chain $Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_k$ of prime ideals in R that contain I; we need to show that $k \leqslant d-n$. Since $Q_0 \supseteq I = \bigcap_{i=1}^m J_i$, Exercise A.5.10(d) shows that Q_0 must contain one of the ideals J_i, and we rearrange the variables if necessary to assume that $Q_0 \supseteq J_i = (X_1^{e_1},\ldots,X_j^{e_j})R$. Note that $n \leqslant j$. Using Exercise A.7.14, in conjunction with Proposition 2.3.2(c), we see that $Q_0 = \operatorname{rad}(Q_0) \supseteq \operatorname{rad}(J_i) = \text{m-rad}(J_i) = (X_1,\ldots,X_j)R$. Since $(X_1,\ldots,X_j)R$ is prime, we can assume without loss of generality that $Q_0 = (X_1,\ldots,X_j)R$. Thus, the given chain of length k corresponds to a chain of prime ideals in $R/(X_1,\ldots,X_j)R$, hence, it corresponds to a chain of prime ideals in $R' = A[X_{j+1},\ldots,X_d]$. As we discussed before this theorem, the chains in R' have length at most equal to the number of variables, so $k \leqslant d-j \leqslant d-n$ as desired. $\square$

For example, if A is a field, the monomial ideal

$$I = (X,Y^2)R \cap (X^2,Z^3)R \cap (X^3,Y^3,Z^3)R$$

in $R = A[X,Y,Z]$ has $\dim(R/I) = 3-2 = 1$.

Remark 5.1.3. When A is a field and I is defined by one of the constructions of Chapter 4, one can use Theorem 5.1.2 to describe $\dim(A[X_1,\ldots,X_d]/I)$. For instance, if G is a graph with vertex set $V = \{v_1,\ldots,v_d\}$, and n is the size of the smallest vertex cover of G, then $\dim(A[X_1,\ldots,X_d]/I_G) = d-n$; see Theorem 4.3.6. If Δ is a simplicial complex on V, then $\dim(A[X_1,\ldots,X_d]/J_\Delta) = \dim(\Delta)+1$; see Theorem 4.5.4.

Exercises

**Exercise 5.1.4.* Let A be a field. Use Theorem 5.1.2 to compute $\dim(R/I)$ for the ideal I in $R = A[X,Y,Z]$ from Example 3.5.6. Do the same for all graphs on $d \leqslant 4$ vertices. Justify your answers. (This is used in Exercise 5.2.11.)

**Exercise 5.1.5.* Let A be a field, and set $R = A[X_1,\ldots,X_d]$. Let G be a graph with vertex set $V = \{v_1,\ldots,v_d\}$, and let n be the size of the smallest vertex cover of G. Use Theorems 4.3.6 and 5.1.2 to prove that $\dim(R/I_G) = d-n$. (This exercise is used in the proofs of Theorem 5.2.7 and Corollary 5.2.9.)

Exercise 5.1.6. Let A be a field, and set $R = A[X_1,\ldots,X_d]$. Let Δ be a simplicial complex with vertex set $V = \{v_1,\ldots,v_d\}$. Use Theorems 4.5.4 and 5.1.2 to prove that $\dim(R/J_\Delta) = \dim(\Delta)+1$.

Exercise 5.1.7. Let A be a field, and let I be a monomial ideal of $R = A[X_1,\ldots,X_d]$. Prove that $\dim(R/I) = 0$ if and only if $\text{m-rad}(I) = (X_1,\ldots,X_d)R$.

Exercise 5.1.8. This exercise involves the construction $\mathrm{V}(I)$ from Exploration Section A.10. Let A be a field, and let I be a monomial ideal of $R = A[X_1,\ldots,X_d]$. Exercise 3.3.11(b) shows that there is an integer $k \in \mathbb{N}$, and there are linear subspaces $V_1,\ldots,V_k \subseteq A^d$ such that $\mathrm{V}(I) = V_1 \cup \cdots \cup V_k$ and such that $V_i \not\subseteq V_j$ whenever $i \neq j$. Prove that $\dim(R/I) = \max\{\dim_A(V_i) \mid i = 1,\ldots,k\}$.

Challenge Exercise 5.1.9. Let Δ be a simplicial complex on the vertex set $V = \{v_1,\ldots,v_d\}$, and set $R = A[X_1,\ldots,X_d]$ where A is a field. Describe $\dim(R/K_\Delta)$ in terms of vertex covers of Δ; see Section 4.6. Justify your answer. (See also Laboratory Exercise 5.1.12.)

Krull Dimension in Macaulay2

As with simplicial complexes, the Krull dimension of a quotient ring is computed using the `dim` command

```
i1 : R = ZZ/101[x,y,z];

i2 : I = monomialIdeal(x^2,x*y,y*z)
                         2
o2 = monomialIdeal (x , x*y, y*z)

o2 : MonomialIdeal of R

i3 : dim I

o3 = 1

i4 : exit
```

Exercises

Exercise 5.1.10. Set $A = \mathbb{Z}_{101}$, and use Macaulay2 to show that $\dim(A[X_1,\ldots,X_d]) = d$ for some values of d and $\dim(A[X,Y]/(Y - X^2)) = 1$.

Exercise 5.1.11. Set $A = \mathbb{Z}_{101}$, and use Macaulay2 to verify the equality $\dim(R/I) = d - n$ from Theorem 5.1.2 for some examples, and similarly for the equality $\dim(R/I_G) = d - n$ from Exercise 5.1.5 and the equality $\dim(R/J_\Delta) = \dim(\Delta) + 1$ from Exercise 5.1.6.

Laboratory Exercise 5.1.12. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate $\dim(R/K_\Delta)$ for some specific complexes Δ, to help with Challenge Exercise 5.1.9.

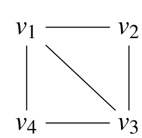
5.2 Vertex Covers and Phasor Measurement Unit (PMU) Placement

One problem in electrical engineering involves finding optimal placements of sensors (called "phasor measurement units" or "PMUs") in an electrical power system to monitor the substations and the transmission lines between the substations. Effective placement of PMUs in a system ensure the secure operation of the power system, and optimal placement helps reduce the cost of running the system. The problem of finding the smallest number of PMUs needed to monitor the entire system (and the placements of the PMUs) is the "PMU Placement Problem". Techniques to attack this optimization problem (which has been shown to be NP-complete) include integer linear programming, genetic algorithms, Gröbner bases, and graph theory.

In this section, we discuss how this problem relates to one that we have already seen: the problem of finding minimal vertex covers of graphs. (In particular, this portion of the text depends on Sections 4.2–4.3 above.) We begin with some basic notions from electrical engineering.

Definition 5.2.1. In an electrical power system, a *bus* is a substation where *transmission lines* meet. (The term "transmission line" is frequently shortened to "line".) Each line connects two buses.

In practice, electrical power systems are represented by diagrams of varying complexity, depending on how much information about the systems is being tracked. Here we are only interested in illustrating the buses and lines, so we model the systems with graphs where the vertices represent buses and the edges represent transmission lines. For instance, the next graph



represents a power system with four buses and five lines.

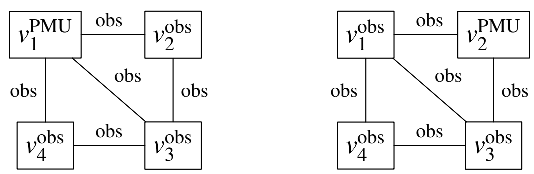
Definition 5.2.2. A *phasor measurement unit (PMU)* is a device placed at a bus in an electrical power system to monitor the voltage at the bus and the current in all lines connected to it. A *PMU placement* is a set of buses where PMUs are placed.

A bus in the system is *observable* if its voltage is known, e.g., by the placement of one or more PMUs. A line in the system is *observable* if its current is known. The power system is *observable* if every bus is observable and every line is observable.

In the preceding definition, the term "phasor measurement unit" indicates that each PMU is a unit or device that tracks the voltage *phasor* (magnitude and phase) at a bus and similarly for the current phasor in the lines.

The placement of a PMU at a bus v_i makes that bus observable as well as every line incident to v_i. Ohm's Law implies that (1) any line incident to two observable buses is also observable, and (2) any bus incident to an observable line is also observable. In addition, Kirchhoff's Current Law implies that if v_i is incident to $k > 1$ lines, $k-1$ of which are observable, then all k of these lines are observable.

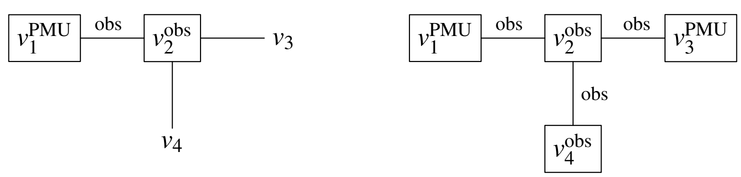
For example, in the power system represented by the graph above, a PMU placed at the bus v_1 makes the entire system observable: the PMU observes the bus v_1 and every line incident to v_1; thus, every other bus is observable by Ohm's Law (2), and Ohm's Law (1) shows that every line is observable. This is illustrated in the first of the following diagrams where observable buses are in boxes, buses with PMUs are labeled "PMU", and other observable buses and observable lines are labeled "obs".



(5.2.2.1)

Similarly, a PMU placed at the bus v_2 observes the entire system, as in the second diagram above: the bus v_2 is observable and each line incident to v_2 is observable; so Ohm's Law (2) implies that v_1 and v_3 are observable, and Ohm's Law (1) shows that the line v_1v_3 is observable; then Kirchhoff's Current Law implies that the remaining two lines are observable, hence so is the remaining vertex.

On the other hand, for the graph sketched next, a single PMU placed at v_1 does not make the entire system observable



(5.2.2.2)

but the following PMU placements do make the entire system observable: $\{v_2\}$ and $\{v_1, v_3\}$ and $\{v_1, v_4\}$ and $\{v_3, v_4\}$.

It is clear that PMU placements are related to vertex covers. Before making this relation explicit, we introduce some definitions.

Definition 5.2.3. A *PMU cover* of an electrical power system is a PMU placement that observes the entire system. A PMU cover V' is *minimal* if it does not properly contain another PMU cover of the system. A *smallest PMU cover* is a PMU cover that has the smallest size among all PMU covers of the system. A *smallest vertex cover* of a graph is a vertex cover that has the smallest size among all vertex covers of the graph.

It is important to note that the terms "smallest PMU cover" and "minimal PMU cover" are not synonymous. Indeed, as we have seen, a given graph can have minimal PMU covers of different size. The power system in (5.2.2.2) above has this property: the sets $\{v_2\}$ and $\{v_1, v_3\}$ are minimal PMU covers of different size. Thus, the first one is a smallest PMU cover: the system requires at least one PMU to be observable, and this PMU cover has exactly one PMU. However, the second vertex cover is not a smallest PMU cover because it has more than one PMU. One reason for having different terminology is the expense involved in placing PMUs at buses. Engineers are interested in minimizing the cost of observing the system, by minimizing the number of PMUs.

Furthermore, note that PMU covers of power systems are not the same as vertex covers of the corresponding graphs. One can see this in (5.2.2.1) above: this power system has minimal PMU covers of size 1, but these are not vertex covers by Example 4.3.4. On the other hand, the next result shows that, under mild hypotheses, every vertex cover is a PMU cover.

Proposition 5.2.4 *Let G be a graph with vertex set V, and assume that G has no isolated vertices, that is, every vertex of G is adjacent to at least one other vertex. If V' is a vertex cover of G, then V' is a PMU cover of the associated power system.*

Proof. Let V' be a vertex cover of G. By definition, this means that every edge of G is covered by V', so the corresponding line of the power system is observable. To show that V' is a PMU cover of the power system, it remains to show that every bus $v_i \in V$ is observable by V'. Our assumption about isolated vertices implies that there is another vertex v_j adjacent to V'. As we have already seen, the line v_iv_j is observable by V', so Ohm's Law implies that v_i and v_j are both observable. □

It is worth noting that the assumption in the preceding proposition about isolated vertices is crucial. Indeed, assume that G has an isolated vertex v_i, that is, this vertex has no other vertices adjacent to it. Then the only way to make v_i observable is to place a PMU at v_i; in other words, every PMU cover of the associated power system must contain v_i. On the other hand, since v_i is not incident to any edges, it is not needed to cover any edges; so v_i is not in any minimal vertex cover of G.

As one can see from the above proof and discussion, it becomes cumbersome to distinguish between a graph and the associated power system. Thus, from this point on, we identify graphs with their associated power systems, so for instance we have PMU covers of graphs and vertex covers of power systems.

Our motivation for the next construction comes from the decomposition result 4.3.6 for edge ideals in terms of vertex covers, and the above connection between PMU covers and vertex covers.

Definition 5.2.5. Let G be a graph with vertex set $V = \{v_1, \ldots, v_d\}$, and set $R = A[X_1, \ldots, X_d]$. The *power edge ideal* of G is

$$I_G^{\mathrm{P}} = \bigcap_{V'} P_{V'}$$

where the intersection is taken over all PMU covers of G.

Note that Proposition 4.1.6 shows that I_G^{P} is a square-free monomial ideal.

In contrast with previous constructions, this one is defined in terms of an m-irreducible decomposition, instead of being determined by generators. The next result gives the irredundant m-irreducible decomposition, which will make it easier to compute examples.

Proposition 5.2.6 *Let G be a graph with vertex set $V = \{v_1, \ldots, v_d\}$, and set $R = A[X_1, \ldots, X_d]$. Then the power edge ideal $I_G^P \subseteq R$ has the following irredundant m-irreducible decomposition*
$$I_G^P = \bigcap_{V' \text{ min.}} P_{V'}$$
where the intersection is taken over all minimal PMU covers of G.

Proof. Exercise 5.2.13. □

For example, for the graph in (5.2.2.1) above, one checks readily that the minimal PMU covers are exactly the singletons $\{v_i\}$ for $i = 1, 2, 3, 4$; so the power edge ideal in this case is
$$I_G^{\mathrm{P}} = \bigcap_{i=1}^{4} (X_i)R = (X_1X_2X_3X_4)R.$$
This shows, for instance, that unlike the edge ideal, the power edge ideal need not be a quadratic monomial ideal. In spite of this difference, the edge and power edge ideals are somewhat related, as we see next.

Theorem 5.2.7 *Let G be a graph with vertex set $V = \{v_1, \ldots, v_d\}$. Let A be a field, and set $R = A[X_1, \ldots, X_d]$. Assume that G has no isolated vertices. Then $I_G^P \subseteq I_G$ and* $\dim(R/I_G^P) \geqslant \dim(R/I_G)$.

Proof. Proposition 5.2.4 implies that every vertex cover of G is a PMU cover of G. This explains the containment in the next display
$$I_G^{\mathrm{P}} = \bigcap_{V' \text{ PMU}} P_{V'} \subseteq \bigcap_{V' \text{ v.c.}} P_{V'} = I_G$$
wherein the first intersection is taken over all PMU covers of G, and the second intersection is taken over all vertex covers of G. The first equality in this display is by definition, and the last equality is from Theorem 4.3.6.

Let $n(G)$ denote the size of the smallest vertex cover of G, and let $n^{\mathrm{P}}(G)$ denote the size of the smallest PMU cover of G. Since every vertex cover of G is a PMU cover of G, we have $n^{\mathrm{P}}(G) \leqslant n(G)$. Thus, Exercises 5.1.5 and 5.2.15 imply that
$$\dim(R/I_G^{\mathrm{P}}) = d - n^{\mathrm{P}}(G) \geqslant d - n(G) = \dim(R/I_G).$$
Alternately, one can use the sketch of the proof of Theorem 5.1.2 with the containment $I_G^{\mathrm{P}} \subseteq I_G$ to prove this inequality. □

We have already seen examples where the containment $I_G^{\mathrm{P}} \subseteq I_G$ is proper and the inequality $\dim(R/I_G^{\mathrm{P}}) \geqslant \dim(R/I_G)$ is strict. Indeed, for the graph in (5.2.2.1), the ideal $I_G^{\mathrm{P}} = (X_1X_2X_3X_4)R$ is not quadratic, so $I_G^{\mathrm{P}} \neq I_G$. Furthermore, Exercise 5.2.15 shows that $\dim(R/I_G^{\mathrm{P}}) = 3$ for this graph, while $\dim(R/I_G) = 2$ by Example 4.3.4 and Exercise 5.1.5.

On the other hand, the same logic shows that the graph from (5.2.2.1) satisfies $\dim(R/I_G^{\mathrm{P}}) = 3 = \dim(R/I_G)$ since $\{v_2\}$ is a minimal vertex cover and a minimal PMU cover in this case. However, the ideals I_G^{P} and I_G are not equal in this case.

The graph $v_1 - v_2$ consisting of a single edge gives an example where $I_G^{\mathrm{P}} = I_G$ because the two singletons $\{v_i\}$ are exactly the minimal PMU covers and the minimal vertex covers. One can build other examples satisfying this equality by taking disjoint copies of this one

$$v_1 \text{———} v_2 \qquad v_3 \text{———} v_4 \qquad \cdots \qquad v_{2n-1} \text{———} v_{2n} \tag{5.2.7.1}$$

As we mention above, a question in electrical engineering asks, given a power system, how to find a smallest PMU cover for it? Haynes, et. al. [34] call this the "Power Dominating Set (PDS) Problem" and show that it is NP-complete. (The same is true of the corresponding problem for vertex covers, the "Vertex Cover Problem" by a classical result of Karp [45].)

On the other hand, considering the problem from a graph-theoretical standpoint, Brueni and Heath [8] prove the following result. In it, the number $\lfloor d/3 \rfloor$ is the "floor" or "round-down" of $d/3$.

Theorem 5.2.8 ([8, Theorem 6]) *Given an electrical power system G with $d \geqslant 3$ buses, there is a PMU cover Π of G such that $|\Pi| \leqslant \lfloor d/3 \rfloor$.*

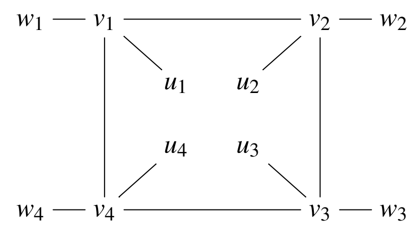
The proof of this theorem is quite technical, so we omit it here. However, we note the following algebraic consequence regarding the Krull dimension wherein $\lceil 2d/3 \rceil$ is the "ceiling" or "round-up" of $2d/3$.

Corollary 5.2.9 *Let G be a graph with vertex set $V = \{v_1, \ldots, v_d\}$ with $d \geqslant 3$. Let A be a field, and set $R = A[X_1, \ldots, X_d]$. Then $\dim(R/I_G^{\mathrm{P}}) \geqslant d - \lfloor d/3 \rfloor = \lceil 2d/3 \rceil$.*

Proof. Let n be the size of the smallest PMU cover of G. Theorem 5.2.8 implies that $n \leqslant \lfloor d/3 \rfloor$, so we conclude from Exercise 5.2.15 that $\dim(R/I_G^{\mathrm{P}}) = d - n \geqslant d - \lfloor d/3 \rfloor = \lceil 2d/3 \rceil$, as desired. $\square$

Brueni and Heath [8] also use the following class of examples, which they denote $B_{\ell,2}$, to show that the bound in Theorem 5.2.8 is sharp. These graphs are special cases of "coronas".

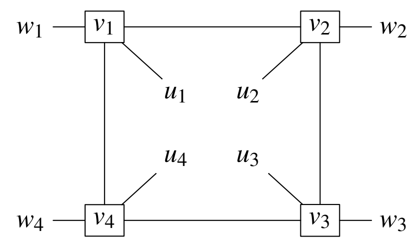
Example 5.2.10 ([8, special case of Theorem 7]). Fix an integer $\ell \geqslant 3$. We build a graph B_ℓ with $d = 3\ell$ vertices such that each smallest PMU cover Π of B_ℓ has $|\Pi| = \ell = \lfloor 3\ell/3 \rfloor$; hence $\dim(R/I_{B_\ell}^{\mathrm{P}}) = 2\ell = d - \lfloor 3\ell/3 \rfloor$. Start with the ℓ-cycle C_ℓ with vertex set $\{v_1, \ldots, v_\ell\}$. For $i = 1, \ldots, \ell$ add two vertices u_i and w_i, and add edges u_iv_i and v_iw_i; the resulting graph is B_ℓ. For instance, the graph B_4 is sketched next.



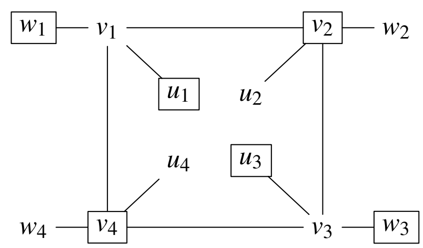
Technically, the graph B_ℓ has vertex set and edge set

$$V = \{v_1, \dots, v_\ell, u_1, \dots, u_\ell, w_1, \dots, w_\ell\}$$
$$E = \{u_1v_1, \dots, u_\ell v_\ell, v_1w_1, \dots, v_\ell w_\ell\} \cup \{v_1v_2, v_2v_3, \dots, v_{\ell-1}v_\ell, v_\ell v_1\}.$$

In particular, B_ℓ has 3ℓ vertices. The set $V' = \{v_1, \dots, v_\ell\}$ is a smallest PMU cover of B_ℓ with size $\ell = 3\ell/3 = d/3 = \lfloor d/3 \rfloor$. (This takes some work to show.) Hence, this is also a minimal PMU cover. For instance, the case $\ell = 4$ is illustrated next.



Note that this is also a smallest (hence, minimal) vertex cover. Furthermore, B_ℓ has minimal vertex covers that have more than ℓ elements. For instance, here is one for the case $\ell = 4$.



Note that this is not a minimal PMU cover because, e.g., w_1 and w_3 can be removed.

See the exercises below for more about B_ℓ.

Exercises

Exercise 5.2.11. Let G be a graph with vertex set $V = \{v_1, \dots, v_d\}$. Let A be a field, and set $R = A[X_1, \dots, X_d]$.

(a) For all such G with $d \leqslant 4$, find all minimal and smallest PMU covers of G, and use these to compute $\dim(R/I_G^{\mathrm{P}})$ and the irredundant generating sequence for I_G^{P}. Use Exercises 5.1.4 and 5.2.15 to verify the conclusion of Corollary 5.2.9 for these graphs.
(b) Repeat part (a) for the graph from (5.2.7.1).

Justify your answers.

Exercise 5.2.12. Prove that the minimal PMU covers for the following graphs are exactly the singletons $\{v_i\}$ for $i = 1, \ldots, d$: the d-cycle C_d, the d-path P_d, and the complete graph K_d. Conclude that $I_{C_d}^{\mathrm{P}} = I_{P_d}^{\mathrm{P}} = I_{K_d}^{\mathrm{P}} = (X_1 \cdots X_d)R$. In particular, distinct (i.e., non-isomorphic) graphs can have the same power edge ideals.

Exercise 5.2.13. Prove Proposition 5.2.6. (Hint: Model your proof on the proof of Theorem 4.3.6.)

Exercise 5.2.14. Prove that every minimal vertex cover of a graph G contains a minimal PMU cover. Does every smallest vertex cover of G contain a smallest PMU cover? Justify you answer.

Exercise 5.2.15. Let A be a field, and set $R = A[X_1, \ldots, X_d]$. Let G be a graph with vertex set $V = \{v_1, \ldots, v_d\}$, and let n be the size of the smallest PMU cover of G. Use Theorem 5.1.2 to prove that $\dim(R/I_G^{\mathrm{P}}) = d - n$.

Exercise 5.2.16.

(a) Prove that each complete bipartite graph $K_{m,n}$ has a PMU cover of size 2. Conclude that $K_{m,n}$ has smallest PMU cover of size 1 or 2.
(b) [Challenge] Characterize the values of m and n for which $K_{m,n}$ has smallest PMU cover of size 1.

Challenge Exercise 5.2.17. Fix an integer $\ell \geqslant 3$, and consider the graph B_ℓ from Example 5.2.10.

(a) Prove that $V' = \{v_1, \ldots, v_\ell\}$ is the unique smallest vertex cover of B_ℓ.
(b) Find all the minimal vertex covers of B_ℓ. Show that B_ℓ has minimal vertex covers of size $> \ell$.
(c) Write out an irredundant monomial generating sequence and an irredundant m-irreducible decomposition for the edge ideal I_{B_ℓ}.
(d) Repeat parts (a)–(c) for PMU covers and the power edge ideal. Conclude that $\dim(R/I_{B_\ell}^{\mathrm{P}}) = 2\ell = \dim(R/I_{B_\ell})$.

Justify your answers. (See also Laboratory Exercise 5.2.22.)

Challenge Exercise 5.2.18. The graph B_ℓ from Example 5.2.10 is connected with vertex set of size $d \geqslant 9$ such that the bound $\dim(R/I_G^{\mathrm{P}}) \geqslant \lceil 2d/3 \rceil$ in Corollary 5.2.9 is sharp. (A graph G is *connected* if, for all distinct vertices v_i and v_j, there is a path of edges in G from v_i to v_j.) For which values of $d < 9$ are there connected graphs with d vertices such that the bound is sharp? Justify your answer.

Challenge Exercise 5.2.19. Let A be a field, and set $R = A[X_1,\ldots,X_d]$. Let G be a graph with vertex set $V = \{v_1,\ldots,v_d\}$.

(a) Assume that v_d is an isolated vertex of G. Let G' denote the graph obtained by deleting the vertex v_d from G. (In other words, G' is the subgraph of G "induced" by the vertices $v_1,\ldots,v_{d-1}$.) Characterize the PMU covers of G' in terms of the PMU covers of G. Do the same for the minimal PMU covers and the smallest PMU covers.
(b) Set $R' = A[X_1,\ldots,X_{d-1}]$ so we have $I_G^{\mathrm{P}} \subseteq R$ and $I_{G'}^{\mathrm{P}} \subseteq R'$. Describe $\dim(R'/I_{G'}^{\mathrm{P}})$ in terms of $\dim(R/I_G^{\mathrm{P}})$.
(c) Let G'' be the graph obtained by deleting all the isolated vertices of G. Repeat parts (a) and (b) for the various PMU covers of G''.
(d) Repeat parts (a)–(c) for vertex covers and edge ideals.
(e) Use parts (a)–(d) to formulate and prove a version of Theorem 5.2.7 for graphs that may have some isolated vertices.

Justify your answers.

Challenge Exercise 5.2.20. Let A be a field, and set $R = A[X_1,\ldots,X_d]$. The graph $v_1 - v_2$ is connected such that every PMU cover is a vertex cover, so $I_G^{\mathrm{P}} = I_G$. Are there any other connected graphs with $d \geqslant 2$ satisfying these conditions? Justify your answer.

B_ℓ *in Macaulay2*

In this tutorial, we will write a method that, for a given $\ell \geqslant 3$, constructs the edge ideal of the graph B_ℓ from Example 5.2.10. This will illustrate how one can use Macaulay2 functions to define rings and ideals.

The input of this function will be an integer `ell` and a list of symbols that will be the names of the variables in the construction (such as u, v and w from the example). Before going further, a few words should be said regarding symbols in Macaulay2.

A `Symbol` object is an alphanumeric string that serves as the name of an object (e.g., a constant, a polynomial, or a ring) in Macaulay2. For instance, in the following example,

```
i1 : foo = 3
o1 = 3
```

`foo` is a (Macaulay2) variable[1] whose value is the constant 3. When we ask the interpreter for the value of `foo` using the command

```
i2 : foo
o2 = 3
```

[1] Note the disctinction between variable in the algebraic sense, and variable in the programming sense here.

the interpreter tells us the value of the variable `foo`. On the other hand, the command `symbol` returns the associated symbol.

```
i3 : symbol foo
o3 = foo
o3 : Symbol
```

Variables that do not yet have data associated to them are automatically symbols once they are provided to the interpreter.

```
i4 : bar
o4 = bar
o4 : Symbol
```

It is good practice to avoid exporting symbols from inside a user-defined function. In this tutorial, we accomplish this by specifying the symbols we use in our output as an argument to our method.

Now we begin to write our method. We first check that the input to the method is valid, and then define the polynomial ring R in 3ℓ generators using the symbols from the list provided. Next, we define three lists of variables (which we call `us`, `vs` and `ws`) of R that correspond to the packets of variables in the construction. Here we use the operator `_` where `R_i` will return the ith variable of R (beginning with index 0). Next, we create the edges $v_i v_{i+1}$ and $v_1 v_\ell$ in the `cycleGens` variable (using `|` to concatenate lists) and the $u_i v_i, v_i w_i$ edges in the `whiskers` variable. Finally, we concatenate these lists again using `|`, call the command `graph`, and return the result.

```
i5 : needsPackage "EdgeIdeals"
o5 = EdgeIdeals
o5 : Package

i6 : sharpPMUExample = method()
o6 = sharpPMUExample
o6 : MethodFunction

i7 : sharpPMUExample(ZZ,List) := (ell,symbList) -> (
if ell < 3 then error "Expected input integer greater than 2.";
if #symbList != 3 or not all(symbList, s -> class s === Symbol) then
error "Expected a list of three Symbols.";
u := symbList_0;
v := symbList_1;
w := symbList_2;
R := QQ[v_1..v_ell,u_1..u_ell,w_1..w_ell];
vs := toList (R_0..R_(ell-1));
us := toList (R_ell..R_(2*ell-1));
ws := toList (R_(2*ell)..R_(3*ell-1));
cycleGens := apply(ell-1, i -> (vs_i)*(vs_(i+1))) | {(vs_0)*(vs_(ell-1))};
whiskers := flatten apply(ell, i -> {us_i*vs_i,vs_i*ws_i});
graph (cycleGens | whiskers)
)
```

```
o7 = {*Function[stdio:7:44-20:20]*}
o7 : FunctionClosure
```

Note our use of the disjunction `or` in the second `if-then` statement.

Now we can create the graph B_4[2]:

```
i8 : G = sharpPMUExample(4,{u,v,w})
o8 = Graph{edges => {{v , v }, {v , v }, {v , v }, {v , v }, {v , u }, ···
                       1   2     2   3     1   4     3   4     1   1   ···
           ring => QQ[v , v , v , v , u , u , u , u , w , w , w , w ]
                       1   2   3   4   1   2   3   4   1   2   3   4
           vertices => {v , v , v , v , u , u , u , u , w , w , w , w }
                         1   2   3   4   1   2   3   4   1   2   3   4
o8 : Graph
```

Note that a subsequent call to `sharpPMUExample` on this same input will fail since `u`, `v`, `w` are no longer symbols. For instance, since `u` is the 'base' of an indexed variable, `u` is an `IndexedVariableTable` object.

Now that we have the graph associated to Example 5.2.10, we can compute the Krull dimension of the quotient by its edge ideal

```
i9 : dim edgeIdeal G
o9 = 8
```

and a smallest vertex cover can be found using the following commands.

```
i10 : vertCovers = vertexCovers G;

i11 : netList pack(vertCovers,3)
      +------------+------------+----------+
o11 = |v v v v     |v v v u w   |v v v u w |
      | 1 2 3 4    | 2 3 4 1 1  | 1 3 4 2 2|
      +------------+------------+----------+
      |v v v u w   |v v u u w w |v v v u w |
      | 1 2 4 3 3  | 2 4 1 3 1 3| 1 2 3 4 4|
      +------------+------------+----------+
      |v v u u w w |            |          |
      | 1 3 2 4 2 4|            |          |
      +------------+------------+----------+

i12 : minDeg = min(vertCovers / degree)
o12 = {4}
o12 : List

i13 : support first select(vertCovers, c -> degree c == minDeg)
o13 = {v , v , v , v }
        1   2   3   4
o13 : List
```

[2] The edge list has been truncated because it is too wide for the page.

Note that we suppressed the output of `vertexCovers` and used the commands `netList` and `pack` as in the tutorial in Section 4.6 to make the output a table with three columns. As predicted by Example 5.2.10, the dimension of the quotient ring is 8, which is the difference $12-4$ between the number of vertices and the size of a smallest vertex cover.

```
i14 : exit
```

Exercises

Exercise 5.2.21. Use Macaulay2 as in the the tutorial above to verify the equality $\dim(R/I_{B_\ell}) = 2\ell$ from Challenge Exercise 5.2.17(d) for $\ell = 5, 6, \ldots, 10$.

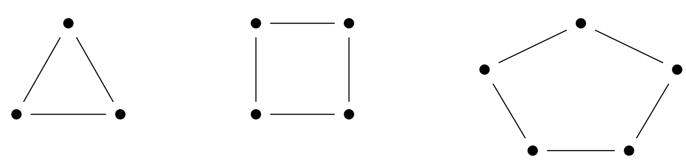
Laboratory Exercise 5.2.22. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate the graphs B_ℓ for some specific values of ℓ, to help with Challenge Exercise 5.2.17.

5.3 Cohen-Macaulayness and the Upper Bound Theorem

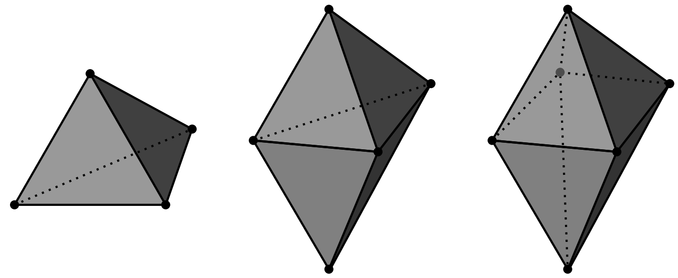
In this section, we discuss a powerful application of monomial ideals to topology. (For basic notions from topology, we refer the reader to any standard textbook, e.g., James Munkres [62]. For simplicial complexes, see Sections 4.4 and 4.5 above.) To get started, we need the following definition.

Definition 5.3.1. A *simplicial sphere* is a simplicial complex whose geometric realization is homeomorphic to a sphere $\mathbb{S}^{q-1}$ in $\mathbb{R}^q$.

For instance, the p-cycle C_p (illustrated here for $p = 3, 4, 5$)



is a simplicial sphere with $q = 2$. With $q = 3$, here are sketches of examples with $p = 4, 5, 6$ that are homeomorphic to $\mathbb{S}^2$.

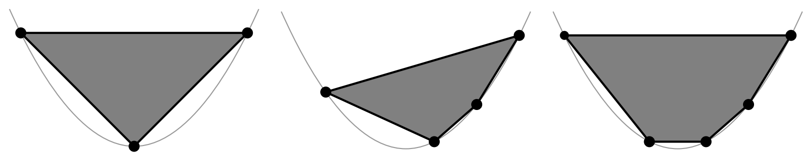


(5.3.1.1)

To state the Upper Bound Conjecture (UBC) for simplicial spheres, we need the next definition.

Definition 5.3.2. Let p and q be positive integers such that $p > q$, and consider the curve $X_q = \{(t, t^2, t^3, \dots, t^q) \mid t \in \mathbb{R}\}$ in $\mathbb{R}^q$. Choose p distinct points in X_q and let $C(p,q)$ denote the convex hull of these points. This forms a geometric object called a *polytope* in $\mathbb{R}^q$, and this particular polytope is called a *cyclic polytope*.

For example, the curve X_2 is the parabola $y = x^2$ in $\mathbb{R}^2$. Some examples of $C(p,2)$ for $p = 3,4,5$ are sketched next.



In general, the object $C(p,q)$ is homeomorphic to a q-dimensional ball in $\mathbb{R}^q$, and its boundary $\Delta(p,q) = \partial C(p,q)$ is homeomorphic to $\mathbb{S}^{q-1}$. (This depends on the condition $p > q$.) Moreover, $C(p,q)$ and $\Delta(p,q)$ are geometric realizations of simplicial complexes, which we also denote $C(p,q)$ and $\Delta(p,q)$. These simplicial complexes are independent of the choice of points in X_q, and only depend on p and q, up to re-labeling the vertices. This allows us to state the Upper Bound Conjecture (UBC) for simplicial spheres.

Conjecture 5.3.3 (UBC for simplicial spheres). Let Δ be a simplicial complex on a set of p vertices whose geometric realization is homeomorphic to a sphere $\mathbb{S}^{q-1}$ in $\mathbb{R}^q$. Then we have $f_i(\Delta) \leqslant f_i(C(p,q))$ for $i = 0, \dots, q-1$ where f is the f-vector from Definition 4.4.10.

Theodore Motzkin [61] first formulated the UBC for the special case of convex simplicial polytopes, and it was proved in this case by Peter McMullen [55]. The general form stated here was formulated by Victor Klee, and proved by Stanley [73].

Theorem 5.3.4 (Upper Bound Theorem) *The Upper Bound Conjecture for simplicial spheres holds.*

Stanley's famous proof of this result is outside the scope of this text. (See, however, the discussion following Definition 5.3.15.) One key feature of his proof is the use of “Cohen-Macaulayness” from commutative algebra (via monomial ideals) which we outline next.

To motivate the following definition, let $R = A[X_1,\dots,X_d]$ for some non-zero commutative ring A with identity. Every square-free monomial ideal $J \subsetneq R$ has an irredundant m-irreducible decomposition $J = \bigcap_{i=1}^k P_{V_i}$ by Proposition 4.1.6 and Algorithm 3.3.6. Also, Theorem 3.3.8 implies that this decomposition is unique up to re-ordering the V_i's. In particular, the list of V_i's is unique up to re-ordering. We have seen several examples where all the V_i's all have the same size, and other examples show that the V_i's can have different sizes. In the first of these cases, J is nicer than in the other case, and we identify this with the following definition.

Definition 5.3.5. Let A be a non-zero commutative ring with identity, and set $R = A[X_1,\dots,X_d]$. Consider a monomial ideal $J \subsetneq R$ with irredundant m-irreducible decomposition $J = \bigcap_{i=1}^k Q_i$. For $i = 1,\dots,k$ we have $\text{m-rad}(Q_i) = P_{V_i}$ for some set $V_i \subseteq \{v_1,\dots,v_d\}$. Then J is *m-unmixed* if $|V_i| = |V_j|$ for all $i \neq j$. We say that J is *m-mixed* if it is not m-unmixed, that is, if there are indices $i \neq j$ such that $|V_i| \neq |V_j|$.

For instance, the edge ideal I_G from Example 4.3.7 is m-mixed, as is the ideal J from Example 4.3.8. On the other hand, the ideal from Example 4.5.6 is m-unmixed. For a graph G, Theorem 4.3.6 tells us that the edge ideal I_G is m-unmixed if and only if every minimal vertex cover of G has the same cardinality. (In the language of Section 5.2, this means that every minimal vertex cover of G is a smallest vertex cover.) For a simplicial complex Δ, Theorem 4.6.5 tells us that the face ideal J_Δ is m-unmixed if and only if it is pure; see Sections 4.4–4.5.

The notion of Cohen-Macaulayness is a souped-up version of unmixedness; see Theorem 5.3.16(a). Outside of commutative algebra, it is incredibly important in areas like algebraic geometry that heavily rely on techniques from commutative algebra. One outcome of Stanley's proof of the UBC for simplicial spheres is that this notion has also become important in topology and combinatorics. The definition requires some additional notions, which we introduce next.

Definition 5.3.6. Let R be a non-zero commutative ring with identity, and let $g \in R$. Then g is *R-regular* if the map $R \xrightarrow{g} R$ given by $p \mapsto gp$ is injective and not surjective. More generally, given an ideal $I \subsetneq R$, the element $g \in R$ is *regular for the quotient* R/I from Section A.8 if the map $R/I \xrightarrow{g} R/I$ given by $r + I \mapsto (gr) + I$ is injective and not surjective; see Exercise A.8.11.

For instance, if $R = A[X_1,\dots,X_d]$ where A is a field and g is a non-constant polynomial in R, then g is R-regular. (More generally, if R is an integral domain and g is a non-zero non-unit of R, then g is R-regular; see Section 1.2.) Here is another example for later use.

Example 5.3.7. Let A be a field, and set $R = A[X_1,X_2,X_3]$ and $I = (X_1X_2X_3)R$. The element X_1 is not regular for R/I because we have $0 \neq X_2X_3 + I \in R/I$ and $0 = X_1(X_2X_3 + I) \in R/I$; see Exercise A.8.11. Moreover, there are no monomials in R that are regular for R/I.

On the other hand, the element $X_3 - X_2$ is regular on R/I. To see this, we first show that the map $R/I \xrightarrow{X_3-X_2} R/I$ is injective, using Exercise A.8.11. Let $r \in R$ such that $(X_3 - X_2)(r+I) = 0$ in R/I; we need to show that $r \in I$. The condition $(X_3 - X_2)(r+I) = 0$ implies that $(X_3 - X_2)r \in I = (X_1X_2X_3)R$. From the unique factorization property in R (this uses the assumption that A is a field) it follows that $r \in (X_1X_2X_3)R = I$, as desired.

To show that the map $R/I \xrightarrow{X_3-X_2} R/I$ is not surjective, argue as follows. Modding out by $X_3 - X_2$ is tantamount to setting X_3 equal to X_2. Thus, we have

$$\begin{aligned}(R/I)/[(X_3-X_2)(R/I)] &\cong R/(I+(X_3-X_2)R)\\ &= A[X_1,X_2,X_3]/(X_1X_2X_3, X_3-X_2)R\\ &\cong A[X_1,X_2]/(X_1X_2^2)\\ &\neq 0\end{aligned}$$

as desired. See also Exercise 5.3.25.

Similar reasoning as above shows that $X_2 - X_1$ is regular for $R/(I+(X_3-X_2)R)$ and that we have

$$R/((I+(X_3-X_2)R)+(X_2-X_1)R) \cong A[X_1,X_2]/(X_1X_2^2, X_2-X_1) \cong A[X_1]/(X_1^3).$$

Regular elements for R are important in commutative algebra because they allow for effective transmission of certain important properties between R and $R/(g)R$. Geometrically, if g is regular for R/I, then the intersection between the vanishing locus of g and the vanishing locus of I (in A^d) is sufficiently nice as to allow similar transfer between the zero-loci of I and $I+(g)R$. A hint of this can be seen in the following result of Maurice Auslander and David Buchsbaum [3, Section 1] which says that modding out by a regular element causes the Krull dimension from Section 5.1 to drop by exactly 1. (See, also [9, Proposition 1.2.12] and Exercise 5.3.35.)

Fact 5.3.8. Let A be a field, and set $R = A[X_1,\ldots,X_d]$. Consider an ideal $I \subsetneq R$ generated by homogeneous polynomials, and let g be a homogeneous polynomial in R that is regular for R/I. Then $\dim(R/(I+(g)R)) = \dim(R/I) - 1$.

Example 5.3.9. Continue with the notation of Example 5.3.7. As A is a field, the ring $R = A[X_1,X_2,X_3]$ has Krull dimension 3; cf. the discussion after Definition 5.1.1.

According to the paragraph preceding Example 5.3.7, the element $X_1X_2X_3$ is R-regular, so with $I = (X_1X_2X_3)R$ we have $\dim(R/I) = 2$, by Fact 5.3.8. One can also obtain this via Theorem 5.1.2.

The element $X_3 - X_2$ is R/I-regular by Example 5.3.7, so as in the previous paragraph, the ring $R/(I+(X_3-X_2)R) \cong A[X_1,X_2]/(X_1X_2^2)$ has Krull dimension 1. Similarly, the ring $R/((I+(X_3-X_2)R)+(X_2-X_1)R) \cong A[X_1]/(X_1^3)$ has Krull dimension 0. Note that one can check directly that $A[X_1]/(X_1^3)$ has Krull dimension 0 by showing that this ring has a unique prime ideal.

The following extension of regularity to sequences with more than one element is key to defining the Cohen-Macaulay property.

Definition 5.3.10. Let A be a field, and set $R = A[X_1,\dots,X_d]$. Consider an ideal $I \subsetneq R$. Then a sequence $g_1,\dots,g_m \in R$ is *regular for* R/I if it satisfies the next conditions:

(1) the polynomial g_1 is regular for R/I, and
(2) for $i = 2,\dots,m$ the polynomial g_i is regular for $R/(I+(g_1,\dots,g_{i-1})R)$.

Example 5.3.11. From Example 5.3.7, the sequence $X_3 - X_2, X_2 - X_1$ is regular for $A[X_1,X_2,X_3]/(X_1X_2X_3)R$.

An important consequence of Fact 5.3.8 is the following.

Lemma 5.3.12. *Let A be a field, and set $R = A[X_1,\dots,X_d]$. Consider an ideal $I \subsetneq R$ generated by homogeneous polynomials, and let $g_1,\dots,g_m$ be non-constant homogeneous polynomials in R. If this sequence is regular for R/I, then $m \leqslant \dim(R/I)$.*

Proof. Exercise. (Use Fact 5.3.8.) □

We are finally prepared to define the Cohen-Macaulay property.

Definition 5.3.13. Let A be a field, and set $R = A[X_1,\dots,X_d]$. Consider an ideal $I \subsetneq R$ generated by homogeneous polynomials. The quotient R/I is *Cohen-Macaulay* if there exists a homogeneous sequence $g_1,\dots,g_m \in R$ that is regular for R/I with $m = \dim(R/I)$.

Example 5.3.14. When A is a field, the quotient $A[X_1,X_2,X_3]/(X_1X_2X_3)R$ is Cohen-Macaulay: this ring has dimension 2 by Example 5.3.9, so it is Cohen-Macaulay by Example 5.3.11. Note that the first paragraph of Example 5.3.7 shows that this cannot be detected by only considering monomial regular sequences.

Before we return to the UBC for simplicial spheres, we need to discuss how the notion of Cohen-Macaulayness applies to simplicial complexes.

Definition 5.3.15. Let A be a field, and set $R = A[X_1,\dots,X_d]$. A simplicial complex Δ with vertex set $\{v_1,\dots,v_d\}$ is *Cohen-Macaulay* over A if the quotient R/J_Δ is Cohen-Macaulay. A graph G with vertex set $\{v_1,\dots,v_d\}$ is *Cohen-Macaulay* over A if the quotient R/I_G is Cohen-Macaulay;[3] see Sections 4.2–4.3.

In general, a simplicial complex Δ is *Cohen-Macaulay* if it is Cohen-Macaulay over every field A. A graph G with vertex set $\{v_1,\dots,v_d\}$ is *Cohen-Macaulay* if it is Cohen-Macaulay over every field A.

For instance, Example 5.3.14 shows that the 3-cycle, considered as a simplicial complex, is Cohen-Macaulay because we have $R/J_{C_3} = A[X_1,X_2,X_3]/(X_1X_2X_3)R$.

Finally, we are in position to discuss (in the broadest terms) Stanley's proof of the UBC for simplicial spheres. (See also Exercise 5.3.34.) First, Stanley shows that a result of James Munkres [63] implies that simplicial spheres are Cohen-Macaulay.

[3] As every graph is a simplicial complex, this terminology is ambiguous. So, when working with graphs, we state explicitly whether we consider them as graphs or as simplicial complexes.

He uses this, with a result of Peter McMullen [55] to show that the desired bounds from Conjecture 5.3.3 hold. For more details, see [9, 73, 74].

Having given some indication of the importance of Cohen-Macaulayness, we are motivated to present more properties of this condition. For instance, the next result contains an important test for non-Cohen-Macaulayness, due to Irvin Cohen [11] and Francis Macaulay [52]. The proof of part (a) is beyond the scope of this text; see Exercise 5.3.36. However, we show how it implies parts (b) and (c). It is worth noting that the converse of part (c) fails, e.g., by Exercise 5.3.31, parts (b) and (e); thus the converses of the other parts fail as well.

Theorem 5.3.16 *Let A be a field, and set $R = A[X_1,\ldots,X_d]$.*

(a) *Let I be a monomial ideal in R. If R/I is Cohen-Macaulay, then I is m-unmixed.*
(b) *Let Δ be a simplicial complex with vertex set $\{v_1,\ldots,v_d\}$. If Δ is not pure, then it is not Cohen-Macaulay over A.*
(c) *Let G be a graph with vertex set $\{v_1,\ldots,v_d\}$. If G has minimal vertex covers of different sizes, then G is not Cohen-Macaulay over A.*

Proof. (b) We prove the contrapositive. Assume that Δ is Cohen-Macaulay over A. Part (a) implies that J_Δ is m-unmixed, so Theorem 4.6.5 tells us that Δ is pure; see also the discussion following Definition 5.3.5.

(c) This follows like part (b), using Theorem 4.3.6. □

It is worth noting that the hypotheses of parts (b) and (c) of the previous result are independent of the field A. On the other hand, the Cohen-Macaulayness of Δ or G can depend on the choice of the field. See, Exercise 5.3.40.

Theorem 5.3.16 shows us how to find a monomial ideal I such that the quotient R/I is not Cohen-Macaulay. Indeed, let A be a field, and set $R = A[X,Y]$ with $I = (X^2,XY)R = (X)R \cap (X^2,Y)R$. The m-irreducible decomposition here is irredundant and shows that I is m-mixed. Thus, Theorem 5.3.16(a) implies that R/I is not Cohen-Macaulay. Note that Theorem 5.1.2 implies that $\dim(R/I) = 1$.

When $\dim(R/I)$ is small, the next result shows that Cohen-Macaulayness can be easy to check. It also shows that the example from the previous paragraph has minimal dimension among non-Cohen-Macaulay examples.

Theorem 5.3.17 *Let A be a field, and set $R = A[X_1,\ldots,X_d]$ with $\mathfrak{X} = (X_1,\ldots,X_d)R$. Let $I \neq R$ be a monomial ideal in R, and let $I = \bigcap_{i=1}^n J_i$ be an irredundant m-irreducible decomposition.*

(a) *If $\dim(R/I) = 0$, then R/I is Cohen-Macaulay.*
(b) *If $f \in R$ is a non-zero linear homogeneous polynomial with $f \notin \bigcup_{i=1}^n \text{m-rad}(J_i)$, then f is regular for R/I.*
(c) *If $\text{m-rad}(J_i) \neq \mathfrak{X}$ for $i = 1,\ldots,n$, then $f = X_1 + \cdots + X_d$ is regular for R/I.*
(d) *If $\dim(R/I) = 1$, then R/I is Cohen-Macaulay if and only if I is m-unmixed.*

Proof. (a) The empty sequence is vacuously regular with length $0 = \dim(R/I)$.

(b) By Exercise 5.3.25, the fact that f is non-constant and homogeneous implies that the map $R/I \xrightarrow{f} R/I$ is not surjective. To show that this map is injective, by

Exercise A.8.11 it suffices to let $p \in R$ be such that the coset $p+I \in R/I$ satisfies $fp+I=0$; we need to show that $p+I=0$. That is, it suffices to assume that $fp \in I$ and prove that $p \in I$.

Case 1: I is m-irreducible. In this case, I is generated by pure powers of the variables by Theorem 3.1.3. Re-order the variables if necessary to assume that $I=(X_k^{e_k},\ldots,X_d^{e_d})R$ for some $e_i \geqslant 1$. By assumption, we have $f \notin \text{m-rad}(I) = (X_k,\ldots,X_d)R$, so Lemma 1.1.10 implies that some monomial occurring in f is not in this ideal. Since f is linear and homogeneous, this means that the monomial X_j occurs in f with non-zero coefficient for some $j<k$. Re-order the variables $X_1,\ldots,X_{j-1}$ if necessary to assume that the monomial X_1 occurs in f, and $1<k$.

To show that $p \in I$, it suffices to show that for every monomial $\underline{X}^{\underline{n}}$ occurring in p is in I. To accomplish this, we use the lexicographical (lex) order from Section A.9. By design, the largest monomial occurring in f with respect to the lex order is X_1. Let $\underline{X}^{\underline{n}}$ be the largest monomial occurring in g, with coefficient α. It is straightforward to show that the largest monomial occurring in the product fp is the product of the largest monomials occurring in f and p, that is, it is $X_1\underline{X}^{\underline{n}}$. (This uses the fact that A is a field, and is akin to the fact that the degree of the product is the sum of the degrees. See Exercise 5.4.15(a) for a souped-up version of this.)

As fp is in I, Lemma 1.1.10 implies that every monomial occurring in fp is in I. In particular, the largest one $X_1\underline{X}^{\underline{n}}$ is in $I=(X_k^{e_k},\ldots,X_d^{e_d})R$. The condition $1<k$ implies that $\underline{X}^{\underline{n}} \in I$. From this, we have $f(p-\alpha\underline{X}^{\underline{n}}) = fp - f\cdot\alpha\underline{X}^{\underline{n}} \in I$. The polynomial $p-\alpha\underline{X}^{\underline{n}}$ has strictly fewer monomials than p, so an induction argument shows that the monomials of $p-\alpha\underline{X}^{\underline{n}}$ are all in I; that is, the remaining monomials of p are in I. This establishes the result in Case 1.

Case 2: the general case. Our assumption on p says that $fp \in I = \bigcap_{i=1}^n J_i$, so we have $fp \in J_i$ for all i. By assumption, we have $f \notin \text{m-rad}(J_i)$ for all i, so Case 1 implies that $p \in J_i$ for all i; that is, we have $p \in \bigcap_{i=1}^n J_i = I$, as desired.

(c) By part (b) it suffices to show that $f \notin \text{m-rad}(J_i)$ for all i. Suppose by way of contradiction that f is in the monomial ideal $\text{m-rad}(J_i)$. Lemma 1.1.10 implies that every monomial occurring in f is in $\text{m-rad}(J_i)$. That is, $\text{m-rad}(J_i)$ contains all the variables of R, contradicting the assumption $\text{m-rad}(J_i) \neq \mathfrak{X}$.

(d) Assume that $\dim(R/I)=1$. Exercise 5.1.7 implies that $\text{m-rad}(J_i) \neq \mathfrak{X}$ for all i. Thus, part (c) provides a homogeneous regular sequence for R/I of length $1=\dim(R/I)$. By definition, this implies that R/I is Cohen-Macaulay. □

We conclude this section with some further discussion regarding regular sequences of homogeneous polynomials.

Definition 5.3.18. Let A be a field, let $R=A[X_1,\ldots,X_d]$, and let $I \subsetneq R$ be an ideal of R generated by homogeneous polynomials. A homogeneous regular sequence $g_1,\ldots,g_m$ for R/I is *maximal* if it cannot be extended to a longer homogeneous regular sequence on R/I.

A remarkable result of Auslander and Buchsbaum [3, 4] regarding regular sequences is the following. (See also [9, Theorem 1.2.5] and Exercise 5.3.37.)

Fact 5.3.19. Let A be a field, let $R = A[X_1, \dots, X_d]$, and let $I \subsetneq R$ be an ideal of R generated by homogeneous polynomials. Every homogeneous regular sequence is a subset of a maximal homogeneous regular sequence, and every maximal homogeneous regular sequence on R/I has the same length. Thus, R/I is Cohen-Macaulay if and only if every (equivalently, some) maximal homogeneous regular sequence for R/I has length equal to $\dim(R/I)$.

The usual proof of this fact uses homological algebra (usually via Ext functors, Koszul homology, or local cohomology) and is beyond the scope of our discussion. As a consequence of this result, though, we have the following "rigidity" property for Cohen-Macaulayness.

Proposition 5.3.20 *Let A be a field, let $R = A[X_1, \dots, X_d]$, and let $I \subsetneq R$ be an ideal of R generated by homogeneous polynomials. Assume that $g_1, \dots, g_m \in R$ is a homogeneous regular sequence for R/I. Then R/I is Cohen-Macaulay if and only if $R/(I + (g_1, \dots, g_m)R)$ is Cohen-Macaulay.*

Proof. For the forward implication, assume that R/I is Cohen-Macaulay. Fact 5.3.19 implies that the sequence $g_1, \dots, g_m$ can be extended to a homogeneous regular sequence $g_1, \dots, g_m, \dots, g_n$ for R/I where $n = \dim(R/I)$. From Fact 5.3.8 we know that $\dim(R/(I + (g_1, \dots, g_m)R)) = n - m$. Since the sequence $g_{m+1}, \dots, g_n$ is regular for $R/(I + (g_1, \dots, g_m)R)$ with length $\dim(R/(I + (g_1, \dots, g_m)R))$, it follows that $R/(I + (g_1, \dots, g_m)R)$ is Cohen-Macaulay, as desired.

The converse is handled similarly. $\square$

Theorem 5.3.16(a) and Proposition 5.3.20 combine to suggest one property that makes Cohen-Macaulay quotients particularly nice. Specifically, if I is m-unmixed and $g \in R$ is regular for R/I, then there is no reason for $I + gR$ to have any nice unmixedness properties; see, e.g., Exercise 5.3.32(e). However, it can be shown that when R/I satisfies the stronger Cohen-Macaulay property, then the ideal $I + (g_1, \dots, g_n)R$ is "unmixed" for every homogeneous sequence $g_1, \dots, g_n$ that is regular for R/I.

Definition 5.3.21. Let A be a field, let $R = A[X_1, \dots, X_d]$, and let $I \subsetneq R$ be an ideal of R generated by homogeneous polynomials. The length of a maximal homogeneous regular sequence on R/I is the *depth* of R/I, and is denoted $\operatorname{depth}(R/I)$.

Using this definition, one obtains compact statements like the following. (See also Exercise 5.3.38.)

Fact 5.3.22. Let A be a field, and set $R = A[X_1, \dots, X_d]$. Consider an ideal $I \subsetneq R$ generated by homogeneous polynomials.

(a) (Lemma 5.3.12) One has $\operatorname{depth}(R/I) \leqslant \dim(R/I)$.
(b) (Definition 5.3.13) One has $\operatorname{depth}(R/I) = \dim(R/I)$ if and only if R/I is Cohen-Macaulay.
(c) (Proof of Proposition 5.3.20) If $g_1, \dots, g_m \in R$ is a homogeneous regular sequence for R/I, then $\operatorname{depth}(R/(I + (g_1, \dots, g_m)R)) = \operatorname{depth}(R/I) - m$.

Exercises

Exercise 5.3.23. Let R be a non-zero commutative ring with identity.

(a) Prove that an element $g \in R$ is R-regular if and only if g is a non-unit in R such that $(0 :_R g) = 0$. See Section 2.6.
(b) Fix an ideal $I \subsetneq R$. Prove that an element $g \in R$ is regular for R/I if and only if $I + (g)R \neq R$ and $(I :_R g) = I$.

Exercise 5.3.24. Prove Lemma 5.3.12. (Hint: Use Fact 5.3.8.)

Exercise 5.3.25. Let A be a field, and set $R = A[X_1, \ldots, X_d]$. Let $I \neq R$ be an ideal generated by homogeneous polynomials. Let $f \in R$ be a non-constant homogeneous polynomial, and prove that the map $R/I \xrightarrow{f} R/I$ is not surjective.

Exercise 5.3.26. Let A be a field, and set $R = A[X_1, \ldots, X_d]$. Prove that $R/(X_1 \cdots X_d)$ is Cohen-Macaulay. (Note that this is the simplest case of Stanley's observation that simplicial spheres are Cohen-Macaulay.) (Hint: It may be easier to prove the following result by induction on d, arguing as in part of Example 5.3.9: For every monomial $f \in [\![R]\!]$, if $f \neq 1$, then R/fR is Cohen-Macaulay.)

Exercise 5.3.27. Decide whether the simplicial complexes from Example 4.4.2 and Exercise 4.4.12 are Cohen Macaulay. Do the same for the simplicial complexes following Definitions 5.3.1 and 5.3.2. (Hint: Theorem 5.3.16 and the proof of Theorem 5.3.4.) Justify your answers.

Exercise 5.3.28. Finish the proof of Proposition 5.3.20 by proving that if the quotient $R/(I + (g_1, \ldots, g_m)R)$ is Cohen-Macaulay, then R/I is Cohen-Macaulay.

Exercise 5.3.29. Let A be a field and set $R = A[X_1, \ldots, X_d]$. Consider the complete graph K_d and its edge ideal I_{K_d}. Prove that I_{K_d} is m-unmixed and $\dim(R/I_{K_d}) = 1$. Conclude that K_d is Cohen-Macaulay.

Exercise 5.3.30. Let A be a field and set $R = A[X_1, \ldots, X_d]$ with $d \geqslant 2$. Consider the path P_d v_1 —— v_2 —— $\cdots$ —— v_d and its edge ideal I_{P_d}. This exercise is a guided proof of the following result of Villarreal [76, Theorem 2.4]: the path P_d is Cohen-Macaulay over A if and only if $d = 2$ or 4.

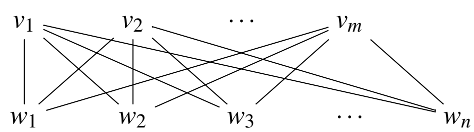
(a) Prove that I_{P_2} is m-unmixed with $\dim(R/I_{P_2}) = 1$. Conclude that the path P_2 is Cohen-Macaulay.
(b) Show that P_3 has minimal vertex covers of size 1 and 2. Conclude that R/I_{P_3} is not Cohen-Macaulay.
(c) Argue as in part (b) to show that R/I_{P_d} is not Cohen-Macaulay when $d \geqslant 5$.
(d) Prove that I_{P_4} is m-unmixed and $\dim(R/I_{P_4}) = 2$. (Note that P_4 has three minimal vertex covers, each of size 2.)
(e) Label the vertices of P_4 as in the sketch above. The following steps show that R/I_{P_4} is Cohen-Macaulay.

(1) Use Theorem 5.3.17(b) to show that $f = X_1 - X_4$ is regular on R/I_{P_4}.
(2) Show that we have

$$R/(I_{P_4} + fR) \cong A[X_1, X_2, X_3]/(X_1X_2, X_2X_3, X_1X_3)R.$$

(Essentially, we are taking R/I_{P_4} and setting $X_4 = X_1$.)
(3) Verify that $R/(I_{P_4} + fR)$ is Cohen-Macaulay, e.g., by Theorem 5.3.17(d).
(4) Use Proposition 5.3.20 to conclude that R/I_{P_4} is Cohen-Macaulay.

Exercise 5.3.31. Let A be a field and set $R = A[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ with $m, n \geqslant 1$. Consider the complete bipartite graph $K_{m,n}$



and its edge ideal $I_{K_{m,n}}$. This exercise is a guided proof of the following result: the graph $K_{m,n}$ is Cohen-Macaulay over A if and only if $m = n = 1$.

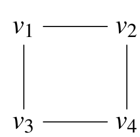
(a) Prove that $K_{1,1}$ is Cohen-Macaulay over A.
(b) Prove that $K_{m,n}$ has exactly two minimal vertex covers, one of size m and the other of size n.
(c) Prove that the polynomial $f = Y_n - X_m$ is regular for $R/I_{K_{m,n}}$.
(d) Assume that $m, n \geqslant 2$. Show that we have $R/(I_{K_{m,n}} + fR) \cong R'/J$ where $R' = A[X_1, \ldots, X_m, Y_1, \ldots, Y_{n-1}]$ and

$$J = (X_1, \ldots, X_m)R' \cap (Y_1, \ldots, Y_{n-1}, X_m)R' \cap (X_1, \ldots, , X_{m-1}, X_m^2, Y_1, \ldots, Y_{n-1})R'.$$

Prove that this decomposition is irredundant.
(e) Prove that $K_{m,n}$ is Cohen-Macaulay over A if and only if $m = n = 1$.

Exercise 5.3.32. Let A be a field and set $R = A[X_1, \ldots, X_d]$ with $d \geqslant 3$. Consider the cycle C_d and its edge ideal I_{C_d}. This exercise is a guided proof of most of the following result of Villarreal [76, Corollary 4.7]: the cycle C_d is Cohen-Macaulay over A if and only if $d = 3$ or 5.

(a) Prove that I_{C_3} is m-unmixed with $\dim(R/I_{C_3}) = 1$. Conclude that the cycle C_3 is Cohen-Macaulay.
(b) Show that C_6 has minimal vertex covers of size 3 and 4. Conclude that R/I_{C_6} is not Cohen-Macaulay.
(c) Argue as in part (b) to show that R/I_{C_d} is not Cohen-Macaulay when $d \geqslant 6$.
(d) Prove that I_{C_4} and I_{C_5} are m-unmixed and $\dim(R/I_{C_4}) = 2 = \dim(R/I_{C_5})$.
(e) Label the vertices of C_4 as follows.



The following steps show that R/I_{C_4} is not Cohen-Macaulay.

