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Written Exam on Basic Numerical Methods, 1DV519, 7,5 hp
Saturday 27th of October 2018, 12.00–17.00.

The solutions should be complete, correct, motivated, well structured and easy to follow.
Aids: Calculator (you may use a scientific calculator but *not* with internet connection)
Grades: 15p-17p⇒E; 18p-20p⇒D; 21p-23p⇒C; 24p-26p⇒B; 27p-30p⇒A.

1. Given the following set of points $(x, y) : (0, 1), (2, 1), (3, 4), (4, 5)$;
   a) determine the corresponding interpolating polynomial (of lowest possible degree), (3p)
   b) find an approximate value of y for $x = 1$. (1p)

   a) Lagrange:

$$L(x) = 1 * \frac{(x-2)*(x-3)*(x-4)}{(0-2)*(0-3)*(0-4)} + 1 * \frac{(x-0)*(x-3)*(x-4)}{(2-0)*(2-3)*(2-4)} + \dots$$
$$4 * \frac{(x-0)*(x-2)*(x-4)}{(3-0)*(3-2)*(3-4)} + 5 * \frac{(x-0)*(x-2)*(x-3)}{(4-0)*(4-2)*(4-3)}$$

   OR Newton-divided differences:

$$\begin{array}{c|cccc} 0 & 1 & & & \\ & & 0 & & \\ 2 & 1 & & 1 & \\ & & 3 & & -1/2 \\ 3 & 4 & & -1 & \\ & & 1 & & \\ 4 & 5 & & & \end{array}$$

$$P(x) = 1 + (x-0)(x-2) - \frac{1}{2}(x-0)(x-2)(x-3)$$

   Control: $P(0) = 1$, $P(2) = 1$, $P(3) = 4$ and $P(4) = 5$.

$$P(x) = 1 - 5x + \frac{7}{2}x^2 - \frac{1}{2}x^3$$

   b) $L(1) = -1$ or $P(1) = -1$

2. Given the same set of points as in the problem above, that is $(x, y) : (0, 1), (2, 1), (3, 4), (4, 5)$;
   a) fit the data with a polynomial of degree *one* using the least square method, (4p)
   b) find an approximate value of y for $x = 1$. (1p)

a)

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 1 \\ 4 \\ 5 \end{bmatrix}$$

Normal equations

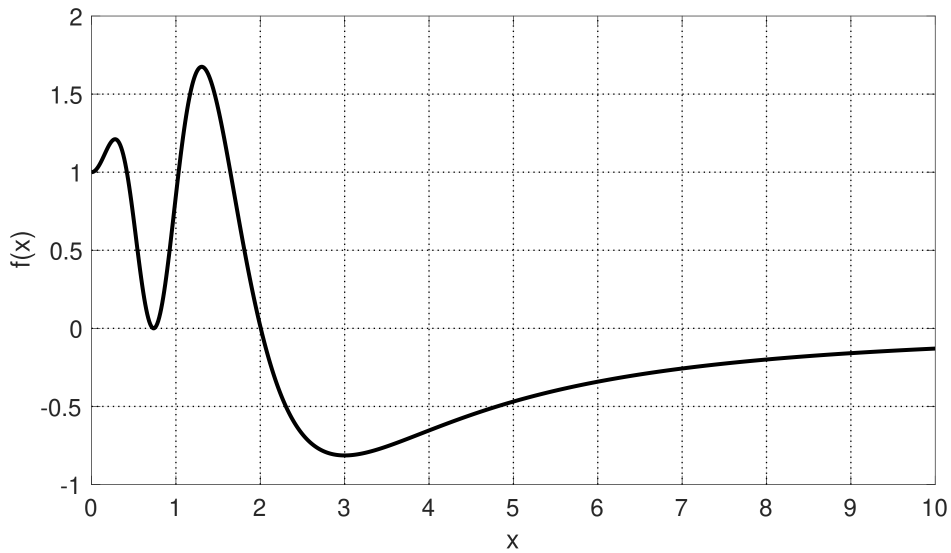
$$A^T A = \begin{bmatrix} 4 & 9 \\ 9 & 29 \end{bmatrix}, \qquad A^T b = \begin{bmatrix} 11 \\ 34 \end{bmatrix}$$

