L6 Elementary matrices and the matrix inverse

1ma901/1ma406 Linear algebra

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Engelsk-svensk ordlista

English	Swedish
Multiplicative inverse	Multiplikativ invers
Elementary matrix	Elementärmatris
Triangular matrix	Triangulär matris

Linear combination

Definition

Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be vectors in $\mathbb{R}^m$, and $c_1, \cdots c_n$ scalars, then a sum of the form

$$c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n$$

is called a *linear combination* of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

Example

$$5\begin{pmatrix}3\\2\\1\end{pmatrix} - 2\begin{pmatrix}1\\0\\-1\end{pmatrix} + 7\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}15-2+7\\10-0+7\\5-1+7\end{pmatrix} = \begin{pmatrix}20\\17\\12\end{pmatrix}$$

The vector $\begin{pmatrix}20\\17\\12\end{pmatrix}$ is a linear combination of the vectors

$$\begin{pmatrix}3\\2\\1\end{pmatrix}, \quad \begin{pmatrix}1\\0\\-1\end{pmatrix}, \quad \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

Identity matrix

Elements of the form a_{ii} of a matrix A are said to be *diagonal elements* and the set of all diagonal elements form the *main diagonal*.

An $n \times n$ matrix *i.e.* having the same number of rows and columns is said to be a *square matrix*, and if it is nonzero only in the main diagonal it is said to be a *diagonal matrix*.

The particular case of a diagonal matrix with every diagonal element is equal to 1, is called the *identity matrix*.

For instance we have for $n = 2$ and $n = 4$ we have

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If it is clear by context we omit the subscript.

The identity matrix serves as the neutral element for matrix multiplication. Let A be an $n \times n$ matrix. Then

$$AI = IA = A.$$

Note that besides being a diagonal matrix, the identity matrix is also what is known as a *symmetric matrix*. We say that a matrix A is symmetric if $A = A^T$.

The multiplicative inverse of a matrix

Let A be a square matrix is size $n \times n$. The multiplicative inverse of A, if it exists, is a matrix B such that

$$AB = BA = I.$$

If such a matrix B exists then A is said to be *nonsingular* or *invertible*. The matrix B is usually denoted by A^{-1}.

Lemma

The multiplicative inverse of a matrix is unique.

Proof.

Assume that A is nonsingular and we have two multiplicative inverses B and C. Then we have

$$B = BI = B(AC) = (BA)C = IC = C.$$

Hence, we conclude that $B = C$. □

In contrast to the nonsingular case, we say that a matrix A without a multiplicative inverse (or simply inverse for short) is said to be *singular* or *noninvertible*.

For non-square matrices we may not define multiplicative inverses in the same way. Although, we may construct either left- or right inverses. However, for this course we restrict to study inverses for square matrices.

Product of nonsingular matrices

Theorem

If A and B are $n \times n$ nonsingular matrices, then AB is also nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof.

We have

$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I,$$

and

$$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

This completes the proof of the theorem. $\square$

Note that this may be extended inductively the any number of nonsingular matrices as

$$(A_1A_2\cdots A_j)^{-1} = A_j^{-1}\cdots A_2^{-1}A_1^{-1}.$$

Equivalent systems

Let

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

be a linear system represented by

$$A\mathrm{x} = \mathrm{b}.$$

If we multiply both sides of the above equation by a nonsingular matrix M of size $m \times m$ we obtain an equivalent system.

$$A\mathrm{x} = \mathrm{b} \sim MA\mathrm{x} = M\mathrm{b}.$$

If x_0 is a solution to the latter system then

$$M^{-1}MA\mathrm{x}_0 = M^{-1}M\mathrm{b} \iff A\mathrm{x}_0 = \mathrm{b}.$$

Thus, it is also a solution to the former system, and hence they are equivalent.

Note that if A is nonsingular. Then

$$A\mathrm{x} = \mathrm{b} \iff A^{-1}A\mathrm{x} = A^{-1}\mathrm{b} \iff \mathrm{x} = A^{-1}\mathrm{b},$$

and the solution is given directly.

Elementary matrices

We recall the following definition of *elementary row operations* on matrices from Lecture 2. An elementary row operation on a matrix is one of the following

(I) Interchange two rows

(II) Multiply a row by a nonzero real number

(III) Replace a row by its sum with a multiple of another row.

Definition

A matrix which differs from the identity matrix by *one* row operation is said to be a elementary matrix.

Example

The matrices

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

are all examples of elementary matrices. We say that they are of type I, II and III respectively, corresponding to the different row operations.

Definition

A matrix B is *row equivalent* to A if there exists a finite sequence of elementary matrices $E_1, E_2, \ldots, E_j$ such that

$$B = E_j \cdots E_2 \cdot E_1 A.$$

Inverse of an elementary matrix

Theorem

If E is an elementary matrix, then E is nonsingular and E^{-1} is an elementary matrix of the same type.

Proof.

1. If E is of type I. Then E is its own inverse $EE = I$, because multiplication by E again will change back the original order of the rows.
2. If E is of type II. Then E is a diagonal matrix with diagonal $d = (1, \ldots, 1, \alpha, 1, \ldots, 1)$, where $d_i = \alpha$ and $d_j = 1$ for all j except for $j = i$. Let E' be the diagonal matrix with diagonal $d' = (1, \ldots, 1, 1/\alpha, 1, \ldots, 1)$, where $d_i = \alpha$ and $d_j = 1$ for all j except for $j = i$. Denote by C their product and then we have
$$(c_{ii}) = \vec{e}_i \mathrm{e}'_i = 1.$$
Hence,
$$E'E = EE' = I.$$
3. If E is of type III. Then we assume that E is obtained by adding m times the ith row to the jth row. Then its inverse is given by subtracting m times the ith row from the jth row.

