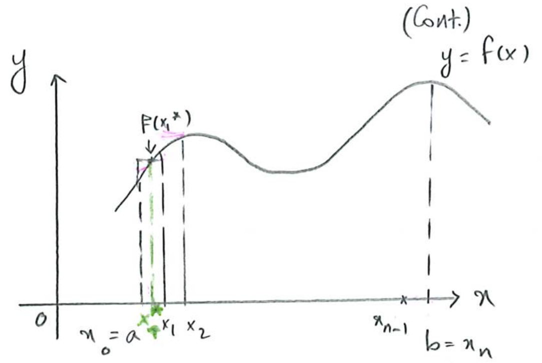
Multiple integration

Recall: Integration in one variable:



Partition: $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$

$\Delta x_i = x_i - x_{i-1}$

$x_i^* \in [x_{i-1}, x_i]$

$I_0 = [a,b]$

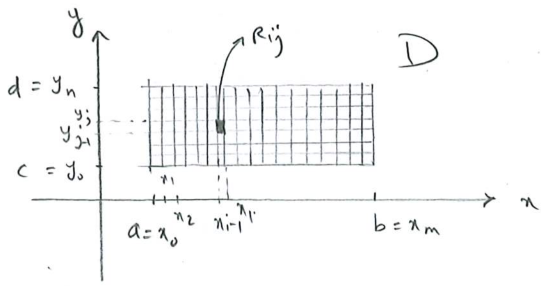
$$\int_a^b f(x)\,dx = \lim_{\max(\Delta x_i) \to 0} \underbrace{\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i}_{\text{Riemann sum}} = \text{area under } f$$

Now, consider $f: \mathbb{R}^2 \to \mathbb{R}$.

Take a rectangle: $D = [a,b] \times [c,d]$ and a partition:

Partition P: $\begin{cases} a = x_0 < x_1 < \cdots < x_{m-1} < x_m = b \\ c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d. \end{cases}$

$R_{ij} = \{ (x,y) \in \mathbb{R}^2 : x_{i-1} < x < x_i \ , \ y_{j-1} < y < y_j \}$ $\quad 1 \le i \le m, \ 1 \le j \le n.$



$\Delta x_i = x_i - x_{i-1}$

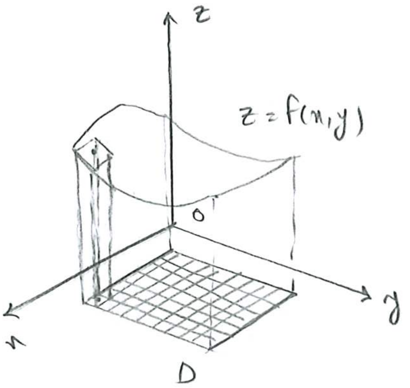
$\Delta y_j = y_j - y_{j-1}$

$|P| = \max_{i,j} \text{diam } R_{ij} = \sqrt{(\Delta x_i)^2 + (\Delta y_j)^2}$

We pick $(x_{ij}^*, y_{ij}^*) \in R_{ij}$.

$\Rightarrow$ Riemann sum: $$R(f,P) = \sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{f(x_{ij}^*, y_{ij}^*)\,\Delta x_i \,\Delta y_j}_{\text{volume of parallelepiped}}$$

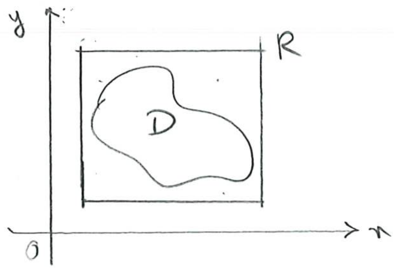
Approximate the volume under f over D.



<u>Definition</u>. f is integrable (Riemann integrable) over D with integral $I \in \mathbb{R}$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|R(f,p) - I| < \varepsilon$ when $|p| < \delta$.

We write $I = \iint_D f(x,y)\, dA = \iint_D f(x,y)\, dx\, dy$.

