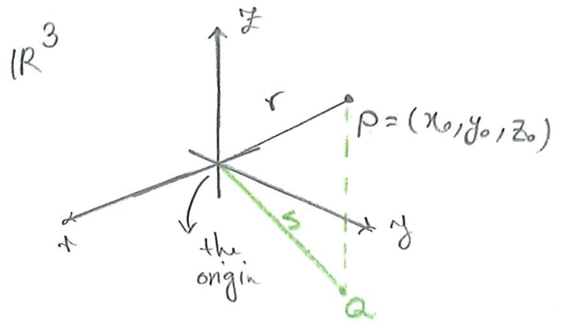
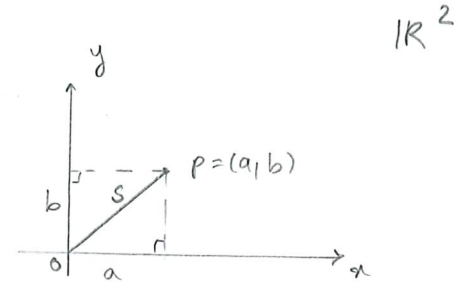
Vectors and coordinate geometry in $\mathbb{R}^3$ (10.1)





$$s=\sqrt{x_0^2+y_0^2}$$

$$r^2=z_0^2+s^2$$

$$r=\sqrt{x_0^2+y_0^2+z_0^2}$$

$$s^2=a^2+b^2$$

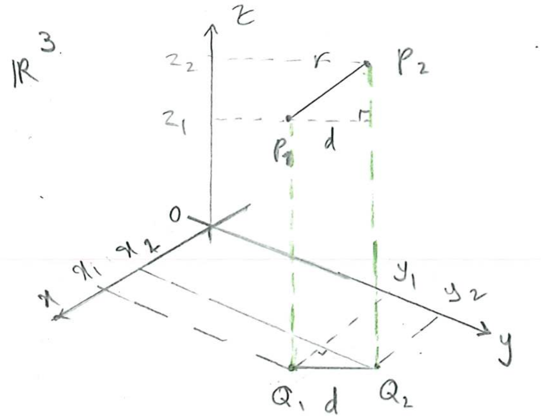
$$s=\sqrt{a^2+b^2}$$

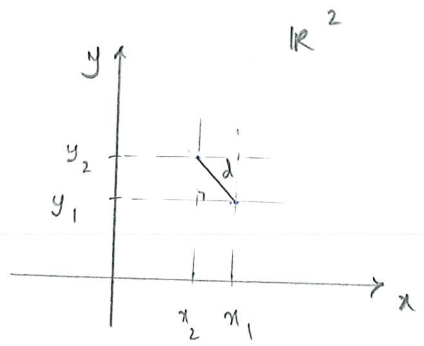
(pythagorean theorem)

Distance

$P_1=(x_1,y_1,z_1)$

$P_2=(x_2,y_2,z_2)$





$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

(pythagorean theorem)

$$r=\sqrt{\underbrace{(x_2-x_1)^2+(y_2-y_1)^2}_{d^2}+(z_2-z_1)^2}$$

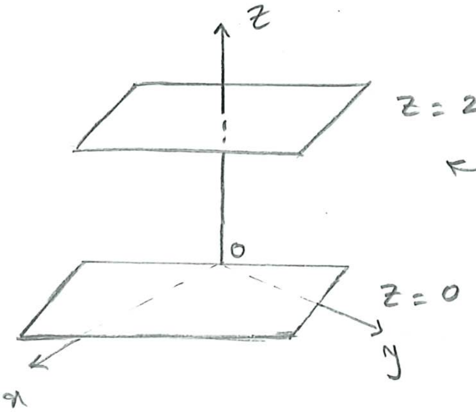
Example. Calculate the lengths of the sides of a triangle with vertices: $A=(1,3,1)$, $B=(1,0,-2)$, and $C=(1,1,1)$.

