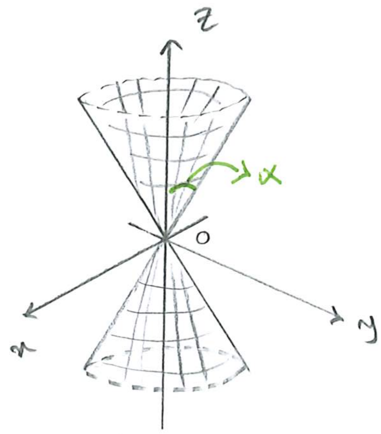
Some quadric surfaces (10.5)

Cone.

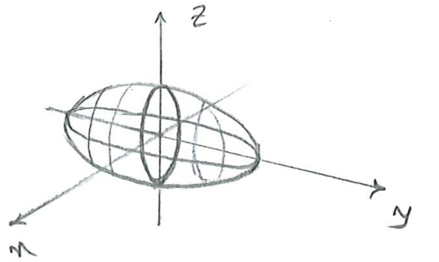
$$a^2 z^2 = x^2 + y^2$$

$$\boxed{\alpha = \tan^{-1} a = \arctan a}$$



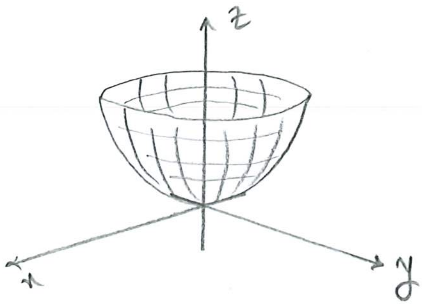
Ellipsoid.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$



Elliptic paraboloid:

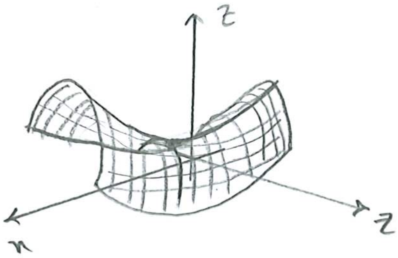
$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



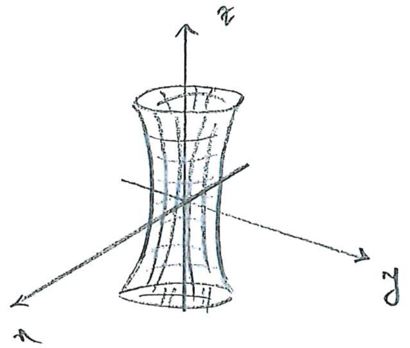
Hyperbolic paraboloid:

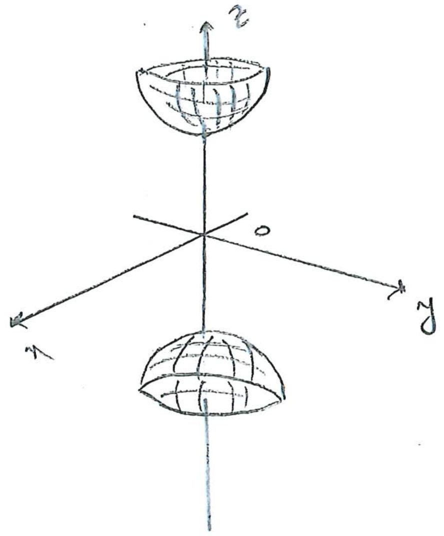
$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

(like saddle)

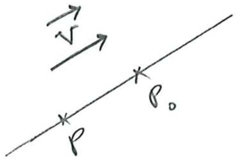


Hyperboloids
- of one sheet $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
- of two sheets $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$





Lines.



$\overrightarrow{PP_0} = t\vec{v} \quad t \in \mathbb{R}$

$\vec{v} = (a, b, c)$, $P_0 = (x_0, y_0, z_0)$, $P = (x, y, z)$.

