Linnæus University
Linear algebra,1ma901/1ma406
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Geometry and Point sets

This is a document consisting of the material from the lectures concerning geometry in $\mathbb{R}^3$ to be seen as a complement to the book.

Equation of a line

The equation of a line is given by a direction vector $\mathbf{u} = (u_1, u_2, u_3)^T$ and a point $P = (x_0, y_0, z_0)$. We write

$$\begin{cases} x = t \cdot u_1 + x_0 \\ y = t \cdot u_2 + y_0 \\ z = t \cdot u_3 + z_0. \end{cases}$$

This is called the parametric equation of a line. We may also solve each of the previous equations for t yielding

$$t = \frac{x - x_0}{u_1}, \quad t = \frac{y - y_0}{u_2}, \quad t = \frac{z - z_0}{u_3}.$$

Equating these gives a parameter free form of the line

$$\frac{x - x_0}{u_1} = \frac{y - y_0}{u_2} = \frac{z - z_0}{u_3}.$$

See Figure 1 for a schematic picture of the line.

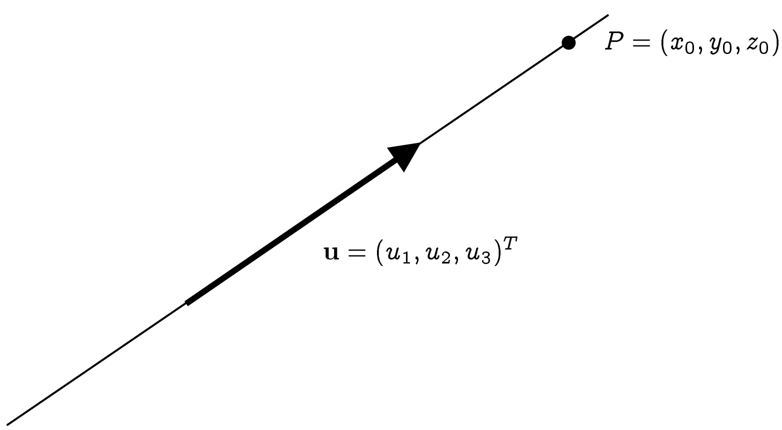

Figure 1: A line in $\mathbb{R}^3$

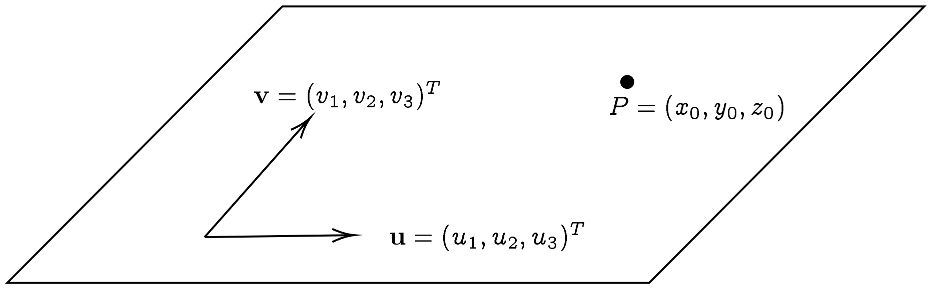

Figure 2: A plane spanned by two vectors $\mathbf{u}$ and $\mathbf{v}$, through the point P

