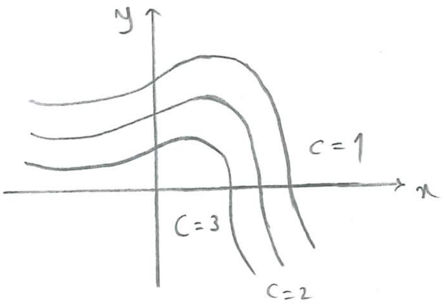
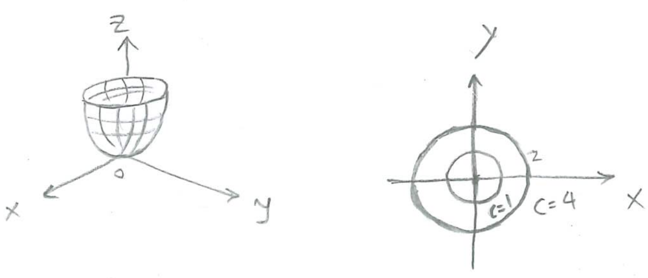
Level curves (topographic map)

$\{(x,y) \in \mathbb{R}^2 : f(x,y) = c\}$ for $c \in \mathbb{R}$ constant.



Example: $f(x,y) = z = x^2 + y^2$



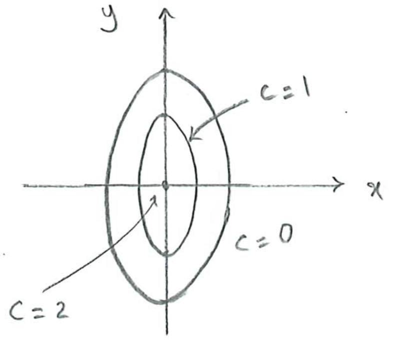
Example. $z = f(x,y) = \frac{1}{3}\sqrt{36-9x^2-4y^2}$

$$\Rightarrow 9x^2+4y^2+9z^2=36 \quad \text{or} \quad \frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{4}=1$$

$z=0 \Longrightarrow \frac{x^2}{4}+\frac{y^2}{9}=1$ ellipse

$z=1 \Longrightarrow \frac{x^2}{4}+\frac{y^2}{9}=\frac{3}{4}$ ellipse

$z=2 \Longrightarrow \frac{x^2}{4}+\frac{y^2}{9}=0 \Longrightarrow (x=y=0)$

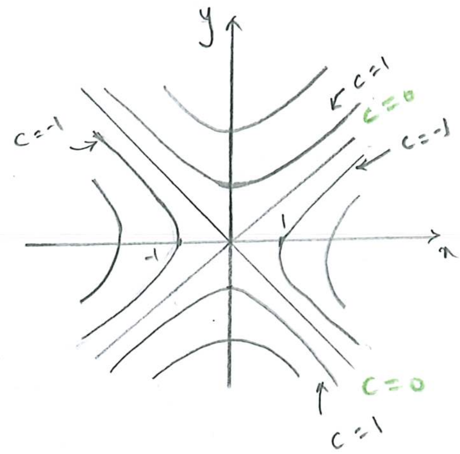


Example. $z = f(x,y) = y^2-x^2 = C$

$C=0$: $y^2-x^2=0 \Longrightarrow y=\pm x$

$C=1$: $y^2-x^2=1$ (hyperbola)

$C=-1$: $y^2-x^2=-1$ (hyperbola)



Example. $z = g(x,y) = xy = C$ $\quad y=\frac{C}{x}$

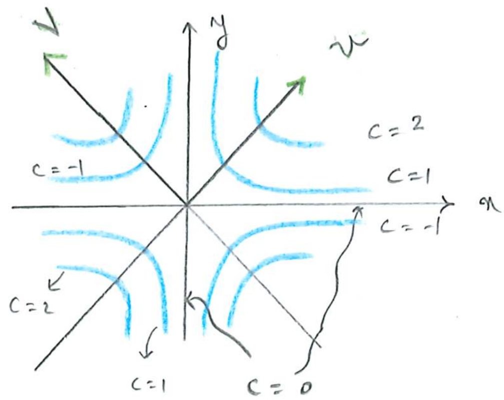
$C=0$: $\quad x=0$ or $y=0$

We note that:

If $x=u-v$ and $y=u+v$, then

$xy = u^2-v^2 = g(u,v)$. Thus:

$$\begin{cases} x+y=2u \\ y-x=2v \end{cases}$$



(13.2)

