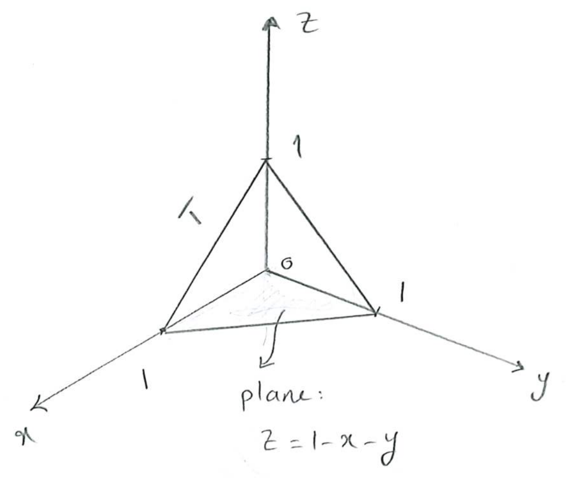
Example: If T is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, evaluate $I = \iiint_T y \, dv$.



$$I = \iiint_T y \, dv = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} y \, dz \right) dy \right) dx$$

$$= \int_0^1 \left(\int_0^{1-x} y(1-x-y) \, dy \right) dx$$

$$= \int_0^1 \left(\int_0^{1-x} \left(y(1-x) - y^2 \right) dy \right) dx$$

$$= \int_0^1 \left[\frac{y^2}{2}(1-x) - \frac{y^3}{3} \right]_0^{1-x} dx$$

$$= \int_0^1 \left(\frac{(1-x)^3}{2} - \frac{(1-x)^3}{3} \right) dx$$

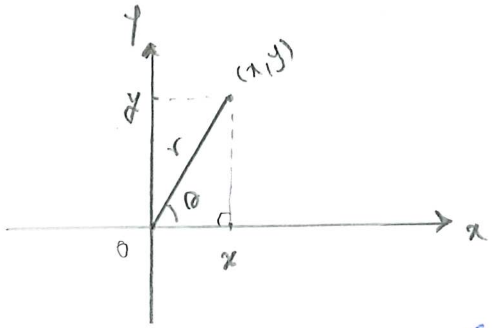
$$= \int_0^1 \frac{(1-x)^3}{6} \, dx$$

$$= \left[-\frac{(1-x)^4}{24} \right]_0^1$$

$$= \frac{1}{24}$$

Cylindrical and spherical coordinates

Recall: **Polar coordinates:**



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \iff \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases}$$

Actually:
$$\theta = \begin{cases} \arctan\left(\frac{y}{x}\right), & x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi, & x < 0,\ y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi, & x < 0,\ y < 0 \\ \frac{\pi}{2}, & x = 0,\ y > 0 \\ -\pi/2, & x = 0,\ y < 0 \\ \text{undefined}, & x = y = 0 \end{cases}$$

Example:

$r = 2a\cos\theta$ is an equation in polar coordinates.

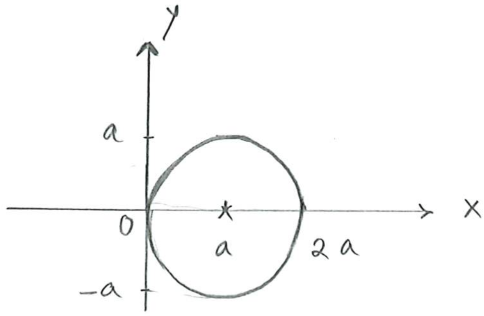
Find the cartesian equation and sketch the graph.

We know $x^2+y^2=r^2$. $\Longrightarrow x^2+y^2=r^2=r\cdot r = r\cdot 2a\cos\theta = 2a\cdot\overset{x}{(r\cos\theta)}$

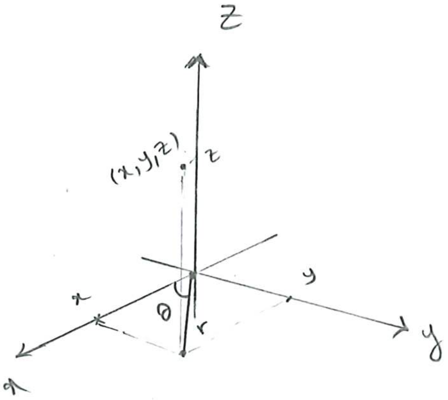
$\iff x^2+y^2=2ax \iff x^2-2ax+y^2=0$

$\iff (x-a)^2+y^2=a^2$

Circle of radius a and centered at $(a,0)$.



