Thm: $N(A) = R(A^T)^\perp$

and

$N(A^T) = R(A)^\perp$

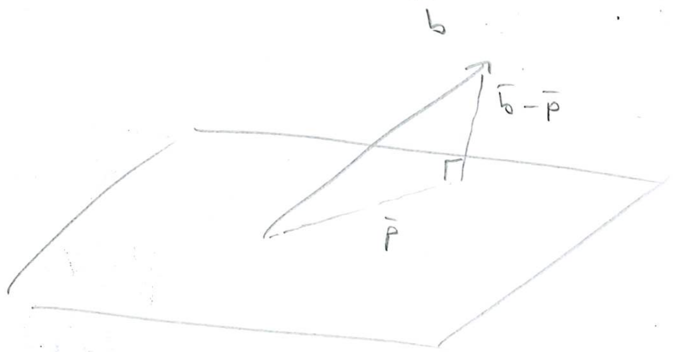
Note: Any vector in $N(A^T)$ is orthogonal to any vector in the column space.

Thm: $S \subseteq \mathbb{R}^n$, for each $\bar{b} \in \mathbb{R}^n$ there is a unique $\bar{p} \in S$ such that

$$\|\bar{b} - \bar{y}\| > \|\bar{b} - \bar{p}\|$$

for any $\bar{y} \neq \bar{p}$ in S.

Note: We say that the vector $\bar{p} \in S$ is the projection of $\bar{b}$ onto S.



$(\bar{b} - \bar{p}) \in S^\perp$.

Suppose that $S = R(A)$, $\bar{b} \in \mathbb{R}^m$.

Consider $A\bar{x} = b$, where $\underline{A}$ is $m \times n$

$m > n$, with residual $r(\bar{x}) = \bar{b} - A\bar{x}$.

$\|r(\bar{x})\|$ is minimized if $r(\bar{x}) \in S^{\perp}$

Recall: $S^{\perp} = R(A)^{\perp} = N(A^T)$

The least squares solution $\hat{x}$ must satisfy

$$0 = A^T r(\hat{x}) = A^T(\bar{b} - A\hat{x}) = A^T\bar{b} - A^T A\hat{x}.$$

Thm: A $m \times n$ matrix, a least squares solution $\hat{x}$ of the system $A\bar{x} = \bar{b}$ is given by any solution $A^T A\bar{x} = A^T b$.

Example: Find a least squares solution to

$$\begin{cases} x_1 + x_2 = 1 \\ x_1 + 2x_2 = 2 \\ x_1 + 3x_2 = 4 . \end{cases}$$

Solution: Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$, $\bar{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$.

- We compute

- $A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$

$A^T \bar{b} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 17 \end{pmatrix}$.

