List of formulas for the exam in Numerical Methods, 2024

These formulas will be attached to the exam. The list is not guaranteed to be complete, and the use, meaning, conditions and assumptions of the formulas are purposely left out.

- **Taylor's formula**

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

(ξ between x and a)

- **Operation count**

$$1+2+3+4+\cdots+n = \frac{n(n+1)}{2}, \qquad 1^2+2^2+3^2+4^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

- **Absolute and relative error**

$$\Delta_x = \tilde{x} - x, \qquad \frac{\Delta_x}{x} \approx \frac{\Delta_x}{\tilde{x}}, \qquad \Delta_{x+y} = \Delta_x + \Delta_y, \qquad \frac{\Delta_{xy}}{xy} \approx \frac{\Delta_x}{x} + \frac{\Delta_y}{y}$$

- **Error propagation formulas, condition number (1D)**

$$\Delta f \approx f'(x)\Delta x, \qquad \left|\frac{\Delta f/f}{\Delta x/x}\right| \approx \left|\frac{xf'(x)}{f(x)}\right|$$

$$\Delta f \approx f''(x)\frac{\Delta x^2}{2}$$

- **Correct decimals**
  $|\Delta x| \le 0.5 \cdot 10^{-t}$

- **Binary numbers**
  $x = x_m 2^m + x_{m-1}2^{m-1} + \ldots + x_0 + x_{-1}2^{-1} + \ldots = (x_m x_{m-1} \ldots x_0.x_{-1} \ldots)_b$

- **Iterative methods**

Bisection method:

```
c=(a+b)/2;
while (b-a)>2*tol
    if f(c)*f(a)>0
        a=c;
    else
        b=c;
    end
    c=(a+b)/2;
end
```

Newton-Raphson: $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad J(\mathbf{x}_n)(\mathbf{x}_{n+1} - \mathbf{x}_n) + \mathbf{f}(\mathbf{x}_n) = 0$$

The secant method: $$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})},$$

$e_n = x_n - x^*, \qquad \lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = c$

- **Equation systems**

$A\mathbf{x} = \mathbf{b}$, residual $\mathbf{r} = \mathbf{b} - A\widetilde{\mathbf{x}}$

LU-factorization:	$A = LU, \quad PA = LU$
QR-factorization:	$A = QR, \quad Q^T Q = I$
(Iterative methods)	$A = D + L + U$
Jacobi methods:	$\mathbf{x}^{(k)} = -D^{-1}(L+U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b}$
Gauss-Seidel:	$\mathbf{x}^{(k)} = -(D+L)^{-1}U\mathbf{x}^{(k-1)} + (D+L)^{-1}\mathbf{b}$

Backward: $\|\mathbf{r}\|_\infty$, forward: $\|\mathbf{x} - \widetilde{\mathbf{x}}\|_\infty$

- **Norms and condition numbers**

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a vector:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}, \quad \|\boldsymbol{x}\|_1 = \sum_{i=1}^n |x_i|, \quad \|\boldsymbol{x}\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}, \quad \|\mathbf{x}\|_\infty = \max_i |x_i|.$$

Let A be a $n \times n$ matrix:

$$\|A\| = \sup_{\mathbf{x}\neq 0} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|}, \qquad \|A\mathbf{x}\| \leq \|A\| \cdot \|\mathbf{x}\|, \qquad \kappa(A) = \|A\| \cdot \|A^{-1}\|$$

$A(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}$

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \leq \kappa(A)\frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|}, \qquad \frac{\|\delta\mathbf{x}\|/\|\mathbf{x}\|}{\|\delta\mathbf{b}\|/\|\mathbf{b}\|} \leq \kappa(A),$$

- **Interpolation**

Let $(x_0, y_0), \ldots, (x_n, y_n)$ be $n+1$ points in the xy-plane.