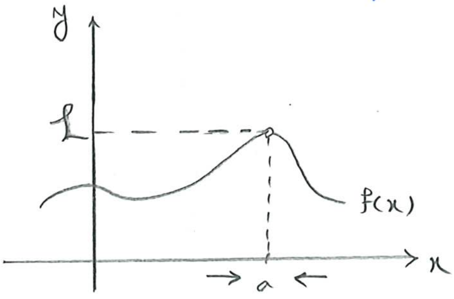
Limit

Recall:

$\lim_{x \to a} f(x) = L$ for $f: \mathbb{R} \to \mathbb{R}$ means:

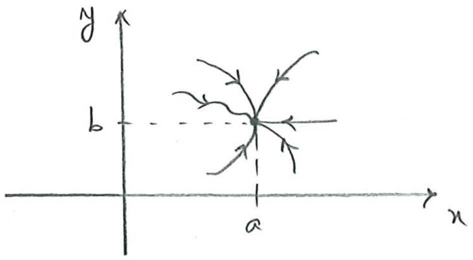
for any $\varepsilon > 0$ there exists $\delta_{(\varepsilon)} > 0$ such that if $0 < |x-a| < \delta$

then $|f(x) - L| < \varepsilon$.



two directions of approach.

In $\mathbb{R}^2$ you can approach (a,b) from infinitely many directions.



Definition

(Limit)

$f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$.

$\lim_{(x,y) \to (a,b)} f(x,y) = L$ if for each $\varepsilon > 0$ there exists

$\delta = \delta(\varepsilon) > 0$ such that $|f(x,y) - L| < \varepsilon$ for all

$(x,y) \in D$, $0 < \|(x,y) - (a,b)\| = \left((x-a)^2 + (y-b)^2\right)^{\frac{1}{2}} < \delta$.

Shorter:

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $(x,y) \in D$, $0 < \|(x,y)-(a,b)\| < \delta \Longrightarrow |f(x,y) - L| < \varepsilon$.

We say: "f has limit $L \in \mathbb{R}$ as (x,y) approaches (a,b)"

Properties of limits

Suppose $\lim\limits_{(x,y)\to(a,b)} f(x,y) = A$, $\lim\limits_{(x,y)\to(a,b)} g(x,y) = B$, and $D(f) \cap D(g) \cap B((a,b),\delta) \neq \varphi$

for any $\delta > 0$. Then:

1. $\lim\limits_{(x,y)\to(a,b)} (f(x,y) + g(x,y)) = A + B$,

2. $\lim\limits_{(x,y)\to(a,b)} (f(x,y)\, g(x,y)) = A \cdot B$,

3. $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{A}{B}$ provided $B \neq 0$.

4. If $F : \mathbb{R} \to \mathbb{R}$ is continuous, then $\lim\limits_{(x,y)\to(a,b)} F(f(x,y)) = F(A)$.
$f : \mathbb{R}^2 \to \mathbb{R}$

---

* $B((a,b),\delta)$ is called an open ball at $(a,b) \in \mathbb{R}^2$ with radius δ
(disk in $\mathbb{R}^2$)
and is defined by:

$$B((a,b),\delta) = \{ (x,y) \in \mathbb{R}^2 : \|(x,y) - (a,b)\| < \delta \}.$$

Squeeze theorem

If $f(x,y) \longrightarrow \ell_1$ and $g(x,y) \longrightarrow \ell_1$ as $(x,y) \longrightarrow (a,b)$, and

$f(x,y) \leq h(x,y) \leq g(x,y)$ holds for all (x,y) close to (a,b), then:

$h(x,y) \longrightarrow \ell_1$ as $(x,y) \longrightarrow (a,b)$.

Definition

$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is continuous at the point $(a,b) \in \mathbb{R}^2$, if $(a,b) \in D_f$

and $\lim\limits_{(x,y) \to (a,b)} f(x,y) = f(a,b)$.

Some methods to calculate a limit or to show that a limit does not exist:

1. See if you have one of the expressions: $\underbrace{\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty}_{\text{Some of the indeterminate forms}}$.
2. Simplification
3. Substitution (change of variable)
4. Estimation
5. Squeeze theorem
6. To show that a limit does not exist: Find two curves Γ_1 and Γ_2 such that: $\lim\limits_{\substack{(x,y) \to (a,b) \\ (x,y) \in \Gamma_1}} f(x,y) = \ell_1 \neq \ell_2 = \lim\limits_{\substack{(x,y) \to (a,b) \\ (x,y) \in \Gamma_2}} f(x,y)$.