** If $D \subseteq \mathbb{R}^2$ is bounded, then $D \subseteq R$ = rectangle.

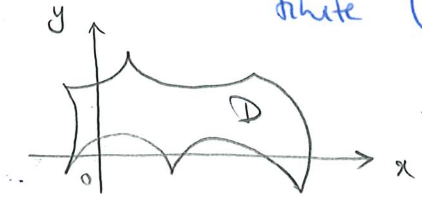


. f is integrable over (on) D if $f\chi_D$ is integrable over R.

$$\chi_D(x,y) = \begin{cases} 1, & (x,y) \in D \\ 0, & (x,y) \notin D \end{cases}$$ (indicator function) (characteristic function)

The function χ_D is sometimes denoted by I_D, 1_D or K_D.

<u>Theorem</u>. If f is continuous on a closed and bounded domain D, whose boundary consists of finitely many curves of finite length, then f is integrable over D.



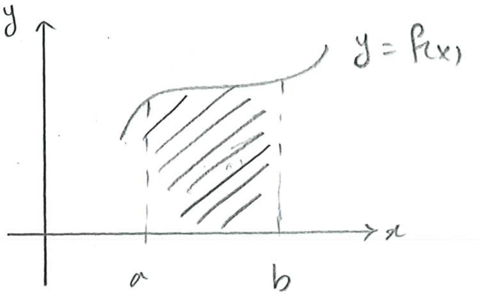
Comparison with single integrals:

$\int_a^a f(x)\,dx = 0$

$\int_a^b dx = b-a = \text{length of } [a,b]$

$f(x) \geq 0 \Rightarrow \int_a^b f(x)\,dx \geq 0$

is the area between $y=0$ and $y=f(x)$ and the lines $x=a$ and $x=b$.

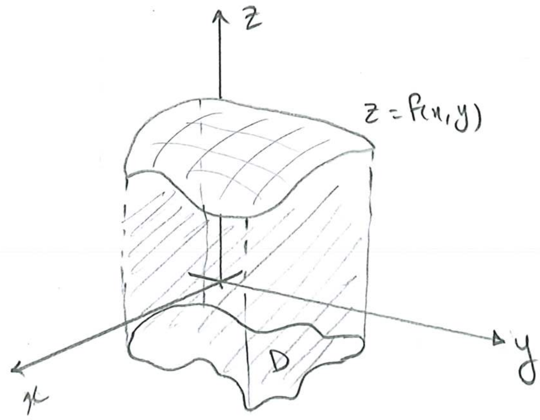


Area$(D)=0 \Rightarrow \iint_D f(x,y)\,dA = 0$

$\iint_D dA = \text{area } (D)$

If $f(x,y) \geq 0$ then

$\iint_D f(x,y)\,dA \geq 0$ is the volume of solid between $z=0$ and $z=f(x,y)$ over D.



Properties of double integrals:

Linearity: $\iint_D \left(a f(x,y) + b g(x,y)\right) dA = a \iint_D f(x,y)\,dA + b \iint_D g(x,y)\,dA$

$(a, b \in \mathbb{R})$

Preservation of inequality: (monotonic property)

$f(x,y) \leq g(x,y), \quad (x,y) \in D \longrightarrow \iint_D f(x,y)\,dA \leq \iint_D g(x,y)\,dA.$

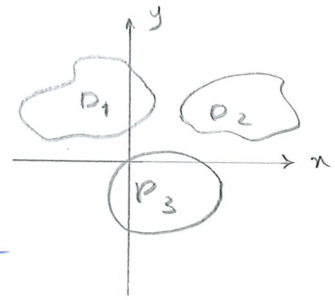
Triangle inequality:

$$\left| \iint_D f(x,y)\, dA \right| \le \iint_D |f(x,y)|\, dA$$

Additivity over domains:

If: $D_1, D_2, \dots, D_k$ pairwise disjoint ($D_i \cap D_j = \varphi$, $i \neq j$), $D = D_1 \cup D_2 \cup \dots \cup D_k$.

then: $\Longrightarrow \displaystyle\iint_D f(x,y)\, dA = \sum_{j=1}^{k} \iint_{D_j} f(x,y)\, dA.$

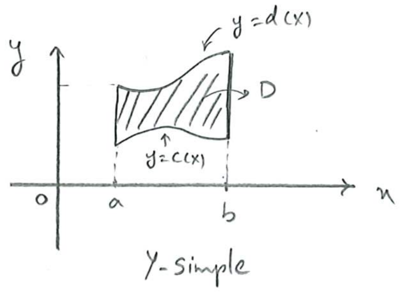


Iteration of double integrals

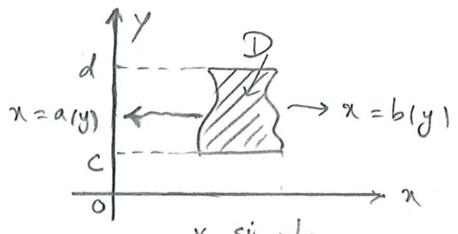
We say that domain D in the xy-plane is y-simple if it is bounded by two vertical lines $x=a$ and $x=b$ and two continuous graphs $y=c(x)$ and $y=d(x)$ between these lines.