(1) Use Theorem 5.3.17(b) to show that $f = X_1 - X_4$ is regular on R/I_{C_4}.
(2) Show that we have

$$R/(I_{C_4} + fR) \cong A[X_1,X_2,X_3]/(X_1X_2,X_2X_3,X_1X_3,X_1^2)R.$$

(Essentially, we are taking R/I_{C_4} and setting $X_4 = X_1$.)
(3) Verify the following irredundant m-irreducible decomposition:

$$(X_1X_2,X_2X_3,X_1X_3,X_1^2)R = (X_1,X_2)R \cap (X_1,X_3)R \cap (X_1^2,X_2,X_3)R.$$

(4) Conclude that the ideal $(X_1X_2,X_2X_3,X_1X_3,X_1^2)R$ is m-mixed.
(5) Conclude that $R/(I_{C_4} + fR)$ is not Cohen-Macaulay.
(6) Use Proposition 5.3.20 to conclude that R/I_{C_4} is Cohen-Macaulay.

The graph C_5 is Cohen-Macaulay, but this is outside the scope of this text. (See, however, Exercise 5.3.39(b).)

Exercise 5.3.33. This exercise involves the construction $\mathrm{V}(I)$ from Exploration Section A.10. Let A be a field, and let I be a monomial ideal of $R = A[X_1,\dots,X_d]$. Exercise 3.3.11(b) shows that there is an integer $k \in \mathbb{N}$, and there are linear subspaces $V_1,\dots,V_k \subseteq A^d$ such that $\mathrm{V}(J_\Delta) = V_1 \cup \cdots \cup V_k$ and such that $V_i \not\subseteq V_j$ whenever $i \neq j$. Prove that I is m-unmixed if and only if the subspaces V_i all have the same dimension over A.

Cohen-Macaulayness and the Upper Bound Theorem in Macaulay2

The Macaulay2 package `Polyhedra` by René Birkner [7] (with contributions by Nathan Ilten) provides functionality for working with rational polyhedra, cones, and fans. In particular, there is a method for constructing the cyclic polytope $C(p,q)$ of dimension q on p vertices. Indeed, this is done using the `cyclicPolytope` method. Warning: the notation for $C(p,q)$ is `cyclicPolytope(q,p)`.

```
i1 : needsPackage "Polyhedra"
o1 = Polyhedra
o1 : Package

i2 : P = cyclicPolytope(3,6)
o2 = {ambient dimension => 3           }
      dimension of lineality space => 0
      dimension of polyhedron => 3
      number of facets => 8
      number of rays => 0
      number of vertices => 6
```

```
o2 : Polyhedron
```

One can also ask for the f-vector of P.

```
i3 : fVector P
o3 = {6, 12, 8, 1}
o3 : List
```

So we see that this polytope has 6 vertices, 12 edges, and 8 triangles. The 1 at the end indicates the interior of the polytope; the empty face is not listed in this output.

The Upper Bound Theorem states that any simplicial complex on 6 vertices whose geometric realization is homeomorphic to a 2-sphere has f-vector bounded above by the f-vector of $C(6,3)$. Note that the octahedron in (5.3.1.1) has the same f-vector as cyclic polytope $C(6,3)$.

One can also use Macaulay2 to verify that a sequence $g_1,\dots,g_m$ is regular. Indeed, consider the sequence X_3-X_2,X_2-X_1 for $R/I=\mathbb{Q}[X_1,X_2,X_3]/(X_1X_2X_3)$. The polynomial X_3-X_2 is regular provided that multiplication by X_3-X_2 is injective on R/I. (Note that homogeneity implies that this multiplication map is not surjective by Exercise 5.3.25.) This holds if and only if there are no non-zero elements $f+I$ of R/I such that $(f+I)((X_3-X_2)+I)=I$, that is, if and only if the *annihilator ideal* $\operatorname{ann}_{R/I}(X_3-X_2)=(0:_{R/I}(X_3-X_2))$ is 0; see Exercise 5.3.23(a). We can check this condition using the command ann.

```
i4 : R = QQ[x_1,x_2,x_3]/ideal{x_1*x_2*x_3}
o4 = R
o4 : QuotientRing

i5 : ann(x_3-x_2) == 0
o5 = true
```

Next, we want to ensure that X_2-X_1 is regular for $R/(I+(X_3-X_2)R)$, which we can do by defining the quotient of R by X_3-X_2, and computing the annihilator of X_2-X_1 there.

```
i6 : S = R/ideal (x_3-x_2)
o6 = S
o6 : QuotientRing

i7 : ann(x_2-x_1) == 0
o7 = true
```

On the other hand, Macaulay2 can check Cohen-Macaulayness directly with the package Depth.

```
i8 : use R
o8 = R
o8 : QuotientRing

i9 : needsPackage "Depth"
o9 = Depth
o9 : Package
```

```
i10 : depth R
o10 = 2

i11 : depth R == dim R
o11 = true

i12 : isCM(R)
o12 = true

i13 : exit
```

Exercises

Exercise 5.3.34. Use the command `cyclicPolytope` as in the above tutorial to verify the Upper Bound Theorem for the other simplicial spheres displayed at the beginning of this section and for some other simplicial spheres of your own devising.

Exercise 5.3.35. Set $A = \mathbb{Z}_{101}$, and use Macaulay2 to verify the equality

$$\dim(R/(I+(g)R)) = \dim(R/I) - 1$$

from Fact 5.3.8 for some examples. (Don't forget to make sure that I and g satisfy the hypotheses of 5.3.8. Theorem 5.3.17 may be useful for finding g once you've chosen I. Alternately, you might try using the command `random` to find a regular g.)

Exercise 5.3.36. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to verify Theorem 5.3.16(a) for some monomial ideals. (Construct some m-mixed ideals as in the discussion following Theorem 5.3.16, and check that they are not Cohen-Macaulay.)

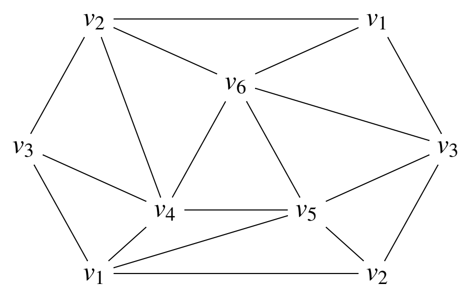
Exercise 5.3.37. Set $A = \mathbb{Z}_{101}$, and use Macaulay2 to explore the conclusions of Fact 5.3.19 for some examples.

Exercise 5.3.38. Set $A = \mathbb{Z}_{101}$, and use Macaulay2 to verify the inequality and the equalities from Fact 5.3.22 for some examples.

Exercise 5.3.39. Use Macaulay2 to verify the following.

(a) The complete graphs K_d are Cohen-Macaulay over $\mathbb{Q}$ of dimension 1 for $d = 2,\ldots,6$. See Exercise 5.3.29.
(b) The cycles C_3 and C_5 are Cohen-Macaulay over $\mathbb{Q}$, and the cycles C_4, C_6, and C_7 are not Cohen-Macaulay over $\mathbb{Q}$. See Exercise 5.3.32.
(c) The paths P_2 and P_4 are Cohen-Macaulay over $\mathbb{Q}$, and the paths P_3, P_5, and P_6 are not Cohen-Macaulay over $\mathbb{Q}$. See Exercise 5.3.30.
(d) The complete bipartite graph $K_{1,1}$ is Cohen-Macaulay over $\mathbb{Q}$, and the graphs $K_{1,2}$, $K_{1,3}$, $K_{2,2}$, $K_{2,3}$, and $K_{3,3}$ are not. See Exercise 5.3.31.

Exercise 5.3.40. Consider the following simplicial complex Δ from [69, Remark 3] or [9, Section 5.3].



Use Macaulay2 to show that Δ is not Cohen-Macaulay over $\mathbb{Z}_2$, but that it is Cohen-Macaulay over $\mathbb{Z}_{41}$ and over $\mathbb{Q}$. Thus, Cohen-Macaulayness depends on the field one chooses. (It is worth noting that the simplicial complex Δ is a triangulation of the projective plane $\mathbb{RP}^2$, providing a further glimpse of the connections between monomial ideals and topology. For instance, this parallels the difference between singular homology over $\mathbb{Z}_2$ and that over $\mathbb{Q}$.)

Exercise 5.3.41. Use the command `isRegularSequence` from the `Depth` package to check that the sequence $X_3 - X_2, X_2 - X_1$ is regular for the quotient ring $R/I = \mathbb{Q}[X_1, X_2, X_3]/(X_1X_2X_3)$.

Exercise 5.3.42. Use Macaulay2 as in the the tutorial from Section 5.2 to compute $\operatorname{depth}(R/I_{B_\ell})$ for $\ell = 5, 6, \ldots, 10$. Is B_ℓ Cohen-Macaulay?

5.4 Hilbert Functions and Initial Ideals

In this section, A is a field.

One of the amazing things about monomial ideals is that they have the power to give us information about other ideals. In short, given an ideal I generated by homogeneous polynomials over A and a monomial order $<$ on R, see Section A.9, one can find a monomial ideal $\operatorname{in}(I) = \operatorname{in}_<(I)$ of the polynomial ring R with many of the same properties as I. The ideal $\operatorname{in}(I)$ is an "initial ideal" of I. In principle, this allows one to transfer problems for non-monomial ideals in R, which can be quite messy, to similar problems for monomial ideals. One can then apply, e.g., combinatorial techniques to the monomial ideal in addition to purely algebraic techniques, as we have already seen.

In this section, we focus on the "Hilbert function" of the quotient R/I from Section A.8, which is the same as that of $R/\operatorname{in}(I)$; see Theorem 5.4.7. Hilbert functions are another extremely important tool in commutative algebra and algebraic geometry. For instance, they are crucial for Stanley's proof of the UBC outlined in Section 5.3. However, our treatment here only scratches the surface. It is also worth noting that the ideas in this section form the starting point for the study of "Gröbner bases" which have applications in many areas including, surprisingly, the study of electrical power systems; see Kavasseri and Nag [46].

Definition 5.4.1. Set $R = A[X_1,\dots,X_d]$, and consider an ideal $I \subsetneq R$ generated by homogeneous polynomials. For each $i \in \mathbb{N}$, we consider the finite-dimensional A-vector space

$$(R/I)_i = \{f + I \in R/I \mid f \in R \text{ is homogeneous of degree } i\} \cup \{0 + I\}$$

and its vector space dimension

$$h_{R/I}(i) = \dim_A((R/I)_i).$$

The function $h_{R/I} \colon \mathbb{N} \to \mathbb{N}$ is the *Hilbert function* of R/I.

It is worth noting that the fact that there are only finitely many monomials of a fixed degree i in $R = A[X_1,\dots,X_d]$ implies that $\dim_A((R/I)_i) < \infty$.

One reason for the importance of Hilbert functions comes from the following result of Hilbert [39] that shows that these functions encode surprising algebraic and geometric data, e.g., the Krull dimension from Section 5.1. (See also [54, Theorem 13.2 and Corollary] and Exercise 5.4.20.)

Theorem 5.4.2 *Set $R = A[X_1,\dots,X_d]$, and consider an ideal $I \subsetneq R$ generated by homogeneous polynomials. There is a polynomial $p_{R/I}$ in one variable over $\mathbb{Q}$ such that $h_{R/I}(i) = p_{R/I}(i)$ for $i \gg 0$. (Here we say that a property $P(i)$ holds for $i \gg 0$ when there is an integer N such that $P(i)$ holds for all $i \geqslant N$.)*

Moreover, $p_{R/I}$ has degree equal to $\dim(R/I) - 1$ and is of the form

$$p_{R/I}(i) = \frac{e(R/I)}{(\dim(R/I)-1)!} i^{\dim(R/I)-1} + \textit{lower degree terms}$$

where $e(R/I)$ is a positive integer. Here $\dim(R/I)$ is the Krull dimension of R/I. In the case $\dim(R/I) = 0$, the degree of $p_{R/I}$ is -1, which we interpret as $p_{R/I} = 0$.

Definition 5.4.3. Set $R = A[X_1,\dots,X_d]$, and consider an ideal $I \subsetneq R$ generated by homogeneous polynomials. The polynomial $p_{R/I}$ in Theorem 5.4.2 is the *Hilbert polynomial* of R/I. If $\dim(R/I) \geqslant 1$, the integer $e(R/I)$ is the *multiplicity* of R/I.

Section 1.5 contains an exploration of the case where $I = 0$, where one shows the first step in the following sequence.

$$
\begin{aligned}
h_{R/0}(i) &= \binom{i+d-1}{d-1} \\
&= \frac{(i+d-1)!}{(d-1)!i!} \\
&= \frac{(i+d-1)(i+d-2)\cdots(i+1)}{(d-1)!} \\
&= \frac{1}{(d-1)!}i^{d-1} + \text{lower degree terms}
\end{aligned}
$$

Thus, Theorem 5.4.2 gives another method for verifying that R has Krull dimension d, and we see that the multiplicity of R is 1.

As we have already remarked, the Krull dimension of R/I is a measure of the size of R/I. Similarly, the multiplicity of R/I is a measure of the complexity of R/I. For instance, given a simplicial complex Δ, it can be shown that the multiplicity of R/J_Δ equals the number of facets of Δ that have maximal dimension; see Sections 4.4–4.5 and Theorem 5.4.4. Similarly, given a graph G, the multiplicity of R/I_G equals the number of minimal vertex covers of G of minimal size, i.e., the number of smallest vertex covers of G; see Sections 4.2–4.3 and 5.2.

The previous paragraph indicates how some information about the Hilbert polynomial $p_{R/I}$ can be obtained combinatorially in the case where I is a square-free monomial ideal. (This is due to the fact such an ideal I is of the form J_Δ, by Remark 4.4.5.) The next result of Stanley [74] shows that, in fact, the entire Hilbert function is determined by combinatorial data in this case. (See also [9, Theorem 5.1.7] and Exercise 5.4.22.)

Theorem 5.4.4 *Let Δ be a simplicial complex of dimension n on a vertex set of size d, and assume that each singleton $\{v_i\}$ is in Δ. Set $R = A[X_1,\ldots,X_d]$. Then, using the f-vector of Definition 4.4.10, we have*

$$
h_{R/J_\Delta}(i) = \begin{cases} 1 & \textit{if } i = 0 \\ \sum_{j=0}^{n} f_j(\Delta)\binom{i-1}{j} & \textit{if } i \geqslant 1. \end{cases}
$$

For instance, if Δ is the 3-cycle C_3, considered as a simplicial complex, then $f(\Delta) = (3,3)$ and $n = 1$. Theorem 5.4.4 tells us that for $i \geqslant 1$ we have the following.

$$
h_{R/J_\Delta}(i) = \sum_{j=0}^{n} f_j(\Delta)\binom{i-1}{j} = 3\binom{i-1}{0} + 3\binom{i-1}{1} = 3(1) + 3(i-1) = 3i
$$

From this we can see that the Hilbert polynomial of R/J_Δ is $p_{R/J_\Delta}(i) = 3i$ and that we have $p_{R/J_\Delta}(i) = h_{R/J_\Delta}(i)$ for all $i \geqslant 1$. This has the form predicted by Theorem 5.4.2 since

(1) the degree of $p_{R/J_\Delta}(i)$ is $\dim(R/J_\Delta) - 1 = \dim(\Delta) = 1$, by Exercise 5.1.6, and
(2) the leading coefficient of p_{R/J_Δ} is $\frac{e(R/J_\Delta)}{(\dim(R/J_\Delta)-1)!} = \frac{3}{0!} = \frac{3}{1} = 3$, by the discussion following Definition 5.4.3.

How can one compute the Hilbert polynomial of the quotient R/I for non-monomial ideals? (We still assume that I is generated by homogeneous polynomials.) Since the value of the Hilbert function of a quotient R/I at i is the dimension of the A-vector space $(R/I)_i$, it should not come as a surprise that this can be done using matrices, as follows.

Consider the matrix M_i where

(1) the rows of M_i correspond to a spanning set of I_i, where I_i is the vector space of homogeneous polynomials of degree i in I (together with the zero polynomial),
(2) the columns of M_i correspond to the monomials of R of degree i, in decreasing ordered via a fixed monomial order from Section A.9, and
(3) the p,q-entry of M_i is the coefficient of the qth monomial in the pth spanning polynomial.

Note that the linear combinations of the rows of M_i correspond exactly to the linear combinations of the spanning polynomials for I_i. In other words, the elements of the row space of M_i are in 1-1 correspondence with the polynomials in I_i; in particular:

$$\dim_A(I_i) = \dim_A(\mathrm{Row}(M_i)) = \mathrm{rank}(M_i). \tag{5.4.4.1}$$

Note also that the leftmost non-zero entry in a row of M_i corresponds to the largest monomial appearing in the associated spanning polynomial, and similarly for any vector in the row space of M_i.

To continue, we consider the special case of the matrix M_2 for the ideal $I = (X-Y, XY-Z^2)R$ in $R = A[X,Y,Z]$. By Exercise 5.4.12, a spanning set for I_2 is

$$\{X(X-Y), Y(X-Y), Z(X-Y), XY-Z^2\} = \{X^2-XY, XY-Y^2, XZ-YZ, XY-Z^2\}.$$

Ordering the degree-2 monomials of R lexicographically, we have

$$X^2 >_{\mathrm{lex}} XY >_{\mathrm{lex}} XZ >_{\mathrm{lex}} Y^2 >_{\mathrm{lex}} YZ >_{\mathrm{lex}} Z^2.$$

Using the coefficients of the spanning polynomials to build M_2, we obtain

$$M_2 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

In this example (and in general), the pivot columns of M_i correspond exactly to the monomials that appear as the largest monomial occurring in any non-zero polynomial of I_i. That is, the monomial corresponding to each pivot column appears as the "leading term" of some polynomial of I_i, and the leading term of each non-zero polynomial of I_i corresponds to a pivot column. In other words, the number of leading terms of polynomials from I_i is exactly $\mathrm{rank}(M_i)$ which, in our example, is 4.

Combining this with (5.4.4.1), we see that $\dim_A(I_i)$ is exactly the number of leading terms of polynomials from I_i. If I'_i is the span of these leading terms, this says that $\dim_A(I_i) = \dim_A(I'_i)$. It follows that

$$\begin{aligned}
h_{R/I}(i) &= \dim_A((R/I)_i) \\
&= \dim_A(R_i) - \dim_A(I_i) \\
&= \dim_A(R_i) - \dim_A(I'_i) \\
&= \dim_A((R/I')_i) \\
&= h_{R/I'}(i).
\end{aligned}$$

where I' is the monomial ideal spanned by all the leading terms of all the polynomials from I.

Note that $\dim_A(R_i)$ is the number of columns of M_i. Thus, this discussion also implies that $h_{R/I}(i)$ is the difference between the number of columns of M_i and the number of pivot columns of M_i. Thus, in our example, we have $h_{R/I}(2) = 6-4 = 2$. Continuing with this example, the matrix M_3 does not have full rank. Indeed, performing the same computation as in the previous case (see Exercise 5.4.13), we obtain a 9×10 matrix of rank 8, so that $h_{R/I}(3) = 10-8 = 2$.

This discussion leads us to the notion of the initial ideal of I, defined next, and its connection to Hilbert functions in Theorem 5.4.7 below. The idea behind this construction is to throw away all but the leading terms of the elements of I.

Definition 5.4.5. Set $R = A[X_1,\ldots,X_d]$, and fix a monomial order $<$ on $[\![R]\!]$; see Section A.9.

For each non-zero polynomial $f \in R$, write $f = \sum_{i=1}^{p} a_i \underline{X}^{\underline{n}_i}$ where we have $0 \neq a_i \in A$ and $\underline{n}_i \in \mathbb{N}^d$ such that $\underline{X}^{\underline{n}_1} < \cdots < \underline{X}^{\underline{n}_p}$. The *leading term* of f is $\operatorname{lt}(f) = \underline{X}^{\underline{n}_p}$.

Let I be a non-zero ideal of R. The *initial ideal of I with respect to* $<$ is the monomial ideal generated by the leading terms of the non-zero polynomials in I:

$$\operatorname{in}_<(I) = (\operatorname{lt}_<(f) \mid 0 \neq f \in I)R.$$

Also, we set $\operatorname{in}_<(0) = 0$. Often, one writes $\operatorname{lt}(f)$ and $\operatorname{in}(I)$ when the order $<$ is understood.

Example 5.4.6. For example, the leading term of a monomial f is just f itself. It follows that the initial ideal of a monomial ideal I is just I; see Exercise 5.4.14. For the polynomial $XZ - Y^2$ in $R = A[X,Y,Z]$, we have

$$\operatorname{lt}_{<_{\text{lex}}}(XZ - Y^2) = \operatorname{lt}_{<_{\text{grlex}}}(XZ - Y^2) = XZ$$
$$\operatorname{lt}_{<_{\text{grevlex}}}(XZ - Y^2) = Y^2$$

because $Y^2 <_{\text{lex}} XZ$ and $XZ <_{\text{grevlex}} Y^2$. From this, one can show that

$$\operatorname{in}_{<_{\text{lex}}}((XZ - Y^2)R) = \operatorname{in}_{<_{\text{grlex}}}((XZ - Y^2)R) = (XZ)R$$
$$\operatorname{in}_{<_{\text{grevlex}}}((XZ - Y^2)R) = (Y^2)R.$$

More generally, see Exercise 5.4.15.

Do be careful, though, trying to use generators of an ideal to find generators of an initial ideal. If $I = (f_1,\ldots,f_m)R$, then one always has

$$\mathrm{in}_<(I) \supseteq (\mathrm{lt}_<(f_1), \ldots, \mathrm{lt}_<(f_m))R$$

but this containment may be strict; see Exercise 5.4.17. In fact, generating sets $f_1, \ldots, f_m$ of I such that $\mathrm{in}_<(I) = (\mathrm{lt}_<(f_1), \ldots, \mathrm{lt}_<(f_m))$ are very special and have many nice computational properties and applications; they are called Gröbner bases.

The next result, due to Francis Macaulay [52], is the one we have been building up to. It provides the connection between the Hilbert functions of homogeneous ideals and their initial ideals. (See also the discussion preceding Definition 5.4.5 as well as Exercises 5.4.24 and 5.4.25 below and [9, Section 4.2].)

Theorem 5.4.7 *Set $R = A[X_1, \ldots, X_d]$. Consider an ideal $I \subsetneq R$ generated by homogeneous polynomials, and fix a monomial order $<$. Then the quotient rings R/I and $R/\mathrm{in}_<(I)$ have the same Hilbert functions, Hilbert polynomials, Krull dimensions, and multiplicities.*

For example, one can apply this to the ideal $I = (XZ - Y^2)R$ where $R = A[X,Y,Z]$. As we have seen, one has $\mathrm{in}_{<_{\mathrm{lex}}}(I) = \mathrm{in}_{<_{\mathrm{grlex}}}(I) = (XZ)R$ as well as $\mathrm{in}_{<_{\mathrm{grevlex}}}(I) = (Y^2)R$. In some ways, the Hilbert function of these initial ideals is simpler to compute than that of I itself:

$$h_{R/I}(i) = h_{R/\mathrm{in}(I)}(i) = \begin{cases} 1 & i = 0 \\ 3 & i = 1 \\ 2i+1 & i \geqslant 2 \end{cases}$$

One point of Theorem 5.4.7 is the following. Certain invariants of ideals are hard to compute in general, but easy to compute for monomial ideals. For instance, Theorem 5.1.2 shows how to find the Krull dimension $\dim(R/I)$ when I is a monomial ideal. Similarly, there are straightforward algorithms for computing the multiplicity $e(R/I)$; see the exercises below. This technique is incredibly powerful; for example, Macaulay used it to characterize all possible Hilbert functions of quotients by homogeneous ideals.

We end this section with a construction for the Macaulay2 tutorial in Section 6.4.

Definition 5.4.8. Set $R = A[X_1, \ldots, X_d]$. Consider an ideal $I \subsetneq R$ generated by homogeneous polynomials. The *Hilbert series* of the quotient R/I is the formal power series with non-negative integer coefficients

$$H_{R/I}(t) = \sum_{i=0}^{\infty} h_{R/I}(i) t^i.$$

It is an amazing result of Hilbert [39] that the Hilbert series of R/I can be expressed as a rational function of the form

$$H_{R/I}(t) = P(t)/(1-t)^d$$

for some polynomial $P(t) \in \mathbb{Z}[t]$; see also, e.g., [54, Theorem 13.2]. One should interpret this equality as meaning that the Maclaurin series of the rational function on the right-hand side is exactly the series on the left-hand side.

Not only is this result surprising, but it is also powerful. For instance, it yields non-trivial recurrence relations for $h_{R/I}(i)$ which are in turn useful for understanding the structure of I and R/I.

Exercises

Exercise 5.4.9. Set $R = A[X_1,\dots,X_d]$, and let I be a monomial ideal of R.

(a) Prove that for $i = 0,1,2,\dots$, the following set is a basis for $(R/I)_i$:

$$\{\underline{X}^{\underline{n}} + I \mid \underline{X}^{\underline{n}} \text{ is a monomial of degree } i \text{ in } R \smallsetminus I\}.$$

Prove also that $h_{R/I}(i) = |\{\underline{X}^{\underline{n}} \mid \underline{X}^{\underline{n}} \text{ is a monomial of degree } i \text{ in } R \smallsetminus I\}|$.

(b) Let $d = 1$. Fix a monomial $\underline{X}^{\underline{m}} \in [\![R]\!]$ of degree e, and set $I = (\underline{X}^{\underline{m}})R$. Show that

$$h_{R/I}(i) = \begin{cases} 1 & \text{if } 0 \leqslant i < e \\ 0 & \text{if } i \geqslant e. \end{cases}$$

Conclude that R/I has dimension e as a vector space over A.

(c) Let $d = 2$. Fix a monomial $\underline{X}^{\underline{m}} \in [\![R]\!]$ of degree e, and set $I = (\underline{X}^{\underline{m}})R$. Show that

$$h_{R/I}(i) = \begin{cases} i+1 & \text{if } 0 \leqslant i < e \\ e & \text{if } i \geqslant e. \end{cases}$$

Conclude that R/I has multiplicity e.

(d) [Challenge] Extend parts (b) and (c) to the case $d \geqslant 3$. In particular, show that the degree of $p_{R/I}(t)$ is $\dim(R/I) = d-1$, as promised by Fact 5.3.8 and Theorem 5.4.2, and $e(R/I) = e$. (See also Laboratory Exercise 5.4.30.)

Exercise 5.4.10. Use Theorem 5.4.4 to compute the Hilbert function for R/J_Δ where Δ is the simplicial complex from Example 4.4.2. Justify your answer.

Exercise 5.4.11. Use Theorem 5.4.4 to prove the following.

(a) Given a simplicial complex Δ, the multiplicity $e(R/J_\Delta)$ is the number of facets of Δ that have maximal dimension; see Sections 4.4–4.5.

(b) Given a graph G, the multiplicity $e(R/I_G)$ is the number of minimal vertex covers of G of minimal size, i.e., the number of smallest vertex covers of G; see Sections 4.2–4.3, Lemma 4.4.8, and Theorem 4.5.7.

Can you prove these without using Theorem 5.4.4?

Exercise 5.4.12. Set $R = A[X_1,\dots,X_d]$ and let I be an ideal generated by the set of homogeneous polynomials $\{f_1,\dots,f_c\}$. Let I_i be the vector space of polynomials in I that are homogeneous of degree i. Prove that a spanning set of I_i is given by

$$\{mf_i \mid m \in [\![R]\!] \text{ and } i = 1,\dots,c \text{ such that } \deg m + \deg f_i = d\}.$$

Exercise 5.4.13. Consider the ideal $I = (X - Y, XY - Z^2)R$ in $R = A[X,Y,Z]$. Compute the matrices M_3 and M_4 (as in the discussion just before Definition 5.4.5) and determine their ranks in order to compute $h_{R/I}(i)$ for $i = 3,4$. Do the same for the ideal $I = (X^2 - XY, Y^2 - XY)$ for $i = 2,3,4$. Justify your answers.

Exercise 5.4.14. Let I be an ideal of $R = A[X_1,\dots,X_d]$. Prove that I is a monomial ideal if and only if it is equal to all (equivalently, at least one) of its initial ideals.

Exercise 5.4.15. Set $R = A[X_1,\dots,X_d]$, and fix a monomial order $<$ for R.

(a) Prove that for non-zero polynomials $f,g \in R$, we have $\operatorname{lt}_<(fg) = \operatorname{lt}_<(f)\operatorname{lt}_<(g)$.
(b) Let f be a non-zero polynomial in R, and prove that $\operatorname{in}_<((f)R) = (\operatorname{lt}_<(f))R$.

Exercise 5.4.16. Set $R = A[X_1,\dots,X_d]$, and let f be a non-constant homogeneous polynomial in R of degree e. Use Theorem 5.4.7 and Exercise 5.4.9 to find the Hilbert function, Hilbert polynomial, and multiplicity of $R/(f)R$.

Exercise 5.4.17. Set $R = A[X,Y]$ and $I = (X^3 - Y^3, X^3)R$.

(a) Prove that $(\operatorname{lt}_{<_{\text{lex}}}(X^3 - Y^3), \operatorname{lt}_{<_{\text{lex}}}(X^3))R = (X^3)R$.
(b) Prove that $Y^3 \in \operatorname{in}_{<_{\text{lex}}}(I)$.
(c) Conclude that $\operatorname{in}_{<_{\text{lex}}}((X^3 - Y^3, X^3)R) \supsetneq (\operatorname{lt}_{<_{\text{lex}}}(X^3 - Y^3), \operatorname{lt}_{<_{\text{lex}}}(X^3))R$.

Exercise 5.4.18. Set $R = A[X,Y,Z]$ and $J = (XZ - Y^3)R$. Prove that $\operatorname{in}_{<_{\text{lex}}}(J) = (XZ)R$ and $\operatorname{in}_{<_{\text{grlex}}}(J) = \operatorname{in}_{<_{\text{grevlex}}}(J) = (Y^3)R$. Conclude that the function $h_{R/\operatorname{in}(J)}(i)$ can depend on the choice of monomial order when J is not generated by homogeneous polynomials.

Challenge Exercise 5.4.19. Let $V = \{v_1,\dots,v_d\}$ and $R = A[X_1,\dots,X_d]$. Let Δ be a simplicial complex on V. Use Theorem 4.6.5 (or your answer to Challenge Exercise 4.6.13) with Theorem 5.4.4 to describe the multiplicity $e(R/K_\Delta)$ in terms of Δ. Justify your answer. (See also Laboratory Exercise 5.4.26.) Can you prove this without using Theorem 5.4.4?

Hilbert Functions and Initial Ideals in Macaulay2

Computations of Hilbert functions and initial ideals form the starting point for many of the applications for which Macaulay2 was written in the first place. Let us consider the quotient ring from Example 5.4.6.

```
i1 : R = QQ[x,y,z];
```

```
i2 : I = ideal {x*z - y^2}
               2
o2 = ideal(- y  + x*z)
o2 : Ideal of R

i3 : S = R/I
o3 = S
o3 : QuotientRing
```

One uses the command `hilbertPolynomial` to compute the Hilbert polynomial of a ring in Macaulay2.

```
i4 : p = hilbertPolynomial S
o4 = - P  + 2*P
        0      1
o4 : ProjectiveHilbertPolynomial
```

Some comments on this output are in order. It turns out that Hilbert polynomials are often written in terms of the binomial polynomials (in the variable i)

$$P_d = \binom{d+i}{i} = \frac{(i+1)(i+2)\cdots(i+d)}{d!}.$$

The primary reason for this is that the polynomials $\{P_0, P_1, \dots P_d\}$ form a basis of all polynomials of degree at most d (called the *binomial basis*), and the coefficients of the Hilbert polynomial when written in the binomial basis have geometric significance. For example, the coefficient of $\binom{\dim(R/I)+i-1}{\dim(R/I)}$ is the multiplicity $e(R/I)$. In addition, P_d is the Hilbert polynomial of the polynomial ring in $d+1$ variables [4].

The output `o4` above tells us that the Hilbert polynomial of R/I is

$$2\binom{i+1}{1} - \binom{i+0}{0} = 2(i+1) - 1 = 2i+1$$

as we saw in the discussion just after Theorem 5.4.7.

We can also use Macaulay2 to do this simplification for us. To this end, first note that the return value of `hilbertPolynomial` in output `o4` above is of type `ProjectiveHilbertPolynomial`. We view the ancestors of this type as follows.

```
i5 : ancestors ProjectiveHilbertPolynomial
o5 = {ProjectiveHilbertPolynomial, HashTable, Thing}
o5 : List
```

Therefore, the type `ProjectiveHilbertPolynomial` has an ancestor `HashTable` object, so it inherits hash table operations. (See Section 3.2 for more about hash tables.) For instance, to see the data making up the underlying hash table, we can use the `pairs` command as we did in Section 3.2.

---

[4] The reason for the subscript discrepancy, as well as the notation P_d, is because P_d is also the Hilbert polynomial of d-dimensional projective space.

```
i6 : pairs p
o6 = {(0, -1), (1, 2)}
o6 : List
```

We see that the keys (i.e. the first coordinates) are the subscripts from the decomposition in output `o4` above, and the values (the second coordinates) are the coefficients. Therefore, we can easily write down the Hilbert polynomial as a function of i by sending each key-value pair in the hash table to the constant times the Hilbert polynomial of the corresponding polynomial ring as follows.

```
i7 : A = QQ[i];

i8 : newp = sum(pairs p, (d,c) -> c * binomial(d+i,d))
o8 = 2i + 1
o8 : A
```

One can use Macaulay2 to compute the multiplicity $e(R/I)$ without computing the entire Hilbert polynomial, using the `degree` command.

```
i9 : use S
o9 = S
o9 : QuotientRing

i10 : degree I
o10 = 2
```

Computing the initial ideal of an ideal with respect to a monomial order is done in Macaulay2 via the command `leadTerm`. However, since the monomial order is part of the data that define the ring, it cannot be changed after the ring's creation. Consider the following ideal.

```
i11 : use R
o11 = R
o11 : PolynomialRing

i12 : I = ideal {x^5+y^4+z^3-1,x^3+y^3+z^2-1}
              5    4    3       3    3    2
o12 = ideal (x  + y  + z  - 1, x  + y  + z  - 1)
o12 : Ideal of R
```

Recall that if a monomial order is not specified, grevlex order is used. As we mentioned in Section B.9, the monomial order a ring in question is using can be obtained via the next command.

```
i13 : (options monoid R)#MonomialOrder
o13 = {MonomialSize => 32  }
      {GRevLex => {1, 1, 1}}
      {Position => Up      }
o13 : VerticalList
```

We compute the initial ideal with the following command.

```
i14 : ltI1 = leadTerm I
o14 = | x3 x2y3 y6 |
                1       3
o14 : Matrix R  <--- R
```

Now we can say more about why grevlex is the default order. It turns out that grevlex seems to do better than other monomial orders 'on average' in that the initial ideal has fewer generators, and those generators are of lower degree. Indeed, in this example, let us compute the lex initial ideal. First, we define a new ring R' with the lex monomial order.

```
i15 : R' = QQ[x,y,z,MonomialOrder=>Lex]
o15 = R'
o15 : PolynomialRing
```

Since I was defined as an ideal of R, we must first view I as an ideal of R' via the command `sub` (a synonym for `substitute`), and then compute its initial ideal.

```
i16 : ltI2 = leadTerm sub(I,R')
o16 = | y15 xz21 69984xyz2 xy4 3888x2z7 7776x2yz x2y3 x3 |
                 1        8
o16 : Matrix R'  <--- R'
```

Note that the lex initial ideal has many more generators of much higher degree than the initial ideal with respect to grevlex. This is not to say that lex initial ideals do not have their merits; they are quite useful for applications such as computations involving ring homomorphisms, elimination theory, and rational implicitization.

```
i17 : exit
```

Exercises

Exercise 5.4.20. Set $A = \mathbb{Z}_{101}$. Use the Macaulay2 commands `degree` and `dim` to compute $e(R/I)$ and $\dim(R/I)$ for some ideals I generated by homogeneous polynomials. Then use Macaulay2 as in the tutorial to show that the Hilbert polynomial $p_{R/I}$ has the form predicted by Theorem 5.4.2.

Exercise 5.4.21. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to verify the conclusions of Theorem 5.4.4 for some examples of your own choosing.

Exercise 5.4.22. Use Macaulay2 to verify the conclusion of Theorem 5.4.4 for some examples.

Exercise 5.4.23. Over the ring $R = \mathbb{Q}[X,Y,Z]$, use Macaulay2 to compute the initial ideal for $I = (XZ - Y^2)R$ with respect to the glex order.

**Exercise 5.4.24.* Consider the ring $R = \mathbb{Q}[X,Y,Z,U,V,W]$ and the ideal

$$I = (XV - YU, XW - ZU, YW - ZV)R$$

generated by the maximal minors of the matrix $\left(\begin{smallmatrix} X & Y & Z \\ U & V & W \end{smallmatrix}\right)$. Use Macaulay2 to perform the following tasks.

(a) Find the Hilbert polynomial $p_{R/I}(t)$.
(b) Find the Krull dimension, depth, and multiplicity of R/I. Determine whether R/I is Cohen-Macaulay.
(c) Find the initial ideals $\mathrm{in}(I)$ with respect to grevlex, lex, and glex. Find the Hilbert polynomial, Krull dimension, depth, and multiplicity of $R/\mathrm{in}(I)$. Determine whether $R/\mathrm{in}(I)$ is Cohen-Macaulay. Compare with what you found in parts (a) and (b); see Theorem 5.4.7.

(This is used in Laboratory Exercise 5.5.23.)

**Laboratory Exercise 5.4.25.* Repeat Exercise 5.4.24(b) for other matrixes. Use the data you gather to make guesses about the Krull dimension, depth, multiplicity, and Cohen-Macaulayness of R/I. Use Macaulay2 to check your guesses for some more examples. Revise your formula and re-experiment as necessary until you come to conjectures for these items. (This is used in Laboratory Exercise 5.5.23.)

Laboratory Exercise 5.4.26. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate $e(R/K_\Delta)$ for some specific complexes Δ, to help with Challenge Exercise 5.4.19.

Laboratory Exercise 5.4.27. Let I be a square-free monomial ideal in the ring $R = \mathbb{Q}[X_1, \ldots, X_d]$ with m-irreducible decomposition $I = \bigcap_{i=1}^n J_i$. Use Macaulay2 to compare the multiplicity $e(R/I)$ with number of ideals J_i such that $\dim(R/J_i) = \dim(R/I)$ for some of your favorite examples. Make a conjecture describing $e(R/I)$.

Laboratory Exercise 5.4.28. Set $R = \mathbb{Q}[X_1, \ldots, X_d]$ and consider an m-irreducible ideal $J = (X_1^{e_1}, \ldots, X_m^{e_m})R$. Use Macaulay2 to compare the multiplicity $e(R/J)$ with the product $e_1 \cdots e_m$ for some examples. Make a conjecture describing $e(R/J)$.

Laboratory Exercise 5.4.29. Set $R = \mathbb{Q}[X_1, \ldots, X_d]$, and let I be a monomial ideal in R with irredundant m-irreducible decomposition $I = \bigcap_{i=1}^n J_i$. Combine your answers to Laboratory Exercises 5.4.27–5.4.28 to make a guess for a formula describing $e(R/I)$. Use Macaulay2 to check your formula for some examples where I is not square-free and not m-irreducible. Revise your formula and re-experiment as necessary until you come to a conjecture describing $e(R/I)$.

Laboratory Exercise 5.4.30. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate $h_{R/I}$ where $I = (\underline{X}^{\underline{m}})R$ for some specific monomials $\underline{X}^{\underline{m}}$ with $d \geqslant 3$, to help with Challenge Exercise 5.4.9(d).

5.5 Resolutions of Monomial Ideals

In this section, A is a field.

Here, we explore some basic aspects of homological algebra due to Diana Taylor [75] as they relate to monomial ideals; see also [18, Exercise 17.11]. The starting point in this area is the following: a list of polynomials $f_1, \ldots, f_n \in R = A[X_1, \ldots, X_d]$ with $n \geqslant 2$ can be linearly independent over A, but it will not be linearly independent over R: we always have the commutativity relation $f_i f_j - f_j f_i = 0$. For an arbitrary list of polynomials, it is difficult to write down all such relations. However, for monomials, one can use combinatorial data to find all such relations with relative ease. To explain how this is accomplished, we start with a couple of concrete examples. Note that the case $d = 1$ is straightforward by Exercise 5.5.11, so we begin by discussing the case $d = 2$.

Example 5.5.1. Set $R = A[X,Y]$ and $I = (X^3, Y^2)R$. In this case, the commutativity relations noted above are $Y^2X^3 - X^3Y^2 = 0$. From this, we derive other relations, namely, for any $g \in R$ we have $gY^2X^3 - gX^3Y^2 = g(Y^2X^3 - X^3Y^2) = 0$. It is straightforward to show that these are the only relations of the form $fX^3 + hY^2 = 0$; see Exercise 5.5.12.

In some senses, the next example shows that, as one increases the number of polynomials and variables, things get more complicated. However, this example is still reasonably manageable.

Example 5.5.2. Set $R = A[X,Y,Z]$ and $I = (X^3, Y^5, Z^4)R$. There are three commutativity relations here: $Y^5X^3 - X^3Y^5 = 0 = Z^4X^3 - X^3Z^4 = Z^4Y^5 - Y^5Z^4$. As in the previous example, one can use these to generate other relations

$$h_1(Y^5X^3 - X^3Y^5) + h_2(Z^4X^3 - X^3Z^4) + h_3(Z^4Y^5 - Y^5Z^4) = 0. \tag{5.5.2.1}$$

We claim that in this case, these are the only relations, meaning that every relation

$$g_1X^3 + g_2Y^5 + g_3Z^4 = 0 \tag{5.5.2.2}$$

can be rewritten in the form (5.5.2.1).

To see how this is done, first consider the special case $g_1 = 0$. In this case, equation 5.5.2.2 has the form $g_2Y^5 + g_3Z^4 = 0$. In particular, the polynomial g_2Y^5 is divisible by Z^4. Since Y^5 and Z^4 have no variable factors in common, it follows that g_2 is divisible by Z^4, so there is a polynomial $h \in R$ such that $g_2 = hZ^4$. Similarly, there is a polynomial $k \in R$ such that $g_3 = kY^5$, and our relation becomes

$$0 = g_2Y^5 + g_3Z^4 = hZ^4Y^5 + kY^5Z^4 = (h+k)Y^5Z^4.$$

It follows that $h + k = 0$, that is $k = -h$, so $g_3 = -hY^5$ and our relation becomes

$$0 = g_2Y^5 + g_3Z^4 = hZ^4Y^5 - hY^5Z^4 = h(Z^4Y^5 - Y^5Z^4)$$

as desired.

Next, consider the special case where $g_1 = aX^pY^qZ^r$ for some non-zero element $a \in A$. In this case, equation (5.5.2.2) has the form

$$0 = aX^pY^qZ^rX^3 + g_2Y^5 + g_3Z^4 = aX^{p+3}Y^qZ^r + g_2Y^5 + g_3Z^4 \tag{5.5.2.3}$$

In particular, the monomial $X^{p+3}Y^qZ^r$ must occur in $g_2Y^5 + g_3Z^4$, so $q \geqslant 5$ or $r \geqslant 4$. We consider the case where we have both $q \geqslant 5$ and $r \geqslant 4$. The remaining cases ($q \geqslant 5$ and $r < 4$; or $q < 5$ and $r \geqslant 4$) are handled similarly; see Exercise 5.5.13(a). In this case, the monomial $X^{p+3}Y^qZ^r$ may occur in both g_2Y^5 and g_3Z^4. Write $g_2 = bX^{p+3}Y^{q-5}Z^r + k_2$ and $g_3 = cX^{p+3}Y^qZ^{r-4} + k_3$ such that $b, c \in A$, the monomial $X^{p+3}Y^{q-5}Z^r$ does not occur in k_2, and $X^{p+3}Y^qZ^{r-4}$ does not occur in k_3. Then 5.5.2.3 becomes

$$\begin{aligned}
0 &= aX^pY^qZ^rX^3 + g_2Y^5 + g_3Z^4\\
0 &= aX^pY^qZ^rX^3 + (bX^{p+3}Y^{q-5}Z^r + k_2)Y^5 + (cX^{p+3}Y^qZ^{r-4} + k_3)Z^4\\
0 &= (a+b+c)X^{p+3}Y^qZ^r + k_2Y^5 + k_3Z^4.
\end{aligned} \tag{5.5.2.4}$$

By the construction of k_2 and k_3, the monomial $X^{p+3}Y^qZ^r$ does not occur in the polynomial $k_2Y^5 + k_3Z^4$, so (5.5.2.4) implies that $(a+b+c)X^{p+3}Y^qZ^r = 0$, hence $a = -b-c$. We conclude that $g_1 = -bX^pY^qZ^r - cX^pY^qZ^r$, so (5.5.2.2) can be rewritten as

$$\begin{aligned}
0 &= (-bX^pY^qZ^r - cX^pY^qZ^r)X^3 + (bX^{p+3}Y^{q-5}Z^r + k_2)Y^5 + (cX^{p+3}Y^qZ^{r-4} + k_3)Z^4\\
&= -bX^pY^{q-5}Z^r(Y^5X^3 - X^3Y^5) - cX^pY^qZ^{r-4}(Z^4X^3 - X^3Z^4) + k_2Y^5 + k_3Z^4.
\end{aligned}$$

It follows that $k_2Y^5 + k_3Z^4 = 0$, so the previous case implies that there is a polynomial $h \in R$ such that $k_2 = hZ^4$ and $k_3 = -hY^5$. Thus, our relation takes the form

$$\begin{aligned}
0 &= -bX^pY^{q-5}Z^r(Y^5X^3 - X^3Y^5) - cX^pY^qZ^{r-4}(Z^4X^3 - X^3Z^4) + hZ^4Y^5 - hY^5Z^4\\
0 &= -bX^pY^{q-5}Z^r(Y^5X^3 - X^3Y^5) - cX^pY^qZ^{r-4}(Z^4X^3 - X^3Z^4) + h(Z^4Y^5 - Y^5Z^4)
\end{aligned}$$

as desired.

The general case follows by induction on the number of monomials occurring in g_1, using a similar argument as in the previous case. See Exercise 5.5.13(b).

The previous example is a bit dodgy since we just keep re-writing things that are zero in terms of other things that are zero, so it isn't entirely clear what we're really accomplishing. To clarify this, we introduce a bit of linear algebra notation.

Example 5.5.3. Consider the set R^3 of column vectors of size 3 over R.

$$R^3 = \left\{ \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} \middle| \, g_1, g_2, g_3 \in R \right\}$$

Each relation (5.5.2.2) determines a column vector $\left(\begin{smallmatrix} g_1 \\ g_2 \\ g_3 \end{smallmatrix}\right) \in R^3$ such that

$$\begin{pmatrix} X^3 & Y^5 & Z^4 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = g_1X^3 + g_2Y^5 + g_3Z^4 = 0. \tag{5.5.3.1}$$

(Here we use the usual multiplication of matrices.) Moreover, a column vector $\left(\begin{smallmatrix} g_1 \\ g_2 \\ g_3 \end{smallmatrix}\right) \in R^3$ satisfies equation (5.5.3.1) if and only if it determines a relation (5.5.2.2). For instance, the commutativity relation $Y^5X^3 - X^3Y^5 = 0$ determines the column vector $\left(\begin{smallmatrix} Y^5 \\ -X^3 \\ 0 \end{smallmatrix}\right)$, and similarly for the remaining commutativity relations. The analysis of Example 5.5.3 shows how to rewrite a given column vector $\left(\begin{smallmatrix} g_1 \\ g_2 \\ g_3 \end{smallmatrix}\right) \in R^3$ satisfying equation (5.5.3.1) in the form

$$\left(\begin{smallmatrix} g_1 \\ g_2 \\ g_3 \end{smallmatrix}\right) = h_1 \left(\begin{smallmatrix} Y^5 \\ -X^3 \\ 0 \end{smallmatrix}\right) + h_2 \left(\begin{smallmatrix} Z^4 \\ 0 \\ -X^3 \end{smallmatrix}\right) + h_3 \left(\begin{smallmatrix} 0 \\ Z^4 \\ -Y^5 \end{smallmatrix}\right).$$

For instance, in the special case where $g_1 = aX^pY^qZ^r$ with $q \geqslant 5$ and $r \geqslant 4$, our analysis yields

$$\left(\begin{smallmatrix} g_1 \\ g_2 \\ g_3 \end{smallmatrix}\right) = -bX^pY^{q-5}Z^r \left(\begin{smallmatrix} Y^5 \\ -X^3 \\ 0 \end{smallmatrix}\right) - cX^pY^qZ^{r-4} \left(\begin{smallmatrix} Z^4 \\ 0 \\ -X^3 \end{smallmatrix}\right) + h \left(\begin{smallmatrix} 0 \\ Z^4 \\ -Y^5 \end{smallmatrix}\right).$$

The point of the next example is that when the generators for an ideal have non-trivial common factors, things are not as simple as in the preceding examples.

Example 5.5.4. Set $R = A[X,Y]$ and $I = (X^2Y, XY^2)R$. In this case, the only commutativity relation is $(XY^2)(X^2Y) - (X^2Y)(XY^2) = 0$. In R^2, this says that the column vector $\left(\begin{smallmatrix} XY^2 \\ -X^2Y \end{smallmatrix}\right)$ satisfies $\left(\begin{smallmatrix} X^2Y & XY^2 \end{smallmatrix}\right)\left(\begin{smallmatrix} XY^2 \\ -X^2Y \end{smallmatrix}\right) = 0$. Unlike in the previous two examples, this is not the only way to generate relations. For instance, we have $Y(X^2Y) - X(XY^2) = 0$. One checks readily that every relation $f(X^2Y) + h(XY^2) = 0$ comes from this one; see Exercise 5.5.14.

One should take two points from the above examples.

First, one can easily find situations where monomials $f_1, \ldots, f_n$ have relations in addition to those coming from the commutativity relations $f_jf_i - f_if_j = 0$, specifically when f_i and f_j have a non-trivial common factor. The point of our next result is that one can actually get a handle on all the relations between these monomials. In essence, it says that every relation $\sum_{i=1}^n g_if_i = 0$ with $g_i \in R$ can be re-written as

$$\sum_{i<j} h_{i,j} \left(\frac{f_i}{\gcd(f_i, f_j)} f_j - \frac{f_j}{\gcd(f_i, f_j)} f_i \right) = 0 \tag{5.5.4.1}$$

for some polynomials $h_{i,j} \in R$. (See Exercise 2.1.14 for GCD's.)

Second, our discussion of these relations is made rigorous using ideas from linear algebra. Accordingly, before stating the next result, we specify some more notation.

Notation 5.5.5. Set $R = A[X_1,\ldots,X_d]$, and let $f_1,\ldots,f_n \in [\![R]\!]$. Let R^n denote the set of column vectors of size n over R.

$$R^n = \left\{ \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix} \middle| g_1,\ldots,g_n \in R \right\}$$

For $i = 1,\ldots,n$ let $\underline{e}_i \in R^n$ denote the ith standard basis vector

$$\underline{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where the 1 occurs in the ith position. Then each relation $\sum_{i=1}^n g_i f_i = 0$ is equivalent to a vector $\sum_{i=1}^n g_i \underline{e}_i$ in R^n such that $\sum_{i=1}^n f_i g_i = 0$, i.e., such that the matrix product $\begin{pmatrix} f_1 & \cdots & f_n \end{pmatrix} (\sum_{i=1}^n g_i \underline{e}_i)$ is 0.

With this notation, we can now rigorously describe all the relations between the monomial generators of a monomial ideal. (The proof of this result, however, is fairly involved, so we postpone it to the end of the section.) See Example 5.5.7 below for a concrete example demonstrating the conclusion of the theorem.

Theorem 5.5.6 *Set $R = A[X_1,\ldots,X_d]$, and let $I \subseteq R$ be a monomial ideal with monomial generating sequence $f_1,\ldots,f_n$ where $n \geqslant 2$. Then for all $g_i \in R$ with $\sum_{i=1}^n g_i f_i = 0$, the associated vector $\sum_{i=1}^n g_i \underline{e}_i \in R^n$ can be re-written as*

$$\sum_{i=1}^n g_i \underline{e}_i = \sum_{i<j} h_{i,j} \left(\frac{f_i}{\gcd(f_i,f_j)} \underline{e}_j - \frac{f_j}{\gcd(f_i,f_j)} \underline{e}_i \right)$$

for some polynomials $h_{i,j} \in R$.

The next example shows that Theorem 5.5.6 does not give a "minimal" set of relations between the f_i in general. See also Exercise 5.5.16.

Example 5.5.7. Set $R = A[X,Y]$ and $I = (Y^2, XY, X^2)R$. Our informal formulation (5.5.4.1) of Theorem 5.5.6 says that every relation $g_1 Y^2 + g_2 XY + g_3 X^2 = 0$ can be re-written in the form

$$\begin{aligned}
0 &= h_{1,2}\left(\frac{Y^2}{\mathrm{gcd}(Y^2,XY)}XY - \frac{XY}{\mathrm{gcd}(Y^2,XY)}Y^2\right)\\
&\quad + h_{1,3}\left(\frac{Y^2}{\mathrm{gcd}(Y^2,X^2)}X^2 - \frac{X^2}{\mathrm{gcd}(Y^2,X^2)}Y^2\right)\\
&\quad + h_{2,3}\left(\frac{XY}{\mathrm{gcd}(XY,X^2)}X^2 - \frac{X^2}{\mathrm{gcd}(XY,X^2)}XY\right)\\
&= h_{1,2}\left(\frac{Y^2}{Y}XY - \frac{XY}{Y}Y^2\right) + h_{1,3}\left(\frac{Y^2}{1}X^2 - \frac{X^2}{1}Y^2\right) + h_{2,3}\left(\frac{XY}{X}X^2 - \frac{X^2}{X}XY\right)\\
&= h_{1,2}(Y\cdot XY - X\cdot Y^2) + h_{1,3}(Y^2\cdot X^2 - X^2\cdot Y^2) + h_{2,3}(Y\cdot X^2 - X\cdot XY).
\end{aligned}$$

Formally, the result says that the corresponding vector $\left(\begin{smallmatrix} g_1\\ g_2\\ g_3\end{smallmatrix}\right) = \sum_{i=1}^3 g_i\underline{e}_i$ can be re-written in the form

$$\begin{aligned}
\sum_{i=1}^3 g_i\underline{e}_i &= h_{1,2}(Y\underline{e}_2 - X\underline{e}_1) + h_{1,3}(Y^2\underline{e}_3 - X^2\underline{e}_1) + h_{2,3}(Y\underline{e}_3 - X\underline{e}_2)\\
\begin{pmatrix} g_1\\ g_2\\ g_3\end{pmatrix} &= h_{1,2}\begin{pmatrix} -X\\ Y\\ 0\end{pmatrix} + h_{1,3}\begin{pmatrix} -X^2\\ 0\\ Y^2\end{pmatrix} + h_{2,3}\begin{pmatrix} 0\\ -X\\ Y\end{pmatrix}. \qquad (5.5.7.1)
\end{aligned}$$

In other words, the vectors $\left(\begin{smallmatrix} -X\\ Y\\ 0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -X^2\\ 0\\ Y^2\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0\\ -X\\ Y\end{smallmatrix}\right)$ span the set of relation vectors in this situation. In these terms the non-minimality mentioned before this example means that one of these vectors is in the span of the others over R, that is, that one of the vectors can be written as an R-linear combination of the others. Indeed:

$$\begin{pmatrix} -X^2\\ 0\\ Y^2\end{pmatrix} = X\begin{pmatrix} -X\\ Y\\ 0\end{pmatrix} + Y\begin{pmatrix} 0\\ -X\\ Y\end{pmatrix}$$

so equation (5.5.7.1) becomes

$$\begin{aligned}
\begin{pmatrix} g_1\\ g_2\\ g_3\end{pmatrix} &= h_{1,2}\begin{pmatrix} -X\\ Y\\ 0\end{pmatrix} + h_{1,3}\left(X\begin{pmatrix} -X\\ Y\\ 0\end{pmatrix} + Y\begin{pmatrix} 0\\ -X\\ Y\end{pmatrix}\right) + h_{2,3}\begin{pmatrix} 0\\ -X\\ Y\end{pmatrix}\\
&= (h_{1,2} + Xh_{1,3})\begin{pmatrix} -X\\ Y\\ 0\end{pmatrix} + (h_{2,3} + Yh_{1,3})\begin{pmatrix} 0\\ -X\\ Y\end{pmatrix}
\end{aligned}$$

Note also that each quotient $f_i/\mathrm{gcd}(f_i,f_j)$ in Theorem 5.5.6 can be re-written as $\mathrm{lcm}(f_i,f_j)/f_j$, by Exercise 2.1.14. The LCM-formulation of this expression is more convenient for the work that follows.

One of the insights of homological algebra is that there is value in understanding the relations between the relations on the f_i, and the relations between the relations between the relations, and so on. For arbitrary ideals, this is an extremely difficult problem. However, for monomial ideals the "Taylor resolution" solves this problem. Note that it uses the relations from Theorem 5.5.6 as a starting point, so it is not minimal in general. However, it gives very useful information about monomial ideals. Moreover, given a non-monomial ideal I one can use the Taylor resolution of the initial ideal $\text{in}(I)$ from Section 5.4 to get useful information about the original ideal I. As a motivation for the general construction, we first identify the relations between the relations from Theorem 5.5.6 for Example 5.5.3.

Example 5.5.8. Set $R = A[X,Y,Z]$, and consider the ideal $(X^3,Y^5,Z^4)R$. In R^3, we have the following relations from Theorem 5.5.6.

$$X^3\underline{e}_2 - Y^5\underline{e}_1 \qquad X^3\underline{e}_3 - Z^4\underline{e}_1 \qquad Y^5\underline{e}_3 - Z^4\underline{e}_2.$$

A relation between these relations is an equation of the form

$$g_{1,2}(X^3\underline{e}_2 - Y^5\underline{e}_1) + g_{1,3}(X^3\underline{e}_3 - Z^4\underline{e}_1) + g_{2,3}(Y^5\underline{e}_3 - Z^4\underline{e}_2) = 0. \tag{5.5.8.1}$$

For instance, we have

$$Z^4(X^3\underline{e}_2 - Y^5\underline{e}_1) - Y^5(X^3\underline{e}_3 - Z^4\underline{e}_1) + X^3(Y^5\underline{e}_3 - Z^4\underline{e}_2) = 0. \tag{5.5.8.2}$$

In fact, we can check that every relation of the form (5.5.8.1) can be re-written as a multiple of the one given in (5.5.8.2). Indeed, combining like terms in (5.5.8.1) yields the following.

$$0 = -(g_{1,2}Y^5 + g_{1,3}Z^4)\underline{e}_1 + (g_{1,2}X^3 - g_{2,3}Z^4)\underline{e}_2 + (g_{1,3}X^3 + g_{2,3}Y^5)\underline{e}_3.$$

It follows that we have

$$0 = g_{1,2}Y^5 + g_{1,3}Z^4 \tag{5.5.8.3}$$
$$0 = g_{1,2}X^3 - g_{2,3}Z^4 \tag{5.5.8.4}$$
$$0 = g_{1,3}X^3 + g_{2,3}Y^5. \tag{5.5.8.5}$$

From equation (5.5.8.3), arguing as in Example 5.5.2 we deduce that there is a polynomial $\tilde{g}_{1,2}$ such that $g_{1,2} = Z^4\tilde{g}_{1,2}$ and $g_{1,3} = -Y^5\tilde{g}_{1,2}$. Similarly, equation (5.5.8.4) implies that $g_{2,3} = X^3\tilde{g}_{1,2}$. It follows that equation (5.5.8.1) can be re-written in the following form.

$$\begin{aligned}0 &= Z^4\tilde{g}_{1,2}(X^3\underline{e}_2 - Y^5\underline{e}_1) - Y^5\tilde{g}_{1,2}(X^3\underline{e}_3 - Z^4\underline{e}_1) + X^3\tilde{g}_{1,2}(Y^5\underline{e}_3 - Z^4\underline{e}_2)\\ &= \tilde{g}_{1,2}[Z^4(X^3\underline{e}_2 - Y^5\underline{e}_1) - Y^5(X^3\underline{e}_3 - Z^4\underline{e}_1) + X^3(Y^5\underline{e}_3 - Z^4\underline{e}_2)]\end{aligned}$$

As claimed, this is a multiple of the relation given in (5.5.8.2).

As in the discussion preceding this example, we make this notion more formal with some more linear algebra. Consider a new copy of R^3. Each column vector

$$\underline{v} = \begin{pmatrix} g_{1,2} \\ g_{1,3} \\ g_{2,3} \end{pmatrix}$$

has the potential to determine a relation as in equation (5.5.8.1). For instance, the relation (5.5.8.2) determines the following column vector.

$$\underline{w} = \begin{pmatrix} Z^4 \\ -Y^5 \\ X^3 \end{pmatrix}$$

What we have shown implies that the vector $\underline{v}$ determines a relation as in equation (5.5.8.1) if and only if there is a polynomial h such that $\underline{v} = h\underline{w}$. See Exercise 5.5.17.

The next result gives a general version of the previous example. Informally, it says that, with $w_{i,j} = \frac{f_i}{\gcd(f_i,f_j)}\underline{e}_j - \frac{f_j}{\gcd(f_i,f_j)}\underline{e}_i$, each relation $\sum_{i<j} g_{i,j}w_{i,j} = 0$ where $g_{i,j} \in R$ can be re-written in the form

$$\sum_{i<j<k} h_{i,j,k}\left(\frac{\operatorname{lcm}(f_i,f_j,f_k)}{\operatorname{lcm}(f_j,f_k)}w_{j,k} - \frac{\operatorname{lcm}(f_i,f_j,f_k)}{\operatorname{lcm}(f_i,f_k)}w_{i,k} + \frac{\operatorname{lcm}(f_i,f_j,f_k)}{\operatorname{lcm}(f_i,f_j)}w_{i,j}\right) = 0$$

for some polynomials $h_{i,j,k} \in R$. (See Exercise 2.1.15 for LCMs of three or more monomials.)

Theorem 5.5.9 *Set $R = A[X_1,\ldots,X_d]$, and let $I \subseteq R$ be a monomial ideal with monomial generating sequence $f_1,\ldots,f_n$. Assume that $n \geqslant 2$. For all i,j with $1 \leqslant i < j \leqslant n$, consider the following vector in R^n.*

$$w_{i,j} = \frac{f_i}{\gcd(f_i,f_j)}\underline{e}_j - \frac{f_j}{\gcd(f_i,f_j)}\underline{e}_i = \frac{\operatorname{lcm}(f_i,f_j)}{f_j}\underline{e}_j - \frac{\operatorname{lcm}(f_i,f_j)}{f_i}\underline{e}_i.$$

Now, consider the set $R^{\binom{n}{2}}$ of column vectors of R of size $\binom{n}{2}$. Denote the standard basis vectors in $R^{\binom{n}{2}}$ as $\underline{e}_{1,2},\ldots,\underline{e}_{1,n},\underline{e}_{2,3},\ldots,\underline{e}_{2,n},\ldots,\underline{e}_{n-1,n}$. Then a vector

$$\underline{v} = \begin{pmatrix} g_{1,2} \\ \vdots \\ g_{n-1,n} \end{pmatrix} = \sum_{i<j} g_{i,j}\underline{e}_{i,j} \in R^{\binom{n}{2}}$$

yields a relation $\sum_{i<j} g_{i,j}w_{i,j} = 0$ in R^n if and only if it can be re-written in the form

$$\underline{v} = \sum_{i<j<k} h_{i,j,k}\left(\frac{\operatorname{lcm}(f_i,f_j,f_k)}{\operatorname{lcm}(f_j,f_k)}\underline{e}_{j,k} - \frac{\operatorname{lcm}(f_i,f_j,f_k)}{\operatorname{lcm}(f_i,f_k)}\underline{e}_{i,k} + \frac{\operatorname{lcm}(f_i,f_j,f_k)}{\operatorname{lcm}(f_i,f_j)}\underline{e}_{i,j}\right)$$

for some polynomials $h_{i,j,k} \in R$.

As one may imagine, one can continue along these lines. For instance, if we consider the vector

$$w_{i,j,k} = \frac{\operatorname{lcm}(f_i,f_j,f_k)}{\operatorname{lcm}(f_j,f_k)}\underline{e}_{j,k} - \frac{\operatorname{lcm}(f_i,f_j,f_k)}{\operatorname{lcm}(f_i,f_k)}\underline{e}_{i,k} + \frac{\operatorname{lcm}(f_i,f_j,f_k)}{\operatorname{lcm}(f_i,f_j)}\underline{e}_{i,j}$$

in $R^{\binom{n}{2}}$, we can ask what all the relations between the $w_{i,j,k}$ look like. It turns out that they look similar to the $w_{i,j,k}$ themselves, just like the relations between the w_i look similar to the w_i. This is formalized, as follows.

