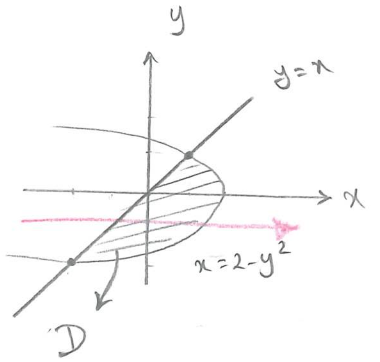
Example. Compute $\iint_D xy \, dx \, dy$, where D is the region limited by $x = 2-y^2$ and $x = y$.



$$\begin{cases} x = y \\ x = 2-y^2 \end{cases} \Rightarrow 2-y^2 = y \Rightarrow y^2+y-2=0$$

$$\Rightarrow (y+2)(y-1)=0$$

$$\Rightarrow y=-2 \text{ and } y=1.$$

$$\Rightarrow (-2,-2) \text{ and } (1,1).$$

$$\Rightarrow \iint_D xy \, dx \, dy = \int_{-2}^{1} \left(\int_{y}^{2-y^2} xy \, dx \right) dy = \int_{-2}^{1} \left(\left[\frac{x^2}{2}\right]_{y}^{2-y^2} \cdot y \right) dy$$

$$= \int_{-2}^{1} \left(\frac{(2-y^2)^2}{2} \cdot y - \frac{y^2}{2} \cdot y \right) dy = \frac{1}{2} \int_{-2}^{1} \left(y(4-4y^2+y^4) - y^3 \right) dy$$

$$= \frac{1}{2} \int_{-2}^{1} (4y - 5y^3 + y^5) \, dy = \frac{1}{2} \left[2y^2 - \frac{5}{4} y^4 + \frac{y^6}{6} \right]_{-2}^{1}$$

$$= \frac{1}{2} \left[2 - \frac{5}{4} + \frac{1}{6} - 8 + 20 - \frac{32}{3} \right] = \frac{9}{8}$$

Exam 29 January 2022

3. Compute the double integral $\iint_{\Omega} (4xy-7)\, dA$ where Ω is the portion of $x^2+y^2=2$ in the first quadrant.

We use the polar coordinates: $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$

Then the requested integral is:

$$\iint_{\Omega} (4xy-7)\, dA = \int_0^{\sqrt{2}} \int_0^{\pi/2} (4r^2 \sin\theta \cos\theta - 7)\, r \, d\theta \, dr$$

$$= \int_0^{\sqrt{2}} \left[2r^3 \sin^2\theta - 7r\theta\right]_0^{\pi/2} dr = \left[\frac{r^4}{2} - \frac{7\pi}{4} r^2\right]_0^{\sqrt{2}}$$

$$= 2 - \frac{7\pi}{2}.$$

$$\int \cos\theta \sin\theta \, d\theta = \int u \, du = \frac{u^2}{2} + C = \frac{\sin^2\theta}{2} + C$$

$u = \sin\theta$
$du = \cos\theta \, d\theta$

Exercise 3, Exam 17 April 2021. (IMA465)

Compute the double integral $\iint_\Omega (x^2+y^2)\,dx\,dy$ where:

$$\Omega = \{(x,y)\in\mathbb{R}^2 : x^2+y^2-4y \leq 0\}.$$

$$\Rightarrow \Omega = \{(x,y)\in\mathbb{R}^2 : x^2+(y-2)^2 \leq 4\}$$

We use polar coordinates with respect to $(0,2)$; it means:

$$\begin{cases} x = r\cos\theta \\ y = 2 + r\sin\theta \end{cases} \Rightarrow \left|\det \frac{\partial(x,y)}{\partial(r,\theta)}\right| = r \quad \text{and we get:}$$

$$\iint_\Omega (x^2+y^2)\,dx\,dy = \int_0^{2\pi}\int_0^2 (r^2\cos^2\theta + (2+r\sin\theta)^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 (r^2\cos^2\theta + r^2\sin^2\theta + 4 + 4r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\int_0^2 (r^3+4r+4r^2\sin\theta)\,dr\right)d\theta = \int_0^{2\pi}\left[\frac{r^4}{4} + 2r^2 + \frac{4}{3}r^3\sin\theta\right]_0^2 d\theta$$

$$= \int_0^{2\pi}\left(4+8+\frac{32}{3}\sin\theta\right)d\theta = \left[12\theta - \frac{32}{3}\cos\theta\right]_0^{2\pi} = \underline{24\pi}.$$