$A^T Ac = A^T b$, $c = [13, 37]^T/35$

Line: $y \approx c_1 + c_2 x = (13 + 37x)/35$

b) $y(1) = (13 + 37)/35 = 50/35 \approx 1.4286$

3. Assume that you seek the roots to a non-linear equation $f(x) = 0$. To get an understanding for the problem, you first plot the function $f(x)$. The result is depicted in the figure below (the roots visible in the figure are the only ones that exist for $x \geq 0$):



a) For each root that you can see, how would you (with some suitable words) describe or characterize it? (make sure that it in each case is clear which root you mean, by stating its approximate value or by marking it in a figure) (1p)

b) With what rate does the Newton-Raphson method converge to the roots? (1p)

c) Without doing any actual calculations, describe what will happen if you use (i) $x_0 = 1$, (ii) $x_0 = 2$, (iii) $x_0 = 3$ or (iv) $x_0 = 4$ as initial guess. (2p)

d) Could you find the roots using the Bisection method? Why/why not? (1p)

a) Root $\alpha_1 \approx 1$: Double root
Root $\alpha_2 \approx 2$: Single root

b) Root $\alpha_1 \approx 1$: Linearly
Root $\alpha_2 \approx 2$: Quadratically

c) (i) The iterations will converge to α_1
(ii) The iterations will converge to α_2
(iii) Division by zero pga $f'(x_k) = 0$ eller nära noll
(iv) The iterations will diverge/converge to $+\infty$

d) Root $\alpha_1 \approx 1$: No, bracketing not possible
Root $\alpha_2 \approx 2$: Yes, bracketing possible

4. The function values y are given for a few points according to

x	0	0.2	0.4	0.6	0.8
$y(x)$	0.10000	0.66800	0.82200	0.45900	−0.13000

a) Compute an approximation to the integral $\int_0^{0.8} y(x)dx$ using Simpson's method. All available function values must be used. (2p)

Answer:

$S(h = 0.2) = \frac{0.2}{3}(0.10000 + 4 * 0.66800 + 2 * 0.82200 + 4 * 0.45900 - 0.13000) \approx 0.4081$

b) Assume that the truncation error in the above computation can be estimated to 0.002. How large would the error be (approximately) if the number of function values would be increased such that the step length is halved. Motivate your answer. (1p)

Answer: The error will be approximately $0.002/16 = 1.25 \cdot 10^{-4}$. Simpsons method has order of accuracy 4, that is the errors are proportional to h^4. Making the step size half as big makes the error proportional to $(h/2)^4 = h^4/2^4 = h^4/16$.

c) Compute an approximation of $y'(x)$ in $x = 0.4$, using the second order accurate central finite difference formula $D_0(y, h)$. First, use $h = 0.4$, then use $h = 0.2$. (2p)

$$\begin{aligned}
\frac{f(x+h) - f(x-h)}{2h} &= f'(x) + f^{(3)}(\xi)\frac{h^2}{6} \\
h = 0.4 : \frac{f(0.8) - f(0)}{2 * 0.4} &= \frac{-0.13000 - 0.10000}{0.8} = -0.2875 \\
h = 0.2 : \frac{f(0.6) - f(0.2)}{2 * 0.2} &= \frac{0.45900 - 0.66800}{0.4} = -0.5225
\end{aligned}$$

d) If you compute $D_0(y, h)$ using a computer, do you expect the error $|D_0(y, h) - y'(x)| \to 0$ when $h \to 0$? Why/why not? (1p)

Answer: No, the error will decrease initially, but then the rounding errors will dominate and the error will increase due to *cancellation/loss of significance*

5. Given an electrical circuit with some voltages V and resistances R, we want to determine the resulting currents i. Applying Ohm's law and Kirchhoff's law to each current-loop in the circuit, we obtain a system of linear equations. Here we consider an example (consisting of three loops, 5 resistances and 2 voltages) described by the system

$$\begin{bmatrix} R_1 + R_2 + R_3 & -R_2 & -R_3 \\ -R_2 & R_2 + R_5 & -R_5 \\ -R_3 & -R_5 & R_3 + R_4 + R_5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -V_1 \\ V_2 \\ 0 \end{bmatrix}.$$

a) Let $R_1 = R_2 = R_3 = 1$, $R_4 = R_5 = 3$, $V_1 = 10$ and $V_2 = 15$. Apply two iterations using the Jacobi method ($\mathbf{x}^{(k)} = -D^{-1}(L+U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b}$ where $A = L+U+D$). Use $i_1 = i_2 = i_3 = 2$ as initial guess. (3p)