Cylindrical coordinates in $\mathbb{R}^3$



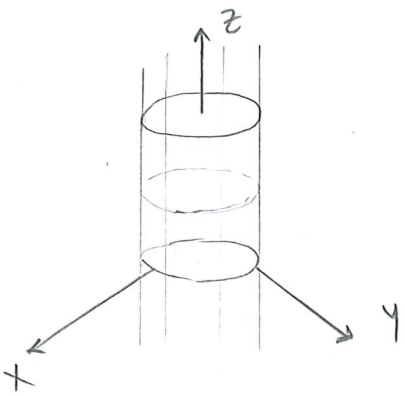
$(x, y, z) \longrightarrow (r, \theta, z)$

$r = \sqrt{x^2+y^2}$

$\theta = \arctan\left(\frac{y}{x}\right)$ $\qquad \theta \in (-\pi, \pi]$

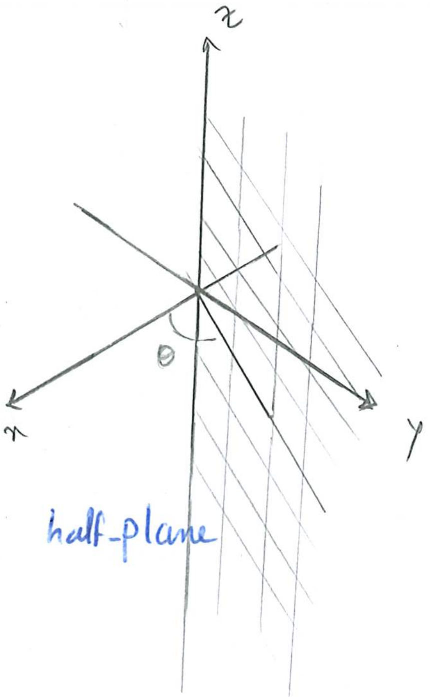
Inverse: $(r, \theta, z) \rightarrow (r\cos\theta, r\sin\theta, z)$

$r =$ Constant.



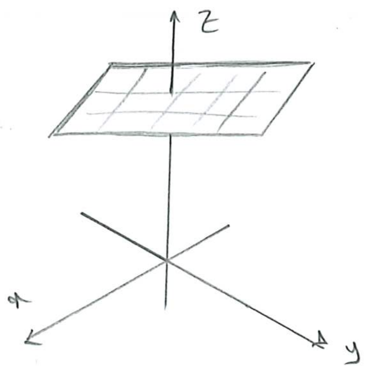
Cylinder

$\theta =$ Constant.



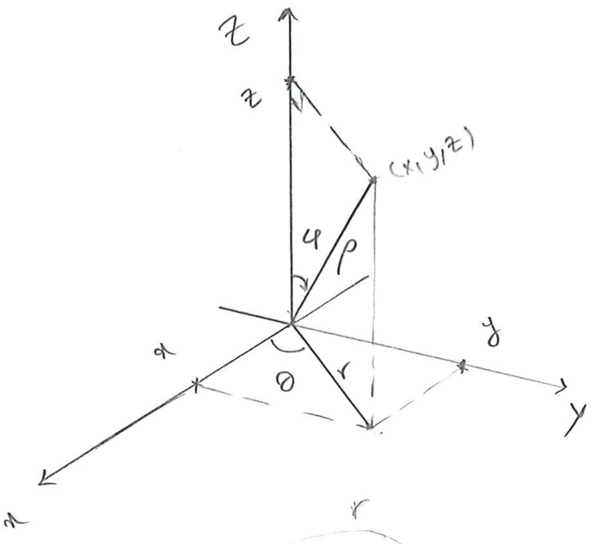
half-plane

$z =$ Constant



plane

Spherical coordinates in $\mathbb{R}^3$



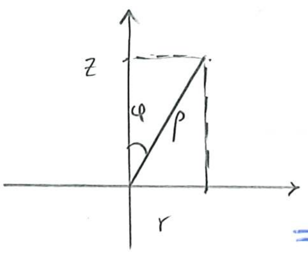
$(x, y, z) \longrightarrow (\rho, \varphi, \theta)$

θ: longitude

φ: colatitude

$$\begin{cases} X = \overbrace{\rho \sin\varphi}^{r} \cos\theta \\ y = \overbrace{\rho \sin\varphi}^{r} \sin\theta \\ Z = \rho \cos\varphi \end{cases}$$