Exam 28 October 2019

3. Compute the volume of the solid that is bounded by the surfaces $z=\sqrt{x^2+y^2}$ and $z=\sqrt{9-2(x^2+y^2)}$.

The surfaces intersect when $x^2+y^2=9-2(x^2+y^2) \iff x^2+y^2=3.$

This gives the volume

$$V=\iint_{x^2+y^2\le 3}\left(\sqrt{9-2(x^2+y^2)}-\sqrt{x^2+y^2}\right)dx\,dy$$

$$=\int_0^{2\pi}\int_0^{\sqrt{3}}\left(\sqrt{9-2r^2}-r\right)r\,dr\,d\theta$$

$$=(2\pi)\left[\left(-\tfrac{1}{4}\right)\left(\tfrac{2}{3}\right)(9-2r^2)^{3/2}-\frac{r^3}{3}\right]_0^{\sqrt{3}}$$

$$=\frac{\pi}{3}\left[-(9-2r^2)^{3/2}-2r^3\right]_0^{\sqrt{3}}=\pi(9-3\sqrt{3})$$

Exam 14 August 2019

2. Compute the double integral $\iint_\Omega x^2y^2\,dx\,dy$ where $\Omega=\left\{(x,y)\in\mathbb{R}^2 : x,y\ge 0,\ 1\le xy\le 2,\ 1\le \frac{y}{x^2}\le 2\right\}$.

We introduce new variables: $\begin{cases} u=xy \\ v=\dfrac{y}{x^2}\end{cases}$

$$\Rightarrow \frac{\partial(u,v)}{\partial(x,y)}=\begin{pmatrix} y & x \\ \frac{-2y}{x^3} & \frac{1}{x^2}\end{pmatrix} \Rightarrow \det\frac{\partial(u,v)}{\partial(x,y)}=\frac{3y}{x^2}=3v.$$

$\Rightarrow$ The requested integral is:

$$I=\iint_\Omega x^2y^2\,dx\,dy=\int_1^2\int_1^2 u^2\left|\det\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

$$\Rightarrow I = \int_1^2 \int_1^2 \frac{u^2}{3v}\, du\, dv = \frac{1}{3}\left[\frac{u^3}{3}\right]_1^2 \left[\ln v\right]_1^2 = \frac{7\ln 2}{9}.$$

$\boxed{\text{Exam 28 October 2019}}$

2. Compute the double integral $\iint_\Omega \sqrt{xy}\, dx\, dy$ where

$$\Omega = \left\{ (x,y) \in \mathbb{R}^2 : x, y \geq 0;\ 1 \leq xy \leq 3,\ 1 \leq \frac{y}{x} \leq 2 \right\}.$$

We introduce new variables: $\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases}$

$$\Rightarrow \left| \frac{\partial(u,v)}{\partial(x,y)} \right| = \left| \begin{matrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{matrix} \right| = \frac{2y}{x} = 2v.$$

$\Rightarrow$ The requested integral is:

$$I = \iint_\Omega \sqrt{xy}\, dx\, dy = \int_1^2 \int_1^3 \sqrt{u}\ \left| \det \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

$$= \int_1^2 \int_1^3 \frac{\sqrt{u}}{2v}\, du\, dv = \frac{1}{2}\left[\frac{2}{3} u^{3/2}\right]_1^3 \left[\ln v\right]_1^2$$

$$= \left(\sqrt{3} - \frac{1}{3}\right) \ln 2.$$