Roots of Polynomials

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A certain rabbit told me up until now he thought that constructing polynomials can only be done through binomial expansion. He is horribly, horribly mistaken.
Roots of polynomials, however large, can be thought of as variables.
For instance the roots of $ax^2 + bx + c = 0$ can be called α and β.
Given this, we can find the polynomial in terms of the roots and vice versa in order to get simultaneous equations in terms of the roots.

Where $ax^2 + bx + c = 0$, we can deduce that:
$\alpha + \beta = \frac{-b}{a}$
and
$\alpha \cdot \beta = \frac{c}{a}$

The same can be done for polynomials of any degree. From now on the notation $\sum \alpha$ will be used for the sum of each root, $\sum \alpha\beta$ will be used for the sum of the first-order products of roots, $\sum \alpha\beta\gamma$ for the sum of the second-order products of roots and so on.

For $ax^3 + bx^2 + cx + d = 0$, we can deduce:
$\sum \alpha = \frac{-b}{a}$, $\sum \alpha\beta = \frac{c}{a}$, $\alpha\beta\gamma = \frac{-d}{a}$
For $ax^4 + bx^3 + cx^2 + dx + e = 0$:
$\sum \alpha = \frac{-b}{a}$, $\sum \alpha\beta = \frac{c}{a}$, $\sum \alpha\beta\gamma = \frac{-d}{a}$, $\alpha\beta\gamma\delta = \frac{e}{a}$

And in theory this can be extrapolated to polynomials of any degree.
When given a relation between roots (such as an arithmetic progression), you can write roots in terms of another for ease of solution.

1 The Farmer's Easy Problems

1) The Certain Rabbit wants to create a quintic equation with roots 1, 2, 3, 4, and 5. Help him!
2) Given that the roots follow an arithmetic distribution, solve $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$.
3) Given that one root is twice the other, solve $\begin{bmatrix} x^3 & -4x^2 & -11x & -14i \end{bmatrix}$
$\begin{bmatrix} 1 \\ 5 \\ -7 \\ -7i \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$

Bonus Questions Good luck!
4) i) Construct a duodecic equation with roots the first 12 terms of the fibonacci sequence.
ii) Given that the equation from part (i) is f(x), find $\frac{d^9 f}{dx^9}$

2 The Farmer's Easy Problems: Solutions

1)$x^5 - 15x^4 + 85x^3 - 225x^2 + 274x - 120 = 0$
2) $x = -4, x = -1, x = 2, x = 5$
3) $x = -2, x = 7, x = 14$
4) i) $1x^{12}+1x^{11}+1x^{10}+2x^9+1536x^8+2254342434324480x^7+27656345068767491604576153420888$
$72347425836164761488154081910888543917512012780640806357889449984000000000000000000$
$17002128607567803637759520208630963495015575509387091447375624455505844893562961920$
$30963454718960054822969246779894642673092903344400531870724683866888280945459200000$
$674129921807822677705190163721626103970522196466131271680000000000000000000000x^2+$
$525673432388469540091299102720000000000x^1 + 1570247078400$
ii)$\frac{d^9 f}{dx^9} = \frac{12!}{3!}x^3 + \frac{11!}{2!}x^2 + 10!x + 9!$