$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \pi.$



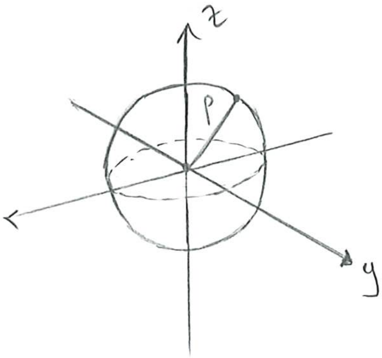
$$\begin{cases} Z = \rho \cos\varphi \\ r = \rho \sin\varphi \end{cases}$$

$\Rightarrow$ from polar coordinates:

$$\begin{cases} x = r \cos\theta = \rho \sin\varphi \cos\theta \\ y = r \sin\theta = \rho \sin\varphi \sin\theta \\ Z = \rho \cos\varphi \end{cases}$$

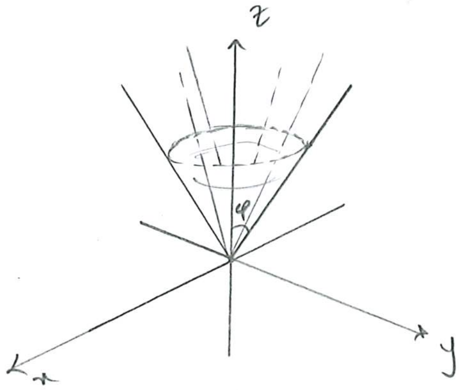
$$\left.\begin{array}{l} * \ \rho^2 = r^2 + z^2 = x^2 + y^2 + z^2 \implies \rho = \sqrt{x^2 + y^2 + z^2} \\ \varphi = \arctan\left(\frac{r}{z}\right) = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) \\ \theta = \arctan\left(\frac{y}{x}\right) \end{array}\right\} \text{Inverse transformation}$$

ρ = Constant



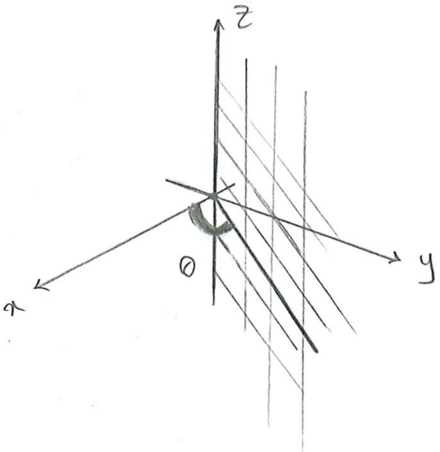
Sphere

φ = Constant



Cone
(half - Cone)

θ = Const



vertical half-plane

Example : (a) Describe the surface with the given cylindrical equation.

$z = r \Rightarrow z = \sqrt{x^2+y^2}$ the equation of a cone.
(half cone)

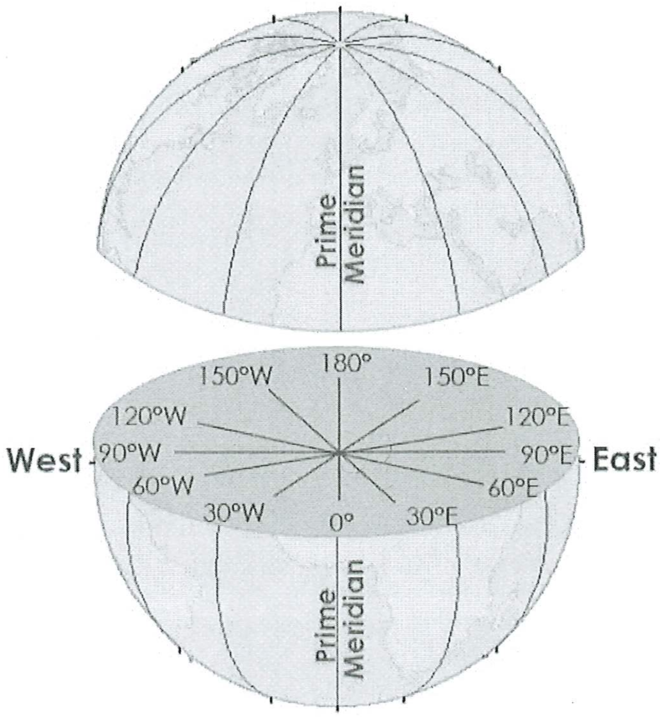
(b) Describe the surface with the given spherical equation.

