Linnaeus University
Department of Mathematics
Sofia Eriksson

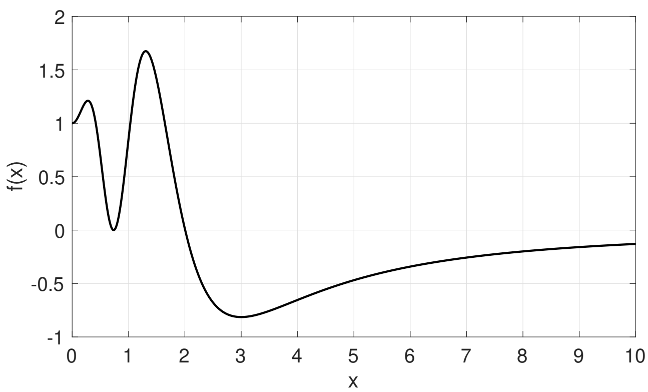
Written Exam on Basic Numerical Methods, 1DV519, 7,5 hp
Saturday 27th of October 2018, 12.00–17.00.

The solutions should be complete, correct, motivated, well structured and easy to follow.
Aids: Calculator (you may use a scientific calculator but *not* with internet connection)
Grades: 15p-17p⇒E; 18p-20p⇒D; 21p-23p⇒C; 24p-26p⇒B; 27p-30p⇒A.

1. Given the following set of points $(x, y) : (0, 1), (2, 1), (3, 4), (4, 5)$;
   a) determine the corresponding interpolating polynomial (of lowest possible degree), (3p)
   b) find an approximate value of y for $x = 1$. (1p)

2. Given the same set of points as in the problem above, that is $(x, y) : (0, 1), (2, 1), (3, 4), (4, 5)$;
   a) fit the data with a polynomial of degree *one* using the least square method, (4p)
   b) find an approximate value of y for $x = 1$. (1p)

3. Assume that you seek the roots to a non-linear equation $f(x) = 0$. To get an understanding for the problem, you first plot the function $f(x)$. The result is depicted in the figure below (the roots visible in the figure are the only ones that exist for $x \geq 0$):



   a) For each root that you can see, how would you (with some suitable words) describe or characterize it? (make sure that it in each case is clear which root you mean, by stating its approximate value or by marking it in a figure) (1p)
   b) With what rate does the Newton-Raphson method converge to the roots? (1p)
   c) Without doing any actual calculations, describe what will happen if you use (i) $x_0 = 1$, (ii) $x_0 = 2$, (iii) $x_0 = 3$ or (iv) $x_0 = 4$ as initial guess. (2p)
   d) Could you find the roots using the Bisection method? Why/why not? (1p)

4. The function values y are given for a few points according to

x	0	0.2	0.4	0.6	0.8
$y(x)$	0.10000	0.66800	0.82200	0.45900	−0.13000

a) Compute an approximation to the integral $\int_0^{0.8} y(x)dx$ using Simpson's method. All available function values must be used. (2p)

b) Assume that the truncation error in the above computation can be estimated to 0.002. How large would the error be (approximately) if the number of function values would be increased such that the step length is halved. Motivate your answer. (1p)

c) Compute an approximation of $y'(x)$ in $x = 0.4$, using the second order accurate central finite difference formula $D_0(y, h)$. First, use $h = 0.4$, then use $h = 0.2$. (2p)

d) If you compute $D_0(y, h)$ using a computer, do you expect the error $|D_0(y, h) - y'(x)| \to 0$ when $h \to 0$? Why/why not? (1p)

