Hubbard

ECE 317
Chapter 2
Homework Solutions

2.1.4) Given $F_X(x) = [1 - 1/(x+1)]\, u(x)$

Then $P(1 < X \leq 2) = F_X(2) - F_X(1)$

$$= \left(1 - \frac{1}{3}\right) - \left(1 - \frac{1}{2}\right) = \frac{2}{3} - \frac{1}{2} = \boxed{\frac{1}{6}}$$

$P(X > 3) = 1 - P(X \leq 3)$

$$= 1 - F_X(3)$$

$$= 1 - \left(1 - \frac{1}{4}\right)$$

$$= 1 - \frac{3}{4} = \boxed{\frac{1}{4}}$$

2.1.8) Given $F_X(x) = [1 - 1/(x+1)]\,u(x)$

$$f_X(x) = \frac{d}{dx}\{F_X(x)\}$$

$$F_X(x) = [1 - (x+1)^{-1}]\,u(x)$$

$$\Rightarrow \frac{d}{dx}\{F_X(x)\} = \frac{d}{dx}\{1-(x+1)^{-1}\}\cdot u(x) + [1-(x+1)^{-1}]\cdot \frac{d}{d\cdot}\{u(x)\}$$

$$= \underbrace{(x+1)^{-2}\,u(x)}\; + \underbrace{\left(1-\frac{1}{x+1}\right)\delta(x)}$$

$$= \frac{1}{(x+1)^2}\,u(x)$$

From sifting property of $\delta(t)$, the term

$$= \left(1-\frac{1}{x+1}\right)\Big|_{x=0}\cdot\delta(x)$$

$$= 0$$

$$\Rightarrow \boxed{f_X(x) = \frac{1}{(x+1)^2}\,u(x)}$$

2.1.11) $F_X(x) = \left(\frac{2}{3} - \frac{1}{2}e^{-2x}\right)u(x) + \frac{1}{3}u(x-1)$

$$f_X(x) = \frac{dF_X(x)}{dx}$$

$$= e^{-2x}u(x) + \underbrace{\left(\frac{2}{3} - \frac{1}{2}e^{-2x}\right)\delta(x)} + \frac{1}{3}\delta(x-1)$$

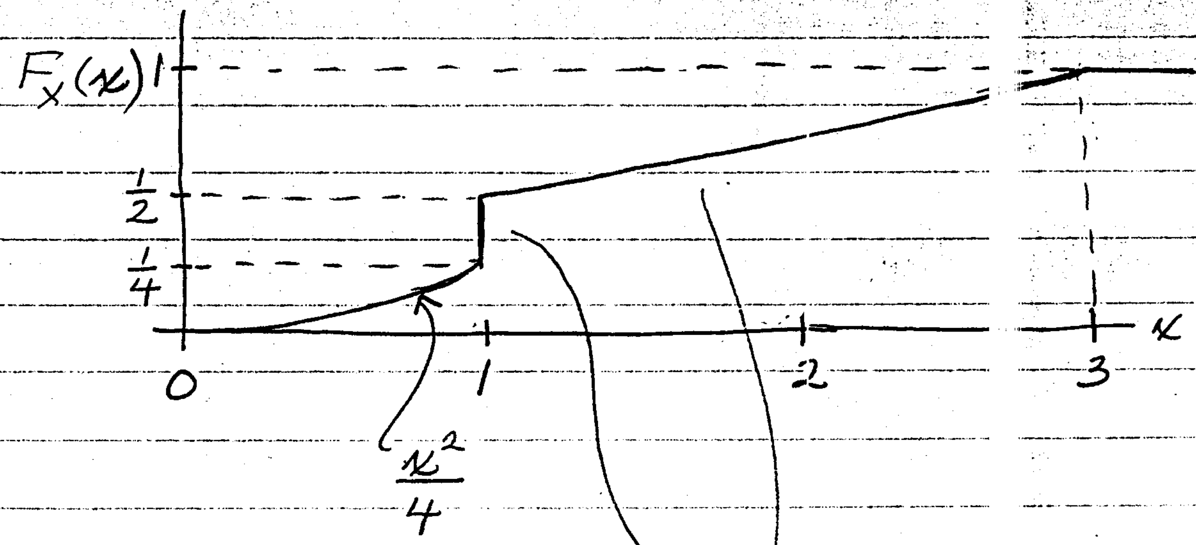
$$= \left(\frac{2}{3} - \frac{1}{2}e^{-2x}\right)\Big|_{x=0} \cdot \delta(x)$$

$$= \left(\frac{2}{3} - \frac{1}{2}\right)\delta(x)$$

$$= \frac{1}{6}\delta(x)$$

$$\Rightarrow \boxed{f_X(x) = e^{-2x}u(x) + \frac{1}{6}\delta(x) + \frac{1}{3}\delta(x-1)}$$

2.1.13) Given $F_X(x)$:

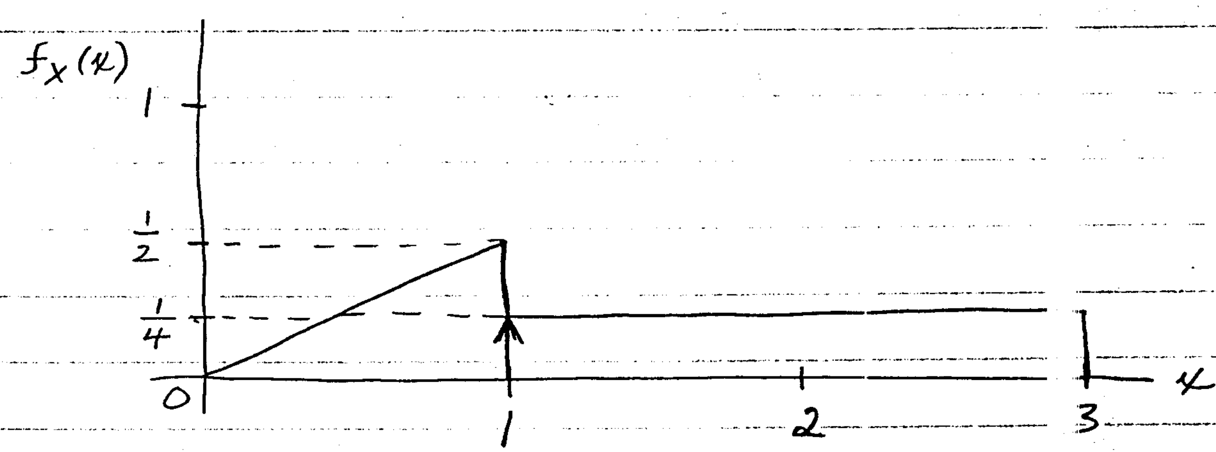


$$f_X(x) = \frac{dF_X(x)}{dx}$$

Note: $\frac{d}{dx}\left\{\frac{x^2}{4}\right\} = \frac{x}{2}$

Slope of line $= \frac{\frac{1}{2}}{2} = \frac{1}{4}$

$\Rightarrow$ Derivative of step at $x=1$ is: $\delta(x-1)$



2.1.18) Given a continuous random variable X, with probability distribution function $F_X(x)$:

$$P[X^2+4X<5] = P[X^2+4X+4<9]$$

$$= P[(X+2)^2<9] = P[|X+2|<3]$$

$$= P[-3<X+2<3]$$

$$= P[-5<X<1]$$

$$= \boxed{F_X(1) - F_X(-5)}$$

2.2.1) Given X is Gaussian, $\mu = 2$, $\sigma^2 = 16$

$\Rightarrow \sigma = 4$

Determine $P(1 < X \leq 10)$

$$P(1 < X \leq 10) = P(X > 1) - P(X > 10)$$

$$= P\left(\frac{X-\mu}{\sigma} > \frac{1-\mu}{\sigma}\right) - P\left(\frac{X-\mu}{\sigma} > \frac{10-\mu}{\sigma}\right)$$

$$= P\left(\frac{X-2}{4} > \frac{1-2}{4}\right) - P\left(\frac{X-2}{4} > \frac{10-2}{4}\right)$$

$$= Q(-0.25) - Q(2)$$

$$= (1 - Q(0.25)) - Q(2)$$

$$= 1 - 0.40129 - 0.02275$$

$$= \boxed{0.57596}$$

2.2.2) Given X is Gaussian, $\mu = 5$, $\sigma^2 = 64$
$\Rightarrow \sigma = 8$

Determine $P(1 < X \leq 15)$

$$P(1 < X \leq 15) = P(X > 1) - P(X > 15)$$

$$= P\left(\frac{X-\mu}{\sigma} > \frac{1-\mu}{\sigma}\right) - P\left(\frac{X-\mu}{\sigma} > \frac{15-\mu}{\sigma}\right)$$

$$= P\left(\frac{X-5}{8} > \frac{1-5}{8}\right) - P\left(\frac{X-5}{8} > \frac{15-5}{8}\right)$$

$$= Q(-0.5) - Q(1.25)$$

$$= \left(1 - Q(0.5)\right) - Q(1.25)$$

$$= 1 - 0.30854 - 0.10565$$

$$= \boxed{0.58581}$$

2.2.3) Given X is Gaussian with $P(X<1) = P(X>13) = Q(2)$

Need to determine μ and σ^2

$$P(X<1) = 1 - P(X \geq 1) = 1 - P\left(\frac{X-\mu}{\sigma} \geq \frac{1-\mu}{\sigma}\right)$$

$$= 1 - Q\left(\frac{1-\mu}{\sigma}\right) = Q\left(\frac{\mu-1}{\sigma}\right) = Q(2)$$

$$\Rightarrow \frac{\mu-1}{\sigma} = 2 \Rightarrow \mu - 1 = 2\sigma$$

$$P(X>13) = P\left(\frac{X-\mu}{\sigma} > \frac{13-\mu}{\sigma}\right) = Q\left(\frac{13-\mu}{\sigma}\right)$$

$$= Q(2) \Rightarrow \frac{13-\mu}{\sigma} = 2 \Rightarrow 13-\mu = 2\sigma$$

2 equations and 2 unknowns:

$$\begin{array}{r} \mu - 1 = 2\sigma \\ -\,(-\mu + 13 = 2\sigma) \\ \hline 2\mu - 14 = 0 \end{array} \Rightarrow \boxed{\mu = 7}$$

