Linnæus University
Linear algebra, 1MA406/1MA901
Jonas Nordqvist
Sofia Eriksson
`jonas.nordqvist@lnu.se`

Computer assignment: Linear algebra Fall 2022

Students aiming for a higher grade (A-B) must hand in individual solutions to the computer assignment.
Students aiming for the grades (C-E) may hand in solutions in groups of two.
Exercises marked with (*) is only mandatory for getting higher grades (A-B).
Deadline: November 6, 2022.

Hand in: A SINGLE pdf-file. This can be produced using *e.g.* Word or similar and then exported to pdf (alternatively use LaTeXif you are used to it).

Start with a heading and your name. Continue with sub headings for the exercises. From each exercises the following needs to be handed in:

1: Code from 1(e). Short answer to 1(b).

2: A short text describing your conclusions from the experminents.

3: The augmented matrix for the system, and preferably a description/figure on the naming, *e.g.* x_1 is the road from A to B, etc. Write down the matrix in (b), a valid solution in 6(c), short answer and some reasoning in 6(d).

4: Code from 2(a-e). The four (or five) figures produced in 2(a, b, c, d, (e)).

5: The code from 3. The value on c in 3(a). The figure in 3(b). Short answer from 3(c).

6: Code from 4(d). Three figures that the code produces from three different angles.

7: Answers to the questions.

Save the Matlab figures in a suitable format, so that they can be inserted in your Word-document. When you paste code in Word copied from an .m-file it may be that the type face is automatically is formatted as in Matlab. If this does not occur automatically, please use a different type face for code (*e.g.* `Courier`) or another color (or both), so that it is easy to separate text and copy-pasted code.

The assignment is divided into two parts. For campus students, Part I *may* be examined by showing the solutions to one of the teacher at the second laboration session (Sep 27). Alternatively, you hand in *all* assignments (Part I (Exercise 1-3) + Part II (Exercise 4-7)) in written form (as described above) by November 6.

Hint: Use the help in Matlab. Write `help plot` etc. in the command window to know how the plot command is used. Test your commands in the command window until you are satisfied and know how they work before you copy-paste them to your .m-files.

Part I

Exercise 1: Matrices and system of linear equations

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 3 \\ 5 \\ 6 \end{pmatrix}.$$

(a) Construct A and b in MATLAB.

(b) Compute the product Ab (using MATLAB). What happens? Why?

(c) Compute $A^T A$ and $A^T b$ (by A^T we mean the transpose of A).

(d) Determine $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in the system $A^T Ax = A^T b$, using the \ (backslash) command in MATLAB.

(e) Use the commands `ones` och `diag` to construct the matrix C_N below $N = 5$, $N = 10$ and $N = 15$. Compute its inverses C_N^{-1} using `inv`.

$$C_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{pmatrix} \qquad C_N = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & 1 & \ldots & 1 & 2 \end{pmatrix}$$

Exercise 2: Solving big systems

In this exercise you will do a small experiment. Set $n = 200$ and generate an $n \times n$ matrix and two vectors in $\mathbb{R}^n$, both having integer entries, by setting

$$A = \texttt{floor}(10 \times \texttt{rand}(n));$$

$$\mathbf{b} = \texttt{sum}(A');$$

$$\mathbf{z} = \texttt{ones}(n, 1);$$

The floor function is the function that takes as input a real number x, and gives as output the greatest integer less than or equal to x, *e.g.* $\texttt{floor}(2.3) = 2$, $\texttt{floor}(\pi) = 3$ and $\texttt{floor}(10) = 10$.

(a) The exact solution of the system $A\mathbf{x} = \mathbf{y}$ is $\mathbf{z}$. Why? Explain this? One could compute the solution using MATLAB using the backslash as in the previous exercise, or by computing A^{-1} and then multiply A^{-1} by $\mathbf{b}$. We will now compare these two methods in terms of speed and accuracy. One can use the commands `tic` and `toc` to measure elapsed time for a computation. We use them as follows

$$\texttt{tic}, \mathbf{x} = A\backslash\mathbf{b}; \texttt{toc}$$

$$\texttt{tic}, \mathbf{y} = \texttt{inv}(A) * \mathbf{b}; \texttt{toc}$$

Which method is faster?

(b) We may compare the accuracy of the methods, by measure how close the solutions are to the real solution $\mathbf{z}$. We can do this as follows

$$\texttt{max}(\texttt{abs}(\mathbf{x} - \mathbf{z}))$$

$$\texttt{max}(\texttt{abs}(\mathbf{y} - \mathbf{z}))$$

Which method is more accurate?

