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Written Exam on Numerical Methods, 2MA903, 1 hp (5 hp)
Thursday 18th of March 2021, 08.00–13.00.

The solutions should be complete, correct, motivated, well structured and easy to follow.
Aids: Calculator (you may use a scientific calculator but *not* with internet connection).
Please begin each question on a new paper.
Preliminary grades: 15p-17p⇒E; 18p-20p⇒D; 21p-23p⇒C; 24p-26p⇒B; 27p-30p⇒A.

1. (a) Compute the two roots of the equation $x^2 + 200x - 10^{-10} = 0$ with 4 significant digitis.

   (b) Reformulate the function $g(x) = \sin x - x$ to avoid cancellation problems for x very close to zero. (4p)

   (a)

$$x^2 + 200x - 10^{-10} = 0$$
$$x = -100 \pm \sqrt{10^4 + 10^{-10}}$$

For the negative root this is fine, with $x \approx -200.0$. For the positive root, we reformulate as

$$x = -100 + \sqrt{10^4 + 10^{-10}} = \frac{\left(-100 + \sqrt{10^4 + 10^{-12}}\right)\left(-100 - \sqrt{10^4 + 10^{-10}}\right)}{\left(-100 - \sqrt{10^4 + 10^{-10}}\right)}$$
$$= \frac{10000 - (10^4 + 10^{-10})}{\left(-100 - \sqrt{10^4 + 10^{-12}}\right)} = \frac{10^{-10}}{100 + \sqrt{10^4 + 10^{-10}}}$$

yielding $x = 5.000 \cdot 10^{-13}$

(b) Use Taylor expansion: $g(x) = \sin x - x = \left(x - \frac{x^3}{6} + \mathcal{O}(x^5)\right) - x = \frac{x^3}{6} + \mathcal{O}(x^5)$

2. (a) Use the Newton-Raphson method to find all solutions to the equation $x^3 - 6x^2 + 17 = 0$. Answer with 4 significant digits.

   (b) In case of double roots, it is better to use any of the modified versions of the Newton-Raphson methods shown below, that is

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}, \qquad \text{or} \qquad x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}.$$

   What is the numerical consequence if you keep using the original Newton-Raphson method instead of using one of the modified versions, and what characteristic of double roots is causing this problem? (5p)

   (a) Let $f(x) = x^3 - 6x^2 + 17$. For Newton-Raphson we need $f'(x) = 3x^2 - 12x$.

   NR: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

   The equation is a third degree polynomial, thus all=3 roots. For suitable starting guesses, do a rough sketch of the curve.

   Evaluating for some choices of x

$$\begin{aligned} f(-10) = -1000 - 600 + 17 = -1583 < 0 \\ f(0) = 17 > 0 \\ f(4) = 4^3 - 6 * 4^2 + 17 = -15 < 0 \\ f(10) = 1000 - 600 + 17 = 417 > 0 \end{aligned}$$

   we find that the equations have roots somewhere in]-10,0[,]0,4[and in]4,10[. Performing the Newton-algoritm gives us the roots -1.505, 2.083 and 5.422.

   (b) Slower convergence or (fatal) division by zero.

   This is caused by the fact that $f'(x_n) \to 0$ when $x_n \to r$, since $f'(r) = 0$ at a double root r.

3. Given the following set of points (x, y): $(-3, -5)$, $(1, 1)$, $(3, 14)$, $(4, 30)$:

(a) determine the corresponding interpolating polynomial (of lowest possible degree),

(b) if more points were to be added to the set of points, what would be the preferred method of finding the interpolating polynomial?

(c) Next, consider the points

x	-4	-3	-2	-1	0	1	2	3	4
y	1	1	1	1	2	1	1	1	1

Without doing any actual computations, make a sketch of the corresponding interpolating polynomial (together with the given points).
The scales on the axes do not need to be very exact, but the essential behaviour of the polynomial should be clearly demonstrated. (5p)

(a) E.g. Lagrange:

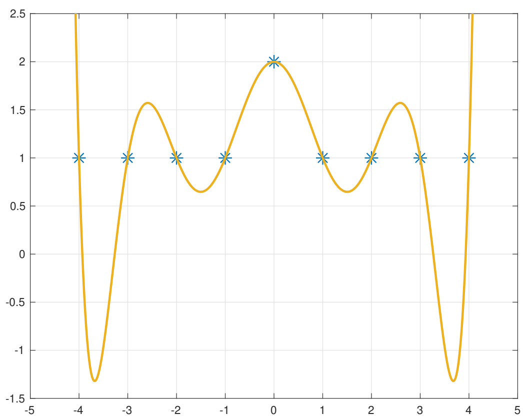
$$\ell_0 = \frac{(x-1)(x-3)(x-4)}{(-3-1)(-3-3)(-3-4)} = \frac{x^3 - 8x^2 + 19x - 12}{-24 * 7}$$
$$\ell_1 = \frac{(x-(-3))(x-3)(x-4)}{(1-(-3))(1-3)(1-4)} = \frac{x^3 - 4x^2 - 9x + 36}{24}$$
$$\ell_2 = \frac{(x-(-3))(x-1)(x-4)}{(3-(-3))(3-1)(3-4)} = \frac{x^3 - 2x^2 - 11x + 12}{-12}$$
$$\ell_3 = \frac{(x-(-3))(x-1)(x-3)}{(4-(-3))(4-1)(4-3)} = \frac{x^3 - x^2 - 9x + 9}{21}$$