$(x - x_0, y - y_0, z - z_0) = (ta, tb, tc)$

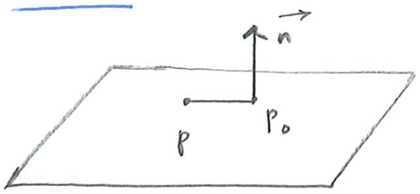
$\Rightarrow \begin{cases} x - x_0 = ta \\ y - y_0 = tb \\ z - z_0 = tc \end{cases} \Rightarrow \begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases} \xrightarrow[\text{if } a=0:]{\text{for example}} \begin{cases} x = x_0 \\ \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c} \end{cases}$

Parametric form

$\rightarrow \begin{cases} \dfrac{x - x_0}{a} = t \\ \dfrac{y - y_0}{b} = t \\ \dfrac{z - z_0}{c} = t \end{cases} \Rightarrow \underbrace{\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}}_{(a, b, c \neq 0)}$

(standard form)

Planes



$P_0 = (x_0, y_0, z_0)$

$P = (x, y, z)$

$\vec{n} = (A, B, C)$ (normal)

$\vec{n} \cdot \vec{PP_0} = 0 \Rightarrow (A, B, C) \cdot (x - x_0, y - y_0, z - z_0) = 0$

$\Rightarrow Ax + By + Cz - (Ax_0 + By_0 + Cz_0) = 0$

$\Rightarrow \underline{Ax + By + Cz = D}$

A line is the intersection of two planes:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \qquad a, b, c \neq 0$$

$$\begin{cases} \dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} \Rightarrow \dfrac{x}{a} - \dfrac{x_0}{a} - \dfrac{y}{b} + \dfrac{y_0}{b} = 0 \Rightarrow \underset{\text{a plane}}{Ax + By + D = 0} \\ \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c} \Rightarrow \dfrac{y}{b} - \dfrac{y_0}{b} - \dfrac{z}{c} + \dfrac{z_0}{c} = 0 \Rightarrow \underset{\text{a plane}}{By + Cz + D' = 0} \end{cases}$$

The Cauchy-Schwarz inequality in $\mathbb{R}^n$:

$\bar{u}, \bar{v} \in \mathbb{R}^n$ $\qquad |\bar{u} \cdot \bar{v}| \leq |\bar{u}||\bar{v}|$

Proof. It is obvious that we have the equality when:

$\bar{u} = \bar{0}$ or $\bar{v} = \bar{0}$.

Assume that $\bar{u} \neq 0$ and $\bar{v} \neq 0$.

We set: $\bar{x} = \hat{u} = \dfrac{\bar{u}}{|\bar{u}|}$, $\bar{y} = \hat{v} = \dfrac{\bar{v}}{|\bar{v}|}$

$\Rightarrow |\bar{x}| = |\hat{u}| = 1$ and $|\bar{y}| = |\hat{v}| = 1$.

Now: $\quad 0 \leq |\bar{x} - \bar{y}|^2 = (\bar{x} - \bar{y}) \cdot (\bar{x} - \bar{y})$

$$= \bar{x} \cdot \bar{x} - \bar{x} \cdot \bar{y} - \bar{y} \cdot \bar{x} + \bar{y} \cdot \bar{y}$$

$$= |\bar{x}|^2 - 2\bar{x} \cdot \bar{y} + |\bar{y}|^2$$

$$= 2 - 2\bar{x} \cdot \bar{y}$$

$$\Rightarrow 2\bar{x} \cdot \bar{y} \leq 2 \Rightarrow \bar{x} \cdot \bar{y} \leq 1$$

$$\Rightarrow \frac{\bar{u}}{|\bar{u}|} \cdot \frac{\bar{v}}{|\bar{v}|} \leq 1 \Rightarrow \underbrace{\bar{u} \cdot \bar{v} \leq |\bar{u}||\bar{v}|}, \quad (*)$$