Example

(indeterminate form)

① $$\lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{x^2+xy-2y^2} \overset{\frac{0}{0}}{=} \lim_{(x,y)\to(1,1)} \frac{(x-y)(x+y)}{(x-y)(x+2y)} = \lim_{(x,y)\to(1,1)} \frac{x+y}{x+2y} = \frac{2}{3}$$

② $$\lim_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2} \overset{\frac{0}{0}}{=} \lim_{t\to0^+} \frac{1-\cos t}{t^2} = \lim_{t\to0^+} \frac{2\sin^2(\frac{t}{2})}{t^2}$$

change of variable $t=x^2+y^2$

$$= \lim_{t\to0^+} \frac{\sin(t/2)\ \sin(t/2)}{t/2 \cdot t/2} \cdot \frac{1}{2} = \frac{1}{2}$$

standard limit $\lim_{x\to0} \frac{\sin x}{x} = 1$

③ Show that $\lim_{(x,y)\to(0,0)} \frac{xy^2 \sin(\frac{1}{x^2+y^2})}{x^2+y^2} = 0$, (with (ε, δ) definition).

Note that $\left|\sin\left(\frac{1}{x^2+y^2}\right)\right| \leq 1$, and $\frac{|2xy|}{x^2+y^2} \leq 1$.

So:

$$\left|\frac{xy^2 \sin(\frac{1}{x^2+y^2})}{x^2+y^2}\right| \leq \frac{|xy^2|}{x^2+y^2} = \frac{1}{2}|y| \frac{|2xy|}{x^2+y^2} \leq \frac{1}{2}|y|$$

$$\leq \frac{1}{2}\sqrt{x^2+y^2} = \frac{1}{2}\|(x,y)-(0,0)\| < \varepsilon$$

$\delta := 2\varepsilon$

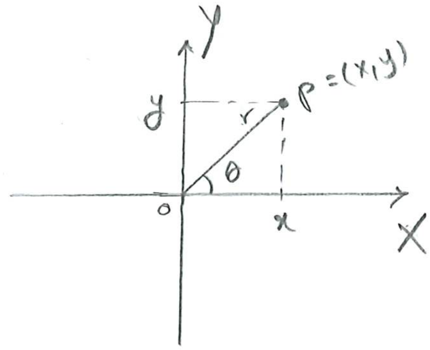
$\Rightarrow \forall \varepsilon > 0\ \exists \delta = 2\varepsilon > 0$ s.t. $\left|\frac{xy^2 \sin(\frac{1}{x^2+y^2})}{x^2+y^2}\right| < \varepsilon$ if $\|(x,y)\| < \delta$. #

The proof is complete.

Recall:

Polar Coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases}$$



$\theta \in (-\pi, \pi]$

Actually:

$$\theta = \begin{cases} \arctan\left(\frac{y}{x}\right), & x > 0 \\ \arctan\left(\frac{y}{x}\right)+\pi, & x<0,\ y \geq 0 \\ \arctan\left(\frac{y}{x}\right)-\pi, & x<0,\ y<0 \\ \frac{\pi}{2}, & x=0,\ y>0 \\ -\pi/2, & x=0,\ y<0 \\ \text{undefined} & x=y=0 \end{cases}$$

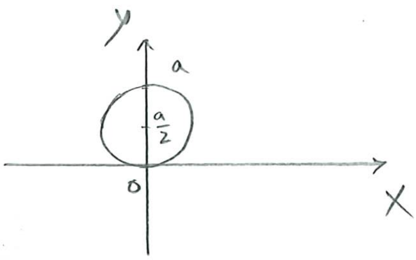
Example:

$r = a\sin\theta$ is an equation in polar coordinates. Find the Cartesian equation and sketch the graph.

$$r = a\sin\theta \Rightarrow r = a\frac{y}{r} \Rightarrow r^2 = ay \Rightarrow x^2+y^2 = ay$$

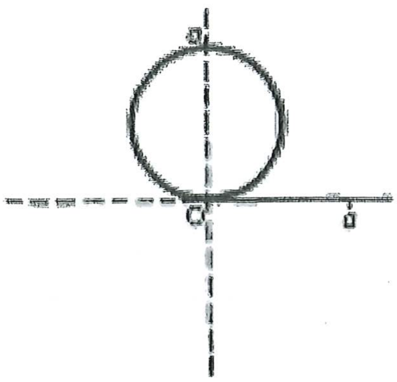
$$\Rightarrow x^2+y^2-ay=0 \Rightarrow x^2+\left(y-\frac{a}{2}\right)^2 = \frac{a^2}{4}$$

This is a circle of radius $\frac{a}{2}$ centered at $(0, \frac{a}{2})$.