and the polynomial is

$$\begin{aligned} -5\ell_0 + \ell_1 + 14\ell_2 + 30\ell_3 &= -5\frac{x^3 - 8x^2 + 19x - 12}{24 * 7} + \frac{x^3 - 4x^2 - 9x + 36}{24} \\ &+ 14\frac{x^3 - 2x^2 - 11x + 12}{-12} + 30\frac{x^3 - x^2 - 9x + 9}{21} \\ &= \frac{2x^3 + 3x^2 + x}{6} \end{aligned}$$

(b) The Newton's divided differences

(c)



4. (a) Use the (composite) trapezoidal rule to compute the integral

$$I = \int_{-1}^{1} \sqrt{1-x^2}\, \mathrm{d}x$$

for 3 different h ($h = 2$, $h = 1$ and $h = 1/2$). Use Romberg iteration to improve the approximation of the integral.

(b) Derive a finite difference formula D_h that uses the values $f(x)$, $f(x+h)$ and $f(x+2h)$ to approximate $f'(x)$. The formula should be as accurate as possible.

Find the error term and the order for the approximation formula D_h.

(c) Draw a figure showing the behaviour of the error $|D_h - f'(x)|$ as a function of h. The axes should be in log-log scale. (6p)

(a) Let $f(x) = \sqrt{1-x^2}$. Use the trapezoidal rule

$$\begin{aligned}
R_{1,1} = T_{h=2} &= \frac{h}{2}\left(f(-1)+f(1)\right) = \frac{2}{2}\left(\sqrt{1-(-1)^2}+\sqrt{1-1^2}\right) = 0 \\
R_{2,1} = T_{h=1} &= \frac{h}{2}\left(f(-1)+2f(0)+f(1)\right) = \frac{1}{2}\left(0+2\cdot 1+0\right) = 1 \\
R_{3,1} = T_{h=1/2} &= \frac{h}{2}\left(f(-1)+2f(-1/2)+2f(0)+2f(1/2)+f(1)\right) \\
&= \frac{1}{4}\left(0+2\sqrt{3/4}+2\cdot 1+2\sqrt{3/4}+0\right) = \\
&= \frac{1}{2}\left(\sqrt{3}+1\right) = 1.366025403784439
\end{aligned}$$

Use Romberg (=Richardson for Trapezoidal)

$$\begin{aligned}
R_{2,2} &= \frac{4R_{2,1}-R_{1,1}}{3} = \frac{4\cdot 1 - 0}{3} = 1.333333333333333 \\
R_{3,2} &= \frac{4R_{3,1}-R_{2,1}}{3} = \frac{4\cdot 1.366025403784439 - 1}{3} = 1.488033871712585
\end{aligned}$$

Use the improved answers again

$$R_{3,3} = \frac{16R_{3,2}-R_{2,2}}{15} = \frac{16\cdot 1.488033871712585 - 1.333333333333333}{15} = 1.498347240937868$$

(b) We want $af(x)+bf(x+h)+cf(x+2h) \approx f'(x)$. Use Taylor expansion:

$$\begin{aligned}
f(x+h) &= f(x)+hf'(x)+\frac{h^2}{2}f''(x)+\dots \\
f(x+2h) &= f(x)+2hf'(x)+4\frac{h^2}{2}f''(x)+\dots
\end{aligned}$$

From

$$af(x)+bf(x+h)+cf(x+2h) = (a+b+c)f(x)+(0+b+2c)hf'(x)+(0+b+4c)\frac{h^2}{2}f''(x)+\dots$$

we get the demands

$$a+b+c=0 \qquad b+2c=1/h \qquad b+4c=0$$

solved by $a = -1.5/h$, $b = 2/h$ and $c = -0.5/h$. This yields

$$\begin{aligned}
D_h &= \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} \\
&= \frac{-3f(x) + 4(f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \ldots)}{2h} \\
&+ \frac{-(f(x) + 2hf'(x) + 4\frac{h^2}{2}f''(x) + 8\frac{h^3}{6}f'''(x) + \ldots)}{2h} \\
&= f'(x) - \frac{h^2}{3}f'''(x) + \ldots
\end{aligned}$$

Second order accurate

(c) See lecture notes

5. Let $y(x)$ be the solution of $y'(x) = y(1-y)$ for which $y(0) = 0.2$.

(a) Find an approximation of $y(2)$ using Euler forward with step length $h = 1$ and another approximation using $h = 0.5$. Use 4 correctly rounded decimals of function values in the written presentation.