If we replace $\bar{u}$ by $(-\bar{u})$ in $(*)$, we get:

$$(-\bar{u}) \cdot \bar{v} \leq |(-\bar{u})||\bar{v}|$$

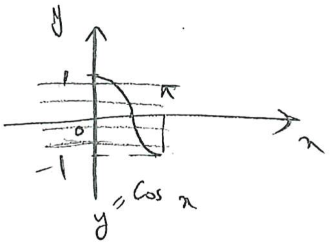
$$\Rightarrow \underbrace{-\bar{u} \cdot \bar{v} \leq |\bar{u}||\bar{v}|}, \quad (**)$$

$$\xrightarrow[\text{and } (**)]{\text{from } (*)} \underbrace{|\bar{u} \cdot \bar{v}| \leq |\bar{u}||\bar{v}|},$$

#

Consequence 1: From the Cauchy-Schwarz inequality, we know that $\frac{|\bar{u}\cdot\bar{v}|}{|\bar{u}||\bar{v}|} \le 1$. It means $A = \frac{\bar{u}\cdot\bar{v}}{|\bar{u}||\bar{v}|}$ is a number between (-1) and (1).

On the other side $\cos\theta$ is an invertible function on $[0,\pi]$, So, for every number between (-1) and (1) like A there exists a unique angle $\theta \in [0,\pi]$ such that $A = \cos\theta$.



$$\Rightarrow \frac{\bar{u}\cdot\bar{v}}{|\bar{u}||\bar{v}|} = \cos\theta \qquad \theta\in[0,\pi]$$

$$\Rightarrow \bar{u}\cdot\bar{v} = |\bar{u}|\,|\bar{v}|\cos\theta$$ for every vectors $\bar{u},\bar{v}\in\mathbb{R}^n$

Consequence 2: The triangle inequality

$$|\bar{u}+\bar{v}| \le |\bar{u}| + |\bar{v}| \qquad (\bar{u},\bar{v}\in\mathbb{R}^n)$$

Proof:
$$
\begin{aligned}
|\bar{u}+\bar{v}|^2 = (\bar{u}+\bar{v})\cdot(\bar{u}+\bar{v}) &= \bar{u}\cdot\bar{u} + \bar{v}\cdot\bar{u} + \bar{v}\cdot\bar{v} + \bar{u}\cdot\bar{v} \\
&= |\bar{u}|^2 + 2\,\bar{u}\cdot\bar{v} + |\bar{v}|^2 \\
&\le |\bar{u}|^2 + 2|\bar{u}||\bar{v}| + |\bar{v}|^2 \\
&= (|\bar{u}| + |\bar{v}|)^2
\end{aligned}
$$

$$\Rightarrow \boxed{|\bar{u}+\bar{v}| \le |\bar{u}| + |\bar{v}|.}$$

#

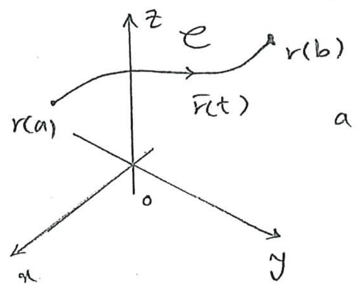
Vector valued functions (12.1)

A vector-valued function is a mathematical function of one or more variables whose range is a set of multidimensional vectors or infinite dimensional vectors.

Here, we discuss vector-valued functions of a single variable:

$$\bar{r} : \mathbb{R} \longrightarrow \mathbb{R}^3 \qquad (\text{Curves in } \mathbb{R}^3)$$

$$\bar{r}(t) = (x(t), y(t), z(t)) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$$



$a \leq t \leq b$

It is natural to interpret $\bar{r}(t)$ as giving the position at time t of a particle moving around in space.

If $z(t) = 0$, then the curve is in the xy-plane.

$\bar{r}(t)$ is continuous $\Longleftrightarrow$ $\left\{\begin{matrix} x(t) \\ y(t) \\ z(t) \end{matrix}\right.$ are continuous.

