L7 Determinants

1ma901/1ma406 Linear algebra

Jonas Nordqvist

Singular matrices

We recall that a matrix is said to be *singular* when it is not invertible.

Theorem

Let A be a $n \times n$ matrix. The following are equivalent:

(a) *A is nonsingular.*

(b) *$A\mathrm{x} = 0$ has only the trivial solution.*

(c) *A is row equivalent to I.*

Hence, by (b) if A is row equivalent to a matrix with row consisting of just zeros then A is singular.

For a 2×2 matrix A assuming $a_{11} \neq 0$ we have

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \sim \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \frac{a_{12}a_{21}}{a_{11}} \end{pmatrix}.$$

Any such matrix is invertible if and only if

$$a_{22} - \frac{a_{12}a_{21}}{a_{11}} \neq 0 \iff a_{11}a_{22} - a_{12}a_{21} \neq 0.$$

For 2×2 matrices we say that $a_{11}a_{22} - a_{12}a_{21}$ is the *determinant* of A, and denote this by $\det(A)$.

Then A is nonsingular if and only if $\det(A) \neq 0$.

Cofactor expansion

In view of the results from the previous side we make the following definition, which will allow us to generalize the notaion of determinants.

Definition

Let $A = (a_{ij})$ be a square $n \times n$ matrix, and let M_{ij} be the $(n-1) \times (n-1)$ matrix obtained when removing the ith row and the jth column of A. The determinant of M_{ij} is called the *minor* of a_{ij} and. We define the *cofactor* A_{ij} of a_{ij} by

$$A_{ij} = (-1)^{i+j} \det(M_{ij}).$$

Example

Let

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 6 & 7 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then

$$M_{23} = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad A_{23} = (-1)^{2+3} \det(M_{23}) = (-1)(-2) = 2.$$

Further, in view of this definition we have for a 2×2 matrix A

$$\det(A) = a_{11}A_{11} + a_{12}A_{12},$$

or alternatively

$$\det(A) = a_{21}A_{21} + a_{22}A_{22}.$$

Determinant of a matrix

This allows us to make a definition of determinants for higher order matrices.

Definition

The *determinant* of an $n \times n$ matrix A denoted $\det(A)$, is a scalar associated with the matrix A that is defined inductively by

$$\det(A) = \begin{cases} a_{11}, & \text{if } n = 1 \\ a_1 1 A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}, & \text{if } n > 1 \end{cases}$$

where

$$A_{ij} = (-1)^{1+j} \det(M_{1j})$$

are the cofactors associated with the entries of the first row of A.

Theorem

If A is an $n \times n$ matrix with $n \geq 2$ then $\det(A)$ can be expressed as a cofactor expansion using any row or column of A, i.e.

$$\begin{aligned} \det(A) &= a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \\ &= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}. \end{aligned}$$

Cofactor expansion example

Example

Let

$$A = \begin{pmatrix} 2 & 2 & 3 & 0 \\ 1 & 1 & 2 & 2 \\ 1 & 3 & 0 & 1 \\ 0 & 3 & -1 & -1 \end{pmatrix}.$$

Compute $\det(A)$.

Solution.

We cofactor expand along the fourth column an obtain

$$\det(A) = 0A_{41} + 2A_{42} + A_{43} - A_{44}$$

or explicitly

$$\begin{aligned} \det(A) = (-1)^{1+4} 0 \begin{vmatrix} 1 & 1 & 2 \\ 1 & 3 & 0 \\ 0 & 3 & -1 \end{vmatrix} + (-1)^{2+4} 2 \begin{vmatrix} 2 & 2 & 3 \\ 1 & 3 & 0 \\ 0 & 3 & -1 \end{vmatrix} \\ + (-1)^{3+4} \begin{vmatrix} 2 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 3 & -1 \end{vmatrix} + (-1)^{4+4}(-1) \begin{vmatrix} 2 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 3 & 0 \end{vmatrix}. \end{aligned}$$

Cont. example

Cont. solution.