For $t = 1,\ldots,n$ consider the set $R^{\binom{n}{t}}$ of column vectors of R of size $\binom{n}{t}$. Denote the standard basis vectors in $R^{\binom{n}{t}}$ as $\underline{e}_{F_1},\ldots,\underline{e}_{F_{\binom{n}{t}}}$ where $F_1,\ldots,F_{\binom{n}{t}}$ are the distinct subsets of $\{1,\ldots,n\}$ of size t. (In other words, the F_i are the distinct $(t-1)$-dimensional faces of the $(n-1)$-simplex Δ_{n-1}.) For each $F_i = \{j_1,\ldots,j_t\}$ with $j_1 < \cdots < j_t$, consider the following vector in $R^{\binom{n}{t-1}}$.

$$w_{F_i} = \sum_{p=1}^{t}(-1)^{p-1}\frac{\mathrm{lcm}(f_{j_1},\ldots,f_{j_t})}{\mathrm{lcm}(f_{j_1},\ldots,f_{j_{p-1}},f_{j_{p+1}},\ldots,f_{j_t})}\underline{e}_{\{j_1,\ldots,j_{p-1},j_{p+1},\ldots,j_t\}}$$

Then we have the following generalization of Theorem 5.5.9.

Theorem 5.5.10 *With the above notation, a vector*

$$\begin{pmatrix} g_{F_1} \\ \vdots \\ g_{F_{\binom{n}{t}}} \end{pmatrix} = \sum_{i=1}^{\binom{n}{t}} g_{F_i}\underline{e}_{F_i} \in R^{\binom{n}{t}}$$

determines a relation $\sum_{i=1}^{\binom{n}{t}} g_{F_i}w_{F_i} = 0$ *in* $R^{\binom{n}{t-1}}$ *if and only if it can be re-written as*

$$\sum_{i=1}^{\binom{n}{t}} g_{F_i}\underline{e}_{F_i} = \sum_{q=1}^{\binom{n}{t+1}} h_{F_q}w_{F_q}$$

for some polynomials $h_{F_q} \in R$.

One translates this into the language of homological algebra as follows. Let $\partial_t \colon R^{\binom{n}{t}} \to R^{\binom{n}{t-1}}$ denote the function given by the rule

$$\partial_t\left(\sum_{i=1}^{\binom{n}{t}} g_{F_i}\underline{e}_{F_i}\right) = \sum_{i=1}^{\binom{n}{t}} g_{F_i}w_{F_i}.$$

In other words, using the standard correspondence between linear transformations and matrices (remembering that elements of $R^{\binom{n}{t}}$ are column vectors) ∂_t is represented by the matrix δ_t whose ith column is w_{F_i}. Each map ∂_t is *R-linear*, meaning that one has $\partial_t(rv+sw) = r\partial_t(v) + s\partial_t(w)$ for all $r,s \in R$ and all $v,w \in R^{\binom{n}{t}}$. (One also says that ∂_t is an *R-module homomorphism*.) The *image* of ∂_t is the set of all outputs of ∂_t

$$\mathrm{Im}(\partial_t) = \left\{\partial_t(v) \mid v \in R^{\binom{n}{t}}\right\}$$

and the *kernel* of ∂_t is

$$\operatorname{Ker}(\partial_t) = \left\{ v \in R^{\binom{n}{t}} \mid \partial_t(v) = 0 \right\}.$$

In linear algebra terms, these correspond to the column space and the null space of the matrix δ_t, respectively. With this notation, Theorem 5.5.10 says that for $t = 1, \dots, n$ we have $\operatorname{Im}(\partial_{t+1}) = \operatorname{Ker}(\partial_t)$. When $t = n$, this means that $\operatorname{Ker}(\partial_n) = 0$.

In the language of homological algebra, we say that the following sequence

$$0 \to R^{\binom{n}{n}} \xrightarrow{\partial_n} R^{\binom{n}{n-1}} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} R^{\binom{n}{1}} \xrightarrow{\partial_1} R^{\binom{n}{0}} \tag{5.5.10.1}$$

is *exact*. Frequently, one focuses on the sequence

$$0 \to R^{\binom{n}{n}} \xrightarrow{\partial_n} R^{\binom{n}{n-1}} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} R^{\binom{n}{1}} \xrightarrow{\partial_1} R^{\binom{n}{0}} \xrightarrow{\tau} R/I \to 0$$

where τ is the function $\tau(r) = r + I$. Such a sequence is called a *free resolution* of R/I. This notion is the starting point of homological algebra, in many ways. For instance, one uses free resolutions to construct the cohomology module $\operatorname{Ext}^i_R(R/I, A)$. This is a finite-dimensional vector space over A whose dimension is the ith *Betti number* of R/I, an important invariant used to study I. On the other hand, one can use $\operatorname{Ext}^i_R(A, R/I)$ to decide when R/I is Cohen-Macaulay. These ideas have far-reaching applications to many areas of mathematics and science.

Proof (Proof of Theorem 5.5.6). Write $f_i = \underline{X}^{\underline{c}_i}$ for $i = 1, \dots, n$.

Let $g_1, \dots, g_n \in R$ be such that $\sum_{i=1}^n g_i f_i = 0$. Note that if $g_i = 0$ for all i, then we can re-write the given relation in the desired form using $h_{i,j} = 0$ for all i, j. Thus, we assume that at least one of the g_i's is non-zero. Note that it follows that at least two of the g_i's are non-zero. Indeed, if g_k is the only non-zero element in the list of g_i's, then we have $0 = \sum_i g_i f_i = g_k f_k$. Since A is a field, the fact that g_k and f_k are non-zero implies that their product $g_k f_k$ is also non-zero, contradiction.

Case 1: For $i = 1, \dots, n$ there are monomials $\underline{X}^{\underline{m}_i} \in [\![R]\!]$ and coefficients $a_i \in A$ such that $g_i = a_i \underline{X}^{\underline{m}_i}$. We argue by induction on the number p of non-zero g_i's.

Base case: $p = 2$. To simplify matters, re-order the f_i's if necessary to assume that $g_i \neq 0$ for $i = 1, 2$ and $g_i = 0$ for $i > 2$. For $l = 1, \dots, d$ set $q_l = \min(c_{1,l}, c_{2,l})$. Then we have $\gcd(f_1, f_2) = \underline{X}^{\underline{q}}$. In this case, we have

$$0 = g_1 f_1 + g_2 f_2 = a_1 \underline{X}^{\underline{m}_1} f_1 + a_2 \underline{X}^{\underline{m}_2} f_2.$$

Since $a_1 \neq 0 \neq a_2$, the linear independence of the monomials of R implies that $a_2 = -a_1$ and $\underline{X}^{\underline{m}_1} f_1 = \underline{X}^{\underline{m}_2} f_2$. Thus, we have

$$\underline{X}^{\underline{m}_1 + \underline{c}_1} = \underline{X}^{\underline{m}_1} \underline{X}^{\underline{c}_1} = \underline{X}^{\underline{m}_1} f_1 = \underline{X}^{\underline{m}_2} f_2 = \underline{X}^{\underline{m}_2} \underline{X}^{\underline{c}_2} = \underline{X}^{\underline{m}_2 + \underline{c}_2}.$$

We conclude that in $\mathbb{N}^d$ one has

$$\underline{m}_1 + \underline{c}_1 = \underline{m}_2 + \underline{c}_2. \tag{5.5.10.2}$$

We claim that $\underline{m}_1 \succcurlyeq \underline{c}_2 - \underline{q}$. To see this, use the definition $q_l = \min(c_{1,l}, c_{2,l})$ to analyze two cases. If $q_l = c_{1,l} \leqslant c_{2,l}$, then equation (5.5.10.2) implies that $m_{1,l} = m_{2,l} + c_{2,l} - c_{1,l} = m_{2,l} + c_{2,l} - q_l \geqslant c_{2,l} - q_l$. On the other hand, if $q_l = c_{2,l} \leqslant c_{1,l}$, then $m_{1,l} \geqslant 0 = c_{2,l} - q_l$. This establishes the claim.

It follows that the expression $\underline{X}^{\underline{m}_1 - (\underline{c}_2 - \underline{q})} = \underline{X}^{\underline{m}_1} / \underline{X}^{\underline{c}_2 - \underline{q}}$ describes a valid monomial in R. Similarly, we have $\underline{m}_2 \succcurlyeq \underline{c}_1 - \underline{q}$, so the expression $\underline{X}^{\underline{m}_2 - (\underline{c}_1 - \underline{q})} = \underline{X}^{\underline{m}_2} / \underline{X}^{\underline{c}_1 - \underline{q}}$ describes a valid monomial in R. Moreover, equation (5.5.10.2) implies that $\underline{m}_2 + \underline{c}_2 - \underline{q} = \underline{m}_1 + \underline{c}_1 - \underline{q}$, so we have $\underline{m}_1 - (\underline{c}_2 - \underline{q}) = \underline{m}_2 - (\underline{c}_1 - \underline{q})$ and therefore $\underline{X}^{\underline{m}_1 - (\underline{c}_2 - \underline{q})} = \underline{X}^{\underline{m}_2 - (\underline{c}_1 - \underline{q})}$ in R. Set $h_{1,2} = a_2 \underline{X}^{\underline{m}_2 - (\underline{c}_1 - \underline{q})}$. The relation $0 = g_1 f_1 + g_2 f_2$ determines the vector $g_1 \underline{e}_1 + g_2 \underline{e}_2 \in R^n$, which we re-write as

$$\begin{aligned}
a_1 \underline{X}^{\underline{m}_1} \underline{e}_1 + a_2 \underline{X}^{\underline{m}_2} \underline{e}_2 &= -a_2 \underline{X}^{\underline{m}_1} \underline{e}_1 + a_2 \underline{X}^{\underline{m}_2} \underline{e}_2 \\
&= a_2(-\underline{X}^{\underline{m}_1} \underline{e}_1 + \underline{X}^{\underline{m}_2} \underline{e}_2) \\
&= a_2 \left(-\underline{X}^{\underline{m}_1 - (\underline{c}_2 - \underline{q})} \underline{X}^{\underline{c}_2 - \underline{q}} \underline{e}_1 + \underline{X}^{\underline{m}_2 - (\underline{c}_1 - \underline{q})} \underline{X}^{\underline{c}_1 - \underline{q}} \underline{e}_2 \right) \\
&= a_2 \left(-\underline{X}^{\underline{m}_2 - (\underline{c}_1 - \underline{q})} \underline{X}^{\underline{c}_2 - \underline{q}} \underline{e}_1 + \underline{X}^{\underline{m}_2 - (\underline{c}_1 - \underline{q})} \underline{X}^{\underline{c}_1 - \underline{q}} \underline{e}_2 \right) \\
&= a_2 \underline{X}^{\underline{m}_2 - (\underline{c}_1 - \underline{q})} \left(-\underline{X}^{\underline{c}_2 - \underline{q}} \underline{e}_1 + \underline{X}^{\underline{c}_1 - \underline{q}} \underline{e}_2 \right) \\
&= h_{1,2} \left(-\frac{\underline{X}^{\underline{c}_2}}{\underline{X}^{\underline{q}}} \underline{e}_1 + \frac{\underline{X}^{\underline{c}_1}}{\underline{X}^{\underline{q}}} \underline{e}_2 \right) \\
&= h_{1,2} \left(-\frac{f_2}{\gcd(f_1, f_2)} \underline{e}_1 + \frac{f_1}{\gcd(f_1, f_2)} \underline{e}_2 \right).
\end{aligned}$$

Thus, the given relation can be re-written in the desired form.

Induction step. Assume that $p > 2$ and that every relation $\sum_{i=1}^n \tilde{g}_i f_i = 0$ with $\tilde{g}_i = \tilde{a}_i \underline{X}^{\underline{m}_i}$ such that at most $p-1$ of the $\tilde{g}_i$'s are non-zero can be re-written in the desired form. Consider the relation $\sum_{i=1}^n g_i f_i = 0$ with $g_i = a_i \underline{X}^{\underline{m}_i}$ such that p of the g_i's are non-zero. To simplify matters, re-order the f_i's if necessary to assume that $g_i \neq 0$ for $i = 1, \ldots, p$ and $g_i = 0$ for $i > p$. Thus, we have

$$0 = \sum_{i=1}^p g_i f_i = \sum_{i=1}^p a_i \underline{X}^{\underline{m}_i} \underline{X}^{\underline{c}_i} = \sum_{i=1}^p a_i \underline{X}^{\underline{m}_i + \underline{c}_i}.$$

Since $a_p \neq 0$, linear independence of the monomials implies that the monomial $\underline{X}^{\underline{m}_p + \underline{c}_p}$ must occur in the shorter sum $\sum_{i=1}^{p-1} a_i \underline{X}^{\underline{m}_i + \underline{c}_i}$. Re-order the f_i's if necessary to assume that $\underline{X}^{\underline{m}_p + \underline{c}_p} = \underline{X}^{\underline{m}_{p-1} + \underline{c}_{p-1}}$. Arguing as in the base case, there is a monomial $\underline{X}^{\underline{b}}$ such that

$$\underline{X}^{\underline{b}} \frac{f_p}{\gcd(f_{p-1}, f_p)} f_{p-1} = \underline{X}^{\underline{m}_p + \underline{c}_p} = \underline{X}^{\underline{m}_{p-1} + \underline{c}_{p-1}} = \underline{X}^{\underline{b}} \frac{f_{p-1}}{\gcd(f_{p-1}, f_p)} f_p.$$

Set $h_{p-1,p} = a_p \underline{X}^{\underline{b}}$, and note that $a_p \underline{X}^{\underline{b}} \frac{f_{p-1}}{\gcd(f_{p-1}, f_p)} = a_p \underline{X}^{\underline{m}_p} = g_p$. Then set $\tilde{g}_{p-1} = (a_{p-1} + a_p) \underline{X}^{\underline{m}_{p-1}}$, and consider the relation

$$
\begin{aligned}
0 &= \left(\sum_{i=1}^{p} g_i f_i\right) - h_{p-1,p}\left(\frac{f_{p-1}}{\gcd(f_{p-1},f_p)} f_p - \frac{f_p}{\gcd(f_{p-1},f_p)} f_{p-1}\right)\\
&= \left(\sum_{i=1}^{p} g_i f_i\right) - a_p \underline{X}^{\underline{b}}\left(\frac{f_{p-1}}{\gcd(f_{p-1},f_p)} f_p - \frac{f_p}{\gcd(f_{p-1},f_p)} f_{p-1}\right)\\
&= \left(\sum_{i=1}^{p} g_i f_i\right) - a_p \underline{X}^{\underline{b}}\frac{f_{p-1}}{\gcd(f_{p-1},f_p)} f_p + a_p \underline{X}^{\underline{b}}\frac{f_p}{\gcd(f_{p-1},f_p)} f_{p-1}\\
&= \left(\sum_{i=1}^{p} g_i f_i\right) - g_p f_p + a_p \underline{X}^{\underline{m}_{p-1}} f_{p-1}\\
&= \left(\sum_{i=1}^{p-2} g_i f_i\right) + \tilde{g}_{p-1} f_{p-1}.
\end{aligned}
$$

The final expression in this display is one where our induction hypothesis applies. Thus, the vector $\left(\sum_{i=1}^{p} g_i \underline{e}_i\right) - h_{p-1,p}\left(\frac{f_{p-1}}{\gcd(f_{p-1},f_p)}\underline{e}_p - \frac{f_p}{\gcd(f_{p-1},f_p)}\underline{e}_{p-1}\right)$ can be re-written in the form $\sum_{i<j} \tilde{h}_{i,j}\left(\frac{f_i}{\gcd(f_i,f_j)}\underline{e}_j - \frac{f_j}{\gcd(f_i,f_j)}\underline{e}_i\right) = 0$. By adding the vector $h_{p-1,p}\left(\frac{f_{p-1}}{\gcd(f_{p-1},f_p)}\underline{e}_p - \frac{f_p}{\gcd(f_{p-1},f_p)}\underline{e}_{p-1}\right)$, we conclude that the original vector $\sum_{i=1}^{p} g_i \underline{e}_i$ can be re-written in the desired form. This concludes the proof in Case 1.

Case 2: The general case. Write each g_i as a linear combination of monomials. Then rewrite the relation $\sum_{i=1}^{n} g_i f_i = 0$ in terms of the monomials occurring in the g_i's. Use the linear independence of the monomials in R to obtain relations $\sum_{i=1}^{n} g_{i,\underline{m}} f_i = 0$ where $\underline{m} \in \mathbb{N}^d$ and each $g_{i,\underline{m}}$ is of the form $a_{i,\underline{m}} \underline{X}^{\underline{p}_i}$ for some $\underline{p}_i \in \mathbb{N}^d$ such that $g_{i,\underline{m}} f_i = a_i \underline{X}^{\underline{m}}$. Apply Case 1 to each of the vectors $\sum_{i=1}^{n} g_{i,\underline{m}} \underline{e}_i$, and add the results to re-write the general vector $\sum_{i=1}^{n} g_i \underline{e}_i$ in the desired form. $\square$

Exercises

Exercise 5.5.11.

(a) Set $R = A[X_1, \ldots, X_n]$, and let $f_1, \ldots, f_n \in [\![R]\!]$ be such that $f_1 \mid f_i$ for $i = 2, \ldots, n$, say $f_i = f_1 h_i$. Without using Theorem 5.5.6, prove that a vector $\sum_{i=1}^{n} g_i \underline{e}_i \in R^n$ yields a relation $\sum_{i=1}^{n} g_i f_i = 0$ if and only if it can be re-written in the form $\sum_{i=2}^{n} g_i(\underline{e}_i - h_i \underline{e}_1)$.

(b) Set $R = A[X]$, and let $f_1, \ldots, f_n \in [\![R]\!]$. Without using Theorem 5.5.6, prove that the f_i can be re-ordered so that a vector $\sum_{i=1}^{n} g_i \underline{e}_i \in R^n$ determines a relation $\sum_{i=1}^{n} g_i f_i = 0$ if and only if it can be re-written in the form $\sum_{i=2}^{n} g_i(\underline{e}_i - h_i \underline{e}_1)$.

Exercise 5.5.12. Consider the set-up of Example 5.5.1. Without using Theorem 5.5.6 prove that a vector $\binom{f}{h} \in R^2$ determines a relation $fX^3 + hY^2 = 0$ if and only if it has the form $g\begin{pmatrix} Y^2 \\ -X^3 \end{pmatrix}$ for some $g \in R$.

Exercise 5.5.13. Finish Example 5.5.2 by performing the following tasks, without using Theorem 5.5.6.

(a) In the special case where $g_1 = aX^pY^qZ^r$, complete the argument in the remaining sub-cases ($q \geqslant 5$ and $r < 4$; or $q < 5$ and $r \geqslant 4$).
(b) Prove the general case by induction on the number of non-zero monomials occurring in g_1.

Exercise 5.5.14. Consider the set-up of Example 5.5.4. Without using Theorem 5.5.6 prove that a column vector $\left(\begin{smallmatrix} f \\ h \end{smallmatrix}\right) \in R^2$ satisfies $\left(\begin{smallmatrix} X^2Y & XY^2 \end{smallmatrix}\right)\left(\begin{smallmatrix} f \\ h \end{smallmatrix}\right) = 0$ if and only if there is a polynomial $g \in R$ such that $\left(\begin{smallmatrix} f \\ h \end{smallmatrix}\right) = g\left(\begin{smallmatrix} Y \\ -X \end{smallmatrix}\right)$.

Exercise 5.5.15. Use Theorem 5.5.6 to give alternate justifications of the conclusions of Examples 5.5.1–5.5.4.

Exercise 5.5.16. Set $R = A[X,Y]$, and let $I \subseteq R$ be a monomial ideal with irredundant monomial generating sequence $f_1, \ldots, f_n$. Assume that $n \geqslant 2$ and that $f_1 <_{\text{lex}} \cdots <_{\text{lex}} f_n$. Then we have $f_i = X^{a_i}Y^{b_i}$ such that $a_1 < \cdots < a_n$ and $b_1 > \cdots > b_n$.

(a) Let $h_i, h_{i+1} \in [\![R]\!]$. Without using Theorem 5.5.6, prove that the vector

$$h_i\underline{e}_i + h_{i+1}\underline{e}_{i+1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ h_i \\ h_{i+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

determines a relation $h_if_i - h_{i+1}f_{i+1} = 0$ if and only if it can be expressed as a monomial multiple of the following vector.

$$v_i = X^{a_{i+1}-a_i}\underline{e}_i - Y^{b_i-i_{p+1}}\underline{e}_{i+1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ X^{a_{i+1}-a_i} \\ -Y^{b_i-b_{i+1}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{5.5.16.1}$$

(b) Let $h_i, h_j \in [\![R]\!]$ and $c_i, c_j \in A$ be such that $i < j$ and $c_ih_if_i + c_jh_jf_j = 0$. Without using Theorem 5.5.6, prove that $c_j = -c_i$ and that there exist $g_i, \ldots, g_{j-1} \in [\![R]\!]$ such that $h_i = g_iX^{a_{i+1}-a_i}$ and $h_j = g_{j-1}Y^{b_{j-1}-b_j}$ and such that the vector $c_ih_i\underline{e}_i + c_jh_j\underline{e}_j$ can be re-written in the form $\sum_{p=i}^{j-1} c_pg_p\underline{v}_p$ where $v_i, \ldots, v_{j-1}$ are from (5.5.16.1). (Hint: Argue by induction on $j - i$.)
(c) Let $h_1, \ldots, h_n \in R$ be such that $\sum_{i=1}^n h_if_i = 0$. Without using Theorem 5.5.6, prove that the vector $\sum_{i=1}^n h_i\underline{e}_i \in R^n$ can be re-written in the form $\sum_{i=1}^{n-1} g_i\underline{v}_i$ for suitable $g_1, \ldots, g_{n-1} \in R$, where the v_i's are from (5.5.16.1). (Hint: Rewrite each h_i as a linear combination of monomials, and collect like terms.)

(d) Prove that the v_i's are minimal in that one cannot write the vector v_i as a linear combination over R of the remaining v_j's. (Hint: Suppose that $v_i = \sum_{j \neq i} g_j v_j$ for some polynomials $g_j \in R$. Rewrite this in the form $0 = \sum_{i=1}^{n-1} g_i v_i$ where $g_i = -1$. Write this out in terms of (5.5.16.1), collect all the terms into a single vector expression, show that $g_j = 0$ for $j = 1, \ldots, n-1$, and derive a contradiction.

Exercise 5.5.17. In the notation of Example 5.5.8, prove that the vector $\underline{v}$ determines a relation as in equation (5.5.8.1) if and only if there is a polynomial h such that $\underline{v} = h\underline{w}$, without using Theorem 5.5.9 or 5.5.10.

Exercise 5.5.18. Use Theorem 5.5.9 to give an alternate justification of the conclusion of Example 5.5.8.

Exercise 5.5.19. Set $R = A[X,Y]$ and $I = (X^5, X^4Y^2, X^2Y^3)R$.

(a) Write out the minimal relations between the generators of I, as in Exercise 5.5.16.
(b) Write out the relations between the generators of I, as in Theorem 5.5.6, and the relations between the relations (etc.) as in Theorems 5.5.9 and 5.5.10.
(c) Repeat part (b) for the ideal $J = (X^3, Y^5, Z^4) \subseteq A[X,Y,Z]$.

Resolutions of Monomial Ideals in Macaulay2

One can compute the Taylor resolution of a monomial ideal using the package `ChainComplexExtras`, which has had contributions at various times by David Eisenbud, W. Frank Moore, Frank-Olaf Schreyer and Gregory Smith. Consider the following situation.

```
i1 : needsPackage "ChainComplexExtras"
o1 = ChainComplexExtras
o1 : Package

i2 : R = QQ[x_1..x_4];

i3 : I = monomialIdeal {x_1*x_2,x_1*x_3,x_1*x_4,x_2*x_3*x_4}
o3 = monomialIdeal (x x , x x , x x , x x x )
                     1 2   1 3   1 4   2 3 4
o3 : MonomialIdeal of R

i4 : T = taylorResolution I
      1      4      6      4      1
o4 = R  <-- R  <-- R  <-- R  <-- R


     0      1      2      3      4
o4 : ChainComplex
```

This output shows that there are four interesting matrices in the Taylor resolution, as in (5.5.10.1) above. Note that Macaulay2 presents the resolution from right to left, in contrast with the usual convention of (5.5.10.1). This is primarily so that a $m \times n$ matrix determines a homomorphism $R^m \longleftarrow R^n$, so that the order in which m and n is written is not reversed under this convention.

One obtains the list of matrices in the resolution using the next command.

```
i5 : T.dd
          1                                           4
o5 = 0 : R  <-------------------------------------- R  : 1
               | x_1x_2 x_1x_3 x_1x_4 x_2x_3x_4 |
          4                                                       6
     1 : R  <-------------------------------------------------- R  : 2
               {2} | -x_3 -x_4 0    -x_3x_4 0       0       |
               {2} | x_2  0    -x_4 0       -x_2x_4 0       |
               {2} | 0    x_2  x_3  0       0       -x_2x_3 |
               {3} | 0    0    0    x_1     x_1     x_1     |
          6                                 4
     2 : R  <---------------------------- R  : 3
               {3} | x_4  x_4 0   0   |
               {3} | -x_3 0   x_3 0   |
               {3} | x_2  0   0   x_2 |
               {4} | 0    -1  -1  0   |
               {4} | 0    1   0   -1  |
               {4} | 0    0   1   1   |
          4                   1
     3 : R  <-------------- R  : 4
               {4} | -1 |
               {4} | 1  |
               {4} | -1 |
               {4} | 1  |
o5 : ChainComplexMap
```

If one only wants a single matrix, say the second one, one can obtain it as follows.

```
i6 : T.dd_2
o6 = {2} | -x_3 -x_4 0    -x_3x_4 0       0       |
     {2} | x_2  0    -x_4 0       -x_2x_4 0       |
     {2} | 0    x_2  x_3  0       0       -x_2x_3 |
     {3} | 0    0    0    x_1     x_1     x_1     |
             4       6
o6 : Matrix R  <--- R
```

Note that this is not the smallest resolution of R/I. In fact, one knows this due to the presence of constants in the third map in line `o5` above. One can obtain the smallest sequence (called the *minimal free resolution* of R/I) via the next command.

```
i7 : res coker gens I
```

```
       1      4      4      1
o7 = R  <-- R  <-- R  <-- R  <-- 0

     0      1      2      3      4
o7 : ChainComplex
```

Note here that the command `gens I` creates a row matrix with entries equal to the generators of I. Considering this matrix as a map into R, we see that the image of this map is I. The command `coker` computes the *cokernel* of this map, that is, the quotient by the image of the map; in this case, the quotient is R/I. The point of writing it this way is that it allows us to compute the resolution over R; if we used the command R/I instead, Macaulay2 would switch to the ring R/I, which is not what we want.

It is a general fact that the minimal free resolution of R/I is unique up to a collection of changes of bases that commute with the homomorphisms in the resolution. The sizes of the matrices that appear in the minimal resolution are important invariants of the ideal I; in fact these are the same as the Betti numbers alluded to above, and comprise an important area of active research.

```
i8 : exit
```

Exercises

Exercise 5.5.20. Use Macaulay2 to generate the Taylor resolutions and the minimal free resolutions of R/I for the ideals from the numbered examples from this section and Exercise 5.5.19. Do your computations match with the claims in the example and your answers to the exercise? Do the Taylor and minimal resolutions have the same shape?

Laboratory Exercise 5.5.21. Set $R = \mathbb{Z}_{101}[X,Y,Z]$.

(a) Use Macaulay2 to generate the Taylor resolutions and the minimal free resolutions of R/I for some ideals of the form $(X^a,Y^b,Z^c)R$. Do the Taylor and minimal resolutions have the same shape and the same matrices? Make a conjecture about the shapes and matrices of these resolutions in general.
(b) Repeat part (a) for more variables.

Laboratory Exercise 5.5.22. Set $R = \mathbb{Q}[X,Y]$ and let I be a monomial ideal of R. The *projective dimension* of R/I over R is the length of the minimal free resolution of R/I; this number is denoted $\operatorname{pdim}_R(R/I)$. The Macaulay2 command for projective dimension is `pdim`.

(a) Use this command to compare the projective dimension $\operatorname{pdim}_R(R/I)$ and the number of variables in the ring R for some examples.
(b) Repeat part (a) for ideals in more variables. The inequality you should find is known as Hilbert's Syzygy Theorem [39]. (Note that this includes the fact that $\operatorname{pdim}_R(R/I)$ is finite, which fails generally for other rings.)

(c) Compare the projective dimension, depth (see Section 5.3), and number of variables for your examples from parts (a)–(b). Make a conjecture relating these numbers. The equality you should find is known as the Auslander-Buchsbaum formula.

Laboratory Exercise 5.5.23. Use Macaulay2 to compare the numbers $\mathrm{pdim}_R(R/I)$ and $\mathrm{pdim}_R(R/\mathrm{in}(I))$ for your examples from Section 5.4. Be sure to use different monomial orders for some computations. Make a conjecture relating these numbers. How does this fit with your conjectures from Exercises 5.4.24, 5.4.25, and 5.5.22?

Laboratory Exercise 5.5.24. Set $R = \mathbb{Q}[X_1,\ldots,X_d]$, let I be a square-free monomial ideal of R, and let $I^\vee$ be its Alexander dual; see Section 4.7. Use Macaulay2 to compute minimal free resolutions of $R/I^\vee$ when R/I is Cohen-Macaulay. Do the entries in the matrices in these resolutions have any special properties? Compare these matrices with ones you get from resolutions of $R/I^\vee$ when R/I is not Cohen-Macaulay. Make a conjecture about the entries in these matrices.

Concluding Notes

The Krull dimension of Section 5.1 was introduced by and named after Wolfgang Krull. It is absolutely fundamental for much of how we understand ideals and rings.

Our treatment of the PMU Placement Problem in Section 5.2 is motivated largely by the article of Dennis Brueni and Lenwood Heath [8]. Other references include the articles of Thomas Baldwin, et. al. [5], Teresa Haynes, et. al. [34], Rajesh Kavasseri and Sudarshan Srinivasan [47], and Arun Phadke [67].

The use of monomial ideals to study simplicial complexes was pioneered by Melvin Hochster [42] and Gerald Reisner [69]. Stanley's proof of the Upper Bound Theorem is one of the most important applications of this idea. Accordingly, these ideals are often called "Stanley-Reisner ideals" in the literature. More information about this can be found, e.g., in the texts of Bruns and Herzog [9] and Stanley [74], or Stanley's original article [73].

As its title suggests, the text [9] contains a wealth of information about Cohen-Macaulayness, as do the texts of David Eisenbud [18] and Hideyuki Matsumura [54], in addition to many of the texts we have mentioned above, e.g., [37, 58, 74, 77]. It is worth noting that the Macaulay in Cohen-Macaulay and Macaulay2 are the same: Francis Macaulay. The name of this condition comes from Cohen's [11] and Macaulay's [52] work on the unmixedness property; see Theorem 5.3.16(a).

The interested reader will find many sources to augment our treatment of initial ideals and Hilbert functions in Section 5.4, including the texts [9, 13, 18, 37, 54, 58, 74, 77] in addition to any text on Gröbner bases. We recommend [13] strongly. Many of these also include significant information about free resolutions, including the notions of Betti numbers and Ext from the end of Section 5.5.

The interplay between the notions in this chapter and the previous ones is significant. Some of this we have already seen. For instance, the definition of Cohen-

Macaulayness is given in terms of Krull dimension, which is described in terms of m-irreducible decompositions; see Theorem 5.1.2. As we saw in the Macaulay2 exercises for Section 5.5, depth and dimension (hence, Cohen-Macaulayness) related to the lengths of free resolutions, by the Auslander-Buchsbaum formula [3] and Hlbert's Syzygy Theorem [39]. (The term "syzygy" is from the Greek "suzugos" meaning "yoked or paired" like oxen. The term is used in astronomy to describe three celestial objects arranged in a line. The use in homological algebra is similar since syzygies are often displayed in exact sequences $0 \to A \to B \to C \to 0$.)

Cohen-Macaulayness is also related to the shapes of free resolutions, via Alexander duality. The connections between initial ideals and linear algebra mentioned in Section 5.4 lead to fast algorithms for computing Gröbner bases, by work of Jean-Charles Faugère [22]; see also [13, Chapter 10]. The dissertation of Taylor [75] deals with both resolutions (she is the "Taylor" in "Taylor resolution") and decompositions, but in a more general context than we consider. The list goes on and on.

Part III
Decomposing Monomial Ideals

In Part I of this text, we established the existence of m-irreducible decompositions of monomial ideals, and in Part II, we showed how such decompositions are connected to other areas. Now, we are prepared for the third major goal of this text: to establish techniques for actually computing m-irreducible decompositions. Our results are of two types: techniques for computing decompositions of general classes of monomial ideals, eventually for arbitrary monomial ideals, and techniques for taking decompositions of two or more input ideals and combining them to construct a decomposition of an ideal built out of the input ideals, e.g., the sum of the ideals. Of course, we have seen some aspects of this already, in Sections 3.5, 4.3, and 4.5.

This part begins with Chapter 6 which treats the special case of "parametric decompositions". These are the m-irreducible decompositions $I = \bigcap_{i=1}^{n} J_i$ where each J_i is a parameter ideal, that is, an ideal generated by powers of *all* the variables. These are the subject of Section 6.1, and most of the rest of the chapter is focused on the computation of parametric decompositions. These are given in terms of "corner elements", which are introduced in Section 6.2. The subsequent Sections 6.3 and 6.4 contain algorithms for finding the corner elements of ideals, and hence their parametric decompositions. The chapter concludes with three explorations, two about computing special decompositions (Sections 6.5 and 6.6), and one about connections to differential operators (Section 6.7). The Macaulay2 material of this chapter focuses on building more methods and running examples.

The follow-up Chapter 7 treats more general m-irreducible decompositions. It begins with Sections 7.1–7.4 showing how m-irreducible decompositions of two monomial ideals I and J can be used to find decompositions of ideals defined in terms of I and J: monomial radicals, bracket powers, sums, and colon ideals. After this comes Section 7.5 which contains two algorithms for computing m-irreducible decompositions of arbitrary monomial ideals. The exploration Section 7.6 applies one of these algorithms to some combinatorially-defined ideals from Chapter 4, and the final Sections 7.7–7.9 return to the theme of the beginning of the chapter, by exploring decompositions of two more constructions.

Most of the Macaulay2 commands relevant to this chapter are introduced in preceding sections. So most of the tutorials in this chapter are very short, with exercises focusing on examples, laboratory work, and writing methods. An exception to this can be found in Section 7.5, where we discuss recursion.

Chapter 6
Parametric Decompositions of Monomial Ideals

This chapter deals with another case of monomial ideals with a reasonable algorithm for computing m-irreducible decompositions. These are the monomial ideals I with monomial radical $\text{m-rad}\,(I)$ equal to the ideal $\mathfrak{X}$ generated by all the variables in R. See Section 2.3 for properties of the monomial radical; the exercises of that section are particularly relevant.

The chapter begins with Section 6.1, discussing properties of the m-irreducible ideals that arise in the decompositions of these ideals. These ideals are called "parameter ideals," and they determine "parametric decompositions." This section explicitly characterizes the monomial ideals that admit parametric decompositions as those monomial ideals I such that $\text{m-rad}\,(I) = \mathfrak{X}$. The rest of the chapter focuses on techniques for computing parametric decompositions. Section 6.2 shows how the "corner elements" determine irredundant parametric decompositions in general, thus translating the problem of finding decompositions of these ideals into the problem of finding their corner elements. The latter problem here, in two variables and in general, is the topic of Sections 6.3 and 6.4, respectively. Note that corner elements are defined in terms of colon ideals; see Section 2.5.

This chapter ends with three explorations. Section 6.5 explores the use of parametric decompositions in two variables to find m-irreducible decompositions in two variables for ideals that do not necessarily have parametric decompositions. Section 6.6 treats decompositions of powers of some ideals, including the special case $\mathfrak{X}^n$. Lastly, Section 6.7 delves into Macaulay's inverse systems, connecting differential operators with parametric decompositions. Along the way, we continue using Macaulay2 to run examples, to develop methods, and to explore these notions.

6.1 Parameter Ideals and Parametric Decompositions

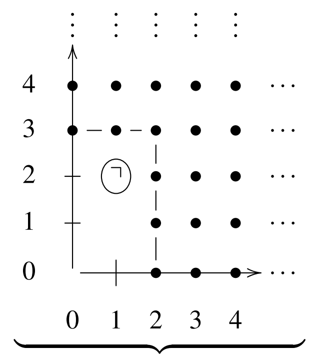
In this section, A is a non-zero commutative ring with identity.

This section deals with special cases of m-irreducible monomial ideals. In the following definition, the "P" stands for "parameter". The term "parameter ideal" comes from the idea that each power of each variable is a parameter, so these ideals are generated by a complete sequence of parameters.

Definition 6.1.1. Set $R = A[X_1,\dots,X_d]$. A *parameter ideal* in R is an ideal of the form $(X_1^{a_1},\dots,X_d^{a_d})R$ with $a_1,\dots,a_d \geqslant 1$. If $f = \underline{X}^{\underline{n}}$ with $\underline{n} \in \mathbb{N}^d$, then set

$$\mathrm{P}_R(f) = (X_1^{n_1+1},\dots,X_d^{n_d+1})R.$$

For example, set $R = A[X,Y]$. Then $\mathrm{P}_R(XY^2) = (X^2,Y^3)R$. From the graph



$\Gamma(\mathrm{P}_R(XY^2)) = \Gamma((X^2,Y^3)R)$

one can see that the "corner" in the graph of $\mathrm{P}_R(XY^2)$ corresponds exactly to the monomial XY^2. (See also Corollary 6.2.12 below.) This partially explains why this ideal is denoted $\mathrm{P}_R(XY^2)$ instead of $\mathrm{P}_R(f)$ for some different monomial f.

Other computations in this situation include $\mathrm{P}_R(1) = (X,Y)R$ and $\mathrm{P}_R(X) = (X^2,Y)R$ and $\mathrm{P}_R(Y) = (X,Y^2)R$.

The next lemma is particularly useful for working with parameter ideals. It explicitly characterizes when a given monomial is in a given parameter ideal.

Lemma 6.1.2. *Set $R = A[X_1,\dots,X_d]$. Let f and g be monomials in R.*

(a) *We have $f \notin \mathrm{P}_R(f)$.*
(b) *We have $g \in \mathrm{P}_R(f)$ if and only if $f \notin (g)R$.*

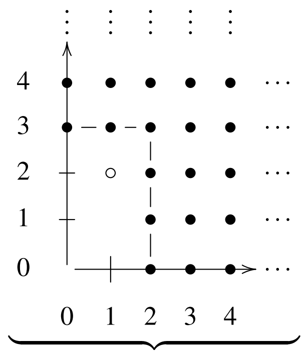
Proof. Write $f = \underline{X}^{\underline{m}}$ and $g = \underline{X}^{\underline{n}}$.

(a) We have $\mathrm{P}_R(f) = (X_1^{m_1+1},\dots,X_d^{m_d+1})R$. Suppose that $f \in \mathrm{P}_R(f)$. Theorem 1.1.9 implies that f is a monomial multiple of some $X_i^{m_i+1}$, and it follows from Lemma 1.1.7 that $m_i \geqslant m_i + 1$, which is impossible.

(b) $\Longrightarrow$: Assume that $g \in \mathrm{P}_R(f)$ and suppose that $f \in (g)R$. The condition $g \in \mathrm{P}_R(f)$ implies $f \in (g)R \subseteq \mathrm{P}_R(f)$, which contradicts part (a).

$\Longleftarrow$: Assume that $g \notin \mathrm{P}_R(f)$. Since $\mathrm{P}_R(f)$ is generated by $X_1^{m_1+1},\dots,X_d^{m_d+1}$ this implies that $g \notin (X_i^{m_i+1})R$ for $i = 1,\dots,d$. Lemma 1.1.7 implies that $n_i < m_i + 1$ for all i, that is, that $n_i \leqslant m_i$, so $\underline{m} \succcurlyeq \underline{n}$. This implies that $f \in (g)R$ by Lemma 1.1.7. $\square$

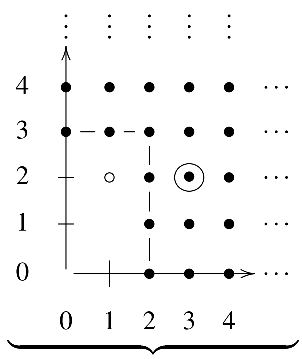
Here is a graphical example of the previous lemma. Set $R = A[X,Y]$. Then $\mathrm{P}_R(XY^2) = (X^2, Y^3)R$. For part (a) one can see from the graph



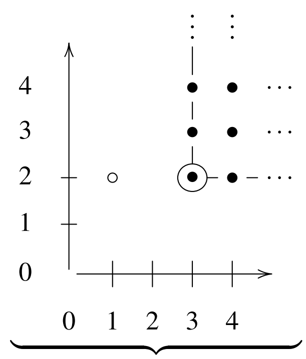
$\Gamma(\mathrm{P}_R(f))=\Gamma((X^2,Y^3)R)$

that $f = XY^2 \notin \mathrm{P}_R(XY^2)$. (The monomial XY^2 is represented by $\circ$ in this graph.)

For part (b) we look at two examples. In the first example, the monomial $g_1 = X^3Y^2 \in \mathrm{P}_R(f)$ is circled

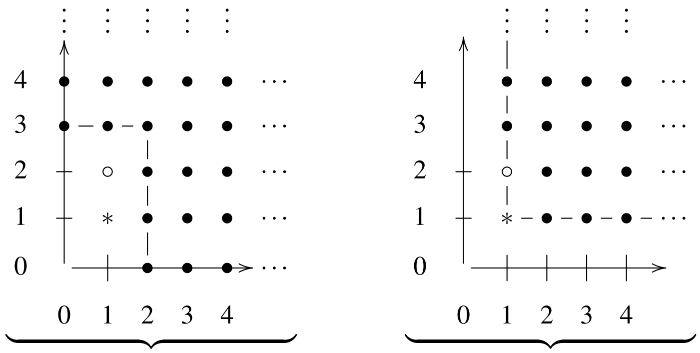


$\Gamma(\mathrm{P}_R(f))=\Gamma((X^2,Y^3)R)$



$\Gamma((g_1)R)$

and we have $g_1 \in \mathrm{P}_R(f)$ and $f \notin (g_1)R$. In the second example, the monomial $g_2 = XY \notin \mathrm{P}_R(f)$ designated with an asterisk $*$



$\Gamma(\mathrm{P}_R(f)){=}\Gamma((X^2,Y^3)R)$ $\qquad$ $\Gamma((g_2)R)$

and we have $g_2 \notin \mathrm{P}_R(f)$ and $f \in g_1 R$.

The next result contains the first step toward characterizing the monomial ideals that decompose as intersections of parameter ideals. It explicitly characterizes the parameter ideals as certain m-irreducible ideals. See Sections 2.3 and 3.1.

Proposition 6.1.3 *Set $R = A[X_1,\ldots,X_d]$ and $\mathfrak{X} = (X_1,\ldots,X_d)R$. A monomial ideal $J \subseteq R$ is a parameter ideal if and only if J is m-irreducible and* $\operatorname{m-rad}(J) = \mathfrak{X}$.

Proof. If J is a parameter ideal, then J is m-irreducible by Theorem 3.1.3, and Corollary 2.3.8 implies that $\operatorname{m-rad}(J) = \mathfrak{X}$.

Conversely, assume that J is m-irreducible and $\operatorname{m-rad}(J) = \mathfrak{X}$. The condition $\operatorname{m-rad}(J) = \mathfrak{X}$ implies that $J \neq 0$, so Theorem 3.1.3 provides positive integers $k, t_1, \ldots, t_k, e_1, \ldots, e_k$ such that $1 \leqslant t_1 < \cdots < t_k \leqslant d$ and $J = (X_{t_1}^{e_1}, \ldots, X_{t_k}^{e_k})R$. By Exercise 2.3.16, the irredundant monomial generating sequence $X_{t_1}^{e_1}, \ldots, X_{t_k}^{e_k}$ for J contains a power of each variable X_i. That is, we have $J = (X_1^{e_1}, \ldots, X_d^{e_d})R$, so J is a parameter ideal. $\square$

The decompositions of interest for this chapter are defined next.

Definition 6.1.4. Set $R = A[X_1,\ldots,X_d]$. Let J be a monomial ideal of R. A *parametric decomposition* of J is a decomposition of J of the form $J = \bigcap_{i=1}^n \mathrm{P}_R(z_i)$. A parametric decomposition $J = \bigcap_{i=1}^n \mathrm{P}_R(z_i)$ is *redundant* if there exists an index j such that $J = \bigcap_{i \neq j} \mathrm{P}_R(z_i)$, where the intersection is taken over $i = 1,\ldots,n$ such that $i \neq j$. A parametric decomposition $J = \bigcap_{i=1}^n \mathrm{P}_R(z_i)$ is *irredundant* if it is not redundant, that is if for all indices j one has $J \neq \bigcap_{i \neq j} \mathrm{P}_R(z_i)$.

A monomial ideal J in the polynomial ring $R = A[X_1,\ldots,X_d]$ may or may not have a parametric decomposition. In fact, we see in Theorem 6.1.5 that J has a parametric decomposition if and only if $\operatorname{m-rad}(J) = (X_1,\ldots,X_d)R$.

Proposition 6.1.3 implies that every parameter ideal in R is m-irreducible. Hence, each parametric decomposition of J is an m-irreducible decomposition; also, such a decomposition is (ir)redundant as a parametric decomposition if and only if it is (ir)redundant as an m-irreducible decomposition. Furthermore, any parametric decomposition can be reduced to an irredundant parametric decomposition, and irredundant parametric decompositions are unique up to re-ordering; see Algorithm 3.3.6 and Theorem 3.3.8.

We next give the aforementioned characterization of the ideals that admit parametric decompositions.

Theorem 6.1.5 *Set $R = A[X_1,\ldots,X_d]$ and $\mathfrak{X} = (X_1,\ldots,X_d)R$. Let J be a monomial ideal of R. Then J has a parametric decomposition if and only if* $\text{m-rad}\,(J) = \mathfrak{X}$.

Proof. $\Longrightarrow$: If J has a parametric decomposition $J = \bigcap_{i=1}^{n} \mathrm{P}_R(z_i)$, then

$$\text{m-rad}\,(J) = \text{m-rad}\left(\bigcap_{i=1}^{n} \mathrm{P}_R(z_i)\right) = \bigcap_{i=1}^{n} \text{m-rad}\,(\mathrm{P}_R(z_i)) = \bigcap_{i=1}^{n} \mathfrak{X} = \mathfrak{X}.$$

See Propositions 2.3.4(b) and 6.1.3.

$\Longleftarrow$: Assume that $\text{m-rad}\,(J) = \mathfrak{X}$. The monomial ideal J has an m-irreducible decomposition $J = \bigcap_{i=1}^{n} J_i$ by Corollary 3.3.7. Exercise 2.3.17 implies that $\text{m-rad}\,(J_i) = \mathfrak{X}$ for each index i, so each J_i is a parameter ideal by Proposition 6.1.3. Thus, the intersection $\bigcap_{i=1}^{n} J_i$ is a parametric decomposition of J. □

Theorem 6.1.5 conspires with Exercise 2.3.16 to give the following useful characterization of ideals admitting parametric decompositions.

Corollary 6.1.6 *Set $R = A[X_1,\ldots,X_d]$, and $\mathfrak{X} = (X_1,\ldots,X_d)R$. Let I be a monomial ideal such that $I \neq R$. Then the following conditions are equivalent:*

(i) *I admits a parametric decomposition;*
(ii) *for each $i = 1,\ldots,d$ there exists an integer $n_i > 0$ such that $X_i^{n_i} \in I$;*
(iii) *an irredundant monomial generating sequence for I contains a power of each variable; and*
(iv) *the set $[\![R]\!] \smallsetminus [\![I]\!]$ is finite.* □

For example, in the notation of the corollary, every power $\mathfrak{X}^n$ admits a parametric decomposition. On the other hand, the ideal $[(X_2,\ldots,X_d)R]^n$ does not.

Now that we know which ideals admit parametric decompositions, it is natural to ask how one goes about computing such decompositions. This is the focus of the remaining sections of this chapter.

Exercises

Exercise 6.1.7. Set $R = A[X_1,\ldots,X_d]$. Compute $\mathrm{P}_R(1)$ and $\mathrm{P}_R(X_i)$ for $i = 1,\ldots,d$. What are the generators for these ideals? Justify your answers.

Exercise 6.1.8. Set $R = A[X_1,\ldots,X_d]$. Let f, g be monomials in $[\![R]\!]$.

(a) Prove that $\mathrm{P}_R(fg) \subseteq \mathrm{P}_R(f) \cap \mathrm{P}_R(g)$.

(b) Prove or disprove: $\mathrm{P}_R(fg) = \mathrm{P}_R(f) \cap \mathrm{P}_R(g)$.
(c) Prove or disprove: $\mathrm{P}_R(fg) \subseteq \mathrm{P}_R(f)\mathrm{P}_R(g)$.
(d) Prove or disprove: $\mathrm{P}_R(fg) = \mathrm{P}_R(f)\mathrm{P}_R(g)$.
(e) Prove that the following conditions are equivalent.

(i) $\mathrm{P}_R(fg) = \mathrm{P}_R(f)$;
(ii) $fg = f$; and
(iii) $g = 1_R$.

Justify your answers.

Exercise 6.1.9. Set $R = A[X_1, \ldots, X_d]$. Let f and g be monomials in R. Prove that if $\deg(f) = \deg(g)$ and $f \neq g$, then $f \in \mathrm{P}_R(g)$.

Exercise 6.1.10. Set $R = A[X_1, \ldots, X_d]$. For a monomial $z \in [\![R]\!]$, prove that $z \notin I$ if and only if $I \subseteq \mathrm{P}_R(z)$.

**Exercise 6.1.11.* Set $R = A[X_1, \ldots, X_d]$. Let w, z be monomials in R. Prove that the following conditions are equivalent:

(i) $z \in (w)R$;
(ii) $w \notin \mathrm{P}_R(z)$;
(iii) $\mathrm{P}_R(z) \subseteq \mathrm{P}_R(w)$; and
(iv) $(\mathrm{P}_R(z) :_R w) \neq R$.

(This exercise is used in the proofs of Proposition 6.2.11, Proposition 6.2.13, Algorithm 6.2.14, and Proposition 6.2.15.)

Exercise 6.1.12. Set $R = A[X_1, \ldots, X_d]$ with $d \geqslant 2$. Let I be a monomial ideal in R. If f and w are monomials in R, show that $(\mathrm{P}_R(fw) :_R f) = \mathrm{P}_R(w)$.

Exercise 6.1.13. Set $R = A[X_1, \ldots, X_d]$. Set $\mathfrak{X} = (X_1, \ldots, X_d)R$. Prove that the following conditions are equivalent.

(i) A is reduced (see Section 2.4 for a treatment of reduced rings);
(ii) for every monomial $z \in R$, one has $\operatorname{rad}(\mathrm{P}_R(z)) = \mathfrak{X}$; and
(iii) there exists a monomial $z \in R$ such that $\operatorname{rad}(\mathrm{P}_R(z)) = \mathfrak{X}$.

Exercise 6.1.14. This exercise involves the construction $\mathrm{V}(I)$ from Exploration Section A.10. Let A be a field, and let I be a monomial ideal of $R = A[X_1, \ldots, X_d]$. Prove that I has a parametric decomposition if and only if $\mathrm{V}(I) = \{\underline{0}\}$ where $\underline{0}$ is the origin in A^d. (See Exercises 2.3.19 and 3.3.11.)

Parameter Ideals in Macaulay2

In this tutorial we show how to define the parameter ideal associated to a monomial of the polynomial ring. For this, we will use the generalized bracket powers introduced in Section 2.8. We start with a polynomial ring, and a monomial.

```
i1 : R = QQ[x,y,z];

i2 : mon = x^5*y^6*z^3
      5 6 3
o2 = x y z
o2 : R
```

Recall that we obtain the exponents on a monomial using the following command.

```
i3 : expo = first exponents mon
o3 = {5, 6, 3}
o3 : List
```

Since we have to add 1 to each of these exponents before taking the power of the corresponding generator, we need a list of 1's of length equal to the number of generators of the ring. We do this with the next command.

```
i4 : ones = toList ((numgens R):1)
o4 = {1, 1, 1}
o4 : List
```

Generalized bracket powers work with an `Array` object, so we must first take the (elementwise) sum of the lists `expo` and `ones`, and then convert it to an array using the `new` command.

```
i5 : paramArray = new Array from (expo + ones)
o5 = [6, 7, 4]
o5 : Array
```

Finally, we define the parameter ideal.

```
i6 : paramIdeal = (monomialIdeal gens R)^paramArray
                        6   7   4
o6 = monomialIdeal (x , y , z )
o6 : MonomialIdeal of R
```

Putting all these steps together, we have the following method, that we will use in Section 6.2. Note our use of the command `ring` to obtain the ring associated to the monomial `mon`.

```
i7 : parameterIdeal = method()
o7 = parameterIdeal
o7 : MethodFunction

i8 : parameterIdeal RingElement := mon -> (
if #(terms mon) > 1 then error "Expected a monomial.";
if mon == 0 then error "Expected nonzero input.";
R := ring mon;
expo := first exponents mon;
ones := toList ((numgens R):1);
paramArray := new Array from (expo + ones);
(monomialIdeal gens R)^paramArray
)
```

```
o8 = {*Function[stdio:8:35-15:24]*}
o8 : FunctionClosure
```

We conclude by evaluating this method on two sample monomials.

```
i9 : parameterIdeal (x^2*y^3*z)
                         3   4   2
o9 = monomialIdeal (x , y , z )
o9 : MonomialIdeal of R

i10 : parameterIdeal (1_R)
o10 = monomialIdeal (x, y, z)
o10 : MonomialIdeal of R
```

Exercises

Exercise 6.1.15. Input the method `parameterIdeal` above, and use it to create the ideal $J = \mathrm{P}_R(x^2y^3z^4)$ in $\mathbb{Q}[x,y,z]$. Set $\mathfrak{X} = (x,y,z)R = \mathrm{P}_R(1)$, and compute $(J : \mathfrak{X})$. Do the same for some other parameter ideals. What do you notice?

Exercise 6.1.16. Use the method `parameterIdeal` to verify any counterexamples you devise for Exercise 6.1.8.

Exercise 6.1.17. Use Macaulay2 to verify the conclusion of Exercise 6.1.9 for some monomials $f,g \in \mathbb{Q}[X,Y,Z]$.

Exercise 6.1.18. Set $R = \mathbb{Q}[X,Y,Z]$ and $I = (XY,Z^2)R$. Consider the monomials $f = YZ$ and $g = XYZ$.

(a) Use Macaulay2 to show that $f \notin I$ and $I \subseteq \mathrm{P}_R(YZ)$.
(b) Use Macaulay2 to show that $g \in I$ and $I \not\subseteq \mathrm{P}_R(XYZ)$.
(c) Use Macaulay2 to verify that $g \in (f)R$ and $f \notin \mathrm{P}_R(g)$ and $\mathrm{P}_R(g) \subseteq \mathrm{P}_R(f)$ and $(\mathrm{P}_R(g) :_R f) \neq R$.
(d) Use Macaulay2 to verify that $f \notin (g)R$ and $g \in \mathrm{P}_R(f)$ and $\mathrm{P}_R(f) \not\subseteq \mathrm{P}_R(g)$ and $(\mathrm{P}_R(f) :_R g) = R$.

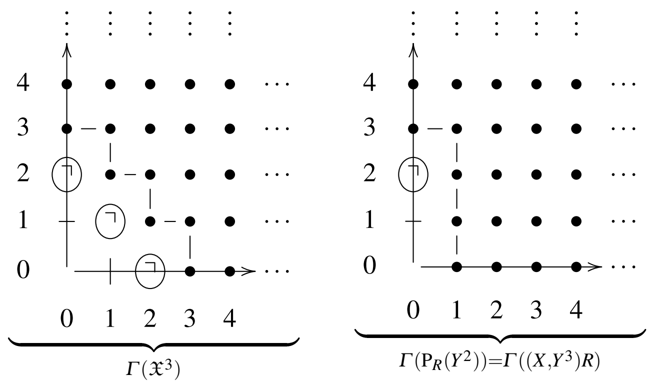
6.2 Corner Elements

In this section, A is a non-zero commutative ring with identity.

This section contains an explicit technique for computing irredundant parametric decompositions; see Theorem 6.2.9. The formula uses the next definition.

Definition 6.2.1. Set $R = A[X_1,\ldots,X_d]$. Let J be a monomial ideal in R. A monomial $z \in [\![R]\!]$ is a *J-corner element* if $z \notin J$ and $X_1z,\ldots,X_dz \in J$. The set of J-corner elements of J in $[\![R]\!]$ is denoted $\mathrm{C}_R(J)$.

Example 6.2.2. Set $R = A[X,Y]$ and $\mathfrak{X} = (X,Y)R$. It is straightforward to check the $\mathfrak{X}^3$-corner elements are exactly X^2, XY, and Y^2; that is, we have $\mathrm{C}_R(\mathfrak{X}^3) = \{X^2, XY, Y^2\}$. Similarly, we have $\mathrm{C}_R(\mathrm{P}_R(Y^2)) = \{Y^2\}$.



The next three results contain tools for the proof of Theorem 6.2.9. Each one exhibits some useful properties of corner elements. The first one uses colon ideals; see Section 2.5.

Proposition 6.2.3 *Set $R = A[X_1,\dots,X_d]$. Set $\mathfrak{X} = (X_1,\dots,X_d)R$, and let J be a monomial ideal in R.*

(a) *The J-corner elements are precisely the monomials in $(J :_R \mathfrak{X}) \smallsetminus J$, in other words, we have $\mathrm{C}_R(J) = [\![(J :_R \mathfrak{X})]\!] \smallsetminus [\![J]\!]$.*
(b) *If z and z' are distinct J-corner elements, then $z \notin (z')R$ and $z' \notin (z)R$.*
(c) *The set $\mathrm{C}_R(J)$ is finite.*

Proof. (a) This follows from Proposition A.6.2(b).

(b) Assume that z and z' are distinct J-corner elements and suppose that $z \in (z')R$. It follows that there is a monomial $f \in [\![R]\!]$ such that $z = fz'$. Since $z \neq z'$, we conclude that $f \neq 1$. Since f is a monomial, it follows that $f \in \mathfrak{X}$. By part (a), we have $z' \in (J :_R \mathfrak{X})$, so the condition $f \in \mathfrak{X}$ implies that $z = fz' \in J$. This contradicts the condition $z \in \mathrm{C}_R(J)$, and so the condition $z \in (z')R$ must be false. Similarly, we conclude that $z' \notin (z)R$ as desired.

(c) The ideal $K = (\mathrm{C}_R(J))R$ is a monomial ideal, so Dickson's Lemma 1.3.1 implies that K is generated by a finite list of monomials $z_1,\dots,z_n \in \mathrm{C}_R(J)$. We claim that $\{z_1,\dots,z_n\} = \mathrm{C}_R(J)$. The containment $\{z_1,\dots,z_n\} \subseteq \mathrm{C}_R(J)$ holds by assumption. For the reverse containment, let $z' \in \mathrm{C}_R(J)$. Then we have $z' \in \mathrm{C}_R(J) \subseteq K = (z_1,\dots,z_n)R$, so z' is a monomial multiple of z_j for some index j. Part (b) implies $z' = z_j$, as desired. $\square$

Example 6.2.4. Continue with the set-up of Example 6.2.2, where the $\mathfrak{X}^3$-corner elements are exactly X^2, XY, and Y^2. As promised by Proposition 6.2.3, this list is finite, and no element in this list divides any other element in this list.

Note that the next result does not usually give a parametric decomposition of J, because (according to Theorem 6.1.5) J will only have such a decomposition if $\text{m-rad}(J) = \mathfrak{X}$. See Example 6.2.6 and Theorem 6.2.9.

Corollary 6.2.5 *Set $R = A[X_1,\ldots,X_d]$, and let J be a monomial ideal in R. Let $z_1,\ldots,z_m$ be the distinct J-corner elements and set $J' = \mathrm{P}_R(z_1) \cap \cdots \cap \mathrm{P}_R(z_m)$.*

(a) *For $i = 1,\ldots,m$ we have $(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_m)R \subseteq \mathrm{P}_R(z_i)$ and $z_i \notin \mathrm{P}_R(z_i)$ and $z_i \notin J'$.*
(b) *The intersection $J' = \bigcap_{i=1}^m \mathrm{P}_R(z_i)$ is irredundant.*
(c) *There is a containment $J \subseteq J'$.*

Proof. (a) The condition $z_i \notin \mathrm{P}_R(z_i)$ is from Lemma 6.1.2(b). From this, the special case $m = 1$ is straightforward, so we assume that $m \geqslant 2$. For indices i and j such that $j \neq i$, we have $z_i \neq z_j$, so $z_i \notin (z_j)R$ by Proposition 6.2.3(b); it follows that $z_j \in \mathrm{P}_R(z_i)$ by Lemma 6.1.2(b). We conclude that $z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_m \in \mathrm{P}_R(z_i)$ so $(z_1,\ldots,z_{i-1},z_{i+1},\ldots,z_m)R \subseteq \mathrm{P}_R(z_i)$.

The condition $z_i \notin J'$ follows because $J' \subseteq \mathrm{P}_R(z_i)$.

(b) This follows from part (a) because $z_i \in \bigcap_{j\neq i} \mathrm{P}_R(z_j)$ and $z_i \notin \mathrm{P}_R(z_i)$.

(c) Exercise 6.2.20(c). $\square$

Example 6.2.6. Continue with the set-up of Example 6.2.4, with $I = \mathfrak{X}^3$, where the I-corner elements are exactly X^2, XY, and Y^2. The ideal I' given by the preceding corollary is

$$I' = \mathrm{P}_R(X^2) \cap \mathrm{P}_R(XY) \cap \mathrm{P}_R(Y^2).$$

Using the LCM's of the generators of the parameter ideals in this intersection, one sees that $I' = I = \mathfrak{X}^3$. One can check easily that this decomposition is irredundant. For instance, the monomial X^2 is in $\mathrm{P}_R(XY) \smallsetminus \mathfrak{X}^3$.

On the other hand, consider the ideal

$$J = XY\mathfrak{X}^3 = (X^4Y, X^3Y^2, X^2Y^3, XY^4)R.$$

It is straightforward to show that the J-corner elements are exactly X^3Y, X^2Y^2, and XY^3. The ideal J' given by the preceding corollary is

$$J' = \mathrm{P}_R(X^3Y) \cap \mathrm{P}_R(X^2Y^2) \cap \mathrm{P}_R(XY^3).$$

Using the LCM's of the generators of the parameter ideals in this intersection, one sees that $J' = (X^4, X^3Y^2, X^2Y^3, Y^4)R$. From this, one checks readily the proper containment $J \subsetneq J'$. Of course, one should not be surprised by the fact that $J \neq J'$, since J does not have a parametric decomposition by Corollary 6.1.6, and J' does have a parametric decomposition by construction.

The following result is the next step in proving that a monomial ideal I with $\text{m-rad}(I) = \mathfrak{X}$ has a parametric decomposition.

Proposition 6.2.7 *Set $R = A[X_1,\ldots,X_d]$. Set $\mathfrak{X} = (X_1,\ldots,X_d)R$, and let I be a monomial ideal in R such that* $\text{m-rad}(I) = \mathfrak{X}$.

(a) *If* $f \in [\![R]\!] \smallsetminus [\![I]\!]$, *then there is a monomial* $g \in [\![R]\!]$ *such that* $fg \in \mathrm{C}_R(I)$.
(b) $\mathrm{C}_R(I) \neq \emptyset$.
(c) *Given a monomial ideal* $J \subseteq R$ *such that* $J \not\subseteq I$ *one has* $\mathrm{C}_R(I) \cap J \neq \emptyset$.

