Farmer Olympiad 4

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You have 45 minutes to complete all questions, justifying your answers.
The paper is out of 60 marks, 20 for each of questions 1-3.

1. Define a set $f : \mathbb{N}^3 \to \mathbb{N}$ by

$$f(a,b,c) = \binom{\binom{a}{c}}{\binom{b}{c}}.$$

Show that f is not one-to-one, i.e. for some $(a,b,c) \in \mathbb{N}^3$ there exists $(d,e,f) \in \mathbb{N}^3$ such that $(a,b,c) \neq (d,e,f)$ and $f(a,b,c) = f(d,e,f)$.

(You are given that $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ for natural numbers n, k, and that for a natural number n, $n! = 1 \times \cdots \times n$)

2. Where σ and μ are constants, and for all $i \in \mathbb{Z}$

$$X_i \sim \mathcal{N}(\mu, \sin(\frac{i\pi}{3})\sigma^2)$$

Define a random variable Y by

$$Y = \sum_{i=0}^{50} X_i.$$

Find the exact value of z such that $\mathbb{P}(Y \leq 100\mu) = \Phi(z)$.

(You are given that $\Phi(z) = \mathbb{P}(Z \leq z)$ where Z is the standard normal random variable defined by $Z \sim \mathcal{N}(0,1)$.)

3. Express the definite integral

$$\int_0^x \frac{e^{-\pi(\arctan(t^{1/49}))^2}\,dt}{49t^{48/49} + 49t^{50/49}}$$

in terms of $\Phi(g(x))$ where $g(x)$ is some function of x, assuming that the integral converges.

(You are given that $\Phi(z) = \mathbb{P}(Z \leq z)$ where Z is the standard normal random variable defined by $Z \sim \mathcal{N}(0,1)$, and that the probability density function of the normal distribution is

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\dfrac{(x-\mu)^2}{2\sigma^2}}$$

Where $X \sim \mathcal{N}(\mu, \sigma^2)$.)

Good luck!