L8 Vector spaces and its subspaces

1MA901/1MA406 Linear algebra

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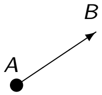
Engelsk-svensk ordlista

English	Swedish
Vector space	Vektorrum
Subspace	Underrum
Nullspace (of a matrix)	Nollrum (till en matris)
Span	Spänner
Spanning set	Linjärt hölje/spannet
Linear independence	Linjärt oberoende

Vector geometry

Definition

A line segment between points A and B with a direction is called *directed line segment*. The directed line segment has the *initial point* A and the *terminal point* B.



Definition

By the *length* of the directed line segment $\vec{AB}$ we mean the length of the line segment AB. The length of $\vec{AB}$ is denoted $|\vec{AB}|$.

Example

If A and B are two different points then $\vec{AB} \neq \vec{BA}$, but $|\vec{AB}| = |\vec{BA}|$.

Definition

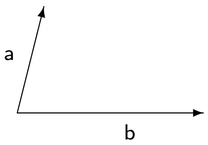
Two line segments are *parallel* if they have the same or opposite direction. This is denoted $||$.

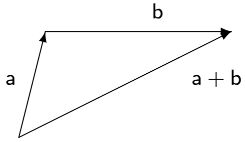
Definition

The set of all directed line segments having the same direction and length is called a vector. An element in this set is called a *representative* for the vector.

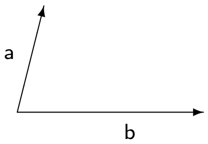
Sum of two vectors

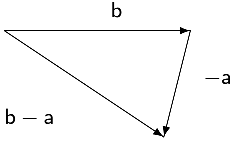
Let a and b be two vectors. The sum $a + b$ is then defined to be one of the diagonals in the parallelogram given by the two vectors.

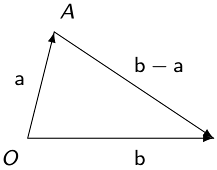




The negative of a vector a is the vector $-a$ given by the vector of equal length as a but opposite direction. The difference between two vectors a and b, denoted $b - a$ is defined as adding the negative vector $-a$ to b. We form the vector $b - a$ according to the figure. below

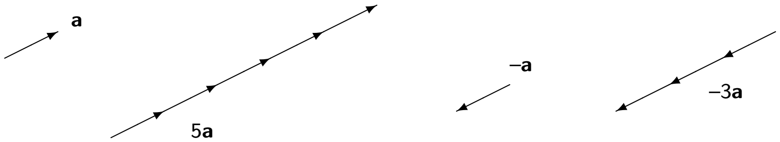






Scalar of a vector

We recall that the product of a scalar and a vector x in $\mathbb{R}^n$ is the vector (αx). In geometric terms this means the following: The product of a scalar $\alpha \in \mathbb{R}$ and a vector x is a vector αx with the length $|\alpha x| = |\alpha||x|$. If $\alpha > 0$ then αx has the same direction as x, and if $\alpha < 0$ then the direction is the opposite of x.



Theorem

The vectors $a \neq 0$ *and* $b \neq 0$ *are parallel if and only if there exists a number* $\alpha \in \mathbb{R}$ *such that* $a = \alpha b$. *If* $a = 0$ *or* $b = 0$, *then* a *and* b *are said to be parallel.*

Vector space

The axioms of a vector space.

Definition

The set V of vectors together with the operations of vector addition and scalar multiplication is said to form a *vector space* if the following axioms are satisfied. Let a, b, c be vectors in V and λ and α be scalars. Then the axioms are as follows

(i) Vector addition is commutative, i.e. $\mathrm{a} + \mathrm{b} = \mathrm{b} + \mathrm{a}$.

(ii) Vector addition is associative, i.e. $(\mathrm{a} + \mathrm{b}) + \mathrm{c} = \mathrm{a} + (\mathrm{b} + \mathrm{c})$.

(iii) Vector addition has a neutral element, i.e. $\mathrm{a} + 0 = \mathrm{a}$.

(iv) The negative of a vector is the additive inverse, i.e. $\mathrm{a} + (-\mathrm{a}) = 0$.

(v, vi) The two operations are combined in the following distributive laws $(\lambda + \alpha)\mathrm{a} = \lambda\mathrm{a} + \alpha\mathrm{a}$ and $\lambda(\mathrm{a} + \mathrm{b}) = \lambda\mathrm{a} + \lambda\mathrm{b}$.