Proof. (a) Set $S = [\![R]\!] \smallsetminus [\![I]\!]$ which is a finite set by Exercise 2.3.16. Set

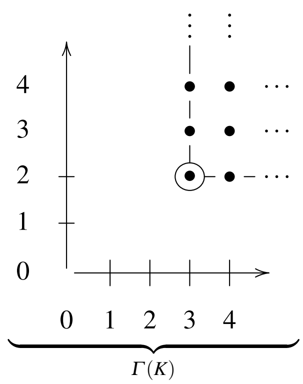
$$T = \{g \in [\![R]\!] \mid fg \notin I\}.$$

If $g \in T$, then $g \notin I$ since $fg \notin I$ and I is an ideal. It follows that $T \subseteq S$, so T is a finite set. Let g be a monomial in T with maximal degree. By the maximality of $\deg(g)$, we have $X_i g \notin T$ for $i = 1, \ldots, d$. In other words, we have $X_i fg \in I$ for $i = 1, \ldots, d$. This says that $fg \in (I :_R \mathfrak{X})$, so the condition $fg \notin I$ implies $fg \in \mathrm{C}_R(I)$, as desired.

(b) Since $\text{m-rad}\,(I) = \mathfrak{X}$, we have $I \neq R$ by Proposition 2.3.3(a). In particular, $f = 1 \in [\![R]\!] \smallsetminus [\![I]\!]$ so part (a) provides a monomial $g \in [\![R]\!]$ such that $g = 1g \in \mathrm{C}_R(I)$.

(c) Fix a monomial $f \in [\![J]\!] \smallsetminus [\![I]\!]$. Part (a) implies that there is a monomial $g \in [\![R]\!]$ such that $fg \in \mathrm{C}_R(I)$. Since J is an ideal and $f \in J$, we have $fg \in J$, and so $fg \in \mathrm{C}_R(I) \cap J$. $\square$

Example 6.2.8. Continue with the set-up of Example 6.2.6, with $I = \mathfrak{X}^3$ and $J = XY\mathfrak{X}^3$. Also, set $K = (X^3Y^2)R$.



It is straightforward to show that $\mathrm{C}_R(K) = \emptyset$; thus, the conclusion of Proposition 6.2.7(b) fails for arbitrary monomial ideals. On the other hand, we have $\text{m-rad}\,(J) \neq \mathfrak{X}$ and $\mathrm{C}_R(J) \neq \emptyset$, so the conclusion of Proposition 6.2.7(b) holds for a larger class of ideals than the one covered by the proposition.

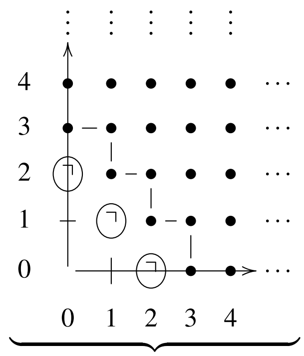
As an example of the behavior documented in part (c) of the proposition, set $L = (X^2, Y^3)R$. Note that we have $L \not\subseteq I$ since $X^2 \in L \smallsetminus I$, and $X^2 \in \mathrm{C}_R(I) \bigcap L$.

Now we are ready to describe irredundant parametric decompositions in terms of corner elements. See the first paragraph of Example 6.2.6 for a sample computation.

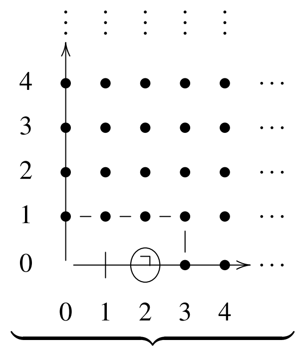
Theorem 6.2.9 *Set* $R = A[X_1, \ldots, X_d]$. *Set* $\mathfrak{X} = (X_1, \ldots, X_d)R$, *and let* J *be a monomial ideal of* R *such that* $\text{m-rad}\,(J) = \mathfrak{X}$. *If the distinct* J*-corner elements are* $z_1, \ldots, z_m$ *then* $J = \bigcap_{j=1}^m \mathrm{P}_R(z_j)$ *is an irredundant parametric decomposition.*

Proof. Note first that Proposition 6.2.7 shows that J has a corner element. Set $J' = \bigcap_{j=1}^{m} \mathrm{P}_R(z_j)$. Corollary 6.2.5 implies that this intersection is irredundant and that $J' \supseteq J$. Thus, it remains to show that $J' \subseteq J$. Since each $\mathrm{P}_R(z_j)$ is a monomial ideal, Theorem 2.1.1 implies that J' is a monomial ideal. Suppose by way of contradiction that $J' \not\subseteq J$. Proposition 6.2.7(c) implies that J' contains a J-corner element, say $z_i \in J'$. This implies that $z_i \in \mathrm{P}_R(z_i)$, which contradicts Lemma 6.1.2(a). $\square$

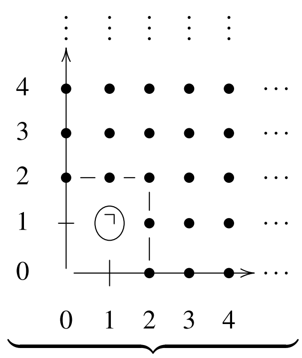
Example 6.2.10. Continue with the set-up of Example 6.2.8, where the $\mathfrak{X}^3$-corner elements are exactly X^2, XY, and Y^2. One can see the parametric decomposition of $\mathfrak{X}^3$ from Theorem 6.2.9 (see Example 6.2.6) in the graphs



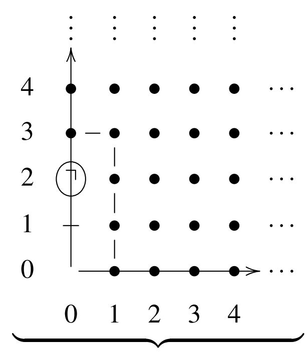
$\Gamma(\mathfrak{X}^3)$



$\Gamma(\mathrm{P}_R(X^2))=\Gamma((X^3,Y)R)$



$\Gamma(\mathrm{P}_R(XY))=\Gamma((X^2,Y^2)R)$



$\Gamma(\mathrm{P}_R(Y^2))=\Gamma((X,Y^3)R)$

Theorem 6.2.9 shows that the J-corner elements determine an irredundant parametric decomposition of J. The next result is the reverse: an irredundant parametric decomposition of J determines the J-corner elements. Note that Proposition 6.2.13 below characterizes the sequences $z_1, \ldots, z_m \in [\![R]\!]$ such that the decomposition $J = \bigcap_{j=1}^{m} \mathrm{P}_R(z_j)$ is irredundant.

Proposition 6.2.11 *Set $R = A[X_1, \ldots, X_d]$. Fix monomials $z_1, \ldots, z_m \in [\![R]\!]$ and assume that $J = \bigcap_{j=1}^{m} \mathrm{P}_R(z_j)$ is an irredundant parametric decomposition of J. Then the distinct J-corner elements are $z_1, \ldots, z_m$.*

Proof. Claim 1: Each z_i is a J-corner element. First, note that $z_i \notin \mathrm{P}_R(z_i)$ by Lemma 6.1.2(a); it follows that $z_i \notin J$ since $J = \bigcap_{j=1}^m \mathrm{P}_R(z_j) \subseteq \mathrm{P}_R(z_i)$. To complete the proof of the claim, we need to show that $X_j z_i \in J$ for each j. By way of contradiction, suppose that $X_j z_i \notin J$. It follows that $X_j z_i \notin \mathrm{P}_R(z_k)$ for some k. Lemma 6.1.2(b) implies that $z_k \in (X_j z_i)R \subseteq (z_i)R$. Exercise 6.1.11 implies that $\mathrm{P}_R(z_k) \subseteq \mathrm{P}_R(z_i)$. The irredundancy of the intersection implies that $\mathrm{P}_R(z_k) = \mathrm{P}_R(z_i)$, so $i = k$. The condition $z_k \in (X_j z_i)R$ then reads as $z_k \in (X_j z_k)R$. A degree argument shows that this is impossible. Thus we must have $X_j z_i \in J$ for each i and j, so $z_i \in \mathrm{C}_R(J)$, as claimed.

Claim 2: The elements $z_1, \ldots, z_m$ are distinct. Indeed, if $z_i = z_j$, then $\mathrm{P}_R(z_i) = \mathrm{P}_R(z_j)$, so the irreduncancy of the intersection implies that $i = j$.

Claim 3: $\mathrm{C}_R(J) \subseteq \{z_1, \ldots, z_m\}$. (Once this claim is established, the proof is complete.) Let $z \in \mathrm{C}_R(J)$. This implies that $z \notin J$, and so there is an index k such that $z \notin \mathrm{P}_R(z_k)$. Lemma 6.1.2(b) implies that $z_k \in (z)R$, so Proposition 6.2.3(b) says that $z = z_k$. This establishes the claim. $\square$

Part (a) of the next result gives a general explanation of some of the computations from Example 6.2.2. Also, part (b) complements Proposition 6.1.3.

Corollary 6.2.12 *Set $R = A[X_1, \ldots, X_d]$. Set $\mathfrak{X} = (X_1, \ldots, X_d)R$, and let J be a monomial ideal in R such that* $\text{m-rad}\,(J) = \mathfrak{X}$.

(a) *For each monomial $z \in [\![R]\!]$ we have* $\mathrm{C}_R(\mathrm{P}_R(z)) = \{z\}$.
(b) *The following conditions are equivalent:*

  (i) *the ideal J is m-irreducible;*
  (ii) *the ideal J is a parameter ideal; and*
  (iii) *there is precisely one J-corner element.*

Proof. (a) The trivial intersection $J = \mathrm{P}_R(z)$ is an irredundant parametric decomposition, so Proposition 6.2.11 implies that $\mathrm{C}_R(\mathrm{P}_R(z)) = \mathrm{C}_R(J) = \{z\}$.

(b) The equivalence (i) $\Longleftrightarrow$ (ii) is from Proposition 6.1.3. The implication (ii) $\Longrightarrow$ (iii) follows from part (a).

(iii) $\Longrightarrow$ (ii): Assume that there is precisely one J-corner element w. Then the decomposition $J = \bigcap_{z \in \mathrm{C}_R(J)} \mathrm{P}_R(z)$ from Theorem 6.2.9 reads as $J = \mathrm{P}_R(w)$, so J is a parameter ideal. $\square$

Checking that a given parametric decomposition is irredundant can be tedious. Imagine how it goes in three or more variables when there is no good visual to help guide you. The next proposition makes it a lot easier.

Proposition 6.2.13 *Set $R = A[X_1, \ldots, X_d]$. Fix monomials $z_1, \ldots, z_m \in [\![R]\!]$, and set $I = \bigcap_{j=1}^m \mathrm{P}_R(z_j)$. The following conditions are equivalent:*

(i) *the intersection $\bigcap_{j=1}^m \mathrm{P}_R(z_j)$ is irredundant; and*
(ii) *for all indices i and j, if $i \neq j$, then $z_i \notin (z_j)R$.*

Proof. Exercise 6.1.11. $\square$

For example, set $R = A[X,Y]$ and

$$\begin{aligned} J &= (X^3,Y^6)R \cap (X^4,Y^4)R \cap (X^5,Y^2)R \\ &= \mathrm{P}_R(X^2Y^5) \cap \mathrm{P}_R(X^3Y^3) \cap \mathrm{P}_R(X^4Y). \end{aligned}$$

To show that the intersection is irredundant, it suffices (by Proposition 6.2.13) to observe that no monomial in the list X^2Y^5, X^3Y^3, X^4Y is a monomial multiple of any other monomial in this list.

Propositions 6.2.11 and 6.2.13 combine to give a method for constructing monomial ideals with corners $z_1,\ldots,z_m$. Specifically, let $z_1,\ldots,z_m \in [\![R]\!]$ be such that $z_i \notin (z_j)R$ for all $i \neq j$. (Note that Proposition 6.2.3(b) shows that we need $z_i \notin (z_j)R$ for all $i \neq j$.) Proposition 6.2.13 guarantees that the decomposition $J = \bigcap_{j=1}^m \mathrm{P}_R(z_j)$ is irredundant, so Proposition 6.2.11 implies that $\mathrm{C}_R(J) = \{z_1,\ldots,z_m\}$.[1]

Proposition 6.2.13 and Exercise 6.1.11 combine to give the following algorithm for transforming redundant parametric decompositions into irredundant ones.

Algorithm 6.2.14. Set $R = A[X_1,\ldots,X_d]$. Fix monomials $z_1,\ldots,z_m \in [\![R]\!]$ and set $I = \bigcap_{j=1}^m \mathrm{P}_R(z_j)$. We assume that $m \geqslant 1$.

Step 1. Check whether the intersection $\bigcap_{j=1}^m \mathrm{P}_R(z_j)$ is irredundant using Proposition 6.2.13.

Step 1a. If, for all indices i and j such that $i \neq j$, we have $z_j \notin (z_i)R$, then the intersection is irredundant; in this case, the algorithm terminates.

Step 1b. If there exist indices i and j such that $i \neq j$ and $z_j \in (z_i)R$, then the intersection is redundant; in this case, continue to Step 2.

Step 2. Remove a parameter ideal that causes a redundancy in the intersection. By assumption, there exist indices i and j such that $i \neq j$ and $z_j \in (z_i)R$. Re-order the indices to assume without loss of generality that $i = m$. Thus, we have $j < m$ and $z_j \in (z_m)R$. Exercise 6.1.11 implies that $\mathrm{P}_R(z_j) \subseteq \mathrm{P}_R(z_m)$, and it follows that $I = \bigcap_{j=1}^m \mathrm{P}_R(z_j) = \bigcap_{j=1}^{m-1} \mathrm{P}_R(z_j)$. Now apply Step 1 to the new list of monomials $z_1,\ldots,z_{m-1}$.

The algorithm will terminate in at most $m-1$ steps because one can remove at most $m-1$ monomials from the list and still form an ideal that is a non-empty intersection of parameter ideals.

As an example of this algorithm, set $R = A[X,Y]$ and

$$\begin{aligned} J &= (X,Y^5)R \cap (X^2,Y^3)R \cap (X^4,Y^4)R \cap (X^3,Y)R \cap (X^5,Y^2)R \\ &= \mathrm{P}_R(Y^4) \cap \mathrm{P}_R(XY^2) \cap \mathrm{P}_R(X^3Y^3) \cap \mathrm{P}_R(X^2) \cap \mathrm{P}_R(X^4Y). \end{aligned}$$

The list of z_i's to consider is $Y^4, XY^2, X^3Y^3, X^2, X^4Y$.

The monomial X^3Y^3 is a multiple of XY^2, so we remove XY^2 from the list.

---

[1] Note that the assumption $z_i \notin (z_j)R$ for all $i \neq j$ says that $z_1,\ldots,z_m$ is an irredundant monomial generating sequence for $(z_1,\ldots,z_m)R$. Thus, the ideal $J = \bigcap_{j=1}^m \mathrm{P}_R(z_j)$ is similar to the $*$-dual $((z_1,\ldots,z_m)R)^*$ of Section 4.7, e.g., Exercise 4.7.4(b)–(c).

The new list of z_i's to consider is Y^4, X^3Y^3, X^2, X^4Y.

The monomial X^4Y is a multiple of X^2, so we remove X^2 from the list.

The new list of z_i's to consider is Y^4, X^3Y^3, X^4Y. No monomial in the list is a multiple of another since the exponent vectors $(0,4)$, $(3,3)$ and $(4,1)$ are incomparable. Hence, the intersection

$$J = \mathrm{P}_R(Y^4) \cap \mathrm{P}_R(X^3Y^3) \cap \mathrm{P}_R(X^4Y) = (X, Y^5)R \cap (X^4, Y^4)R \cap (X^5, Y^2)R.$$

is an irredundant parametric decomposition of J.

Here is a one-step procedure for transforming redundant parametric intersections into irredundant parametric intersections.

Proposition 6.2.15 *Set $R = A[X_1, \ldots, X_d]$, and let $m \geqslant 1$. Fix distinct monomials $z_1, \ldots, z_m \in [\![R]\!]$, and set $I = \bigcap_{j=1}^m \mathrm{P}_R(z_j)$. For $j = 1, \ldots, m$ write $z_j = \underline{X}^{\underline{n}_j}$ with $\underline{n}_j \in \mathbb{N}^d$. Set $\Delta = \{\underline{n}_1, \ldots, \underline{n}_m\} \subseteq \mathbb{N}^d$ and consider the order $\succcurlyeq$ on $\mathbb{N}^d$ from Definition A.9.3. Let Δ' denote the set of maximal elements of Δ under this order. Then $I = \bigcap_{\underline{n}_j \in \Delta'} \mathrm{P}_R(z_j)$ is an irredundant parametric decomposition of I and $\mathrm{C}_R(I) = \{z_j \mid \underline{n}_j \in \Delta'\} \subseteq \{z_1, \ldots, z_n\}$.*

Proof. Note that the set Δ has maximal elements since Δ is finite.

The maximality of the elements of Δ' implies that for each $\underline{n}_i \in \Delta$, there is an element $\underline{n}_j \in \Delta'$ such that $\underline{n}_j \succcurlyeq \underline{n}_i$. It follows that $z_j \in (z_i)R$, so Exercise 6.1.11 implies that $\mathrm{P}_R(z_j) \subseteq \mathrm{P}_R(z_i)$. From this, we conclude that $I = \bigcap_{\underline{n}_j \in \Delta'} \mathrm{P}_R(z_j)$.

For each $\underline{n}_j, \underline{n}_k \in \Delta'$ such that $j \neq k$, we have $\underline{n}_j \not\succcurlyeq \underline{n}_k$ since $\underline{n}_j$ and $\underline{n}_k$ are both maximal among the elements of Δ and they are distinct. It follows that $z_j \notin (z_k)R$. Proposition 6.2.13 then implies that the intersection $I = \bigcap_{\underline{n}_j \in \Delta'} \mathrm{P}_R(z_j)$ is irredundant, and the equality $\mathrm{C}_R(I) = \{z_j \mid \underline{n}_j \in \Delta'\}$ comes from Proposition 6.2.11. $\square$

Again as an example of this proposition, consider $R = A[X, Y]$ and

$$\begin{aligned} J &= (X, Y^5)R \cap (X^2, Y^3)R \cap (X^4, Y^4)R \cap (X^3, Y)R \cap (X^5, Y^2)R \\ &= \mathrm{P}_R(Y^4) \cap \mathrm{P}_R(XY^2) \cap \mathrm{P}_R(X^3Y^3) \cap \mathrm{P}_R(X^2) \cap \mathrm{P}_R(X^4Y). \end{aligned}$$

The list of exponent vectors is $\Delta = \{(0,4), (1,2), (3,3), (2,0), (4,1)\}$. The list of maximal elements in Δ is $\Delta' = \{(0,4), (3,3), (4,1)\}$. Hence, the intersection

$$J = \mathrm{P}_R(Y^4) \cap \mathrm{P}_R(X^3Y^3) \cap \mathrm{P}_R(X^4Y) = (X, Y^5)R \cap (X^4, Y^4)R \cap (X^5, Y^2)R.$$

is an irredundant parametric decomposition of J. Compare this to the decomposition preceding the proposition.

Exercises

Exercise 6.2.16. Set $R = A[X_1, \ldots, X_d]$ with $d \geqslant 2$. Let f be a monomial in $[\![R]\!]$ and prove that $\mathrm{C}_R((f)R) = \emptyset$.

Exercise 6.2.17. Verify any unsubstantiated claims in the examples (numbered and unnumbered) in this section.

Exercise 6.2.18. Set $R = A[X_1,\ldots,X_d]$ and $\mathfrak{X} = (X_1,\ldots,X_d)R$. Let J be a monomial ideal in R, and prove that the following conditions are equivalent:

(i) $1 \in \mathrm{C}_R(J)$;
(ii) $J = \mathfrak{X}$; and
(iii) $\mathrm{C}_R(J) = \{1\}$.

Exercise 6.2.19. Set $R = A[X_1,\ldots,X_d]$. Let J be a monomial ideal in R, and fix a monomial $f = \underline{X}^{\underline{n}} \in [\![R]\!]$. For $i = 1,\ldots,d$ set $\underline{e_i} = (0,\ldots,0,1,0,\ldots,0) \in \mathbb{N}^d$ where the 1 occurs in the ith position. Prove that f is a J-corner element if and only if $\underline{n} \notin \Gamma(J)$ and $\underline{n} + \underline{e_i} \in \Gamma(J)$ for each $i = 1,\ldots,d$; see Definition 1.1.11.

Exercise 6.2.20. Set $R = A[X_1,\ldots,X_d]$. Let J be a monomial ideal in R, and fix a monomial $w \in [\![R]\!]$.

(a) Prove that if $w \notin J$, then $J \subseteq \mathrm{P}_R(w)$.
(b) Prove that if w is a J-corner element, then $J \subseteq \mathrm{P}_R(w)$.
(c) In the notation of Corollary 6.2.5, prove that $J \subseteq J'$.

Exercise 6.2.21. Set $R = A[X,Y]$ and $\mathfrak{X} = (X,Y)R$. Find monomial ideals I and J in R such that $\mathrm{rad}\,(I) = \mathrm{rad}\,(\mathfrak{X}) = \mathrm{rad}\,(J)$ and $I \subseteq J$ and $\mathrm{C}_R(I) \cap \mathrm{C}_R(J) = \emptyset$; in particular, such an example has $\mathrm{C}_R(I) \not\subseteq \mathrm{C}_R(J)$ and $\mathrm{C}_R(I) \not\supseteq \mathrm{C}_R(J)$. Justify your answers.

Exercise 6.2.22. Set $R = A[X,Y,Z]$. Consider the monomial ideal

$$J = (Z^4, Y^2Z^3, Y^3, XYZ, XY^2, X^2)R$$

and set $f = X$. Show that $f \notin J$ and find a monomial g such that $fg \in \mathrm{C}_R(I)$. (See Proposition 6.2.7(c).) Justify your answer.

Exercise 6.2.23. Set $R = A[X,Y,Z]$. For each of the following sets of monomials, decide whether there is a monomial ideal I with $\mathrm{C}_R(I)$ equal to the given set. If the answer is no, prove it. If the answer is yes, give the irredundant parametric decomposition of I and the minimal monomial generating sequence of I.

(a) $\{X^2Y, Y^2Z, XZ^2, XYZ\}$
(b) $\{X^2Y, Y^2Z, XZ^2, XY^2Z\}$

Exercise 6.2.24. Set $R = A[X_1,\ldots,X_d]$ and $\mathfrak{X} = (X_1,\ldots,X_d)R$. Let I be a monomial ideal in R. Prove that if $I \subsetneq \mathfrak{X}$, then $\mathrm{C}_R(I) \subseteq \text{m-rad}\,(I)$.

Exercise 6.2.25. Set $R = A[X,Y,Z]$. Set

$$I = (X^2,Y,Z)R \cap (X,Y^2,Z)R \cap (X^3,Y,Z^2)R \cap (X,Y^2,Z^3)R \cap (X^2,Y^2,Z^2)R.$$

(a) Find a finite set $\{z_1,\ldots,z_5\}$ of monomials such that

$$I = \mathrm{P}_R(z_1) \cap \cdots \cap \mathrm{P}_R(z_5).$$

(b) Find an irredundant parametric decomposition for I and list the I-corner elements using:

(1) Algorithm 6.2.14.
(2) Proposition 6.2.15.

Justify your answers.

Exercise 6.2.26. Set $R = A[X_1, \ldots, X_d]$, and let J be a monomial ideal of R with $J \neq R$. Prove that $\text{m-rad}(J) \neq \mathfrak{X}$ if and only if there is a descending chain of monomial ideals $J_1 \supsetneq J_2 \supsetneq \cdots$ containing J. (The existence of such a descending chain says that the quotient ring R/J is not "artinian". The artinian property is dual to the noetherian property of Section 1.4.) (Hint: for the forward direction, there is a variable X_i such that $X_i^n \notin J$ for $n = 1, 2, \ldots$; let J' be as in Corollary 6.2.5, and set $J_n = J' \cap \mathrm{P}_R(X_i^n)$. For the converse, argue by contrapositive; assume that $\text{m-rad}(J) = \mathfrak{X}$ and use the finiteness of $[\![R]\!] \smallsetminus [\![J]\!]$.)

**Exercise 6.2.27.* Set $R = A[X_1, \ldots, X_d]$. Set $\mathfrak{X} = (X_1, \ldots, X_d)R$ and let J be a monomial ideal in R such that $J \subseteq \mathfrak{X}$. Let $z_1, \ldots, z_m$ be the distinct J-corner elements. Prove that $(J :_R \mathfrak{X}) = (z_1, \ldots, z_m)R + J$. (This exercise is used in the proof of Lemma 7.4.6.)

Exercise 6.2.28. Set $R = A[X_1, \ldots, X_d]$, and let $J \neq R$ be a monomial ideal in R. Consider the quotient $\overline{R} = R/J$, and set $x_i = X_i + J \in \overline{R}$ for $i = 1, \ldots, d$ and $\mathfrak{m} = (x_1, \ldots, x_d)\overline{R}$.

(a) Show that $\mathrm{C}_R(J) \neq \emptyset$ if and only if $(0 :_{\overline{R}} \mathfrak{m}) \neq 0$.
(b) Show that the set $\{f + J \in \overline{R} \mid f \in \mathrm{C}_R(J)\}$ is an irredundant generating sequence for the ideal $(0 :_{\overline{R}} \mathfrak{m})$.

If A is a field, then the ideal $(0 :_{\overline{R}} \mathfrak{m}) \subseteq \overline{R}$ is called the *socle* of $\overline{R}$, named after the architectural term for the block at the base of a pillar.

Corner Elements in Macaulay2

In this tutorial we verify several of the results proved in this section, using the `parameterIdeal` method from Section 6.1. We begin by defining a polynomial ring and a parameter ideal.

```
i11 : M = ideal vars R
o11 = ideal (x, y, z)
o11 : Ideal of R

i12 : I = parameterIdeal(x^4*y^2*z^5)
                       5   3   6
o12 = monomialIdeal (x , y , z )
o12 : MonomialIdeal of R
```

To compute the I-corner elements, we find all the monomials in $(I :_R \mathfrak{X}) \smallsetminus I$ using the next command.

```
i13 : flatten entries mingens ((I:M)/I)

        4 2 5
o13 = {x y z }

o13 : List
```

Comparing this with i12, we see that this verifies Corollary 6.2.12 in this case. Let us now examine a larger example by using Corollary 6.2.5 to produce an ideal with prescribed corner elements. First, we define the potential corners.

```
i14 : monList = {x^4*y^2*z^5, x^3*y*z^7, x^6*y^5*z}

        4 2 5   3   7   6 5
o14 = {x y z , x y*z , x y z}

o14 : List
```

Then we compute the parameter ideal associated to each one.

```
i15 : idealList = apply(monList, mon -> parameterIdeal mon);

i16 : netList pack(idealList / toString,2)

      +---------------------------+---------------------------+
o16 = |monomialIdeal (x^5,y^3,z^6)|monomialIdeal (x^4,y^2,z^8)|
      +---------------------------+---------------------------+
      |monomialIdeal (x^7,y^6,z^2)|                           |
      +---------------------------+---------------------------+
```

Note that we used the command toString here to convert each ideal to a string before presenting the output to the user. This has the advantage that the output of toString can be read into another Macaulay2 session, so that the output of lengthy computations can be used as input later.

Finally we intersect these parameter ideals, and find the corner elements.

```
i17 : J = intersect idealList

                      7   6   5 2   3 2   4 6   2 6   8
o17 = monomialIdeal (x , y , x z , y z , x z , y z , z )

o17 : MonomialIdeal of R

i18 : flatten entries mingens ((J:M)/J)

        3   7   4 2 5   6 5
o18 = {x y*z , x y z , x y z}

o18 : List
```

Compare this with o14.

```
i19 : exit
```

Exercises

Exercise 6.2.29. Use Macaulay2 to verify any unsubstantiated claims in the examples (numbered and unnumbered) in this section.

Exercise 6.2.30. Use Macaulay2 to verify that the examples you devised for Exercises 6.2.21 and 6.2.22 satisfy the desired properties.

Exercise 6.2.31. Use Macaulay2 to check your answers for Exercise 6.2.25.

6.3 Finding Corner Elements in Two Variables

In this section, A a non-zero commutative ring with identity and $R = A[X,Y]$.

Here we show how to find corner elements, and hence (according to Theorem 6.2.9) irredundant parametric decompositions, for monomial ideals in two variables. See also Sections 3.5 and 6.5. The idea here is to determine the corner elements for a monomial ideal in terms of its generators, using the lexicographical order from Definition A.9.8(a). Note that the case $n = 1$ is handled in Exercise 6.2.16.

Theorem 6.3.1 *Let J be a non-zero monomial ideal in R and let $f_1,\ldots,f_n \in [\![J]\!]$ be an irredundant monomial generating sequence for J with $n \geqslant 2$. For $i = 1,\ldots,n$ write $f_i = X^{a_i}Y^{b_i}$, and order the f_i's so that $a_i < a_j$ and $b_i > b_j$ whenever $i < j$. For $i = 1,\ldots,n-1$ set $z_i = X^{a_{i+1}-1}Y^{b_i-1}$. Then the monomials $z_1,\ldots,z_{n-1}$ are the distinct J-corner elements. Hence J has exactly $n-1$ corner elements.*

Proof. First, we note that the f_i's can be ordered as in the statement. Since the generating sequence $f_1,\ldots,f_n$ is irredundant, we have $f_i \neq f_j$ whenever $i \neq j$. It follows from Exercise A.9.13 that we have either $f_i <_{\text{lex}} f_j$ or $f_j <_{\text{lex}} f_i$ whenever $i \neq j$. Thus, we may always re-order the list $f_1,\ldots,f_n$ to assume that $f_1 <_{\text{lex}} f_2 <_{\text{lex}} \cdots <_{\text{lex}} f_n$. By Lemma 1.3.10, the inequality $f_i <_{\text{lex}} f_j$ for $i < j$ implies that $a_i < a_j$ and $b_i > b_j$ whenever $i < j$.

In particular, the inequalities $0 \leqslant a_1 < a_j$ for $j \geqslant 2$ imply that $1 \leqslant a_j$ when $j \geqslant 2$, and the inequalities $0 \leqslant b_n < b_i$ for $i < n$ imply that $1 \leqslant b_i$ when $i < n$.

Claim 1: For $i = 1,\ldots,n-1$ we have $z_i \notin J$. Suppose by way of contradiction that $z_i \in J$. Theorem 1.1.9 then implies that z_i is a monomial multiple of f_j for some j. Comparing exponents, we have $a_{i+1} - 1 \geqslant a_j$ and $b_i - 1 \geqslant b_j$. If $j \leqslant i$, then this implies the first inequality in the following sequence

$$b_i - 1 \geqslant b_j \geqslant b_i$$

while the second inequality is from our ordering of the f_i's; this is impossible. If $j > i$, then we have $j \geqslant i+1$, so similar reasoning explains the sequence

$$a_{i+1} - 1 \geqslant a_j \geqslant a_{i+1}$$

which is also impossible. This establishes the claim.

Claim 2: For $i=1,\dots,n-1$ we have $z_i \in \mathrm{C}_R(J)$. Since we have already seen that $z_i \notin J$, it suffices to show that $Xz_i, Yz_i \in J$. By construction, we have

$$\begin{aligned} Xz_i &= XX^{a_{i+1}-1}Y^{b_i-1} = X^{a_{i+1}}Y^{b_i-1} = X^{a_{i+1}}Y^{b_{i+1}}Y^{b_i-1-b_{i+1}} \\ &= f_{i+1}Y^{b_i-1-b_{i+1}} \in (f_{i+1})R \subseteq J. \end{aligned}$$

Note that the element $Y^{b_i-1-b_{i+1}}$ is a bona fide element of R because the condition $b_i > b_{i+1}$ implies that $b_i - 1 - b_{i+1} \geqslant 0$. Similarly, we have

$$\begin{aligned} Yz_i &= YX^{a_{i+1}-1}Y^{b_i-1} = X^{a_{i+1}-1}Y^{b_i} = X^{a_i}Y^{b_i}X^{a_{i+1}-1-a_i} \\ &= f_iX^{a_{i+1}-1-a_i} \in (f_i)R \subseteq J. \end{aligned}$$

This establishes the claim.

Claim 3: For indices i,j such that $i \neq j$ we have $z_i \neq z_j$. This comes from a direct comparison of X-exponents when $i < j$: the inequality $a_i < a_j$ implies that $a_i - 1 < a_j - 1$, so $z_i <_{\text{lex}} z_j$.

Claim 4: For each J-corner element $z \in \mathrm{C}_R(J)$, there is an index i such that $z = z_i$. (Once this claim is established, the proof will be complete.) Write $z = X^aY^b$. We have $z \notin J$ by assumption, and $Xz, Yz \in J$. Suppose by way of contradiction that $z \neq z_i$ for $i = 1,\dots,n-1$. Since the lexicographical order on $[\![R]\!]$ is a total order, we know that one of the following three cases must occur: $z <_{\text{lex}} z_1$ or $z_i <_{\text{lex}} z <_{\text{lex}} z_{i+1}$ for some $i = 1,\dots,n-1$ or $z_{n-1} <_{\text{lex}} z$.

Case 1: $z <_{\text{lex}} z_1$. By definition, this condition implies that either $(a < a_2 - 1)$ or $(a = a_2 - 1$ and $b < b_1 - 1)$.

If $a < a_2 - 1$, then $a+1 < a_2 \leqslant a_i$ for all $i \geqslant 2$. It follows that Xz is not a monomial multiple of f_i for all $i \geqslant 2$. Hence, the condition $Xz \in J$ implies that Xz is a monomial multiple of f_1. This implies that $a_1 \leqslant a+1$ and $b_1 \leqslant b$. However, since $z \notin J$, we know that z is not a monomial multiple of f_1, and so either $a_1 > a$ or $b_1 > b$. The condition $b_1 \leqslant b$ implies that $b_1 \not> b$, so we must have $a_1 > a$. Hence, we have $a < a_1 \leqslant a+1$, and so $a_1 = a+1$. This implies that $a = a_1 - 1 < a_1 \leqslant a_i$ for all $i \geqslant 1$. Comparing X-exponents, we conclude that Yz is not a monomial multiple of any f_i, so $Yz \notin J$, a contradiction.

It follows that we must have $a = a_2 - 1$ and $b < b_1 - 1$. In this case, the Y-exponent of Yz is $b+1 < b_1$ which shows that Yz is not a monomial multiple of f_1. The X-exponent of Yz is $a_2 - 1 < a_i$ for all $i \geqslant 2$, and this shows that Yz is not a monomial multiple of f_i for $i = 2,\dots,n$. This shows that $Yz \notin J$, a contradiction. Thus, the condition $z <_{\text{lex}} z_1$ is impossible.

The remaining cases similarly result in contradictions. Thus, the supposition $z \neq z_i$ for $i = 1,\dots,n-1$ is false. This establishes the claim and completes the proof. □

Example 6.3.2. We compute an irredundant parametric decomposition of the ideal $J = (X^6, X^4Y, X^3Y^2, Y^6)R$. The distinct J-corner elements are X^2Y^5, X^3Y, X^5, by Theorem 6.3.1. So, Theorem 6.2.9 yields

$$J = \mathrm{P}_R(X^2Y^5) \cap \mathrm{P}_R(X^3Y) \cap \mathrm{P}_R(X^5) = (X^3, Y^6)R \cap (X^4, Y^2)R \cap (X^6, Y)R.$$

Exercises

Exercise 6.3.3. Verify the list of corner elements in Example 6.3.2, using Theorem 6.3.1. Sketch the graph of this ideal, indicating the corner elements. Verify the decomposition

$$J = (X^3, Y^6)R \cap (X^4, Y^2)R \cap (X^6, Y)R$$

in this example using Theorem 2.1.5.

Exercise 6.3.4. Set $J = (Y^9, XY^7, X^3Y^4, X^5Y^2, X^{11})R$.

(a) Use Theorem 6.3.1 to find the corner elements of J.
(b) Sketch the graph of J and check that your answer from part (a) agrees with the graph.
(c) Use Theorem 6.2.9 to find an irredundant parametric decomposition of J.
(d) Verify that your decomposition from part (c) is correct using Theorem 2.1.5.

Justify your answers.

Exercise 6.3.5. Finish the proof of Theorem 6.3.1 by deriving contradictions in the remaining cases.

Finding Corner Elements in Two Variables in Macaulay2

In this section, we write a method that computes the J-corner elements of the monomial ideal $J = (f_1, \ldots, f_n)$ in two variables where $n \geqslant 2$ via Theorem 6.3.1. In order to use the technique in this result, we must be working in a ring with the monomial order lex. However, we would like this method to be robust enough so that the monomial ideal passed as input does not need to be defined in a ring with lex order. So, in our method, we first define a new ring used only in the method that uses lex order, place the input ideal in this new ring, perform the computation there, and bring the answer back to the original ring. The subtle point in this method will be that it is considered bad form to allow a ring to 'escape' a method, so we have to take steps to prevent this from happening.

First, let us create a ring and a monomial ideal.

```
i1 : R = QQ[x,y];

i2 : I = monomialIdeal {x^5,x^4*y^4,x^3*y^6,y^8}
                         5   4 4   3 6   8
o2 = monomialIdeal (x , x y , x y , y )

o2 : MonomialIdeal of R
```

As you can see, if we sort the generators of I, the monomials are listed in increasing grevlex order, not lex order; see the Macaulay2 tutorials in Sections 3.5, 5.4 and B.9.

```
i3 : sort I_*
       5   8   4 4   3 6
o3 = {x , y , x y , x y }
o3 : List
```

The following is a method function that implements the algorithm in Theorem 6.3.1. A description of the commands will follow.

```
i4 : cornerElements = method()
o4 = cornerElements
o4 : MethodFunction

i5 : cornerElements MonomialIdeal := J -> (
R := ring J;
if (numgens R != 2) then error "Expected a MonomialIdeal
in a ring with two variables.";
xx := symbol xx;
tempRing := QQ[xx_1,xx_2,MonomialOrder=>Lex];
phi := map(tempRing,R, gens tempRing);
newJGens := (phi J)_*;
newJGens = sort newJGens;
exps := newJGens / exponents // flatten;
cornerExps := apply(#exps-1, i -> {exps#(i+1)#0 - 1,exps#i#1 - 1});
monomialIdeal apply(cornerExps, e -> R_e)
)
o5 = {*Function[stdio:5:35-16:40]*}
o5 : FunctionClosure
```

In the method, we first check that the monomial ideal J is in a ring with only two variables. Next, we create the temporary symbol `xx` to use as the base of a set of indexed variables, which we use in the definition of the lexicographic ring `tempRing` on the following line.

Next, we bring J to `tempRing`. The way to do this is to use the command `map` to define a *ring homomorphism* `phi` from R to `tempRing` that sends the ith variable of R to $\mathtt{xx}_i$. In this case, the ring homomorphism is nothing more than a relabeling of the variables but this command can create more general homomorphisms as well. Next, we obtain the generators of the image of J in `tempRing` under `phi`, sort them, and obtain the exponent vectors of the monomials in this sorted list. On the second to last line, we use Theorem 6.3.1 to compute the corner elements and then finally use the command `R_e` to obtain the monomial of R corresponding to each exponent vector `e` in `cornerExps`.

Let us test our new method.

```
i6 : cornersI = cornerElements I
                    4 3   3 5   2 7
```

```
o6 = monomialIdeal (x y , x y , x y )

o6 : MonomialIdeal of R
```

As in the previous section, we can check our work using colon ideals and the following command.

```
i7 : flatten entries mingens ((I:ideal vars R) / I)

        4 3   3 5   2 7
o7 = {x y , x y , x y }

o7 : List

i8 : exit
```

Exercises

Exercise 6.3.6. Consider the ideal J from Example 6.3.2.

(a) Use the two methods from the above tutorial to verify the J-corner elements.
(b) Use the technique of Section 6.2 to find a parametric decomposition of J.
(c) Verify that the command `irreducibleDecomposition` gives the same decomposition as in part (b).
(d) Repeat parts (a)–(c) for the ideal of Exercise 6.3.3.

Laboratory Exercise 6.3.7. Consider the ideals J and K from Example 6.2.8.

(a) Use Macaulay2 to find the corner elements and irredundant parametric decomposition for the ideal $J+(X^a,Y^a)R$ for $a=6,\dots,10$. Compare this with the corner elements and irredundant m-irreducible decomposition of J.
(b) Repeat part (a) for the ideal K.
(c) Combine your answers to parts (a)–(b) to make a guess for a decomposition of an arbitrary monomial ideal I in terms of parametric decompositions of $I+(X^a,Y^a)R$. Use Macaulay2 to check your guess for some examples where $\text{m-rad}(I) \subsetneq (X,Y)R$. Revise your formula and re-experiment as necessary until you come to a conjecture describing a decomposition of I.

6.4 Finding Corner Elements in General

In this section, A is a non-zero commutative ring with identity.

The algorithm from the previous section does not generalize easily to the case of three or more variables. However, the next result covers the case of any number of variables, when $\text{m-rad}(I) = \mathfrak{X}$.

Proposition 6.4.1 *Set $R = A[X_1,\dots,X_d]$. Set $\mathfrak{X} = (X_1,\dots,X_d)R$, and let I be a monomial ideal in R such that* m-rad$(I) = \mathfrak{X}$. *Set $S = [\![R]\!] \smallsetminus [\![I]\!]$, the set of monomials in R that are not in I, and set $w = \max\{\deg(f) \mid f \in S\}$. For $j = 0,\dots,w$ set $D_j = \{f \in S \mid \deg(f) = j\}$. For $j = 0,\dots,w-1$ set*

$$C_j = \{f \in D_j \mid \textit{for } i = 1,\dots,d \textit{ we have } X_i f \notin D_{j+1}\}$$

and set $C_w = D_w$. Then $\mathrm{C}_R(I)$ is the disjoint union $\mathrm{C}_R(I) = \bigcup_{j=0}^{w} C_j$.

Proof. Note that $\mathrm{C}_R(I) \neq \emptyset$ by Proposition 6.2.7. Also, the set S is finite by Exercise 2.3.16. In particular, the number w is well-defined as it is the maximum element of a finite set of natural numbers. Also, for $i \neq j$ we have $C_i \cap C_j = \emptyset$ because the elements of C_i have different degrees from the elements of C_j.

Claim 1: $\bigcup_{j=0}^{w} C_j \subseteq \mathrm{C}_R(I)$. To verify this containment, we fix a monomial $f \in C_j$ for some j and show that $f \in \mathrm{C}_R(I)$. The containments $C_j \subseteq D_j \subseteq S \subseteq R \smallsetminus I$ show that $f \notin I$. Since $\deg(f) = j$ by definition, we have $\deg(X_i f) = j+1$ for $i = 1,\dots,d$.

If $j = w$, then $f \in C_w = D_w$, and so the condition

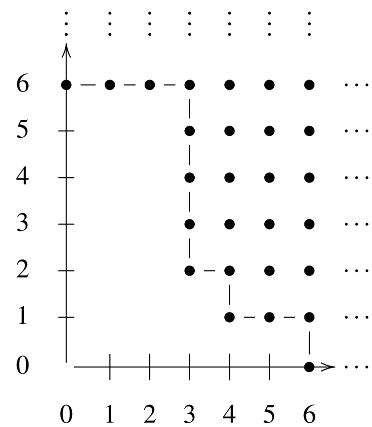
$$\deg(X_i f) = j+1 = w+1 > w = \max\{\deg(g) \mid g \in S\}$$

implies that $X_i f \notin S$ for $i = 1,\dots,d$. On the other hand, if $j < w$, then $X_i f \notin D_{j+1}$ by definition of C_j, so the definition of D_{j+1} implies that $X_i f \notin S$. In either case, the elements $X_1 f,\dots,X_d f$ are monomials of R that are not in $S = [\![R]\!] \smallsetminus [\![I]\!]$. It follows that $X_1 f,\dots,X_d f \in I$ and so $f \in \mathrm{C}_R(I)$. This establishes the claim.

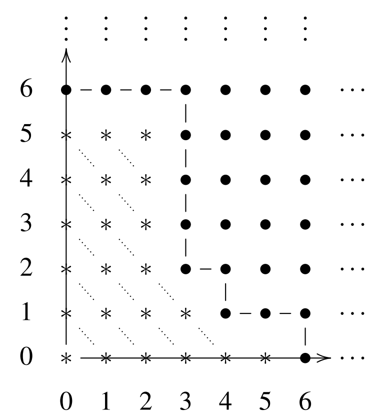
Claim 2: $\bigcup_{j=0}^{w} C_j \supseteq \mathrm{C}_R(I)$. To verify this containment, we fix a monomial $g \in \mathrm{C}_R(I)$ and set $j = \deg(g)$; we show that $g \in C_j$. The assumption $g \in \mathrm{C}_R(I)$ implies that g is a monomial in R that is not in I, and so $g \in S$. By definition, we have $g \in D_j$. When $j = w$, this implies $g \in C_j$, so we assume that $j < w$. For $i = 1,\dots,d$ we have $X_i g \in I$ and so $X_i g \notin S$. It follows that $X_i g$ cannot be in D_{j+1}. This shows that $g \in C_j$, thus completing the proof of the claim and the proof of the proposition. □

Here is an example of Proposition 6.4.1 in action.

Example 6.4.2. Set $R = A[X,Y]$. Then the ideal $J = (X^6, X^4Y, X^3Y^2, Y^6)R$ has the following graph.



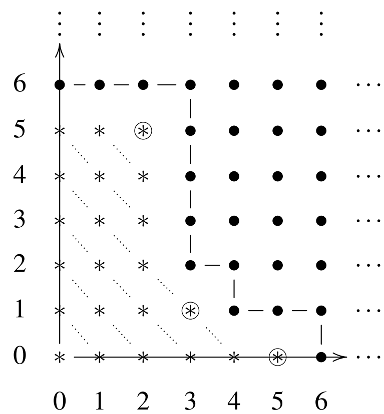
The monomials in the set S are designated with $*$'s, and the elements of D_j are the represented by the $*$'s on the diagonal line of slope -1 and Y-intercept j.



In this example, the set S contains 22 monomials, and the largest degree occurring is $w = 7$. The sets D_j are

$$D_0 = \{1\} \qquad D_1 = \{X, Y\}$$
$$D_2 = \{X^2, XY, Y^2\} \qquad D_3 = \{X^3, X^2Y, XY^2, Y^3\}$$
$$D_4 = \{X^4, X^3Y, X^2Y^2, XY^3, Y^4\} \qquad D_5 = \{X^5, X^2Y^3, XY^4, Y^5\}$$
$$D_6 = \{X^2Y^4, XY^5\} \qquad D_7 = \{X^2Y^5\}$$

so $C_7 = D_7 = \{X^2Y^5\}$. For $j < w = 7$, the elements of C_j are the monomials of degree j represented by $*$'s such that (a) the point one unit to the right is a $\bullet$, and (b) the point one unit up is a $\bullet$. Such points are designated in the next graph as $\circledast$'s



and so we have

$$\begin{array}{llll} C_0 = \emptyset & C_1 = \emptyset & C_2 = \emptyset & C_3 = \emptyset \\ C_4 = \{X^3Y\} & C_5 = \{X^5\} & C_6 = \emptyset & C_7 = \{X^2Y^5\}. \end{array}$$

It follows that $\mathrm{C}_R(J) = \{X^3Y, X^5, X^2Y^5\}$, so Theorem 6.2.9 yields the following irredundant m-irreducible decomposition

$$J = \mathrm{P}_R(X^3Y) \cap \mathrm{P}_R(X^5) \cap \mathrm{P}_R(X^2Y^5) = (X^4, Y^2)R \cap (X^6, Y)R \cap (X^3, Y^6)R.$$

While this gives us a longer algorithm for finding corner elements in the case of two variables, as compared to the previous section, it also works in more variables.

Exercises

Exercise 6.4.3. Verify the decomposition

$$J = (X^4, Y^2)R \cap (X^6, Y)R \cap (X^3, Y^6)R$$

from Example 6.4.2 using Theorem 2.1.5.

Exercise 6.4.4. Set $R = A[X, Y, Z]$ and

$$J = (Z^4, Y^2Z^3, Y^3, XYZ, XY^2, X^2)R.$$

Use Proposition 6.4.1 to find the J-corner-elements. State the value of w and list the elements in each C_i and D_i. Use Theorem 6.2.9 to find an irredundant parametric decomposition of J. Justify your answers.

Exercise 6.4.5. Set $R = A[U, X, Y, Z]$. Set

$$J = (Z^5, YZ^4, Y^2Z^2, Y^3, XZ^2, XYZ, X^3Z, X^3Y^2, X^4, U)R.$$

Use Proposition 6.4.1 to find the J-corner-elements. State the value of w and list the elements in each C_i and D_i. Use Theorem 6.2.9 to find an irredundant parametric decomposition of J. Justify your answers.

Challenge Exercise 6.4.6. Set $R = A[U, X, Y, Z]$. Set

$$J = (Z^d, Y^c, X^b, UXYZ, U^a)R$$

where a, b, c, and d are integers that are greater than 1. Use Proposition 6.4.1 to find the J-corner-elements. State the value of w and list the elements in each C_i and D_i. Use Theorem 6.2.9 to find an irredundant parametric decomposition of J. Justify your answers. (See also Laboratory Exercise 6.4.13.)

Finding Corner Elements in General in Macaulay2

In this section, we introduce some commands that aid in implementing the algorithm in Theorem 6.4.1. First, let's generate a monomial ideal in three variables.

```
i1 : R = QQ[x,y,z];

i2 : I = monomialIdeal {x^4,x^3*y^3*z,x^2*y^2*z^2,x*y^2*z^3,y*z^4,y^5,z^5}
                         4   5   3 3    2 2 2     2 3     4   5
o2 = monomialIdeal (x , y , x y z, x y z , x*y z , y*z , z )
o2 : MonomialIdeal of R
```

Since $x^4, y^5, z^5 \in I$, we have that $\text{m-rad}(I) = \mathfrak{X}$, so the theorem applies.

As part of the algorithm, we must find the complete list of all monomials of R that are not in I, and collect them by degree. To know what degrees are necessary, one can use the Hilbert series of the quotient ring R/I; see the end of Section 5.4. To perform this computation, we first define the quotient R/I.

```
i3 : S = R/I
o3 = S
o3 : QuotientRing
```

We can ask Macaulay2 for the Hilbert series of S using the next command.

```
i4 : hilbertSeries S
          4     5     6     7      8      9     10
     1 - T  - 3T  - T  + T  + 3T  + 6T  - 6T
o4 = -----------------------------------------
                               3
                        (1 - T)
o4 : Expression of class Divide
```

However, this looks like a rational function rather than a polynomial. This is because, by default, Macaulay2 always expresses the Hilbert Series of R/I as a rational function with denominator $(1-t)^n$, where n is the number of variables of R. To reduce this rational function to lowest terms, we use the following command.

```
i5 : reduceHilbert hilbertSeries S
                 2      3      4      5      6     7
     1 + 3T + 6T  + 10T  + 14T  + 15T  + 12T  + 6T
o5 = --------------------------------------------
                               1

o5 : Expression of class Divide
```

Now it is a rational function with constant denominator, i.e. a polynomial. Thus, we can obtain the numerator via the next command.

```
i6 : hilbS = numerator reduceHilbert hilbertSeries S
                 2      3      4      5      6     7
o6 = 1 + 3T + 6T  + 10T  + 14T  + 15T  + 12T  + 6T

o6 : ZZ[T]
```

This says that there is 1 monomial of degree 0 not in I, 3 monomials of degree 1 not in I, 6 monomials of degree 2 not in I, and so on.

For each power of T appearing in the Hilbert series, we can ask Macaulay2 for a list of the monomials of R not in I in that degree. This is done using the `basis` command, for example.

```
i7 : basis(6,S)

o7 = | x3y3 x3y2z x3yz2 x3z3 x2y4 x2y3z x2yz3 x2z4 xy4z xy3z2 y4z2 y3z3 |

             1       12
o7 : Matrix S  <--- S
```

In total, there are only finitely many monomials of R not in I, and we can also ask Macaulay2 for all these monomials at once with the next command. We again utilize `netList` and `pack` to save room.

```
i8 : basisS = basis(S);

             1       67
o8 : Matrix S  <--- S

i9 : netList pack(flatten entries basisS, 10)

     +------+------+------+-----+------+------+-----+------+-----+------+
     |      |      | 2    | 3   | 3    | 3 2  | 3 3 | 3 4  | 3 2 | 3    |
o9 = |1     |x     |x     |x    |x y   |x y   |x y  |x y   |x y z|x y*z |
     +------+------+------+-----+------+------+-----+------+-----+------+
     | 3   2| 3   3| 3    | 3 2 | 3 3  | 3 4  | 2   | 2 2  | 2 3 | 2 4  |
     |x y*z |x y*z |x z   |x z  |x z   |x z   |x y  |x y   |x y  |x y   |
     +------+------+------+-----+------+------+-----+------+-----+------+
     | 2 4  | 2 3  | 2 2  | 2   | 2   2| 2   3| 2   | 2 2  | 2 3 | 2 4  |
     |x y z |x y z |x y z |x y*z|x y*z |x y*z |x z  |x z   |x z  |x z   |
     +------+------+------+-----+------+------+-----+------+-----+------+
     |      |   2  |   3  |   4 |   4  |   4 2|   3 |   3 2|   2 |   2 2|
     |x*y   |x*y   |x*y   |x*y  |x*y z |x*y z |x*y z|x*y z |x*y z|x*y z |
     +------+------+------+-----+------+------+-----+------+-----+------+
```

```
|      |     2|     3|     |   2  |   3  |   4 |      | 2   | 3    |
|x*y*z |x*y*z |x*y*z |x*z  |x*z   |x*z   |x*z  |y     |y    |y     |
+------+------+------+-----+------+------+-----+------+-----+------+
| 4    | 4    | 4 2  | 4 3 | 3    | 3 2  | 3 3 | 2    | 2 2 | 2 3  |
|y     |y z   |y z   |y z  |y z   |y z   |y z  |y z   |y z  |y z   |
+------+------+------+-----+------+------+-----+------+-----+------+
|      |   2  |   3  |     | 2    | 3    | 4   |      |     |      |
|y*z   |y*z   |y*z   |z    |z     |z     |z    |      |     |      |
+------+------+------+-----+------+------+-----+------+-----+------+
```

Note, however, that the `basis` command should only be called in this manner when the user *knows* that I is a parameter ideal, otherwise the function will return an error.

Other functions that you may find useful for implementing Theorem 6.4.1 include `all` or `any` as well as `member`, which is used to determine if an object appears as an element of a list or set. If any of these commands are not clear, use the Macaulay2 documentation to learn more about them.

```
i10 : exit
```

Exercises

Exercise 6.4.7. For R and I as in the code example above, use the `basis` and `hilbertSeries` commands to create a list D whose ith element is the list of monomials of R not in I of degree i.

Exercise 6.4.8. Use the list D from the previous exercise to define a list C whose ith element is the set of monomials f of `D#i` such that xf is not in `D#(i+1)` for all variables x of R.

Exercise 6.4.9. What function can you use to turn the list of lists C from the previous exercise into a single list that is the union of all the lists in C? Find the list of the corner elements of I using this function.

Exercise 6.4.10. Check that your answer to Exercise 6.4.9 is correct by computing the corner elements of I using the command:

```
flatten entries (mingens ((I:ideal vars R) / I)).
```

Coding Exercise 6.4.11. Write a method that uses ideas from Exercises 6.4.7–6.4.9 to compute the corner elements of a monomial ideal I with $\text{m-rad}(I) = \mathfrak{X}$ in general, in any number of variables.

Exercise 6.4.12. Consider the ideal J from Example 6.4.2.

(a) Apply the method from Coding Exercise 6.4.11 to find the J-corner elements.
(b) Use the command from Exercise 6.4.10 to check your answer for part (a).

(c) Use your answer to part (a) to find an irredundant parametric decomposition for the ideal J.
(d) Repeat parts (a)–(c) for the ideals in Exercises 6.4.3–6.4.5.
(e) Use the ideas from this exercise to explore Challenge Exercise 6.4.6.

Laboratory Exercise 6.4.13. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate the J-corner elements for some ideals of the form $J = (Z^d, Y^c, X^b, UXYZ, U^a)R$, to help with Challenge Exercise 6.4.6

6.5 Exploration: Decompositions in Two Variables II

In this section, A is a non-zero commutative ring with identity, and $R = A[X,Y]$.

This section outlines how to find m-irreducible decompositions for arbitrary monomial ideals in two variables, using the parametric decompositions from Section 6.3. Contrast this with Section 3.5 where we found parametric decompositions in two variables, without explicitly using corner elements. We begin with the case of principal ideals, which have no corner elements, then move to the general case.

Exercise 6.5.1. Fix a monomial $f = X^aY^b \in [\![(X,Y)R]\!]$.

(a) Prove that $(f)R = (X^a)R \cap (Y^b)R$. This is an m-irreducible decomposition of the ideal $(f)R$.
(b) Prove that if $a, b \geqslant 1$, then the decomposition from part (a) is irredundant.
(c) Prove that if $a = 0$ or $b = 0$, then $(f)R$ is m-irreducible, so the trivial intersection $(f)R$ is an irredundant m-irreducible decomposition of $(f)R$.

The point in part (c) of the next result is the following. Each non-principal monomial ideal in two variables has a corner element, so one can construct a parametric decomposition of some ideal using those corners, but as we have seen this will not in general be a decomposition of the original ideal. Graphically, the problem is that one may have a gap between the ideal and the X- and Y-axes. Part (c) of the next result shows how to modify that parametric decomposition to obtain an m-irreducible decomposition of the original ideal. Graphically, one takes the parametric decomposition and carves out the gaps between the axes. Part (a) of the exercise asks you to explore this graphically.

Challenge Exercise 6.5.2. Let J be a monomial ideal of R such that $0 \neq J \neq R$, and let $f_1, \ldots, f_n \in [\![J]\!]$ be an irredundant monomial generating sequence for J. Assume that $n \geqslant 2$ and $f_1 <_{\text{lex}} f_2 <_{\text{lex}} \cdots <_{\text{lex}} f_n$. For $i = 1, \ldots, n$ write $f_i = X^{a_i}Y^{b_i}$. For $i = 1, \ldots, n-1$ set $z_i = X^{a_{i+1}-1}Y^{b_i-1}$; see Theorem 6.3.1.

(a) Compare the graphs of the ideals J and $\bigcap_{i=1}^{n-1} \mathrm{P}_R(z_i)$ and $(X^{a_1})R \cap (Y^{b_n})R$ in some special cases.

(b) Use part (a) to make a conjecture about an irredundant monomial generating sequence for $\bigcap_{i=1}^{n-1} \mathrm{P}_R(z_i)$. Prove your conjecture. (If you need some help, see Section 3.5.)
(c) Prove that $J = (X^{a_1})R \cap (Y^{b_n})R \cap \mathrm{P}_R(z_1) \cap \cdots \cap \mathrm{P}_R(z_{n-1})$. This is an m-irreducible decomposition of J. (Note that this is not in general a parametric decomposition of J, contrary to the theme of this chapter.)
(d) Prove that if $a_1, b_n \geqslant 1$, then the decomposition from part (c) is irredundant.
(e) If $a_1 = 0$ or $b_n = 0$, find an irredundant m-irreducible decomposition of J. Justify your answer.

(See also Laboratory Exercise 6.5.5.)

Decompositions in Two Variables II in Macaulay2, Exercises

See the tutorials in Sections 3.3, 6.1, and 6.3 for helpful reminders for this section.

Coding Exercise 6.5.3. Write a method that implements the technique from Exercise 6.5.2 to compute an irredundant m-irreducible decomposition of any monomial ideal in two variables. Test your method on some examples, comparing your output and run time with the command `irreducibleDecomposition`.

Laboratory Exercise 6.5.4. Set $R = \mathbb{Q}[X,Y]$ and $I = (X^5, X^4Y^2, X^2Y^3, Y^6)R$.

(a) Use any technique in Macaulay2 to explore the corners and decompositions of the ideals of the form X^aY^bI with $a, b \in \mathbb{N}$.
(b) Use the data you collect in part (a) to make a conjecture about the corners and decompositions of the ideals from part (a).
(c) Prove your conjecture from part (b).
(d) State and prove a similar conjecture for arbitrary monomial ideals of the form fJ, in any number of variables.

Laboratory Exercise 6.5.5. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate Challenge Exercise 6.5.2.

6.6 Exploration: Decompositions of Powers of Ideals

In this section, A is a non-zero commutative ring with identity. Set $R = A[X_1, \ldots, X_d]$ and $\mathfrak{X} = (X_1, \ldots, X_d)R$.

The point of this section is to look at some patterns in decompositions of powers of ideals, beginning with the ideals $\mathfrak{X}^n$. See also Section 7.9.

**Exercise 6.6.1.* Let n be a positive integer.

(a) Prove that the ideal $\mathfrak{X}^n$ has a parametric decomposition.
(b) Prove that the $\mathfrak{X}^n$-corner elements are exactly the monomials of degree $n-1$, so we have the irredundant parametric decomposition $\mathfrak{X}^n = \bigcap_{\deg(f)=n-1} \mathrm{P}_R(f)$.
(c) Given a monomial ideal I of R, let $\mathrm{c}_R(I)$ denote the number of I-corner elements: $\mathrm{c}_R(I) = |\mathrm{C}_R(I)|$. Use Exercise 1.6.3 to find a formula for $\mathrm{c}_R(\mathfrak{X}^n)$ as a function of n. Justify your answer. Prove that $\mathrm{c}_R(\mathfrak{X}^n)$ is a polynomial in n of degree $d-1$ with leading coefficient $1/(d-1)!$.

(This exercise is used in Example 7.5.10.)

Exercise 6.6.2. Set $R = A[X,Y,Z]$.

(a) Use Exercise 6.6.1 to find an irredundant parametric decomposition of the ideal $((X,Y,Z)R)^4$. Justify your answer.
(b) Verify that your decomposition from part (a) is correct using Theorem 2.1.5.

Next, we have a souped-up version of Exercise 6.6.1. It gives a similar decomposition for powers of more general ideals, specifically, powers of ideals generated by some of the variables.

Exercise 6.6.3. Fix positive integers k and n with $n \leqslant d$, and set $I = (X_1,\ldots,X_n)R$. For each $\underline{e} = (e_1,\ldots,e_n) \in \mathbb{N}^n$, set $I_{\underline{e}} = (X_1^{e_1},\ldots,X_n^{e_n})R$. The goal of this exercise is to verify the irredundant m-irreducible decomposition

$$I^k = \bigcap_{e_1+\cdots+e_n=k+n-1} (X_1^{e_1},\ldots,X_n^{e_n})R$$

where the intersection runs over all sequences $e_1,\ldots,e_n$ of positive integers such that $e_1+\cdots+e_n = k+n-1$. (Note that this is not a parametric decomposition when $n<d$. However, it has the same shape as the parametric decomposition from Exercise 6.6.1.)

(a) Prove that $I^k \subseteq \bigcap_{|\underline{e}|=k+n-1} I_{\underline{e}}$. (Hint: The ideal I^k is generated by monomials of the form $f = X_1^{m_1}\cdots X_n^{m_n}$ such that $m_1+\cdots+m_n = k$. Suppose that $f \notin I_{\underline{e}}$. Conclude that $e_i > m_i$ for all $i = 1,\ldots,n$, and use this to deduce that $k+n-1 = \sum_{i=1}^n e_i \geqslant k+n$, a contradiction.)
(b) Next, verify the containment $I^k \supseteq \bigcap_{|\underline{e}|=k+n-1} I_{\underline{e}}$. (Hint: Consider the ring $R' = A[X_1,\ldots,X_n] \subseteq R$ and the ideal $I' = (X_1,\ldots,X_n)R'$. For each n-tuple $\underline{e}$ such that $|\underline{e}| = k+n-1$, set $I'_{\underline{e}} = (X_1^{e_1},\ldots,X_n^{e_n})R'$. Let p be the number of n-tuples $\underline{e}$ such that $|\underline{e}| = k+n-1$. Using an induction argument based on Theorem 2.1.5, show that the ideals $\bigcap_{|\underline{e}|=k+n-1} I_{\underline{e}}$ and $\bigcap_{|\underline{e}|=k+n-1} I'_{\underline{e}}$ have the same generating sets, namely, the set of all monomials of the form $\mathrm{lcm}(f_1,\ldots,f_p)$ where each f_j is a generator of I_j; see Exercise 2.1.15. Use Exercise 6.6.1 to show that each of these generators is in $(I')^k \subseteq I^k$.)
(c) Prove that the decomposition $I^k \subseteq \bigcap_{|\underline{e}|=k+n-1} I_{\underline{e}}$ is irredundant. (Hint: By way of contradiction, suppose that the intersection is redundant. Conclude that there are n-tuples $\underline{e} \neq \underline{e}'$ such that $I_{\underline{e}} \subseteq I_{\underline{e}'}$. In particular, each generator of $I_{\underline{e}}$ is in $I_{\underline{e}'}$.