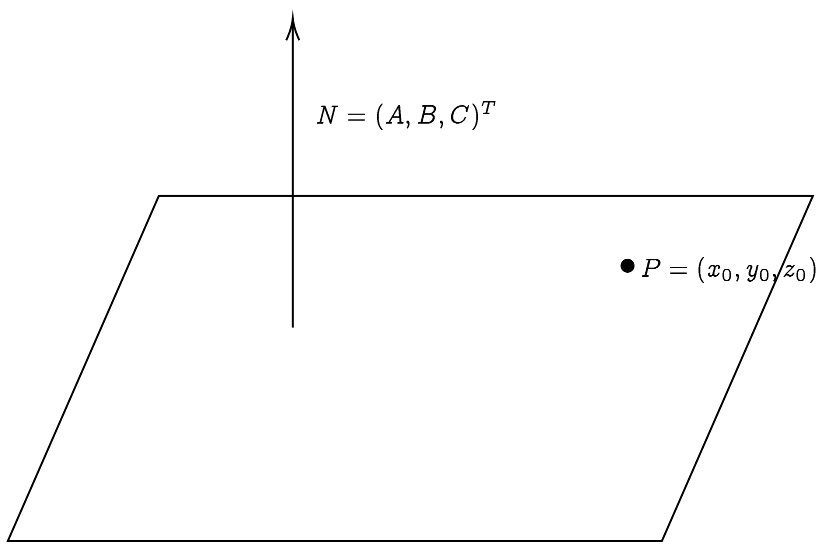

Figure 3: A plane Π and a point $P = (x_0, y_0, z_0)$. The plane has the normal $N = (A, B, C)^T$.

Equation of a plane

A plane is given by two direction vectors $\mathbf{u} = (u_1, u_2, u_3)^T$ and $\mathbf{v} = (v_1, v_2, v_3)^T$ (linearly independent) and a point $P = (x_0, y_0, z_0)$ (see Figure 2). The paramteric equation can be written as

$$\begin{cases} x = t \cdot u_1 + s \cdot v_1 + x_0 \\ y = t \cdot u_2 + s \cdot v_2 + y_0 \\ z = t \cdot u_3 + s \cdot v_3 + z_0. \end{cases}$$

For planes in $\mathbb{R}^3$ we also have the so-called normal equation $Ax + By + Cz + D = 0$. This is covered in the book in chapter 5.1.

Distance between point and a plane

Given a plane $\Pi : Ax + By + Cz + D = 0$, and a point $P = (x_0, y_0, z_0)$. What is the shortest distance between P and Π?

We find the shortest distance by constructing a line ℓ with direction vector $N = (A, B, C)^T$, through the point P. This line intersects Π in exactly one point we call Q (see Figure 4). The shortest distance between P and Π is then given by the length of the vector $\overline{PQ}$. However, since Q is a point on the line it must satisfy its equation for some value of t, the vector $\overline{PQ}$ can thus be written as

$$\overline{PQ} = (A, B, C)t + (x_0, y_0, z_0) - (x_0, y_0, z_0) = (A, B, C)t. \tag{1}$$

However, Q is also a point in Π and must therefore satisy its equation and we get by insertion of Q into the normal equation of Π

$$A(At + x_0) + B(Bt + y_0) + C(Ct + z_0) + D = 0$$

Solving for t yields

$$t = \frac{-Ax_0 - By_0 - Cz_0 - D}{A^2 + B^2 + C^2}.$$

First we note that $A^2 + B^2 + C^2 = ||N||^2$. Now, plugging this value for t back into $\overline{PQ}$ and computing its length we get

$$||\overline{PQ}|| = ||(A, B, C)t|| = |t| \cdot ||N|| = \frac{|Ax_0 + By_0 + Cz_0|}{||N||^2}||N|| = \frac{|Ax_0 + By_0 + Cz_0|}{||N||}.$$

So, the shortest distance d between P and the plane Π is given by

$$d = \frac{|Ax_0 + By_0 + Cz_0|}{||N||} = \frac{|Ax_0 + By_0 + Cz_0|}{\sqrt{A^2 + B^2 + C^2}}.$$

Distance between two lines in $\mathbb{R}^3$

Given two lines ℓ_1 and ℓ_2 in $\mathbb{R}^3$. In the case that the two lines intersect or are overlapping the shortest distance between them is clearly 0. In other cases there is always a shortest distance. In case the lines have parallel direction vectors the problem reduces to finding the distance between a point (any point will do) on ℓ_1 and find the shortest distance from that point to ℓ_2.

