L12 Row space, column space and the Rank-Nullity Theorem

1MA901/1MA406 Linear algebra

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Engelsk-svensk ordlista

English	Swedish
Vector space	Vektorrum
Subspace	Underrum
Nullspace (of a matrix)	Nollrum (till en matris)
Span	Spänner
Spanning set	Linjärt hölje/spannet
Linear independence	Linjärt oberoende
Basis	Bas
Change of basis	Basbyte
Row space (of a matrix)	Radrummet (till en matris)
Column space (of a matrix)	Kolonnrummet (till en matris)
Rank	Rang
Rand-nullity theorem	Dimensionssatsen

Row space and column space

Definition

Given a matrix A of size $m \times n$ the subspace spanned by its row vectors is called the *row space* of A and is a subspace of R^n. The subspace spanned by its columns is called the *column space* of A and is a subspace of $\mathbb{R}^m$.

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Example

Find the row and column space of the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & -1 & 4 \end{pmatrix}.$$

The row space consists of all vectors of the form

$$\alpha(2,1,0) + \beta(-1,-1,4) = (2\alpha - \beta, \alpha - \beta, 4\beta).$$

The column space is $\mathbb{R}^2$ since we have that every vector in the column space may be written of the form

$$\alpha(2,-1)^T + \beta(1,-1)^T + \gamma(0,4)^T.$$

Rank

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Two row equivalent matrices have the same row space.

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Proof.

Since B is row equivalent to A its row space must be a subspace of the row space of A, since the row vectors of B are linear combinations of the rows in A. Since the opposite is also true, their subspaces must coincide. □

Note! In general note true for the column space.

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Theorem (Rank-nullity theorem (Dimensionssatsen))

If A is an $m \times n$ matrix then the sum of its rank and its nullity is equal to n.

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Proof.

Let U be the reduced row echelon form of A. Then $A\mathrm{x} = 0 \sim U\mathrm{x} = 0$. If A has rank r then U will have r nonzero rows, and the system will have r leading variables and $n - r$ free variables. The dimension of $N(A)$ is equal to the number of free variables. □

Dimension of column space

Theorem

Let A be a $m \times n$ matrix. The linear system $A\mathrm{x} = \mathrm{b}$ is consistent for every $\mathrm{b} \in \mathbb{R}^m$ if and only if the column vectors of A span $\mathbb{R}^m$. The system of linear equations $A\mathrm{x} = \mathrm{b}$ has at most one solution for every $\mathrm{b} \in \mathbb{R}^m$ if and only if the columns of A are linearly independent.

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Remark

If A is $m \times n$, then

$$\dim(\textit{col space}) = \dim(\textit{row}) = n - \dim(\textit{null space}).$$

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The columns of U with leading ones are linearly independent, but they are not necessarily a basis for the columns space of A. Denote by U_L and A_L the matrices in which we remove the columns without leading ones in U and the corresponding columns of A. Then $A_L \sim U_L$.

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This implies that any solution x to $A\mathrm{x} = 0$ must necessarily also be a solution of $U\mathrm{x} = 0$. We've concluded that the columns of U must be linearly independent, and thus $\mathrm{x} = 0$ is the only solution (according to the previous theorem). As a consequence (of said previous theorem) we must have that the columns of A_L are linearly independent.

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Hence, we may conclude that the columns space of A is at least of dimension r, since there are r linearly independent columns of A. So, we've shown that

$$\dim(\text{col space}) \geq \dim(\text{row space}).$$

However, we also have

$$\begin{aligned}
\dim(\text{row space of } A) &= \dim(\text{col space of } A^T) \\
&\geq \dim(\text{row space of } A^T) \\
&= \dim(\text{col space of } A).
\end{aligned}$$

This completes the proof of the theorem.

Example

Let A be the matrix

$$A = \begin{pmatrix} 4 & 3 & 6 \\ 4 & 5 & 10 \\ 1 & 1 & 2 \end{pmatrix}.$$

Then we have

$$A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} = U.$$

Note that

$$A_L = \begin{pmatrix} 4 & 3 \\ 4 & 5 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = U_L.$$

The columns of leading ones are $(1,0,0)^T$ and $(0,1,0)^T$ which clearly does not span the colum space since *e.g.* $(4,4,1)^T$ cannot be written as a linear combination of these two.

Although the corresponding vectors in A, *i.e.* $(4,4,1)^T$ and $(3,5,1)^T$ are linearly independent, and form a basis for the column space.

Example

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Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 & 0 \\ -1 & 1 & -1 & 1 & -1 \\ 2 & 3 & 4 & 6 & 1 \end{pmatrix}$$

Find the rank of A and the dimension of its column space and null space. Further, find a basis for the corresponding subspaces.