Compare exponent vectors to show that $e_i \geqslant e_i'$ for $i = 1, \ldots, n$. Use the assumption $\underline{e} \neq \underline{e}'$ to conclude that $e_i > e_i'$ for some i. Deduce that $|\underline{e}| > |\underline{e}'|$, and derive a contradiction.)

(d) Fix positive integers $t_1, \ldots, t_n$ and set $J = (X_{t_1}, \ldots, X_{t_n})R$. Observe, by re-ordering the variables in R, that we have the irredundant m-irreducible decomposition

$$J^k = \bigcap_{e_1+\cdots+e_n=k+n-1} (X_{t_1}^{e_1}, \ldots, X_{t_n}^{e_n})R.$$

(e) Find a formula for the number $f(k)$ of ideals in the irredundant m-irreducible decomposition of J^k. Justify your answer. Prove that $f(k)$ is a polynomial in k of degree $n-1$ with leading coefficient $1/(n-1)!$.

Next, we extend Exercise 6.6.1 in a different direction: whereas that exercise deals with powers of the specific parameter ideal $\mathfrak{X} = \mathrm{P}_R(1)$, this one deals with powers of arbitrary parameter ideals.

Exercise 6.6.4. Set $R = A[X_1, \ldots, X_d]$, and let $f = \underline{X}^{\underline{a}} \in [\![R]\!]$.

(a) Prove that for each integer $n \geqslant 1$, we have

$$\mathrm{P}_R(f)^n = \bigcap_{|\underline{m}|=n-1} \mathrm{P}_R(X_1^{a_1m_1+a_1+m_1} \cdots X_d^{a_dm_d+a_d+m_d})$$

where the intersection runs over all d-tuples $\underline{m} \in \mathbb{N}^d$ such that $|\underline{m}| = n-1$.

(b) Prove that this intersection is irredundant.

(c) Set $R = A[X,Y]$ and $I = (X^2, Y^3)R$. Use parts (a)–(b) to find an irredundant m-irreducible decomposition of the ideal I^3. Justify your answer.

(d) Verify that your decomposition from part (c) is correct using Theorem 2.1.5.

Next, we have a hybrid of the previous two exercises involving ideals generated by pure powers of some (but not necessarily all) of the variables.

Challenge Exercise 6.6.5. Fix positive integers $n, t_1, \ldots, t_n, e_1, \ldots, e_n$, and set $R = A[X_1, \ldots, X_d]$ and $I = (X_{t_1}^{e_1}, \ldots, X_{t_n}^{e_n})R$.

(a) Describe an irredundant m-irreducible decomposition of I^k. Justify your answer. (Hint: Mimic the proof of Exercise 6.6.3, using Exercise 6.6.4 in place of Exercise 6.6.1.)

(b) Set $R = A[X,Y,Z]$ and $I = (X^2, Z^3)R$. Use part (a) to find an irredundant m-irreducible decomposition of the ideal I^3. Justify your answer.

(c) Verify that your decomposition from part (b) is correct using Theorem 2.1.5.

(See also Laboratory Exercise 6.6.7.)

Decompositions of Powers of Ideals in Macaulay2, Exercises

See the tutorials in Sections B.3, B.5, and 6.1 for helpful reminders for this section.

Exercise 6.6.6. Use Macaulay2 to verify the conclusions of the above exercises for some examples.

Laboratory Exercise 6.6.7. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate I^k for some ideals of the form $I = (X_{t_1}^{e_1}, \ldots, X_{t_n}^{e_n})R$, to help with Challenge Exercise 6.6.5.

Challenge Exercise 6.6.8. Set $R = \mathbb{Q}[X,Y]$ and $I = \mathrm{P}_R(XY^2) \cap \mathrm{P}_R(X^2Y)$.

(a) Use any technique in Macaulay2 to explore the corners and decompositions of the ideals I^n with $n \in \mathbb{N}$.
(b) Use the data you collect in part (a) to make a conjecture about the corners and decompositions of the ideals from part (a).
(c) Prove your conjecture from part (b).
(d) Repeat parts (a)–(c) for other ideals in R of the form $J = \mathrm{P}_R(f) \cap \mathrm{P}_R(g)$.
(e) Use the results of part (d) to make and prove a conjecture about the corners and decompositions of powers of arbitrary ideals in R of the form $J = \mathrm{P}_R(f) \cap \mathrm{P}_R(g)$.
(f) Repeat the above for intersections of three or more parameter ideals in two variables and/or two or more parameter ideals in d variables.

6.7 Exploration: Macaulay Inverse Systems

In this section, A is a field of characteristic 0, e.g., $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, and $R = A[X,Y]$.

This section outlines how two parametric decompositions are related to the following formal differential operators.

Definition 6.7.1. Define the operators ∂_X and ∂_Y on monomials in R as follows: for $a,b \in \mathbb{N}$, set

$$\partial_X(X^aY^b) = aX^{a-1}Y^b \qquad\qquad \partial_Y(X^aY^b) = bX^aY^{b-1}.$$

Note that ∂_X and ∂_Y are the usual partial derivatives from multi-variable calculus. For instance, these satisfy $\partial_X(Y^b) = 0 = \partial_Y(X^a)$.

Next, for $i,j \in \mathbb{N}$, set

$$\begin{aligned}\partial_X^i(X^aY^b) &= a(a-1)\cdots(a-i+1)X^{a-i}Y^b\\ \partial_Y^j(X^aY^b) &= b(b-1)\cdots(b-j+1)X^aY^{b-j}.\end{aligned}$$

More generally, given a polynomial $f \in R$, write $f = \sum_{a,b\in\mathbb{N}}^{\text{finite}} r_{a,b}X^aY^b$ and set

$$\begin{aligned}\partial_X^i(f) &= \partial_X^i\left(\sum_{a,b\in\mathbb{N}}^{\text{finite}} r_{a,b}X^aY^b\right) = \sum_{a,b\in\mathbb{N}}^{\text{finite}} r_{a,b}\partial_X^i(X^aY^b)\\ \partial_Y^j(f) &= \partial_Y^j\left(\sum_{a,b\in\mathbb{N}}^{\text{finite}} r_{a,b}X^aY^b\right) = \sum_{a,b\in\mathbb{N}}^{\text{finite}} r_{a,b}\partial_Y^j(X^aY^b).\end{aligned}$$

As an example, we have

$$\partial_Y^2(X^3+X^2Y+XY^2+Y^3)=2X+6Y.$$

Also, ∂_X^i and ∂_Y^j are linear transformations $R \to R$, such that $\partial_X^0=\partial_Y^0$ is the identity transormation $R \to R$.

One point of this section is to show how to endow a certain set of differential operators with the structure of a commutative ring with identity. The next exercise starts our work on this task.

Exercise 6.7.2. For $i,j,p,q \in \mathbb{N}$ prove that

$$\partial_X^i = \underbrace{\partial_X \circ \cdots \circ \partial_X}_{i \text{ factors}} \qquad \partial_X^i \circ \partial_X^p = \partial_X^{i+p} = \partial_X^p \circ \partial_X^i$$

$$\partial_Y^j = \underbrace{\partial_Y \circ \cdots \circ \partial_Y}_{j \text{ factors}} \qquad \partial_Y^j \circ \partial_Y^q = \partial_Y^{j+q} = \partial_Y^q \circ \partial_Y^j$$

$$\partial_X^i \circ \partial_Y^j = \partial_Y^j \circ \partial_X^i.$$

Here are the differential operators we will use to form our ring. The subsequent exercise documents the Commutative Laws for addition and composition.

Definition 6.7.3. For all $i,j \in \mathbb{N}$ let $s_{i,j}, t_{i,j} \in A$ be such that only finitely many of the elements $s_{i,j}, t_{i,j}$ are non-zero. Define $\sum_{i,j\in\mathbb{N}}^{\text{finite}} s_{i,j}\partial_X^i\partial_Y^j$ by the formula

$$\left(\sum_{i,j\in\mathbb{N}}^{\text{finite}} s_{i,j}\partial_X^i\partial_Y^j\right)(f) = \sum_{i,j\in\mathbb{N}}^{\text{finite}} s_{i,j}\partial_X^i(\partial_Y^j(f))$$

for all $f \in R$. For example, we have

$$(3\partial_Y^2+\partial_X\partial_Y-\partial_X^2)(X^3+X^2Y+XY^2+Y^3)=2X+18Y$$

and

$$\left(\sum_{i,j\in\mathbb{N}}^{\text{finite}} s_{i,j}\partial_X^i\partial_Y^j\right)(X) = s_{0,0}X+s_{1,0}.$$

Note that $\sum_{i,j\in\mathbb{N}}^{\text{finite}} s_{i,j}\partial_X^i\partial_Y^j$ is also a linear transformation $R \to R$. Also, define

$$\left(\sum_{i,j\in\mathbb{N}}^{\text{finite}} s_{i,j}\partial_X^i\partial_Y^j\right) + \left(\sum_{i,j\in\mathbb{N}}^{\text{finite}} t_{i,j}\partial_X^i\partial_Y^j\right) = \left(\sum_{i,j\in\mathbb{N}}^{\text{finite}} (s_{i,j}+t_{i,j})\partial_X^i\partial_Y^j\right).$$

Exercise 6.7.4. Verify the following equalities.

$$\left(\sum_{i,j\in\mathbb{N}}^{\text{finite}} s_{i,j}\partial_X^i\partial_Y^j\right)+\left(\sum_{i,j\in\mathbb{N}}^{\text{finite}} t_{i,j}\partial_X^i\partial_Y^j\right)=\left(\sum_{i,j\in\mathbb{N}}^{\text{finite}} t_{i,j}\partial_X^i\partial_Y^j\right)+\left(\sum_{i,j\in\mathbb{N}}^{\text{finite}} s_{i,j}\partial_X^i\partial_Y^j\right)$$

$$\left(\sum_{i,j\in\mathbb{N}}^{\text{finite}} s_{i,j}\partial_X^i\partial_Y^j\right)\circ\left(\sum_{p,q\in\mathbb{N}}^{\text{finite}} t_{p,q}\partial_X^p\partial_Y^q\right)=\left(\sum_{p,q\in\mathbb{N}}^{\text{finite}} t_{p,q}\partial_X^p\partial_Y^q\right)\circ\left(\sum_{i,j\in\mathbb{N}}^{\text{finite}} s_{i,j}\partial_X^i\partial_Y^j\right)$$

Now we are in position to define our ring of differential operators.

Definition 6.7.5. Set

$$D=A[\partial_X,\partial_Y]=\left\{\textstyle\sum_{i,j\in\mathbb{N}}^{\text{finite}} s_{i,j}\partial_X^i\partial_Y^j \mid s_{i,j}\in A\right\}.$$

Exercise 6.7.6. Prove that D is a non-zero commutative ring with identity. Bonus: prove that D is isomorphic (as a ring or as an A-algebra) to a polynomial ring over A in two variables.

As we wrote in the introduction to this section, our point here is to connect these differential operators to parametric decompositions. The connection comes from the following notion.

Definition 6.7.7. Let $S\subseteq R$ be given. Set

$$Z(S)=\{\delta\in D\mid \delta(f)=0 \text{ for all } f\in S\}.$$

This is the *Macaulay inverse system* associated to S. For notational simplicity, we use the shorthand $Z(f_1,\ldots,f_n)$ in place of $Z(\{f_1,\ldots,f_n\})$.

For instance, set $\delta=\sum_{i,j\in\mathbb{N}}^{\text{finite}} s_{i,j}\partial_X^i\partial_Y^j$. From the computation $\delta(X)=s_{0,0}X+s_{1,0}$, we have $\delta\in Z(X)$ if and only if $s_{0,0}=0=s_{1,0}$, that is, if and only if δ has no constant term and no ∂_X-term, in other words, if and only if $\delta\in(\partial_X^2,\partial_Y)D$. In summary, this shows that $Z(X)$ is the monomial ideal $(\partial_X^2,\partial_Y)D$. Exercise 6.7.9 below expands on this. First, however, here are a couple of useful properties of this construction.

Exercise 6.7.8. Let $S,T\subseteq R$ be given.

(a) Prove that the set $Z(S)$ is an ideal of D.
(b) Prove that if $S\subseteq T$, then $Z(S)\supseteq Z(T)$.

Exercise 6.7.9. Let $a,b\in\mathbb{N}$ be given. Prove that

$$Z(X^aY^b)=(\partial_X^{a+1},\partial_Y^{b+1})D=\mathrm{P}_D(\partial_X^a\partial_Y^b).$$

Note that this is a parameter ideal in D.

Now that we know how to realize parameter ideals from this construction, our next goal is to realize parametric decompositions similarly. This is achieved in Exercise 6.7.11 which follows from the next result combined with the previous one.

Exercise 6.7.10. Let $\{S_\lambda\}_{\lambda\in\Lambda}$ be a set of subsets of R. Prove that

$$Z\left(\bigcup_{\lambda\in\Lambda} S_\lambda\right) = \bigcap_{\lambda\in\Lambda} Z(S_\lambda).$$

Exercise 6.7.11. For $i = 1,\ldots,n$, let $a_i, b_i \in \mathbb{N}$ be given. Prove that

$$Z(\{X^{a_1}Y^{b_1},\ldots,X^{a_n}Y^{b_n}\}) = \bigcap_{i=1}^{n}(\partial_X^{a_i+1},\partial_Y^{b_i+1})D = \bigcap_{i=1}^{n}\mathrm{P}_D(\partial_X^{a_i}\partial_Y^{b_i}).$$

Note that this is a parametric decomposition in D.

One might guess that $Z(S) = Z((S)R)$ for each subset $S \subseteq R$. However, this fails in general, as we see next.

Exercise 6.7.12. Let I be a non-zero monomial ideal in R. Prove that $Z(I) = Z([\![I]\!]) = 0$. Explain how this shows in general that we have $Z(S) \neq Z((S)R)$, and we can have $Z(S) = Z(T)$ with $S \neq T$.

Up to this point in this section, we have worked in the case $d = 2$, to ease us into this concept. However, everything passes through to the general case, as the next exercise shows.

Challenge Exercise 6.7.13. Repeat the exercises of this section for the polynomial rings $A[X,Y,Z]$ and $A[X_1,\ldots,X_d]$. (See also Laboratory Exercise 6.7.15.)

Macaulay Inverse Systems in Macaulay2

The Macaulay2 command that computes $Z(I)$, where I is an ideal in the polynomial ring, is `fromDual`. The difference between the function in Macaulay2 and our treatment is that, in Macaulay2, the ring D and the ring R are identified via the isomorphism alluded to in Exercise 6.7.6.

Below is an example that illustrates Exercise 6.7.11 in three variables.

```
i1 : R = QQ[x,y,z];

i2 : f1 = x^2*y^3*z^4
      2 3 4
o2 = x y z
o2 : R

i3 : f2 = x*y^5*z^2
        5 2
o3 = x*y z
o3 : R
```

```
i4 : I1 = ideal fromDual matrix {{f1}}
             3   4   5
o4 = ideal (x , y , z )
o4 : Ideal of R

i5 : I2 = ideal fromDual matrix {{f2}}
             2   3   6
o5 = ideal (x , z , y )
o5 : Ideal of R

i6 : I3 = ideal fromDual matrix {{f1,f2}}
             3   5   6   2 4   4 3
o6 = ideal (x , z , y , x y , y z )
o6 : Ideal of R
```

As one can check, `I3` is the intersection of `I1` and `I2`.

```
i7 : I3 == intersect(I1,I2)
o7 = true

i8 : exit
```

Exercises

Exercise 6.7.14. Set $f = XY^2 \in \mathbb{Q}[X,Y]$. Use Macaulay2 to explore the ideals $Z(X^aY^bf)$ for $a,b \in \mathbb{N}$. Use the data you collect to help with Exercise 6.7.12.

Laboratory Exercise 6.7.15. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate Challenge Exercise 6.7.13.

Challenge Exercise 6.7.16. Set $f = XY^2 \in \mathbb{Q}[X,Y]$.

(a) What does the command `ideal fromDual fromDual matrix` $\{\{$`f`$\}\}$ compute? How about `ideal fromDual fromDual fromDual matrix` $\{\{$`f`$\}\}$?
(b) Use Macaulay2 to explore higher iterations of the commands from part (a).
(c) Use the data you collect in part (b) to make a conjecture of about the limit of this procedure.
(d) Prove your conjecture from part (c).
(e) Repeat parts (b)–(d) for an arbitrary monomial in R.
(f) Repeat part (e) for $\mathbb{Q}[X,Y,Z]$ and $\mathbb{Q}[X_1,\dots,X_d]$.

Concluding Notes

Our treatment of the material in Sections 6.1–6.6 is heavily influenced by the paper of Heinzer, Ratliff, and Shah [36]. However, the idea of corner elements determining decompositions is much older than this, going back at least to a paper of Bass [6], and such elements were used as early as 1903 by Ferdinand Frobenius [24].

Macaulay introduced his inverse systems in 1916 [53]. Since then, they have found numerous applications. It is again worth noting that this is the same Macaulay as in “Macaulay2” and “Cohen-Macaulay”.

Chapter 7
Computing M-Irreducible Decompositions

This chapter deals with two aspects of the problem of computing m-irreducible decompositions. First, the bulk of the chapter consists of results of the following type: given monomial ideals I and J, use m-irreducible decompositions $I = \bigcap_{j=1}^{n} I_j$ and $J = \bigcap_{i=1}^{m} J_i$ to find m-irreducible decompositions of other ideals obtained from I and J. For instance, the intersection $I \cap J$ decomposes easily as $I \cap J = (\bigcap_{k=1}^{n} I_k) \cap (\bigcap_{k=1}^{m} J_k)$. (Of course, this also works for intersections of more than two ideals.) Less obvious is how to find decompositions of the monomial radical m-rad(I), the bracket powers $I^{[n]}$, the sum $I+J$, the colon ideal $(I :_R J)$, the saturation $(I :_R J^\infty)$, the generalized bracket powers $I^{[\underline{e}]}$, and the product IJ. These are the topics of Sections 7.1, 7.2, 7.3, 7.4, 7.7, 7.8, and 7.9, respectively; note that the last three of these are explorations.

Second, we present two algorithms for computing m-irreducible decompositions for arbitrary monomial ideals in Section 7.5. One of these algorithms extends two-variable ideas from Sections 3.5 and 6.5. The other algorithm allows one to give alternate proofs of the decomposition results from Chapter 4; this is the subject of the exploration Section 7.6.

Most of the Macaulay2 commands for this material come from preceding chapters, so the tutorials here tend to be short, with the focus on exercises, including a number of coding exercises and laboratory exercises. Section 7.5 is the sole exception to this; here we show how Macaulay2 deals with recursion.

7.1 M-Irreducible Decompositions of Monomial Radicals

In this section, A is a non-zero commutative ring with identity.

Here we show how to use an m-irreducible decomposition of J to find an m-irreducible decomposition of m-rad(J) from Section 2.3.

Proposition 7.1.1 *Set $R = A[X_1, \dots, X_d]$. Let I be a monomial ideal of R with m-irreducible decomposition $I = \bigcap_{j=1}^n I_j$.*

(a) *Each ideal* $\operatorname{m-rad}(I_j)$ *is m-irreducible.*
(b) *An m-irreducible decomposition of* $\operatorname{m-rad}(I)$ *is* $\operatorname{m-rad}(I) = \bigcap_{j=1}^n \operatorname{m-rad}(I_j)$.
(c) *If the decomposition $I = \bigcap_{j=1}^n I_j$ is redundant, then so is the decomposition* $\operatorname{m-rad}(I) = \bigcap_{j=1}^n \operatorname{m-rad}(I_j)$.

Proof. (a) Exercise 3.1.6(a).

(b) The first step in the next sequence is by assumption:

$$\operatorname{m-rad}(I) = \operatorname{m-rad}\left(\bigcap_{j=1}^n I_j\right) = \bigcap_{j=1}^n \operatorname{m-rad}(I_j).$$

The second step is from Proposition 2.3.4(b). Part (a) shows that each ideal $\operatorname{m-rad}(I_j)$ is m-irreducible, so this is an m-irreducible decomposition.

(c) Assume that the decomposition $I = \bigcap_{j=1}^n I_j$ is redundant. Then there are indices $j \neq j'$ such that $I_j \subseteq I_{j'}$. Proposition 2.3.3(c) implies that $\operatorname{m-rad}(I_j) \subseteq \operatorname{m-rad}(I_{j'})$, so the decomposition $\operatorname{m-rad}(I) = \bigcap_{j=1}^n \operatorname{m-rad}(I_j)$ is redundant. $\square$

For example, set $R = A[X,Y,Z]$. The ideal

$$J = (X^2Z^2, Y^4, Y^3Z^2)R = (X^2, Y^3)R \cap (Y^4, Z^2)R$$

has $\operatorname{m-rad}(J) = (XZ, Y)R$ with m-irreducible decomposition

$$\operatorname{m-rad}(J) = (X,Y)R \cap (Y,Z)R.$$

See Theorem 2.3.7 and Proposition 7.1.1.

Exercises

**Exercise 7.1.2.* Set $R = A[X,Y,Z]$, and consider the monomial ideal

$$J = (X^3Y^4, X^2Y^4Z^3, X^2Z^5, Y^4Z^3, Y^3Z^5)R.$$

(a) Find an irredundant m-irreducible decomposition of $\operatorname{m-rad}(J)$. (Hint: Use Theorem 2.1.5 to show that $J = (X^2, Y^3)R \cap (X^3, Z^3)R \cap (Y^4, Z^5)R$.)
(b) Use Theorem 2.3.7 to cook up an irredundant monomial generating sequence for $\operatorname{m-rad}(J)$, and verify that your decomposition from part (a) is correct using Theorem 2.1.5.

Justify your answers. (This exercise is used in Example 7.2.2.)

Exercise 7.1.3. Set $R = A[X,Y]$. Find a non-zero monomial ideal $I \subsetneq R$ with irredundant m-irreducible decomposition $I = \bigcap_{j=1}^n I_j$ such that the induced decomposition

$\text{m-rad}(I) = \bigcap_{j=1}^{n} \text{m-rad}(I_j)$ is redundant. Can this be done in one variable? Justify your answers.

Exercise 7.1.4. Set $R = A[X,Y]$. Find non-zero monomial ideals $I, J \subsetneq R$ with irredundant m-irreducible decompositions $I = \bigcap_{j=1}^{n} I_j$ and $J = \bigcap_{i=1}^{m} J_i$ such that the decomposition $I \cap J = \left(\bigcap_{j=1}^{n} I_j\right) \cap \left(\bigcap_{i=1}^{m} J_i\right)$ is redundant. Can this be done for monomial ideals in one variable? Justify your answers.

Exercise 7.1.5. Set $R = A[X_1,\ldots,X_d]$, and let J be a monomial ideal of R that has a parametric decomposition. Prove that $\mathrm{C}_R(\text{m-rad}(J)) = \{1\}$. (See Chapter 6 for our treatment of parametric decompositions and corner elements.)

Exercise 7.1.6. This exercise involves the construction $\mathrm{V}(I)$ from Section A.10. Let A be a field, and let I be a monomial ideal of $R = A[X_1,\ldots,X_d]$. Use Exercise 3.1.13 and Proposition 7.1.1(b) to give an alternate proof of Exercise 2.3.19.

**Challenge Exercise 7.1.7.* Let J be a monomial ideal of $R = A[X_1,\ldots,X_d]$.

(a) The monomial radical $\text{m-rad}(J)$ is square-free, so it is of the form J_Δ for some simplicial complex Δ by Exercise 4.4.19. Describe Δ in terms of J.
(b) Similarly, we have $J = K_\Lambda$ for some simplicial complex Λ by Exercise 4.6.12. Describe Λ in terms of J.
(c) Characterize the monomial ideals J such that there is a graph G such that $J = I_G$, and describe G in terms of J. See Sections 4.2–4.3.

Justify your answers. (This is used in Challenge Exercise 7.4.21(d).) (See also Laboratory Exercise 7.1.12.)

M-Irreducible Decompositions of Monomial Radicals in Macaulay2, Exercises

The commands relevant for this section appear in the Macaulay2 tutorial for Section 2.3; see the command `mRadical` given there, or (over a field) one can also use the built-in command `radical`.

Exercise 7.1.8. Use Macaulay2 to verify the example after Proposition 7.1.1 as well as Exercise 7.1.2.

Exercise 7.1.9. Use Macaulay2 to verify your examples for Exercises 7.1.3–7.1.4.

Exercise 7.1.10. Use Macaulay2 to verify the conclusion of Exercise 7.1.5 for some examples of your own devising.

Exercise 7.1.11. The command from Exercise 6.4.10 may be useful for this exercise.

(a) Use Macaulay2 to compute $\mathrm{C}_R(\text{m-rad}(J))$ for some monomial ideals J that do not have parametric decompositions. (See Chapter 6 for our treatment of parametric decompositions and corner elements.)

(b) Use the data you collect in part (a) to make a conjecture about $\mathrm{C}_R(\text{m-rad}\,(J))$ when J does not have a parametric decomposition.
(c) Prove your conjecture from part (b).

Laboratory Exercise 7.1.12. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate $\text{m-rad}\,(J)$ for some specific monomial ideals J, to help with Challenge Exercise 7.1.7.

Challenge Exercise 7.1.13. This exercise investigates the behavior of monomial radicals with respect to some topics from Chapter 5: dimension, depth, and Cohen-Macaulayness.

(a) Use Macaulay2 to compute $\dim(R/J)$ and $\dim(R/\text{m-rad}\,(J))$ for some monomial ideals J.
(b) Use the data from part (a) to make a conjecture about $\dim(R/\text{m-rad}\,(J))$.
(c) Prove your conjecture from part (b).
(d) Repeat part (a) for $\operatorname{depth}(R/J)$ and $\operatorname{depth}(R/\text{m-rad}\,(J))$. Does there seem to be a nice pattern here? If so, can you prove it? If not, can you identify any nice special cases where there is a nice pattern and prove it?
(e) Repeat part (d) for the Cohen-Macaulay property.

7.2 M-Irreducible Decompositions of Bracket Powers

In this section, A is a non-zero commutative ring with identity.

Here we consider bracket powers of monomial ideals, as discussed in Section 2.6.

Proposition 7.2.1 *Set $R = A[X_1,\ldots,X_d]$. Let I be a monomial ideal of R with m-irreducible decomposition $I = \bigcap_{j=1}^{n} I_j$, and let k be a positive integer.*

(a) *The ideal I is m-irreducible if and only if $I^{[k]}$ is m-irreducible.*
(b) *An m-irreducible decomposition of $I^{[k]}$ is $I^{[k]} = \bigcap_{j=1}^{n} I_j{}^{[k]}$.*
(c) *The decomposition $I = \bigcap_{j=1}^{n} I_j$ is irredundant if and only if the decomposition $I^{[k]} = \bigcap_{j=1}^{n} I_j{}^{[k]}$ is irredundant.*

Proof. (a) $\Longrightarrow$: Assume that I is m-irreducible. If $I = 0$, then $I^{[k]} = 0$, which is m-irreducible. So, assume that $I \neq 0$. Theorem 3.1.3 provides positive integers $m, t_1,\ldots,t_m, e_1,\ldots,e_m$ such that $1 \leqslant t_1 < \cdots < t_m \leqslant d$ and $I = (X_{t_1}^{e_1},\ldots,X_{t_m}^{e_m})R$. From Proposition 2.6.5, we have $I^{[k]} = (X_{t_1}^{ke_1},\ldots,X_{t_m}^{ke_m})R$ so Theorem 3.1.3 implies that $I^{[k]}$ is m-irreducible.

$\Longleftarrow$: Assume that $I^{[k]}$ is m-irreducible. As in the previous paragraph, assume without loss of generality that $I \neq 0$. Let $f_1,\ldots,f_m$ be an irredundant monomial generating sequence for I. Then an irredundant monomial generating sequence for $I^{[k]}$ is $f_1^k,\ldots,f_m^k$ by Proposition 2.6.5. Since $I^{[k]}$ is m-irreducible, Theorem 3.1.3 implies that for $i = 1,\ldots,m$ there is an index j_i and an exponent e_i such that $f_i^k = X_{j_i}^{e_i}$. A comparison of exponent vectors shows that this implies that for $i = 1,\ldots,m$ there is an exponent a_i such that $e_i = ka_i$ and $f_i = X_{j_i}^{a_i}$. It follows from Theorem 3.1.3 that I is m-irreducible.

(b) Proposition 2.6.7 shows that $I^{[k]} = \bigcap_{j=1}^n I_j{}^{[k]}$, and part (a) shows that each ideal $I_j{}^{[k]}$ is m-irreducible.

(c) "$\Longleftarrow$": If the decomposition $I = \bigcap_{j=1}^n I_j$ is redundant, then there are indices $i \neq i'$ such that $I_i \subseteq I_{i'}$. Lemma 2.6.6(a) implies that $I_i{}^{[k]} \subseteq I_{i'}{}^{[k]}$, so the decomposition $I^{[k]} = \bigcap_{j=1}^n I_j{}^{[k]}$ is redundant.

"$\Longrightarrow$": If the decomposition $I^{[k]} = \bigcap_{j=1}^n I_j{}^{[k]}$ is redundant, then there are indices $i \neq i'$ such that $I_i{}^{[k]} \subseteq I_{i'}{}^{[k]}$. Lemma 2.6.6(a) implies that $I_i \subseteq I_{i'}$, so the decomposition $I = \bigcap_{j=1}^n I_j$ is redundant. $\square$

Example 7.2.2. Set $R = A[X,Y,Z]$, and consider the monomial ideal

$$\begin{aligned} J &= (X^3Y^4, X^2Y^4Z^3, X^2Z^5, Y^4Z^3, Y^3Z^5)R \\ &= (X^2,Y^3)R \cap (X^3,Z^3)R \cap (Y^4,Z^5)R. \end{aligned}$$

See Exercise 7.1.2. This is an irredundant m-irreducible decomposition of J. Then we have the following irredundant m-irreducible decomposition of $J^{[3]}$:

$$\begin{aligned} J^{[3]} &= (X^9Y^{12}, X^6Y^{12}Z^9, X^6Z^{15}, Y^{12}Z^9, Y^9Z^{15})R \\ &= ((X^2,Y^3)R)^{[3]} \cap ((X^3,Z^3)R)^{[3]} \cap ((Y^4,Z^5)R)^{[3]} \\ &= (X^6,Y^9)R \cap (X^9,Z^9)R \cap (Y^{12},Z^{15})R. \end{aligned}$$

Exercises

For the exercises of this section, see Chapter 6 for our treatment of parametric decompositions and corner elements.

Exercise 7.2.3. Set $R = A[X,Y]$ and $J = (X^3, X^2Y, Y^3)R$.

(a) Find the J-corner elements and use them to compute an irredundant parametric decomposition of J.
(b) Use Proposition 7.2.1 with your answer from part (a) to find an irredundant parametric decomposition of $J^{[3]}$.
(c) Use Proposition 2.6.5 to find an irredundant monomial generating sequence for $J^{[3]}$. Verify that your decomposition from part (b) is correct using Theorem 2.1.5.

Justify your answers.

Exercise 7.2.4. Set $R = A[X_1,\ldots,X_d]$. Let I be a monomial ideal of R, and let k be a positive integer.

(a) Prove that I is a parameter ideal if and only if $I^{[k]}$ is a parameter ideal.

(b) Prove that if I has a parametric decomposition $I = \bigcap_{j=1}^{n} I_j$, then $I^{[k]} = \bigcap_{j=1}^{n} I_j{}^{[k]}$ is a parametric decomposition of $I^{[k]}$.
(c) Prove that I has a parametric decomposition if and only if $I^{[k]}$ has one.

Exercise 7.2.5. Set $R = A[X_1,\ldots,X_d]$. Let I be a monomial ideal of R, and let k be a positive integer. For each monomial $f = X_1^{n_1}\cdots X_d^{n_d}$ in R, set

$$f^{(k)} = X_1^{k(n_1+1)-1}\cdots X_d^{k(n_d+1)-1}.$$

(a) Prove that $\mathrm{C}_R(I^{[k]}) = \{f^{(k)} \mid f \in \mathrm{C}_R(I)\}$.
(b) Use part (a) to find the $J^{[3]}$-corner elements for the ideal from Exercise 7.2.3.
(c) Verify your answer to (b) using the decomposition of $J^{[3]}$ from Exercise 7.2.3(b).

Justify your answers.

Exercise 7.2.6. This exercise involves the construction $\mathrm{V}(I)$ from Section A.10. Let A be a field, and let I be a monomial ideal of $R = A[X_1,\ldots,X_d]$. Use Exercise 3.1.13 and Proposition 7.2.1(b) to give an alternate proof of Exercise 2.6.18.

M-Irreducible Decompositions of Bracket Powers in Macaulay2, Exercises

The commands relevant for this section are in the Macaulay2 tutorial for Section 2.6.

Exercise 7.2.7. Use Macaulay2 to verify the example after Proposition 7.2.1 as well as Exercise 7.2.3 and Exercise 7.2.5(b)–(c).

Exercise 7.2.8. Use Macaulay2 to verify the conclusion of Exercise 7.2.5(a) for some examples of your own devising.

Challenge Exercise 7.2.9. This exercise investigates the behavior of bracket powers with respect to some topics from Chapter 5: dimension, depth, and the Cohen-Macaulay property.

(a) Use Macaulay2 to compute $\dim(R/I)$ and $\dim(R/I^{[n]})$ for some monomial ideals I and some positive integers n.
(b) Use the data you collect in part (a) to make a conjecture about $\dim(R/I^{[n]})$.
(c) Prove your conjecture from part (b).
(d) Repeat part (a) for $\operatorname{depth}(R/I)$ and $\operatorname{depth}(R/I^{[n]})$. Does there seem to be a nice pattern here? If so, can you prove it? If not, can you identify any nice special cases where there is a nice pattern and prove it?
(e) Repeat part (d) for the Cohen-Macaulay property.

7.3 M-Irreducible Decompositions of Sums

In this section, A is a non-zero commutative ring with identity.

Next, we look at sums of monomial ideals. Recall that Exercise 1.3.13 shows that each sum of monomial ideals is a monomial ideal. We begin by showing that each sum of m-irreducible monomial ideals is m-irreducible.

Proposition 7.3.1 *Set $R = A[X_1,\ldots,X_d]$. If $J_1,\ldots,J_n$ are m-irreducible monomial ideals of R, then the sum $J_1+\cdots+J_n$ is m-irreducible.*

Proof. We prove the result by induction on n. The case $n=1$ is evident.

Base case: $n=2$. Assume that I and J are m-irreducible monomial ideals; we show that $I+J$ is m-irreducible. If $I=0$, then $I+J=0+J=J$ which is m-irreducible. Similarly, if $J=0$, then we are done, so we assume that I and J are both non-zero. Theorem 3.1.3 shows that there are positive integers $j,k,s_1,\ldots,s_j,t_1,\ldots,t_k,d_1,\ldots,d_j,e_1,\ldots,e_k$ such that $1\leqslant s_1<\cdots<s_j\leqslant d$ and $1\leqslant t_1<\cdots<t_k\leqslant d$ and $I=(X_{s_1}^{d_1},\ldots,X_{s_j}^{d_j})R$ and $J=(X_{t_1}^{e_1},\ldots,X_{t_k}^{e_k})R$. Exercise A.4.6(a) implies that

$$I+J=(X_{s_1}^{d_1},\ldots,X_{s_j}^{d_j},X_{t_1}^{e_1},\ldots,X_{t_k}^{e_k})R.$$

It is straightforward but tedious to show that it follows that $I+J=(X_{u_1}^{f_1},\ldots,X_{u_l}^{f_l})R$ for appropriate positive integers $l,u_1,\ldots,u_l,f_1,\ldots,f_l$. Theorem 3.1.3 implies that this ideal is m-irreducible.

Induction step: Exercise. □

For instance, in the ring $R=A[X,Y,Z]$, one has

$$(X^2,Y^3)R+(X^3,Z^3)R=(X^2,Y^3,Z^3)R.$$

The next result is a Distributive Law for intersections and sums of monomial ideals, based on the corresponding Distributive Law for intersections and unions.

Lemma 7.3.2. *Set $R=A[X_1,\ldots,X_d]$. Given monomial ideals $I_1,\ldots,I_n$ and $J_1,\ldots,J_m$ of R, one has*

$$\left(\bigcap_{p=1}^{n} I_p\right)+\left(\bigcap_{q=1}^{m} J_q\right)=\bigcap_{p=1}^{n}\bigcap_{q=1}^{m}(I_p+J_q).$$

Proof. The ideals $(\bigcap_{p=1}^{n} I_p)+(\bigcap_{q=1}^{m} J_q)$ and $\bigcap_{p=1}^{n}\bigcap_{q=1}^{m}(I_p+J_q)$ are monomial ideals, so we need only show that the sets of monomials in each ideals are the same. To this end, The first and fourth steps below are from Exercise 1.3.13(b).

$$\begin{aligned}
\left[\!\left[\left(\bigcap_{p=1}^{n} I_p\right)+\left(\bigcap_{q=1}^{m} J_q\right)\right]\!\right] &= \left[\!\left[\bigcap_{p=1}^{n} I_p\right]\!\right] \cup \left[\!\left[\bigcap_{q=1}^{m} J_q\right]\!\right] \\
&= \left(\bigcap_{p=1}^{n} [\![I_p]\!]\right) \cup \left(\bigcap_{q=1}^{m} [\![J_q]\!]\right) \\
&= \bigcap_{p=1}^{n}\bigcap_{q=1}^{m}([\![I_p]\!] \cup [\![J_q]\!]) \\
&= \bigcap_{p=1}^{n}\bigcap_{q=1}^{m}[\![I_p+J_q]\!] \\
&= \left[\!\left[\bigcap_{p=1}^{n}\bigcap_{q=1}^{m}(I_p+J_q)\right]\!\right].
\end{aligned}$$

The second and fifth steps are from Theorem 2.1.1. The third step is from the Distributive Laws for intersections and unions. □

The next result shows how to build m-irreducible decompositions for sums of monomial ideals.

Theorem 7.3.3 *Set $R = A[X_1,\dots,X_d]$. Let I and J be monomial ideals of R with m-irreducible decompositions $I = \bigcap_{j=1}^{n} I_j$ and $J = \bigcap_{i=1}^{m} J_i$. Then an m-irreducible decomposition of $I+J$ is*

$$I+J = \bigcap_{j=1}^{n}\bigcap_{i=1}^{m}(I_j+J_i).$$

Proof. From Lemma 7.3.2 we have

$$I+J = \left(\bigcap_{j=1}^{n} I_j\right)+\left(\bigcap_{i=1}^{m} J_i\right) = \bigcap_{j=1}^{n}\bigcap_{i=1}^{m}(I_j+J_i)$$

and Proposition 7.3.1 shows that each ideal I_j+J_i is m-irreducible. □

Example 7.3.4. Set $R = A[X,Y,Z]$, and consider the monomial ideals

$$\begin{aligned}
I &= (X^3,X^2Z^3,X^3Y^3,Y^3Z^3)R = (X^2,Y^3)R \cap (X^3,Z^3)R \\
J &= (Y^4,Z^5)R.
\end{aligned}$$

Then we have the following irredundant m-irreducible decomposition of $I+J$:

$$\begin{aligned}
I+J &= (X^3,X^2Z^3,X^3Y^3,Y^3Z^3,Y^4,Z^5)R \\
&= ((X^2,Y^3)R+(Y^4,Z^5)R) \cap ((X^3,Z^3)R+(Y^4,Z^5)R) \\
&= (X^2,Y^3,Z^5)R \cap (X^3,Y^4,Z^3)R.
\end{aligned}$$

Exercises

Exercise 7.3.5. Complete the induction step of Proposition 7.3.1.

Exercise 7.3.6. Verify that the decomposition from Example 7.3.4 is correct using Theorem 2.1.5.

Exercise 7.3.7. Set $R = A[X_1, \dots, X_d]$. Prove or disprove the following: if $J_1, \dots, J_n$ are monomial ideals of R such that the sum $J_1 + \cdots + J_n$ is an m-irreducible monomial ideal, then J_i is m-irreducible for $j = 1, \dots, n$. Justify your answer.

Exercise 7.3.8. Set $R = A[X,Y]$. Use Theorem 7.3.3 to find an irredundant m-irreducible decomposition of the ideal $I + J$ where $I = (X^3, XY^2, Y^3)R$ and $J = (X^3, X^2Y, Y^3)R$. Verify that your decomposition is correct using Theorem 2.1.5. Justify your answer.

Exercise 7.3.9. Set $R = A[X,Y]$. Find non-zero monomial ideals $I, J \subsetneq R$ with irredundant m-irreducible decompositions $I = \bigcap_{j=1}^n I_j$ and $J = \bigcap_{i=1}^m J_i$ such that the decomposition $I + J = \bigcap_{j=1}^n \bigcap_{i=1}^m (I_j + J_i)$ is redundant. Can this be done for monomial ideals in one variable? Justify your answers.

**Exercise 7.3.10.* Let $K_1, \dots, K_p \subsetneq R$ be monomial ideals of R. For $i = 1, \dots, p$ fix an m-irreducible decomposition $K_i = \bigcap_{j=1}^{s_i} K_{i,j}$. Prove that

$$K_1 + \cdots + K_p = \bigcap_{l_1=1}^{s_1} \cdots \bigcap_{l_p=1}^{s_p} (K_{1,l_1} + \cdots + K_{p,l_p})$$

and prove that this is an m-irreducible decomposition. (This exercise is used in Exercise 7.9.4 and Example 7.9.5.)

Exercise 7.3.11. Set $R = A[X,Y]$. Use Exercise 7.3.10 to find an m-irreducible decomposition of $(X^2, XY^5, Y^6)R + (X^4, X^3Y^3, Y^5)R + (X^7, X^3Y^2, Y^3)R$. Verify that your decomposition is correct using Theorem 2.1.5. Justify your answer.

Exercise 7.3.12. Set $R = A[X_1, \dots, X_d]$. (See Chapter 6 for our treatment of parameter ideals, parametric decompositions, and corner elements.)

(a) Prove that if $f, g \in [\![R]\!]$, then $\mathrm{P}_R(f) + \mathrm{P}_R(g) = \mathrm{P}_R(\gcd(f,g))$; cf. Exercise 2.1.14 for the definition of $\gcd(f,g)$.)
(b) Prove that if $J_1, \dots, J_n$ are parameter ideals of R, then the sum $J_1 + \cdots + J_n$ is a parameter ideal.
(c) Let I and J be monomial ideals of R with parametric decompositions $I = \bigcap_{j=1}^n I_j$ and $J = \bigcap_{i=1}^m J_i$. Prove that a parametric decomposition of $I + J$ is $I + J = \bigcap_{j=1}^n \bigcap_{i=1}^m (I_j + J_i)$.
(d) Let I and J be monomial ideals of R with parametric decompositions. Prove that $\mathrm{C}_R(I + J) \subseteq \{\gcd(f,g) \mid f \in \mathrm{C}_R(I) \text{ and } g \in \mathrm{C}_R(J)\}$.

(e) Find monomial ideals I and J of R with parametric decompositions such that $\mathrm{C}_R(I+J) \subsetneq \{\gcd(f,g) \mid f \in \mathrm{C}_R(I) \text{ and } g \in \mathrm{C}_R(J)\}$. Justify your answer.
(f) Does the containment in part (d) hold if I or J does not have a parametric decomposition? Justify your answer.

Challenge Exercise 7.3.13. Set $R = A[X_1,\ldots,X_d]$, and let I and J be square-free monomial ideals of R.

(a) Assume that there are graphs G and G' such that $I = I_G$ and $J = I_{G'}$. Prove that there is a graph G'' such that $I+J = I_{G''}$, and describe G'' in terms of G and G'. See Sections 4.2–4.3.
(b) There are simplicial complexes Δ and Δ' such that $I = J_\Delta$ and $J = J_{\Delta'}$ by Exercise 4.4.19. The sum $I+J$ is square-free by Exercise 4.1.10(a), so there is a simplicial complex Δ'' such that $I+J = J_{\Delta''}$. Describe Δ'' in terms of Δ and Δ'.
(c) Similarly, we have $I = K_\Lambda$ and $J = K_{\Lambda'}$ and $I+J = K_{\Lambda''}$ for some simplicial complexes Λ, Λ', and Λ'' by Exercise 4.6.12. Describe Λ'' in terms of Λ and Λ'.

Justify your answers. (See also Laboratory Exercise 7.3.18.)

Challenge Exercise 7.3.14. This exercise involves the construction $\mathrm{V}(I)$ from Exploration Section A.10. Let A be a field, and let I and J be monomial ideals of $R = A[X_1,\ldots,X_d]$. Exercise 3.3.11(b) shows that there is an integer $k \in \mathbb{N}$, and there are linear subspaces $V_1,\ldots,V_k \subseteq A^d$ such that $\mathrm{V}(I) = V_1 \cup \cdots \cup V_k$ and such that $V_i \not\subseteq V_j$ whenever $i \neq j$. This exercise also yields such irredundant subspace decompositions $\mathrm{V}(J) = W_1 \cup \cdots \cup W_l$ and $\mathrm{V}(I+J) = U_1 \cup \cdots \cup U_m$. Describe the linear subspaces U_n in terms of the subspaces V_i and W_j. Justify your answer.

M-Irreducible Decompositions of Sums in Macaulay2

In this tutorial, we illustrate how to add your own implementation of operators between objects for which Macaulay2 does not have a default implementation. This is useful for user-defined objects as well as Macaulay2 objects.

Although many operations for `List` objects are already built into Macaulay2, one operation that is missing is the analogue of cartesian product of sets. That is, given two lists $\{x_1,\ldots,x_n\}$ and $\{y_1,\ldots,y_m\}$ we would like to return a list of sequences of the form (x_i,y_j) for $i = 1,\ldots,m$ and $j = 1,\ldots,n$.

Since the `**` operator is used for cartesian product of `Set` objects, we will illustrate how one can extend the definition of this operator to `List` objects as well. This function is a straightforward application of the `apply` function, but it can be handy to have around in order to simplify code.

```
i1 : List ** List := (X,Y) -> flatten apply(X, x -> apply(Y, y -> (x,y)))
o1 = {*Function[stdio:1:23-1:65]*}
o1 : FunctionClosure
```

Let us see this function in action.

```
i2 : L1 = {1,2}
o2 = {1, 2}
o2 : List

i3 : L2 = {x,y,x}
o3 = {x, y, x}
o3 : List

i4 : L1 ** L2
o4 = {(1, x), (1, y), (1, x), (2, x), (2, y), (2, x)}
o4 : List
```

This function may be useful when implementing Proposition 7.3.3 for yourself, as in the exercises below. A list of all operators that can be overloaded in this way can be obtained from the Macaulay2 documentation for `operators`.

```
i5 : exit
```

Exercises

Exercise 7.3.15. Use Macaulay2 to verify the following:

(a) the conclusions of Example 7.3.4;
(b) any examples you devise for Exercises 7.3.7, 7.3.9, and 7.3.12(f); and
(c) your calculations for Exercises 7.3.8 and 7.3.11.

Coding Exercise 7.3.16. Write a method that takes as input two lists of ideals $\{I_1,\ldots,I_n\}$ and $\{J_1,\ldots,J_m\}$ and returns the list of ideals

$$\{I_j + J_i \mid 1 \leqslant i \leqslant m \text{ and } 1 \leqslant j \leqslant n\}.$$

Coding Exercise 7.3.17. Use the method from Coding Exericise 7.3.16 together with the method `makeIrredundant` from Section 3.3 to write a method that computes an irredundant m-irreducible decomposition of the ideal $I+J$, assuming irredundant m-irreducible decompositions of I and J are passed as input. Test your method using examples and exercises from the section.

Laboratory Exercise 7.3.18. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate $I+J$ for some specific square-free monomial ideals I and J, to help with Challenge Exercise 7.3.13.

Challenge Exercise 7.3.19. This exercise investigates the behavior of sums with respect to some topics from Chapters 5 and 6: dimension, depth, Cohen-Macaulayness, and corner elements.

(a) Use Macaulay2 to compute $\dim(R/I)$ and $\dim(R/J)$ and $\dim(R/(I+J))$ for some monomial ideals I, J.
(b) Use the data from part (a) to make a conjecture about $\dim(R/(I+J))$.

(c) Prove your conjecture from part (b).
(d) Repeat part (a) for $\operatorname{depth}(R/I)$ and $\operatorname{depth}(R/J)$ and $\operatorname{depth}(R/(I+J))$. Does there seem to be a nice pattern here? If so, can you prove it? If not, can you identify any nice special cases where there is a nice pattern and prove it?
(e) Repeat part (d) for the Cohen-Macaulay property.
(f) Repeat part (d) for $\mathrm{C}_R(I+J)$. (See Exercise 6.4.10.)

7.4 M-Irreducible Decompositions of Colon Ideals

In this section, A is a non-zero commutative ring with identity.

Next, we look at decompositions of colon ideals of monomial ideals; see Theorem 7.4.4. Recall that Theorem 2.5.1 implies that the colon ideal of two monomial ideals is a monomial ideal.

Proposition 7.4.1 *Set $R = A[X_1,\ldots,X_d]$. Let $k,t_1,\ldots,t_k,e_{t_1},\ldots,e_{t_k}$ be positive integers, and set $J = (X_{t_1}^{e_{t_1}},\ldots,X_{t_k}^{e_{t_k}})R$. Given a monomial $f = \underline{X}^{\underline{n}} \in [\![R]\!]$, we have*

$$(J :_R f) = \begin{cases} R & \text{if there is an index } i \text{ such that } n_{t_i} \geqslant e_{t_i} \\ (X_{t_1}^{e_{t_1}-n_{t_1}},\ldots,X_{t_k}^{e_{t_k}-n_{t_k}})R & \text{if for } i=1,\ldots,k \text{ we have } n_{t_i} < e_{t_i} \end{cases}$$
$$= \begin{cases} R & \text{if } f \in J \\ (X_{t_1}^{e_{t_1}-n_{t_1}},\ldots,X_{t_k}^{e_{t_k}-n_{t_k}})R & \text{if } f \notin J. \end{cases}$$

Proof. We know that $f \in J$ if and only if if there is an index i such that $f \in (X_{t_i}^{e_{t_i}})R$. By comparing exponent vectors, this says that $f \in J$ if and only if there is an index i such that $n_{t_i} \geqslant e_{t_i}$.

If there is an index i such that $n_{t_i} \geqslant e_{t_i}$, then $f \in J$, so $(J :_R f) = R$ by Proposition A.6.2(c).

Assume now that for $i = 1,\ldots,k$ we have $n_{t_i} < e_{t_i}$. For $i = 1,\ldots,k$ the monomial $X_{t_i}^{e_{t_i}-n_{t_i}}$ is in $(J :_R f)$ because

$$X_{t_i}^{e_{t_i}-n_{t_i}} f = X_1^{n_1}\cdots X_{t_i}^{e_{t_i}-n_{t_i}+n_{t_i}}\cdots X_d^{n_d} \in (X_{t_i}^{e_{t_i}})R \subseteq J.$$

To complete the proof, we need to fix a monomial $g \in (J :_R f)$ and show that $g \in (X_{t_i}^{e_{t_i}-n_{t_i}})R$ for some index i. (This uses the fact that $(J :_R f)$ is a monomial ideal; see Theorem 2.5.1.) Let $g = \underline{X}^{\underline{m}} \in [\![(J :_R f)]\!]$. Then $fg \in J$, so there is an index i such that $1 \leqslant i \leqslant n$ and $fg \in (X_{t_i}^{e_{t_i}})R$. A comparison of exponent vectors shows that this implies that $n_{t_i} + m_{t_i} \geqslant e_{t_i}$, so $m_{t_i} \geqslant e_{t_i} - n_{t_i}$. Another comparison of exponent vectors implies that $g \in (X_{t_i}^{e_{t_i}-n_{t_i}})R$ as desired. $\square$

Corollary 7.4.2 *Set $R = A[X_1,\dots,X_d]$, and let J be an m-irreducible monomial ideal of R. Given a monomial $f \in [\![R]\!]$, either the ideal $(J :_R f)$ is m-irreducible or $(J :_R f) = R$. The ideal $(J :_R f)$ is m-irreducible if and only if $f \notin J$.*

Proof. If $J = 0$, then $(J :_R f) = 0$ which is m-irreducible. If $J \neq 0$, then the result follows from Theorem 3.1.3 and Proposition 7.4.1. □

Example 7.4.3. Set $R = A[X,Y,Z]$ and $J = (X^2,Z^3)R$. Proposition 7.4.1 explains the following computations:

$$(J :_R XY) = (X,Z^3)R \qquad\qquad (J :_R XYZ^4) = R.$$

Next, we have our general decomposition result for colon ideals. It uses a decomposition of one of the ideals and generators for the other ideal.

Theorem 7.4.4 *Set $R = A[X_1,\dots,X_d]$. Let I be a monomial ideal of R with monomial generating sequence $f_1,\dots,f_t$. Let J be a monomial ideal of R with m-irreducible decomposition $J = \bigcap_{i=1}^m J_i$. Assume that $I \not\subseteq J$. Then an m-irreducible decomposition of $(J :_R I)$ is*

$$(J :_R I) = \bigcap_{f_j \notin J_i} (J_i :_R f_j)$$

where the intersection is taken over the set of all ordered pairs (i,j) such that $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant t$ and $f_j \notin J_i$.

Proof. The first and third equalities below are by assumption:

$$\begin{aligned}
(J :_R I) &= (J :_R (f_1,\dots,f_t)R) \\
&= \bigcap_{j=1}^{t} (J :_R f_j) \\
&= \bigcap_{j=1}^{t} \left(\bigcap_{i=1}^{m} J_i :_R f_j \right) \\
&= \bigcap_{j=1}^{t} \bigcap_{i=1}^{m} (J_i :_R f_j) \\
&= \bigcap_{f_j \notin J_i} (J_i :_R f_j).
\end{aligned}$$

The second equality is from Proposition A.6.2(b), and the fourth equality is from Proposition A.6.3(b). The fifth equality is from Corollary 7.4.2. Another application of Corollary 7.4.2 shows that when $f_j \notin J_i$, the ideal $(J_i :_R f_j)$ is m-irreducible. □

Example 7.4.5. Set $R = A[X,Y,Z]$, and consider the monomial ideals

$$\begin{aligned}
I &= (Y^4,Z^5)R \\
J &= (X^3,X^2Z^3,X^3Y^3,Y^3Z^3)R = (X^2,Y^3)R \cap (X^3,Z^3)R.
\end{aligned}$$

In the notation of Theorem 7.4.4, we have $f_1 = Y^4$, $f_2 = Z^5$, $J_1 = (X^2, Y^3)R$, and $J_2 = (X^3, Z^3)R$. To find an m-irreducible decomposition of $(J :_R I)$, we first find the ordered pairs (i, j) such that $f_j \notin J_i$: $f_1 \in J_1$, $f_1 \notin J_2$, $f_2 \notin J_1$, and $f_2 \in J_2$. Thus, the first step in the next sequence is from Theorem 7.4.4:

$$(J :_R I) = (J_2 :_R f_1) \cap (J_1 :_R f_2) = (X^2, Y^3)R \cap (X^3, Z^3)R = J.$$

The second step is from Proposition 7.4.1.

To conclude this section, we address the question of which monomial ideals admit corner elements. (See Chapter 6 for our treatment of parameter ideals and corner elements.) By definition, this is intimately related with computations of colon ideals. We begin with the next lemma showing that an m-irreducible monomial ideal admits a corner element if and only if it is a parameter ideal.

Lemma 7.4.6. *Set $R = A[X_1, \ldots, X_d]$ and $\mathfrak{X} = (X_1, \ldots, X_d)R$, and let J be an m-irreducible monomial ideal of R.*

(a) *If* m-rad$(J) \neq \mathfrak{X}$*, then* $(J :_R \mathfrak{X}) = J$ *and* $\mathrm{C}_R(J) = \emptyset$.
(b) *If* m-rad$(J) = \mathfrak{X}$*, then J is a parameter ideal, say* $J = \mathrm{P}_R(z)$*, and there are equalities* $(J :_R \mathfrak{X}) = J + (z)R$ *and* $\mathrm{C}_R(J) = \{z\}$.

Proof. If $J = 0$, then the result is straightforward. Thus, we assume that $J \neq 0$. Theorem 3.1.3 provides positive integers $k, t_1, \ldots, t_k, e_1, \ldots, e_k$ such that $J = (X_{t_1}^{e_1}, \ldots, X_{t_k}^{e_k})R$.

(a) Assume that m-rad$(J) \neq \mathfrak{X}$. By Exercise 6.2.27, it suffices to show that $\mathrm{C}_R(J) = \emptyset$. By definition, we have $\mathrm{C}_R(J) = [\![(J :_R \mathfrak{X})]\!] \smallsetminus [\![J]\!]$, so we need to show that $[\![(J :_R \mathfrak{X})]\!] \subseteq J$. Fix a monomial $f = \underline{X}^{\underline{m}} \in [\![(J :_R \mathfrak{X})]\!]$.

The assumption m-rad$(J) \neq \mathfrak{X}$. provides an index j such that $X_j \notin$ m-rad(J). Since $f \in (J :_R \mathfrak{X})$, we have $X_j f \in J$. Theorem 1.1.9 implies that there is an index p such that $X_j f \in (X_{t_p}^{e_p})R$. Since $X_j \notin$ m-rad(J), we have $X_j \neq X_{t_p}$. Thus, a comparison of exponent vectors shows that $m_{t_p} \geqslant e_p$, and it follows that $f = \underline{X}^{\underline{m}} \in (X_{t_p}^{e_p})R \subseteq J$.

(b) Assume that m-rad$(J) = \mathfrak{X}$. Then Corollary 2.3.8 shows that J is a parameter ideal, say $J = \mathrm{P}_R(z)$. Corollary 6.2.12(a) implies that $\mathrm{C}_R(J) = \{z\}$, and it follows from Exercise 6.2.27 that $(J :_R \mathfrak{X}) = J + (z)R$. □

Next, we explicitly describe the corner elements of an arbitrary monomial ideal, in terms of the ideal's irredundant m-irreducible decomposition.

Proposition 7.4.7 *Set $R = A[X_1, \ldots, X_d]$ and $\mathfrak{X} = (X_1, \ldots, X_d)R$. Let J be a monomial ideal of R with irredundant m-irreducible decomposition $J = \bigcap_{i=1}^m J_i$ with $m \geqslant 1$. Assume that the ideals J_i are ordered so that* m-rad$(J_i) = \mathfrak{X}$ *if and only if* $1 \leqslant i \leqslant n$. *Then* $\mathrm{C}_R(J) = \bigcup_{i=1}^n \mathrm{C}_R(J_i)$. *In other words, if $J_i = \mathrm{P}_R(z_i)$ for $i = 1, \ldots, n$ then the distinct J-corner elements are $z_1, \ldots, z_n$.*

Proof. The case where J is m-irreducible is covered by Lemma 7.4.6. Thus, we assume without loss of generality that J is not m-irreducible, that is, that $m \geqslant 2$.

The irredundancy of the intersection $\bigcap_{i=1}^m J_i$ implies that for indices $i \neq j$ we have $J_j \not\subseteq J_i$. When $1 \leqslant i \leqslant n$, we have $\text{m-rad}\,(J_i) = \mathfrak{X}$, so Proposition 6.2.7(c) implies that $\mathrm{C}_R(J_i) \cap J_j \neq \emptyset$. Since $\mathrm{C}_R(J_i) = \{z_i\}$, this means that $z_i \in J_j$.

Claim: For $i = 1, \ldots, n$ we have $z_i \in \mathrm{C}_R(J)$. Since $z_i \notin J_i$ and $J \subseteq J_i$, we have $z_i \notin J$. Thus, we need only show that $X_k z_i \in J$ for $k = 1, \ldots, d$. Since $J = \bigcap_{j=1}^m J_j$, it suffices to show that $X_k z_i \in J_j$ for $j = 1, \ldots, m$. When $j \neq i$, this follows from the condition $z_i \in J_j$ established in the previous paragraph. When $j = i$, this follows from the fact that $z_i \in \mathrm{C}_R(J_i)$.

Claim: We have $\mathrm{C}_R(J) \subseteq \{z_1, \ldots, z_n\}$. The first equality in the next sequence is by assumption:

$$(J :_R \mathfrak{X}) = \left(\left[\bigcap_{i=1}^m J_i\right] :_R \mathfrak{X}\right) = \bigcap_{i=1}^m (J_i :_R \mathfrak{X}) = \left(\bigcap_{i=1}^n [J_i + (z_i)R]\right) \cap \left(\bigcap_{j=n+1}^m J_j\right).$$

The second equality is by Proposition A.6.3(b), and the third one is by Lemma 7.4.6. This explains the first equality in the next sequence:

$$\begin{aligned}
[\![(J :_R \mathfrak{X})]\!] &= \left[\!\!\left[\bigcap_{i=1}^n [J_i + (z_i)R] \cap \left[\bigcap_{j=n+1}^m J_j\right]\right]\!\!\right] \\
&= \left[\bigcap_{i=1}^n [\![J_i + (z_i)R]\!]\right] \cap \left[\bigcap_{j=n+1}^m [\![J_j]\!]\right] \\
&= \left[\bigcap_{i=1}^n [\![J_i]\!] \cup \{z_i\}\right] \cap \left[\bigcap_{j=n+1}^m [\![J_j]\!]\right].
\end{aligned}$$

The second equality is from Theorem 2.1.1. For the third equality, use the fact that $\{z_i\} = \mathrm{C}_R(J_i) = [\![(J_i :_R \mathfrak{X})]\!] \smallsetminus [\![J_i]\!]$ to conclude that $[\![J_i + (z_i)R]\!] = [\![J_i]\!] \cup \{z_i\}$.

Given an element $z \in \mathrm{C}_R(J) \subseteq [\![(J :_R \mathfrak{X})]\!]$, we conclude from the previous displayed sequence that $z \in [\![J_i]\!] \cup \{z_i\}$ for $i = 1, \ldots, n$ and that $z \in [\![J_j]\!]$ for $j = n+1, \ldots, m$. On the other hand, since $z \notin [\![J]\!]$, the sequence

$$[\![J]\!] = \left[\!\!\left[\bigcap_{i=1}^m J_i\right]\!\!\right] = \bigcap_{i=1}^m [\![J_i]\!]$$

shows that there is an index i' such that $z \notin [\![J_{i'}]\!]$. Since we have $z \in [\![J_j]\!]$ for $j = n+1, \ldots, m$ it follows that $i' \leqslant n$, that is, we have $z \in ([\![J_{i'}]\!] \cup \{z_i\}) \smallsetminus [\![J_{i'}]\!] = \{z_{i'}\}$. We conclude that $z = z_{i'}$, as desired. □

Here is our characterization of monomial ideals admitting corner elements.

Corollary 7.4.8 *Set $R = A[X_1, \ldots, X_d]$ and $\mathfrak{X} = (X_1, \ldots, X_d)R$. Let J be a monomial ideal of R with irredundant m-irreducible decomposition $J = \bigcap_{i=1}^m J_i$ with $m \geqslant 1$. Then J has a corner element if and only if there is an index i such that* $\text{m-rad}\,(J_i) = \mathfrak{X}$.

Proof. In the notation of Proposition 7.4.7, there is a J-corner element if and only if $n \geqslant 1$, that is, if and only if there is an index i such that $\text{m-rad}\,(J_i) = \mathfrak{X}$. $\square$

Example 7.4.9. Set $R = A[X,Y,Z]$ and $J = (XY, XZ, YZ)R$. We show that $\mathrm{C}_R(J) = \emptyset$. Example 7.5.10 provides an irredundant m-irreducible decomposition.

$$J = (Y,Z)R \cap (X,Z)R \cap (X,Y)R.$$

As this decomposition has no parameter ideals, Corollary 7.4.8 says that $\mathrm{C}_R(J) = \emptyset$.

On the other hand, consider the next ideal from Example 7.3.4.

$$\begin{aligned} K &= (X^3, X^2Z^3, X^3Y^3, Y^3Z^3, Y^4, Z^5)R \\ &= (X^2, Y^3, Z^5)R \cap (X^3, Y^4, Z^3)R \\ &= \mathrm{P}_R(XY^2Z^4) \cap \mathrm{P}_R(X^2Y^3Z^2) \end{aligned}$$

Then Proposition 7.4.7 shows that $\mathrm{C}_R(K) = \{XY^2Z^4, X^2Y^3Z^2\}$.