$\rho = \sin\theta \sin\varphi$

$\Rightarrow \rho^2 = \rho \sin\theta \sin\varphi \Rightarrow x^2+y^2+z^2 = y \Rightarrow x^2+y^2-y+z^2=0$

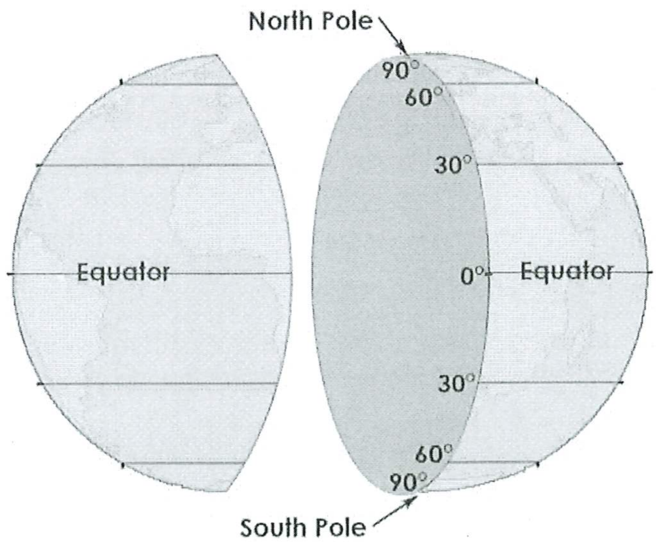
$\Rightarrow x^2+(y-\frac{1}{2})^2+z^2 = \frac{1}{4} \Rightarrow$ The equation describes a sphere centered at point $(0, \frac{1}{2}, 0)$ with radius $\frac{1}{2}$.

As shown in the image below, **lines of longitude** have X-coordinates between -180 and +180 degrees.



Longitude Coordinates

And on the other hand, **lines of latitudes** have Y-values that are between -90 and +90 degrees.



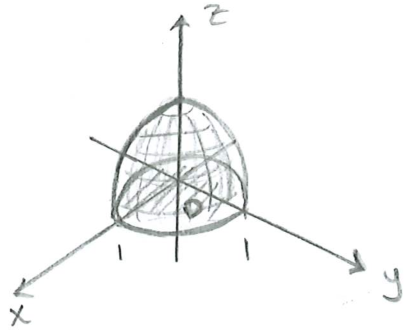
Latitude Coordinates

Double integrals with polar coordinates

The domain and/or the integrand may have a better/easier formulation in polar coordinates.

Example.

Compute the volume V of the solid bounded by the xy-plane and the paraboloid $z = 1 - x^2 - y^2$.



$$V = \iint_D (1 - x^2 - y^2)\, dA$$

$D: x^2 + y^2 \leq 1$

$$= \int_{x=-1}^{x=1} \left(\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy \right) dx$$

$$= \int_{-1}^{1} \left[y - yx^2 - \frac{y^3}{3} \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \int_{-1}^{1} \left(2\sqrt{1-x^2}\,(1-x^2) - \frac{2}{3}(1-x^2)^{3/2} \right) dx$$

$$= \frac{4}{3} \int_{-1}^{1} (1-x^2)^{3/2}\, dx = \cdots$$

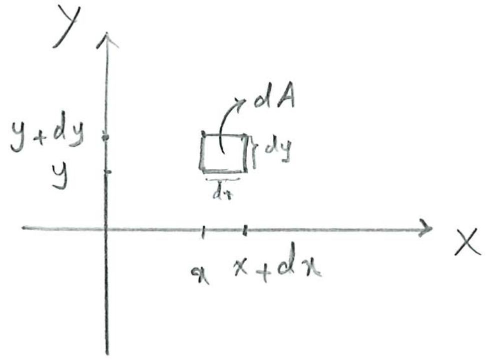
We need to use substitution method to solve it.

