Planck-Bound Unified Framework (PBUF) — Mathematical Supplement (v9.0)

Derivation of $\sigma_{\mu\nu}$, conservation proof, and stability analysis

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Notation & Conventions

Signature (-,+,+,+). Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. $\sigma_{\mu\nu}$ is defined so that $G_{\mu\nu} + \sigma_{\mu\nu} = 8\pi G\, T_{\mu\nu}$.

Effective Action & Definition of $\sigma_{\mu\nu}$

$S = \int\sqrt{-g}\,[\, R/(16\pi G) + L_{elastic}(g; \text{invariants}) + L_m \,]$. $\sigma_{\mu\nu} \equiv -2/\sqrt{-g}\; \delta(\sqrt{-g}\, L_{elastic})/\delta g^{\mu\nu}$. Field eqs: $G_{\mu\nu} + \sigma_{\mu\nu} = 8\pi G\, T_{\mu\nu}$.

Bounded f(R) Realization

Let $L_{elastic} = (f(R) - R)/(16\pi G)$. Then $\sigma_{\mu\nu} = (f_R - 1)G_{\mu\nu} + \frac{1}{2} g_{\mu\nu}(f - R\, f_R) + \nabla_\mu\nabla_\nu f_R - g_{\mu\nu}$ ■ f_R. Choose $f_R>0$ and $f_{RR}>0$ with a smooth saturation to enforce the Planck Bound.

Covariant Conservation

Diffeomorphism invariance and Bianchi identity imply $\nabla\cdot G=0$. With minimal coupling, $\nabla\cdot T=0$, therefore $\nabla\cdot\sigma=0$.

FLRW Effective Fluid

On FLRW, define $\rho_\sigma = -\sigma^0{}_0/(8\pi G)$, $p_\sigma = (\sigma^i{}_i)/(24\pi G)$. Conservation gives ρ■σ + $3H(\rho_\sigma+p_\sigma)=0$.

Linear Perturbations — Stability

Scalar sector: $S^{(2)} = \frac{1}{2}\int a^3\, [Q_S$ ■$^2 - c_S^2\, (k^2/a^2)\, x^2]$. Require $Q_S>0$, $c_S^2>0$. Sufficient conditions in bounded f(R): $f_R>0$, $f_{RR}>0$.

Tensor Sector & GW

Tensor modes satisfy $\ddot{h} + (3H+\nu)$ h■ $+ c_T^2\, k^2/a^2\, h = 0$ with $c_T^2=1$ for curvature-only elastic sectors; GW170817-safe.

Hamiltonian Positivity

Legendre map to scalar–tensor with $\varphi=f_R$ gives $\omega_{BD}>0$ and bounded potential $V(\varphi)$; avoids Ostrogradsky instabilities.

Summary

$\sigma_{\mu\nu}$ is covariantly conserved; background closes as an effective fluid; perturbations are ghost/gradient stable and $c_T=1$. Mathematical well-posedness supports the v9.0 empirical calibration.