Exercises

Exercise 7.4.10. Verify directly the equalities in Example 7.4.3. Be sure to justify your answers.

Exercise 7.4.11. Verify directly the equality $(J :_R I) = J$ in Example 7.4.5. Justify your answer.

Exercise 7.4.12. Verify the equalities $\mathrm{C}_R(J) = \emptyset$ and $\mathrm{C}_R(K) = \{XY^2Z^4, X^2Y^3Z^2\}$ in Example 7.4.9 directly. Justify your answer.

Exercise 7.4.13. Set $R = A[X,Y]$. Use Proposition 7.4.1 to identify the ideals $(J :_R f)$ and $(J :_R g)$ where $J = (X^3, X^2Y, Y^3)R$, $f = XY^2$, and $g = X^2Y^2$. Justify your answer.

Exercise 7.4.14. Set $R = A[X_1, \ldots, X_d]$. Let J be a monomial ideal of R, and let $f \in [\![R]\!]$. If $(J :_R f)$ is m-irreducible, must J be m-irreducible? Justify your answer.

Exercise 7.4.15. Set $R = A[X,Y]$, and use the ideals $I = (X^3, XY^2, Y^3)R$ and $J = (X^3, X^2Y, Y^3)R$.

(a) Use Theorem 7.4.4 to find an irredundant m-irreducible decomposition $\bigcap_{i=1}^m J_i$ of the ideal $(J :_R I)$.
(b) Compute directly a monomial generating sequence for $(J :_R I)$ and verify the decomposition $(J :_R I) = \bigcap_{i=1}^m J_i$ from part (a) by computing the generators for $\bigcap_{i=1}^m J_i$ using least common multiples.

Justify your answers.

Exercise 7.4.16. Set $R = A[X,Y]$. Find non-zero monomial ideals $I, J \subsetneq R$ such that the decomposition $(J :_R I) = \bigcap_{f_j \notin J_i}(J_i :_R f_j)$ from Theorem 7.4.4 is redundant. Can this be done for monomial ideals in one variable? Justify your answers.

For the next few exercises, see Chapter 6 for our treatment of parameter ideals, parametric decompositions, and corner elements.

Exercise 7.4.17. Set $R = A[X,Y,Z]$ and $J = (X^2Y, Y^2Z, XZ^2, XYZ)R$. Use Corollary 7.4.8 to show that $\mathrm{C}_R(J) = \emptyset$.

Exercise 7.4.18. Set $R = A[X_1,\ldots,X_d]$, and let J be a parameter ideal of R. Fix a monomial $f \in [\![R]\!]$.

(a) Prove that either $(J :_R f)$ is a parameter ideal or $(J :_R f) = R$.
(b) Prove that $(J :_R f)$ is a parameter ideal if and only if $f \notin J$.
(c) Prove that if $f = \underline{X}^{\underline{m}}$ and $g = \underline{X}^{\underline{n}} \notin (f)R$, then $(\mathrm{P}_R(g) :_R f) = \mathrm{P}_R(\underline{X}^{\underline{p}})$ where $p_i = m_i - n_i$ for $i = 1,\ldots,d$.

Exercise 7.4.19. Set $R = A[X_1,\ldots,X_d]$, and let I and J be monomial ideals of R. Assume that J has a parametric decomposition $J = \bigcap_{i=1}^{m} \mathrm{P}_R(z_i)$ and that $I \not\subseteq J$.

(a) Prove that the decomposition of the colon ideal $(J :_R I)$ from Theorem 7.4.4 is a parametric decomposition.
(b) Let $f_1,\ldots,f_n$ be a monomial generating sequence of I, and for $j = 1,\ldots,n$ write $f_j = \underline{X}^{\underline{n}_j}$ where $\underline{n}_j = (n_{j,1},\ldots,n_{j,d}) \in \mathbb{N}^d$. For $i = 1,\ldots,m$ write $z_i = \underline{X}^{\underline{m}_i}$ where $\underline{m}_i = (m_{i,1},\ldots,m_{i,d}) \in \mathbb{N}^d$. When $f_j \notin \mathrm{P}_R(z_i)$, write $g_{i,j} = \underline{X}^{\underline{p}}$ where $p_i = m_i - n_i$. Prove that $\mathrm{C}_R((J :_R I)) \subseteq \{g_{i,j} \mid f_j \notin \mathrm{P}_R(z_i)\}$.
(c) Find ideals I and J such that $\mathrm{C}_R((J :_R I)) \subsetneq \{g_{i,j} \mid f_j \notin \mathrm{P}_R(z_i)\}$.
(d) Does the containment in part (b) hold in the case where J does not have a parametric decomposition?

Justify your answers.

Exercise 7.4.20. Set $R = A[X_1,\ldots,X_d]$, and let J be a monomial ideal of R with irredundant monomial generating sequence $f_1,\ldots,f_t$. Prove that if J has a corner element, then $t \geqslant d$.

Challenge Exercise 7.4.21. Set $R = A[X_1,\ldots,X_d]$, and let I and J be square-free monomial ideals of R. There are simplicial complexes Δ and Δ' such that $I = J_\Delta$ and $J = J_{\Delta'}$ by Exercise 4.4.19, and there are simplicial complexes Λ and Λ' such that $I = K_\Lambda$ and $J = K_{\Lambda'}$ by Exercise 4.6.12.

(a) The colon ideal $(I :_R J)$ is square-free by Exercise 4.1.11(a), so there is a simplicial complex Δ'' such that $(I :_R J) = J_{\Delta''}$. Describe Δ'' in terms of the simplicial complexes Δ, Δ', Λ, and Λ'.
(b) Similarly, we have $(I :_R J) = K_{\Lambda''}$ for some simplicial complex Λ''. Describe Λ'' in terms of Δ, Δ', Λ, and Λ'.
(c) Provide conditions under which there is a graph G such that $(I :_R J) = I_G$, and describe G in terms of data from I and J. See Sections 4.2–4.3.

(d) Use Exercise 4.1.11(a) and Challenge Exercise 7.1.7 to formulate and prove versions of your results for parts (a)–(c) in case J is not square-free.

Justify your answers. (See also Laboratory Exercise 7.4.23.)

Challenge Exercise 7.4.22. This exercise involves the construction $\mathrm{V}(I)$ from Exploration Section A.10. Let A be a field, and let I and J be monomial ideals of $R = A[X_1,\ldots,X_d]$. Exercise 3.3.11(b) shows that there is an integer $k \in \mathbb{N}$, and there are linear subspaces $V_1,\ldots,V_k \subseteq A^d$ such that $\mathrm{V}(I) = V_1 \cup \cdots \cup V_k$ and such that $V_i \not\subseteq V_j$ whenever $i \neq j$. This exercise also yields such irredundant subspace decompositions $\mathrm{V}(J) = W_1 \cup \cdots \cup W_l$ and $\mathrm{V}((I :_R J)) = U_1 \cup \cdots \cup U_m$. Describe the linear subspaces U_n in terms of the subspaces V_i, W_j and any other data about J you need. Justify your answer.

M-Irreducible Decompositions of Colon Ideals in Macaulay2, Exercises

As described in Theorem 7.4.4, we can use a generating set for I, together with an m-irredicible decomposition of J to compute an m-irreducible decomposition of the ideal $(I :_R J)$, each component of which is also a colon ideal computed by Proposition 7.4.1. Similar in spirit to other computations you have done, this can be done with some nested `apply` functions.

Laboratory Exercise 7.4.23. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate the colon ideal $(I :_R J)$ for some specific square-free monomial ideals I and J, to help with Challenge Exercise 7.4.21.

Coding Exercise 7.4.24. Write a method that takes an m-irreducible monomial ideal J and a ring element f as input, and returns the monomial ideal $(J :_R f)$ via Proposition 7.4.1. The method should verify that f is a monomial and that J is m-irreducible, and provide a suitable error if either condition fails. The function `isMIrreducible` from Section 3.1 may be helpful here. Test your method on some examples. Compare the output of your method with that of the command `J:f`.

Coding Exercise 7.4.25. Use the method you developed in the previous problem to provide a method that takes a list L of m-irreducible monomial ideals (which should be thought of as an irredundant m-irreducible decomposition of some ideal J) and a monomial ideal I as input, and uses Theorem 7.4.4 to return a list which is an m-irreducible decomposition of the ideal $(J :_R I)$. Test your method on some examples, and compare the output of your method with the output of the command `J:I` where `J = intersect L`.

Coding Exercise 7.4.26. Write a method whose input is a list L of m-irreducible monomial ideals, and whose output is a list of the corner elements of the intersection of the input list, using Proposition 7.4.7 as a guide. Test your method on some examples, and compare the output of your method with the output of the command `flatten entries mingens ((J:M)/J)` where `J = intersect L` and `M = ideal vars R`.

Challenge Exercise 7.4.27. This exercise investigates the behavior of colon ideals with respect to some topics from Chapters 5 and 6: dimension, depth, Cohen-Macaulayness, and corner elements.

(a) Use Macaulay2 to compute $\dim(R/I)$ and $\dim(R/J)$ and $\dim(R/(J:_R I))$ for some monomial ideals I, J.
(b) Use the data from part (a) to make a conjecture about $\dim(R/(J:_R I))$.
(c) Prove your conjecture from part (b).
(d) Repeat part (a) for $\operatorname{depth}(R/I)$ and $\operatorname{depth}(R/J)$ and $\operatorname{depth}(R/(J:_R I))$. Does there seem to be a nice pattern here? If so, can you prove it? If not, can you identify any nice special cases where there is a nice pattern and prove it?
(e) Repeat part (d) for the Cohen-Macaulay property.
(f) Repeat part (d) for $\mathrm{C}_R((J:_R I))$. (See Exercise 6.4.10.)

Challenge Exercise 7.4.28. This exercise explores decompositions of saturations; see Section 2.7.

(a) Use the command `irreducibleDecomposition saturate` to compute irredundant m-irreducible decompositions of saturations $(J:_R I^\infty)$ for some monomial ideals I, J.
(b) Analyze the output from (a) with Proposition 7.4.1 and Theorem 7.4.4 in mind.
(c) Use your results from part (b) to make a conjecture about irredundant m-irreducible decompositions of saturations.
(d) Prove your conjecture from part (c).

7.5 Computing General M-Irreducible Decompositions

In this section, A is a non-zero commutative ring with identity.

Here we describe some techniques for computing m-irreducible decompositions of arbitrary monomial ideals, beginning with another Distributive Law. Recall that the definitions of LCM and support are in 2.1.3 and 2.3.5.

Lemma 7.5.1. *Set $R = A[X_1, \ldots, X_d]$, and let J be a monomial ideal of R. Given monomials $f, g \in [\![R]\!]$, one has*

$$[J + (f)R] \cap [J + (g)R] = J + [(f)R \cap (g)R] = J + (\operatorname{lcm}(f, g))R.$$

In particular, if $\operatorname{Supp}(f) \cap \operatorname{Supp}(g) = \emptyset$, then $[J + (f)R] \cap [J + (g)R] = J + (fg)R$.

Proof. Exercise 2.1.12(a) explains the first two steps in the next sequence:

$$\begin{aligned}
[J+(f)R]\cap[J+(g)R] &= (J\cap[J+(g)R])+[(f)R\cap[J+(g)R]] \\
&= (J\cap J)+[J\cap(g)R]+[(f)R\cap J]+[(f)R\cap(g)R] \\
&= J+[J\cap(g)R]+[(f)R\cap J]+[(f)R\cap(g)R] \\
&\subseteq J+[(f)R\cap(g)R] \\
&\subseteq [J+(f)R]\cap[J+(g)R].
\end{aligned}$$

The third step is from the equality $J\cap J=J$, and the fourth step follows from the containments $J\cap(f)R\subseteq J$ and $J\cap(g)R\subseteq J$. The fifth step follows from the containments $J+[(f)R\cap(g)R]\subseteq J+(f)R$ and $J+[(f)R\cap(g)R]\subseteq J+(g)R$. This explains the equation $[J+(f)R]\cap[J+(g)R]=J+[(f)R\cap(g)R]$, and the equality $J+[(f)R\cap(g)R]=J+(\mathrm{lcm}(f,g))R$ follows from Lemma 2.1.4.

Assume that $\mathrm{Supp}(f)\cap\mathrm{Supp}(g)=\emptyset$. It follows that $\mathrm{lcm}(f,g)=fg$, hence the equality $[J+(f)R]\cap[J+(g)R]=J+(fg)R$ follows. $\square$

Example 7.5.2. We show how to use Lemma 7.5.1 to decompose the ideal $I=(X^2Y^3,XZ^2,Y^2Z)R$ in $R=A[X,Y,Z]$.

$$\begin{aligned}
I &\overset{(1)}{=} (X^2Y^3,XZ^2)R+(Y^2Z)R \\
&\overset{(2)}{=} [(X^2Y^3,XZ^2)R+(Y^2)R]\cap[(X^2Y^3,XZ^2)R+(Z)R] \\
&\overset{(1)}{=} (X^2Y^3,XZ^2,Y^2)R\cap(X^2Y^3,XZ^2,Z)R \\
&\overset{(3)}{=} (XZ^2,Y^2)R\cap(X^2Y^3,Z)R \\
&\overset{(1)}{=} [(XZ^2)R+(Y^2)R]\cap[(X^2Y^3)R+(Z)R] \\
&\overset{(2)}{=} [(X)R+(Y^2)R]\cap[(Z^2)R+(Y^2)R]\cap[(X^2)R+(Z)R]\cap[(Y^3)R+(Z)R] \\
&\overset{(1)}{=} (X,Y^2)R\cap(Z^2,Y^2)R\cap(X^2,Z)R\cap(Y^3,Z)R
\end{aligned}$$

Each step (1) here is by Exercise A.4.6(a), and the steps labeled (2) are from Lemma 7.5.1. Step (3) is by redundancy-removal.

A compressed version of this that avoids the repeated uses of Exercise A.4.6(a) is as follows:

$$\begin{aligned}
I &= (X^2Y^3,XZ^2,Y^2Z)R && \text{split the last generator} \\
&= (X^2Y^3,XZ^2,Y^2)R\cap(X^2Y^3,XZ^2,Z)R && \text{remove redundancies} \\
&= (XZ^2,Y^2)R\cap(X^2Y^3,Z)R && \text{split each first generator} \\
&= (X,Y^2)R\cap(Z^2,Y^2)R\cap(X^2,Z)R\cap(Y^3,Z)R.
\end{aligned}$$

Here is a formal statement of the compressed technique from this example.

Lemma 7.5.3. *Set $R=A[X_1,\ldots,X_d]$, and let J be a monomial ideal of R with monomial generating sequence $h_1,\ldots,h_t$. Let $f,g\in[\![R]\!]$ be such that $h_t=fg$ and* $\mathrm{Supp}(f)\cap\mathrm{Supp}(g)=\emptyset$. *Then one has*

$$J = (h_1,\dots,h_{t-1},fg)R = (h_1,\dots,h_{t-1},f)R \cap (h_1,\dots,h_{t-1},g)R.$$

Proof. Exercise. $\square$

The next results are souped-up versions of Lemmas 7.5.1 and 7.5.3.

Lemma 7.5.4. *Set $R = A[X_1,\dots,X_d]$, and let J be a monomial ideal of R. Fix positive integers $m, i_1,\dots,i_m, a_1,\dots,a_m$ such that $1 < m \leqslant d$ and $1 \leqslant i_1 < \dots < i_m \leqslant d$. Then*

$$J + (X_{i_1}^{a_1} \cdots X_{i_m}^{a_m})R = \bigcap_{j=1}^{m} [J + (X_{i_j}^{a_j})R].$$

Proof. Proceed by induction on m. The base case $m = 2$ follows from Lemma 7.5.1. The induction step is an exercise. $\square$

Lemma 7.5.5. *Set $R = A[X_1,\dots,X_d]$, and let J be a monomial ideal of R with monomial generating sequence $h_1,\dots,h_t$. Write $h_t = X_{i_1}^{a_1} \cdots X_{i_m}^{a_m}$ for positive integers $m, i_1,\dots,i_m, a_1,\dots,a_m$ such that $1 < m \leqslant d$ and $1 \leqslant i_1 < \dots < i_m \leqslant d$. Then one has*

$$J = (h_1,\dots,h_{t-1},X_{i_1}^{a_1} \cdots X_{i_m}^{a_m})R = \bigcap_{j=1}^{m} (h_1,\dots,h_{t-1},X_{i_j}^{a_j})R.$$

Proof. Exercise. $\square$

Example 7.5.6. Set $R = A[X,Y,Z]$ and $I = (XY^3Z^5, X^6Y^4Z^2)R$. By splitting generators as in Lemma 7.5.5, we have the first and third steps in the next sequence:

$$\begin{aligned}
I &= (XY^3Z^5,X^6)R \cap (XY^3Z^5,Y^4)R \cap (XY^3Z^5,Z^2)R \\
&= (XY^3Z^5,X^6)R \cap (XY^3Z^5,Y^4)R \cap (Z^2)R \\
&= (X,X^6)R \cap (Y^3,X^6)R \cap (Z^5,X^6)R \cap (X,Y^4)R \\
&\quad \cap (Y^3,Y^4)R \cap (Z^5,Y^4)R \cap (Z^2)R \\
&= (X)R \cap (Y^3,X^6)R \cap (Z^5,X^6)R \cap (X,Y^4)R \cap (Y^3)R \cap (Z^5,Y^4)R \cap (Z^2)R \\
&= (X)R \cap (Y^3)R \cap (X^6,Z^5)R \cap (Y^4,Z^5)R \cap (Z^2)R.
\end{aligned}$$

The remaining steps follow by redundancy-removal.

The next result is a one-step version of the techniques from the above examples.

Theorem 7.5.7 *Set $R = A[X_1,\dots,X_d]$, and let I be a monomial ideal of R with monomial generating sequence $f_1,\dots,f_t$. For $i = 1,\dots,t$ write $f_i = \underline{X}^{\underline{a}_i}$ where $\underline{a}_i = (a_{i,1},\dots,a_{i,d}) \in \mathbb{N}^d$. Then we have*

$$I = \bigcap_{i_1=1}^{d} \cdots \bigcap_{i_t=1}^{d} (X_{i_1}^{a_{1,i_1}},\dots,X_{i_t}^{a_{t,i_t}})R.$$

Before proving this result, we give an example.

Example 7.5.8. Set $R = A[X,Y,Z]$ and $I = (XY^3Z^5, X^6Y^4Z^2)R$, as in Example 7.5.6. From Theorem 7.5.7 we have the first step in the next sequence:

$$\begin{aligned}
I &= (X,X^6)R \cap (X,Y^4)R \cap (X,Z^2)R \\
&\quad \cap (Y^3,X^6)R \cap (Y^3,Y^4)R \cap (Y^3,Z^2)R \\
&\quad \cap (Z^5,X^6)R \cap (Z^5,Y^4)R \cap (Z^5,Z^2)R \\
&= (X)R \cap (X,Y^4)R \cap (X,Z^2)R \\
&\quad \cap (X^6,Y^3)R \cap (Y^3)R \cap (Y^3,Z^2)R \\
&\quad \cap (X^6,Z^5)R \cap (Y^4,Z^5)R \cap (Z^2)R \\
&= (X)R \cap (Y^3)R \cap (X^6,Z^5)R \cap (Y^4,Z^5)R \cap (Z^2)R.
\end{aligned}$$

The remaining steps follow by redundancy-removal.

Proof (Proof of Theorem 7.5.7). We proceed by induction on t.

Base case: $t = 1$. In this case, we have

$$I = (f_1)R = (X_1^{a_{1,1}} \cdots X_d^{a_{1,d}})R = (X_1^{a_{1,1}})R \cap \cdots \cap (X_d^{a_{1,d}})R = \bigcap_{i_1=1}^{d} (X_{i_1}^{a_{1,i_1}})R$$

by Lemma 2.1.4. This is the desired formula.

Induction step: Assume that $t \geqslant 2$ and that the result holds for monomial ideals generated by $t-1$ monomials. Set $J = (f_2, \ldots, f_t)R$. By the induction hypothesis

$$J = \bigcap_{i_2=1}^{d} \cdots \bigcap_{i_t=1}^{d} (X_{i_2}^{a_{2,i_2}}, \ldots, X_{i_t}^{a_{t,i_t}})R$$

and this explains the fourth step in the next sequence:

$$\begin{aligned}
I &= J + (f_1)R \\
&= J + (X_1^{a_{1,1}} \cdots X_d^{a_{1,d}})R \\
&= \bigcap_{i_1=1}^{d} [J + (X_{i_1}^{a_{1,i_1}})R] \\
&= \bigcap_{i_1=1}^{d} \left[\left[\bigcap_{i_2=1}^{d} \cdots \bigcap_{i_t=1}^{d} (X_{i_2}^{a_{2,i_2}}, \ldots, X_{i_t}^{a_{t,i_t}})R \right] + (X_{i_1}^{a_{1,i_1}})R \right] \\
&= \bigcap_{i_1=1}^{d} \left[\bigcap_{i_2=1}^{d} \cdots \bigcap_{i_t=1}^{d} [(X_{i_2}^{a_{2,i_2}}, \ldots, X_{i_t}^{a_{t,i_t}})R + (X_{i_1}^{a_{1,i_1}})R] \right] \\
&= \bigcap_{i_1=1}^{d} \cdots \bigcap_{i_t=1}^{d} (X_{i_1}^{a_{1,i_1}}, \ldots, X_{i_t}^{a_{t,i_t}})R.
\end{aligned}$$

The first step is from Exercise A.4.6(a), and the second step is by assumption. The third step is from Lemma 7.5.4, and the fifth step follows from Lemma 7.3.2. The final step is by the associativity of intersection. □

Here is another technique for computing m-irreducible decompositions in general. The point is that, given a monomial ideal I, one creates an ideal J having a parametric decomposition (which we can compute via corner elements), and the result shows how to transform an irredundant parametric decomposition of J into an irredundant m-irreducible decomposition of I. (See Chapter 6 for our treatment of parameter ideals, parametric decompositions, and corner elements.)

Theorem 7.5.9 *Set $R = A[X_1,\ldots,X_d]$, and let I be a monomial ideal of R with monomial generating sequence $f_1,\ldots,f_t$. For $j = 1,\ldots,t$ write $f_j = \underline{X}^{\underline{a}_j}$ where $\underline{a}_j = (a_{j,1},\ldots,a_{j,d}) \in \mathbb{N}^d$. Fix an integer*

$$n > \max\{a_{j,i} \mid i = 1,\ldots,d \text{ and } j = 1,\ldots,t\}.$$

Set $J = (X_1^n,\ldots,X_d^n)R + I$, and let $J = \bigcap_{k=1}^{l} Q_k$ be an irredundant parametric decomposition of J. For $k = 1,\ldots,l$ let Q_k' be the m-irreducible ideal obtained by removing $X_1^n,\ldots,X_d^n$ from the irredundant generators of Q_k. Then $I = \bigcap_{k=1}^{l} Q_k'$ is an irredundant m-irreducible decomposition of I.

Before proving this result, we present an example.

Example 7.5.10. Set $R = A[X,Y,Z]$ and $I = (XY,XZ,YZ)R$. Following the notation of Theorem 7.5.9, we may set $n = 2$ and

$$J = (XY,XZ,YZ)R + (X^2,Y^2,Z^2)R = (XY,XZ,YZ,X^2,Y^2,Z^2)R = \mathfrak{X}^2$$

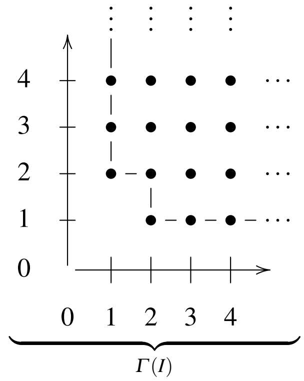
where $\mathfrak{X} = (X,Y,Z)R$. Exercise 6.6.1(b) shows that the J-corner elements are X,Y,Z and Theorem 6.2.9 yields the decomposition

$$J = \mathrm{P}_R(X) \cap \mathrm{P}_R(Y) \cap \mathrm{P}_R(Z) = (X^2,Y,Z)R \cap (X,Y^2,Z)R \cap (X,Y,Z^2)R.$$

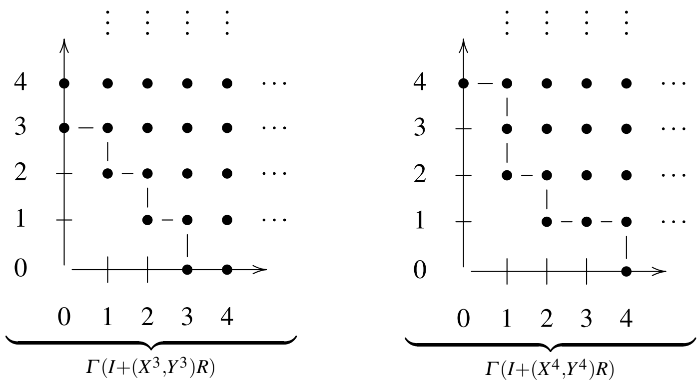
We remove the monomials X^2,Y^2,Z^2 from these ideals to obtain

$$I = (Y,Z)R \cap (X,Z)R \cap (X,Y)R.$$

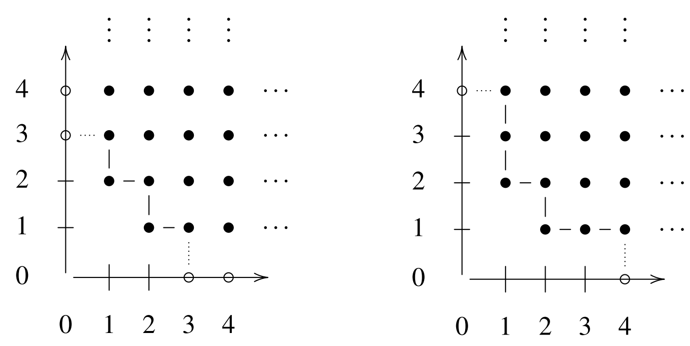
Graphically, the idea behind Theorem 7.5.9 is as follows. Set $R = A[X,Y]$, and, for example, consider the ideal $I = (X^2Y,XY^2)R$.



Following the notation of Theorem 7.5.9, we may choose any value $n \geqslant 3$. The next two graphs exhibit the ideal $I+(X^n,Y^n)R$ for the values $n=3,4$.



The point is that the ideal I can be seen as the part of J obtained by removing the generators X^n, Y^n.



Proof (Proof of Theorem 7.5.9). Recall that $I = (f_1,\dots,f_t)R$ with $f_i = \underline{X}^{\underline{a}_i}$ for $i = 1,\dots,t$. Given an integer $n > \max\{a_{j,i} \mid i = 1,\dots,d \text{ and } j = 1,\dots,t\}$, set

$$J = (X_1^n,\dots,X_d^n)R + I = (X_1^n,\dots,X_d^n,f_1,\dots,f_t)R;$$

see Exercise A.4.6(a). For each t-tuple $\underline{i} = (i_1,\dots,i_t)$ such that $1 \leqslant i_k \leqslant d$, set

$$Q_{\underline{i}} = (X_1^n,\dots,X_d^n,X_{i_1}^{a_{1,i_1}},\dots,X_{i_t}^{a_{t,i_t}})R$$
$$Q'_{\underline{i}} = (X_{i_1}^{a_{1,i_1}},\dots,X_{i_t}^{a_{t,i_t}})R.$$

Applying Theorem 7.5.7, we have

$$J = \bigcap_{i_1=1}^{d}\cdots\bigcap_{i_t=1}^{d}(X_1^n,\dots,X_d^n,X_{i_1}^{a_{1,i_1}},\dots,X_{i_t}^{a_{t,i_t}})R = \bigcap_{\underline{i}} Q_{\underline{i}} \tag{7.5.10.1}$$

$$I = \bigcap_{i_1=1}^{d}\cdots\bigcap_{i_t=1}^{d}(X_{i_1}^{a_{1,i_1}},\dots,X_{i_t}^{a_{t,i_t}})R = \bigcap_{\underline{i}} Q'_{\underline{i}}. \tag{7.5.10.2}$$

Exercise A.4.6(a) implies that $Q_{\underline{i}} = (X_1^n,\dots,X_d^n)R + Q'_{\underline{i}}$. Furthermore, since $n > a_{j,i_j}$, the ideal $Q'_{\underline{i}}$ is obtained by removing $X_1^n,\dots,X_d^n$ from the generators of $Q_{\underline{i}}$, as in the statement of the theorem.

Claim: Given t-tuples $\underline{i}$ and $\underline{j}$, one has $Q'_{\underline{i}} \subseteq Q'_{\underline{j}}$ if and only if $Q_{\underline{i}} \subseteq Q_{\underline{j}}$. For the forward implication, if $Q'_{\underline{i}} \subseteq Q'_{\underline{j}}$, then

$$Q_{\underline{i}} = (X_1^n,\dots,X_d^n)R + Q'_{\underline{i}} \subseteq (X_1^n,\dots,X_d^n)R + Q'_{\underline{j}} = Q_{\underline{j}}.$$

For the converse, assume that $Q_{\underline{i}} \subseteq Q_{\underline{j}}$. Since each monomial $X_{i_p}^{a_{p,i_p}}$ is in $Q_{\underline{i}}$, it follows that $X_{i_p}^{a_{p,i_p}} \in Q_{\underline{j}}$. To prove that $Q'_{\underline{i}} \subseteq Q'_{\underline{j}}$, it suffices to show that $X_{i_p}^{a_{p,i_p}} \in Q'_{\underline{j}}$. Theorem 1.1.9 implies that $X_{i_p}^{a_{p,i_p}}$ is in the ideal generated by one of the monomials from the list $X_1^n,\dots,X_d^n,X_{j_1}^{a_{1,j_1}},\dots,X_{j_t}^{a_{t,j_t}}$ of generators of $Q_{\underline{j}}$. Since $n > a_{p,i_p}$, a comparison of exponent vectors shows that $X_{i_p}^{a_{p,i_p}} \notin (X_k^n)R$ for $k = 1,\dots,n$. Thus, there is an index q such that $X_{i_p}^{a_{p,i_p}} \in (X_{j_q}^{a_{q,j_q}})R \subseteq Q'_{\underline{j}}$

Recall that an irredundant m-irreducible decomposition of I can be obtained from (7.5.10.2) by removing the ideals $Q'_{\underline{j}}$ such that there is an ideal $Q'_{\underline{i}}$ contained in $Q'_{\underline{j}}$. Similarly, an irredundant m-irreducible decomposition of J can be obtained from (7.5.10.1) by removing the ideals $Q_{\underline{j}}$ such that there is an ideal $Q_{\underline{i}}$ contained in $Q_{\underline{j}}$. From the above claim, we see that the ideals $Q_{\underline{i}}$ removed from the decomposition (7.5.10.1) are in 1-1 correspondence with the ideals $Q'_{\underline{i}}$ removed from the decomposition (7.5.10.2). In summary, if S is a set such that $J = \bigcap_{\underline{i}\in S} Q_{\underline{i}}$ is an irredundant m-irreducible decomposition of J, then $I = \bigcap_{\underline{i}\in S} Q'_{\underline{i}}$ is an irredundant m-irreducible decomposition of I. This establishes the theorem. $\square$

Exercises

Exercise 7.5.11. Prove Lemmas 7.5.3, 7.5.4, and 7.5.5.

Exercise 7.5.12. Verify the decompositions in the examples of this section using Theorem 2.1.5. Justify your answers.

Exercise 7.5.13. Use Theorem 7.5.9 to decompose the ideals in Examples 7.5.2 and 7.5.6. Justify your answer.

Exercise 7.5.14. Use Lemma 7.5.3 to decompose the ideal in Example 7.5.10. Justify your answer.

Exercise 7.5.15. Set $R = A[X,Y,Z]$ and $I = (X^2YZ, XY^2Z, XYZ^2)R$. Find an irredundant m-irreducible decomposition of I by using Lemma 7.5.5; then do it using Theorem 7.5.7, and and then using Theorem 7.5.9. Verify that your decomposition is correct using Theorem 2.1.5. Justify your answers.

Exercise 7.5.16. Set $R = A[X_1,\ldots,X_d]$, and let J be a non-zero monomial ideal of R with irredundant m-irreducible decomposition $J = \bigcap_{i=1}^m J_i$. Let $f_1,\ldots,f_t$ be a monomial generating sequence for J. Use Theorem 7.5.7 to prove that each ideal J_i has a generating sequence consisting of t monomials.

Exercise 7.5.17. Set $R = A[X_1,\ldots,X_d]$, and let I be a monomial ideal of R with monomial generating sequence $f_1,\ldots,f_t$. For $j = 1,\ldots,t$ write $f_j = \underline{X}^{\underline{a}_j}$ where $\underline{a}_j = (a_{j,1},\ldots,a_{j,d}) \in \mathbb{N}^d$. For $i = 1,\ldots,d$, fix an integer

$$n_i > \max\{a_{j,i} \mid j = 1,\ldots,t\}.$$

Set $J = (X_1^{n_1},\ldots,X_d^{n_d})R + I$, and let $J = \bigcap_{k=1}^l Q_k$ be an irredundant parametric decomposition of J. For $k = 1,\ldots,l$ let Q'_k be the m-irreducible ideal obtained by removing $X_1^{n_1},\ldots,X_d^{n_d}$ from the irredundant generators of Q_k. Then $I = \bigcap_{k=1}^l Q'_k$ is an irredundant m-irreducible decomposition of I. (This is a souped-up version of Theorem 7.5.9.)

Computing General M-Irreducible Decompositions in Macaulay2

The fact that Theorem 7.5.7 includes a variable number of nested intersections makes programming the algorithm slightly more complicated than usual. There are several approaches to implementing this, but perhaps the easiest way is via a recursive function.

A function is *recursive* if it is defined in terms of itself, together with some base cases guaranteeing that the function definition is not circular. The prototypical example of a recursive function is the factorial function, since for all positive integers n, one has $n! = n(n-1)!$ where $0! = 1$. One can implement this in Macaulay2 via the following code.

```
i1 : factorial = method()
o1 = factorial
o1 : MethodFunction

i2 : factorial ZZ := n -> if n == 0 then 1 else n*factorial(n-1)
o2 = {*Function[stdio:2:19-2:58]*}
o2 : FunctionClosure

i3 : factorial 4
o3 = 24
```

The way in which Macaulay2 evaluates `factorial 4` is by successively calling `factorial` on smaller and smaller integers until we finally reach `factorial 0` which is 1 by definition. Of course, Macaulay2 already includes the command `!` for computing $n!$, so we can test our method easily.

```
i4 : factorial 9 == 9!
o4 = true
```

In our case, we wish to use recursion to control the depth of the nested intersections in Theorem 7.5.7. We can do this via the following code.

```
i5 : mIrredDecomp = method()
o5 = mIrredDecomp
o5 : MethodFunction

i6 : mIrredDecomp MonomialIdeal := M -> (
gensM := M_*;
R := ring M;
firstMon := gensM#0;
if #gensM == 1 then
apply(numgens R, i -> monomialIdeal (R_i)^((first exponents firstMon)#i))
else (
flatten for i from 0 to (numgens R)-1 list (
flatten for c in mIrredDecomp monomialIdeal drop(gensM,1) list (
(monomialIdeal (R_i)^((first exponents firstMon)#i)) + c
)
)
)
)
o6 = {*Function[stdio:6:33-15:56]*}
o6 : FunctionClosure

i7 : exit
```

Exercises

Coding Exercise 7.5.18. Work through the definition of `mIrredDecomp` to make sure that you understand all the steps in the function. Where is the base case of the recursion? What is the purpose of the various `flatten` commands used in the code? Test this method with examples and exercises from the section.

Coding Exercise 7.5.19. Use the command from Exercise 6.4.10 together with Theorem 7.5.9 to implement a method that computes an irredundant m-irreducible decomposition of a monomial ideal I. Test your method with examples and exercises from the section.

Coding Exercise 7.5.20. Compare run times of the method implemented in the tutorial against run times of the method from the previous exercise. Which one is faster? Can you make improvements that speed up either of the methods?

7.6 Exploration: Edge Ideals, Face Ideals, and Facet Ideals Revisited

In this section, A is a non-zero commutative ring with identity. Set $R = A[X_1,\ldots,X_d]$. Here we investigate how Theorem 7.5.9 connects with the decomposition results from Chapter 4, beginning with the most natural case.

Exercise 7.6.1. Let Δ be a simplicial complex on $V = \{v_1,\ldots,v_d\}$; see Sections 4.4, 4.5, 4.6. Consider the face ideal J_Δ, and set $J = (X_1^2,\ldots,X_d^2)R + J_\Delta$.

(a) Prove that the faces of Δ are in bijection with the monomials of $[\![R]\!] \smallsetminus [\![J]\!]$. (Hint: Start with some examples. Don't forget to account for the fact that Δ may not contain each vertex $\{v_i\}$.) In other words, the faces of Δ are in 1-1 correspondence with the monomials in R/J. (In fact, this is one of the reasons that J_Δ is defined the way it is.)
(b) Prove that the facets of Δ are in 1-1 correspondence with the J-corner elements.
(c) Use parts (a)–(b) above with Theorems 6.2.9 and 7.5.9 to give a different proof of Theorem 4.5.4.

Next, we consider similar results for edge ideals of graphs and facet ideals of simplicial complexes.

Exercise 7.6.2. Let G be a graph on $V = \{v_1,\ldots,v_d\}$. Consider the edge ideal I_G from Sections 4.2–4.3, and set $J = (X_1^2,\ldots,X_d^2)R + I_G$.

(a) Prove that the vertex covers of G are in 1-1 correspondence with the monomials of $[\![R]\!] \smallsetminus [\![J]\!]$.
(b) Prove that the minimal vertex covers of G are in 1-1 correspondence with the J-corner elements.

(c) Use parts (a)–(b) above with Theorems 6.2.9 and 7.5.9 to give a different proof of Theorem 4.3.6.

Challenge Exercise 7.6.3. Formulate and prove a version of Exercise 7.6.2 for the facet ideal K_Δ of a simplicial complex Δ, in particular, to give a different proof of Theorem 4.6.5. (Hint: Theorem 4.6.5 tells you what data about Δ describes the monomials of $[\![R]\!] \smallsetminus [\![J]\!]$ where $J = (X_1^2, \ldots, X_d^2)R + K_\Delta$.) (See also Coding Exercise 7.6.7.)

As with the previous exercise, in the next result, look to the corresponding decomposition theorems from Chapter 4 to figure out which combinatorial data should describe the monomials of the appropriate $[\![R]\!] \smallsetminus [\![J]\!]$.

Challenge Exercise 7.6.4. Repeat Challenge Exercise 7.6.3 for the following decomposition results:

(a) The decomposition of the face ideal J_G of a graph G considered as a simplicial complex in Exercise 4.5.12(a).
(b) The decomposition of the face ideal $J_{\Delta(\Pi)}$ of the order complex $\Delta(\Pi)$ of a poset Π in Exercise 4.5.14(a).
(c) The decomposition of the facet ideal $K_{\Delta(\Pi)}$ of the order complex $\Delta(\Pi)$ of a poset Π in Challenge Exercise 4.6.14(c).

(See also Coding Exercise 7.6.8.)

Edge Ideals, Face Ideals, and Facet Ideals Revisited in Macaulay2, Exercises

Useful commands for the following exercises can be found in Chapter 4.

Coding Exercise 7.6.5. Given a `SimplicialComplex` object Δ, write a function that computes the set of faces of Δ using Exercise 7.6.1(a). (See Sections 4.4–4.5 for some relevant commands.) Compare your output for some examples and your code to that of the `faces` command in the `SimplicialComplexes` package; use the command `code` here. Use your code to explore the connection between facets and corner elements as in Exercise 7.6.1(b).

Coding Exercise 7.6.6. Repeat Coding Exercise 7.6.5 for the edge ideal of a graph G (*à la* Exercise 7.6.2), using `Graph` objects from the `EdgeIdeals` package. (See Sections 4.2–4.3 for some relevant commands.) Again compare your output and code to the existing code in the `EdgeIdeals` package, and explore the connection between minimal vertex covers and corner elements as in Exercise 7.6.2(b).

Coding Exercise 7.6.7. Repeat Coding Exercise 7.6.5 for the facet ideal (*à la* Challenge Exercise 7.6.3). (See Section 4.6 for relevant commands.)

Coding Exercise 7.6.8. Repeat Coding Exercise 7.6.5 for the constructions of Challenge Exercise 7.6.4.

7.7 Exploration: Decompositions of Saturations

In this section, A is a non-zero commutative ring with identity.

We continue the exploration of saturations from Section 2.7 with a view to their m-irreducible decompositions; see also Section 7.4. As with decompositions of colon ideals, we begin with a special case, then use general properties of the saturation to understand the general case.

Exercise 7.7.1. Set $R = A[X_1,\ldots,X_d]$. Let $k,t_1,\ldots,t_k,e_{t_1},\ldots,e_{t_k}$ be positive integers, and set $J = (X_{t_1}^{e_{t_1}},\ldots,X_{t_k}^{e_{t_k}})R$. Given a monomial $f = \underline{X}^{\underline{n}} \in [\![R]\!]$, use Proposition 7.4.1 to prove that

$$(J :_R (fR)^\infty) = \begin{cases} R & \text{if there is an index } i \text{ such that } n_{t_i} \geqslant 1 \\ J & \text{if for } i = 1,\ldots,k \text{ we have } n_{t_i} = 0 \end{cases}$$
$$= \begin{cases} R & \text{if } f \in \text{m-rad}\,(J) \\ J & \text{if } f \notin \text{m-rad}\,(J). \end{cases}$$

In particular, either the ideal $(J :_R f)$ is m-irreducible or $(J :_R f) = R$, moreover, the ideal $(J :_R f)$ is m-irreducible if and only if $f \notin J$.

Exercise 7.7.2. Use Exercise 7.7.1 to compute the saturation $(J :_R (fR)^\infty)$ for the following examples.

(a) $R = A[X,Y]$ and $J = (X^3,Y^4)R$ and $f = XY$
(b) $R = A[X,Y]$ and $J = (X^3)R$ and $f = Y^2$
(c) $R = A[X,Y,Z]$ and $J = (X^3,Y^4)R$ and $f = XY$
(d) $R = A[X,Y,Z]$ and $J = (X^3,Z^4)R$ and $f = Y^2$

Justify your answers.

Here is the general decomposition result for saturations. Note that the assumption $I \not\subseteq \text{m-rad}\,(J)$ guarantees that $(J :_R I^\infty) \neq R$.

Exercise 7.7.3. Set $R = A[X_1,\ldots,X_d]$. Let I be a monomial ideal of R with monomial generating sequence $f_1,\ldots,f_t$. Let J be a monomial ideal of R with m-irreducible decomposition $J = \bigcap_{i=1}^m J_i$. Assume that $I \not\subseteq \text{m-rad}\,(J)$. Use Exercises 2.7.6 and 7.7.1 to prove that an m-irreducible decomposition of $(J :_R I^\infty)$ is

$$(J :_R I^\infty) = \bigcap_{f_j \notin \text{m-rad}(J_i)} (J_i :_R (f_jR)^\infty)$$

where the intersection is taken over the set of all ordered pairs (i,j) such that $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant t$ and $f_j \notin \text{m-rad}\,(J_i)$.

Exercise 7.7.4. Use Exercise 7.7.3 to compute an irredundant m-irreducible decomposition of the saturation $(J :_R I^\infty)$ for each of the following examples.

(a) $R = A[X,Y]$ and $J = (X^3)R$ and $I = (XY, Y^2)R$
(b) $R = A[X,Y]$ and $J = (X^3, Y^4)R$ and $I = (XY, Y^2)R$
(c) $R = A[X,Y,Z]$ and $J = (X^3, Z^4)R$ and $I = (XY, Y^2)R$
(d) $R = A[X,Y,Z]$ and $J = (X^3, Y^4)R$ and $I = (XY, Y^2)R$
(e) $R = A[X,Y]$ and $J = (X^3, XY)R$ and $I = (XY, Y^2)R$
(f) $R = A[X,Y,Z]$ and $J = (X^3, YZ, Z^4)R$ and $I = (XY, Y^2)R$
(g) $R = A[X,Y,Z]$ and $J = (X^3Y^4, XZ^2, Y^3Z)R$ and $I = (XY, Y^2, YZ)R$

Justify your answers.

Next, we investigate the connections between saturations and the combinatorial constructions of Chapter 4.

Exercise 7.7.5. Set $R = A[X_1, \ldots, X_d]$. Let I be a monomial ideal of R, and let J be a square-free monomial ideal of R.

(a) Prove that $(J :_R I^\infty)$ is square-free.
(b) [Challenge] Repeat Challenge Exercise 7.4.21 for the saturation $(J :_R I^\infty)$. See Exercise 2.7.3(i). (See also Laboratory Exercise 7.7.8.)

The next exercise involves the construction $\mathrm{V}(I)$ from Exploration Section A.10.

Challenge Exercise 7.7.6. Let A be a field, and let I and J be monomial ideals of $R = A[X_1, \ldots, X_d]$. Exercise 3.3.11(b) shows that there is an integer $k \in \mathbb{N}$, and there are linear subspaces $V_1, \ldots, V_k \subseteq A^d$ such that $\mathrm{V}(I) = V_1 \cup \cdots \cup V_k$ and such that $V_i \not\subseteq V_j$ whenever $i \neq j$. This exercise also yields such irredundant subspace decompositions $\mathrm{V}(J) = W_1 \cup \cdots \cup W_l$ and $\mathrm{V}((J :_R I^\infty)) = U_1 \cup \cdots \cup U_m$. Describe the linear subspaces U_n in terms of the subspaces V_i and W_j and any other reasonable data about I you need. Justify your answer.

Decompositions of Saturations in Macaulay2, Exercises

See Section 2.7 for the Macaulay2 commands relevant to the next exercises.

Exercise 7.7.7. Use Macaulay2 to verify your computations from Exercise 7.7.2 and Exercise 7.7.4.

Laboratory Exercise 7.7.8. Set $A = \mathbb{Z}_{101}$. Use Macaulay2 to investigate $(I :_R J^\infty)$ for some specific monomial ideals I and J where J is square-free, to help with Challenge Exercise 7.7.5(b).

Challenge Exercise 7.7.9. This exercise investigates the behavior of saturations with respect to some topics from Chapters 5 and 6: dimension, depth, the Cohen-Macaulay property, and corner elements.

(a) Use Macaulay2 to compute $\dim(R/I)$ and $\dim(R/J)$ and $\dim(R/(J :_R I^\infty))$ for some monomial ideals I, J.

(b) Use the data from part (a) to make a conjecture about $\dim(R/(J :_R I^\infty))$.
(c) Prove your conjecture from part (b).
(d) Repeat part (a) for $\operatorname{depth}(R/I)$ and $\operatorname{depth}(R/J)$ and $\operatorname{depth}(R/(J :_R I^\infty))$. Does there seem to be a nice pattern here? If so, can you prove it? If not, can you identify any nice special cases where there is a nice pattern and prove it?
(e) Repeat part (d) for the Cohen-Macaulay property.
(f) Repeat part (d) for $\mathrm{C}_R((J :_R I^\infty))$. (See Exercise 6.4.10.)

7.8 Exploration: Decompositions of Generalized Bracket Powers

In this section, A is a non-zero commutative ring with identity. Set $R = A[X_1,\ldots,X_d]$, and fix a d-tuple $\underline{e} \in \mathbb{N}^d$ such that $e_1,\ldots,e_d \geqslant 1$. Our results for generalized bracket powers are very similar to the ones for regular bracket powers; see Section 2.8 for the basic properties of the generalized construction. We dive right in with the general decomposition result in this setting.

Exercise 7.8.1. Let I be a monomial ideal of R with m-irreducible decomposition $I = \bigcap_{j=1}^n I_j$.

(a) Prove that the ideal I is m-irreducible if and only if $I^{[\underline{e}]}$ is m-irreducible.
(b) Prove that an m-irreducible decomposition of $I^{[\underline{e}]}$ is $I^{[\underline{e}]} = \bigcap_{j=1}^n I_j{}^{[\underline{e}]}$.
(c) Prove that the decomposition $I = \bigcap_{j=1}^n I_j$ is irredundant if and only if the decomposition $I^{[\underline{e}]} = \bigcap_{j=1}^n I_j{}^{[\underline{e}]}$ is irredundant.

Exercise 7.8.2. Set $R = A[X,Y]$ and $\underline{e} = (2,3)$. Set $J = (X^3, X^2Y, Y^3)R$, and use Exercise 7.8.1 to find an irredundant m-irreducible decomposition of the ideal $J^{[\underline{e}]}$. Verify your decomposition using Theorem 2.1.5. Justify your answers.

For the next exercises, see Chapter 6 for our treatment of parameter ideals, parametric decompositions, and corner elements.

Exercise 7.8.3. Let I be a monomial ideal of R.

(a) Prove that I is a parameter ideal if and only if $I^{[\underline{e}]}$ is a parameter ideal.
(b) Prove that if I has a parametric decomposition $I = \bigcap_{j=1}^n I_j$, then $I^{[\underline{e}]} = \bigcap_{j=1}^n I_j{}^{[\underline{e}]}$ is a parametric decomposition of $I^{[\underline{e}]}$.
(c) Prove that I has a parametric decomposition if and only if $I^{[\underline{e}]}$ has one.

Exercise 7.8.4. Let I be a monomial ideal of R. For each monomial $f = \underline{X}^{\underline{n}}$ in R, set $f^{(\underline{e})} = X_1^{e_1(n_1+1)-1}\cdots X_d^{e_d(n_d+1)-1}$. Prove that the set of $I^{[\underline{e}]}$-corner elements is $\mathrm{C}_R(I^{[\underline{e}]}) = \{f^{(\underline{e})} \mid f \in \mathrm{C}_R(I)\}$.

Exercise 7.8.5. Use the results of this section to solve Exercise 6.6.4 and Challenge Exercise 6.6.5.

The following exercise involves the geometric construction $\mathrm{V}(I)$ from Exploration Section A.10.

Exercise 7.8.6. Let A be a field, and let I be a monomial ideal of $R = A[X_1, \ldots, X_d]$. Use Exercises 3.1.13 and 7.8.1(b) to give an alternate proof of Exercise 2.8.14.

Decompositions of Generalized Bracket Powers in Macaulay2, Exercises

See Section 2.8 for the Macaulay2 commands relevant to the next exercises.

Coding Exercise 7.8.7. Use Exercise 7.8.1 to devise a method that takes as input a list L of m-irreducible monomial ideals (which should be thought of as an irredundant m-irreducible decomposition of some ideal J) and a vector $\underline{e}$, and whose output is a list that is an irredundant m-irreducible decomposition of the generalized bracket power $I^{[\underline{e}]}$.

Coding Exercise 7.8.8.

(a) Devise another method that takes a monomial ideal J and a vector $\underline{e}$ as its input, uses the command `irreducibleDecomposition` to compute an irredundant m-irreducible decomposition of J, then uses your method from the previous exercise to compute an irredundant m-irreducible decomposition of the generalized bracket power $I^{[\underline{e}]}$.
(b) Devise another method that takes a monomial ideal J and a vector $\underline{e}$ as its input, computes the generalized bracket power $I^{[\underline{e}]}$ as in Section 2.8, and then uses the command `irreducibleDecomposition` to compute an irredundant m-irreducible decomposition of $I^{[\underline{e}]}$.
(c) Compare the outputs and run times of your methods from parts (a) and (b). Which is faster? Why do you think this is so?

Challenge Exercise 7.8.9. This exercise investigates the behavior of generalized bracket powers with respect to some topics from Chapter 5: dimension, depth, and Cohen-Macaulayness.

(a) Use Macaulay2 to compute $\dim(R/I)$ and $\dim(R/I^{[\underline{e}]})$ for some monomial ideals I and some vectors $\underline{e}$.
(b) Use the data you collect in part (a) to make a conjecture about $\dim(R/I^{[\underline{e}]})$.
(c) Prove your conjecture from part (b).
(d) Repeat part (a) for $\operatorname{depth}(R/I)$ and $\operatorname{depth}(R/I^{[\underline{e}]})$. Does there seem to be a nice pattern here? If so, can you prove it? If not, can you identify any nice special cases where there is a nice pattern and prove it?
(e) Repeat part (d) for the Cohen-Macaulay property.

7.9 Exploration: Decompositions of Products of Monomial Ideals

In this section, A is a non-zero commutative ring with identity.

The last section of this chapter investigates the problem of finding m-irreducible decompositions of products of monomial ideals; see Exercise 1.3.12. We provide an algorithm for computing such decompositions, but it is highly redundant; see Example 7.9.5. Recall that Section 6.6 deals with some special cases where nice decomposition behavior can be exhibited. We begin with the situation where one of the factor-ideals is principal.

Exercise 7.9.1. Let $J_1,\ldots,J_n,I$ be monomial ideals of R, and let $f \in [\![R]\!]$.

(a) Prove that $f(\bigcap_{i=1}^n J_i) = \bigcap_{i=1}^n (fJ_i)$.
(b) Must one have $I(\bigcap_{i=1}^n J_i) = \bigcap_{i=1}^n (IJ_i)$? Justify your answer.

Exercise 7.9.2. Fix integers $k,t_1,\ldots,t_k,e_1,\ldots,e_k \geqslant 1$ such that $t_1 < \cdots < t_k \leqslant d$, and set $J = (X_{t_1}^{e_1},\ldots,X_{t_k}^{e_k})R$. Let $f = \underline{X}^{\underline{a}} \in [\![R]\!]$, and set $J' = (X_{t_1}^{e_1+a_{t_1}},\ldots,X_{t_k}^{e_k+a_{t_k}})R$. Prove that $fJ = (f)R \cap J'$. (Hint: Compute a generating sequence for $(f)R \cap J'$.)

Exercise 7.9.3. Set $R = A[X,Y,Z]$. In the notation of Exercise 7.9.2, compute J' where $J = (X^2,Z^3)R$ and $f = Y^3Z^2$. Justify your answer.

Here is an algorithm for computing an m-irreducible decomposition of the product of two monomial ideals. The subsequent example shows how messy it is even for two relatively simple ideals.

Exercise 7.9.4. Let I and J be non-zero monomial ideals in R such that $J \neq R$. The following algorithm combines a monomial generating sequence for I with an m-irreducible decomposition of J to produce an m-irreducible decomposition of IJ.

Step 1. Let $f_1,\ldots,f_m$ be a monomial generating sequence for I. Note that Exercise A.4.6(b) implies that $I = \sum_{j=1}^m (f_j)$.

Step 2. For $j = 1,\ldots,m$ write $f_j = \underline{X}^{\underline{a}_j} \in [\![R]\!]$ where $\underline{a}_j = (a_{j,1},\ldots,a_{j,d}) \in \mathbb{N}^d$.

Step 3. Write each f_j in terms of positive exponents: For $j = 1,\ldots,m$ fix positive integers $l_j,s_{j,1},\ldots,s_{j,l_j}$ such that the exponents $a_{j,s_{j,1}},\ldots,a_{j,s_{j,l_j}}$ are positive and $f_j = X_{s_{j,1}}^{a_{j,s_{j,1}}} \cdots X_{s_{j,l_j}}^{a_{j,s_{j,l_j}}}$. Note that Theorems 2.1.5 and 3.1.3 imply that $(f_j)R = \bigcap_{p=1}^{l_j} (X_{s_{j,p}}^{a_{j,s_{j,p}}})R$ is an irredundant m-irreducible decomposition.

Step 4. Fix an m-irreducible decomposition $J = \bigcap_{i=1}^n J_i$ and positive integers $k_i,t_{i,1},\ldots,t_{i,k_i},e_{i,1},\ldots,e_{i,k_i}$ such that $t_{i,1} < \cdots < t_{i,k_i} \leqslant d$ and $J_i = (X_{t_{i,1}}^{e_{i,1}},\ldots,X_{t_{i,k_i}}^{e_{i,k_i}})R$. Set $J_{i,j} = (X_{t_{i,1}}^{e_{i,1}+a_{j,t_{i,1}}},\ldots,X_{t_{i,k_i}}^{e_{i,k_i}+a_{j,t_{i,k_i}}})R$.

Step 5. Decompose IJ as follows:

$$\begin{aligned}
IJ &= \left(\sum_{j=1}^{m}(f_j)R\right)\left(\bigcap_{i=1}^{n}J_i\right) && \text{by assumption (see Step 1)}\\
&= \sum_{j=1}^{m}\left(f_j\left(\bigcap_{i=1}^{n}J_i\right)\right) && \text{Exercise A.5.5(d)}\\
&= \sum_{j=1}^{m}\bigcap_{i=1}^{n}(f_jJ_i) && \text{Exercise 7.9.1(a)}\\
&= \sum_{j=1}^{m}\bigcap_{i=1}^{n}((f_j)R\cap J_{i,j}) && \text{Exercise 7.9.2}\\
&= \sum_{j=1}^{m}\left((f_j)R\cap\left(\bigcap_{i=1}^{n}J_{i,j}\right)\right) && \text{basic properties of intersections}\\
&= \sum_{j=1}^{m}\left(\left(\bigcap_{p=1}^{l_j}(X_{s_{j,p}}^{a_{j,s_{j,p}}})R\right)\cap\left(\bigcap_{i=1}^{n}J_{i,j}\right)\right) && \text{Step 3.}
\end{aligned}$$

Step 6. Use Exercise 7.3.10 to find an m-irreducible decomposition of the ideal $IJ = \sum_{j=1}^{m}((\bigcap_{p=1}^{l_j}(X_{s_{j,p}}^{a_{j,s_{j,p}}})R)\cap(\bigcap_{i=1}^{n}J_{i,j}))$.

Example 7.9.5. Set $R = A[X,Y]$. We use Exercise 7.9.4 to find an m-irreduci-ble decomposition of the ideal IJ where $I = (X,Y)R$ and $J = (X^2,XY,Y^2)R$. We use the monomial generating sequence X,Y for I, and the m-irreducible decomposition $J = (X^2,Y)R\cap(X,Y^2)R$. In the following computation, the equalities (1)–(5) follow the sequence of equalities in Exercise 7.9.4:

$$
\begin{aligned}
IJ &\overset{(1)}{=} [(X)R+(Y)R][(X^2,Y)R\cap(X,Y^2)R]\\
&\overset{(2)}{=} X[(X^2,Y)R\cap(X,Y^2)R]+Y[(X^2,Y)R\cap(X,Y^2)R]\\
&\overset{(3)}{=} [X(X^2,Y)R\cap X(X,Y^2)R]+[Y(X^2,Y)R\cap Y(X,Y^2)R]\\
&\overset{(4)}{=} [(X)R\cap(X^3,Y)R\cap(X)R\cap(X^2,Y^2)R]\\
&\quad +[(Y)R\cap(X^2,Y^2)R\cap(Y)R\cap(X,Y^3)R]\\
&\overset{(5)}{=} [(X)R\cap(X^3,Y)R\cap(X^2,Y^2)R]+[(Y)R\cap(X^2,Y^2)R\cap(X,Y^3)R]\\
&\overset{(6)}{=} [(X)R+(Y)R]\cap[(X)R+(X^2,Y^2)R]\cap[(X)R+(X,Y^3)R]\\
&\cap[(X^3,Y)R+(Y)R]\cap[(X^3,Y)R+(X^2,Y^2)R]\cap[(X^3,Y)R+(X,Y^3)R]\\
&\cap[(X^2,Y^2)R+(Y)R]\cap[(X^2,Y^2)R+(X^2,Y^2)R]\cap[(X^2,Y^2)R+(X,Y^3)R]\\
&\overset{(7)}{=} (X,Y)R\cap(X,Y^2)R\cap(X,Y^3)R\\
&\quad\cap(X^3,Y)R\cap(X^2,Y)R\cap(X,Y)R\\
&\quad\cap(X^2,Y)R\cap(X^2,Y^2)R\cap(X,Y^2)R\\
&\overset{(8)}{=} (X,Y^3)R\cap(X^3,Y)R\cap(X^2,Y^2)R
\end{aligned}
$$

The equality (6) is from Exercise 7.3.10, and (7) is from Exercise A.4.6(b). The equality (8) follows from an application of Algorithm 3.3.6.

(Note that $IJ=\mathfrak{X}^3$ where $\mathfrak{X}=(X,Y)R=I$, and this decomposition agrees with the one obtained in Example 6.2.6.)

Exercise 7.9.6. Set $R=A[X,Y]$, and use Exercise 7.9.4 as in the preceding example to find an m-irreduci-ble decomposition of the ideal IJ in each of the next cases:

(a) $I=(X,Y)R$ and $J=(X^3,XY,Y^2)R$.
(b) $I=(X^2,XY,Y^2)R=J$.

Our next exercise involves the construction $\mathrm{V}(I)$ from Exploration Section A.10.

Challenge Exercise 7.9.7. Let A be a field, and let I and J be monomial ideals of $R=A[X_1,\ldots,X_d]$. Exercise 3.3.11(b) shows that there is an integer $k\in\mathbb{N}$, and there are linear subspaces $V_1,\ldots,V_k\subseteq A^d$ such that $\mathrm{V}(I)=V_1\cup\cdots\cup V_k$ and such that $V_i\not\subseteq V_j$ whenever $i\neq j$. This exercise also yields such irredundant subspace decompositions $\mathrm{V}(J)=W_1\cup\cdots\cup W_l$ and $\mathrm{V}(IJ)=U_1\cup\cdots\cup U_m$. Describe the linear subspaces U_n in terms of the subspaces V_i and W_j. Justify your answer. (Hint: The decomposition results of this section may not be the most helpful tools here.)

Decompositions of Products of Monomial Ideals in Macaulay2, Exercises

Coding Exercise 7.9.8. Implement the algorithm outlined in Exercise 7.9.4. As in the tutorial to Section 7.5, one can either use nested `for` loops, `apply`s, recursion or a mixture of the three to achieve the desired result. Check your method against Example 7.9.5 and Exercise 7.9.6 to ensure its correctness, and compare the run times of this example to those implemented in Section 7.5.

Challenge Exercise 7.9.9. This exercise investigates the behavior of powers with respect to some topics from Chapter 5: dimension, depth, and Cohen-Macaulayness.

(a) Use Macaulay2 to compute $\dim(R/I)$ and $\dim(R/J)$ and $\dim(R/(IJ))$ for some monomial ideals I and J.
(b) Use the data you collect in part (a) to make a conjecture about $\dim(R/(IJ))$.
(c) Prove your conjecture from part (b).
(d) Repeat part (a) for $\operatorname{depth}(R/I)$ and $\operatorname{depth}(R/I^n)$. Does there seem to be a nice pattern here? If so, can you prove it? If not, can you identify any nice special cases where there is a nice pattern and prove it?
(e) Repeat part (d) for the Cohen-Macaulay property.

Challenge Exercise 7.9.10. If you have not already done so, complete Challenge Exercise 6.6.8.

Concluding Notes

Our presentation of Section 7.5 is based on lectures of Jung-Chen Liu [51]. Other algorithms can be found, e.g., in [25, 70].

The algorithm presented in Section 7.9 is particularly inefficient, in the sense that Step 5 of Exercise 7.9.4 generally produces a decomposition with many redundancies. (One can see this, in particular, in Example 7.9.5.) However, for instance, using a result of Andrew Crabbe, Daniel Katz, Janet Striuli, and Emanoil Theodorescu [14, Corollary 3.5], based on work of Vijay Kodiyalam [48], one can show the following: if I is a monomial ideal of $R = A[X_1,\ldots,X_d]$ where A is a field and $\text{m-rad}(I) = \mathfrak{X}$, then there is a polynomial $f(t) \in \mathbb{Q}[t]$ of degree $d-1$ such that $c_R(I^n) = f(n)$ for all $n \gg 0$. See Section 6.6 for some special cases of this. It would be interesting to know the leading coefficient of $f(t)$ in terms of basic invariants of I, and similarly for the least number n such that $c_R(I^n) = f(n)$.

Part IV
Commutative Algebra and Macaulay2

This part of the text consists of two appendices of fundamental material for use throughout the text. The first of these is Appendix A, which introduces important algebraic notions: rings, ideals, and related constructions. We do not expect that readers will see much of this material for the first time here, especially Sections A.1–A.3. Instead, we include these topics mostly for ease of reference and review. For this reason, much of this appendix is presented encyclopedically. On the other hand, we assume that other topics here are less familiar, especially the material on colon ideals and radicals in Sections A.6 and A.7, so these are presented somewhat more thoroughly. Note that these more specialized topics are only used in certain places in main part of the text, as is discussed in the Introduction.

