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Written Exam on Numerical Methods, 2MA903, 1 hp (5 hp)
Tuesday 22th of March 2022, 14.00–19.00.

The solutions should be complete, correct, motivated, well structured and easy to follow.
Aids: Calculator (you may use a scientific calculator but *not* with internet connection).
Please begin each question on a new paper.
Preliminary grades: 15p-17p⇒E; 18p-20p⇒D; 21p-23p⇒C; 24p-26p⇒B; 27p-30p⇒A.

1. (a) Find the largest integer k for which $fl(35 + 2^{-k}) > fl(35)$ in double precision floating point representation.

   (b) Find the roots of the quadratic equation $x^2 + 6x - 7^{-14} = 0$ with four significant digits accuracy (combining calculations by hand and evaluation on calculator). (5p)

2. (a) The equation $x^3 - 6x^2 + 11x - 5 = 0$ has one real root, located in the interval $[0, 1]$. Do three iterations using the Bisection method. Report the answers and estimated errors in each iteration.

   (b) How many iterations would we have to do using the Bisection method in order to guarantee four (4) correct decimals? (5p)

3. (a) Interpolate the function $f(x) = \sin(x)$ at 4 equally spaced points on $[0, \pi/2]$.

   (b) Find an upper bound of the interpolation error at $x = \pi/4$. (5p)

4. (a) Use the trapetzoidal method to calculate approximate values of the integral

   $$I = \int_1^2 \ln(x^3)dx,$$

   for 3 different step lengths: $h = 1, 0.5, 0.25$. (2p)

   (b) Use Romberg's method on the approximate values of I obtained in a) to find an improved approximation of I. (3p)

5. Let A be a 6×6 matrix with eigenvalues $\lambda_1 = -7$, $\lambda_2 = -6$, $\lambda_3 = -3$, $\lambda_4 = -2$, $\lambda_5 = 1$ and $\lambda_6 = 5$. Each eigenvalue λ_i is associated to a eigenvector $\mathbf{v}_i$, for $i = 1, 2, 3, 4, 5, 6$.

   To which eigenvector $\mathbf{v}_i$ (if any) does the algorithm converge to, when using

   (a) Power iteration,

   (b) Inverse Power Iteration,

   (c) Inverse Power Iteration with shift $s = 3$?

   Now let $\mathbf{v}$ be one of the eigenvectors of A such that $A\mathbf{v} = \lambda\mathbf{v}$, and assume that we have found an approximation $\widetilde{\mathbf{v}}$ of this eigenvector.

   (d) Derive the Rayleigh quotient, that is find the best approximation of λ in a least square sense. (5p)

Please turn, the questions continue on next page!

6. Let $y(x)$ be the solution of $y'(x) = t - ty$ for which $y(0) = 2$.

   (a) Find an approximation of $y(2)$ using Euler backward with step length $h = 1$ and another approximation using $h = 0.5$. Answer using 4 correctly rounded decimals.

   (b) Sketch the corresponding slope field for $y'(x) = t - ty$, for $x \in [0, 2]$. Include the two approximative solutions from (a) in your slope field picture.

   (c) Using Richardson extrapolation, calculate an improved approximation of y(2) using the results obtained in (a). (5)

Good luck!

List of formulas for the exam in Numerical Methods, 2022

These formulas will be attached to the exam. The list is not guaranteed to be complete, and the use, meaning, conditions and assumptions of the formulas are purposely left out.