We have the similar definitions for x-simple domain.

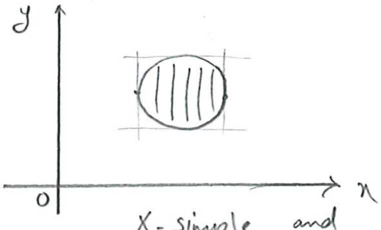
Rectangles, triangles and disks are both y-simple and x-simple.



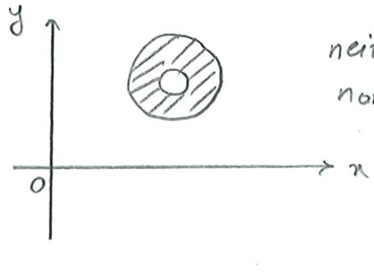
Y-simple



x-simple

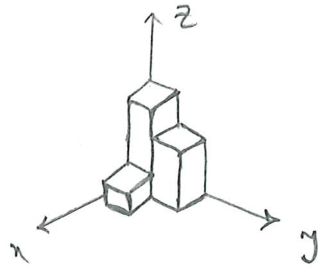


X-simple and y-simple



neither x-simple nor y-simple

* Let $\Phi(x,y) = c_{ij}$, $(x,y) \in R_{ij}$, $1 \le i \le n$, $1 \le j \le m$, be a <u>step</u> function on $D = [a,b] \times [c,d]$.



$$\iint_D \Phi(x,y)\, dA = \sum_{i,j} c_{ij}\, \Delta x_i\, \Delta y_j$$

$$= \sum_i \Delta x_i \sum_j c_{ij}\, \Delta y_j$$

$\Delta x_i = x_i - x_{i-1}$

$\Delta y_j = y_j - y_{j-1} = \int_{y_{j-1}}^{y_j} dy$

$$= \sum_i \Delta x_i \sum_j \int_{y_{j-1}}^{y_j} c_{ij}\, dy$$

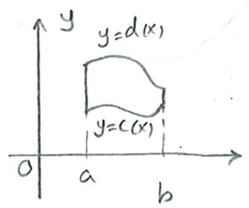
$$= \sum_i \left(\int_{x_{i-1}}^{x_i} \sum_j c_{ij} \int_{y_{j-1}}^{y_j} dy \right) dx = \int_a^b \left(\int_c^d \Phi(x,y)\, dy \right) dx$$

$$= \underbrace{\int_c^d}_{\text{outer integral}} \Big(\underbrace{\int_a^b \Phi(x,y)\, dx}_{\text{inner integral}} \Big)\, dy \longrightarrow \text{iterated integrals}$$

Using approximation techniques, this leads to:

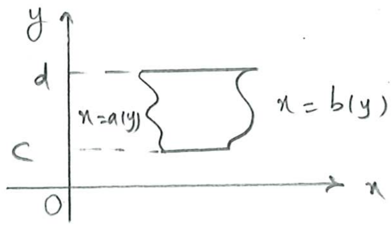
<u>Theorem</u>. If D is y-simple and f is continuous, then:

$$\iint_D f(x,y)\, dA = \int_a^b \left(\int_{c(x)}^{d(x)} f(x,y)\, dy \right) dx$$