Substitute for μ in 1st equation:

$$7 - 1 = 2\sigma$$

$$\Rightarrow \sigma = 3$$

$$\Rightarrow \boxed{\sigma^2 = 9}$$

2.2.6) Given X is Gaussian, $\mu = 9$, $\sigma^2 = 25$ and $P(X < a) = Q(1.2)$.
Determine a:

$$P(X<a) = 1 - P(X \geq a)$$

$$= 1 - P\left(\frac{X-\mu}{\sigma} \geq \frac{a-\mu}{\sigma}\right)$$

(Note $\sigma = 5$)

$$\Rightarrow P(X<a) = 1 - P\left(\frac{X-9}{5} \geq \frac{a-9}{5}\right)$$

Note: We can replace "$\geq$" with "$>$", because t is is a continuous rand m variable.

$$\Rightarrow P(X<a) = 1 - Q\left(\frac{a-9}{5}\right)$$

$$= Q\left(\frac{9-a}{5}\right) = Q(1.2) \quad \text{(given)}$$

$$\Rightarrow \frac{9-a}{5} = 1.2$$

$$\Rightarrow \boxed{a = 3}$$

2.3.4) Given information in 7-bit block v,
bit error probability $p = 0.2$,
block error if 3 or more bits in a block are incorrect.

This is a binomial random vari ble with $N = 7$ and $p = 0.2$

$\Rightarrow P(Y=i) = C_i^7 p^i (1-p)^{7-i}, \quad i = 0, 1, -, \dots, 7$

$P(0 \text{ bit errors in block}) = P(Y=0)$
$= C_0^7 (0.2)^0 (0.8)^7 = (1)(1)(0.21) = \boxed{0.210}$

$P(1 \text{ bit error in block}) = P(Y=1)$
$= C_1^7 (0.2)^1 (0.8)^6 = (7)(0.2)(0.262) \quad \boxed{0.367}$

$P(2 \text{ bit errors in block}) = P(Y=2)$
$= C_2^7 (0.2)^2 (0.8)^5 = (21)(0.04)(0.328) = \boxed{0.275}$

$P(\text{block error}) = P(Y = 3, 4, 5, 6, \text{ or } 7)$
$= 1 - P(Y = 0, 1, \text{ or } 2)$
$= 1 - P\{(Y=0) \cup (Y=1) \cup (Y=2)\}$
mutually exclusive
$= 1 - P(Y=0) - P(Y=1) - P(Y=2)$
$= 1 - 0.210 - 0.367 - 0.275$
$= \boxed{0.148}$

2.3.6) Number of photons (X) is a Poisson random variable, P(0 or 1 photon) = 0.3

$$\Rightarrow P(X=i) = \frac{a^i e^{-a}}{i!}, \quad i = 0, 1, 2 \ldots$$

Need to determine parameter (a)

$P\{(X=0) \cup (X=1)\} = 0.3$

(These are mutually exclusive events)

$\Rightarrow P(X=0) + P(X=1) = 0.3$

$$\Rightarrow \frac{a^0 e^{-a}}{0!} + \frac{a^1 e^{-a}}{1!} = 0.3$$

$$e^{-a} + a e^{-a} = 0.3 \quad \Rightarrow (1+a) e^{-a} = 0.3$$

$$\Rightarrow (1+a)\left(\frac{1}{e^a}\right) = 0.3 \quad \Rightarrow \frac{1+a}{0.3} = e^a$$

$$\Rightarrow \ln\left(\frac{1+a}{0.3}\right) = a$$ Solving by iteration:

$$a_{new} = \ln\left(\frac{1+a_{old}}{0.3}\right)$$ yields $\boxed{a = 2.439}$

P(0, 1, or 2 photons)

$$= P\{(X=0) \cup (X=1) \cup (X=2)\}$$

$$= P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{a^0 e^{-a}}{0!} + \frac{a^1 e^{-a}}{1!} + \frac{a^2 e^{-a}}{2!}$$

$$= 0.087 + 0.213 + 0.259 = \boxed{0.559}$$

2.4.1) Given $F_{XY}(x,y) = (1-e^{-x})(1-e^{-2y})$, $x \geq 0$, $y \geq 0$

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \, \partial y} = \frac{\partial}{\partial x}\left\{\frac{\partial F_{XY}(x,y)}{\partial y}\right\}$$

$$= \frac{\partial}{\partial x}\left\{(1-e^{-x})(2e^{-2y})\right\} = (e^{-x})(2e^{-2y})$$

$$= \boxed{2e^{-x}e^{-2y}}$$

$$P(1 < X \leq 3,\ 1 < Y \leq 2) = \int_1^2\int_1^3 f_{XY}(x,y)\,dx\,dy$$

$$= \int_1^2\int_1^3 2e^{-x}e^{-2y}\,dx\,dy = \int_1^2 2e^{-2y}\int_1^3 e^{-x}\,dx\,dy$$

$$= \int_1^2 2e^{-2y}\left(-e^{-x}\right)\Big|_{x=1}^{x=3} dy = \int_1^2 2e^{-2y}(e^{-1}-e^{-3})\,dy$$