The second appendix here is an introduction to the computer algebra system Macaulay2, presented roughly in parallel with Appendix A. While this material is not necessary for understanding the theorems and proofs in the bulk of the text, it is an incredibly useful tool for studying rings and ideals. Readers who plan to work through any of the Macaulay2 tutorials or exercises in the above chapters will probably want to work through much of this appendix, in particular Sections B.1–B.3, for the fundamental syntax needed to work in Macaulay2. If nothing else, this appendix serves as a handy reference to augment the online documentation

`http://www.math.uiuc.edu/Macaulay2/Documentation/`.

Appendix A
Foundational Concepts

This chapter contains a review of certain fundamental concepts in abstract algebra for use throughout the text. Section A.1 deals with the basic properties of commutative rings with identity, and Section A.2 focuses on polynomial rings. Section A.3 introduces ideals, while Sections A.4–A.7 investigate several methods for building new ideals from old ones. The chapter continues with Sections A.8 and A.9, dealing with quotient rings and relations on sets. It concludes with the Exploration Section A.10 which introduces fundamental constructions from algebraic geometry.

A.1 Rings

Monomial ideals, the focus of this text, are by definition subsets of rings. Loosely speaking, a ring is a set with addition and multiplication that models the basic properties of the sets of integers, rational numbers, real numbers, and so on. In this section, we briefly review some fundamental properties of rings, including important examples to keep in mind throughout the text. We begin with the definition.

Definition A.1.1. A *commutative ring with identity* is a set R equipped with two binary operations (addition and multiplication) satisfying the following axioms:

(1) (Additive Closure Law) for all $r,s \in R$ we have $r+s \in R$;
(2) (Additive Associative Law) for all $r,s,t \in R$ we have $(r+s)+t = r+(s+t)$;
(3) (Additive Commutative Law) for all $r,s \in R$ we have $r+s = s+r$;
(4) (additive identity) there exists an element $z \in R$ such that for all $r \in R$, we have $z+r = r$;
(5) (additive inverse) for each $r \in R$ there exists an element $s \in R$ such that $r+s = z$ where z is the additive identity;
(6) (Mulitplicative Closure Law) for all $r,s \in R$ we have $rs \in R$;
(7) (Mulitplicative Associative Law) for all $r,s,t \in R$ we have $(rs)t = r(st)$;
(8) (Mulitplicative Commutative Law) for all $r,s \in R$ we have $rs = sr$;

(9) (multiplicative identity) there exists an element $m \in R$ such that for all $r \in R$, we have $mr = r$;
(10) (Distributive Law) for all $r,s,t \in R$, we have $r(s+t) = rs + rt$.

Here are the aforementioned examples. You should convince yourself of any properties here that are unfamiliar. Note that one important example is missing: polynomial rings. We treat these in the next section.

(a) The set of integers $\mathbb{Z}$ with the usual addition and multiplication, and with identities $z = 0$ and $m = 1$, is a commutative ring with identity.
(b) The set of rational numbers $\mathbb{Q}$ with the usual addition and multiplication, and with $z = 0$ and $m = 1$, is a commutative ring with identity.
(c) The set of real numbers $\mathbb{R}$ with the usual addition and multiplication, and with $z = 0$ and $m = 1$, is a commutative ring with identity.
(d) The set of complex numbers $\mathbb{C}$ with the usual addition and multiplication, and with $z = 0$ and $m = 1$, is a commutative ring with identity.
(e) The set of natural numbers $\mathbb{N} = \{n \in \mathbb{Z} \mid n \geqslant 0\}$ with the usual addition and multiplication is not a commutative ring with identity because it does not have additive inverses.
(f) The set of even integers $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ with the usual addition and multiplication is not a commutative ring with identity because it does not have a multiplicative identity.
(g) The set $\mathrm{M}_2(\mathbb{R})$ of 2×2 matrices with entries in $\mathbb{R}$, with the usual addition and multiplication is not a commutative ring with identity because multiplication is not commutative.
(h) Fix an integer $n \geqslant 2$ and let $\mathbb{Z}_n = \{m \in \mathbb{Z} \mid 0 \leqslant m < n\}$. For $r,s \in \mathbb{Z}_n$ we define operations on $\mathbb{Z}_n$ by the following formulas:

$$\begin{aligned} r \oplus s &= \text{the remainder after } r+s \text{ is divided by } n \\ r \odot s &= \text{the remainder after } rs \text{ is divided by } n. \end{aligned}$$

With $z = 0$ and $m = 1$, this makes $\mathbb{Z}_n$ into a commutative ring with identity. Note that when $0 < m < n$, the additive inverse of m in $\mathbb{Z}_n$ is $n - m$.
(i) The set $\mathrm{C}(\mathbb{R})$ of continuous functions $f\colon \mathbb{R} \to \mathbb{R}$, with pointwise addition and multiplication, and with the constant functions $z = 0$ and $m = 1$, is a commutative ring with identity.
(j) The set $\mathrm{D}(\mathbb{R})$ of differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$, with pointwise addition and multiplication, and with the constant functions $z = 0$ and $m = 1$, is a commutative ring with identity.

For the rest of this section, let R be a commutative ring with identity.

The following facts (and others in this chapter) are standard exercises showing that arithmetic in a commutative ring with identity is very similar to arithmetic in $\mathbb{Z}$. One point to take away from this is that these properties follow formally from the ring-axioms, not from specific properties of special examples.

(a) The additive and multiplicative identities for R are unique; we denote them 0_R and 1_R, respectively, or 0 and 1 when the context is clear.
(b) For each $r \in R$, the additive inverse of r in R is unique; we denote it $-r$.
(c) (Cancellation Law) Let $r,s,t \in R$. If $r+s = r+t$, equivalently, if $s+r = t+r$, then $s = t$.
(d) For all $r \in R$, we have $0_R r = 0_R$.
(e) For all $r,s \in R$, we have $(-r)s = -(rs) = r(-s)$.
(f) For all $r \in R$, we have $-r = (-1_R)r$. This implies that $-0_R = 0_R$.
(g) For all $r \in R$, we have $-(-r) = r$. This implies that for all $r,s \in R$, we have $(-r)(-s) = rs$.

Where there are addition and additive inverses, there is also subtraction, which we define next.

Definition A.1.2. For all $r,s \in R$ we set $r - s = r + (-s)$.

Here are a few properties of subtraction. Again, these follow from the axioms (and from the above properties), not just for examples.

(a) (Subtractive Closure Law) For all $r,s \in R$ we have $r - s \in R$.
(b) (Distributive Law) For all $r,s,t \in R$ we have $r(s-t) = rs - rt$.
(c) For all $r,s \in R$ we have $-(r-s) = s - r$.
(d) For all $r \in R$ we have $r - r = 0_R$.

Now that we have a general context where it makes sense to add or multiply two elements, it also makes sense to add or multiply more than two elements. To be careful, one defines these operations inductively, and we begin this with nr and r^n for appropriate integers n, then continue with more general sums and products.

Definition A.1.3. Let $r \in R$.

(a) Set $0r = 0_R$ and $1r = r$. Inductively, for each $n \in \mathbb{N}$, define $(n+1)r = (nr) + r$.
(b) Set $(-1)r = -r$. Inductively, for each $n \in \mathbb{N}$ with, define $(-n-1)r = (-n)r - r$.
(c) Set $r^0 = 1_R$ and $r^1 = r$. Inductively, for each $n \in \mathbb{N}$ with, define $r^{n+1} = r^n r$.

Here are a few properties of these notions. If you verify any of these, be careful to treat both cases $n \geqslant 0$ and $n \leqslant 0$ when $n \in \mathbb{Z}$, and work to be formal in your proofs by using induction.

(a) (Associative Law) For all $r,s \in R$ and $n \in \mathbb{Z}$ we have $(nr)s = n(rs) = r(ns)$.
(b) (Distributive Law) For all $r \in R$ and $m,n \in \mathbb{Z}$ we have $(m+n)r = mr + nr$ and $(mn)r = m(nr)$.
(c) For all $r,s \in R$ and $n \in \mathbb{N}$ we have $(rs)^n = r^n s^n$.
(d) For all $r \in R$ and $m,n \in \mathbb{N}$ we have $r^m r^n = r^{m+n}$ and $(r^m)^n = r^{mn}$.

Definition A.1.4. Let $n \geqslant 1$ be an integer, and let $r_1, \ldots, r_n \in R$.

We define the sum $\sum_{i=1}^n r_i = r_1 + \cdots + r_n$ inductively. For $n = 1,2$ we have $\sum_{i=1}^1 r_i = r_1$ and $\sum_{i=1}^2 r_i = r_1 + r_2$. For $n \geqslant 3$, we define $\sum_{i=1}^n r_i = (\sum_{i=1}^{n-1} r_i) + r_n$.

We define the product $\prod_{i=1}^{n} r_i = r_1 \cdots r_n$ inductively. For $n = 1,2$ we have $\prod_{i=1}^{1} r_i = r_1$ and $\prod_{i=1}^{2} r_i = r_1 r_2$. For $n \geqslant 3$, we define $\prod_{i=1}^{n} r_i = (\prod_{i=1}^{n-1} r_i) r_n$.

As degenerate cases of these definitions, we define the *empty sum* $\sum_{i=1}^{0} r_i = 0$ and *empty product* $\prod_{i=1}^{0} r_i = 1$. Note that these are compatible with the definitions of $0r$ and r^0 above.

Given an integer $n \geqslant 1$ and elements $r_1, \ldots, r_n, s_1, \ldots, s_n \in R$, we have the following properties.

(a) (Generalized Closure Laws) We have $\sum_{i=1}^{n} r_i \in R$ and $\prod_{i=1}^{n} r_i \in R$.
(b) (Generalized Associative Laws) Any two "meaningful sums" of the elements $r_1, \ldots, r_n$ in this order are equal. For instance, we have

$$((r+s)+t)+u = (r+s)+(t+u) = r+((s+t)+u)$$

for all $r,s,t,u \in R$. Any two "meaningful products" of the elements $r_1, \ldots, r_n$ in this order are equal.
(c) (Generalized Commutative Law) Given any permutation $i_1, \ldots, i_n$ of the numbers $1, \ldots, n$ we have $r_1 + \cdots + r_n = r_{i_1} + \cdots + r_{i_n}$ and $r_1 \cdots r_n = r_{i_1} \cdots r_{i_n}$.
(d) (Generalized Distributive Law) For all sequences $r_1, \ldots, r_m, s_1, \ldots, s_n \in R$ we have $(\sum_{i=1}^{m} r_i)(\sum_{j=1}^{n} s_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} r_i s_j$. (There are more general Generalized Distributive Laws for products of more than two sums. One verifies them by induction on the number of sums.)

(If you find parts (b) and (c) to be difficult, consult [44, I.1.6–7].)

The next notation is absolutely fundamental for this text, as it contains the fundamental building blocks for our monomial ideals.

Definition A.1.5. A *monomial* in the elements $X_1, \ldots, X_d \in R$ is an element of the form $X_1^{n_1} \cdots X_d^{n_d} \in R$ where $n_1, \ldots, n_d \in \mathbb{N}$. For short, we write $\underline{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ and $\underline{X}^{\underline{n}} = X_1^{n_1} \cdots X_d^{n_d}$.

Given elements $X_1, \ldots, X_d \in R$, it is straightforward, for instance, to show the following: for all $\underline{m}, \underline{n} \in \mathbb{N}^d$ and $p \in \mathbb{N}$ we have $\underline{X}^{\underline{m}} \underline{X}^{\underline{n}} = \underline{X}^{\underline{m}+\underline{n}}$ and $(\underline{X}^{\underline{m}})^p = \underline{X}^{p\underline{m}}$. Here we define addition and scalar multiplication in $\mathbb{N}^d$ coordinate-wise: $\underline{m} + \underline{n} = (m_1 + n_1, \ldots, m_d + n_d) \in \mathbb{N}^d$ and $p\underline{n} = (pn_1, \ldots, pn_d) \in \mathbb{N}^d$.

It is worth noting that the axioms for commutative rings with identity allow for subtraction but not division. This omission is not accidental, in fact, it is crucial for us to be able to study rings like $\mathbb{Z}$ which are not closed under division. Of course, there are some elements that we can always divide by, namely ± 1. The next definition identifies the elements of a ring that we can divide by.

Definition A.1.6. An element $r \in R$ is a *unit* in R if there exists an element $s \in R$ such that $sr = 1_R$; such an element s is a *multiplicative inverse* for r.

Here is what happens for some of our fundamental examples.

(a) Every non-zero element of $\mathbb{Q}$ is a unit, and similarly for $\mathbb{R}$ and $\mathbb{C}$.

(b) An integer m is a unit in $\mathbb{Z}$ if and only if $m = \pm 1$.
(c) Let $n \in \mathbb{Z}$ with $n \geqslant 2$, and let $m \in \mathbb{Z}_n$. Then m is a unit in $\mathbb{Z}_n$ if and only if $\gcd(m,n) = 1$.

On the other hand, in a general commutative ring with identity, we have the following properties. Note that item (b) says that one cannot divide by 0 in a non-zero ring.

(a) If r is a unit in R then it has a unique multiplicative inverse; we denote the multiplicative inverse of r by r^{-1}.
(b) If $1_R \neq 0_R$, i.e., if $R \neq 0$, and r is a unit in R, then $r \neq 0_R$.

Rings where one can divide by any non-zero element are particularly special. We use these heavily in Chapter 5.

Definition A.1.7. The ring R is a *field* if $1_R \neq 0_R$ and every non-zero element of R is a unit in R.

Here is what happens for our standard examples.

(a) The rings $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are fields.
(b) The ring $\mathbb{Z}$ is not a field.
(c) Let $n \in \mathbb{Z}$ with $n \geqslant 1$. The ring $\mathbb{Z}_n$ is a field if and only if n is prime.

It is frequently useful in mathematics when a new object can be defined in terms of another more familiar one. In algebra, one way this is accomplished is via subrings, defined next.

Definition A.1.8. A subset $S \subseteq R$ is a *subring* of R provided that it is a commutative ring under the operations of R with identity $1_S = 1_R$.

For example, the ring $\mathbb{Z}$ is a subring of $\mathbb{Q}$, and $\mathbb{Q}$ is a subring of $\mathbb{R}$, which is a subring of $\mathbb{C}$. Also, in an arbitrary commutative ring R with identity, given a subring $S \subseteq R$, we always have $0_S = 0_R$.

On the other hand, we now show that the condition $1_S = 1_R$ in the preceding definition is not automatic. Let $R = \{\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mid a,b \in \mathbb{R}\}$ and $S = \{\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) \mid a \in \mathbb{R}\}$. Then R and S are commutative rings with identity, under the standard addition and multiplication of matrices. However, we have $1_R = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $1_S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, so $1_R \neq 1_S$.

The last notion of of this section is divisibility. We'll see in the main body of the text that many questions about monomial ideals boil down to questions of divisibility of monomials.

Definition A.1.9. Let $r,s \in R$. We say that r *divides* s when there is an element $t \in R$ such that $s = rt$. When r divides s, we write $r \big| s$.

For example, in $\mathbb{Z}$ we have $2 \big| 4$ but $3 \nmid 4$. On the other hand, if r is a unit in R, then $r \big| s$ for all $s \in R$. Thus, if R is a field and $0 \neq r \in R$, then $r \big| s$ for all $s \in R$.

Exercises

Exercise A.1.10. Verify any unfamiliar facts from this section that we left unproved.

Exercise A.1.11. Define $F_R\colon \mathbb{Z} \to R$ by the formula $F_R(n) = n1_R$. Prove that for all $m,n \in \mathbb{Z}$ we have $F_R(m+n) = F_R(m) + F_R(n)$ and $F_R(mn) = F_R(m)F_R(n)$.

**Exercise A.1.12 (Binomial Theorem).* Let $r,s \in R$. Prove that for each positive integer n, there is an equality $(r+s)^n = \sum_{i=0}^{n} \binom{n}{i} r^i s^{n-i}$. (This exercise is used in the proof of Proposition A.7.2.)

Exercise A.1.13. Let $f,g,h \in R$. Prove or disprove the following: If $h \neq 0_R$ and $fh = gh$, then $f = g$. Justify your answer.

Exercise A.1.14 (subring test). Let $S \subseteq R$. Prove that S is a subring of R if and only if $1_R \in S$ and S is closed under the subtraction and multiplication of R.

A.2 Polynomial Rings

In this section, A is a commutative ring with identity.

This section introduces, the main rings of study in this text: polynomial rings. We assume that the reader has some familiarity with polynomials from college algebra and calculus, in one, two, and three variables.

The first point of this section is to places these objects on a somewhat rigorous footing. Intuitively, we know that a polynomial in one variable X with coefficients in $\mathbb{R}$ are the functions that are finite sums $a_0 + a_1X + a_2X^2 + \cdots + a_nX^n$ with $a_0, \ldots, a_n \in \mathbb{R}$. But what does this mean if we wish to take coefficients in another ring like $\mathbb{Z}_2$? Here we have to be careful because $x^2 = x$ for all $x \in \mathbb{Z}_2$; does this mean that the polynomial $x^2 - x$ should be 0 in this setting? No. So, usually we define a polynomial in one variable X with coefficients in $\mathbb{Z}_2$ as a finite "formal sum" $a_0 + a_1X + a_2X^2 + \cdots + a_nX^n$ with $a_0, \ldots, a_n \in \mathbb{Z}_2$. But one must deal with the question of existence here. If these polynomials are different from the functions they represent, where do the polynomials live? The answer (described in the sketch of proof for the next result) showcases the power of the formalism of abstract algebra.

Theorem A.2.1 *There is a commutative ring with identity $A[X]$ such that:*

(a) *The ring A is a subring of $A[X]$;*
(b) *There is an element $X \in A[X]$ such that for every $f \in A[X]$, there exist $n \in \mathbb{N}$ and $a_0, a_1 \ldots, a_n \in A$ such that $f = a_0 + a_1X + \cdots + a_nX^n$;*
(c) *The elements $1_A, X, X^2, X^3, \ldots$ are linearly independent over A, that is, we have $a_0 + a_1X + \cdots + a_nX^n = 0_A$ if and only if $a_0 = a_1 = \cdots = a_n = 0_A$; and*
(d) *If $A \neq 0$, then X is in the complement $A[X] \smallsetminus A$, that is $X \in A[X]$ and $X \notin A$.*

Proof (Sketch of proof). Let $A[X]$ be the set of sequences in A that are eventually 0:

$$A[X] = \{(a_0, a_1, a_2, \ldots) \mid a_0, a_1, a_2, \ldots \in A \text{ and } a_i = 0_A \text{ for all } i \gg 0\}.$$

(Intuitively, a sequence $(a_0, a_1, a_2, \ldots, a_n, 0_A, 0_A, \ldots)$ corresponds to the list of coefficients of the polynomial $a_0 + a_1X + a_2X^2 + \cdots + a_nX^n$.)

We define addition and multiplication in $A[X]$ as follows:

$$\begin{aligned}(a_0, a_1, a_2, \ldots) + (b_0, b_1, b_2, \ldots) &= (a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots)\\ (a_0, a_1, a_2, \ldots)(b_0, b_1, b_2, \ldots) &= (c_0, c_1, c_2, \ldots)\end{aligned}$$

where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$. It is straightforward to show that the axioms for a commutative ring with identity are satisfied with $0_{A[X]} = (0_A, 0_A, 0_A, \ldots)$ and $1_{A[X]} = (1_A, 0_A, 0_A, \ldots)$.

For each $a \in A$, we identify a with the sequence $(a, 0_A, 0_A, \ldots)$. This identification makes A into a subset of $A[X]$. It is straightforward to show that the addition and multiplication are compatible under this identification.

Set $X = (0_A, 1_A, 0_A, 0_A, \ldots)$ and note that

$$X^n = (\underbrace{0_A, 0_A, \ldots, 0_A}_{n \text{ terms}}, 1_A, 0_A, 0_A, \ldots).$$

It is straightforward to show that properties (b)–(d) are satisfied. $\square$

We now define some terms in this context that should mostly be familiar for polynomials with coefficients in $\mathbb{R}$.

Definition A.2.2. The ring $A[X]$ is the *polynomial ring* in one variable with coefficients in A. An element $f \in A[X]$ is a *polynomial* in one variable with coefficients in A. The (unique) elements $a_0, a_1 \ldots, a_n \in A$ such that $f = a_0 + a_1X + \cdots + a_nX^n$ are the *coefficients* of f. The element $X \in A[X]$ is the *variable* or *indeterminate*. If $0 \neq f \in A[X]$, then the smallest $n \in \mathbb{N}$ such that f can be written in the form $f = a_0 + a_1X + \cdots + a_nX^n$ is the *degree* of f; the corresponding coefficient a_n is the *leading coefficient* of f. If the leading coefficient of f is 1_A, then f is *monic*. The coefficient a_0 is the *constant term* of f. Elements of A are sometimes called *constant polynomials*. We often employ the notation $f = \sum_{i \in \mathbb{N}}^{\text{finite}} a_i X^i$ for elements $f \in A[X]$.

Here are a few elementary remarks about $A[X]$.

(a) The constant polynomial $0_A \in A[X]$ does not have a well-defined degree. (On the other hand, it can be useful to define $\deg(0)$ to be -1 or $-\infty$.)
(b) Theorem A.2.1(c) implies that if $a_0 + a_1X + \cdots + a_mX^m = b_0 + b_1X + \cdots + b_nX^n$ in $A[X]$ and $m \leqslant n$, then $a_i = b_i$ for $i = 0, \ldots, m$ and $b_i = 0_A$ for $i = m+1, \ldots, n$. In other words, we have $\sum_{i \in \mathbb{N}}^{\text{finite}} a_i X^i = \sum_{i \in \mathbb{N}}^{\text{finite}} b_i X^i$ if and only if $a_i = b_i$ for all i.
(c) The ring A is identified with the set of constant polynomials in $A[X]$:

$$A = \{a + 0_AX + 0_AX^2 + \cdots \mid a \in A\}.$$

(d) A polynomial $f = a_0 + a_1X + \cdots + a_nX^n \in A[X]$ gives rise to a well-defined function $f\colon A \to A$ by the rule $f(b) = a_0 + a_1b + \cdots + a_nb^n$.

An important property of $A[X]$ is the following *Division Algorithm*.

Fact A.2.3. Let $f, g \in A[X]$ be such that f is monic. Then there are unique polynomials $q, r \in A[X]$ such that $g = fq + r$ and either $r = 0$ or $\deg(r) < \deg(f)$.

For the next definition, see A.1.5 for the notation $\underline{X}^{\underline{n}}$.

Definition A.2.4.

(a) Inductively, for $d \geqslant 2$, the *polynomial ring* in d variables $X_1, \ldots, X_d$ with coefficients in A is $A[X_1, \ldots, X_d] = A[X_1, \ldots, X_{d-1}][X_d]$. For a small number of variables, we sometimes write $A[X,Y]$ and $A[X,Y,Z]$.
(b) The *polynomial ring* in infinitely many variables $X_1, X_2, X_3, \ldots$ with coefficients in A is $A[X_1, X_2, X_3, \ldots] = \bigcup_{d=1}^{\infty} A[X_1, \ldots, X_d]$.
(c) The *(total) degree* of a monomial $f = \underline{X}^{\underline{n}} \in A[X_1, \ldots, X_d]$ is the integer $\deg(f) = |\underline{n}| = n_1 + \cdots + n_d$.

One sometimes defines the "degenerate" polynomial ring $A[X_1, \ldots, X_d]$ with $d = 0$ to be A, i.e., the polynomial ring with no variables. This possibly allows for slightly more general results, but we implicitly assume $d \geqslant 1$ throughout this text.

Corollary A.2.5 *The set $A[X_1, \ldots, X_d]$ is a commutative ring with identity satisfying the following properties:*

(a) *The ring A is a subring of $A[X_1, \ldots, X_d]$.*
(b) *For every $f \in A[X_1, \ldots, X_d]$, there is a finite collection of indices $\underline{n} \in \mathbb{N}^d$ and elements $a_{\underline{n}} \in A$ such that*

$$f = \sum_{\underline{n} \in \mathbb{N}^d}^{\textit{finite}} a_{\underline{n}} \underline{X}^{\underline{n}} = \sum_{n_1, \ldots, n_d \in \mathbb{N}}^{\textit{finite}} a_{n_1, \ldots, n_d} X_1^{n_1} \cdots X_d^{n_d}.$$

(c) *The set of monomials $\{\underline{X}^{\underline{n}} \mid \underline{n} \in \mathbb{N}^d\}$ is linearly independent over A, that is, we have $\sum_{\underline{n} \in \mathbb{N}^d}^{\textit{finite}} a_{\underline{n}} \underline{X}^{\underline{n}} = 0_A$ if and only if each $a_{\underline{n}} = 0_A$.*

Proof. By induction on d. The base case $d = 1$ is in Theorem A.2.1. □

Definition A.2.6. For a polynomial $f \in A[X_1, \ldots, X_d]$, we say that a monomial $\underline{X}^{\underline{n}}$ *occurs* in f if the corresponding coefficient $a_{\underline{n}}$ is non-zero. The *total degree* of f is the maximum degree among the monomials occurring in f. The polynomial f is *homogeneous (of degree e)* if every monomial occurring in f has degree e. (Note that the constant polynomial 0 is vacuously homogeneous of degree e for all e, so again it does not have a well-defined degree.)

For example, in the ring $R = \mathbb{Z}[X,Y,Z]$, the monomials occurring in the polynomial $3XY - 7X^2Z^3$ are XY and X^2Z^3; this polynomial is not homogeneous because monomials of degree 2 and 5 occur in it. On the other hand, the polynomial $X^2Y^2 + XYZ^2$ is homogeneous of degree 4.

Exercises

Exercise A.2.7. Perform the following polynomial computations, showing your steps and simplifying your answers.

(a) In $\mathbb{Q}[X,Y,Z]$: $(3XY+7Z^2-XY^2Z+5)(X+Z-Y^2+X^3Y^2Z)$.
(b) In $\mathbb{Z}_4[X]$: $(2X^2+2)^2$. This shows that one can have $f^n=0$ even when $f\neq 0$.
(c) In $\mathbb{Z}_4[X]$: $(2X^2+1)^2$. This shows that $\deg(f^n)$ can be strictly smaller than $n\deg(f)$; see Exercise A.2.10.

**Exercise A.2.8.* Consider polynomials $f_1,\dots,f_n\in A[X]$. Assume that each polynomial has degree at most N, that is, we have $f_i=a_{i,0}+a_{i,1}X+\cdots+a_{i,N}X^N$ for $i=1,\dots,n$. Prove that if $\sum_{i=1}^n a_{i,N}\neq 0_A$, then $\sum_{i=1}^n f_i$ has degree N and leading coefficient $\sum_{i=1}^n a_{i,N}$. (This exercise is used in the proof of Theorem 1.4.5.)

Exercise A.2.9. Prove Fact A.2.3. (Hint: For existence, argue by strong induction on $\deg(g)$. If $g=0$ or $\deg(g)<\deg(f)$, use $q=0$ and $r=g$. If $n=\deg(g)\geqslant\deg(f)=m$, let b_n be the leading coefficient of g, and apply the induction hypothesis to $\widehat{g}=g-b_nX^{n-m}f$.)

Exercise A.2.10. Let $f,g\in A[X_1,\dots,X_d]$.

(a) Prove that if $fg\neq 0$, then $\deg(fg)\leqslant\deg(f)+\deg(g)$.
(b) Prove that if $f^n\neq 0$, then $\deg(f^n)\leqslant n\deg(f)$.
(c) Prove that if $f,g,f+g\neq 0$, then $\deg(f+g)\leqslant\max\{\deg(f),\deg(g)\}$.

Exercise A.2.11. Under what conditions can $A[X]$ be a field? Justify your answer with a proof.

Exercise A.2.12. Let p be a prime number and set $R=\mathbb{Z}_p[X_1,\dots,X_d]$. Let $f,g\in R$. Prove that for each integer $e\geqslant 1$ one has $(f+g)^{p^e}=f^{p^e}+g^{p^e}$. Show that the analogous result for $(f+g)^k$ need not hold when k is not a power of p.

Exercise A.2.13. Which of the next polynomials in $\mathbb{Q}[X,Y,Z]$ are homogeneous? $3XY+7Z^2-XY^2Z+5$ and $XZ^5+Y^5Z-XY^2Z^2+X^3Y^2Z$. Justify your answers.

Exercise A.2.14. Prove that every product of two homogeneous polynomials in $A[X_1,\dots,X_d]$ is homogeneous.

Exercise A.2.15. Prove that a polynomial $f=\sum_{\underline{n}\in\mathbb{N}^d}^{\text{finite}} a_{\underline{n}}\underline{X}^{\underline{n}}\in A[\underline{X}]$ gives rise to a well-defined evaluation-function $f\colon A^d\to A$ by the rule $f(\underline{x})=\sum_{\underline{n}\in\mathbb{N}^d}^{\text{finite}} a_{\underline{n}}\underline{x}^{\underline{n}}$.

Exercise A.2.16. Find a commutative ring with identity A and a non-zero $f\in A[X]$ such that the induced function $f\colon A\to A$ is the zero function. Justify your answer.

A.3 Ideals and Generators

In this section, R is a commutative ring with identity.

The following definition was first made by Richard Dedekind, as a generalization of Ernst Kummer's "ideal numbers". The definition extends the well-known properties of even integers under addition and multiplication, namely that the sum of two even integers is even, and the product of any integer and an even integer is even.

Definition A.3.1. An *ideal* of R is a subset $I \subseteq R$ satisfying the following:

(1) $I \neq \emptyset$;
(2) (Additive Closure Law) for all $a, b \in I$ we have $a + b \in I$;
(3) (External Multiplicative Closure Law) for all $r \in R$ and $a \in I$ we have $ra \in I$.

Here is some of what happens for our standard examples.

(a) For each $n \in \mathbb{Z}$, the set $n\mathbb{Z} = \{nm \mid m \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z}$. For instance, if $n = 2$, then we have the set of even integers $2\mathbb{Z}$, which is an ideal of $\mathbb{Z}$. On the other hand, the set of odd integers is not an ideal of $\mathbb{Z}$ because it is not closed under addition.
(b) The ring $\mathbb{Q}$ has precisely two ideals: 0 and $\mathbb{Q}$; and similarly for $\mathbb{R}$ and $\mathbb{C}$.
(c) If R is the polynomial ring $A[X_1, \ldots, X_d]$ where A is some commutative ring with identity, then the set of polynomials in R with constant term 0 is an ideal of R.
(d) The set $I_2 = \{f \in \mathrm{C}(\mathbb{R}) \mid f(2) = 0\}$ is an ideal of $\mathrm{C}(\mathbb{R})$, but the similar set $\{f \in \mathrm{C}(\mathbb{R}) \mid f(2) = 1\}$ is not an ideal of $\mathrm{C}(\mathbb{R})$ as it is not closed under addition.

Next, we describe some fundamental properties of ideals. As in the previous sections of this chapter, any unfamiliar items should be treated as exercises.

Fact A.3.2. Let $I \subseteq R$ be an ideal.

(a) We have $0_R \in I$.
(b) (Additive Inverse Closure Law) If $a \in I$, then $-a \in I$.
(c) (Subtractive Closure Law) For all $a, b \in I$ we have $a - b \in I$.
(d) For all $r \in R$ and all $a \in I$ we have $r + a \in I$ if and only if $r \in I$.
(e) (Generalized Closure Law) For all $r_1, \ldots, r_n \in I$ we have $\sum_{i=1}^{n} r_i \in I$.
(f) The sets $\{0_R\}$ and R are ideals of R. Moreover $\{0_R\}$ is the unique smallest ideal of R and R is the unique largest ideal of R. We often write 0 for the ideal $\{0_R\}$.
(g) We have $I = R$ if and only if $1_R \in I$.

Much of this text is built on the fact that intersections of ideals are also ideals. Indeed, the first major goal of this text is to show that every monomial ideal can be written as a finite intersection of ideals that are somehow simpler than most ideals. (We formalize this in Part I of the text.)

Fact A.3.3. Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be a set of ideals of R.

(a) The intersection $\bigcap_{\lambda\in\Lambda} I_\lambda$ is an ideal of R. Note that in the case $\Lambda = \emptyset$, we use the convention $\bigcap_{\lambda\in\emptyset} I_\lambda = R$, that is, the empty intersection in R is R.
(b) The union $\bigcup_{\lambda\in\Lambda} I_\lambda$ need not be an ideal of R. For instance, if I and J are ideals of R, then $I \cup J$ is an ideal of R if and only if either $I \subseteq J$ or $J \subseteq I$.
(c) If $\{I_\lambda\}_{\lambda\in\Lambda}$ is a chain of ideals of R, then $\bigcup_{\lambda\in\Lambda} I_\lambda$ is an ideal of R. (Recall that the set of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ is a *chain* if, for every $\lambda,\mu \in \Lambda$ we have either $I_\lambda \subseteq I_\mu$ or $I_\mu \subseteq I_\lambda$. For instance, any increasing sequence $I_1 \subseteq I_2 \subseteq \cdots$ is a chain, though chains can generally be more exotic than this.)

The importance of the next concept cannot be overstated. Generators of ideals are akin to spanning vectors of a vector space. For instance, each of these concepts has the capability of turning infinite problems into finite ones. We will see in the main body of this text that, for monomial ideals, nice generating sequences turn complicated algebraic problems into relatively simple combinatorial exercises.

Definition A.3.4.

(a) For each subset $S \subseteq R$, the *ideal generated by* S is the intersection of all the ideals of R containing S:
$$(S)R = \bigcap_{S\subseteq J} J.$$
The subset $S \subseteq R$ is a *generating set* for an ideal $I \subseteq R$ when $I = (S)R$.
(b) For each sequence $s_1,\ldots,s_n \in R$, we simplify notation by writing $(s_1,\ldots,s_n)R$ in place of $(\{s_1,\ldots,s_n\})R$. An ideal is *finitely generated* if it can be generated by a finite list of elements, that is, by a finite set.
(c) An ideal is *principal* if it can be generated by a single element, i.e., it is of the form $(s)R$ (also written as sR) for some $s \in R$.

For each subset S, there is an ideal J containing S, namely $J = R$. Thus, the intersection defining $(S)R$ is not the empty intersection. Moreover, every ideal J containing S also contains 0_R, so we have $0_R \in (S)R$. Given another subset $T \subseteq R$, if $S \subseteq T$, then $(S)R \subseteq (T)R$.

Example A.3.5. One has $R = (1_R)R$ and $\{0_R\} = (0_R)R = (\emptyset)R$.

If R is the polynomial ring $A[X_1,\ldots,X_d]$ where A is some commutative ring with identity, we frequently write $\mathfrak{X}$ for the ideal $(X_1,\ldots,X_d)R$ generated by the variables. Note that $\mathfrak{X}$ is exactly the set of polynomials in R with constant term 0; see, e.g., the next proposition.

Let $S \subseteq R$ be a subset. As in the previous section, we will often find it convenient to use the non-standard notation $t = \sum_{s\in S}^{\text{finite}} sr_s$ instead of writing out $t = \sum_{i=1}^{n} s_i r_i$ for some $n \geqslant 0$ and $s_1,\ldots,s_n \in S$ and $r_1,\ldots,r_n \in R$. Here, the elements r_s are in R, and the superscript "finite" signifies that we are assuming that all but finitely many of the r_s are equal to 0_R.

Note that in part (e) of the next result, we are not claiming that every ideal has a finite generating set.

Proposition A.3.6 *Let $S \subseteq R$.*

(a) *The set $(S)R$ is the smallest ideal of R containing S. In particular, the set $(S)R$ is an ideal of R and, for each ideal $I \subseteq R$, we have $S \subseteq I$ if and only if $(S)R \subseteq I$.*
(b) *We have*
$$(S)R = \left\{ \sum_{s \in S}^{\textit{finite}} s r_s \in R \mid r_s \in R \right\}.$$
(c) *For each sequence $s_1, \ldots, s_n \in R$, we have*
$$(s_1, \ldots, s_n)R = \{\sum_{i=1}^{n} s_i r_i \in R \mid r_1, \ldots, r_n \in R\}.$$
(d) *For each element $s \in R$, we have*
$$(s)R = sR = \{sr \in R \mid r \in R\}.$$
In other words, an element $t \in R$ is in $(s)R$ if and only if $s \big| t$.
(e) *If I is an ideal, then $(I)R = I$. In particular, every ideal has a generating set.*
(f) *If $I = (S)R$ is finitely generated, then there exist elements $s_1, \ldots, s_n \in S$ such that $I = (s_1, \ldots, s_n)R$.*

Proof. We verify parts (a) and (b). Parts (c)–(d) are special cases of (b), and (e)–(f) are left as exercises.

(a) Fact A.3.3(a) shows that $(S)R$ is an ideal of R. By construction, we have $S \subseteq (S)R$; in particular, if $(S)R \subseteq I$, then $S \subseteq (S)R \subseteq I$. Moreover, if I is an ideal of R containing S, then I occurs in the intersection defining $(S)R$, so we have $(S)R \subseteq I$. It follows that $(S)R$ is the unique smallest ideal of R containing S.

(b) Set $T = \{\sum_{s \in S}^{\text{finite}} s r_s \in R \mid r_s \in R\}$. In the case $S = \emptyset$, recall that the empty sum $\sum_{i=1}^{0} s_i r_i$ is 0_R by convention. From this, it follows directly that $T = 0$, so we have $T = 0 = (\emptyset)R$, by Example A.3.5. Thus, we assume $S \neq \emptyset$ for the rest of this part.

We next show that T is an ideal of R. From the equality $0_R = \sum_{s \in S}^{\text{finite}} s 0_R$, we conclude that $0_R \in T$, and so $T \neq \emptyset$. To show that T is closed under addition and external multiplication, fix elements $a, b \in T$ and $r \in R$. Write $a = \sum_{s \in S}^{\text{finite}} s t_s$ and $b = \sum_{s \in S}^{\text{finite}} s u_s$ for elements $t_s, u_s \in R$. Then, we have
$$\begin{aligned} a + b &= (\textstyle\sum_{s \in S}^{\text{finite}} s t_s) + (\textstyle\sum_{s \in S}^{\text{finite}} s u_s) = \textstyle\sum_{s \in S}^{\text{finite}} s(t_s + u_s) \in T \\ ra &= r(\textstyle\sum_{s \in S}^{\text{finite}} s t_s) = \textstyle\sum_{s \in S}^{\text{finite}} s(r t_s) \in T. \end{aligned}$$
Hence the set T is an ideal of R.

Next, we observe that $S \subseteq T$. For this, fix an element $s_0 \in S$ and set
$$r_s = \begin{cases} 1_R & \text{when } s = s_0 \\ 0_R & \text{when } s \neq s_0. \end{cases}$$
With these choices, it follows directly that $s_0 = \sum_{s \in S}^{\text{finite}} s r_s \in T$, as desired.

Next, let $I \subseteq R$ be an ideal; we show that $S \subseteq I$ if and only if $T \subseteq I$. One implication is straightforward: if $T \subseteq I$, then the previous paragraph implies that $S \subseteq T \subseteq I$.

For the converse, assume that $S \subseteq I$. Since I is closed under external multiplication, we conclude that $sr_s \in I$ for all $s \in S$ and all $r_s \in R$. Since I is closed under finite sums, it follows that $\sum_{s\in S}^{\text{finite}} sr_s \in I$ for all $s \in S$ and all $r_s \in R$. That is, every element of T is in I, as desired.

It now follows directly that T is the unique smallest ideal of R containing S. Thus, we have $T = (S)R$ by part (a). $\square$

Exercises

Exercise A.3.7. Verify any unfamiliar facts from this section that we left unproved.

Exercise A.3.8. Let $R = \mathbb{Z}[X]$. Prove or disprove:

(a) The set K of all constant polynomials in R is an ideal of R.
(b) The set I of all polynomials $\sum_i^{\text{finite}} a_i X^i \in R$ with a_0 even is an ideal of R.

Justify your answers.

Exercise A.3.9. Verify the following equalities for ideals in $R = \mathbb{Q}[X,Y]$:

(a) $(X+Y, X-Y)R = (X,Y)R$.
(b) $(X+XY, Y+XY, X^2, Y^2)R = (X,Y)R$.
(c) $(2X^2+3Y^2-11, X^2-Y^2-3)R = (X^2-4, Y^2-1)R$.

This shows that the same ideal can have many different generating sets and that different generating sets may have different numbers of elements.

Exercise A.3.10. For each of the sets in Exercise A.3.8 that is an ideal, find a finite generating set. Prove that the set actually generates the ideal.

Exercise A.3.11. Let A be a non-zero commutative ring with identity. Set $R = A[X,Y]$ and prove that the ideal $(X,Y)R$ is not principal.

Exercise A.3.12. Finish the proof of Proposition A.3.6.

**Exercise A.3.13.* If $I \subseteq \mathbb{Z}$ is an ideal, then there exists an integer $m \in \mathbb{Z}$ such that $I = m\mathbb{Z}$. (Hint: If $I = 0$ then $m = 0$, so you may assume that $I \neq 0$. In this case, show that I has a smallest positive element m; then use the Division Algorithm to show that $I = m\mathbb{Z}$.) (This exercise is used in Example 1.4.4.)

Exercise A.3.14. Given integers m,n that are not both zero, prove that the ideal $(m,n)\mathbb{Z}$ is generated by $\gcd(m,n)$. (Hint: Use the Euclidean Algorithm.)

Exercise A.3.15. Prove the following:

(a) An element $r \in R$ is a unit in R if and only if $rR = R$.
(b) Let $I \subseteq R$ be an ideal. Then $I = R$ if and only if I contains a unit. (For this reason, the ideal $R = (1_R)R$ is often called the *unit ideal*.)

(c) If $1_R \neq 0_R$, then R is a field if and only if the only ideals of R are 0 and R.

Exercise A.3.16. Let $f, g, f_1, \ldots, f_n \in R$. Prove or disprove the following:

(a) If $f \in (f_1, \ldots, f_n)R$, then $f \in (f_i)R$ for some $i = 1, \ldots, n$.
(b) If $f \in (f_i)R$ for some $i = 1, \ldots, n$, then $f \in (f_1, \ldots, f_n)R$
(c) If $ff_j = gf_i$ and $f_i \neq f_j$, then $f \in (f_i)R$ and $g \in (f_j)R$.

Justify your answers.

Exercise A.3.17. Prove or disprove the following: Fix $f_1, \ldots, f_m, g_1, \ldots, g_n \in R$ and an integer $k \geqslant 1$. If $(f_1, \ldots, f_m)R = (g_1, \ldots, g_n)R$, then $(f_1^k, \ldots, f_m^k)R = (g_1^k, \ldots, g_n^k)R$. Justify your answers.

Exercise A.3.18. Prove or disprove the following: given subsets $S, T \subseteq R$, if $(S)R \subseteq (T)R$, then $S \subseteq T$. Justify your answer.

A.4 Sums of Ideals

In this section, R is a commutative ring with identity.

Recall from Fact A.3.3(b) that the union of ideals may not be an ideal; that is, the set of ideals of R is not closed under unions in general. The sum of ideals, defined next, serves as a substitute for unions. For us, one of the most important properties for sums is how they interact with generators; see Exercise A.4.6.

Definition A.4.1.

(a) Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be a set of ideals of R. The *sum* of these ideals is the ideal generated by the union of these ideals:

$$\sum_{\lambda \in \Lambda} I_\lambda = \left(\bigcup_{\lambda \in \Lambda} I_\lambda \right) R.$$

(b) Let n be a positive integer and let $I_1, I_2, \ldots, I_n$ be ideals of R. We sometimes write $I_1 + I_2 + \cdots + I_n$ in place of $\sum_{j=1}^n I_j$. In particular, when $n = 2$ we often write $I_1 + I_2$ instead of $\sum_{j=1}^2 I_j$.

For example, if m and n are integers that are not both zero, then $m\mathbb{Z} + n\mathbb{Z} = \gcd(m,n)\mathbb{Z}$. This can be verified directly with the Euclidean Algorithm, or using Exercises A.3.14 and A.4.6(a).

Theorem A.4.2 *Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be a collection of ideals of R.*

(a) $\sum_{\lambda \in \Lambda} I_\lambda$ *is the intersection of all the ideals of R containing* $\bigcup_{\lambda \in \Lambda} I_\lambda$.
(b) $\sum_{\lambda \in \Lambda} I_\lambda$ *is the unique smallest ideal of R containing* $\bigcup_{\lambda \in \Lambda} I_\lambda$.
(c) *For each ideal $K \subseteq R$, we have $\sum_{\lambda \in \Lambda} I_\lambda \subseteq K$ if and only if $\bigcup_{\lambda \in \Lambda} I_\lambda \subseteq K$.*

(d) *We have*

$$\sum_{\lambda\in\Lambda} I_\lambda = \{\sum_{\lambda\in\Lambda}^{\mathit{finite}} a_\lambda \mid a_\lambda \in I_\lambda\}.$$

(e) *Given a positive integer n and ideals $I_1, I_2, \ldots, I_n$ of R, we have*

$$\sum_{j=1}^n I_j = I_1 + I_2 + \cdots + I_n = \{\sum_{i=1}^n a_i \mid a_i \in I_i \textit{ for } i = 1,2,\ldots,n\}.$$

(f) *Given ideals I and J of R, we have*

$$I + J = \{a + b \mid a \in I, b \in J\}.$$

Proof. Part (a) is by definition. Parts (b)–(c) follow by Proposition A.3.6. Part (d) is proved like Proposition A.3.6(a). Parts (e)–(f) are special cases of part (d). □

Exercises

Exercise A.4.3. Prove Theorem A.4.2(d).

**Exercise A.4.4.*

(a) Let I and J be ideals of R. Prove that $I + J = J$ if and only if $I \subseteq J$.
(b) Let n be a positive integer and let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$ be ideals of R. Prove that $\sum_{j=1}^n I_j = I_n$.
(c) Let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a chain of ideals of R. Prove that $\sum_{\lambda\in\Lambda} I_\lambda = \bigcup_{\lambda\in\Lambda} I_\lambda$.

(This exercise is used in the proof of Theorem 1.3.3.)

Exercise A.4.5.

(a) Prove that if I is an ideal in R, then $0 + I = I$ and $R + I = R$.
(b) (Commutative Law) Prove that if I and J are ideals of R, then $I + J = J + I$. More generally, prove that if $\{I_\lambda\}_{\lambda\in\Lambda}$ is a collection of ideals of R and $f\colon \Lambda \to \Lambda$ is a bijection, then $\sum_{\lambda\in\Lambda} I_\lambda = \sum_{\lambda\in\Lambda} I_{f(\lambda)}$.
(c) (Associative Law) Prove that if I, J and K are ideals of R, then $(I + J) + K = I + J + K = I + (J + K)$. (More general Associative Laws, using more than three ideals, hold by induction on the number of ideals.)

**Exercise A.4.6.*

(a) Let $I = (f_1, \ldots, f_n)R$ and let $J = (g_1, \ldots, g_m)R$. Prove that $I + J$ is generated by the set $\{f_1, \ldots, f_n, g_1, \ldots, g_m\}$.
(b) Let n be a positive integer and let $S_1, S_2, \ldots, S_n$ be subsets of R. Prove that $\sum_{j=1}^n (S_j)R = (\bigcup_{j=1}^n S_j)R$.
(c) Prove that if $\{S_\lambda\}_{\lambda\in\Lambda}$ is a collection of subsets of R, then $\sum_{\lambda\in\Lambda} (S_\lambda)R = (\bigcup_{\lambda\in\Lambda} S_\lambda)R$.

(This is used in Example 1.1.13 Theorem 1.3.3, Theorem 2.5.4, Proposition 7.3.1, Example 7.5.2, Theorem 7.5.7, Theorem 7.5.9, Exercise 7.9.4, and Example 7.9.5.)

A.5 Products and Powers of Ideals

In this section, R is a commutative ring with identity.

Given ideals I,J of R, Theorem A.4.2(f) says that the sum $I+J$ is equal to the set $\{a+b \mid a \in I \text{ and } b \in J\}$. One might be tempted to define the product IJ by a similar formula $\{ab \mid a \in I \text{ and } b \in J\}$; however, this set is rarely an ideal. This is similar to the issue we had with unions of ideals, and it is solved in the following similar manner.

Definition A.5.1. Let I and J be ideals of R. Define the *product* of I and J to be the ideal IJ generated by all products xy, where $x \in I$ and $y \in J$:

$$IJ = (\{xy \mid x \in I, y \in J\})R.$$

Similarly, we define the product of any finite family of ideals $I_1,\ldots,I_n$ of R to be the ideal $I_1I_2\cdots I_n$ generated by all products $x_1x_2\cdots x_n$, where $x_i \in I_i$ for $i=1,2,\ldots,n$:

$$I_1I_2\cdots I_n = (\{x_1x_2\cdots x_n \mid x_i \in I_i \text{ for } i=1,2,\ldots,n\})R.$$

Given an ideal I of R and a positive integer n, we define

$$I^n = \underbrace{II\cdots I}_{n \text{ factors}}$$

that is, I^n is the ideal $I_1I_2\cdots I_n$, where $I_i = I$ for each $i=1,2,\ldots,n$. When $I \neq 0$, we define $I^0 = R$. We leave 0^0 undefined.

For example, if m and n are integers, then $(m\mathbb{Z})(n\mathbb{Z}) = (mn)\mathbb{Z}$. This can be proved directly, or as a consequence of Exercise A.5.6(a).

The following fundamental properties of products come from Proposition A.3.6.

Fact A.5.2. Let $I,J,I_1,\ldots,I_n$ be ideals of R.

(a) The ideal IJ is the smallest ideal of R that contains the set $\{ab \mid a \in I \text{ and } b \in J\}$. In particular, an ideal K contains the product IJ if and only if it contains each product of the form ab where $a \in I$ and $b \in J$.
(b) The product ideal $I_1\cdots I_n$ is the unique smallest ideal of R containing the set $\{a_1\cdots a_n \mid a_j \in I_j \text{ for } j=1,\ldots,n\}$. In particular, an ideal K contains the product $I_1\cdots I_n$ if and only if it contains each product of the form $a_1\cdots a_n$ where $a_j \in I_j$ for $j=1,\ldots,n$.
(c) There are equalities

$$IJ = \left\{\sum_k^{\text{finite}} a_kb_k \mid a_k \in I \text{ and } b_k \in J \text{ for each } k\right\}$$
$$I_1\cdots I_n = \left\{\sum_k^{\text{finite}} a_{1,k}\cdots a_{n,k} \mid a_{j,k} \in I_j \text{ for each } j \text{ and each } k\right\}.$$

Even though the notation for the nth power of an ideal is the same as the notation for the n-fold cartesian product of I with itself, these objects are not the same. We work to be clear which construction we mean at a given location.

Exercises

Exercise A.5.3. Let I and J be ideals of R. Set $K = \{ab \mid a \in I, b \in J\}$. Prove or disprove: K is an ideal of R. Justify your answer.

Exercise A.5.4. Let $I, J, I_1, \ldots, I_n$ be ideals of R. Prove that there are containments $IJ \subseteq I \cap J$ and $I_1 \cdots I_n \subseteq I_1 \cap \cdots \cap I_n$ and $I^n \subseteq I$. Show by example that these containments may be proper or not. Justify your answers

**Exercise A.5.5.* Let $I, J, K, I_1, \ldots, I_n$ be ideals of R.

(a) (0 and 1) Prove that $RI = I$ and $0I = 0$.
(b) (Commutative Law) Prove that $IJ = JI$. Moreover, prove that if $i_1, \ldots, i_n$ is a permutation of the numbers $1, \ldots, n$, then $I_1 \cdots I_n = I_{i_1} \cdots I_{i_n}$.
(c) (Associative Law) Prove that $(IJ)K = IJK = I(JK)$. (More general Associative Laws hold by induction on the number of ideals.)
(d) (Distributive Law) Prove that $(I+J)K = IK + JK = K(I+J)$. (More general Distributive Laws hold by induction on the number of ideals involved.)

(This exercise is used in Exercise 7.9.4.)

Exercise A.5.6.

(a) Prove that if $I = (f_1, \ldots, f_n)R$ and $J = (g_1, \ldots, g_m)R$, then

$$IJ = (\{f_i g_j \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\})R.$$

(b) Prove that if $I_j = (S_j)R$ for $j = 1, \ldots, n$ then

$$I_1 \cdots I_n = (\{s_1 \cdots s_n \mid s_i \in S_i \text{ for } i = 1, \ldots, n\})R.$$

(c) Prove that if n is a positive integer and $I = (f_1, \ldots, f_m)R$, then

$$I^n = (\{f_{i_1} \cdots f_{i_n} \mid 1 \leqslant i_j \leqslant m \text{ for } j = 1, \ldots, n\})R.$$

Exercise A.5.7. Set $R = \mathbb{Z}_2[X,Y]$ and $I = (X,Y)R$. Prove that

$$I^2 = (X^2, XY, Y^2)R \supsetneq (X^2, Y^2)R = (\{f^2 \mid f \in I\})R.$$

This shows that the power I^n is not usually generated by the nth powers of elements of I, that is $I^n \supsetneq (\{f^n \mid f \in I\})R$.

Exercise A.5.8. Let J be an ideal of R, and let $r \in R$. Set

$$rJ = \{rb \mid b \in J\}.$$

Prove that $rJ = (rR)J$. In particular, this shows that the set rJ is an ideal of R such that $rJ \subseteq J$. Show by example that we can have $rJ \subsetneq J$. Justify your answer.

Exercise A.5.9. Let A be a commutative ring with identity, and set $R = A[X_1,\ldots,X_d]$ and $\mathfrak{X} = (X_1,\ldots,X_d)R$. Prove that if $f \in R$, then $f \in \mathfrak{X}^n$ if and only if each monomial occurring in f has degree at least n. Prove that $\mathfrak{X} \neq R$. (This exercise is used in the proof of Proposition 6.2.7.)

**Exercise A.5.10.* In a non-zero commutative ring A with identity, an ideal $J \subsetneq A$ is *prime* if it satisfies the following condition: for all $f,g \in A$ if $fg \in J$, then either $f \in J$ or $g \in J$.

(a) Prove that the ideal 0 of $\mathbb{Z}$ is prime.
(b) Prove that the non-zero prime ideals of $\mathbb{Z}$ are the ideals of the form $p\mathbb{Z}$ where p is a prime number. (This is where the name “prime ideal” comes from.)
(c) Let n be an integer such that $n \geqslant 2$. Prove that the prime ideals of $\mathbb{Z}_n$ are the ideals $p\mathbb{Z}_n$ where p is a prime integer such that $p\big|n$.
(d) Let $P,J_1,\ldots,J_n$ be ideals of A such that P is prime. Prove that the following conditions are equivalent:

  (i) the intersection $\bigcap_{i=1}^n J_i$ is contained in P,
  (ii) the product $J_1 \cdots J_n$ is contained in P, and
  (iii) there is an integer i such that $J_i \subseteq P$.

(This is used in Exercises A.7.14, 1.2.14, 3.2.14 and 4.1.12, and throughout Chapter 5, starting with Definition 5.1.1.)

A.6 Colon Ideals

In this section, R is a commutative ring with identity.

This section deals with a construction of ideals that may be less familiar than the previous ones, namely the colon ideal. However, it is important for several constructions in the main body of the text, like parametric decompositions.

Definition A.6.1. Let S be a subset of R, and let J be an ideal of R. For each element $r \in R$, set $rS = \{rs \mid s \in S\}$. The *colon ideal* of J with S is

$$(J :_R S) = \{r \in R \mid rS \subseteq J\} = \{r \in R \mid rs \in J \text{ for all } s \in S\}.$$

For each $s \in R$, we set $(J :_R s) = (J :_R \{s\})$.

For example, in $\mathbb{Z}$ we have $(6\mathbb{Z} :_{\mathbb{Z}} 15) = (6\mathbb{Z} :_{\mathbb{Z}} 15\mathbb{Z}) = 2\mathbb{Z}$.

The basic properties of colon ideals are contained in the next two results.

Proposition A.6.2 *Let I, J, K be ideals of R, and let S, T be subsets of R.*

(a) *The set $(J :_R S)$ is an ideal of R.*
(b) *We have $(J :_R (S)R) = (J :_R S) = \bigcap_{s\in S}(J :_R s)$.*
(c) *We have $(J :_R S) = R$ if and only if $S \subseteq J$.*
(d) *There are containments $I(J :_R I) \subseteq J \subseteq (J :_R S)$.*
(e) *If $K \subseteq J$, then $(K :_R S) \subseteq (J :_R S)$.*
(f) *If $S \subseteq T$, then $(J :_R S) \supseteq (J :_R T)$.*

Proof. (a) For every $s \in S$, we have $0s = 0 \in J$; so $0 \in (J :_R S)$ and $(J :_R S) \neq \emptyset$.

To show that $(J :_R S)$ is closed under addition, let $r, r' \in (J :_R S)$. For all $s \in S$ we then have $rs, r's \in J$. The Distributive Law combines with the fact that J is closed under addition to show that $(r+r')s = rs + r's \in J$, and so $r + r' \in (J :_R S)$, as desired.

To show that $(J :_R S)$ is closed under external multiplication, let $r \in (J :_R S)$ and $t \in R$. For all $s \in S$ we then have $rs \in J$. The Associative Law combines with the fact that J is closed under external multiplication to show that $(tr)s = t(rs) \in J$, so $tr \in (J :_R S)$, as desired.

(c) For the forward implication, if $(J :_R S) = R$, then $1_R \in (J :_R S)$, so $S = 1_R S \subseteq J$. For the converse, if $S \subseteq J$, then $1_R \in (J :_R S)$; since $(J :_R S)$ is an ideal in R, it follows from Fact A.3.2(g) that $(J :_R S) = R$.

The proofs of the remaining statements are left as exercises. □

Proposition A.6.3 *Let I, J, K be ideals of R, and let S be a subset of R. Let $\{I_\lambda\}_{\lambda\in\Lambda}$ and $\{J_\lambda\}_{\lambda\in\Lambda}$ be sets of ideals of R, and let $\{S_\lambda\}_{\lambda\in\Lambda}$ be a set of subsets of R.*

(a) *There are equalities $((J :_R I) :_R K) = (J :_R IK) = ((J :_R K) :_R I)$.*
(b) *There is an equality $(\bigcap_{\lambda\in\Lambda} J_\lambda :_R S) = \bigcap_{\lambda\in\Lambda}(J_\lambda :_R S)$.*
(c) *There is an equality $(J :_R \bigcup_{\lambda\in\Lambda} S_\lambda) = \bigcap_{\lambda\in\Lambda}(J :_R S_\lambda)$.*
(d) *There is an equality $(J :_R \sum_{\lambda\in\Lambda} I_\lambda) = \bigcap_{\lambda\in\Lambda}(J :_R I_\lambda)$.*

Proof. (a) We first show $((J :_R K) :_R I) \subseteq (J :_R IK)$. Fix elements $r \in ((J :_R K) :_R I)$ and $x \in IK$. It follows that there are elements $b_1, \ldots, b_n \in I$ and $c_1, \ldots, c_n \in K$ such that $x = \sum_{i=1}^n b_i c_i$. Since $r \in ((J :_R K) :_R I)$ and $b_i \in I$, we have $rb_i \in (J :_R K)$. The condition $c_i \in K$ implies that $rb_i c_i \in J$. It follows that we have

$$rx = r\textstyle\sum_{i=1}^n b_i c_i = \sum_{i=1}^n rb_i c_i \in J$$

because J is an ideal, so $r \in (J :_R IK)$, as desired.

We next show that $((J :_R K) :_R I) \supseteq (J :_R IK)$. Let $s \in (J :_R IK)$ and $b \in I$. To show that $s \in ((J :_R K) :_R I)$, it suffices to show that $sb \in (J :_R K)$. Let $c \in K$. One needs to show that $(sb)c \in J$. The conditions $b \in I$ and $c \in K$ imply $bc \in IK$. Hence, the assumption $s \in (J :_R IK)$ implies $(sb)c = s(bc) \in J$, as desired.

The other equality follows from the next sequence

$$((J :_R K) :_R I) = (J :_R IK) = (J :_R KI) = ((J :_R I) :_R K).$$

The proofs of the remaining statements are left as exercises. □

Exercises

Exercise A.6.4. Let J be an ideal of R, and let S and T be subsets of R. Prove that $((J :_R S) :_R T) = ((J :_R T) :_R S)$.

Exercise A.6.5. Let $m,n \in \mathbb{Z}$ and compute $(m\mathbb{Z} :_{\mathbb{Z}} n\mathbb{Z})$; justify your answer.

Exercise A.6.6. Prove parts (b) and (d)–(f) of Proposition A.6.2.

Exercise A.6.7. Prove parts (b)–(d) of Proposition A.6.3.

Exercise A.6.8. Let A be a commutative ring with identity. Consider the ring $R = A[X_1,\ldots,X_d]$ and the ideal $\mathfrak{X} = (X_1,\ldots,X_d)R$.

(a) Prove that $(\mathfrak{X}^n : \mathfrak{X}) = \mathfrak{X}^{n-1}$.
(b) List the monomials in $\mathfrak{X}^{n-1} \smallsetminus \mathfrak{X}^n$; justify your answer.

A.7 Radicals of Ideals

In this section, R is a commutative ring with identity.

This section deals with another construction of ideals that may be less familiar, namely radicals. Again, though, this one is very important for our understanding of square-free monomial ideals, which form the foundation of most of Part II. Loosely speaking, the radical of an ideal I is the set of “roots” or “radicals” of elements of I, hence the name.

Definition A.7.1. Let I be an ideal of R. The *radical* of I is the set

$$\operatorname{rad}(I) = \{x \in R \mid x^n \in I \text{ for some } n \geqslant 1\}.$$

Other common notations include $\sqrt{I}$ and $\mathrm{r}(I)$.

For example, in the ring $\mathbb{Z}$ we have $\operatorname{rad}(12\mathbb{Z}) = 6\mathbb{Z}$. In the ring $\mathbb{Z}_8$ we have $\operatorname{rad}(0\mathbb{Z}_8) = \operatorname{rad}(4\mathbb{Z}_8) = 2\mathbb{Z}_8$.

The next two results contain fundamental properties of the radical operator.

Proposition A.7.2 *Let I be an ideal of R.*

(a) *The set* $\operatorname{rad}(I)$ *is an ideal of R.*
(b) *There is a containment* $I \subseteq \operatorname{rad}(I)$.
(c) *If $I \subseteq J$, then* $\operatorname{rad}(I) \subseteq \operatorname{rad}(J)$.
(d) *There is an equality* $\operatorname{rad}(I) = \operatorname{rad}(\operatorname{rad}(I))$.
(e) *We have* $\operatorname{rad}(I) = R$ *if and only if* $I = R$.
(f) *For each integer $n \geqslant 1$, there is an equality* $\operatorname{rad}(I^n) = \operatorname{rad}(I)$.

Proof. (a) We have $0^1 = 0 \in I$; it follows that $0 \in \mathrm{rad}(I)$, so $\mathrm{rad}(I) \neq \emptyset$.

To show that $\mathrm{rad}(I)$ is closed under addition, let $r, s \in \mathrm{rad}(I)$. There are integers $m, n \geqslant 1$ such that $r^m, s^n \in I$. The Binomial Theorem A.1.12 implies that

$$(r+s)^{m+n} = \textstyle\sum_{i=0}^{m+n} \binom{m+n}{i} r^i s^{m+n-i}.$$

Note that for each $i = 0, \ldots, m+n$ we have either $i \geqslant m$ or $m+n-i \geqslant n$. It follows that each term $\binom{m+n}{i} r^i s^{m+n-i}$ is in I. This implies $(r+s)^{m+n} \in I$, so $r+s \in \mathrm{rad}(I)$.

To show that $\mathrm{rad}(I)$ is closed under external multiplication, let $r \in \mathrm{rad}(I)$ and $t \in R$, and let n be a positive integer such that $r^n \in I$. It follows that $(rt)^n = r^n t^n \in I$, so $rt \in \mathrm{rad}(I)$.

(c) Assume that $I \subseteq J$ and let $r \in \mathrm{rad}(I)$. There is an integer $m \geqslant 1$ such that $r^m \in I \subseteq J$, so $r \in \mathrm{rad}(J)$.

(e) For the forward implication, if $\mathrm{rad}(I) = R$, then there exists a positive integer n such that $1_R = 1_R^n \in I$, so $I = R$. For the converse, if $I = R$, then we have $R = I \subseteq \mathrm{rad}(I) \subseteq R$ by part (b), so $\mathrm{rad}(I) = R$.