Velocity: $\bar{v}(t) = \frac{d}{dt}\bar{r}(t) = (x'(t), y'(t), z'(t)) = r'(t)$

Speed: $v(t) = |\bar{v}(t)|$

Acceleration: $\bar{a}(t) = \frac{d\bar{v}}{dt} = \frac{d^2\bar{r}}{dt^2} = r''(t)$

Theorem. Let $\bar{u}(t)$ and $\bar{v}(t)$ be differentiable vector-valued functions and let $\lambda(t)$ be a differentiable scalar-valued function. Then $\bar{u}(t)+\bar{v}(t)$, $\lambda(t)\bar{u}(t)$, $\bar{u}(t)\cdot\bar{v}(t)$, $\bar{u}(t)\times\bar{v}(t)$, and $\bar{u}(\lambda(t))$ are differentiable, and

(a) $\frac{d}{dt}(\bar{u}(t)+\bar{v}(t)) = \bar{u}'(t)+\bar{v}'(t)$

Product rules $\rightarrow$

(b) $\frac{d}{dt}(\lambda(t)\ \bar{u}(t)) = \lambda'(t)\bar{u}(t)+\lambda(t)\bar{u}'(t)$

(c) $\frac{d}{dt}(\bar{u}(t)\cdot\bar{v}(t)) = \bar{u}'(t)\cdot\bar{v}(t)+\bar{u}(t)\cdot\bar{v}'(t)$

(d) $\frac{d}{dt}(\bar{u}(t)\times\bar{v}(t)) = \bar{u}'(t)\times\bar{v}(t)+\bar{u}(t)\times\bar{v}'(t)$

Chain rule $\rightarrow$ (e) $\frac{d}{dt}(\bar{u}(\lambda(t))) = \lambda'(t)\ u'(\lambda(t))$

Also, at any point where $\bar{u}(t)\neq 0$:

$$\frac{d}{dt}|\bar{u}(t)| = \frac{\bar{u}(t)\cdot\bar{u}'(t)}{|\bar{u}(t)|}$$

Example. $\bar{u}(t)=(2t,3t,t^2)$ If $t\neq 0 \Rightarrow \bar{u}(t)\neq 0$

$\Rightarrow$ For $t\in\mathbb{R}\setminus\{0\}$: $\frac{d}{dt}|\bar{u}(t)| = \frac{\bar{u}(t)\cdot\bar{u}'(t)}{|\bar{u}(t)|}$

$|\bar{u}(t)| = \sqrt{4t^2+9t^2+t^4} = \sqrt{13t^2+t^4} \Rightarrow LS = \frac{d}{dt}|\bar{u}(t)| = \frac{26t+4t^3}{2\sqrt{13t^2+t^4}} = \underline{\frac{13t+2t^3}{\sqrt{13t^2+t^4}}}$

$\bar{u}(t)\cdot\bar{u}'(t) = (2t,3t,t^2)\cdot(2,3,2t) = 4t+9t+2t^3 = 13t+2t^3$

$$\Rightarrow RS = \frac{\bar{u}(t)\cdot\bar{u}'(t)}{|\bar{u}(t)|} = \frac{13t + 2t^3}{\sqrt{13t^2 + t^4}} \Rightarrow RS = LS$$

Example. Show that the speed of a moving particle remains constant over an interval of time $\Longleftrightarrow$ the acceleration is perpendicular to the velocity throughout that interval.

We have:

$$\frac{d}{dt}(\bar{v}(t)\cdot\bar{v}(t)) = \bar{v}'(t)\cdot\bar{v}(t) + \bar{v}(t)\cdot\bar{v}'(t)$$

$$= 2\,\bar{v}'(t)\cdot\bar{v}(t) = 2\,a(t)\cdot\bar{v}(t)$$

$\Rightarrow$ $\bar{a}(t)$ is perpendicular to the velocity $\Longleftrightarrow \bar{a}(t)\cdot\bar{v}(t) = 0$

$\Longleftrightarrow \frac{d}{dt}(\bar{v}(t)\cdot\bar{v}(t)) = 0 \Longleftrightarrow \frac{d}{dt}(|v(t)|^2) = 0$

$\Longleftrightarrow |v(t)|$ is constant. (Speed is constant).