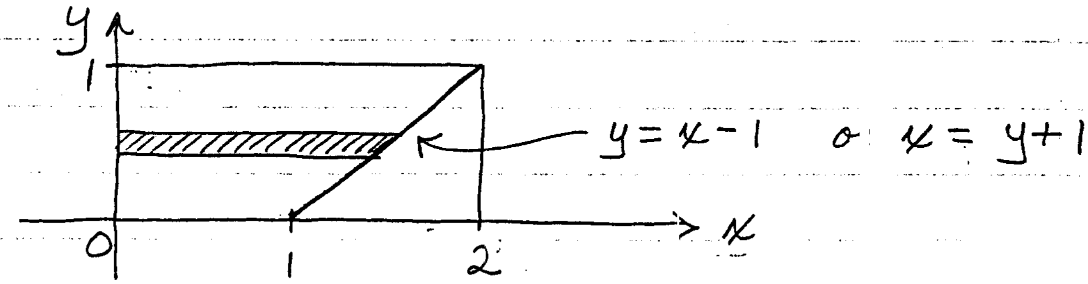
$$= 2(e^{-1}-e^{-3})\int_1^2 e^{-2y}\,dy = 2(e^{-1}-e^{-3})\left(-\tfrac{1}{2}e^{-2y}\right)\Big|_{y=1}^{y=2}$$

$$= \cancel{2}(e^{-1}-e^{-3})\left(\cancel{\tfrac{1}{2}}\right)(e^{-2}-e^{-4})$$

$$= (e^{-1}-e^{-3})(e^{-2}-e^{-4}) = (0.318)(0.117)$$

$$= \boxed{0.0372}$$

2.4.6) Given $f_{XY}(x,y) = xy$, $0 \le x \le 2$, $0 \le y \le 1$



Need to determine $P(X - Y < 1)$

Note: this is probability of all points x, y such that $y > x - 1$ (i.e., all x, y to left of line $y = x - 1$ in figure.

$$\Rightarrow P(X-Y<1) = \int_0^1 \int_0^{y+1} f_{XY}(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{y+1} xy\,dx\,dy = \int_0^1 y \int_0^{y+1} x\,dx\,dy$$

$$= \int_0^1 y\left(\frac{x^2}{2}\right)\Bigg|_{x=0}^{x=y+1} dy = \int_0^1 \left(\frac{y}{2}\right)(y^2+2y+1)\,dy$$

$$= \frac{1}{2}\int_0^1 (y^3+2y^2+y)\,dy = \frac{1}{2}\left(\frac{1}{4}y^4 + \frac{2}{3}y^3 + \frac{1}{2}y^2\right)\Bigg|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{4}+\frac{2}{3}+\frac{1}{2}\right) = \frac{17}{24} = \boxed{0.708}$$

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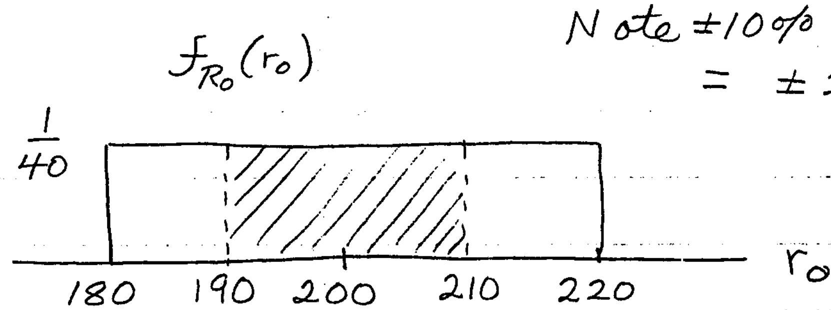
2.4.9) First, determine probability that a single 200Ω 10% resistor is within 5% of 200Ω (Assume R is uniformly distributed over tolerance range):

Let R_0 = resistance

Note $\pm 10\%$ of 200Ω
$= \pm 20\,\Omega$



$$f_{R_0}(r_0) = \frac{1}{40} = 0.025, \quad 180 \le r_0 \le 220$$

$\Rightarrow P(R_0 \text{ within } 5\% \text{ of } 200\Omega)$

$$= P(190 \le R_0 \le 210) = \int_{190}^{210} (0.025)\,dr_0$$

$$= (0.025)(210-190) = \boxed{0.5}$$

Now, put two 100Ω 10% resistor in series to form 200Ω resistor. Let R_1 and R_2 denote their resistances and assume R_1 and R_2 are statistically independent and uniform.

