L5 Matrix arithmetic and matrix algebra

1MA901/1MA406 Linear algebra

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Engelsk-svensk ordlista

English	**Swedish**
Matrix	Matris
Vector	Vektor
Transpose	Transponat
Identity matrix	Enhetsmatris
Singular matrix	Singulär matris
Nonsingular matrix	Inverterbar matris

Matrix notation

Throughout we will use capital letters to denote matrices, such as the following $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

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Matrices consisting of only one row or one column is of special interest and are called *vectors*. For a vector with n entries we may specify whether it is a *row vector* or *column vector* depending on whether it is a $1 \times n$ vector or $n \times 1$ vector respectively.

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Following the book, and standard notation, we use boldface lowercase letters to denote column vectors such as $\mathbf{u}$, and $\vec{v}$ for row vectors.

Matrices

For an $m \times n$ matrix A we denote by $\mathbf{a}_j$ its jth column vector which is

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

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Using this notation we may write

$$A = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n) = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_m \end{pmatrix}.$$

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Definition

Two matrices A and B are said to be equal if $a_{ij} = b_{ij}$ for every i and j.

Definition

Let $\alpha \in \mathbb{R}$, and A a matrix then we define by αA the matrix (αa_{ij}), *i.e.* we multiply each entry in A by α.

Matrix arithmetic

Definition

Given two matrices $A = (a_{ij})$ and $B = (b_{ij})$, which both are $m \times n$ matrices, then their sum $A + B$ is equal to the $m \times n$ matrix such that its (i, j) entry is equal to $a_{ij} + b_{ij}$.

As an example we have

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 5 & 5 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+2 & 2-1 \\ 3+5 & 4+5 \\ 5+0 & 6+1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 8 & 9 \\ 5 & 7 \end{pmatrix}.$$

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Example

Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -3 & 0 \\ 2 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Compute, if possible, $A + B$, $B + C$.

Matrix arithmetic

We note that if O is an $m \times n$ matrix consisting of only zeros, and A is an $m \times n$ matrix then
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For each matrix A there exists another matrix B such that $A + B = O$. The matrix B is called the additive inverse of A. Note that

$$A + (-1)A = O = (-1)A + A.$$

The additive inverse is thus obtained by changing sign in each element.

Linear combination

Definition

Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be vectors in $\mathbb{R}^m$, and $c_1, \cdots c_n$ scalars, then a sum of the form

$$c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n$$

is called a *linear combination* of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$.

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Example

$$5\begin{pmatrix}3\\2\\1\end{pmatrix} - 2\begin{pmatrix}1\\0\\-1\end{pmatrix} + 7\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}15-2+7\\10-0+7\\5+2+7\end{pmatrix} = \begin{pmatrix}20\\17\\14\end{pmatrix}$$

The vector $\begin{pmatrix}20\\17\\14\end{pmatrix}$ is a linear combination of the vectors

$$\begin{pmatrix}3\\2\\1\end{pmatrix}, \quad \begin{pmatrix}1\\0\\-1\end{pmatrix}, \quad \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

Example

Is $\begin{pmatrix}1\\2\end{pmatrix}$ a linear combination of the vectors $\begin{pmatrix}0\\5\end{pmatrix}$ and $\begin{pmatrix}4\\1\end{pmatrix}$?

Matrix multiplication

Inspired by the linear equation $ax = b$, we wish to define matrix-vector multiplication in such a way that $A\mathbf{x} = \mathbf{b}$ defines the system of linear equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}.$$

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Recall that A may be represented as a row vector of columns vectors $A = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$. We define the matrix-vector multiplication as

$$A\mathbf{x} := x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n,$$

where the resulting sum is a $m \times 1$ column vector.

We have defined the matrix-vector multiplication. Note that this is only valid if the number of columns in A coincide with the number of rows in $\mathbf{x}$.

Theorem (Consistency theorem for linear system of equations)

A linear system $A\mathbf{x} = \mathbf{b}$ *is consistent if and only* $\mathbf{b}$ *can be written as a linear combination of the column vectors of* A*.*

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Example

Compute $A\mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & -1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

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Solution.

$$A\mathbf{b} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1+4-3 \\ 4+0+1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$\square$

We extend the definition of matrix-vector multiplication to matrix-matrix multiplication. Assume that A is an $(m \times n)$ matrix, and $B = (\mathbf{b}_1, \ldots, \mathbf{b}_p)$ is $(n \times p)$ matrix. Then we let

$$AB := (A\mathbf{b}_1, A\mathbf{b}_2, \ldots, A\mathbf{b}_p).$$

For matrix multiplication to be well-defined we require that the number of columns of A is equal to the number of rows in B.

Examples

Example

Compute (if possible) AB, BA where

$$A = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 3 & 9 \\ 1 & -2 & 3 \end{pmatrix}.$$

Solution.