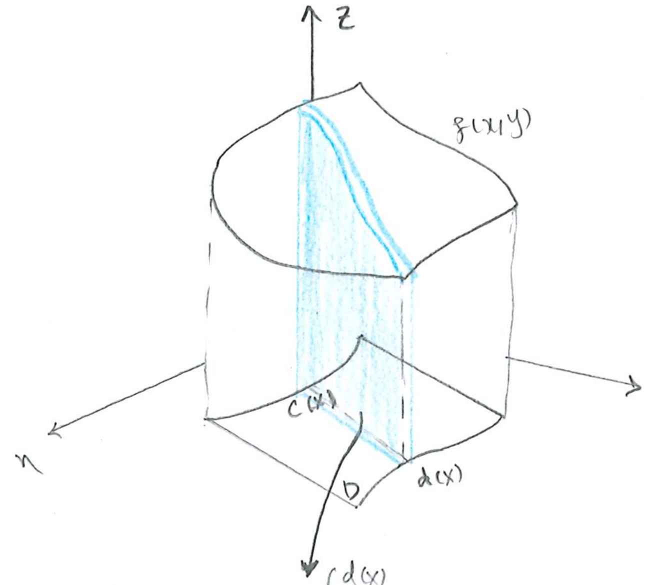


Correspondingly, for x-simple domains:

$$\iint_D f(x,y)\, dA = \int_c^d \left(\int_{a(y)}^{b(y)} f(x,y)\, dx \right) dy$$



The formula is usually called : "Bread slicing formula".

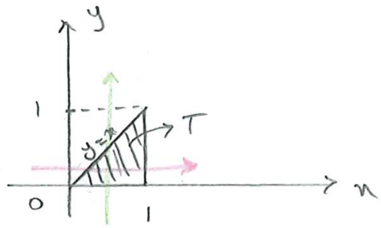


$$\int_{c(x)}^{d(x)} f(x,y)\,dy = A(x) = \text{area of "slice"}$$

$$\int_a^b A(x)\,dx = \int_a^b \left(\int_{c(x)}^{d(x)} f(x,y)\,dy \right) dx = \iint_D f(x,y)\,dA = \text{volume}.$$

Example.

$$I = \iint_T xy\,dx\,dy$$



$T: \begin{cases} 0 \le x \le 1 \\ 0 \le y \le x \end{cases}$ y-simple

and

$T: \begin{cases} 0 \le y \le 1 \\ y \le x \le 1 \end{cases}$ x-simple

$$\Rightarrow I = \int_0^1 \left(\int_0^x xy\,dy \right) dx$$

$$= \int_0^1 x \left[\frac{y^2}{2} \right]_{y=0}^{y=x} dx = \int_0^1 \frac{x^3}{2}\,dx = \frac{1}{2} \left[\frac{x^4}{4} \right]_0^1 = \frac{1}{8}.$$

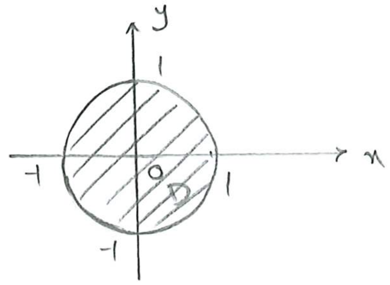
or $$I = \int_0^1 \left(\int_y^1 xy\,dx \right) dy = \int_0^1 y \left[\frac{x^2}{2} \right]_{x=y}^{x=1} dy = \frac{1}{2} \int_0^1 (y - y^3)\,dy$$

$$= \frac{1}{2} \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 = \frac{1}{2} \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{1}{8}.$$

Example. $$\iint_{x^2+y^2\le 1} (x+y^3+2)\, dA = \iint_D x\, dA + \iint_D y^3\, dA + 2\iint_D dA$$

$D: x^2+y^2 \le 1$

$$= 0 + 0 + 2\ \text{area}(D) = 2\pi .$$



$\iint_D x\, dA = 0 \to$ Since x is an odd function and D is symmetric with respect to x.

$\iint_D y^3\, dA = 0 \to$ Since y^3 is an odd function and D is symmetric with respect to y.

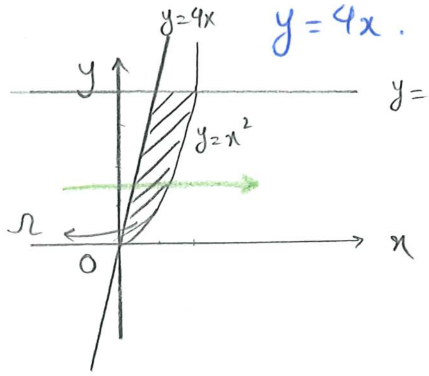
Recall: $f: \mathbb{R} \to \mathbb{R}$ is an odd function if $f(-x) = -f(x)$ for every $x \in D_f$.