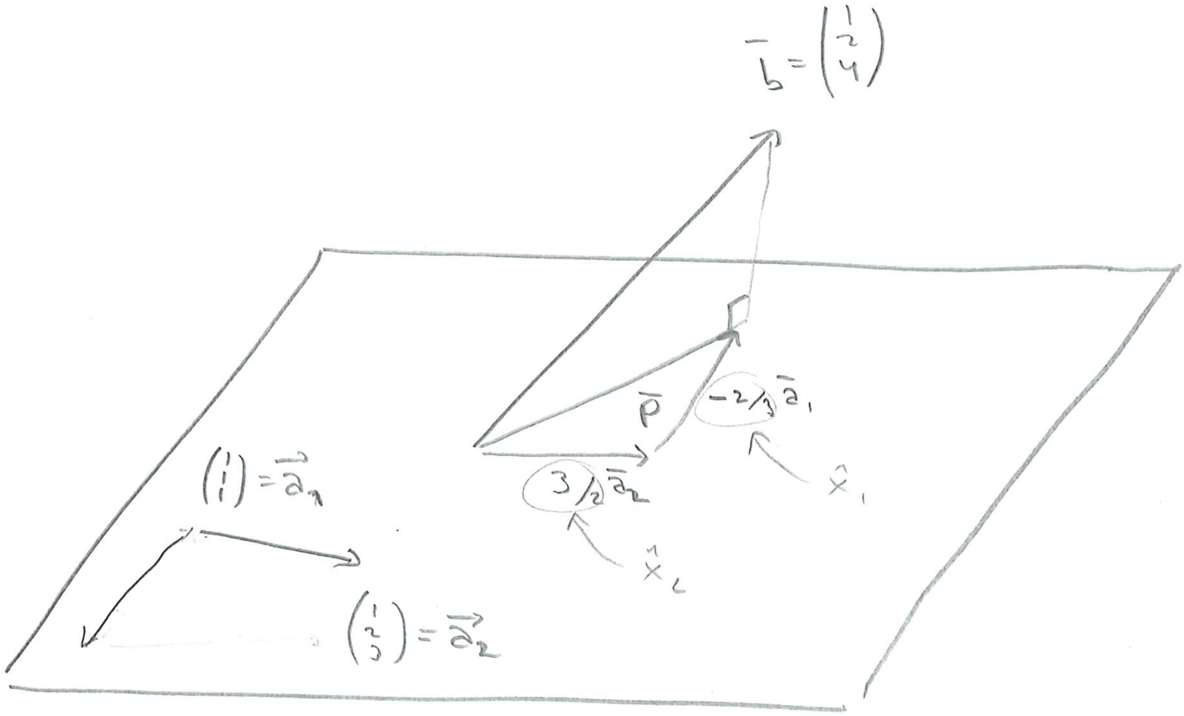
$\hat{x}$ is any solution to $A^T A \bar{x} = A^T b$.

We use Gauss-Jordan

$$\left(\begin{array}{cc|c} 3 & 6 & 7 \\ 6 & 14 & 17 \end{array}\right) (-2) \sim \left(\begin{array}{cc|c} 3 & 6 & 7 \\ 0 & 2 & 3 \end{array}\right) (-3)$$

$$\sim \left(\begin{array}{cc|c} 3 & 0 & -2 \\ 0 & 2 & 3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -2/3 \\ 0 & 1 & 3/2 \end{array}\right).$$

$\hat{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 3/2 \end{pmatrix}$ is the least squares solution.



$$\bar{p} = \hat{x}_1 \bar{a}_1 + \hat{x}_2 \bar{a}_2 = -\frac{2}{3}\bar{a}_1 + \frac{3}{2}\bar{a}_2$$

vector closest to $\bar{b}$
(in (s))

A least squares solution or <u>the</u> least squares solution?

In previous example it was unique.

Example:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 14 & 28 \\ 28 & 56 \end{pmatrix}, \quad A^T \bar{b} = \begin{pmatrix} 9 \\ 18 \end{pmatrix}$$

Solve the system: $A^T A \bar{x} = A^T \bar{b}$.

$$\left(\begin{array}{cc|c} 14 & 28 & 9 \\ 28 & 56 & 18 \end{array}\right) \overset{-2}{\sim} \left(\begin{array}{cc|c} 14 & 28 & 9 \\ 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow$ There are infinitely many least squares solutions.

Thm: If A is an $m \times n$ matrix of rank n, then the normal equation

$$A^T A \bar{x} = A^T \bar{b}$$

have a unique solution, and

$$\hat{x} = (A^T A)^{-1} A^T \bar{b}$$ is this unique solution.

Proof follows by the following theorem.

Thm:

If A is $m \times n$ matrix of rank n, then $A^T A$ is non-singular.

Proof: Let $\bar{z}$ be a solution to $A^T A \bar{x} = \bar{0}$ then $A\bar{z} \in N(A^T)$. Also, $A\bar{z} \in R(A) = N(A^T)^{\perp}$.

Since $N(A^T) \cap N(A^T)^{\perp} = \{\vec{0}\}$, it follows that $A\bar{z} = \vec{0}$.