(vii) Multiplication by scalar has the property $(\lambda\alpha)\mathrm{a} = \lambda(\alpha\mathrm{a})$.

(viii) Multiplication by scalar has a neutral element, i.e. $1\mathrm{a} = \mathrm{a}$.

Further, we note that we require a *closure* property of the vector space as well. This means that for any two vectors a and b their sum $\mathrm{a} + \mathrm{b}$ must also be an element of the vector space. The same must be true for $\alpha\mathrm{a}$.

In the next slide we will see some examples of vector spaces

Examples (and nonexamples) of vector spaces

Example

Vectors of size $n \times 1$ with real coefficients form a vector space. This is commonly denoted by $\mathbb{R}^n$.

Example

The space of continous functions $C[a, b]$ on the interval $[a, b]$ form a vector space under the operations

$$(f + g)(x) = f(x) + g(x),$$

and

$$(\alpha f)(x) = \alpha f(x).$$

Example

The set of all polynomials of degree less than n. For two polynomials p and q we define the operations $p + q$ and αp by $(p + q)(x) = p(x) + q(x)$ and $(\alpha p)(x) = \alpha p(x)$.

Example

A nonexample is all tuples of the form $(a, 1)$, where $a \in \mathbb{R}$. Denote this set by W, and endow it with ordinary vector addition and scalar multiplication. This is a nonexample since

$$(2, 1) + (4, 1) = (6, 2) \notin W.$$

Subspaces

Definition

A subset W of a vector space V is called a subspace of V if W is contained in V, and closed under the operations of vector addition and scalar multiplication.

Example

Let

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| x_2 = 2x_1 \right\}.$$

S is clearly a subset of $\mathbb{R}^2$, so the question is if it is closed under addition and scalar multiplication. If

$$\mathrm{x} = \begin{pmatrix} a \\ 2a \end{pmatrix}$$

is a given element in S and α is any scalar. Then clearly

$$\alpha \mathrm{x} = \alpha \begin{pmatrix} a \\ 2a \end{pmatrix} = \begin{pmatrix} \alpha a \\ 2\alpha a \end{pmatrix}$$

and if

$$\mathrm{y} = \begin{pmatrix} b \\ 2b \end{pmatrix}$$

is another element then

$$\mathrm{x} + \mathrm{y} = \begin{pmatrix} a + b \\ 2(a + b) \end{pmatrix} \in S.$$

Null space of a matrix

Let A be an $m \times n$ matrix, and denote by $N(A)$ the solution set the homogeneous system $A\mathrm{x} = 0$ *i.e.*

$$N(A) = \{\mathrm{x} \in \mathbb{R}^n | A\mathrm{x} = 0\}.$$

Clearly $0 \in N(A)$. Furthermore, for any $\alpha \in \mathbb{R}$ and $\mathrm{x}, \mathrm{y} \in \mathrm{N(A)}$ we have

$$A(\alpha \mathrm{x}) = \alpha(A\mathrm{x}) = \alpha 0 = 0$$

and

$$A(\mathrm{x} + \mathrm{y}) = A(\mathrm{x}) + A(\mathrm{y}) = 0 + 0 = 0.$$

The set $N(A)$ is a subspace of $\mathbb{R}^n$ and is usually called the *null space* of A.

Example

Let

$$A = \begin{pmatrix} 1 & 3 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Gauss-Jordan reduction of $A\mathrm{x} = 0$ yields

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 4 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1/2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 1/2 & 0 \end{pmatrix}.$$

Thus, x_3 is free and we have that any solution is of the form

$$(x_1, x_2, x_3) = (1/2t, -1/2t, t).$$

Hence, $N(A)$ is the set of all vectors of the form $(1/2t, -1/2t, t)$ for any real value t.

The span of a set of vectors

Henceforth, we denote by e_i the ith column vector of the identity matrix, *i.e.* in $\mathbb{R}^3$ we have $\mathsf{e}_1 = (1, 0, 0)^T$

Definition

Let $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n$ be vectors in a vector space V. A sum of the form

$$\alpha_1 \mathsf{v}_1 + \alpha_2 \mathsf{v}_2 + \cdots + \alpha_n \mathsf{v}_n$$

for scalars $\alpha_1, \ldots, \alpha_n$ is called a *linear combination* of $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n$. The set of all linear combinations of $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n$ is called the *span* of $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n$, and is denoted by $\mathrm{Span}(\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n)$.