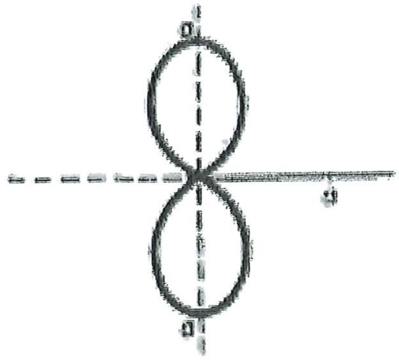


Typical polar curves

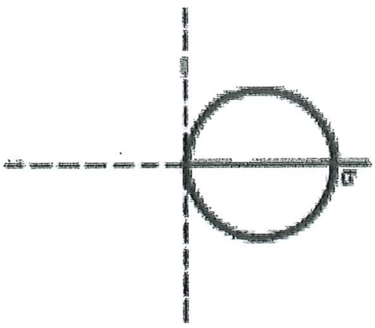
1. $r = a \sin\theta$



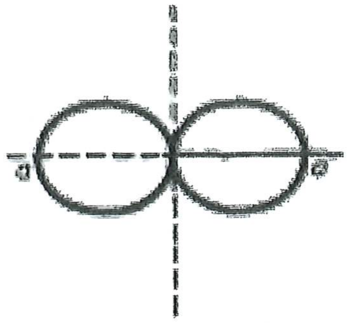
2. $r = a \sin^2\theta$



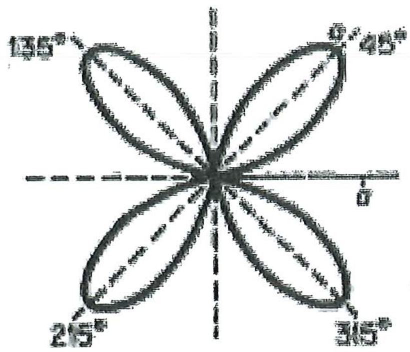
3. $r = a \cos\theta$



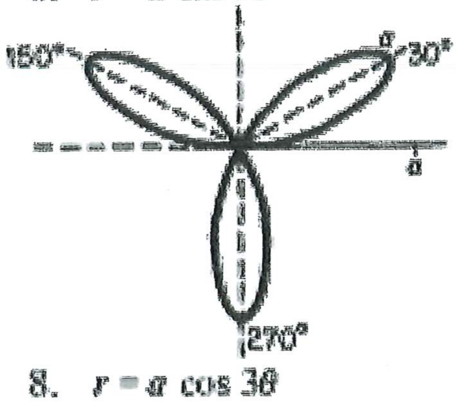
4. $r = a \cos^2\theta$



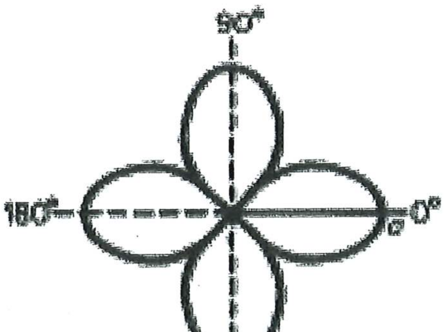
5. $r = a \sin 2\theta$



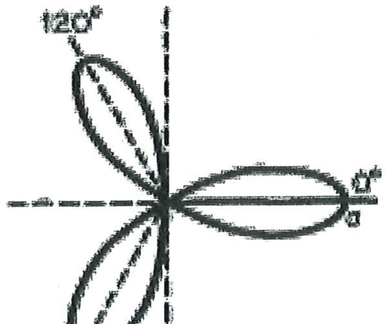
6. $r = a \sin 3\theta$

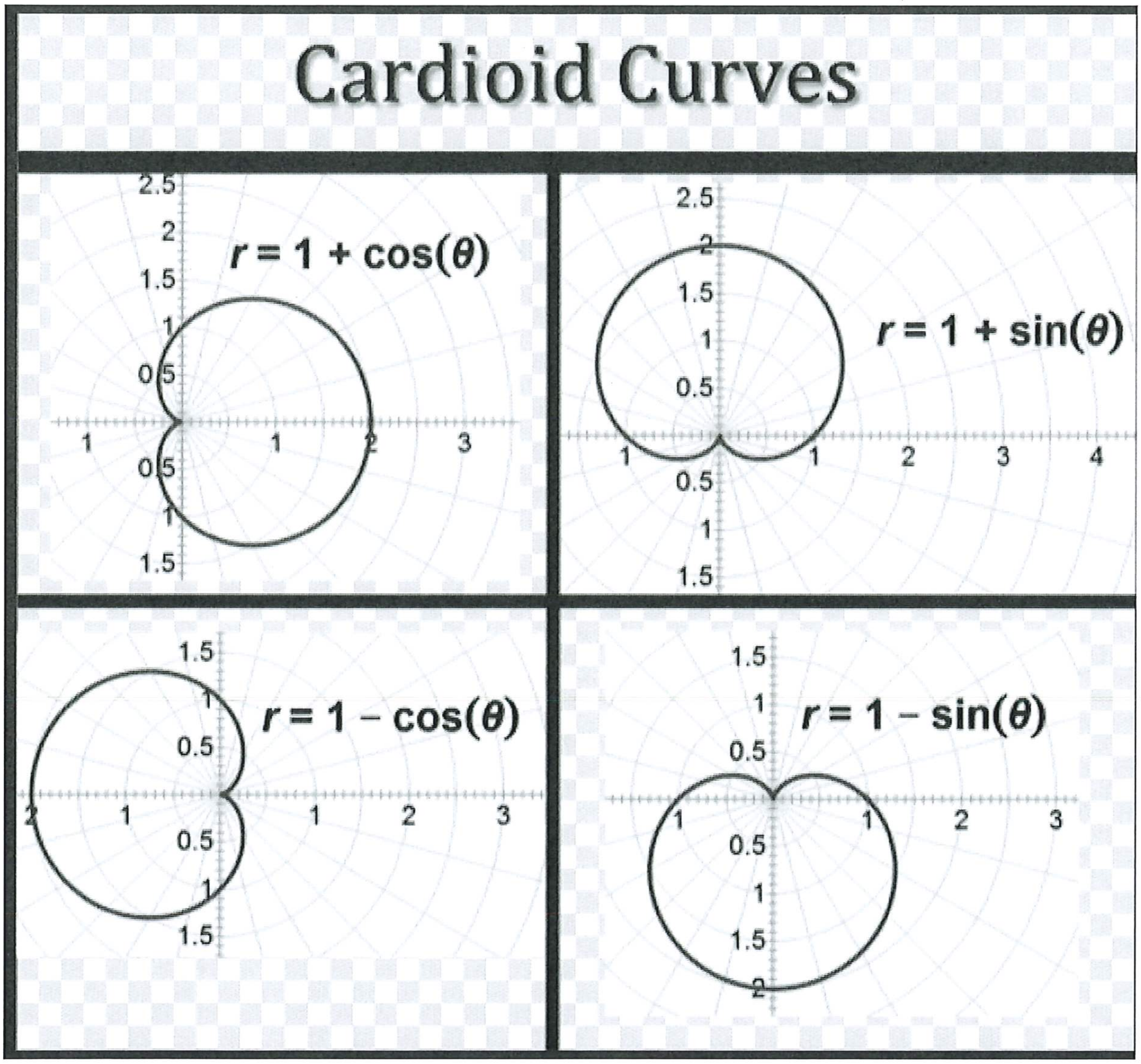


7. $r = a \cos 2\theta$



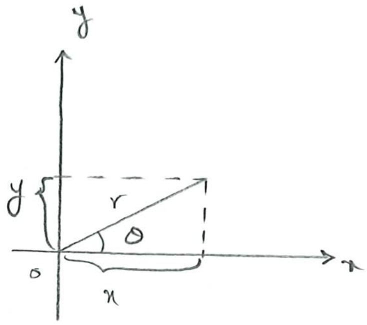
8. $r = a \cos 3\theta$





Cardioid Curves
r = 1 + cos(θ)
r = 1 + sin(θ)
r = 1 − cos(θ)
r = 1 − sin(θ)

④ $$\lim_{(x,y)\to(0,0)} \frac{x^4 y}{(x^2+y^2)} \overset{\frac{0}{0}}{=} ?$$



We use polar coordinates in this example.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$(x,y) \longrightarrow (0,0) \iff r \longrightarrow 0^+$$

We have: $$\frac{x^4 y}{x^2+y^2} = \frac{r^5 \cos^4\theta \sin\theta}{r^2} = r^3 \cos^4\theta \sin\theta$$ and

$$-r^3 \le r^3 \cos^4\theta \sin\theta \le r^3 .$$

Squeeze theorem implies that $\lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^2+y^2} = 0$.