Further the rank is n so all n columns are linearly independent, and $A\bar{x} = \bar{0}$ only have the trivial solution $\Rightarrow \bar{z} = \bar{0}$

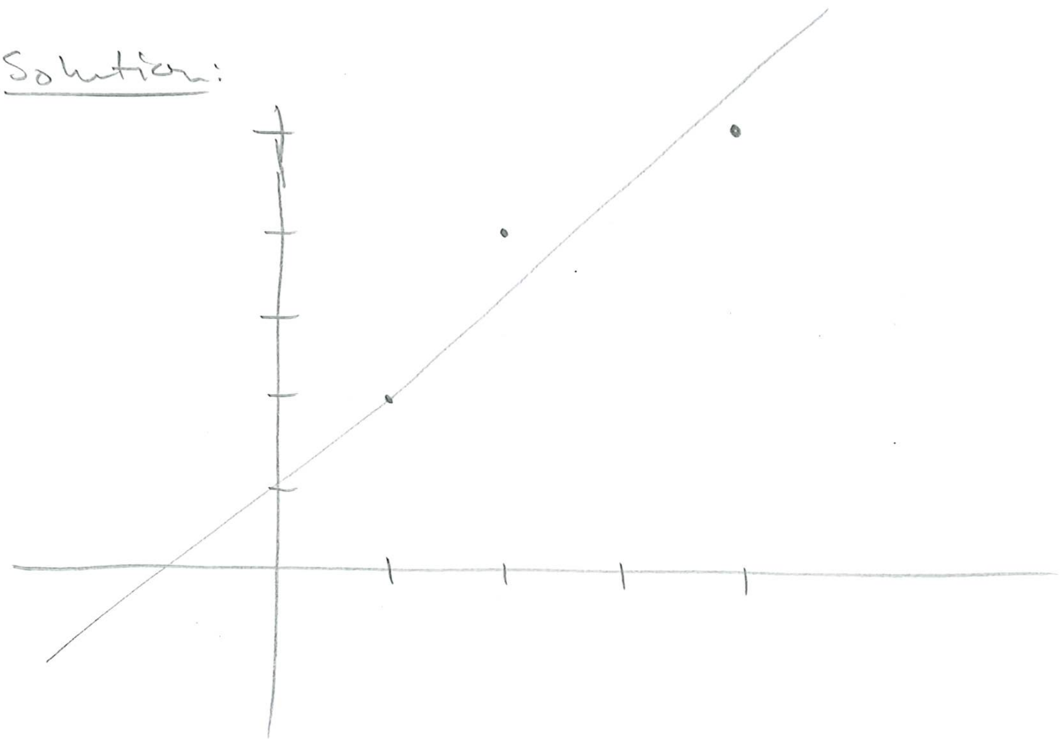
$\Rightarrow (A^T A)\bar{x} = \bar{0}$ only has the trival solution

$\Rightarrow A^T A$ is nonsingular (by Thm 1.5.2) ∎

Example: Find the straight line closests to the following data in a least squares sense.

x	1	2	4
y	2	4	5

Solution:



Any straight line (not vertical) is of the form

$y = kx + m$, we want to find k, m.

Note

$$\begin{cases} 2 = k \cdot 1 + m \\ 4 = k \cdot 2 + m \\ 5 = k \cdot 4 + m \end{cases}$$

overdetermined system

find (k, m) which minimize the error of the system in a least squares sense.

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 21 & 7 \\ 7 & 3 \end{pmatrix}, \quad A^T \bar{b} = \begin{pmatrix} 30 \\ 11 \end{pmatrix}$$

Solve the system $A^T A \bar{x} = A^T \bar{b}$

$$\left(\begin{array}{cc|c} 21 & 7 & 30 \\ 7 & 3 & 11 \end{array}\right) \sim \left(\begin{array}{cc|c} 7 & 3 & 11 \\ 21 & 7 & 30 \end{array}\right) -3$$

$$\sim \left(\begin{array}{cc|c} 7 & 3 & 11 \\ 0 & -2 & -3 \end{array}\right) \tfrac{-1}{2} \sim \left(\begin{array}{cc|c} 7 & 3 & 11 \\ 0 & 1 & 3/2 \end{array}\right) -3$$

$$\sim \left(\begin{array}{cc|c} 7 & 0 & 13/2 \\ 0 & 1 & 3/2 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 13/14 \\ 0 & 1 & 3/2 \end{array}\right).$$

The straight line closest to the data is

$$y = \frac{13}{14} x + \frac{3}{2}$$

~~Quadratic ? ?~~