$$|\vec{BC}|=\sqrt{(1-1)^2+(1-0)^2+(1-(-2))^2}=\sqrt{10}$$

$$|\vec{AC}|=\sqrt{(1-1)^2+(1-3)^2+(1-1)^2}=\sqrt{4}=2$$

$$|\vec{AB}|=\sqrt{(1-1)^2+(0-3)^2+(-2-1)^2}=\sqrt{18}=3\sqrt{2}.$$

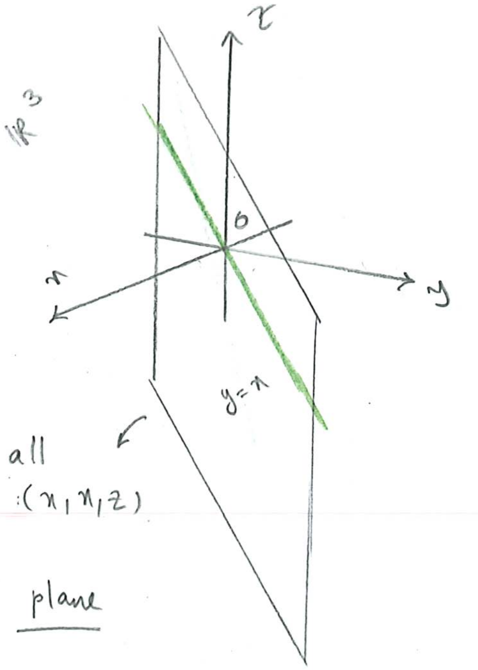
Some equations and surfaces in $\mathbb{R}^3$:



horizontal planes

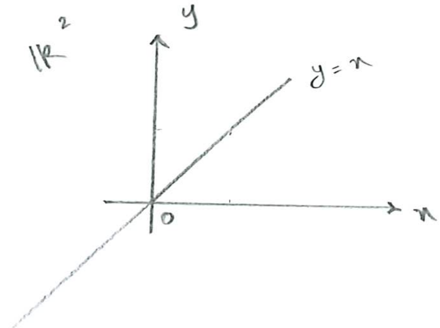
all points $(x,y,2)$

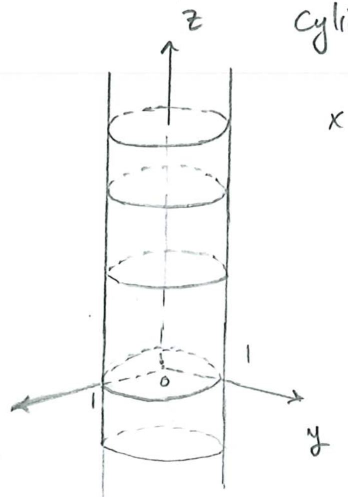
all $(x,y,0)$



all (x,x,z)

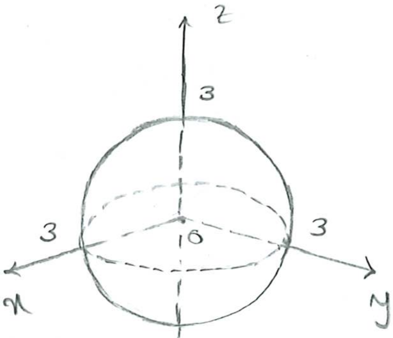
plane





Cylinder

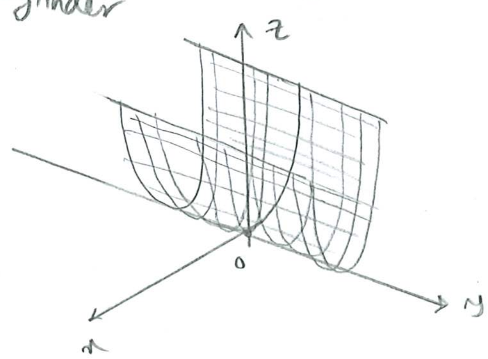
$x^2+y^2=1$



Sphere $\quad x^2+y^2+z^2=3^2$

radius $=3$

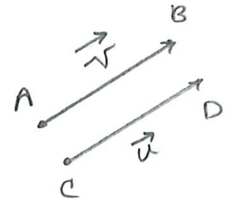
Cylinder $\quad z=x^2$



Some tools from linear algebra: (10.2)