Arc length

We calculate the length of a curve by approximation.
Approximation by a polygon path.

$t_0 = a < t_1 < t_2 < \cdots < t_{n-1} < t_n = b \qquad \bar{r}(t_j) = \bar{r}_j$

Polygon length: $\quad S_n = \sum_{j=1}^{n} |\bar{r}_j - \bar{r}_{j-1}| = \sum_{j=1}^{n} |\Delta \bar{r}_j|$

A curve is rectifiable if there exists a constant $A > 0$ such that $S_n \leq A$ for all $\{t_j\}_{j=0}^{n}$ and all n. The smallest possible $A > 0$ is the length s of the curve.

$$S_n = \sum_{j=1}^{n} |\Delta \bar{r}_j| = \sum_{j=1}^{n} \left| \frac{\Delta \bar{r}_j}{\Delta t_j} \right| \cdot \Delta t_j$$

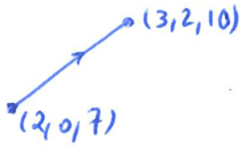
$\Delta t_j = t_j - t_{j-1}$

$$S = \lim_{\substack{n \to \infty \\ \max \Delta t_j \to 0}} S_n = \int_a^b \left| \frac{d\bar{r}}{dt} \right| dt = \int_a^b |\bar{v}(t)|\, dt = \int_{\mathcal{C}} ds, \text{ where}$$

$\mathcal{C}$ is the curve, and $ds = \left| \frac{d\bar{r}}{dt} \right| dt$ is the arc length element.

$*$ S does not depend on parametrization of the curve.

Example. We have a line along the intersection of the planes $y = 2x - 4$, $z = 3x + 1$. Find the length of the curve (line) by two parametrizations. (A line is a straight curve.)



Parametrization 1: We set $y = t$.

$\Rightarrow x = \frac{t+4}{2} = 2 + \frac{t}{2}$ and $z = 3(2 + t/2) + 1 = 7 + \frac{3}{2} t$

$$\Rightarrow \bar{r}(t) = (x(t), y(t), z(t)) = \left(2+\frac{t}{2}, t, 7+\frac{3}{2}t\right)$$

$$= (2, 0, 7) + t\left(\frac{1}{2}, 1, 3/2\right) \qquad \underline{0 \le t \le 2}$$

Parametrization 2: We set $x = t$

$$\Rightarrow y = 2t - 4 \quad \text{and} \quad z = 3t + 1$$

$$\Rightarrow \bar{r}(t) = (t, 2t-4, 3t+1) = (0, -4, 1) + t(1, 2, 3) \qquad \underline{2 \le t \le 3}$$

- Parametrization of a curve is not unique.

Arc length:

① $\bar{r}(t) = (2 + t/2, t, 7 + 3/2\, t) \qquad 0 \le t \le 2 \qquad \frac{d\bar{r}}{dt} = (1/2, 1, 3/2)$

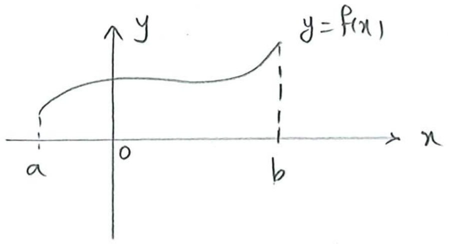
$$S = \int_0^2 \left|\frac{d\bar{r}}{dt}\right| dt = \frac{\sqrt{14}}{2}\int_0^2 dt = \underline{\underline{\sqrt{14}}}.$$

② $\bar{r}(t) = (t, 2t-4, 3t+1) \qquad 2 \le t \le 3 \qquad \frac{d\bar{r}}{dt} = (1, 2, 3)$

$$S = \int_2^3 \left|\frac{d\bar{r}}{dt}\right| dt = \sqrt{14}\int_2^3 dt = \underline{\underline{\sqrt{14}}}.$$