We have: $\int_{-a}^{a} f(x)\, dx = 0$ when f is an odd function. $(a \in \mathbb{R})$

In the example above we have:

$$\iint_D x\, dA = \int_{y=-1}^{1} \underbrace{\left(\int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x\, dx \right)}_{=0} dy = 0 .$$

Example. Calculate the area of $\Omega \subseteq \mathbb{R}^2$ limited by: $y=x^2$, $y=4$, $y=4x$.



$$\text{Area}(\Omega) = \iint_\Omega dA = \iint_\Omega dx\,dy$$

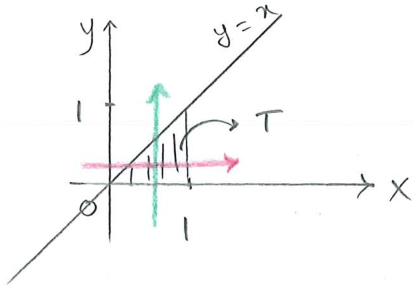
$$= \int_{y=0}^{4} \left(\int_{x=y/4}^{\sqrt{y}} dx \right) dy$$

$$= \int_0^4 \left(\sqrt{y} - \frac{y}{4} \right) dy$$

$$= \left[\frac{2}{3} y^{\frac{3}{2}} - \frac{y^2}{8} \right]_0^4 = \frac{10}{3}.$$

Example. Calculate the integral of $F(x,y) = e^{-x^2}$ over $T \subseteq \mathbb{R}^2$.

$$T = \{ (x,y) \in \mathbb{R}^2 : \ 0 \le x \le 1, \ 0 \le y \le x \}$$
$$= \{ (x,y) \in \mathbb{R}^2 : \ 0 \le y \le 1, \ y \le x \le 1 \}.$$



$$I = \iint_T e^{-x^2} dx\,dy = \int_{y=0}^{y=1} \underbrace{\left(\int_{x=y}^{x=1} e^{-x^2} dx \right)}_{\text{This cannot be calculated easily!}} dy$$

We use the other iterated integrals:

$$I = \iint_T e^{-x^2} dx\,dy = \int_{x=0}^{x=1} \left(\int_{y=0}^{y=x} e^{-x^2} dy \right) dx = \int_0^1 e^{-x^2} \left(\int_0^x dy \right) dx$$

$$= \int_0^1 x e^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2} \right]_0^1 = -\frac{e^{-1}}{2} + \frac{1}{2} = \frac{1 - \frac{1}{e}}{2}$$

Triple integrals

* Volume $(\Omega) = \iiint_\Omega dv = \iiint_\Omega dx\,dy\,dz$
  $\Omega \subseteq \mathbb{R}^3$

* if density $= \delta(x,y,z) \implies$ mass $= \iiint_\Omega \delta(x,y,z)\,dv$

Example. $I = \iiint_{\underbrace{x^2+y^2+z^2\le r^2}} (2+x+\sin(z))\,dv = 2\iiint_B dv + \underbrace{\iiint_B (x+\sin(z))\,dv}_{=0}$

B: Ball of radius $r>0$

since x and $\sin(z)$ are odd functions and B is symmetric with respect to x and z.

$$\implies I = 2\,\text{vol}(B) = 2\cdot\frac{4\pi r^3}{3} = \frac{8\pi r^3}{3}.$$

Example. $\iiint (xy^2+z^3)\,dv = \int_0^c\left(\int_0^b\left(\int_0^a (xy^2+z^3)\,dx\right)dy\right)dz$

$0\le x\le a$
$0\le y\le b$
$0\le z\le c$

$$= \int_0^c\left(\int_0^b \left[\frac{x^2y^2}{2}+xz^3\right]_0^a dy\right)dz = \int_0^c\left(\int_0^b\left(\frac{a^2y^2}{2}+az^3\right)dy\right)dz$$

$$= \int_0^c \left[\frac{a^2y^3}{6}+az^3y\right]_0^b dz = \int_0^c\left(\frac{a^2b^3}{6}+abz^3\right)dz$$

$$= \left[\frac{a^2b^3}{6}z+\frac{ab}{4}z^4\right]_0^c = \frac{a^2b^3c}{6}+\frac{abc^4}{4} = \frac{abc}{2}\left(\frac{ab^2}{3}+\frac{c^3}{2}\right).$$

Example.