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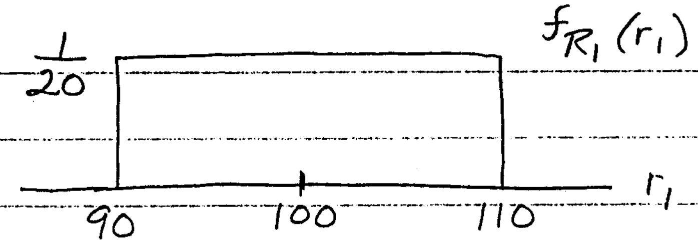
2.4.9
p.2

2.4.9)
Cont.)

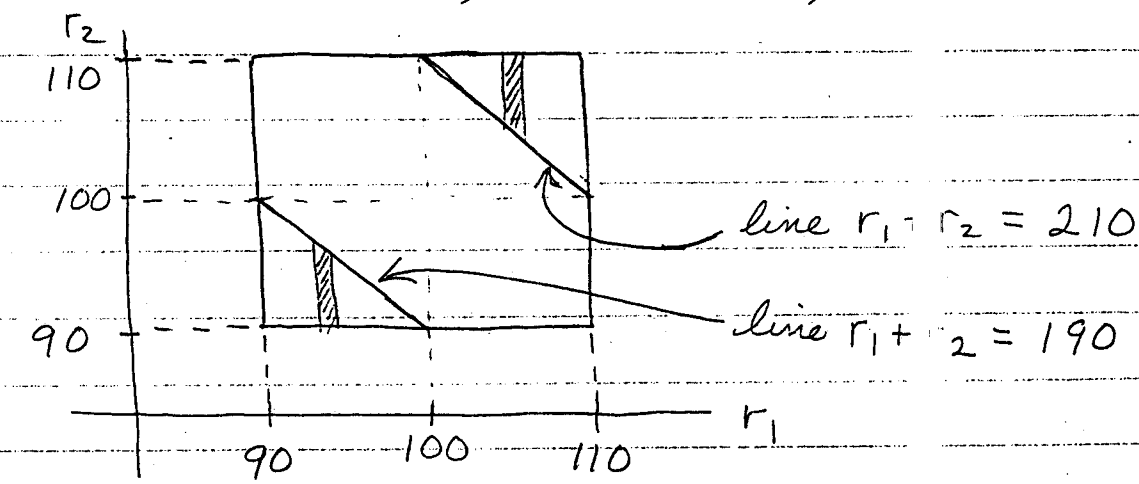


(Graph of $f_{R_2}(r_2)$ is identical).

$$\Rightarrow f_{R_i}(r_i) = \frac{1}{20} = 0.05, \quad 90 \le r_i \le 110$$
$$i = 1, 2$$

Statistical independence of R_1 and R

$$\Rightarrow f_{R_1R_2}(r_1, r_2) = f_{R_1}(r_1) f_{R_2}(r_2)$$
$$= (0.05)^2, \quad 90 \le R_1 \le 110, \; 90 \; R_2 \le 110$$



Need to find probability that $R_1 + R_2$ is within 5% of $200\,\Omega$:

$P(190 \le R_1 + R_2 \le 210)$ = integral of $f_{R_1\,\,2}(r_1, r_2)$ over region between lines $r_1 + r_2 = 210$ and $r_1 + r_2 = 190$

OR, = 1 − (integal over the two t angular regions outside the line)

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2.4.9) Cont.) $\Rightarrow P(190 \le R_1 + R_2 \le 210)$

$$= 1 - P(R_1 + R_2 < 190) - P(R_1 + R_2 > 210)$$

$$= 1 - \int_{90}^{100} \int_{90}^{190-r_1} (0.05)^2 \, dr_2 \, dr_1 - \int_{100}^{110} \int_{210-r_1}^{110} (0.05)^2 \, dr_2 \, dr_1$$

$$= 1 - (0.0025) \int_{90}^{100} r_2 \Big|_{90}^{190-r_1} dr_1 - (0.0025) \int_{100}^{110} r_2 \Big|_{210-r_1}^{110} dr_1$$

$$= 1 - (0.0025) \left[\int_{90}^{100} (100 - r_1) \, dr_1 + \int_{100}^{110} (r_1 - 100) \, dr_1 \right]$$

$$= 1 - (0.0025) \left[-\frac{1}{2}(100 - r_1)^2 \Big|_{90}^{100} + \frac{1}{2}(r_1 - 100)^2 \Big|_{100}^{110} \right]$$

$$= 1 - (0.0025) \left[-\frac{1}{2}(0 - 100) + \frac{1}{2}(100 - 0) \right]$$

$$= 1 - (0.0025)[50 + 50] = 1 - 0.25$$

$$= \boxed{0.75}$$

2.4.9) Cont.) Note: the integral could be computed more easily because the joint density function is constant (= 0.0025) over the square area:

$$P(190 \le R_1 + R_2 \le 210) = 1 - \overset{\text{2 triangles}}{2}(0.0025)(\text{Triangle area})$$

$$= 1 - \cancel{2}(0.0025)(\cancel{\tfrac{1}{2}})(10)^2$$

$$= 1 - 0.25 = \boxed{0.75}$$

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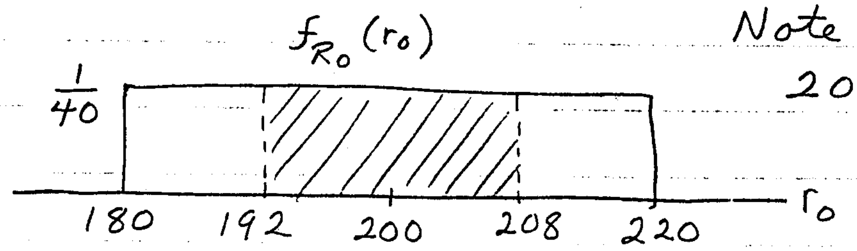
2.4.10) Determine probability that a single 200Ω 10% resistor is within 4% of 200Ω (Assume R is uniformly distributed over tolerance range):