Performing cofactor expansion on the 3×3 matrices yields

$$\begin{vmatrix} 2 & 2 & 3 \\ 1 & 3 & 0 \\ 0 & 3 & -1 \end{vmatrix} = 3 \begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} + (-1) \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 9 - 4 = 5,$$

$$\begin{vmatrix} 2 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 3 & -1 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} - \begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -14 + 11 = -3$$

and

$$\begin{vmatrix} 2 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 3 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -2.$$

Hence, combining everything we obtain

$$\det(A) = 2 \cdot 5 - (-3) - (-2) = 15.$$

$\square$

Some observations

Theorem

If A is a square matrix. Then $\det(A) = \det(A^T)$.

Proof.

The cofactor expansion in A in the first row is the same as the cofactor expansion in the first column of A^T. □

Theorem

If A is a triangular matrix, then the determinant is given by the product of the diagonal elements of A.

Proof.

Without loss of generality we may assume that A is upper triangular. Assume that A is an $n \times n$ matrix, and denote its diagonal entries by $d_1, \ldots, d_n$. Then by cofactor expansion along the first column we obtain

$$\det(A) = d_1 A_{11} + 0A_{21} + \ldots 0A_{n1} = d_1 A_{11}.$$

Now the determinant of A is given by $d_1 A_{11}$. If we continue in the same way to cofactor along the first columns of the matrix A' corresponding to A_{11}, we get $\det(A') = d_2 A'_{11}$, and hence

$$\det(A) = d_1 \det(A') = d_1 d_2 A'_{11}.$$

Repeating the same argument for all n rows we obtain

$$\det(A) = d_1 d_2 \cdots d_n.$$

□

Determinant of an elementary matrix

Recall that there are three types of elementray matrices

(I) Interchanging rows

(II) Multiplying row by a nonzero scalar α

(III) Replacing a row by its sum with a multiple of another row

Theorem

Let E be an elementary matrix. Then

$$\det(E) = \begin{cases} -1, & \textit{if } E \textit{ is of type I} \\ \alpha, & \textit{if } E \textit{ is of type II} \\ 1, & \textit{otherwise.} \end{cases}$$

Proof.

If E is of type I. We ignore the proof of this here, and defer reading to the examples in the course literature.

If E is of type II, then E is triangular (or in particular diagonal) with ones in all diagoanl entries except for one, where it is α. Hence $\det(E) = 1 \cdot 1 \cdots 1 \cdot \alpha = \alpha$ in this case.

If E is of type III, then E is a triangular matrix with ones in its main diagonal. □

Nonsingularity

Now we can prove the main result of this section, but first a lemma which we proved implicitly earlier.

Lemma

Let A be square matrix and E an elementary matrix of the same size. Then

$$\det(EA) = \det(E)\det(A).$$

Theorem

Let A be a square matrix, then A is singular if and only if $\det(A) = 0$.

Proof.

The matrix A may be reduced to reduced row echelon form with a finite number of row operations. Thus

$$U = E_k E_{k-1} \cdots E_1 A,$$

where U is in reduced row echelon form and every E_i is an elementary matrix. Hence we have

$$\det(U) = \det(E_k E_{k-1} \cdots E_1 A) = \det(E_k)\det(E_{k-1}) \cdots \det(E_1)\det(A).$$

The right hand side is nonzero if and only if $\det(A) \neq 0$, and the left hand side is nonzero if and only if U (which is triangular) have diagonal entries 1. This last statement is equivalent to saying that A is nonsingular, *i.e.* it is row equivalent to I. $\square$

Determinant of a product

Theorem

Let A and B be square matrices of the same size. Then

$$\det(AB) = \det(A)\det(B).$$

Proof of Theorem.

First we assume that A is singular, and we claim that this implies that AB is also singular. Otherwise there exists a matrix C such that $(AB)C = I$, and then by the associativity of matrix multiplication $A(BC) = I$, and hence BC would be an inverse of A. Hence $\det(AB) = 0 = \det(A)\det(B)$.

Thus we may assume that neither A nor B is singular. Then by the previous lemma we have

$$\begin{aligned}\det(AB) &= \det(AE_k E_{k-1} \cdots E_1) \\ &= \det(A)\det(E_k)\det(E_{k-1}) \cdots \det(E_1) \\ &= \det(A)\det(E_k E_{k-1} \cdots E_1) \\ &= \det(A)\det(B),\end{aligned}$$

which proves the theorem. $\square$