(c) Repeat part (a) and (b) with $n = 500$ and $n = 1000$.

(d) As noted none of these methods produce the exact solution. However it is possible to let MATLAB find the correct solution, but then $n = 200$ is a bit large (it works but it takes time). Set $n = 100$, and generate new A, $\mathbf{b}$ and $\mathbf{z}$. Further, set $C = \mathtt{sym}(A)$. Now, MATLAB treats B symbolically, we can redo the steps of (a) and (b) using the matrix C instead of A. What changes?

Exercise 3: Traffic flow

In Figure 1 we see 9 (`A-I`) intersections of one-way roads in a city where they must have hired a monkey for the traffic planning. Nonetheless, you've been hired by this monkey to measure the traffic situation. For every entrance and exit to this small part of the city there is a sensor, measuring the amount of cars going by, but inside there is no way to measure the traffic.

Your mission is the following

(a) Think about the system of equations which describes the traffic flow in the figure. Is it overdetermined, underdetermined or none of these? What do you expect in terms of the solution set?

(b) Construct the augmented matrix, which represents the system of linear equations which describes the traffic flow in Figure 1.

(c) Solve the system in (b) by finding the reduced row echelon form of the system. Is the solution unique? Why? Why not?

(d) (*) Suppose that intersection `E` is in need of maintenance and is completely closed. Is there a unique solution now? If so, what is the solution? Is this scenario realisitic or do we require negative solutions for the current traffic flow if intersection `E` is closed down?

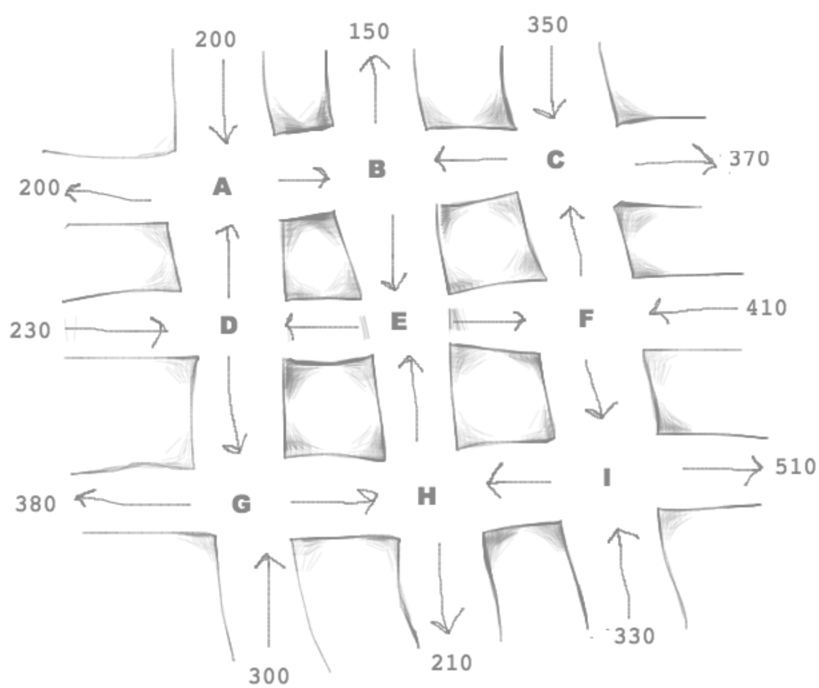

Figure 1: A scheme of 9 intersections of one-way roads in a messy city

Part II

Exercise 4: Computer graphics

In this exercise you are given an image consisting of 200 by 200 pixels, shown in Figure 2. The image is

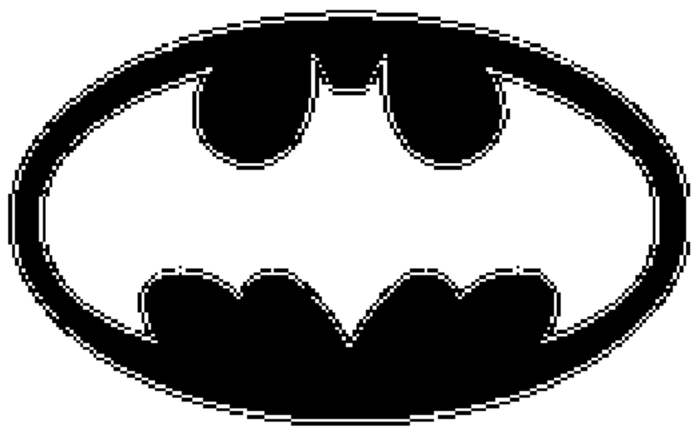

Figure 2: Image to use in Exercise 2

represented by a 2-by-10095-matrix where the first row represent the x-coordinates, the second the y-coordinates of all the pixels which are black in the image. The data is found on MyMoodle and can be imported into MATLAB using the command `load('batman.mat')` if you've placed the file in the same directory as your code.