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 4 & -3 \\ -1 & -3 & 7 \end{bmatrix}, \quad b = \begin{bmatrix} -10 \\ 15 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix},$$

$$L + U = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

x=-inv(D)*(L+U)*x+inv(D)*b

x0=[2;2;2]

x1 =

[-2.0000, 23/4, 8/7]

x2 =

[-29/28, 115/28, 61/28]

Alternative solution (more effective when programming):

Solve for the individual currents:

$$I_1 = \frac{-10 + I_2 + I_3}{3} \quad I_2 = \frac{15 + I_1 + 3I_3}{4} \quad I_1 = \frac{I_1 + 3I_2}{7}$$

and put $x^{(k)} = [I_1^{(k)}, I_2^{(k)}, I_3^{(k)}]^T$ and solve thereafter (gives same result).

b) Without doing any more iterations, can you tell if the method will converge? (Why/why not?) (1p)

The method converges if A is strictly diagonal dominant, i.e. if $|a_{ii}| > \sum_{i \neq j} |a_{ij}|$

In this case A is only diagonal dominant, since $4 = |-1| + |-3|$. So using this test we can not really tell. (However, since it is *almost* strictly diagonal dominant it is at least not unlikely that it converges. To know for sure we have to investigate the eigenvalues of $-D^{-1}(L+U)$ but that is not required in this question.)

c) Assume that the voltage V_1 and V_2 can be measured using a voltmeter with 0.5% error tolerance, and assume that you have obtained a solution of the currents $i_{1,2,3}$ (using some numerical method). Approximately how accurate can you expect the computed currents to be, given that the condition number of A is 11? (1p)

$$\frac{\|\delta \mathbf{x}\|}{\|\mathbf{x}\|} \leq \kappa(A) \frac{\|\delta \mathbf{b}\|}{\|\mathbf{b}\|} = 11 * 0.5\% = 5.5\%$$

6. You have just started a new job and take over an old computational code from a colleague. The program computes the numerical solution to a certain ordinary differential equation (ODE) at time $t = T$. The only input parameter the program needs is the time step h, so it is very easy to use.

The only problem is that your colleague forgot to tell you the convergence order of the method before leaving for a long vacation. Unfortunately the ODE has no known analytical solution so you can not compare with the exact answer.

a) How can you find out what convergence order the method in the code has?

Hint: Let y_* be the (unknown) exact solution at time $t = T$, and let $y(h) = y_* + ch^p + \mathcal{O}(h^r)$, where $r > p$, be the output from the computer code. Derive a formula to estimate the convergence order approximately, using a few solutions $y(h)$ with different inputs h. (4p)

Answer:

$$
\begin{aligned}
y(h) &= y_* + ch^p + \mathcal{O}(h^r) \\
y(2h) &= y_* + c(2h)^p + \mathcal{O}(h^r) \\
y(2h) - y(h) &= ch^p(2^p - 1) + \mathcal{O}(h^r) \\
y(4h) - y(2h) &= c(2h)^p(2^p - 1) + \mathcal{O}(h^r) \\
\frac{y(4h) - y(2h)}{y(2h) - y(h)} &= \frac{c(2h)^p(2^p - 1) + \mathcal{O}(h^r)}{ch^p(2^p - 1) + \mathcal{O}(h^r)} = 2^p + \mathcal{O}(h^{r-p}) \approx 2^p \\
p &\approx \log_2\left(\frac{y(4h) - y(2h)}{y(2h) - y(h)}\right)
\end{aligned}
$$

b) When the convergence order is known, describe how you can combine already computed solutions $y(h)$ (with different inputs h) to get an improved answer. (1p)

Answer: Using Richardson extrapolation with $n = p$

$$
\begin{aligned}
2^p y(h) - y(2h) &= 2^p y_* + c2^p h^p + \mathcal{O}(h^r) - (y_* + c(2h)^p + \mathcal{O}(h^r)) = (2^p - 1)y_* + \mathcal{O}(h^r) \\
y_* &= \frac{2^p y(h) - y(2h)}{2^p - 1} + \mathcal{O}(h^r)
\end{aligned}
$$