# Collection of formulas, 1MA165

## Trigonometry:

$\tan x = \frac{\sin x}{\cos x}$

$\sin^2 x + \cos^2 x = 1 \quad \sin(-x) = -\sin x \quad \cos(-x) = \cos x$

$\sin(\pi - x) = \sin x \quad \cos(\pi - x) = -\cos x \quad \sin(\frac{\pi}{2} - x) = \cos x$

$\cos(\frac{\pi}{2} - x) = \sin x \quad \sin^2 x = \frac{1-\cos 2x}{2} \quad \cos^2 x = \frac{1+\cos 2x}{2}$

$\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \cos^2 x - \sin^2 x$

| $x$ | $0$ | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ |
|---|---|---|---|---|---|---|
| $\cos x$ | $1$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | $0$ | $-1$ |
| $\sin x$ | $0$ | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | $1$ | $0$ |

## Power and logarithm rules:

If $a, b > 0$ is :

$a^0 = 1 \quad a^{x+y} = a^x a^y \quad a^{x-y} = \frac{a^x}{a^y}$

$(a^x)^y = a^{xy} \quad (ab)^x = a^x b^x \quad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

If $a > 0$, $a \neq 1$, $b > 0$ och $b \neq 1$ is :

$\log_a 1 = 0 \quad \log_a(xy) = \log_a x + \log_a y$

$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y \quad \log_a(x^y) = y \log_a x$

$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$

## Useful limits:

$\lim_{x\to\infty} x^\alpha = \infty \quad \text{if } \alpha > 0 \qquad \lim_{x\to\infty} x^\alpha = 0 \quad \text{if } \alpha < 0$

$\lim_{x\to\infty} a^x = \infty \quad \text{if } a > 1 \qquad \lim_{x\to\infty} a^x = 0 \quad \text{if } 0 < a < 1$

$\lim_{x\to\infty} \log_a x = \infty \quad \text{if } a > 1 \qquad \lim_{x\to\infty} \log_a x = -\infty \quad \text{if } 0 < a < 1$

$\lim_{x\to\infty} \frac{a^x}{x^\alpha} = \infty \quad \text{if } a > 1 \qquad \lim_{x\to\infty} \frac{x^\alpha}{\log_a x} = \infty \quad \text{if } \alpha > 0, a > 1$

$\lim_{n\to\infty} \frac{a^n}{n!} = 0, \quad a \in \mathbf{R} \qquad \lim_{x\to 0^+} x^\alpha \log_a x = 0 \quad \text{if } \alpha > 0, a > 1$

$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1 \qquad \lim_{x\to 0} \frac{e^x - 1}{x} = 1$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$

## Rules of differentiation:

$\frac{d}{dx}\big(f(x) + g(x)\big) = f'(x) + g'(x) \qquad \frac{d}{dx}\big(cf(x)\big) = cf'(x)$

$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x) \qquad \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}$

$\frac{d}{dx} f\big(g(x)\big) = f'\big(g(x)\big)g'(x)$

## Rules of integration:

$\int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \qquad \int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

$\int_a^b f(x)g(x)\,dx = \big[F(x)g(x)\big]_a^b - \int_a^b F(x)g'(x)\,dx \quad \text{if } F'(x) = f(x)$

## Derivatives and antiderivatives:

| $g(x)$ | $g'(x)$ | $g(x)$ | $g'(x)$ |
|---|---|---|---|
| $x^a$ | $ax^{a-1}$ | $\ln x$ | $\frac{1}{x}$ |
| $e^x$ | $e^x$ | $a^x$ | $a^x \ln a, \quad a > 0$ |
| $\sin x$ | $\cos x$ | $\cos x$ | $-\sin x$ |
| $\tan x$ | $1 + \tan^2 x = \frac{1}{\cos^2 x}$ | $\cot x$ | $-1 - \cot^2 x = -\frac{1}{\sin^2 x}$ |
| $\arcsin x$ | $\frac{1}{\sqrt{1-x^2}}$ | $\arccos x$ | $-\frac{1}{\sqrt{1-x^2}}$ |
| $\arctan x$ | $\frac{1}{1+x^2}$ | $\operatorname{arccot} x$ | $-\frac{1}{1+x^2}$ |

**Taylor expansion:**

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + O(x^{n+1})$

$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + O(x^{n+1})$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + O(x^{n+1})$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + O(x^{n+1})$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + O(x^{2n+2})$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + O(x^{2n+3})$

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n\frac{x^{2n+1}}{2n+1} + O(x^{2n+3})$

**Summation formulas:**

$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}, \quad r \neq 1$

**Tangent and tangent plane equationer:**

The tangent to the curve $y = f(x)$ at the point $\big(a, f(a)\big)$:

$$y = f'(a)(x-a) + f(a).$$

The tangent plane at the surface $z = f(x,y)$ at the point $\big(a, b, f(a,b)\big)$:

$$z = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + f(a,b).$$

**Second order differential equations with constant coefficients :**

Consider the differential equation

$$ay''(x) + by'(x) + cy(x) = 0, \qquad (1)$$

where $a, b$ and $c$ are real constants. Let $r_1, r_2$ be the roots to the characteristic equation

$$ar^2 + br + c = 0.$$

(i) If $r_1$ and $r_2$ are real and $r_1 \neq r_2$, then

$$y(x) = Ae^{r_1 x} + Be^{r_2 x},$$

is the general solution to (1).

(ii) If $r_1$ and $r_2$ are real and $r_1 = r_2$, then

$$y(x) = (Ax + B)e^{r_1 x},$$

is the general solution to (1).

(iii) If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, $\beta \neq 0$, then

$$y(x) = e^{\alpha x}\big(A\cos(\beta x) + B\sin(\beta x)\big),$$

is the general solution to (1).