L9 Basis and dimensions

1MA901/1MA406 Linear algebra

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Engelsk-svensk ordlista

English	**Swedish**
Vector space	Vektorrum
Subspace	Underrum
Nullspace (of a matrix)	Nollrum (till en matris)
Span	Spänner
Spanning set	Linjärt hölje/spannet
Linear independence	Linjärt oberoende
Basis	Bas
Change of basis	Basbyte
Row space (of a matrix)	Radrummet (till en matris)
Column space (of a matrix)	Kolonnrummet (till en matris)
Rank	Rang
Rand-nullity theorem	Dimensionssatsen

Span and independence $\iff$ basis

Definition

The vectors $v_1, v_2, \dots v_n$ form a *basis* for the vector space V if and only if

1. $v_1, v_2, \dots, v_n$ are linearly independent
2. $v_1, v_2, \dots, v_n$ span V

Example

The set

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for $\mathbb{R}^3$.

Solution.

The set is linearly independent, why?, and it spans $\mathbb{R}^3$, why?. $\square$

Theorem

If $v_1, v_2, \dots v_n$ is a spanning set for a vector space V,and U is any collection of m vectors m such that $m > n$, then the set of vectors U is linearly dependent.

Examples of basis vectors for different vector spaces

Example

The set $e_1, \dots, e_n$ is a basis for $\mathbb{R}^n$.

Example

The set $\{1, x, x^2, \dots, x^n\}$ is a basis for the vectors space of all polynomials of degree less than or equal to n.

Example

The set of of matrices

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and

$$E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

spans $\mathbb{R}^{2\times 2}$.

All these are examples of so-called *standard bases* for the respective vector spaces.

Some results

Theorem

If $\{v_1, v_2, \dots v_n\}$ *is a spanning set for* V*. Then any set of* $m > n$ *vectors in* V *is linearly dependent.*

Sketch of proof.

Assume that the m vectors are denoted by $u_1, u_2, \dots u_m$. Since the set $v_1, v_2, \dots v_n$ spans V we must have that every vector u_i can be written as a linear combination of vectors in $\{v_1, v_2, \dots v_n\}$, and since $m > n$ the vectors u_i must be linearly dependent. □

Theorem

If $\{v_1, v_2, \dots v_n\}$ *and* $\{u_1, u_2, \dots u_m\}$ *are bases for* V *then* $n = m$.

Dimension of a vector space

Definition

The *dimension* of a vector space is equal to the the number of basis vectors.

Example

The dimension of $\mathbb{R}^n$ is n.

Example

The space spanned by the vectors

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \right\}$$

is 2-dimensional since

$$\begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix},$$

and thus the vectors are not linearly independent.

Example

The vector space $C[a, b]$ is infinite dimensional.

Some results

Theorem

If V is a vector space of dimension $n \geq 1$, then

1. *any set of n linearly independent vectors spans V*
2. *any n vectors that span V are linearly independent.*

Theorem

If V is a vector space of dimension $n \geq 1$, then

1. *no set of fewer than n vectors can span V*
2. *any subset of fewer than n linearly independent vectors can be extended to form a basis for V*
3. *any spanning set of more than n vectors can be pared down to form a basis for V.*

Change of basis

In $\mathbb{R}^2$ every vector can be written as a linear combination of the standard bases $\{\mathrm{e}_1, \mathrm{e}_2\}$, *i.e.*

$$\mathrm{x} = x_1 \mathrm{e}_1 + x_2 \mathrm{e}_2.$$

It is thus natural to write x as $\mathrm{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where x_1 and x_2 are called the coordinates of x.

Assume further that $\{\mathrm{a}_1, \mathrm{a}_2\}$ is another basis for $\mathbb{R}^2$, then since this is a basis x can be represented as a linear combination by those vectors, *i.e.*

$$\mathrm{x} = x_1' \mathrm{a}_1 + x_2' \mathrm{a}_2.$$

Note that the coordinates are not necessarily the same as for the previous basis. Hence, we say that $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}_{\mathrm{a}}$ is the coordinates of x with respect to the new basis. Note further the subscript if we want to highlight the basis.