---

* A particular curve: The graph of a function.



$$\bar{r}(x) = (x, f(x)) \qquad a \le x \le b$$

The arc length: $\displaystyle\int_a^b \left|\frac{d\bar{r}}{dx}\right| dx$

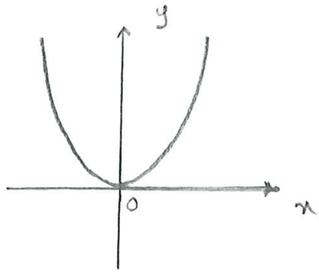
$$= \int_a^b |(1, f'(x))|\, dx$$

$$= \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

the length of the graph of a function.

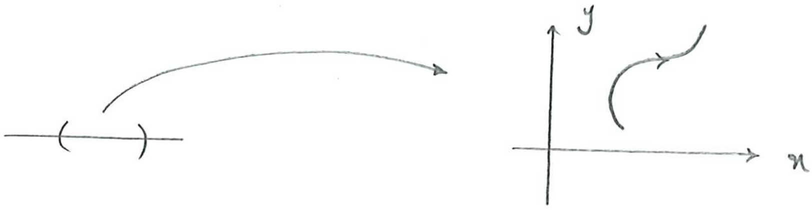
Functions of several variables (13.1)

Known concepts:



Functions $\mathbb{R} \to \mathbb{R}$, e.g. $f(x) = x^2$

Vector-valued functions $\mathbb{R} \to \mathbb{R}^2$, e.g. $f(x) = (x^2, e^x)$



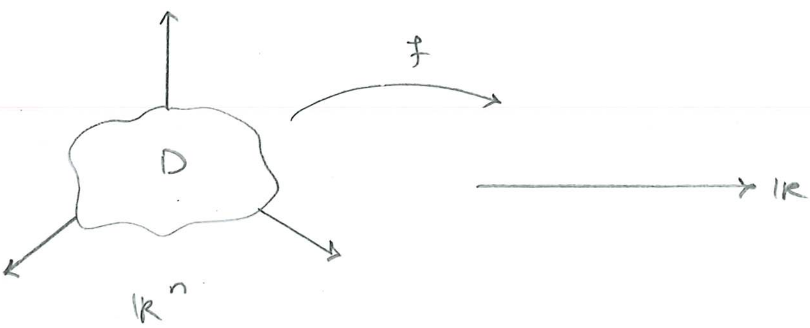
parametrized curve

New concept:

Let $D \subseteq \mathbb{R}^n$. A function of several variables f is a map $f: D \to \mathbb{R}$.

D is called domain.

The set $\{f(\vec{x}), \vec{x} \in D\} \subseteq \mathbb{R}$ is called the range of f.



Mostly we study $n = 2$ or $n = 3$.