Vector: $\vec{v} = \vec{AB}$ $\qquad |\vec{v}| = |\vec{AB}| = |\vec{CD}|$

$\Rightarrow \vec{u} = \vec{v}$

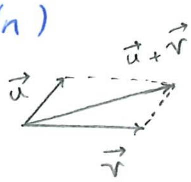


Vector addition:

$\vec{u} = (u_1, u_2, \dots, u_n)$

$\vec{v} = (v_1, v_2, \dots, v_n)$

$\Rightarrow \vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$



Scalar multiplication:

$t \in \mathbb{R}$, $\vec{v} = (v_1, v_2, \dots, v_n)$ $\qquad t\vec{v} = (tv_1, tv_2, \dots, tv_n)$

Dot (inner) product: (Scalar product)

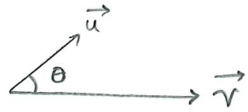
$\vec{u} = (u_1, u_2, \dots, u_n)$

$\vec{v} = (v_1, v_2, \dots, v_n)$

$\Rightarrow \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$

Theorem. $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos\theta$

$(n = 2, 3)$



θ: the angle between the directions of $\vec{u}$ and $\vec{v}$. $(0 \leq \theta \leq \pi)$

Properties:

$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

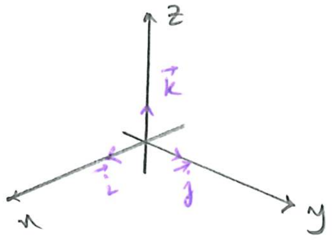
$(t\vec{u}) \cdot (\vec{v}) = \vec{u} \cdot (t\vec{v}) = t(\vec{u} \cdot \vec{v}) \qquad (t \in \mathbb{R})$

$\vec{u} \cdot \vec{u} = |\vec{u}|^2$

$\vec{u} \cdot \vec{v} = 0 \quad (\vec{u}, \vec{v} \neq \vec{0}) \iff \vec{u}$ and $\vec{v}$ are perpendicular.

$\vec{u} \perp \vec{v}$ "normal"

Unit vector: $\vec{u} \neq 0 \quad \hat{u} = \frac{1}{|\vec{u}|}\vec{u} \Rightarrow |\hat{u}| = 1.$



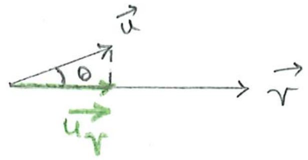
$|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$

$\vec{i} = (1,0,0)$

$\vec{j} = (0,1,0) \qquad \vec{k} = (0,0,1).$

Definition

Scalar projection



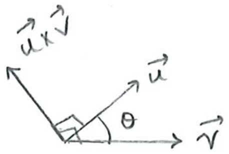
$S = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = |\vec{u}|\cos\theta$

Vector projection: $\vec{u_v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\cdot\hat{v} = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\cdot\vec{v}$ ($\vec{u_v}$ parallel to $\vec{v}$)

$\Rightarrow S = |\vec{u_v}|$

(10.3)

Cross product:



$\vec{u}\times\vec{v}$ orthogonal to $\vec{u}$ and $\vec{v}$.

For any vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$, the cross product $\vec{u}\times\vec{v}$ is the unique vector satisfying:

(i) $(\vec{u}\times\vec{v})\cdot\vec{u} = 0$ and $(\vec{u}\times\vec{v})\cdot\vec{v} = 0$

(ii) $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$

(iii) $\vec{u}, \vec{v}$ and $\vec{u}\times\vec{v}$ are positively oriented. (form a right-handed triad).