Let R_0 = resistance



Note ± 10% of 200Ω is $\pm 20\Omega$

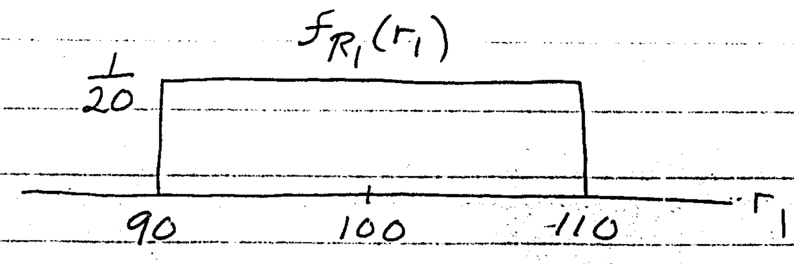
$$f_{R_0}(r_0) = \frac{1}{40} = 0.025 \quad , \quad 180 \le r_0 \le 220$$

$\Rightarrow P(R_0 \text{ within } 4\% \text{ of } 200\Omega)$

$$= P(192 \le R_0 \le 208) = \int_{192}^{208} (0.025)\, dr_0$$

$$= (0.025)(208 - 192) = \boxed{0.4}$$

- Now connect two 100Ω 10% resistors in series to form a 200Ω resistor. Let R_1 and R_2 represent the resistances of the 100Ω resistors.
- Assume R_1 and R_2 are statistically independent and uniform.



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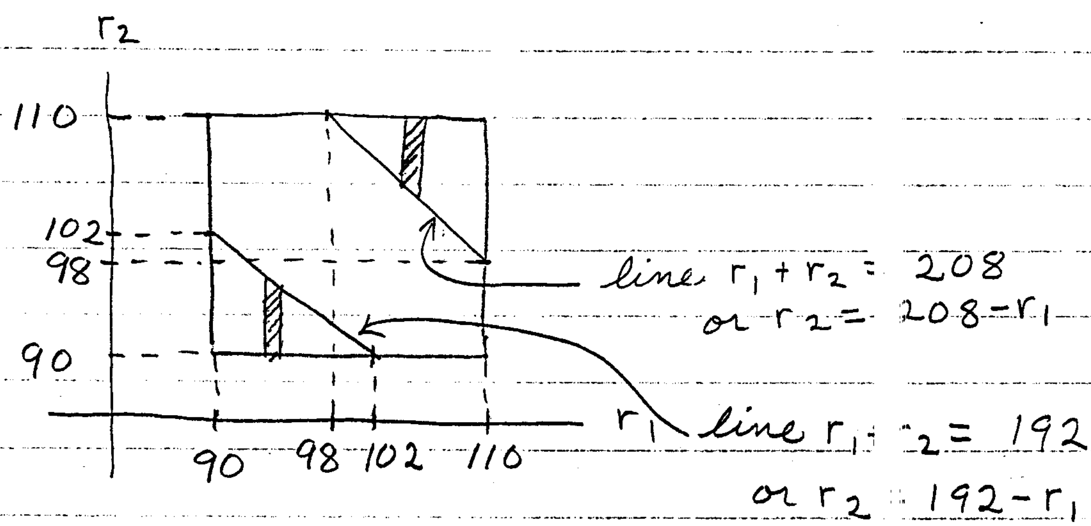
2.4.10 Cont.) Note: graph of $f_{R_2}(r_2)$ is identical to graph of $f_{R_1}(r_1)$

$\Rightarrow f_{R_i}(r_i) = \frac{1}{20} = 0.05$, $90 \le r_i \le 110$, $i = 1, 2$

Statistical independence of R_1 and R_2

$\Rightarrow f_{R_1 R_2}(r_1, r_2) = f_{R_1}(r_1) f_{R_2}(r_2)$

$= (0.05)^2$, $90 \le R_1 \le 110$, $90 \le R_2 \le 110$



Need to find probability that $R_1 + R_2$ is within 4% of 200Ω:

$P(192 \le R_1 + R_2 \le 208)$ = integral of $f_{R_1 R_2}(r_1, r_2)$ over region between lines $r_1 + r_2 = 208$ and $r_1 + r_2 = 190$

OR, = 1 − (integral over the two triangular regions outside the lines).