The perhaps most natural way of thinking about this in linear algebraic terms is that the data consists of 10095 *position vectors* $(x_0, y_0)^T$ in $\mathbb{R}^2$ each describing a position of a black pixel relative to the origin, and it is the image of these vectors under linear transformations that we want to study.

Your task is the following: find the linear transformation matrices for the transformations stated below, and produce the four (or five) plots of the image after the given linear transformations

(a) reflection in the x-axis

(b) counter-clockwise rotation by $\theta = 120°$

(c) stretching out the image by a factor 2 in the x-direction

(d) *first* counter-clockwise rotation by $\theta = 60°$ *and then* reflection in the y-axis (recall that matrix multiplication in general is *not* commutative)

(e) ($*$) projecting the image onto the plane, through the origin, with normal $N = (1, 2, 3)$, (for this exercise you probably have to consider `plot3` in order to make a proper visualization). Include in your plot also a plot of a position vector at the origin parallel to the normal.

When you are plotting the images you could for instance use

```
plot(T(1, :), T(2, :), 'k.','MarkerSize', 1),
```

where T is your transformed matrix.

Exercise 5: The method of least squares

In matrix A data points closely resembling a circle are given. Your assignment to find the circle which is 'closest' to the data in a least squares sense. Recall that the equation of a circle is given by

$$(x - c_1)^2 + (y - c_2)^2 = r^2,$$

or equivalently

$$2xc_1 + 2yc_2 + c_3 = x^2 + y^2,$$

where $c_3 = r^2 - c_1^2 - c_2^2$. The data points are seen in Figure 3. You can access the data in the matrix A using

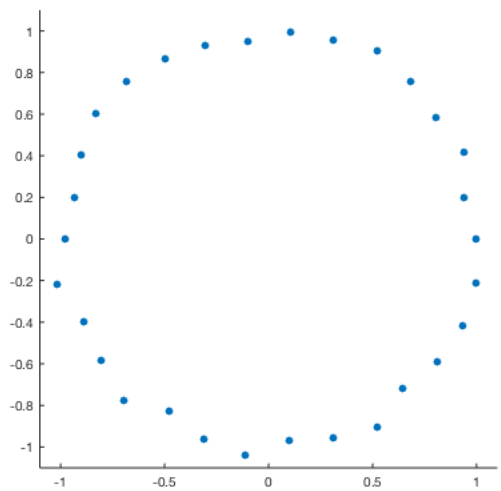

Figure 3: Data points closely resembeling a circle in Exercise 3

the MATLAB code given in Figure 4.

```
rng(7)
n = 30;
center = [0 0];
radius = 1;
eps = 0.1;
r = eps*rand(n,1) + radius-eps/2;
A = [center(1) + r.*cos(2*pi*(1:n)'/n) center(2) + r.*sin(2*pi*(1:n)'/n)];
```

Figure 4: Code for generating the data in the matrix A

Given the solution $c = (c_1, c_2, c_3)$ you can find the radius by solving for r, and thus this gives a way of measuring how close the solution is to being a circle by considering the so-called mean-squared-error (MSE), which for n data points, is given by

$$\text{MSE} = \frac{1}{n} \sum_{j=1}^{n} \left(r^2 - (x_j - c_1)^2 - (y_j - c_2)^2\right)^2.$$

Your task is the following

(a) Find the least squares solution $c = (c_1, c_2, c_3)$

(b) Plot the data together with the least squares solution (the commands `hold on` and `hold off` will probably come in handy when combining two plots)

(c) Compute the MSE for the given problem, and interpret the result. What would be the optimal result here, and what would that imply?

Exercise 6: Plotting a plane in 3D.

(a) Let $\mathbf{u} = (3, 3, -1)$ and $\mathbf{v} = (2, 4, -1)$ be two vectors in $\mathbb{R}^3$.

Determine a vector $\mathbf{w}$ which is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Either do this by hand or using the Matlab command `cross`.