(b) Sketch the corresponding slope field for $y'(x) = y(1-y)$. Include the two approximative solutions from (a), for $x \in [0, 2]$, in your slope field picture.

c) Using Richardson extrapolation, calculate an improved approximation of y(2) using the results obtained in (a). (5)

(a)

Euler forward: $y_{n+1} = y_n + hf(x_n, y_n)$, here with $f(x, y) = y(1-y)$

$\boxed{h = 1}$

$$y_0 = 0.2$$
$$y_1 = y_0 + hf(x_0, y_0) = 0.2 + 1 * 0.2(1-0.2) = 0.2(1+0.8) = 0.36$$
$$y_2 = y_1 + hy_1(1-y_1) = 0.36 + 0.36(1-0.36) = 0.5904$$

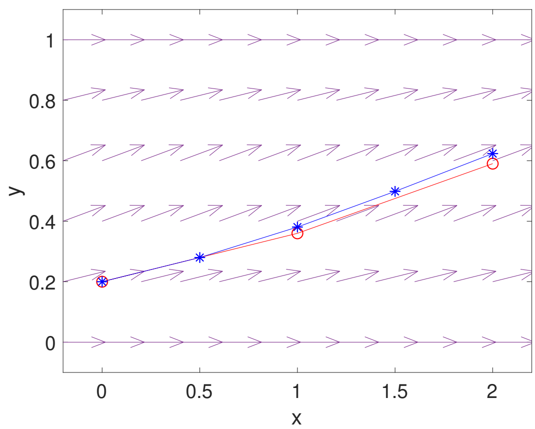
$y(2) \approx y_2 = 0.5904$

$\boxed{h = 0.5}$

$$y_0 = 0.2$$
$$y_1 = y_0 + hf(x_0, y_0) = 0.2 + 0.5 * 0.2(1-0.2) = 0.2(1+0.4) = 0.28$$
$$y_2 = y_1 + hy_1(1-y_1) = 0.28 + 0.5 * 0.28(1-0.28) = 0.3808$$
$$y_3 = 0.49869568$$
$$y_4 = 0.623694829374669$$

$y(2) \approx y_4 = 0.6237$

(b)



(c)

Euler forward is first order accurate, so we use $y(2) = \frac{2y_{h=0.5} - y_{h=1}}{2-1} = 2*0.6237... - 0.5904 = 0.6570$

6. A 10 cm long rod is exposed to a source of heat along its axis. The temperature $T(x)$ of the rod can be modelled with Poisson's equation

$$-\frac{d^2T}{dx^2} = f(x), \qquad\qquad x \in [0,\ 10],$$

where $f(x) = x$ is the heat source. At the ends of the rod the temperature is fixed, such that the boundary conditions are $T(x = 0) = 40$ and $T(x = 10) = 200$.

(a) Approximate the boundary value problem described above as a finite difference problem with step size $\Delta x = h = 2$, and present the resulting system of equations in matrix form.

(You don't have to solve the system of equations.)

(b) Reformulate the problem such that it can be solved using the Shooting method.

(c) Write some Matlab code (or pseudo-code) that shows how the reformulated problem in (b) can be solved using the Shooting method. You may use/refer to existing Matlab functions such as `fzero`, `ode45` or similar. (6)

(a) Let $T_i \approx T(x_i)$ and $f_i = f(x_i)$ where $x_i = ih$ with $i = 0, 1, \dots, n$. This gives

$$\begin{aligned} -\frac{T_{i+1} - 2T_i + T_{i-1}}{h^2} &= f_i, \qquad\qquad i = 1, 2, 3, \dots, n-1 \\ T_0 &= 40 \\ T_n &= 200 \end{aligned}$$

with $f_i = x_i$.

In matrix form, for $h = 2$ (that is $n = 5$), we have $x_0 = 0$, $x_1 = 2$, ..., $x_5 = 10$.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 40 \\ 0 \\ 0 \\ 200 \end{bmatrix} + h^2 \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 48 \\ 16 \\ 24 \\ 232 \end{bmatrix}.$$

(b) Introduce a new variable $v = T'$. Now $-T'' = f$ is rewritten as $-v' = f$ and we obtain a system of differential equations

$$\begin{cases} T' = v \\ v' = -f \end{cases} \qquad\qquad \begin{cases} T(0) = 40 \\ T(10) = 200 \end{cases}$$

To use a solver for IVPs we need to replace the boundary condition $T(10) = 200$ by an initial condition $v(0) = s$, yielding

$$\begin{cases} T' = v \\ v' = -f \end{cases} \qquad\qquad \begin{cases} T(0) = 40 \\ v(0) = s \end{cases}$$

(c) Let $y = [T, v]^T$ and $y' = F$ with $F = [v, -f]^T$ and initial condition $y(0) = [40, s]^T$.

Shooting method: The BVP is rewritten as an IVP. Vary the modified initial condition until the boundary conditions are satisfied with desired accuracy.

s. page 353 in Sauer.