To prove the correctness of the Tree Diameter Finding Algorithm, in which we start from a vertex in a tree, first find another vertex farthest from it, and then start from that vertex to find a vertex in the tree farthest from that. The last two vertices, according to the algorithm, should be the two endpoints of a diameter in the tree.

To show this, it is apparently sufficient to prove the following theorem.

Theorem 0.1 For any vertex in the tree, the farthest vertex from it is an endpoint of a diameter.

Proof For any two vertices x, y in the tree, denote $\text{dis}(x, y)$ the length of the shortest path between them.

Take any vertex x in the tree. Assume on the contrary that the theorem does not hold, then there exists a vertex y such that y is one of the farthest vertices from x in the tree and y is not the end of a diameter.

Suppose that A and B are the two endpoints of a diameter in the tree (by definition they must exist); note that it is immediately clear that A and B are leaf nodes. Imagine reshaping the structure of the tree so that diameter AB becomes horizontal with A on the left, so that the remaining vertices extend upwards or downwards from a vertex on this diameter. There are now two cases to consider.

In the first case, the path from x to y does not intersect with diameter A-B at all, which means that the path is contained within a subtree that is extended upwards or downwards from the horizontal diameter. Let z be the last vertex that this path shares with the path connecting x and A. By the definition of y, $\text{dis}(x, z) + \text{dis}(z, y) = \text{dis}(x, y) \geq \text{dis}(x, A) = \text{dis}(x, z) + \text{dis}(z, A)$, which means $\text{dis}(z, y) \geq \text{dis}(z, A)$. Let w be the vertex on path z-A that first lies on diameter AB, we then have

$$\begin{aligned}
\text{dis}(y, B) &= \text{dis}(y, z) + \text{dis}(z, B) \\
&\geq \text{dis}(z, A) + \text{dis}(z, B) \\
&= (\text{dis}(z, w) + \text{dis}(w, A)) + (\text{dis}(z, w) + \text{dis}(w, B)) \\
&= \text{dis}(w, A) + \text{dis}(w, B) + 2\text{dis}(z, w) \\
&\geq \text{dis}(w, A) + \text{dis}(w, B) \\
&= \text{dis}(A, B),
\end{aligned}$$

meaning that y-B must be a diameter as well, contradicting the definition of y.

In the second case, the path from x to y intersects diameter A-B. Let z be the first vertex on the path from x to y that lies on diameter A-B, and let w be the last. At vertex z, the path either goes left towards A, or right towards B. W.L.O.G., assume that the path from z to y goes left, then we have $\text{dis}(x, z) + \text{dis}(z, w) + \text{dis}(w, y) = \text{dis}(x, y) \geq \text{dis}(x, A) = \text{dis}(x, z) + \text{dis}(z, w) + \text{dis}(w, A)$. This means $\text{dis}(w, y) \geq \text{dis}(w, A)$, after which $\text{dis}(y, B) = \text{dis}(y, w) + \text{dis}(w, B) \geq \text{dis}(w, A) + \text{dis}(w, B) = \text{dis}(A, B)$, again bringing a similar contradiction as in the first case.

Combining the two cases, it is clear that the theorem to be proved must be true.

□