The final case is when we have non-parallel lines that do not intersect so-called skewed lines. The distance between these can be computed according to the following method.

1. Construct an arbitrary vector $\mathbf{w}$ which has initial point on one line and terminal point on the other. This vector will have two free parameters say s and t (one parameter from each line).
2. Condition this vector $\mathbf{w}$ on being orthogonal to the direction of ℓ_1 and the direction of ℓ_2. This will give conditions on s and t in terms of a system of equations.
3. Solve the system of equations for s and t and plug in the values for s and t into $\mathbf{w}$ and compute its length.

We will demonstrate this method by means of an example.

Example

Compute the shortest distance between ℓ_1 and ℓ_2 where these are given by

$$\ell_1 : (x, y, z) = (0, 1, 2)t + (1, 1, 1),$$

and

$$\ell_2 : (x, y, z) = (1, 0, -1)s + (8, 0, 2).$$

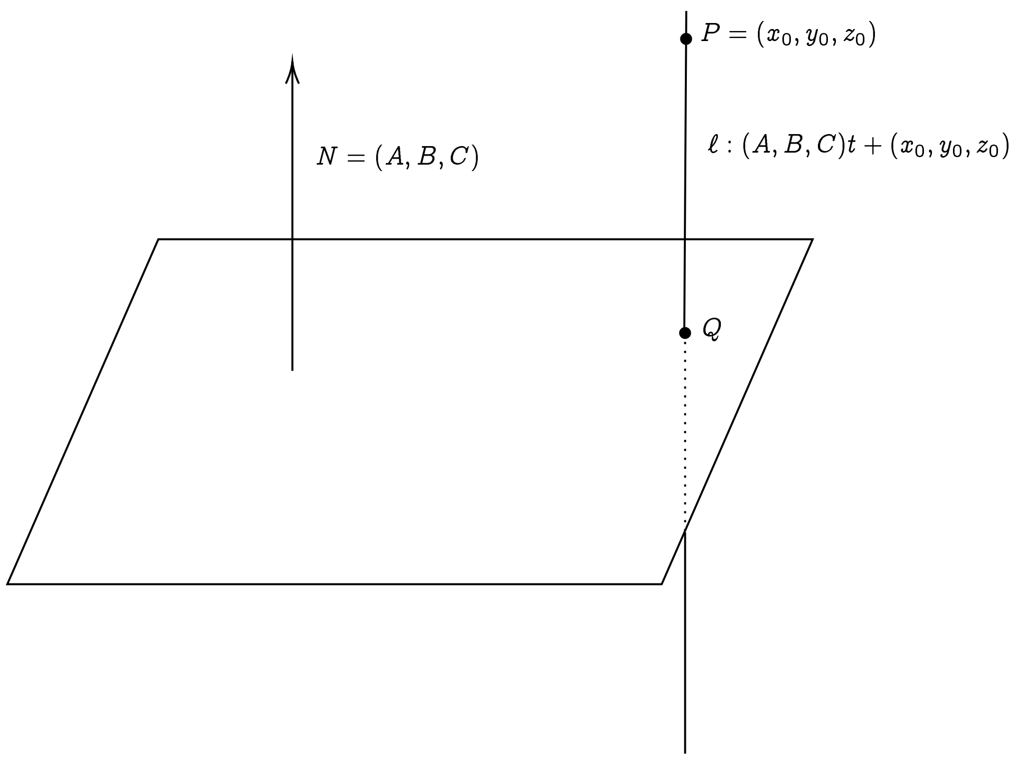

Figure 4: The plane Π, the point P together with the line ℓ that passes through P and is parallel to N. The point of intersection between ℓ and Π is Q.