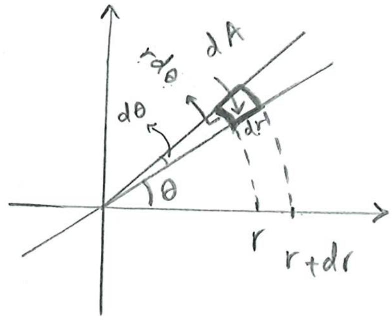
But, by using polar coordinates, it will be easier:

$$V = \iint_{\substack{\text{unit} \\ \text{disk} \\ D}} (1 - r^2)\, dA$$

dA: area element

Area elements in Cartesian and Polar coordinates :





Cartesian

$dA = dx\,dy$

Polar

$dA = r\,dr\,d\theta$

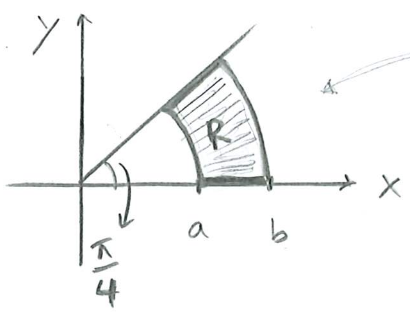
$$\Rightarrow V = \iint_{x^2+y^2\le 1} (1-x^2-y^2)\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (1-r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (r-r^3)\,dr = (2\pi)\left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1 = (2\pi)\left(\frac{1}{2}-\frac{1}{4}\right)$$

$$= \frac{\pi}{2}$$

Example.

$$\iint_R \frac{y^2}{x^2}\,dA = ?$$



$$\iint_R \frac{y^2}{x^2}\,dA = \int_{\theta=0}^{\frac{\pi}{4}}\int_{r=a}^{b} \frac{(r\sin\theta)^2}{(r\cos\theta)^2}\,r\,dr\,d\theta$$

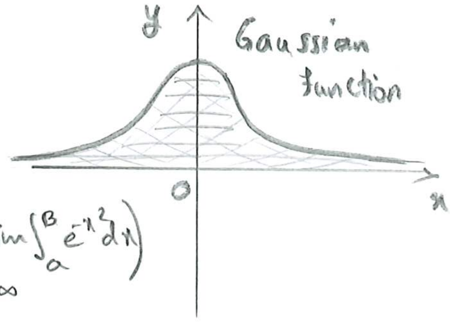
$$= \int_{r=a}^{b} r\,dr \int_{\theta=0}^{\pi/4} \tan^2\theta\,d\theta = \left[\frac{r^2}{2}\right]_a^b \cdot \int_{\theta=0}^{\pi/4} (\tan^2\theta+1-1)\,d\theta$$

$$= \left(\frac{b^2-a^2}{2}\right)\cdot\left[\tan\theta-\theta\right]_0^{\pi/4} = \left(\frac{b^2-a^2}{2}\right)\cdot\left(1-\frac{\pi}{4}\right).$$

*** Example.** $$I=\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \lim_{A\to+\infty}\int_{-A}^{A} e^{-x^2}\,dx$$

$$\left(=\lim_{A\to+\infty}\int_{-A}^{a} e^{-x^2}dx + \lim_{B\to\infty}\int_{a}^{B} e^{-x^2}dx\right)$$

$$I=\,?$$



$$I^2=\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy$$

$$=\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\,d\theta$$

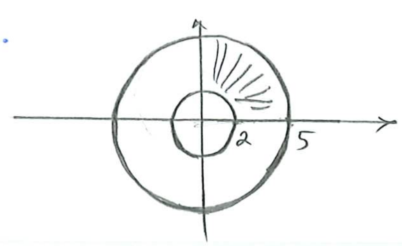
$$=2\pi\left(\lim_{A\to\infty}\int_{r=0}^{A} e^{-r^2} r\,dr\right) = 2\pi\left(\lim_{A\to\infty}\left[\frac{-e^{-r^2}}{2}\right]_0^A\right)$$

$$=2\pi\left(\lim_{A\to\infty}\frac{-e^{-A^2}+1}{2}\right) = 2\pi\cdot\frac{1}{2}=\pi$$

$$\Longrightarrow I=\int_{-\infty}^{\infty} e^{-x^2}\,dx=\sqrt{\pi}$$

Example. Evaluate the following integral by converting it into polar coordinates.

$I=\iint_D 2xy\,dA$, D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.



$$I=\int_{\theta=0}^{\pi/2}\int_{r=2}^{5} 2r^2(\cos\theta)(\sin\theta)\,r\,dr\,d\theta$$

$$\Rightarrow I = 2\int_{\theta=0}^{\pi/2} \int_{r=2}^{5} r^{3} (\sin\theta)(\cos\theta)\, dr\, d\theta$$

$$= 2\int_{\theta=0}^{\pi/2} (\sin\theta)(\cos\theta)\, d\theta \int_{r=2}^{5} r^3\, dr$$

$$= \left[\sin^2\theta\right]_0^{\pi/2} \left[\frac{r^4}{4}\right]_2^5 = \frac{609}{4}$$