2.4.10) Cont.)

$$\Rightarrow P(192 \le R_1 + R_2 \le 208)$$

$$= 1 - P(R_1 + R_2 < 192) - P(R_1 + R_2 \quad 210)$$

$$= 1 - \int_{90}^{102} \int_{90}^{192 - r_1} (0.05)^2 \, dr_2 \, dr_1 - \int_{98}^{110} \int_{208 - r_1}^{110} (0.05)^2 \, dr_2 \, dr_1$$

$$= 1 - (0.0025) \int_{90}^{102} r_2 \Big|_{90}^{192 - r_1} - (0.0025) \int_{98}^{110} r_2 \Big|_{208 - r_1}^{110} dr_1$$

$$= 1 - (0.0025) \left[\int_{90}^{102} (102 - r_1) \, dr_1 + \int_{98}^{110} (r_1 - 98) \, dr_1 \right]$$

$$= 1 - (0.0025) \left[-\frac{1}{2} (102 - r_1)^2 \Big|_{90}^{102} + \frac{1}{2} (r_1 - 98)^2 \Big|_{98}^{110} \right]$$

$$= 1 - (0.0025 \left[-\frac{1}{2} (0 - 12^2) + \frac{1}{2} (12^2 - 0) \right]$$

$$= 1 - (0.0025) [72 + 72]$$

$$= 1 - 0.36 = \boxed{0.64}$$

2.4.10) Cont.) Note: the integral could be computed more easily because $f_{R_1 R_2}(r_1, r_2)$ is constant over the square area

$P(192 \le R_1 + R_2 \le 208)$

$= 1 - \underbrace{2}_{\text{2 triangles}}(0.0025)(\text{Triangle area})$

$= 1 - \cancel{2}(0.0025)(\cancel{\tfrac{1}{2}})(12)^2$

$= 1 - 0.36 = \boxed{0.64}$

2.4.13) Given 2 statistically independent components with failure times modeled by exponential random variable:
Let X_1 = failure time of component 1
X_2 = " " " " 2

Given $P(X_1 > 100) = e^{-2}$
and $P(X_2 > 100) = e^{-3}$

Note: exponential distribution function is $F_{X_i}(x_i) = 1 - e^{-a_i x_i}$, $x_i \geq 0$
where $i = 1$ or 2

$$P(X_1 > 100) = 1 - P(X_1 \leq 100) = 1 - F_{X_1}(100)$$

$$= \cancel{1} - (\cancel{1} - e^{-a_1(100)}) = e^{-100a_1} = e^{-2}$$

$$\Rightarrow a_1 = 0.02$$

$$P(X_2 > 100) = 1 - P(X_2 \leq 100) = 1 - F_{X_2}(100)$$

$$= \cancel{1} - (\cancel{1} - e^{-a_2(100)}) = e^{-100a_2} = e^{-3}$$

$$\Rightarrow a_2 = 0.03$$

Two components are in parallel
$\Rightarrow$ {Failure} = {Failure of component 1} $\cap$ {Failure of component 2}

2.4.13) Cont.) $\Rightarrow P(\text{Failure time} < 30)$

$= P(X_1 < 30 \cap X_2 < 30)$

From statistical independence

$= P(X_1 < 30)\, P(X_2 < 30)$

$= F_{X_1}(30)\, F_{X_2}(30)$

$= \left(1 - e^{-(0.02)(30)}\right)\left(1 - e^{-(0.03)(30)}\right)$

$\cong \boxed{0.268}$

2.5.1) Given the bivariate Gaussian density function

$$f_{XY}(x,y) = \frac{1}{2\pi} e^{-(2x^2-6xy+5y^2)/2},$$
$$-\infty < x < \infty, \quad -\infty < y < \infty$$

Need to determine $f_X(x)$ and $f_{Y|X}(y|x)$

Recall that $f_{XY}(x,y) = f_{Y|X}(y|x)\, f_X(x)$

$\Rightarrow$ We can separate the exponential in $f_{XY}(x,y)$ into the product of two exponentials, one for $f_{Y|X}(y|x)$, and the other for $f_X(x)$ by completing the square of the exponent on y:

$$5y^2 - 6xy + 2x^2 = 5\left[\underbrace{y^2 - \frac{6}{5}xy}\right] + 2x^2$$

Complete the square on

i.e., recall that $(y-a)^2 = y^2 - 2ay + a^2$

$$\Rightarrow \cancel{-}\frac{6}{5}x\cancel{y} = \cancel{-}2a\cancel{y} \Rightarrow a = \frac{3}{5}x$$

add a^2

$$\Rightarrow 5y^2 - 6xy + 2x^2 = 5\Big[\underbrace{y^2 - \frac{6}{5}xy + \left(\frac{3}{5}x\right)^2}_{(y-a)^2}\Big] + 2x^2 \underbrace{- 5\left(\frac{3}{5}x\right)^2}$$

Subtract a^2 to compensate

$$= 5\left(y - \frac{3}{5}x\right)^2 + \frac{1}{5}x^2$$

2.5.1) Cont.)

$$\Rightarrow f_{XY}(x,y) = \frac{1}{2\pi} e^{-\left[5\left(y-\frac{3}{5}x\right)^2 + \frac{1}{5}x^2\right]/2}$$

$$= \frac{1}{2\pi} \underbrace{e^{-\frac{5}{2}\left(y-\frac{3}{5}x\right)^2}} \; \underbrace{e^{-\frac{x^2}{10}}}$$