Example

The vector $\mathrm{x} = (1, 9)^T$ can be represented as a vector using the basis vectors $\mathrm{u}_1 = (5, 1)$ and $\mathrm{u}_2 = (1, -2)$ as

$$\mathrm{x} = \mathrm{u}_1 - 4\mathrm{u}_2.$$

Hence the coordinates in the new basis is $(1, -4)^T$.

Changing coordinates

Let $x = (x_1, x_2)^T$ be a vector with its coordinates given by the standard basis. How can we easily find its coordinates in the basis

$$v_1 = (1, 2)^T, \quad v_2 = (3, -2)^T.$$

Also, suppose we have a vector represented as $\alpha_1 v_1 + \alpha_2 v_2$. What is its representation in the standard basis?

To solve the latter problem first we note that

$$\begin{array}{ccccc} e_1 & + & 2e_2 & = & v_1 \\ 3e_1 & - & 2e_2 & = & v_2 \end{array}.$$

Hence,

$$\alpha_1 v_1 + \alpha_2 v_2 = \alpha_1(e_1 + 2e_2) + \alpha_2(3e_1 - 2e_2) = (\alpha_1 + 3\alpha_2)e_1 + (2\alpha_1 - 2\alpha_2)e_2.$$

Or equivalently, with respect to $\{e_1, e_2\}$ we have

$$\alpha_1 v_1 + \alpha_2 v_2 = \begin{pmatrix} \alpha_1 + 3\alpha_2 \\ 2\alpha_1 - 2\alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = (v_1, v_2) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

Changing coordinates

If y is a vector with coordinates $(y_1, y_2)^T$ with respect to a basis $\{\mathrm{a}_1, \mathrm{a}_2\}$, then its representation in the basis $\{\mathrm{e}_1, \mathrm{e}_2\}$ is given by

$$\mathrm{y}_\mathrm{e} = (\mathrm{a}_1, \mathrm{a}_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The matrix $A = (\mathrm{a}_1, \mathrm{a}_2)$, which has the basis as column vectors is called a *transition matrix* from the basis $\mathrm{a} = \{\mathrm{a}_1, \mathrm{a}_2\}$ to $\mathrm{e} = \{\mathrm{e}_1, \mathrm{e}_2\}$.

The transition from e to a is given by A^{-1}.

Example

Let $\mathrm{u}_1 = (5, 1)^T$ and $\mathrm{u}_2 = (1, -2)^T$. Let $\mathrm{x} = (1, 9)^T$ in the standard basis, and let $\mathrm{y} = (1, 9)^T$ in the basis $\{\mathrm{u}_1, \mathrm{u}_2\}$. What are the coordinates of x and y is the other basis?

Solution.

The transition matrices are given by

$$U = \begin{pmatrix} 5 & 1 \\ 1 & -2 \end{pmatrix}, \quad U^{-1} = -\frac{1}{11}\begin{pmatrix} -2 & 1 \\ 1 & 5 \end{pmatrix}.$$

We get

$$\begin{pmatrix} 5 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 9 \end{pmatrix} = \begin{pmatrix} 14 \\ -17 \end{pmatrix}, \text{ and } \begin{pmatrix} 5 & 1 \\ 1 & -2 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

Row space and column space

Definition

Given a matrix A of size $m \times n$ the subspace spanned by its row vectors is called the *row space* of A and is a subspace of R^n. The subspace spanned by its columns is called the *column space* of A and is a subspace of $\mathbb{R}^m$.

Example

Find the row and column space of the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & -1 & 4 \end{pmatrix}.$$

The row space consists of all vectors of the form

$$\alpha(2,1,0) + \beta(-1,-1,4) = (2\alpha - \beta, \alpha - \beta, 4\beta).$$

The column space is $\mathbb{R}^2$ since we have that every vector in the column space may be written of the form

$$\alpha(2,-1)^T + \beta(1,-1)^T + \gamma(0,4)^T.$$