($n=2$ or $n=3$)

Theorem. Components of cross product:

If $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k} = (u_1, u_2, u_3)$ and

$\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k} = (v_1, v_2, v_3)$, then:

$$\Rightarrow \vec{u}\times\vec{v} = (u_2v_3 - u_3v_2)\vec{i} + (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - v_1u_2)\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}.$$

Properties of cross-product:

$\vec{u}\times\vec{u} = 0$.

$\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$

$\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v} + \vec{u}\times\vec{w}$

$(\vec{u}+\vec{v})\times\vec{w} = \vec{u}\times\vec{w} + \vec{v}\times\vec{w}$

$(t\vec{u})\times\vec{v} = \vec{u}\times(t\vec{v}) = t(\vec{u}\times\vec{v}) \qquad t\in\mathbb{R}$.

$\vec{u}\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\vec{u}\times\vec{v}) = 0$.

In general: $(\vec{u}\times\vec{v})\times\vec{w} \neq \vec{u}\times(\vec{v}\times\vec{w})$

Example. $\quad 0 = (\vec{u}\times\vec{u})\times\vec{w} \neq \vec{u}\times(\vec{u}\times\vec{w})$

$$|\vec{u}\times(\vec{u}\times\vec{w})| = |\vec{u}|\,|\vec{u}\times\vec{w}|\cdot\sin\tfrac{\pi}{2} = |\vec{u}|^2|\vec{w}|\sin\theta$$

$\theta \neq 0, \pi$

$\Rightarrow |\vec{u}\times(\vec{u}\times\vec{w})| \neq 0$.

Recall. $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$,

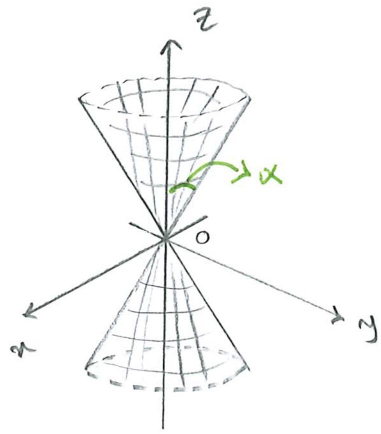
$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}, \qquad \begin{vmatrix} a & b & c \\ a & b & c \\ d & e & f \end{vmatrix} = 0,$$

for all $t\in\mathbb{R}$: $\begin{vmatrix} a & b & c \\ d+ta & e+tb & f+tc \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$.

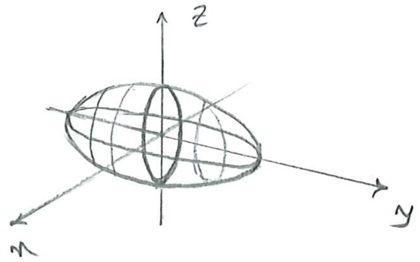
Some quadric surfaces (10.5)

Cone. $a^2 z^2 = x^2 + y^2$

$$\boxed{\alpha = \tan^{-1} a = \arctan a}$$

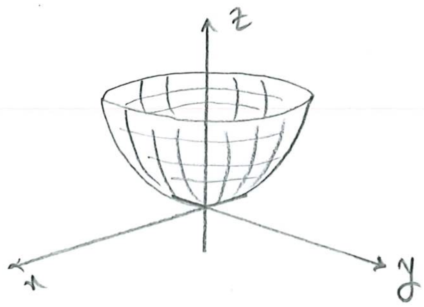


Ellipsoid. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$



Elliptic paraboloid:

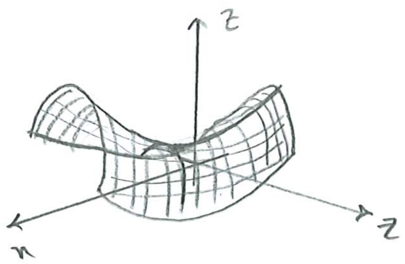
$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



Hyperbolic paraboloid:

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

(like saddle)



Hyperboloids
- of one sheet $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$
- of two sheets $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1$