This completes the proof of the theorem. $\square$

Conditions on nonsingularity

When is a matrix nonsingular? This is an important question, and by the theorem below we obtain some insight into this.

Theorem

Let A be a $n \times n$ matrix. The following are equivalent:

(a) *A is nonsingular.*

(b) *$A\mathrm{x} = 0$ has only the trivial solution.*

(c) *A is row equivalent to I.*

Proof.

We start to prove that $(a) \implies (b)$. If A is nonsingular and x_0 is a solution to $A\mathrm{x} = 0$, then

$$\mathrm{x}_0 = I\mathrm{x}_0 = (A^{-1}A)\mathrm{x}_0 = A^{-1}(A\mathrm{x}_0) = A^{-1}0 = 0.$$

Next we prove that $(b) \implies (c)$. We may transform the equation to $U\mathrm{x} = 0$, where U is in echelon form. Further, we may assume that U is strictly triangular otherwise we would have a nontrivial solution. Hence, I must be the reduced echelon form of A, and thus A is row equivalent to I. We conclude by proving that $(c) \implies (a)$. If A is row equivalent to I we have $A = E_j \cdots E_2 \cdot E_1 I = E_j \cdots E_2 \cdot E_1$. However, since every E_i is invertible we have that their product must be and in particular we have

$$A^{-1} = (E_j \cdots E_2 \cdot E_1)^{-1} = E_1^{-1} \cdot E_2^{-1} \cdots E_j^{-1},$$

and thus A is nonsingular. $\square$

Unique solution to a $n \times n$ linear system

Theorem

The system $A\mathrm{x} = \mathrm{b}$ *of* n *linear equations in* n *unknowns has a unique solution if and only if* A *is nonsingular.*

Proof.

($\Rightarrow$): Assuming that the system has the unique solution x_0. We make a proof by contradiction, by assuming that A is singular. Then by the previous theorem $A\mathrm{x} = 0$ has nontrivial solutions. Let z be any of these. However, then $\mathrm{y} = \mathrm{x}_0 + \mathrm{z}$ must be a solution to the linear system as

$$A\mathrm{y} = A(\mathrm{x}_0 + \mathrm{z}) = A\mathrm{x}_0 + A\mathrm{z} = \mathrm{b} + 0 = \mathrm{b}.$$

However, this is a contradiction since we assumed that x_0 was unique. Hence, A is nonsingular.

($\Leftarrow$): We assume that A is nonsingular, and let x_0 be any solution to the system. Then

$$A\mathrm{x}_0 = \mathrm{b} \iff A^{-1}A\mathrm{x}_0 = A^{-1}\mathrm{b} \iff \mathrm{x}_0 = A^{-1}\mathrm{b},$$

and the solution is unique. $\square$

Matrix inverse example

Let

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}.$$

The inverse of A is found by finding the matrix B such that $(A|I) \sim (I|B)$. We proceed by putting up $(A|I)$

$$(A|I) = \begin{pmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Only by use of row operations we want to get I on left part of the above matrix.

$$\begin{pmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 2 & 1 & 0 \\ 0 & 3 & 4 & 3 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 3 & 0 & -1 & 4 & -3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & 0 & 0 & 1/3 & -1/3 & 0 \\ 0 & 3 & 0 & -1 & 4 & -3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix} \begin{array}{l} |\cdot(-1) \\ |\cdot 1/3 \\ \end{array} \sim \begin{pmatrix} 1 & 0 & 0 & -1/3 & 1/3 & 0 \\ 0 & 1 & 0 & -1/3 & 4/3 & -1 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix}.$$

This means that we've found A^{-1} and it is given by

$$A^{-1} = \begin{pmatrix} -1/3 & 1/3 & 0 \\ -1/3 & 4/3 & -1 \\ 1 & -1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 & 1 & 0 \\ -1 & 4 & -3 \\ 3 & -3 & 3 \end{pmatrix}.$$

Triangular matrices

Definition
An $n \times n$ matrix is said to be *upper triangular* if $a_{ij} = 0$ for all $i > j$ and *lower triangular* if $a_{ij} = 0$ for all $i < j$.

Example
The matrices

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 4 & 3 & 0 \\ 5 & 6 & 1 \end{pmatrix},$$

are upper and lower triangular respectively.

Remark
An alternative definition to diagonal matrices are matrices that are both upper and lower triangular.

Every $n \times n$ matrix A may be written in the form (up to reordering of the rows) as

$$A = LU,$$

where L is a lower triangular matrix, and U is an upper triangular matrix.

Example of LU-factorization

Let

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 5 \\ 6 & 4 & 1 \end{pmatrix}.$$

Find L and U such that $A = LU$. We have

$$\begin{pmatrix} 2 & 2 & 3 \\ 1 & 2 & 5 \\ 6 & 4 & 1 \end{pmatrix} \begin{array}{l} \rceil^{-1/2} \quad \rceil^{-3} \\ \leftarrow + \\ \longleftarrow\!\!-\!\!-\!\!- + \end{array} \sim \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 7/2 \\ 0 & -2 & -8 \end{pmatrix} \begin{array}{l} \\ \rceil^{2} \\ \leftarrow + \end{array} \sim \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 7/2 \\ 0 & 0 & -1 \end{pmatrix}$$

The row operations above corresponds to multiplication by the elementary matrices

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix},$$

and we have

$$L := (E_3 E_2 E_1)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix}, \text{ and put } U := \begin{pmatrix} 2 & 2 & 3 \\ 0 & 1 & 7/2 \\ 0 & 0 & -1 \end{pmatrix}.$$

By construction we have

$$A = LU.$$

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Thank you for today!