This term goes with $f_{Y\mid X}(y\mid x)$	This term goes with $f_X(x)$
$\Rightarrow -\frac{5}{2}\left(y-\frac{3}{5}x\right)^2 = -\frac{(y-\mu_{y\mid x})^2}{2\sigma_{y\mid x}^2}$	$\Rightarrow -\frac{x^2}{10} = \frac{-(x-\mu_x)^2}{2\sigma_x^2}$
$\Rightarrow \mu_{y\mid x} = \frac{3}{5}x$	$\Rightarrow \mu_x = 0,\ \sigma_x^2 = 5$
$\sigma_{y\mid x}^2 = \frac{1}{5}$	

$$\boxed{f_{Y\mid X}(y\mid x)} = \frac{1}{\sqrt{2\pi\sigma_{y\mid x}^2}} e^{\frac{-(y-\mu_{y\mid x})^2}{2\sigma_{y\mid x}^2}}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{1}{5}\right)}} e^{\frac{-\left(y-\frac{3}{5}x\right)^2}{2\left(\frac{1}{5}\right)}}$$

$$= \boxed{\frac{1}{\sqrt{2\pi(0.2)}} e^{\frac{-(y-0.6x)^2}{2(0.2)}}}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}}$$

$$= \frac{1}{\sqrt{2\pi(5)}} e^{\frac{-(x-0)^2}{2(5)}} = \boxed{\frac{1}{\sqrt{2\pi(5)}} e^{-\frac{x^2}{2(5)}}}$$

2.5.2) Given the bivariate Gaussian density function

$$f_{XY}(x,y) = \frac{1}{2\pi} e^{-(2x^2-6xy+5y^2)/2},$$
$$-\infty < x < \infty, \quad -\infty < y < \infty$$

Need to determine $f_Y(y)$ and $f_{Y|X}(y|x)$

Recall that $f_{XY}(x,y) = f_{X|Y}(x|y)\, f_Y(y)$

$\Rightarrow$ We can separate the exponential in $f_{XY}(x,y)$ into the product of two exponentials, one for $f_{XY}(x|y)$, and the other for $f_Y(y)$ by completing the square of the exponent on x:

$$2x^2 - 6xy + 5y^2 = 2\underbrace{[x^2 - 3xy]}_{\text{Complete the square on } x} + 5y^2$$

i.e., recall that $(x-a)^2 = x^2 - 2ax + a^2$

$$\Rightarrow \cancel{-}3\cancel{x}y = \cancel{-}2a\cancel{x} \Rightarrow a = \frac{3}{2}y$$

add a — subtract to compensate

$$\Rightarrow 2x^2 - 6xy + 5y^2 = 2\underbrace{[x^2 - 3xy + (\tfrac{3}{2}y)^2]}_{(x-a)^2} + 5y^2 - 2(\tfrac{3}{2}y)^2$$

$$= 2(x - \tfrac{3}{2}y)^2 + \frac{1}{2}y^2$$

2.5.2) Cont.) $\Rightarrow f_{XY}(x,y) = \frac{1}{2\pi} e^{-\left[2\left(x-\frac{3}{2}y\right)^2 + \frac{1}{2}y^2\right]/2}$

$$= \frac{1}{2\pi} \underbrace{e^{-\left(x-\frac{3}{2}y\right)^2}}_{\text{This term goes with } f_{X|Y}(x,y)} \; \underbrace{e^{-\frac{y^2}{4}}}_{\text{This term goes with } f_Y(y)}$$

This term goes with $f_{X|Y}(x,y)$

$\Rightarrow -\left(x-\frac{3}{2}y\right)^2 = -\frac{(x-\mu_{x|y})^2}{2\sigma_{x|y}^2}$

$\Rightarrow \mu_{x|y} = \frac{3}{2}y$

$\sigma_{x|y}^2 = \frac{1}{2}$

This term goes with $f_Y(y)$

$\Rightarrow -\frac{y^2}{4} = -\frac{(y-\mu_y)^2}{2\sigma_y^2}$

$\Rightarrow \mu_y = 0, \; \sigma_y^2 = 2$

$$\boxed{f_{X|Y}(x|y)} = \frac{1}{\sqrt{2\pi\sigma_{x|y}^2}} e^{-\frac{(x-\mu_{x|y})^2}{2\sigma_{x|y}^2}}$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{1}{2}\right)}} e^{-\frac{\left(x-\frac{3}{2}y\right)^2}{2\left(\frac{1}{2}\right)}}$$

$$= \boxed{\frac{1}{\sqrt{\pi}} e^{-\left(x-\frac{3}{2}y\right)^2}}$$

$$\boxed{f_Y(y)} = \frac{1}{\sqrt{2\pi\sigma_y^2}} e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}$$

$$= \frac{1}{\sqrt{2\pi(2)}} e^{-\frac{(y-0)^2}{2(2)}} = \boxed{\frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}}}$$