- **Taylor's formula**

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

(ξ between x and a)

- **Absolute and relative error**

$$\Delta_x = \tilde{x} - x, \qquad \frac{\Delta_x}{x} \approx \frac{\Delta_x}{\tilde{x}}, \qquad \Delta_{x+y} = \Delta_x + \Delta_y, \qquad \frac{\Delta_{xy}}{xy} \approx \frac{\Delta_x}{x} + \frac{\Delta_y}{y}$$

- **Error propagation formulas, condition number (1D)**

$$\Delta f \approx f'(x)\Delta x, \qquad \left|\frac{\Delta f/f}{\Delta x/x}\right| \approx \left|\frac{xf'(x)}{f(x)}\right|$$

$$\Delta f \approx f''(x)\frac{\Delta x^2}{2}$$

- **Correct decimals**
  $|\Delta x| \leq 0.5 \cdot 10^{-t}$

- **Numbers in base B**
  $x = x_m B^m + x_{m-1}B^{m-1} + \ldots + x_0 B_0 + x_{-1}B^{-1} + \ldots = (x_m x_{m-1} \ldots x_0 . x_{-1} \ldots)_B$

- **Iterative methods**

Bisection method:

```
c=(a+b)/2;
while (b-a)>2*tol
    if f(c)*f(a)>0
        a=c;
    else
        b=c;
    end
    c=(a+b)/2;
end
```

Newton-Raphson: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}, \qquad J(\mathbf{x}_n)(\mathbf{x}_{n+1} - \mathbf{x}_n) + \mathbf{f}(\mathbf{x}_n) = 0$

The secant method: $x_{n+1} = x_n - f(x_n)\dfrac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})},$

$e_n = x_n - x^*, \qquad |x_{n+1} - x^*| < \bar{c}|x_n - x^*|^p, \qquad \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = c$

- **Equation systems**

$A\mathbf{x} = \mathbf{b}$, residual $\mathbf{r} = \mathbf{b} - A\widetilde{\mathbf{x}}$

LU-factorization: $A = LU, \quad PA = LU$

QR-factorization: $A = QR, \quad Q^TQ = I$

(Iterative methods) $A = D + L + U$

Jacobi methods:
Gauss-Seidel:

$$\begin{cases} \mathbf{x}^{(k)} = -D^{-1}(L+U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b} \\ \mathbf{x}^{(k)} = -(D+L)^{-1}U\mathbf{x}^{(k-1)} + (D+L)^{-1}\mathbf{b} \end{cases}$$

Backward: $\|\mathbf{r}\|_\infty$, forward: $\|\mathbf{x} - \widetilde{\mathbf{x}}\|_\infty$

- **Norms and condition numbers**

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a vector:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}, \quad \|\boldsymbol{x}\|_1 = \sum_{i=1}^n |x_i|, \quad \|\boldsymbol{x}\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}, \quad \|\mathbf{x}\|_\infty = \max_i |x_i|.$$

Let A be a $n \times n$ matrix:

$$\|A\| = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}, \qquad \|A\mathbf{x}\| \leq \|A\| \cdot \|\mathbf{x}\|, \qquad \kappa(A) = \|A\| \cdot \|A^{-1}\|$$

$A(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}$

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \kappa(A)\frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}, \qquad econd(A) = \frac{\|\delta\mathbf{x}\|/\|\mathbf{x}\|}{\|\delta\mathbf{b}\|/\|\mathbf{b}\|} \leq \kappa(A),$$

- **Interpolation**

Let $(x_0, y_0), \ldots, (x_n, y_n)$ be $n+1$ points in the xy-plane.

Monomial: $P(x) = a_0 + a_1x + \ldots + a_nx^n$

Lagrangre: $P(x) = \displaystyle\sum_{j=0}^n y_j\ell_j(x), \qquad \ell_j(x) = \prod_{\substack{0\leq m\leq n \\ m\neq j}} \frac{x - x_m}{x_j - x_m}$

Newton's divided differences:

$$P(x) = [y_0] + [y_0, y_1](x - x_0) + \cdots + [y_0, \ldots, y_k](x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$

$$[y_0] = f(x_0), \quad [y_0, y_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \quad [y_0, y_1, y_2] = \frac{[y_1, y_2] - [y_0, y_1]}{x_2 - x_0}, \ldots$$

Interpolation errors:

$$R(x) = f(x) - P(x) = (x - x_0)(x - x_1)\cdots(x - x_n)\frac{f^{n+1}(\xi)}{(n+1)!}, \qquad x_0 < \xi < x_n,$$