5. Given an electrical circuit with some voltages V and resistances R, we want to determine the resulting currents i. Applying Ohm's law and Kirchhoff's law to each current-loop in the circuit, we obtain a system of linear equations. Here we consider an example (consisting of three loops, 5 resistances and 2 voltages) described by the system

$$\begin{bmatrix} R_1 + R_2 + R_3 & -R_2 & -R_3 \\ -R_2 & R_2 + R_5 & -R_5 \\ -R_3 & -R_5 & R_3 + R_4 + R_5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -V_1 \\ V_2 \\ 0 \end{bmatrix}.$$

a) Let $R_1 = R_2 = R_3 = 1$, $R_4 = R_5 = 3$, $V_1 = 10$ and $V_2 = 15$. Apply two iterations using the Jacobi method ($\mathbf{x}^{(k)} = -D^{-1}(L+U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b}$ where $A = L+U+D$). Use $i_1 = i_2 = i_3 = 2$ as initial guess. (3p)

b) Without doing any more iterations, can you tell if the method will converge? (Why/why not?) (1p)

c) Assume that the voltage V_1 and V_2 can be measured using a voltmeter with 0.5% error tolerance, and assume that you have obtained a solution of the currents $i_{1,2,3}$ (using some numerical method). Approximately how accurate can you expect the computed currents to be, given that the condition number of A is 11? (1p)

6. You have just started a new job and take over an old computational code from a colleague. The program computes the numerical solution to a certain ordinary differential equation (ODE) at time $t = T$. The only input parameter the program needs is the time step h, so it is very easy to use.

The only problem is that your colleague forgot to tell you the convergence order of the method before leaving for a long vacation. Unfortunately the ODE has no known analytical solution so you can not compare with the exact answer.

a) How can you find out what convergence order the method in the code has?

Hint: Let y_* be the (unknown) exact solution at time $t = T$, and let $y(h) = y_* + ch^p + \mathcal{O}(h^r)$, where $r > p$, be the output from the computer code. Derive a formula to estimate the convergence order approximately, using a few solutions $y(h)$ with different inputs h. (4p)

b) When the convergence order is known, describe how you can combine already computed solutions $y(h)$ (with different inputs h) to get an improved answer. (1p)

List of math formulas for the exam in Basic Numerical Methods

These formulas will be attached to the exam. The list is not guaranteed to be complete, and the names, use, meaning, conditions and assumptions of the formulas are purposely left out.

- $$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

  (ξ between x and a)

- $$\Delta_x = \tilde{x} - x, \qquad \frac{\Delta_x}{x} = \frac{\tilde{x} - x}{x}, \qquad \Delta_{x+y} = \Delta_x + \Delta_y, \qquad \frac{\Delta_{xy}}{xy} \approx \frac{\Delta_x}{x} + \frac{\Delta_y}{y}$$

- $$\Delta f \approx f'(x)\Delta x, \qquad \left|\frac{\Delta f/f}{\Delta x/x}\right| \approx \left|\frac{xf'(x)}{f(x)}\right|$$

  $$\Delta f \approx f''(x)\frac{\Delta x^2}{2}$$

  $|\Delta x| \leq 0.5 \cdot 10^{-t}$

  $x = x_m B^m + x_{m-1}B^{m-1} + \ldots + x_0 B_0 + x_{-1}B^{-1} + \ldots = (x_m x_{m-1} \ldots x_0 . x_{-1} \ldots)_B$

- ```
  c=(a+b)/2;
  while (b-a)>2*tol
      if f(c)*f(a)>0
          a=c;
      else
          b=c;
      end
      c=(a+b)/2;
  end
  ```

- $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad J(\mathbf{x}_n)(\mathbf{x}_{n+1} - \mathbf{x}_n) + \mathbf{f}(\mathbf{x}_n) = 0$$

- $e_n = x_n - x^* = x_n - r, \qquad |x_{n+1} - x^*| < \bar{c}|x_n - x^*|^p, \qquad \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = c$

- $A\mathbf{x} = \mathbf{b}, \quad \mathbf{r} = \mathbf{b} - A\widetilde{\mathbf{x}}$

  $$PA = LU$$

  $$A = QR, \quad Q^T Q = I$$

  $$A = D + L + U: \qquad \begin{cases} \mathbf{x}^{(k)} = -D^{-1}(L+U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b} \\ \mathbf{x}^{(k)} = -(D+L)^{-1}U\mathbf{x}^{(k-1)} + (D+L)^{-1}\mathbf{b} \end{cases}$$