$$\iiint_{\substack{0\le x\le 1\\0\le y\le 1\\0\le z\le 1}} z^2 y e^{-xyz}\,dV = \int_{z=0}^{z=1}\left(\int_{y=0}^{y=1}\left(\int_{x=0}^{x=1} yz^2 e^{-xyz}\,dx\right)dy\right)dz$$

$$= \int_{z=0}^{z=1}\left(\int_{y=0}^{y=1}\left[-z e^{-xyz}\right]_{x=0}^{x=1} dy\right)dz$$

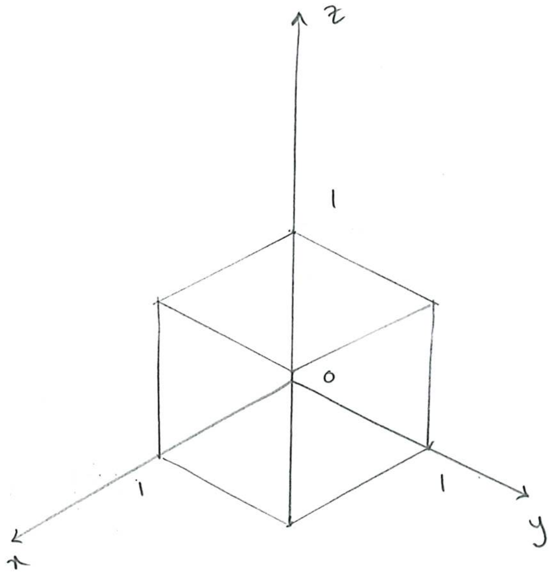
$$= \int_{z=0}^{z=1}\left(\int_{y=0}^{y=1}\left(-z e^{-yz}+z\right)dy\right)dz$$

$$= \int_{z=0}^{z=1}\left[e^{-yz}+zy\right]_{y=0}^{y=1} dz$$

$$= \int_0^1 \left(e^{-z}+z-1\right)dz$$

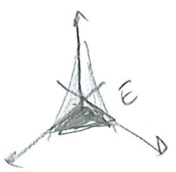
$$= \left[-e^{-z}+\frac{z^2}{2}-z\right]_0^1$$

$$= -e^{-1}+\frac{1}{2}-1+1 = \underline{\frac{1}{2}-e^{-1}} = \underline{\frac{1}{2}-\frac{1}{e}}$$



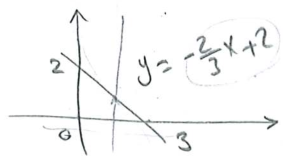
Unit cube

Example. Evaluate $\iiint_E 2x\,dV$ where E is the region under the plane $2x+3y+z=6$ that lies in the first octant.



$0 \le z \le 6-2x-3y$

$$\iiint_E 2x\,dV = \int_0^3\left(\int_0^{-\frac{2}{3}x+2}\left(\int_0^{6-2x-3y} 2x\,dz\right)dy\right)dx$$



$z=0 \longrightarrow 2x+3y=6$

$y = 2-\frac{2x}{3}$

$$\boxed{\begin{cases} x=0 \longrightarrow y=2 \\ y=0 \longrightarrow x=3 \end{cases}}$$

$0 \le x \le -\frac{3}{2}y+3$

$0 \le y \le 2$

(x-simple)

or

$0 \le x \le 3$

$0 \le y \le -\frac{2}{3}x+2$

(y-simple)

$\Leftarrow$

$$\Rightarrow \iiint_E 2x \, dV = \int_0^3 \int_0^{\left(-\frac{2}{3}x+2\right)} \left[2xz\right]_0^{6-2x-3y} dy\, dx$$

$$= \int_0^3 \int_0^{\left(-\frac{2}{3}x+2\right)} \overbrace{(2x)(6-2x-3y)}^{12x-4x^2-6xy} dy\, dx$$

$$= \int_0^3 \left[12xy - 4x^2y - 3xy^2\right]_0^{\left(-\frac{2}{3}x+2\right)} dx$$

$$= \int_0^3 \left(12x\left(-\tfrac{2}{3}x+2\right) - 4x^2\left(-\tfrac{2}{3}x+2\right) - 3x\left(-\tfrac{2}{3}x+2\right)^2\right) dx$$

$$= \int_0^3 \left(12x - 8x^2 + \frac{4}{3}x^3\right) dx$$

$$= \left[6x^2 - \tfrac{8}{3}x^3 + \frac{x^4}{3}\right]_0^3 = 9.$$