- **Least squares, normal equations** $A^TA\mathbf{x} = A^T\mathbf{b}$, residual $\mathbf{r} = \mathbf{b} - A\mathbf{x}$

- **Finite differences**

$$\frac{f(x+h)-f(x)}{h} = f'(x) + f''(\xi)\frac{h}{2} \qquad \xi \in [x, x+h]$$
$$\frac{f(x)-f(x-h)}{h} = f'(x) - f''(\xi)\frac{h}{2} \qquad \xi \in [x-h, x]$$
$$\frac{f(x+h)-f(x-h)}{2h} = f'(x) + f^{(3)}(\xi)\frac{h^2}{6} \qquad \xi \in [x-h, x+h]$$
$$\frac{f(x+h)-2f(x)+f(x-h)}{h^2} = f''(x) + f^{(4)}(\xi)\frac{h^2}{12} \qquad \xi \in [x-h, x+h]$$

- **Trapezoidal rule, Simpson's rule**

$$\int_a^b f(x)dx = \frac{h}{2}\Big(f(x_0) + 2\sum_{k=1}^{n-1} f(x_k) + f(x_n)\Big) - \frac{(b-a)h^2}{12}f''(\xi), \qquad h = \frac{b-a}{n}$$
$$\int_a^b f(x)dx = \frac{h}{3}\Big(f(x_0) + 4\sum_{k=1}^{n} f(x_{2k-1}) + 2\sum_{k=1}^{n-1} f(x_{2k}) + f(x_{2n})\Big) - \frac{(b-a)h^4}{180}f^{(4)}(\xi), \qquad h = \frac{b-a}{2n}$$

$a < \xi < b$

- **Richardson extrapolation**
  $Q = F(h) + kh^n + \mathcal{O}(h^{n+1})$, $\qquad Q = \frac{2^n F(h/2) - F(h)}{2^n - 1} + \mathcal{O}(h^{n+1})$

- **Romberg** $R_{i,1} = T(h/2^{i-1})$, $R_{ij} = \frac{4^{j-1}R_{i,j-1} - R_{i-1,j-1}}{4^{j-1}-1}$

- **Numerical solutions of differential equations**

Differential equation $y' = f(x,y)$ with initial condition $y(x_0) = y_0$

Euler forward ($g_i \sim \mathcal{O}(h)$): $\quad y_{n+1} = y_n + hf(x_n, y_n)$

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Heun's method ($g_i \sim \mathcal{O}(h^2)$):
$$\begin{cases} y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) \\ k_1 = hf(x_n, y_n) \\ k_2 = hf(x_n + h, y_n + k_1) \end{cases}$$

RK4 ($g_i \sim \mathcal{O}(h^4)$):
$$\begin{cases} y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\ k_1 = hf(x_n, y_n) \\ k_2 = hf(x_n + h/2, y_n + k_1/2) \\ k_3 = hf(x_n + h/2, y_n + k_2/2) \\ k_4 = hf(x_n + h, y_n + k_3) \end{cases}$$

where $x_{n+1} = x_n + h$.

- **Boundary value problems**

Two-point boundary problem $y'' = f(x, y, y')$ with initial condition $y(a) = \alpha$ and $y(b) = \beta$.

Shooting method: The BVP is rewritten as a IVP. Vary the modified initial condition until the boundary conditions are satisfied with desired accuracy.

Finite difference method: Derivatives are approximated by finite difference quotients. Exact solution y is replaced by y_i such that $y_i \approx y(x_i)$

- **Eigenvalue problems**

The power method: $\mathbf{v}_{k+1} = A\mathbf{v}_k/\|A\mathbf{v}_k\|$ and $\lambda_1 \approx \mathbf{v}_k^T A\mathbf{v}_k$.
The QR-method. Let $A = Q_0R_0$ be a QR-decomposition of a real matrix A. Set $A_1 = R_0Q_0$ and inductively (if $A_{n-1} = Q_{n-1}R_{n-1}$ is a QR-decomposition) $A_n = R_{n-1}Q_{n-1}$.