The number of columns in A is equal to the number of rows in B so it's well-defined. The same is not true for the product BA.

$$AB = \left(A\begin{pmatrix} 4 \\ 1 \end{pmatrix}, A\begin{pmatrix} 3 \\ -2 \end{pmatrix}, A\begin{pmatrix} 9 \\ 3 \end{pmatrix} \right)$$

We have

$$A\begin{pmatrix} 4 \\ 1 \end{pmatrix} = 4\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} 11 \\ -1 \end{pmatrix}, \quad A\begin{pmatrix} 3 \\ -2 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 13 \end{pmatrix},$$

and

$$A\begin{pmatrix} 9 \\ 3 \end{pmatrix} = 9\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} 27 \\ -6 \end{pmatrix}.$$

Hence, we obtain

$$AB = \begin{pmatrix} 11 & 0 & 27 \\ -1 & 13 & -6 \end{pmatrix}.$$

□

Matrix multiplication

Another way of writing out the matrix multiplication is as follows. This definition is equivalent to the previous.

Definition

If $A = (a_{ij})$ is $m \times n$ and $B = (b_{ij})$ is $n \times p$, then the product $AB = C$ is the $m \times p$ matrix with entries

$$c_{ij} = \vec{a}_i \mathbf{b}_j = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

We further note that $\vec{a}_i \mathbf{b}_j$ is simply a sum of the element-wise multiplications in $\vec{a}_i$ and $\mathbf{b}_j$.

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Example

Let A and B be 2×2 matrices defined by

$$A = \begin{pmatrix} 2 & 2 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} \vec{a}_1 \mathbf{b}_1 & \vec{a}_1 \mathbf{b}_2 \\ \vec{a}_2 \mathbf{b}_1 & \vec{a}_2 \mathbf{b}_2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + 2 \cdot 0 & 2 \cdot (-1) + 2 \cdot 3 \\ 1 \cdot 1 + 0 \cdot 0 & 1 \cdot (-1) + 0 \cdot 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix}.$$

Matrix transpose

Definition

The transpose of a $m \times n$ matrix A is the $n \times m$ matrix B defined by $b_{ji} = a_{ij}$ for all $j \in \{1, \ldots, n\}$ and all $i \in \{1, \ldots, m\}$. We denote the transpose of a matrix by A^T.

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Example

Let

$$A = \begin{pmatrix} 1 & 6 & 7 \\ 4 & 3 & 1 \end{pmatrix}.$$

Then

$$A^T = \begin{pmatrix} 1 & 4 \\ 6 & 3 \\ 7 & 1 \end{pmatrix}.$$

If a matrix equals its transpose, i.e. $A = A^T$. Then A is said to be *symmetric*.

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Let A and B be matrices of size $m \times n$ and C of size $n \times p$. Then the following rules hold

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(AC)^T = C^T A^T$.

Example

Note that for a column vector $\mathbf{c}$, we have $\mathbf{c}^T = \vec{c}$.

Matrix algebra

The following result gives the main rules of matrix algebra

Theorem

Each of the following statements is valid for any scalars α and β, and for any matrices A, B and C for which the indicated operations are defined.

1. $A + B = B + A$
2. $(A + B) + C = A + (B + C)$
3. $(AB)C = A(BC)$
4. $A(B + C) = AB + AC$
5. $(A + B)C = AC + BC$
6. $(\alpha\beta)A = \alpha(\beta A)$
7. $\alpha(AB) = (\alpha A)B = A(\alpha B)$
8. $(\alpha + \beta)A = \alpha A + \beta A$
9. $\alpha(A + B) = \alpha A + \alpha B.$

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Example

Find all matrices that commutes with

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

Proof of $(AB)C = A(BC)$

A proof is found on page 64 in the course book.

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This further means that we may drop the parenthesis all together and we have

$$A(BC) = (AB)C = ABC.$$

Further if $A = B = C$ it is meaningful to use exponential notation $AAA = A^3$. Hence we have

$$A \cdot A \cdots A = A^k,$$

for A multiplied by itself k times.

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For instance we have for $n = 2$ and $n = 4$ we have

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

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If it is clear by context we omit the subscript.

The identity matrix serves as the neutral element for matrix multiplication. Let A be an $n \times n$ matrix. Then

$$AI = IA = A.$$

Note that besides being a diagonal matrix, the identity matrix is also what is known as a *symmetric matrix*. We say that a matrix A is symmetric if $A = A^T$.

The multiplicative inverse of a matrix

Let A be a square matrix is size $n \times n$. The multiplicative inverse of A, if it exists, is a matrix B such that

$$AB = BA = I.$$

If such a matrix B exists then A is said to be *nonsingular* or *invertible*. The matrix B is usually denoted by A^{-1}.

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Lemma

The multiplicative inverse of a matrix is unique.

Proof.

Assume that A is nonsingular and we have two multiplicative inverses B and C. Then we have

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Hence, we conclude that $B = C$. $\square$

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In contrast to the nonsingular case, we say that a matrix A without a multiplicative inverse (or simply inverse for short) is said to be *singular* or *noninvertible*.

For non-square matrices we may not define multiplicative inverses in the same way. Although, we may construct either left- or right inverses. However, for this course we restrict to study inverses for square matrices.

Product of nonsingular matrices

Theorem

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We have

$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I,$$

and

$$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

This completes the proof of the theorem. □

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Note that this may be extended inductively to any number of nonsingular matrices as

$$(A_1A_2\cdots A_j)^{-1} = A_j^{-1}\cdots A_2^{-1}A_1^{-1}.$$

Preparation for next exercise session: Read and do the recommended exercises.
Next lecture we consider so-called elementary matrices and how to find the inverse of a matrix.
Thank you for today!