Examples: $f(x,y) = x^2 + 3y^2$, $D = \mathbb{R}^2$

$g(x,y) = 2x\sqrt{y}$, $D = \{(x,y) : x \in \mathbb{R}, y \geq 0\}$

$z = f(x,y)$

z: dependent variable

x, y: independent variables

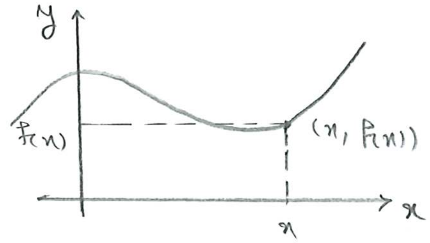
Notation: $n = 2$: $f(x,y)$

$n = 3$: $f(x,y,z)$

n general: $f(x_1, x_2, \ldots, x_n)$

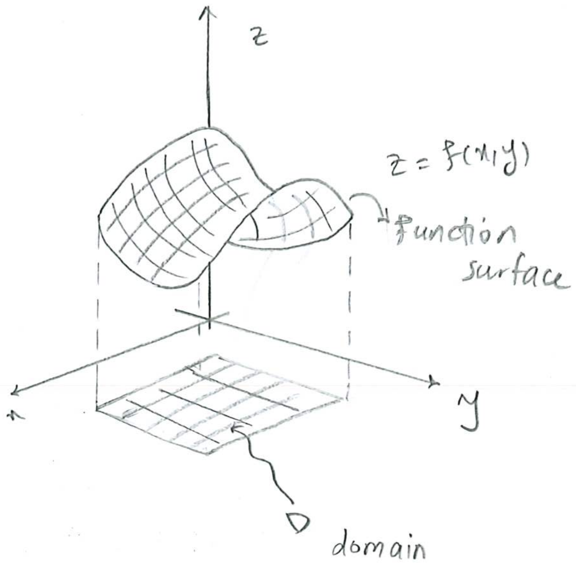
<u>Graphs</u>.

The graph of a function $f: \mathbb{R} \longrightarrow \mathbb{R}$ is $\{(x, f(x)), x \in D\} \subseteq \mathbb{R}^2$.

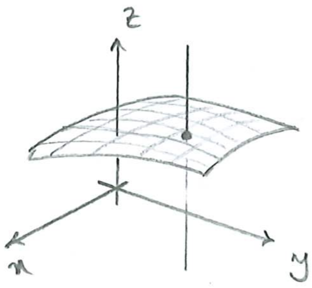


Similarly for $f: \mathbb{R}^n \longrightarrow \mathbb{R}$, the graph is defined by $\{(\underbrace{\vec{x}}_{\in \mathbb{R}^n}, \overbrace{f(\vec{x})}^{\in \mathbb{R}}), \vec{x} \in D\} \subseteq \mathbb{R}^{n+1}$.

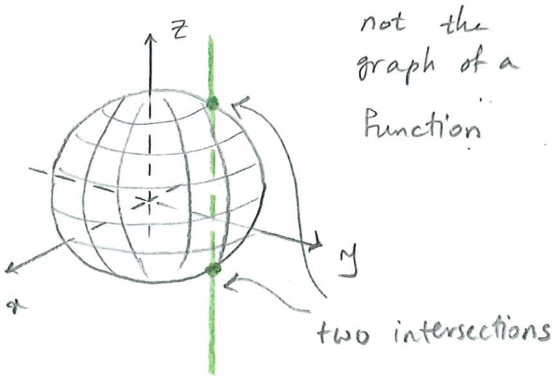
For $n=2$, the graph $\{(x, y, f(x,y)) : (x,y) \in D\} \subseteq \mathbb{R}^3$ can be visualized as a <u>surface</u>.



Each line perpendicular to the xy-plane intersects the surface in at most <u>one</u> point.



<u>function surface</u>



<u>two functions</u>

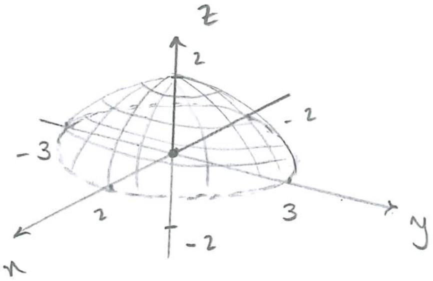
Example: $f(x,y) = \frac{1}{3}\sqrt{36-9x^2-4y^2}$ (Is it a function?)

$z = f(x,y) \geq 0$ $\qquad z^2 = \frac{1}{9}(36-9x^2-4y^2)$

$\Rightarrow 9z^2 = 36-9x^2-4y^2 \Rightarrow 9x^2+4y^2+9z^2 = 36$

$\Rightarrow \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{4} = 1$ ellipsoid

The graph of f is the upper half of the ellipsoid. $f(x,y)$ is a function.



Example. What is the domain for $f(x,y) = \frac{\sqrt{y-x^2}}{x^2+(y-1)^2}$ ≥ 0 ? $\neq 0$

We must exclude $\{(x,y): y<x^2\}$ and $(x,y)=(0,1)$ from $\mathbb{R}^2$.