Determine the equation of the place Π spanned by $\mathbf{u}$ and $\mathbf{v}$ and which goes through the origin.

Next, let $P_1 = (6, 6, z_1)$, $P_2 = (6, -6, z_2)$, $P_3 = (-6, -6, z_3)$ and $P_4 = (-6, 6, z_4)$ be four points in $\mathbb{R}^3$. Determine the four z-coordinates $z_{1,2,3,4}$ so that all four points are in the plane Π.

(b) Let a representative of $\mathbf{u}$ from 4(a) originate in the origin. Plot a figure in Matlab where $\mathbf{u}$ is illustrated by a line with thickness 2, and where the initial point and terminal point of $\mathbf{u}$ is marked with circles. (Useful command: `plot3` and the help from `plot`.

Continue in the same figure as before without removing the plot of $\mathbf{u}$ Plot the other two vectors $\mathbf{v}$ and $\mathbf{w}$, also with initial point in the origin. These two vectors terminal points should be marked using other symbols (*e.g* stars or squares). Make sure to specify 'Markersize' so that the terminal points are not too small.

(c) Consider the four points $P_{1,2,3,4}$ you investigated in 4(a). Construct a vector X containing all their x-coordinates, another vector Y containing their y-coordinates, and a vector Z containing the z-coordinates. These X-, Y- and Z-coordinates describe the corners of a surface (a subset of the plane Π) in 3D. Plot this surface using the command `fill3`, with blue color, in the same figure used earlier. Using the extra property `'facealpha',0.4`, the surface is made transparent.

(d) When it all works properly produce an .m-file with all commands from 4(b-c), which should produce a figure containing the plane Π, two vectors spanning this plane and a normal to the plane.

Make the figure interpretable and nice using commands such as `xlabel('x')`, `ylabel('y')` and `zlabel('z')` followed by the commands `grid on` and `box on` and `set(gca,'fontsize',16)` and (important for this exercise!) `axis equal`.

Exercise 7∗: Leontief input output model

In the closed Leontief input-output model we assume that there are n industries that produces n different goods, and each industry requires input in the form of goods from other industries (including its own). In this case we consider the closed model, in which we do not allow for an external demand (such as foreign export etc.). The problem is to determine the necessary output (in relation to all other industries) of each industry to meet the total demand.

The matrix M given below (and found in `leontief.mat` on MyMoodle) gives the proportion of goods needed from each sector for each sector to produce one unit of its goods

$$M = \begin{pmatrix} 0.034 & 0.11 & 6.2e-3 & 0.062 & 0.17 & 6.0e-3 & 2.8e-3 & 0.036 & 0.17 & 8.8e-3 \\ 0.094 & 0.13 & 0.28 & 0.064 & 0.21 & 0.13 & 4.3e-3 & 0.058 & 0.15 & 0.15 \\ 0.043 & 0.062 & 0.068 & 0.054 & 0.054 & 0.14 & 0.19 & 0.21 & 0.015 & 0.098 \\ 0.071 & 0.32 & 0.16 & 0.066 & 0.27 & 0.033 & 0.15 & 0.16 & 0.043 & 0.039 \\ 0.12 & 0.12 & 0.11 & 0.2 & 0.017 & 0.25 & 0.28 & 9.5e-3 & 0.063 & 0.083 \\ 0.19 & 0.033 & 1.2e-3 & 0.12 & 0.023 & 0.079 & 0.18 & 0.19 & 0.15 & 0.079 \\ 0.26 & 0.17 & 0.058 & 0.18 & 0.02 & 8.6e-4 & 0.061 & 0.11 & 0.014 & 0.15 \\ 0.043 & 6.6e-3 & 0.18 & 0.077 & 0.14 & 0.03 & 4.3e-3 & 0.032 & 0.034 & 0.19 \\ 0.057 & 9.9e-3 & 0.035 & 0.13 & 0.04 & 0.071 & 0.011 & 0.022 & 0.27 & 0.059 \\ 0.096 & 0.038 & 0.092 & 0.051 & 0.058 & 0.25 & 0.11 & 0.18 & 0.088 & 0.14 \end{pmatrix}.$$

If you were to introduce a fair monetary system on this economy which sector would price its goods the highest (as discussed briefly in the course literature), and which the lowest? Why? You can read more on the Leontief input-output model in the course literature. Further, assuming you've introduced a fair monetary system to this economy and assume that Industry 1 (corresponding to the first column of M) good is worth \$1. What would then be the worth of the goods or services produced by third and seventh industries?