The proofs of the remaining statements are left as exercises. $\square$

Without the commutative hypothesis on R, the set $\mathrm{rad}(I)$ need not be an ideal. Indeed, let $\mathrm{M}_2(\mathbb{R})$ denote the set of all 2×2 matrices with entries in $\mathbb{R}$. This is a non-commutative ring with identity under the standard definitions of matrix addition and matrix multiplication. The set $\mathrm{rad}(0)$ is not an ideal because it is not closed under addition. Indeed, we have $M = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{rad}(0)$ because $M^2 = 0$. Similarly, we have $N = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{rad}(0)$, but $M + N = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \notin \mathrm{rad}(0)$ because $M + N$ is invertible. Also, we have $MN = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \notin \mathrm{rad}(0)$ so $\mathrm{rad}(0)$ is not closed under external multiplication. (Moreover, it is not even closed under internal multiplication.)

Back in commutative-land, though, the radical behaves pretty nicely, as we see next for products and intersections.

Proposition A.7.3 *Fix an integer $n \geqslant 1$, and let $I, J, I_1, I_2, \ldots, I_n$ be ideals of R.*

(a) *There are equalities* $\mathrm{rad}(IJ) = \mathrm{rad}(I \cap J) = \mathrm{rad}(I) \cap \mathrm{rad}(J)$.
(b) *There are equalities*

$$\begin{aligned} \mathrm{rad}(I_1 I_2 \cdots I_n) &= \mathrm{rad}(I_1 \cap I_2 \cap \cdots \cap I_n) \\ &= \mathrm{rad}(I_1) \cap \mathrm{rad}(I_2) \cap \cdots \cap \mathrm{rad}(I_n). \end{aligned}$$

(c) $\mathrm{rad}(I+J) = \mathrm{rad}(\mathrm{rad}(I) + \mathrm{rad}(J))$.
(d) $\mathrm{rad}(I_1 + I_2 + \cdots + I_n) = \mathrm{rad}(\mathrm{rad}(I_1) + \mathrm{rad}(I_2) + \cdots + \mathrm{rad}(I_n))$.

Proof. (a) We show that

$$\mathrm{rad}(IJ) \subseteq \mathrm{rad}(I \cap J) \subseteq \mathrm{rad}(I) \cap \mathrm{rad}(J) \subseteq \mathrm{rad}(IJ).$$

The containment $\mathrm{rad}(IJ) \subseteq \mathrm{rad}(I \cap J)$ follows from Proposition A.7.2(c) since $IJ \subseteq I \cap J$. The containment $\mathrm{rad}(I \cap J) \subseteq \mathrm{rad}(I) \cap \mathrm{rad}(J)$ also follows from Proposition A.7.2(c): the containments $I \cap J \subseteq I$ and $I \cap J \subseteq J$ imply that $\mathrm{rad}(I \cap J) \subseteq$

rad (I) and rad $(I \cap J) \subseteq$ rad (J). For the containment rad $(I) \cap$ rad $(J) \subseteq$ rad (IJ), let $r \in$ rad $(I) \cap$ rad (J). There are integers $l, m \geqslant 1$ such that $r^l \in I$ and $r^m \in J$, so $r^{l+m} = r^l r^m \in IJ$. It follows that $r \in$ rad (IJ), as desired.

(b) The case $n = 2$ follows from part (a); the remaining cases are proved by induction on n.

The proofs of the remaining statements are left as exercises. □

The following example shows that we may have rad $(I + J) \neq$ rad $(I) +$ rad (J); compare to Proposition A.7.3(c).

Example A.7.4. Set $R = \mathbb{C}[X,Y]$ and $I = (X^2 + Y^2)R$ and $J = (X)R$. Then rad $(I) = I$ and rad $(J) = J$, so rad $(I) +$ rad $(J) = I + J = (X, Y^2)R$. On the other hand, we have rad $(I + J) =$ rad $\big((X,Y^2)R\big) = (X,Y)R$, so $X \in$ rad $(I+J) \smallsetminus$ rad $(I) +$ rad (J).

Ideals of the form rad (I) for some ideal I have an additional property that can be very useful.

Definition A.7.5. Let R be a commutative ring with identity. An ideal I of R is said to be *radical* provided for all $f \in R$ and all $k \geqslant 1$, if $f^k \in I$ then $f \in I$.

Exercise A.7.13 below explores some properties of radical ideals.

Exercises

Exercise A.7.6. Let $n \in \mathbb{Z}$ and compute rad $(n\mathbb{Z})$; justify your answer.

Exercise A.7.7. Let I and J be ideals of R. Prove or disprove: If rad $(I) \subseteq$ rad (J), then $I \subseteq J$. Justify your answer.

Exercise A.7.8. Prove parts (b), (d), and (f) of Proposition A.7.2.

Exercise A.7.9. Prove parts (b)–(d) of Proposition A.7.3.

Exercise A.7.10. Let J be an ideal of R, and assume that rad (J) is finitely generated. Prove that there is a integer $n \geqslant 1$ such that rad $(J)^n \subseteq J$.

Exercise A.7.11. Let I and J be ideals of R.

(a) Assume that $I = (f_1, \ldots, f_s)R$ with $s \geqslant 1$. Prove that rad $(I) \subseteq$ rad (J) if and only if for each $i = 1, 2, \ldots, s$ there exists a positive integer n_i such that $f_i^{n_i} \in J$.
(b) Assume that $I = (f_1, \ldots, f_s)R$ and $J = (g_1, \ldots, g_t)R$ with $s, t \geqslant 1$. Prove that rad $(I) =$ rad (J) if and only if for each $i = 1, 2, \ldots, s$ there exists a positive integer n_i such that $f_i^{n_i} \in J$, and for each $j = 1, 2, \ldots, t$ there exists a positive integer m_j such that $g_j^{m_j} \in I$.
(c) Suppose $I \subseteq J$ and that $J = (g_1, \ldots, g_t)R$. Prove that rad $(I) =$ rad (J) if and only if for each $j = 1, 2, \ldots, t$ there exists an integer m_j such that $g_j^{m_j} \in I$.

Exercise A.7.12. Let A be a commutative ring with identity. Set $R = A[X,Y]$ and $I = (X^3, Y^4)R$ and $J = (XY^2, X^2Y)R$.

(a) Use Exercise A.7.11 to prove that $\operatorname{rad}(I) \supsetneq \operatorname{rad}(J)$.
(b) Assume that A is a field. Prove that $\operatorname{rad}(I) = (X,Y)R$ and $\operatorname{rad}(J) = (XY)R$. Use this to give another proof that $\operatorname{rad}(I) \supsetneq \operatorname{rad}(J)$.

Exercise A.7.13. Let I and J be ideals of R.

(a) Prove that the ideal $\operatorname{rad}(I)$ is radical.
(b) Prove that if $I \subseteq J$ and J is radical, then $\operatorname{rad}(I) \subseteq J$.
(c) Prove that $I = \operatorname{rad}(I)$ if and only if I is radical.
(d) Prove or disprove: If I and J are radical, then IJ is radical.
(e) Prove or disprove: If I and J are radical, then $I \cap J$ is radical.
(f) Prove or disprove: If I and J are radical, then $I + J$ is radical.

Justify your answers.

**Exercise A.7.14.* Let I, P be ideals of R such that P is prime (see Exercise A.5.10). Prove that $I \subseteq P$ if and only if $\operatorname{rad}(I) \subseteq P$. (This exercise is used in the proof of Theorem 5.1.2.)

A.8 Quotient Rings

In this section, R is a commutative ring with identity.

One of the most powerful aspects of abstract algebra is its ability to formalize certain notions. We have already seen this in action in Section A.2. For another example, consider the field $\mathbb{C}$ of complex numbers. We all know that complex numbers are of the form $a + bi$ where $a, b \in \mathbb{R}$ and $i^2 = -1$. But how do we know that complex numbers exist? Given that we have $a^2 \geqslant 0$ for all $a \in \mathbb{R}$ but $i^2 = -1$, this is a non-trivial and important question. Quotient rings give us the ability to answer these questions rigorously.

Definition A.8.1. Let I be an ideal of R.

(a) For each element $r \in R$, the *coset* of I determined by r is the next subset of R:

$$r + I = \{r + a \mid a \in I\}.$$

(b) The *quotient* R/I is the set of all cosets of I:

$$R/I = \{r + I \mid r \in R\}.$$

For example, in $\mathbb{Z}$ with $I = 5\mathbb{Z} = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$, we have

$$\begin{aligned}
0+5\mathbb{Z} &= \{\dots,-10,-5,0,5,10,\dots\} = 5\mathbb{Z}\\
1+5\mathbb{Z} &= \{\dots,-9,-4,1,6,11,\dots\}\\
2+5\mathbb{Z} &= \{\dots,-8,-3,2,7,12,\dots\}\\
3+5\mathbb{Z} &= \{\dots,-7,-2,3,8,13,\dots\}\\
4+5\mathbb{Z} &= \{\dots,-6,-1,4,9,14,\dots\} = -1+5\mathbb{Z}\\
5+5\mathbb{Z} &= \{\dots,-5,0,5,10,15,\dots\} = 0+5\mathbb{Z}\\
6+5\mathbb{Z} &= \{\dots,-4,1,6,11,16,\dots\} = 1+5\mathbb{Z}
\end{aligned}$$

and so on. In particular, in this example one checks readily using the Division Algorithm that

$$\mathbb{Z}/5\mathbb{Z} = \{5\mathbb{Z}, 1+5\mathbb{Z}, 2+5\mathbb{Z}, 3+5\mathbb{Z}, 4+5\mathbb{Z}\}$$

is a set with five elements.

In general, if one uses the trivial ideal $I = R$, then the quotient R/R is a set with one element, namely $R/R = \{0+R\} = \{R\}$. On the other trivial hand, if one uses the ideal $I = 0$, then the elements of $R/0$ are in bijection with the elements of R by the function $R \to R/0$ given by $r \mapsto r+0$.

Regardless of the choice of I, one always has the following.

Proposition A.8.2 *Let I be an ideal of R.*

(a) *If $r \in R$, then $r \in r+I$. Hence, we have $R = \bigcup_{r\in R} r+I$.*
(b) *For elements $r, s \in R$, the following conditions are equivalent:*

  (i) $r+I = s+I$
  (ii) $r \in s+I$
  (iii) $s \in r+I$
  (iv) $r-s \in I$
  (v) $s-r \in I$
  (vi) $r+I \subseteq s+I$
  (vii) $r+I \supseteq s+I$
  (viii) $(r+I)\cap(s+I) \neq \emptyset$.

(c) *The ring R is the disjoint union of the distinct cosets of I. In other words, the distinct cosets of I form a partition of R.*
(d) *Write $r \sim s$ whenever the equivalent conditions of item* (b) *above are satisfied. Then the relation $\sim$ is an equivalence relation on R.*

Proof. We prove a few parts of this, and leave the others as exercises.

(b) (viii) $\Longrightarrow$ (iv) Assume that $(r+I)\cap(s+I) \neq \emptyset$, and let $t \in (r+I)\cap(s+I)$. Then there are elements $a, b \in I$ such that $r+a = t = s+b$, and it follows that $r-s = b-a \in I$.

(iv) $\Longrightarrow$ (vi) Assume that $r-s \in I$; we show that $r+I \subseteq s+I$. An arbitrary element of $r+I$ has the form $r+a$ for some $a \in I$; we need to show that $r+a \in s+I$. Since I is an ideal with $r-s, a \in I$, we have $(r-s)+a \in I$. Hence the element $r+a = s+[(r-s)+a]$ is in $s+I$, as desired.

(c) This follows from items (a) and (b). $\square$

One of the main points of the construction of R/I is the next result.

Theorem A.8.3 *Let I be an ideal of R. The set R/I is a commutative ring with identity where*

$$
\begin{aligned}
(r+I)+(s+I) &= (r+s)+I & 0_{R/I} &= 0_R+I \\
(r+I)-(s+I) &= (r-s)+I & -(r+I) &= (-r)+I \\
(r+I)(s+I) &= (rs)+I & 1_{R/I} &= 1_R+I
\end{aligned}
$$

Proof. First, we show that multiplication in R/I is well-defined. Let $r,r',s,s' \in R$ such that $r+I = r'+I$ and $s+I = s'+I$. By Proposition A.8.2(b), it follows that $r-r', s-s' \in I$. We need to show that $(rs)+I = (r's')+I$, that is, that $rs-r's' \in I$. Since I is an ideal, the condition $r-r', s-s' \in I$ implies that $r(s-s'), (r-r')s' \in I$ and consequently $rs-r's' = r(s-s')+(r-r')s' \in I$, as desired.

The proof that addition and subtraction are well-defined is similar. To see that 1_R+I is a multiplicative identity, we compute by definition

$$(1_R+I)(r+I) = (1_R r)+I = r+I.$$

One shows similarly that 0_R+I is an additive identity in R/I. Commutativity of multiplication in R/I follows from the fact that multiplication in R is commutative:

$$(r+I)(s+I) = (rs)+I = (sr)+I = (s+I)(r+I).$$

One verifies the remaining axioms similarly. $\square$

We now informally address the question about $\mathbb{C}$ raised in the opening of this section. Consider the polynomial ring $R = \mathbb{R}[X]$ and the ideal $I = (X^2+1)R$. First, we note that the element $i = X+I \in R/I$ satisfies the equation $i^2 = -1$. For this, we note that $X^2-(-1) = X^2+1 \in I$, so we have

$$i^2 = (X+I)^2 = X^2+I = -1+I$$

by Proposition A.8.2(b). This is the desired equality in R/I.

Using the Division Algorithm, it can be shown that every element $f+I \in R/I$ can be written uniquely in the form

$$(a+bX)+I = (a+I)+b(X+I) = (a+I)+bi = a+bi.$$

The last step here uses the fact that we identify $\mathbb{R}$ with the subset $\{a+I \mid a \in \mathbb{R}\} \subseteq R/I$. In conclusion, this sketches the fact that every element of R/I is uniquely of the form $a+bi$ where $i^2 = -1$. In other words, following vigorous hand-waiving, we have $R/I = \mathbb{C}$.

Exercises

Exercise A.8.4. Show that the function $\phi\colon \mathbb{Z}_5 \to \mathbb{Z}/5\mathbb{Z}$ given by $\phi(a) = a+5\mathbb{Z}$ is a well-defined bijection such that $\phi(1) = 1_{\mathbb{Z}/5\mathbb{Z}}$ and satisfying $\phi(a+b) = \phi(a)+\phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$ for all $a,b \in \mathbb{Z}_5$. (This says that the map ϕ is an "isomorphism" of rings and that the rings $\mathbb{Z}_5$ and $\mathbb{Z}/5\mathbb{Z}$ are "isomorphic".) State and prove the analogous result for $\mathbb{Z}_n$ for any $n \in \mathbb{N}$.

Exercise A.8.5. Let I be an ideal of R. Prove that $R/I = 0$ if and only if $R = I$.

Exercise A.8.6. Finish the proof of Proposition A.8.2.

Exercise A.8.7. Finish the proof of Theorem A.8.3.

Exercise A.8.8. Let $f \in R[X]$ be monic of degree e, and set $I = (f)A[X]$. Use the Division Algorithm to prove that every element in $A[X]/I$ can be uniquely written in the form $a_0 + a_1X + \cdots + a_{e-1}X^{e-1} + I$ with $a_i \in A$.

**Exercise A.8.9.*

(a) Use the Division Algorithm to prove that $\mathbb{R}[X]/(X^2+1)\mathbb{R}[X]$ is a field.
(b) Are the following rings also fields?

(1) $\mathbb{Q}[X]/(X^2+1)\mathbb{Q}[X]$
(2) $\mathbb{Z}[X]/(X^2+1)\mathbb{Z}[X]$
(3) $\mathbb{C}[X]/(X^2+1)\mathbb{C}[X]$

(c) For which prime numbers p is the ring $\mathbb{Z}_p[X]/(X^2+1)\mathbb{Z}_p[X]$ a field? (Hint: When p is odd, the element -1 is a perfect square in $\mathbb{Z}_p$, that is, -1 is a "quadratic residue mod p", if and only if $p \equiv 1 \pmod 4$.[1])

Justify your answers. (This exercise is used in Exercise B.8.2.)

Exercise A.8.10. Let A be a commutative ring with identity, and set $R = A[X_1,\ldots,X_d]$ with $\mathfrak{X} = (X_1,\ldots,X_d)R$. Prove that $R/\mathfrak{X}$ is a field if and only if A is a field.

**Exercise A.8.11.* Let A be a commutative ring with identity, let I be an ideal of A, and let $f \in A$. Let $\nu\colon A/I \to A/I$ be given by $\nu(a+I) = (fa)+I$, and recall that the *kernel* of ν is the set $\operatorname{Ker}(\nu) = \{z \in A/I \mid \nu(z) = 0\}$.

(a) Prove that ν is a homomorphism of additive abelian groups.
(b) Prove that the following conditions are equivalent:

(i) ν is injective (that is, 1-1);
(ii) $\operatorname{Ker}(\nu) = 0$; and
(iii) $(I :_A f) = I$.

(c) Prove that ν is surjective (that is, onto) if and only if $A = I + fA$.

This is used in Example 5.3.7.

[1] This is a consequence of "quadratic reciprocity".

A.9 Partial Orders and Monomial Orders

This section contains a brief review of the notion of partial orders, including some specific relations that are very useful for the study of monomial ideals.

Definition A.9.1. Let A be a set. A *relation* on A is a subset $\sim \subseteq A\times A$ where $A\times A$ is the cartesian product of A with itself; that is, $A\times A = \{(a,b) \mid a,b\in A\}$. If $\sim \subseteq A\times A$ is a relation, then we write "$a\sim b$" instead of "$(a,b)\in\sim$".

For example, every equivalence relation on a set A is a relation on A.

Definition A.9.2. Let A be a non-empty set and let $\leqslant$ be a relation on A. The relation $\leqslant$ is a *partial order* on A if it satisfies the following properties:

(a) (reflexivity) For all $a\in A$ we have $a\leqslant a$;
(b) (transitivity) For all $a,b,c\in A$, if $a\leqslant b$ and $b\leqslant c$, then $a\leqslant c$;
(c) (antisymmetry) For all $a,b\in A$, if $a\leqslant b$ and $b\leqslant a$, then $a=b$.

We write $a<b$ when $a\leqslant b$ and $a\neq b$. Also, we write $b\geqslant a$ when $a\leqslant b$, and we write $b>a$ when $a<b$.

Two elements $a,b\in A$ are *comparable* if either $a\leqslant b$ or $b\leqslant a$. A *total order* is a partial order such that every $a,b\in A$ are comparable. A *well-ordering* on A is a total order such that every non-empty subset of A has a least element.

The negation of "$a\leqslant b$" is written "$a\nleqslant b$". Note that this is not the same as $a>b$ because a and b may not be comparable.

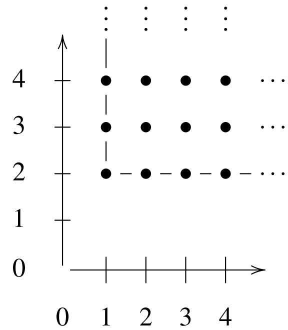
For example, the usual orders $\leqslant$ on the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ are total orders; the "Well-Ordering Principle" states that this is a well-ordering on $\mathbb{N}$. The "divisibility order" on the set of positive integers $\mathbb{N}_+$, given by $n\leqslant_{\text{div}} m$ when $n\big|m$, is a partial order on $\mathbb{N}$. On the other hand, the divisibility order on $\mathbb{N}_+$ is not a total order because $2\nmid 3$ and $3\nmid 2$. The divisibility order on $\mathbb{Z}$ is not even a partial order because it is not antisymmetric: We have $1\big|-1$ and $-1\big|1$, but $1\neq -1$.

Sets with partial orders are often called "partially ordered sets", or "posets" for short. The adjective "partial" is used for these relations because there is no guarantee it is a total order.

Here is the main example for this text, though we will encounter other important examples below.

Definition A.9.3. Fix an integer $d\geqslant 1$ and define a relation $\succcurlyeq$ on $\mathbb{N}^d$ as follows: $(a_1,\ldots,a_d)\succcurlyeq(b_1,\ldots,b_d)$ if for $i=1,\ldots,d$ we have $a_i\geqslant b_i$ in the usual order on $\mathbb{N}$.

For example, when $d=2$, we graph the set $\{\underline{n}\in\mathbb{N}^2 \mid \underline{n}\succcurlyeq(1,2)\}$ as follows.

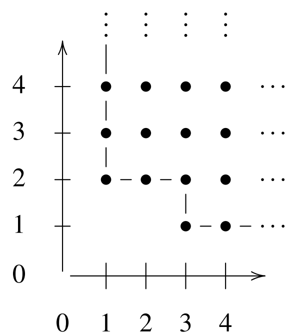


This set can be described as the “translate” or the coset $(1,2)+\mathbb{N}^2$.

Definition A.9.4. Let d be a positive integer. For each $\underline{n} \in \mathbb{N}^d$, set

$$\langle \underline{n} \rangle = \{\underline{m} \in \mathbb{N}^d \mid \underline{m} \succcurlyeq \underline{n}\} = \underline{n} + \mathbb{N}^d.$$

Again, when $d = 2$, the graph of $\langle (1,2) \rangle \cup \langle (3,1) \rangle$ is the following.



Other important examples for us are the “monomial orders”, defined next.

Definition A.9.5. Set $R = A[X_1,\ldots,X_d]$. Let $[\![R]\!]$ denote the set

$$[\![R]\!] = \{\underline{X}^{\underline{n}} \mid \underline{n} \in \mathbb{N}^d\}$$

of monomials in R from Definition A.1.5. (Here $\mathbb{N}^d$ is the d-fold cartesian product $\mathbb{N}^d = \mathbb{N} \times \cdots \times \mathbb{N} = \{(n_1,\ldots,n_d) \mid n_1,\ldots,n_d \in \mathbb{N}\}$.) A *monomial order* on the set $[\![R]\!]$ of monomials of R is a well-ordering $\leqslant$ on $[\![R]\!]$ such that

(1) for every $f \in [\![R]\!]$, we have $1 \leqslant f$, and
(2) for all $f,g,h \in [\![R]\!]$, if $f \leqslant g$, then $fh \leqslant gh$.

Note that the "divisibility order" (where $f \leqslant g$ provided that $f \mid g$) is not a monomial order, unless $d = 1$, since it is not a total order. Neither is the "reverse divisibility order" (where $f \leqslant g$ provided that $g \mid f$) a monomial order, since $g \nmid 1$ for all $g \neq 1$ in $[\![R]\!]$.

Among the monomial orders, an important example is the "lexicographical order". The name of this order comes from the fact that it is modeled on the order of words in the dictionary. (The word "lexicon" means "dictionary".) We first define it in the case $d = 2$, since this is relatively straightforward to define and visualize.

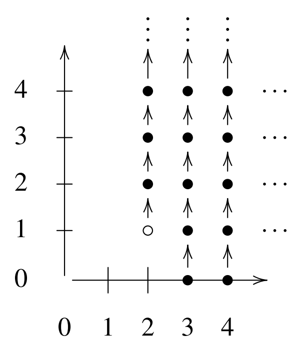
Definition A.9.6. Consider the case $R = A[X,Y]$. We define the *lexicographical order* on the set of monomials $[\![R]\!]$ as follows. For monomials $f = X^aY^b$ and $g = X^cY^d$ write $f <_{\text{lex}} g$ if either ($a < c$) or ($a = c$ and $b < d$). In other words, we have $X^aY^b \leqslant X^cY^d$ provided that the first non-zero entry of the vector $(c,d) - (a,b)$ is positive. We also write $f \leqslant_{\text{lex}} g$ if either $f <_{\text{lex}} g$ or $f = g$.

For example, we have $X^2Y^3 <_{\text{lex}} X^3Y^2$ because of the X-exponents. We also have $X^2Y^3 <_{\text{lex}} X^2Y^5$.

Remark A.9.7. The monomials in $[\![R]\!]$ can be listed in order as follows:

$$1 <_{\text{lex}} Y <_{\text{lex}} Y^2 <_{\text{lex}} Y^3 <_{\text{lex}} \cdots <_{\text{lex}} X <_{\text{lex}} XY <_{\text{lex}} XY^2 <_{\text{lex}} XY^3 <_{\text{lex}} \cdots.$$

We can visualize this order graphically as follows. Given a monomial $f \in [\![R]\!]$, the monomials $g \in [\![R]\!]$ such that $f <_{\text{lex}} g$ are exactly the monomials represented by points that are to the right of f or directly above f.



In the graph, we have represented the order with sequential vertical arrows.

To conclude this section, we define the lexicographical order for arbitrary d, in addition to other monomial orders of interest.

Definition A.9.8. Set $R = A[X_1,\ldots,X_d]$, and fix $\underline{X}^{\underline{m}}, \underline{X}^{\underline{n}} \in [\![R]\!]$.

(a) Write $\underline{X}^{\underline{m}} <_{\text{lex}} \underline{X}^{\underline{n}}$ when, for some i, we have $m_j = n_j$ for $j = 1,\ldots,i$ and $m_{i+1} < n_{i+1}$. This is the *lexicographical order* (or *lex order*) on $[\![R]\!]$.

(b) Write $\underline{X}^{\underline{m}} <_{\text{grlex}} \underline{X}^{\underline{n}}$ when either $\deg(\underline{X}^{\underline{m}}) < \deg(\underline{X}^{\underline{n}})$, or $\deg(\underline{X}^{\underline{m}}) = \deg(\underline{X}^{\underline{n}})$ and $\underline{X}^{\underline{m}} <_{\text{lex}} \underline{X}^{\underline{n}}$. This is the *graded lexicographical order* (or *glex order*) on $[\![R]\!]$.
(c) Write $\underline{X}^{\underline{m}} <_{\text{grevlex}} \underline{X}^{\underline{n}}$ when either $\deg(\underline{X}^{\underline{m}}) < \deg(\underline{X}^{\underline{n}})$, or $\deg(\underline{X}^{\underline{m}}) = \deg(\underline{X}^{\underline{n}})$ and for some i, we have $m_j = n_j$ for $j = i, \ldots, d$ and $m_{i-1} > n_{i-1}$. This is the *graded reverse lexicographical order* (or *grevlex order*) on $[\![R]\!]$.

For example, when $d = 3$, we have $X_1^3X_2^4X_3^2 < X_1^5X_2^3X_3$ for any of the above monomial orders, but the reason for $<_{\text{grevlex}}$ is different: $X_1^3X_2^4X_3^2$ has a higher power of X_3 than does $X_1^5X_2^3X_3$.

Exercises

**Exercise A.9.9.* Prove that the relation from Definition A.9.3 is a partial order. Prove that it is a total order if and only if $d = 1$. (This is used in Remark 1.1.8.)

Exercise A.9.10. Let R be a commutative ring with identity. Prove that the divisibility order on R is reflexive and transitive.

Exercise A.9.11. Set $R = A[X_1, X_2, X_3]$. Show that $X_1^3X_2^4X_3^2 < X_1^5X_2^3X_3$ for any of the monomial orders from Definition A.9.8.

Exercise A.9.12. Set $R = A[X, Y]$. Sketch the glex and grevlex orders as we did for the lex order in Remark A.9.7 above.

**Exercise A.9.13.* Set $R = A[X_1, \ldots, X_d]$. Prove that the lex, glex, and grevlex orders on $[\![R]\!]$ are monomial orders. (This result is used in the proofs of Proposition 2.3.2 and Theorem 6.3.1.)

Exercise A.9.14. Set $R = A[X_1, \ldots, X_d]$, and consider the ordering on $[\![R]\!]$ given by $\underline{X}^{\underline{m}} <_{\text{revlex}} \underline{X}^{\underline{n}}$ if for some i, one has $m_j = n_j$ for $j = 1, \ldots, i$ and $m_{i+1} > n_{i+1}$. (This is the *reverse lexicographical order* or *revlex order*.)

(a) Sketch the revlex order for $d = 2$ as we did for the lex order in Remark A.9.7 above.
(b) Show that $\leqslant_{\text{revlex}}$ is not a monomial order according to Definition A.9.5. What goes wrong?

Exercise A.9.15. Set $R = A[X_1, \ldots, X_d]$, and let $<$ be a monomial order on $[\![R]\!]$. Formulate and prove a version of the Division Algorithm in this context.

A.10 Exploration: Algebraic Geometry

In this section, A is a field and $R = A[X_1, \ldots, X_d]$.

Algebraic geometry is the study of the interplay between algebra and geometry. One uses algebraic tools to study geometric objects and vice versa. This idea is so fundamental that sometimes we don't distinguish between the algebra and the geometry. For instance, much of our understanding of solutions of polynomial equations (algebra) comes from graphs of the the equations (geometry), and of course one way we understand graphs is by algebraically solving the equation $f'(x) = 0$. The fundamental construction of algebraic geometry, defined next, comes directly from this perspective: it is the solution set of a system of polynomial equations.

Definition A.10.1. We work in the d-fold cartesian product $A^d = A \times \cdots \times A$, sometimes called the *$d$-dimensional affine space over A*.

Let $S \subseteq R$. The *algebraic subset* of A^d defined by S is the zero-locus

$$\mathrm{V}(S) = \{\underline{a} \in A^d \mid f(\underline{a}) = 0 \text{ for all } f \in S\}$$

also known as the "vanishing locus" of S. For a finite set $S = \{f_1, \ldots, f_n\}$, we sometimes write $\mathrm{V}(f_1, \ldots, f_n)$ in place of $\mathrm{V}(S)$.

For instance, given a polynomial $f \in \mathbb{R}[X]$, the set $\mathrm{V}(f) \subseteq \mathbb{R}$ is just the solution set $\{a \in \mathbb{R} \mid f(a) = 0\} \subseteq \mathbb{R}$, and $\mathrm{V}(y - f) \subseteq \mathbb{R}^2$ is the graph of the equation $y = f$.

The next exercise outlines the fundamental properties of this construction.

Exercise A.10.2. Let $S, T \subseteq R$.

(a) Prove that $\mathrm{V}(0) = A^d$ and $\mathrm{V}(1) = \emptyset$.
(b) Prove that if $S \subseteq T$, then $\mathrm{V}(S) \supseteq \mathrm{V}(T)$.
(c) Prove or disprove the converse to part (b).
(d) Prove that if $I = (S)R$, then $\mathrm{V}(S) = \mathrm{V}(I)$.
(e) Prove that $\mathrm{V}(S \cup T) = \mathrm{V}(S) \cap \mathrm{V}(T)$. More generally, given a set $\{S_\lambda\}_{\lambda \in \Lambda}$ of subsets of R, prove that $\mathrm{V}(\bigcup_{\lambda \in \Lambda} S_\lambda) = \bigcap_{\lambda \in \Lambda} \mathrm{V}(S_\lambda)$. In particular, for $f_1, \ldots, f_n \in R$, one has $\mathrm{V}(f_1, \ldots, f_n) = \mathrm{V}(f_1) \cap \cdots \cap \mathrm{V}(f_n)$.

Part (d) of the preceding exercise shows that one can focus exclusively on algebraic subsets defined by ideals. This has certain advantages. For example, note that part (c) of the next result does not have a companion in the previous one.

Exercise A.10.3. Let I and J be ideals of R.

(a) Prove that $\mathrm{V}(I) = \mathrm{V}(\mathrm{rad}(I))$.
(b) Prove that $\mathrm{V}(I + J) = \mathrm{V}(I) \cap \mathrm{V}(J)$. More generally, given a collection $\{I_\lambda\}_{\lambda \in \Lambda}$ of ideals of R, prove that $\mathrm{V}(\sum_{\lambda \in \Lambda} I_\lambda) = \bigcap_{\lambda \in \Lambda} \mathrm{V}(I_\lambda)$.
(c) Prove that $\mathrm{V}(I \cap J) = \mathrm{V}(IJ) = \mathrm{V}(I) \cup \mathrm{V}(J)$. More generally, given ideals $I_1, \ldots, I_n$ of R, prove that $\mathrm{V}(I_1 \cap \cdots \cap I_n) = \mathrm{V}(I_1 \cdots I_n) = \mathrm{V}(I_1) \cup \cdots \cup \mathrm{V}(I_n)$. In particular, for $n = 1, 2, \ldots$, one has $\mathrm{V}(I^n) = \mathrm{V}(I)$.

The next construction is, in some ways, the reverse of the previous one: instead of looking at the points that make some polynomials vanish, we look at the polynomials that vanish at some points.

Definition A.10.4. Let $Y \subseteq A^d$. The *geometric ideal* of R defined by Y is

$$\mathrm{I}(Y) = \{f \in R \mid f(\underline{a}) = 0 \text{ for all } \underline{a} \in Y\}.$$

This construction has similar properties to the previous one, documented next.

Exercise A.10.5. Let Y and Z be subsets of A^d.

(a) Prove that $\mathrm{I}(\emptyset) = R$.
(b) Prove that if A is infinite, then $\mathrm{I}(A^d) = 0$. (Note that this conclusion fails if A is finite. For instance, if $A = \mathbb{Z}_p$, then every $a \in A$ is a root of the non-zero polynomial $X^p - X$.)
(c) Prove that if $Y \subseteq Z$, then $\mathrm{I}(Y) \supseteq \mathrm{I}(Z)$.
(d) Prove or disprove the converse to part (c).
(e) Prove that $\mathrm{I}(Y)$ is a radical ideal of R.
(f) Prove that $\mathrm{I}(Y \cup Z) = \mathrm{I}(Y) \cap \mathrm{I}(Z)$. More generally, given a collection $\{Y_\lambda\}_{\lambda \in \Lambda}$ of subsets of A^d, prove that $\mathrm{I}(\bigcup_{\lambda \in \Lambda} Y_\lambda) = \bigcap_{\lambda \in \Lambda} \mathrm{I}(Y_\lambda)$.
(g) Prove that $\mathrm{I}(Y \cap Z) \supseteq \mathrm{I}(Y) + \mathrm{I}(Z)$. Then prove that $\mathrm{I}(Y \cap Z) \supseteq \operatorname{rad}(\mathrm{I}(Y) + \mathrm{I}(Z))$.
(h) Show that the containments in part (g) can fail to be equalities.

Next, we look at the interactions between our two constructions.

Exercise A.10.6. Let Y be a subset of A^d, and let I be an ideal of R.

(a) Prove that $\mathrm{V}(\mathrm{I}(Y)) \supseteq Y$.
(b) Prove that $\mathrm{I}(\mathrm{V}(I)) \supseteq I$. Then prove that $\mathrm{I}(\mathrm{V}(I)) \supseteq \operatorname{rad}(I)$.
(c) Show that the containments in parts (a) and (b) can fail to be equalities. It is worth noting that *Hilbert's Nullstellensatz* [40] shows that, if $A = \mathbb{C}$ (more generally, if A is algebraically closed), then one does have the equality $\mathrm{I}(\mathrm{V}(I)) = \operatorname{rad}(I)$; see also, e.g., [54, Theorem 5.4]. From this, one can deduce that if V is an algebraic subset of A^d, then $\mathrm{V}(\mathrm{I}(V)) = V$.

Given an algebraic subset $V \subseteq A^d$, algebraists and geometers often think of the quotient ring $R/\mathrm{I}(V)$ as representing V; see Section A.8 for background on quotient rings. One reason for this is the fact (from the discussion in the previous exercise) that, when $A = \mathbb{C}$, one can recover V from $\mathrm{I}(V)$. Another reason for this is the tradition of frequently representing mathematical objects by certain sets of functions. In the last exercise of this section, we describe how this connects to $R/\mathrm{I}(V)$.

Exercise A.10.7. Let $Y \subseteq A^d$. A function $f\colon Y \to A$ is *regular* if there is a polynomial $p \in R$ such that $p(y) = f(y)$ for all $y \in Y$. Let $A[Y]$ denote the set of regular functions $f\colon Y \to A$.

(a) Prove that two polynomials $p, q \in R$ determine the same regular function on Y if and only if $p - q \in \mathrm{I}(Y)$.
(b) Prove that $A[Y]$ is a commutative ring with identity.
(c) Prove that there is a ring isomorphism $A[Y] \cong R/\mathrm{I}(Y)$.

Concluding Notes

The study of commutative rings and their ideals has a long history. We touch on only a few highlights here.

The term “ideal” was introduced in a special case by Richard Dedekind in his publication of Peter Dirichlet’s Lectures on Number Theory [17]. Interestingly, the term “ring” was coined after this by David Hilbert [41], again in a special case. The general definitions of these terms (at least, in our situation) are due to Emmy Noether [64]. Dedekind, Hilbert, and Noether are all giants in this field of mathematics and others. As we see in other chapters of this text, both Hilbert and Noether had profound impacts on the specific topics of this text.

Appendix B
Introduction to Macaulay2

The computer algebra system Macaulay2 is a powerful tool for understanding rings and ideals, among other things. This appendix introduces the basic syntax for this system, roughly following the topics of the previous appendix. See the Macaulay2 website `http://www.math.uiuc.edu/Macaulay2/` for free download and installation information.

B.1 Rings

In this tutorial, we show how to declare rings and perform basic computations in Macaulay2. Below is the annotated output of a Macaulay2 session. To begin, type `M2` in a terminal command line,[1] which will produce some initial output, ending with the input prompt.

```
i1 :
```

Let's perform some integer arithmetic: at the input prompt, type `1+2*3^2-4` and hit the return key.[2]

```
i1 : 1+2*3^2-4
o1 = 15
```

The second line here is the output. Note that Macaulay2 uses the standard order of operations, so that `1+2*3^2-4` is `1+2*(3^2)-4`, not `1+(2*3)^2-4`.

For integers, the operators `//` and `%`. compute integer quotients and remainders, respectively, as in the Division Algorithm.

---

[1] One can also run Macaulay2 through emacs or texmacs. For in-depth computations, programming, etc., these options are usually preferable. See the link 'getting started' at the main Macaulay2 website `http://www.math.uiuc.edu/Macaulay2/` for set-up instructions for these options.

[2] Here we are referring to the primary carriage return key, which is marked 'return' on some keyboards and 'enter' on others.

```
i2 : 7 // 4
o2 = 1

i3 : 7 % 4
o3 = 3
```

One can refer to previous lines of output using `o1`, `o2`, etc. Also, you can refer to the most previous output as `oo`, the second most previous output as `ooo`, and the third most previous output as `oooo`.

```
i4 : 1+o1
o4 = 16

i5 : 1/3
     1
o5 = -
     3
o5 : QQ
```

The second line of output `o5` indicates that the output is in the field of rational numbers $\mathbb{Q}$, represented as `QQ`.

Macaulay2 has several rings that are built in, such as $\mathbb{Z}$ (`ZZ`), $\mathbb{Q}$ (`QQ`), $\mathbb{R}$ (`RR`) and $\mathbb{C}$ (`CC`). One can also work over a finite field; for example, in $\mathbb{Z}_{41}$ the command

```
i6 : R = ZZ/41
o6 = R
o6 : QuotientRing
```

defines the field of 41 elements over which one can work. One can work over the field $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ for any $p < 2^{64}$ (the largest of which is $2^{64} - 59$). Note that $\mathbb{Z}_n$ is not allowed when n is composite. To perform arithmetic in the ring above, use the command `_R`.

```
i7 : 12_R^2
o7 = -20
o7 : R

i8 : 11_R/12_R
o8 = 18
o8 : R
```

One can even define Galois fields of order p^n for a prime $p < 2^{29}$ and $n \geqslant 1$ using the command `GF(p,n)`.

```
i9 : GF(32003,4)
o9 = GF 1048969271299456081
o9 : GaloisField

i10 : exit
```

Macaulay2 has a few different ways for users to find documentation. First, one can search or browse the online documentation at the website

`http://www.math.uiuc.edu/Macaulay2/Documentation/`.

Second, one can use the command `help` at the Macaulay2 prompt to view the documentation in the Macaulay2 window; e.g., for help with the command `ideal`, type `help ideal`. Third, the command `viewHelp`, as in `viewHelp ideal`, opens the HTML version of the documentation in your computer's default browser.

Exercises

Exercise B.1.1. Work with Macaulay2 to perform some calculations in $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{Z}_p$ for your favorite prime number p.

Exercise B.1.2. Using the Macaulay2 documentation, learn how to calculate binomial coefficients $\binom{n}{m}$ and factorials $n!$ in $\mathbb{Z}$. (Search for `binomial` and `!`.) Then perform some computations using these commands.

B.2 Polynomial Rings

In this tutorial, we show how to declare polynomial rings and perform basic computations with polynomials in Macaulay2. These are done, as one would expect, first in one variable.

```
i1 : R=ZZ/7[x];
```

Notice that the output specifies the name of the ring and the type of ring. Similarly, with computations, the output not only simplifies for you, but also gives the name of the ring.

```
i2 : (x+1)^3

      3     2
o2 = x  + 3x  + 3x + 1

o2 : R

i3 : (x+1)^7

      7
o3 = x  + 1

o3 : R
```

For multiplication, you need to use `*`. For instance, in the next computation, if you were to type `(x+1)(x+2)`, you would receive an error.

```
i4 : f=(x+1)*(x+2)

      2
o4 = x  + 3x + 2

o4 : R
```

One can identify the leading coefficient of a polynomial in Macaulay2 using the command `leadCoefficient`.

```
i5 : leadCoefficient(f)
o5 = 1
     ZZ
o5 : --
      7
```

As with integers, the operators // and % compute quotients and remainders, respectively, as in the Division Algorithm.

```
i6 : (x^2+x+1)//(x+1)
o6 = x
o6 : R

i7 : (x^2+x+1)%(x+1)
o7 = 1
o7 : R
```

It is similarly easy to work in several variables.

```
i8 : B=ZZ/11[y,z]
o8 = B
o8 : PolynomialRing

i9 : (3*y+7*z)^2*(y+z)-(y-z)^3
         3    2      3
o9 = - 3y  - y z - 5z
o9 : B
```

Note that coefficient multiplication requires the * in the input.

One can use ellipses .. to adjoin several variables. Macaulay2 is quite intelligent in the types of objects one places on either end of an ellipsis.

```
i10 : C = QQ[w_1..w_5]
o10 = C
o10 : PolynomialRing

i11 : D = QQ[a..d]
o11 = D
o11 : PolynomialRing

i12 : E = QQ[t_(1,1)..t_(2,3)]
o12 = E
o12 : PolynomialRing

i13 : gens E
o13 = {t   , t   , t   , t   , t   , t   }
        1,1   1,2   1,3   2,1   2,2   2,3
o13 : List
```

Any ring that can be defined in Macaulay2 can be used as coefficients of a ring of polynomials, but some functions will not work (i.e. are not yet implemented) unless the (ultimate) base ring is the prime field $\mathbb{Z}_p$ for p a prime less than 2^{64}, or $\mathbb{Q}$.

Computations over $\mathbb{Q}$ can be a bit unwieldy due to the presence of arbitrarily large numerators and denominators, and so it is common to replace $\mathbb{Q}$ with a moderately sized prime field in order to speed computations a bit.

```
i14 : exit
```

Exercises

Exercise B.2.1. Use Macaulay2 to perform some calculations in $\mathbb{Z}_p[x,y,z]$ for your favorite prime number p.

Exercise B.2.2. Using the Macaulay2 documentation, learn how to compute the degree of a polynomial in one variable and the degree of a monomial in several variables. (Search for `degree`.)

B.3 Ideals and Generators

In this tutorial, we show how to declare ideals and perform basic computations with ideals in Macaulay2. We begin with ideals in $\mathbb{Z}$, defined using the function `ideal` with a generating sequence.

```
i1 : I = ideal(3)
o1 = ideal 3
o1 : Ideal of ZZ

i2 : J = ideal(4)
o2 = ideal 4
o2 : Ideal of ZZ
```

The command `intersect` is used to intersect ideals.

```
i3 : K=intersect(I,J)
o3 = ideal(-12)
o3 : Ideal of ZZ
```

Notice that the output is given using the same format as one would use to define the ideal. The commands `isSubset` and == test whether ideals are contained in one another and if they are equal.

```
i4 : isSubset(I,J)
o4 = false

i5 : isSubset(K,I)
o5 = true

i6 : I == K
o6 = false
```

Longer lists of generators can also be used for ideals, though Macaulay2 does not automatically reduce the generating list to a minimal one.

```
i7 : L = ideal(6,9)
o7 = ideal (6, 9)
o7 : Ideal of ZZ

i8 : I == L
o8 = true
```

Ideals are handled similarly in polynomial rings.

```
i9 : R = ZZ/31[x..z]
o9 = R
o9 : PolynomialRing
```

Note the manner in which the variables are created. While using the `..` notation is not required for multiple variables, it can be sometimes be cumbersome to list out many variables.

```
i10 : I = ideal(x^2+6*x+9)
             2
o10 = ideal(x  + 6x + 9)
o10 : Ideal of R

i11 : J = ideal(x^2+5*x+6,y+z)
              2
o11 = ideal (x  + 5x + 6, y + z)
o11 : Ideal of R

i12 : K = intersect(I,J)
                2     2                              3     2
o12 = ideal (- x y - x z - 6x*y - 6x*z - 9y - 9z, - x  - 8x  + 10x + 13)
o12 : Ideal of R

i13 : H = ideal(x,y,z)
o13 = ideal (x, y, z)
o13 : Ideal of R
```

In this setting, one can find the generators of an ideal, using the command `_*`

```
i14 : K_*
          2     2                              3     2
o14 = {- x y - x z - 6x*y - 6x*z - 9y - 9z, - x  - 8x  + 10x + 13}
o14 : List
```

and one can reduce modulo an ideal, using the `%` operator.

```
i15 : x^2 % I
o15 = - 6x - 9
o15 : R
```

```
i16 : (x^3-19*x-30) % J
o16 = 0
o16 : R

i17 : exit
```

The output `o15` tells us that $x^2 \notin I$, and the output `o16` tells us that $x^3 - 19x - 30 \in J$; this is a handy way to check if a given polynomial is in a particular ideal. Note the presence of the parenthesis in the second example. This is because `%` has higher operator precedence than `+` in Macaulay2.

Exercises

Exercise B.3.1. Use Macaulay2 to verify the following for ideals in $R = \mathbb{Z}_{41}[X,Y]$:

(a) $(X+Y, X-Y)R = (X,Y)R$.
(b) $(X+XY, Y+XY, X^2, Y^2)R = (X,Y)R$.
(c) $(2X^2+3Y^2-11, X^2-Y^2-3)R = (X^2-4, Y^2-1)R$.

Use Macaulay2 to determine if the same equalities hold in $\mathbb{Z}_2[X,Y]$. In cases where the ideals are not equal, determine if one of the ideals is contained in the other.

Exercise B.3.2. Use Macaulay2 to verify any counterexamples you devised for Exercises A.3.16, A.3.17, or A.3.18.

Exercise B.3.3. Use Macaulay2 to find a generating sequence for the ideal $I = (X^2, Y^3, Z^4)R \cap (X^4, Y, Z^2)R \cap (X^3, Y^2, Z^5)R$ in $R = \mathbb{Z}_{53}[X,Y,Z]$. Is either of the polynomials X^2YZ and X^2Y^2Z in this ideal?

B.4 Sums of Ideals

In this tutorial, we show how to work with sums of ideals in Macaulay2. The relevant command is `+`.

```
i1 : R = ZZ/41[x]
o1 = R
o1 : PolynomialRing

i2 : f = (x+3)^3
      3     2
o2 = x  + 9x  - 14x - 14
o2 : R

i3 : g = (x+3)*(x+2)
      2
o3 = x  + 5x + 6
o3 : R
```

```
i4 : I = ideal f
             3     2
o4 = ideal(x  + 9x  - 14x - 14)
o4 : Ideal of R

i5 : J = ideal g
             2
o5 = ideal(x  + 5x + 6)
o5 : Ideal of R

i6 : K = I+J
              3     2              2
o6 = ideal (x  + 9x  - 14x - 14, x  + 5x + 6)
o6 : Ideal of R
```

Notice that the generating sequence Macaulay2 produces is redundant: K is generated by the greatest common divisor of the generators of I and J.

```
i7 : K == ideal gcd(f,g)
o7 = true
```

For homogeneous ideals, the commands `mingens` and `trim` remove the redundancies since in this case, we have a well-behaved notion of irredundant generating set. See Section 1.3 for more information about these commands.

Exercises

Exercise B.4.1. Set $R = \mathbb{Z}_{41}[X,Y]$, and consider the ideals $I = (X^3,Y)R$ and $J = (X^2,XY,Y^2)R$ and $K = (X,Y^4)R$.

(a) Use Macaulay2 to verify the proper containments $I \cap J \subsetneq I \subsetneq I+J$.
(b) Use Macaulay2 to verify the equalities $I+J = J+I$ and $(I+J)+K = I+(J+K)$.
(c) Use Macaulay2 to verify that $I+J$ is generated by X^2,XY,Y^2,X,Y^4.

B.5 Products and Powers of Ideals

In this tutorial, we show how to work with products and powers of ideals in Macaulay2. These are built using the commands `*` and `^`, respectively. We demonstrate these with a continuation of the previous section's example.

```
i8 : I*J
             5      4     3      2
o8 = ideal(x  + 14x  - 4x  + 11x  + 10x - 2)
o8 : Ideal of R
```

```
i9 : J^3

             6      5      4      3      2
o9 = ideal(x  + 15x  + 11x  + 18x  - 16x  + 7x + 11)

o9 : Ideal of R
```

As in Section B.3, one can use the commands `isSubset` and `==` to compare products and intersections.

```
i10 : isSubset(I*J,intersect(I,J))

o10 = true

i11 : I*J == intersect(I,J)

o11 = false

i12 : exit
```

Exercises

Exercise B.5.1. Set $R = \mathbb{Z}_{41}[X,Y]$, and consider the ideals $I = (X^3, Y)R$ and $J = (X^2, XY, Y^2)R$ and $K = (X, Y^4)R$.

(a) Use Macaulay2 to verify the proper containment $IJ \subsetneq I \cap J$.
(b) Use Macaulay2 to verify the three equalities $IJ = JI$ and $(IJ)K = I(JK)$ and $I(J+K) = IJ + IK$.
(c) Use Macaulay2 to verify that the conclusions of Exercise A.5.6 parts (a) and (c) hold for the ideals I, J, and K above for $n = 2, \ldots, 6$.

Exercise B.5.2. Use Macaulay2 to verify Exercise A.5.7.

Exercise B.5.3. Use Macaulay2 to verify any examples you built for Exercises A.5.4 or A.5.8.

B.6 Colon Ideals

The relevant Macaulay2 command for colon ideals is `:`.

```
i1 : R = ZZ/41[x]

o1 = R

o1 : PolynomialRing

i2 : I = ideal((x+3)^3)

             3     2
o2 = ideal(x  + 9x  - 14x - 14)

o2 : Ideal of R
```

```
i3 : J = ideal((x+2)*(x+3))
            2
o3 = ideal(x  + 5x + 6)
o3 : Ideal of R

i4 : I : J
            2
o4 = ideal(x  + 6x + 9)
o4 : Ideal of R

i5 : J : I
o5 = ideal(x + 2)
o5 : Ideal of R

i6 : exit
```

Exercises

Exercise B.6.1. Set $R = \mathbb{Z}_{41}[X,Y]$, and consider the ideals $I = (X^3,Y)R$ and $J = (X^2,XY,Y^2)R$ and $K = (X,Y^4)R$.

(a) Use Macaulay2 to verify the containments $J(I :_R J) \subseteq I \subseteq (I :_R J)$. Does equality hold in either containment?
(b) Use Macaulay2 to verify the equalities

$$\begin{aligned}((I :_R J) :_R K) = (I :_R JK) &= ((I :_R K) :_R J)\\ (I \cap J :_R K) &= (I :_R K) \cap (J :_R K)\\ (I :_R J + K) &= (I :_R J) \cap (I :_R K).\end{aligned}$$

B.7 Radicals of Ideals

In this tutorial, we show how to work with radicals of ideals in Macaulay2. The relevant command is `radical`.

```
i1 : R = ZZ/41[x]
o1 = R
o1 : PolynomialRing

i2 : I = ideal((x+2)^3*(x-3)^2)
            5      3      2
o2 = ideal(x  - 15x  - 10x  + 19x - 10)
o2 : Ideal of R
```

```
i3 : radical I
                  2
o3 = ideal(- 8x  + 8x + 7)
o3 : Ideal of R
```

One can factor the generators of I and $\operatorname{rad}(I)$ to see an indication of the fact that $\operatorname{rad}(I)$ is obtained from a root-taking process.

```
i4 : factor first I_*
              2       3
o4 = (x - 3) (x + 2)
o4 : Expression of class Product
```

```
i5 : factor first (radical I)_*
o5 = (x - 3)(x + 2)(-8)
o5 : Expression of class Product
```

(The command `first` here is giving us the first (and only, in this case) generator of each ideal.) Note that `intersect` (which `radical` calls) sometimes introduces units, so you may get an answer you do not expect.

```
i6 : exit
```

Exercises

Exercise B.7.1. Use Macaulay2 to verify your answer for Exercise A.7.6 for $n = 1,\ldots,10$.

Exercise B.7.2. Set $R = \mathbb{Z}_{41}[X,Y]$, and consider the ideals $I = (X^3,Y)R$ and $J = (X^2,XY,Y^2)R$ and $K = (X,Y^4)R$.

(a) Use Macaulay2 to verify the containment $I \subseteq \operatorname{rad}(I)$. Does equality hold?
(b) Use Macaulay2 to verify the equalities

$$\operatorname{rad}(\operatorname{rad}(I)) = \operatorname{rad}(I)$$
$$\operatorname{rad}\left(I^3\right) = \operatorname{rad}(I)$$
$$\operatorname{rad}(IJ) = \operatorname{rad}(I\cap J) = \operatorname{rad}(I)\cap\operatorname{rad}(J)$$
$$\operatorname{rad}(I+J) = \operatorname{rad}(\operatorname{rad}(I)+\operatorname{rad}(J)) = \operatorname{rad}(I)+\operatorname{rad}(J).$$

Note that the last equality is not predicted by Proposition A.7.3(c).
(c) Use Macaulay2 to verify that the ideals $\operatorname{rad}(I)$, $\operatorname{rad}(J)$, and $\operatorname{rad}(K)$ are finitely generated. Then use Macaulay2 to find positive integers satisfying the conclusion of Exercise A.7.10 for each ideal I, J, K.

Exercise B.7.3. Use Macaulay2 to verify any counterexamples you devised for Exercises A.7.7, or A.7.13.

Exercise B.7.4. Use Macaulay2 to verify Exercise A.7.12 with $A = \mathbb{Z}_{41}$.

B.8 Quotient Rings

In this section, we illustrate how to work with quotient rings in Macaulay2. Often, one wishes to work in a quotient of a polynomial ring. There are several ways one can go about defining such a ring. One can define it in three steps, as follows.

```
i1 : R = QQ[x,y,z]
o1 = R
o1 : PolynomialRing

i2 : I = ideal (x^2 - y*z)
               2
o2 = ideal(x  - y*z)
o2 : Ideal of R

i3 : S = R/I
o3 = S
o3 : QuotientRing
```

One can also define this ring in one step.

```
i4 : S = QQ[x,y,z]/ideal{x^2-y*z}
o4 = S
o4 : QuotientRing
```

Note that the objects created in this way are of type `QuotientRing`, which derives from type `Ring`. In either case, computations involving `x`,`y` or `z` now take place in the quotient ring `S`.

```
i5 : x^2
o5 = y*z
o5 : S
```

In particular, the user should not use coset notation when working in quotient rings. Ring elements know what ring they are in, and the appropriate addition and multiplication operations are used in their computation.

Notice that in the above calculation, Macaulay2 changed x^2 to yz. This is because Macaulay2 is using something called a *Gröbner basis* of the ideal I to rewrite x^2 to yz by means of the Division Algorithm. The reason for this is that when working with quotient rings, it can be difficult to check that two elements (i.e. cosets) are equal. Proposition A.8.2(b) helps, but it can be difficult to verify that any of the conditions hold. Gröbner bases provide an algorithmic framework to answer this question, as well as many others in computational algebra; see Section 5.4 for more on this topic.

One can also consider quotients of quotient rings using the same syntax.

```
i6 : T = S/ideal{x^5}
o6 = T
o6 : QuotientRing
```

If S is a QuotientRing object, then one can obtain the ring of which it is a quotient using the ambient command.

```
i7 : ambient T
o7 = S
o7 : QuotientRing

i8 : ambient S
o8 = QQ[x, y, z]
o8 : PolynomialRing
```

When working with multiple rings, one must take care that elements entered in the command prompt are in the right place. For example note which ring x is in now that we have defined T.

```
i9 : x
o9 = x
o9 : T
```

In order to work in S again, we use the command use.

```
i10 : use S
o10 = S
o10 : QuotientRing

i11 : x
o11 = x
o11 : S

i12 : exit
```

One can also use the commands substitute or promote to move elements between rings. See their documentation nodes in Macaulay2 for details.

Exercises

Exercise B.8.1. Use Macaulay2 to perform some computations in the rings $\mathbb{Z}[i] = \mathbb{Z}[X]/(X^2+1)\mathbb{Z}[X]$ and $\mathbb{Q}[i] = \mathbb{Q}[X]/(X^2+1)\mathbb{Q}[X]$, e.g., show that $i^2 = -1$ and compute $(1+i)^{1999}$.

Exercise B.8.2. Use Exercise A.8.9(c) to choose two prime numbers p such that the ring $\mathbb{Z}_p[X]/(X^2+1)\mathbb{Z}_p[X]$ is not a field. Use Macaulay2 to and to verify that these rings are not fields.

B.9 Monomial Orders

In this section we show how to use the wide variety of monomial orders available for use in Macaulay2. Note that the monomial order of a ring is defined when the ring is first created, and *cannot* be changed. Therefore, if an application requires the use of a certain monomial order the user must either define the original ring using the desired monomial order, or define a different ring with the desired monomial order, perform the computation there, and then pass back to the original ring.

As a first example, let us consider the default monomial order.

```
i1 : R = QQ[x,y,z]
o1 = R
o1 : PolynomialRing
```

To view the monomial order associated with a polynomial ring in Macaulay2, one uses the next command.

```
i2 : (options monoid R)#MonomialOrder
o2 = {MonomialSize => 32  }
     {GRevLex => {1, 1, 1}}
     {Position => Up      }
o2 : VerticalList
```

(This use of the command `#` accesses the values in the "hash table" `MonomialOrder`, as described in Section 3.2.) The monomial order information is contained in the second entry of this list. As we can see, the default monomial order used here is graded reverse lexicographic (grevlex) order, as given in Definition A.9.8(c). This order first compares total degree, and then favors monomials that have the *smallest* power of the lexicographically *last* variable. For example, we have the following.

```
i3 : z^3 > x^2
o3 = true

i4 : x^2*z > z^3
o4 = true

i5 : x^2*z > x*y*z
o5 = true
```

Note that for all monomial orders in Macaulay2, the lexicographic order of the variables is determined by the order in which they are listed in the definition of the ring. In this case, one has $x > y > z$.

To define the lexicographic order, one uses the next command.

```
i6 : R = QQ[x,y,z,MonomialOrder=>Lex]
o6 = R
o6 : PolynomialRing
```

This order favors monomials with the *largest* power of the lexicographically *first* variable, ignoring any information about the total degree of the monomial.

```
i7 : z^3 > x^2
o7 = false

i8 : x^2*z > z^3
o8 = true

i9 : x^2*z > x*y*z
o9 = true
```

The graded lexicographic order is like the lexicographic order, except the total degree of the monomials is compared first and the ordering compares lexicographically when there is a tie. One can define a ring using graded lex order using the following command.

```
i10 : R = ZZ[x,y,z, MonomialOrder=>{Weights=>{1,1,1},Lex}]
o10 = R
o10 : PolynomialRing

i11 : z^3 > x^2
o11 = true

i12 : x^2*z > z^3
o12 = true

i13 : x^2*z > x*y*z
o13 = true
```

Note that while the comparison results are the same as in the graded lex example above, the last two inequalities hold for different reasons.

Some explanation is warranted regarding the syntax in the specification of the monomial order. This is an example of a *block order*, where several (partial) orders can be specified and combined to give a new order. In a block order, a pair of monomials are compared using the first order in the list, and only when the monomials with respect to that order tie[3] is the next order in the list used. In this example, the weight of a monomial is computed by assigning each variable weight 1, and summing the total weight of the monomial. Only when there is a tie is lexicographic order used. Note that one can use any nonnegative integers one likes as the weight vector in the above example.

```
i14 : R = ZZ[x,y,z,MonomialOrder=>{Weights=>{2,3,1},Lex}]
o14 = R
o14 : PolynomialRing

i15 : z^3 > x^2
o15 = false

i16 : x^2*z > z^3
o16 = true
```

[3] Note that the possibility of a tie means that the 'orders' that make up a block order need not satisfy the antisymmetry condition. However, as long as the last order in the list does satisfy antisymmetry, the resulting block order will indeed be a monomial order.

```
i17 : x^2*z > x*y*z
o17 = false
```

Other monomial orders (such as elimination orders) are available in Macaulay2. Further examples can be found via the documentation for `monomial orderings` and `examples of specifying alternate monomial orders`.

```
i18 : exit
```

Exercises

Exercise B.9.1. Given a list of Macaulay2 objects, the command `sort` will sort the elements of the list in increasing order, as long as a comparison operator is defined for the objects in the list. When the objects are polynomials, the monomial order is used to compare the leading terms of the polynomials.

Use the `sort` command to sort the monomials $z^3, x^2, x^2z, xyz \in \mathbb{Z}[x,y,z]$ with respect to each of the following monomial orders: grevlex, lex, and glex.

Concluding Notes

The software package Macaulay2 began first as Macaulay, which was written and maintained by Michael Stillman and David Bayer from 1986 to 1993[4]. The computational engine for Macaulay was written in C and was cutting edge at the time. However, the interface was not exceedingly user friendly.

In 1993, Stillman and Daniel Grayson undertook the creation of Macaulay2 which included a very robust type system and interpreter that is exposed to the user. It consists of two layers: the engine (written in C++) which does the low-level algebraic manipulations, and the interpreter which serves as an intermediary between the user and the engine. David Eisenbud later joined the development team in 2008. Many features have been added since the initial release, including one of the more important features: packages and easy to write documentation. Macaulay2 is still under active development and has had continuous support from the National Science Foundation (NSF) since its inception.

Many other individuals have contributed to the Macaulay2 system either by adding their packages to the distribution, or submitting their own bug fixes and/or improvements for consideration using the version control system git. The github repository is located at

`https://github.com/Macaulay2/M2.git`.

---

[4] The website for Macaulay still exists: `http://www.math.columbia.edu/~bayer/Macaulay/`.

To help enable the greater mathematics community reach the point where this is possible, the Macaulay2 community organizes workshops on Macaulay2 focused on package development at least once a year. Past organizers of such workshops include Hirotachi Abo, Adam Boocher, Anton Leykin, Sonja Mapes, Sam Payne, Sonja Petrović, Karl Schwede, Gregory Smith, David Swinarski, Amelia Taylor, Zach Teitler and the first author of this text. These workshops have been funded by various agencies and institutions such as the NSF, National Security Agency (NSA), and the Institute for Mathematics and its Applications (IMA).

As the reader of this text may attest, the Macaulay2 language is quite expressive and has many features. The documentation accompanying Macaulay2 provides many examples of language constructs. If one desires more examples, there are two excellent sources. First, the book [19] edited by Eisenbud, Grayson, Stillman and Bernd Sturmfels provides a tutorial of the language, as well as many 'real-world' examples of performing computations in Macaulay2 and even some research articles. More recently, beginning in 2009 the *Journal for Software in Algebra and Geometry* began publishing brief articles accompanied by new software for research in algebra and geometry. Both the article and the code are peer-reviewed, and those Macaulay2 packages that are published in the journal are "certified" and listed with a star on the official package page

`http://www.math.uiuc.edu/Macaulay2/Packages/`.

These packages are considered as excellent examples of how to write and document your Macaulay2 packages. See Section 4.2 for information about using packages.

Further Reading

We know of several fine graduate texts that are devoted (wholly or partially) to the subject of monomial ideals. The texts of Herzog and Hibi [37], Hibi [38], Miller and Sturmfels [58], and Stanley [74] are devoted to the study of monomial ideals. Also, the texts of Bruns and Herzog [9] and Villarreal [77] contain significant material about monomial ideals. It should be noted that these books are more advanced than the current text. See also the Concluding Notes from each chapter above for additional suggestions.

Lastly, the papers of Heinzer, Ratliff, and Shah [35, 36] treat monomial ideals determined by "regular sequences," based in part on the dissertation of Taylor [75]. The article [36] holds a special place in our heart, as we began our work on this text because of it.

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