Example

The span of the vectors

$$\mathsf{a} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathsf{b} = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$$

is the set of vectors of the form

$$\alpha \mathsf{a} + \beta \mathsf{b} = \begin{pmatrix} 2\alpha - \beta \\ \alpha \\ 5\beta \end{pmatrix}.$$

For instance the vector $\mathsf{c} = (1, 1, 5)^T$ is in the span of a and b with $\alpha = \beta = 1$.

Spanning set for a vector space

Definition

The set $\{\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n\}$ is called a spanning set of V if and only if every vector in V can be written as a linear combination of the vectors $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n$.

Example

The set $\{\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n\}$ is a spanning set for $\mathbb{R}^n$.

Example

The set

$$\mathrm{a} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathrm{b} = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}, \quad \mathrm{c} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

spans $\mathbb{R}^3$. Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 2 \\ 1 & 0 & 2 \\ 0 & 5 & 2 \end{pmatrix}$$

it has determinant equal to -8 and is thus nonsingular. Hence, by a result from lecture 6 we have that for every vector d there is a unique solution $\mathbf{x}$ to the system $A\mathrm{x} = \mathrm{d}$ or equivalently there exists $\mathrm{x} = (x_1, x_2, x_3)^T$ such that

$$x_1\mathrm{a} + x_2\mathrm{b} + x_3\mathrm{c} = \mathrm{d},$$

for every $\mathrm{d} \in \mathbb{R}^3$.

Minimal spanning set

We seek to find the smallest possible spanning set for a given vector space V.

Theorem

If $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n$ span a vector space V and one of these vectors can be written as a linear combination of the other $n-1$ vectors, then those $n-1$ vectors span V.

Proof.

Suppose that v_n can be written as a linear combination of the vectors $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_{n-1}$, *i.e.*

$$\mathrm{v}_n = \beta_1 \mathrm{v}_1 + \beta_2 \mathrm{v}_2 + \ldots \beta_{n-1} \mathrm{v}_{n-1}.$$

Let x be any element of V. Since $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_n$ span V we may write

$$\begin{aligned} \mathrm{x} &= \alpha_1 \mathrm{v}_1 + \alpha_2 \mathrm{v}_2 + \cdots + \alpha_n \mathrm{v}_n \\ &= \alpha_1 \mathrm{v}_1 + \alpha_2 \mathrm{v}_2 + \cdots + \alpha_n (\beta_1 \mathrm{v}_1 + \beta_2 \mathrm{v}_2 + \ldots \beta_{n-1} \mathrm{v}_{n-1}) \\ &= (\alpha_1 + \alpha_n \beta_1) \mathrm{v}_1 + (\alpha_2 + \alpha_n \beta_2) \mathrm{v}_2 + \cdots + (\alpha_{n-1} + \alpha_n \beta_{n-1}) \mathrm{v}_{n-1}. \end{aligned}$$

Hence, any vector in V can be written as a linear combination of $\mathrm{v}_1, \mathrm{v}_2, \ldots, \mathrm{v}_{n-1}$ and these vectors thus spans V. $\square$

A smallest set of vectors which spans a vector space is called the *minimal spanning set*. Another name for the minimal spanning set is a *basis*.

Linear independence

Definition

The vectors $v_1, v_2, \ldots, v_n$ in a vector space V are said to be *linearly independent* if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0,$$

implies that all the scalars $\alpha_1, \ldots, \alpha_n$ must equal 0.

A spanning set is minimal if and only if it is linearly independent. And on the contrary:

Definition

The vectors $v_1, v_2, \ldots, v_n$ in a vector space V are said to be *linearly dependent* if there exists scalar $\alpha_1, \ldots, \alpha_n$ not all zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0.$$

Theorem

If $v_1, v_2, \ldots, v_n$ *are n vectors in* $\mathbb{R}^n$*. Then the vectors are linearly dependent if and only if the matrix* $A = (v_1, v_2, \ldots, v_n)$ *is singular.*

Proof.

The equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0,$$

can be written as a matrix equation $Ac = 0$, and this equation will have a nontrivial solution if and only if A is singular as we've seen from earlier. □