Monomial:	$P(x) = a_0 + a_1 x + \ldots + a_n x^n$
Lagrangre:	$P(x) = \sum_{j=0}^n y_j \ell_j(x), \qquad \ell_j(x) = \prod_{\substack{0\leq m\leq n \\ m\neq j}} \frac{x - x_m}{x_j - x_m}$

Newton's divided differences:

$$P(x) = f[x_1] + f[x_1\,x_2](x - x_1) + \cdots + f[x_1\,\cdots\,x_{k+1}](x-x_1)(x-x_2)\cdots(x-x_k).$$

$$f[x_1] = f(x_1), \quad f[x_1\,x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \quad f[x_1\,x_2\,x_3] = \frac{f[x_2\,x_3] - f[x_1\,x_2]}{x_3 - x_1}, \ldots$$

Interpolation remainder:

$$f(x) - P(x) = (x - x_1)(x - x_2)\cdots(x - x_n)\frac{f^{(n)}(\xi)}{(n)!}, \qquad x_1 < \xi < x_n,$$

- **Least squares, normal equations** $A^T A\mathbf{x} = A^T\mathbf{b}$, residual $\mathbf{r} = \mathbf{b} - A\mathbf{x}$

• **Finite differences**

$$
\begin{aligned}
\frac{f(x+h)-f(x)}{h} &= f'(x)+f''(\xi)\frac{h}{2} && \xi\in[x,x+h]\\
\frac{f(x)-f(x-h)}{h} &= f'(x)-f''(\xi)\frac{h}{2} && \xi\in[x-h,x]\\
\frac{f(x+h)-f(x-h)}{2h} &= f'(x)+f^{(3)}(\xi)\frac{h^2}{6} && \xi\in[x-h,x+h]\\
\frac{f(x+h)-2f(x)+f(x-h)}{h^2} &= f''(x)+f^{(4)}(\xi)\frac{h^2}{12} && \xi\in[x-h,x+h]
\end{aligned}
$$

• **Trapezoidal rule, Simpson's rule**

$$\int_a^b f(x)dx = \tfrac{h}{2}\Big(f(x_0)+2\textstyle\sum_{k=1}^{n-1} f(x_k)+f(x_n)\Big) - \tfrac{(b-a)h^2}{12} f''(\xi), \qquad h=\frac{b-a}{n}$$

$$\int_a^b f(x)dx = \tfrac{h}{3}\Big(f(x_0)+4\textstyle\sum_{k=1}^{n} f(x_{2k-1})+2\sum_{k=1}^{n-1} f(x_{2k})+f(x_{2n})\Big) - \tfrac{(b-a)h^4}{180} f^{(4)}(\xi), \qquad h=\frac{b-a}{2n}$$

$a<\xi<b$

• **Richardson extrapolation**
$Q = F(h) + Kh^n + \mathcal{O}(h^{n+1}), \qquad Q = \frac{2^n F(h/2)-F(h)}{2^n-1} + \mathcal{O}(h^{n+1})$

• **Romberg** $R_{i,1} = T(h/2^{i-1})$, $R_{ij} = \frac{4^{j-1}R_{i,j-1}-R_{i-1,j-1}}{4^{j-1}-1}$

• **Numerical solutions of differential equations**
Differential equation $y' = f(x,y)$ with initial condition $y(x_0) = y_0$

Euler forward ($g_i \sim \mathcal{O}(h)$): $\quad y_{n+1} = y_n + hf(x_n, y_n)$

Euler backward ($g_i \sim \mathcal{O}(h)$): $\quad y_{n+1} = y_n + hf(x_{n+1}, y_{n+1})$

RK4 ($g_i \sim \mathcal{O}(h^4)$): $\quad \begin{cases} y_{n+1} = y_n + \frac{1}{6}(k_1+2k_2+2k_3+k_4)\\ k_1 = hf(x_n, y_n)\\ k_2 = hf(x_n+h/2, y_n+k_1/2)\\ k_3 = hf(x_n+h/2, y_n+k_2/2)\\ k_4 = hf(x_n+h, y_n+k_3)\end{cases}$

where $x_{n+1} = x_n + h$.

• **Boundary value problems**
Two-point boundary problem $y'' = f(x,y,y')$ with initial condition $y(a)=\alpha$ and $y(b)=\beta$.

Shooting method: The BVP is rewritten as a IVP. Vary the modified initial condition until the boundary conditions are satisfied with desired accuracy.

Finite difference method: Derivatives are approximated by finite difference quotients. Exact solution y is replaced by y_i such that $y_i \approx y(x_i)$