  Backward: $\|\mathbf{r}\|_\infty$, forward: $\|\mathbf{x} - \widetilde{\mathbf{x}}\|_\infty$
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- Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a vector:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad \|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |x_i|, \quad \|\boldsymbol{x}\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}, \quad \|\mathbf{x}\|_\infty = \max_i |x_i|.$$

Let A be a $n \times n$ matrix:

$$\|A\| = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}, \qquad \|A\mathbf{x}\| \leq \|A\| \cdot \|\mathbf{x}\|, \qquad \kappa(A) = \|A\| \cdot \|A^{-1}\|$$

$A(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}$

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \kappa(A)\frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}, \qquad econd(A) = \frac{\|\delta\mathbf{x}\|/\|\mathbf{x}\|}{\|\delta\mathbf{b}\|/\|\mathbf{b}\|} \leq \kappa(A),$$

- Let $(x_0, y_0), \ldots, (x_n, y_n)$ be $n+1$ points in the xy-plane.

$$P(x) = a_0 + a_1 x + \ldots + a_n x^n$$
$$P(x) = \sum_{j=0}^{n} y_j \ell_j(x), \qquad \ell_j(x) = \prod_{\substack{0\leq m\leq n \\ m\neq j}} \frac{x - x_m}{x_j - x_m}$$
$$P(x) = [y_0] + [y_0, y_1](x - x_0) + \cdots + [y_0, \ldots, y_k](x - x_0)(x - x_1)\cdots(x - x_{n-1}).$$
$$[y_0] = f(x_0), \quad [y_0, y_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \quad [y_0, y_1, y_2] = \frac{[y_1, y_2] - [y_0, y_1]}{x_2 - x_0}$$
$$R(x) = f(x) - P(x) = (x - x_0)(x - x_1)\cdots(x - x_n)\frac{f^{n+1}(\xi)}{(n+1)!}, \qquad x_0 < \xi < x_n,$$

- $A^T A x = A^T b$

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$$\frac{f(x+h) - f(x)}{h} = f'(x) + f''(\xi)\frac{h}{2} \qquad \xi \in [x, x+h]$$
$$\frac{f(x) - f(x-h)}{h} = f'(x) - f''(\xi)\frac{h}{2} \qquad \xi \in [x-h, x]$$
$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + f^{(3)}(\xi)\frac{h^2}{6} \qquad \xi \in [x-h, x+h]$$
$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x) + f^{(4)}(\xi)\frac{h^2}{12} \qquad \xi \in [x-h, x+h]$$

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$$\int_a^b f(x)dx = \tfrac{h}{2}\left(f(x_0) + 2\textstyle\sum_{k=1}^{n-1} f(x_k) + f(x_n)\right) - \tfrac{(b-a)h^2}{12} f''(\xi), \qquad h = \frac{b-a}{n}$$
$$\int_a^b f(x)dx = \tfrac{h}{3}\left(f(x_0) + 4\textstyle\sum_{k=1}^{n} f(x_{2k-1}) + 2\sum_{k=1}^{n-1} f(x_{2k}) + f(x_{2n})\right) - \tfrac{(b-a)h^4}{180} f^{(4)}(\xi), \quad h = \frac{b-a}{2n}$$

$a < \xi < b$

- $Q = F(h) + Kh^n + \mathcal{O}(h^{n+1}), \qquad Q = \frac{2^n F(h/2) - F(h)}{2^n - 1} + \mathcal{O}(h^{n+1})$

$R_{i,1} = T(h/2^{i-1})$, $R_{ij} = \frac{4^{j-1} R_{i,j-1} - R_{i-1,j-1}}{4^{j-1} - 1}$