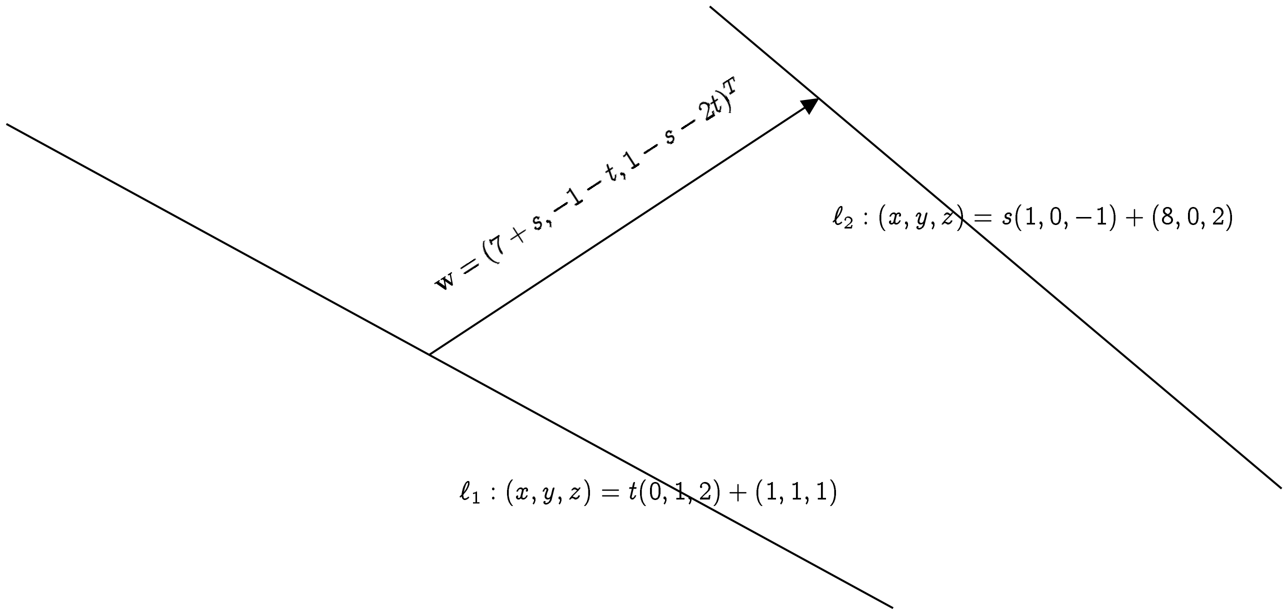

Figure 5: Two lines ℓ_1 and ℓ_2 and a vector $\mathbf{w}$ with terminal point on ℓ_2 and initial point on ℓ_1

Solution. First we denote by $\mathbf{u} = (0,1,2)^T$ and $\mathbf{v} = (1,0,-1)^T$ the direction vectors of ℓ_1 and ℓ_2 respectively. Further, let $\mathbf{w}$ be an vector having its initial point on ℓ_1 and terminal point on ℓ_2. Hence,

$$\mathbf{w} = (7+s, -1-t, 1-s-2t)^T.$$

In Figure 5 you'll find a schematic picture of the vector and the lines.

Next, we condition $\mathbf{w}$ on being orthogonal to $\mathbf{u}$ and $\mathbf{v}$. That is we say that

$$\mathbf{w}^T\mathbf{u} = 1 - 2s - 5t = 0,$$

and

$$\mathbf{w}^T\mathbf{v} = 6 + 2s + 2t = 0.$$

This gives a system of linear equations namely

$$\begin{cases} 2s + 5t = 1 \\ 2s + 2t = -6 \end{cases}.$$

Solving this system yields

$$s = -\frac{16}{3} \quad \text{and} \quad t = \frac{7}{3}.$$

Insertion of these values for s and t yields

$$\mathbf{w} = (5/3, -10/3, 5/3),$$

so the shortest distance between ℓ_1 and ℓ_2 is given by

$$||\mathbf{w}|| = \frac{1}{3}\sqrt{2 \cdot 5^2 + 10^2} = \frac{1}{3}\sqrt{150} = 5\sqrt{\frac{2}{3}}.$$

$\square$