#

⑤. Does the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+5y^2}$ exist?

The answer is no!

If we approach $(0,0)$ along the curve Γ_1: $y=0, x=t$, we get:

$$\lim_{\substack{(x,y)\in\Gamma_1 \\ (x,y)\to(0,0)}} \frac{xy}{x^2+5y^2} = \lim_{t\to 0} \frac{t\cdot 0}{t^2+0} = 0,$$ but if we approach $(0,0)$

Along the curve Γ_2: $x=y=t$, we get:

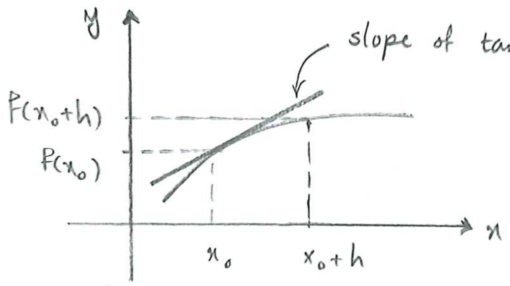
$$\lim_{\substack{(x,y)\in\Gamma_2 \\ (x,y)\to(0,0)}} \frac{xy}{x^2+5y^2} = \lim_{t\to 0} \frac{t^2}{t^2+5t^2} = \frac{1}{6} \neq 0.$$ So, the limit does not exist.

#

Partial derivatives and differentiability: (13.3)

Recall:

The derivative of $f: \mathbb{R} \longrightarrow \mathbb{R}$ at $x = x_0$ is $f'(x_0) = \lim\limits_{h \to 0} \dfrac{f(x_0+h) - f(x_0)}{h}$.



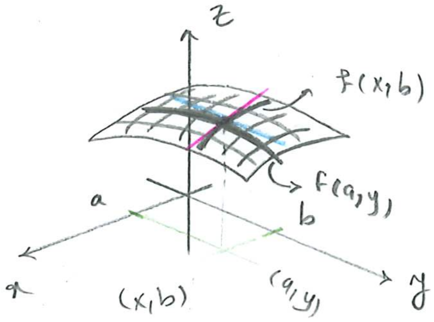
n in general

For $f: \mathbb{R}^{2} \longrightarrow \mathbb{R}$, the partial derivatives at (x,y) are defined by:

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}, \text{ and } \frac{\partial f}{\partial y}(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

$\dfrac{\partial f}{\partial x}$ gives the rate of change in the x-direction, for y fixed.

$\dfrac{\partial f}{\partial y}$ gives the rate of change in the y-direction, for x fixed.



Alternative notations: $\dfrac{\partial f}{\partial x} = f'_x = f_1 = D_1 f = f_x$,

$$\frac{\partial f}{\partial y} = f'_y = f_2 = D_2 f = f_y.$$

Example. ① $f(x,y) = xy^2$

$$f'_x(x,y) = y^2 \qquad f'_y(x,y) = 2xy.$$

② $f(x,y) = \sin^2(2x^3y)$

$$f'_x(x,y) = 2\sin(2x^3y)\cdot\cos(2x^3y)\cdot 6x^2y = \sin(4x^3y).6x^2y.$$

$$f'_y(x,y) = 2\sin(2x^3y)\cos(2x^3y)\cdot 2x^3 = \sin(4x^3y)\cdot 2x^3.$$

③ $f(x,y) = \sqrt{\ln(1+x^2+3y^2)}$

$$f_1(x,y) = f'_x(x,y) = \frac{1}{2}\left(\ln(1+x^2+3y^2)\right)^{-\frac{1}{2}} \cdot \frac{2x}{1+x^2+3y^2}$$

$$= \frac{x}{(1+x^2+3y^2)\left(\ln(1+x^2+3y^2)\right)^{\frac{1}{2}}}$$

$$f_2(x,y) = f'_y(x,y) = \frac{1}{2}\left(\ln(1+x^2+3y^2)\right)^{-\frac{1}{2}} \frac{6y}{1+x^2+3y^2}$$

$$= \frac{3y}{(1+x^2+3y^2)\left(\ln(1+x^2+3y^2)\right)^{\frac{1}{2}}}.$$