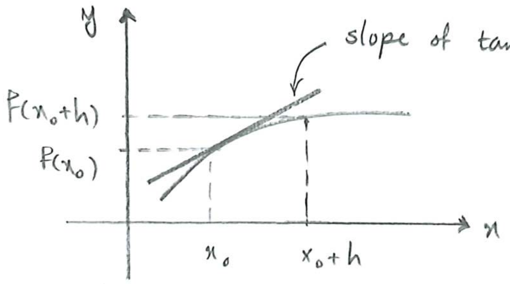
Partial derivatives and differentiability: (13.3)

Recall:

The derivative of $f: \mathbb{R} \to \mathbb{R}$ at $x = x_0$ is $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$.

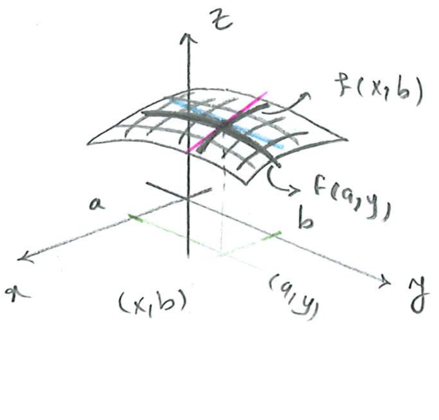


For $f: \mathbb{R}^2 \to \mathbb{R}$ (n in general), the <u>partial derivatives</u> at (x,y) are defined by:

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}, \text{ and } \frac{\partial f}{\partial y}(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

$\frac{\partial f}{\partial x}$ gives the rate of change in the x-direction, for y fixed.

$\frac{\partial f}{\partial y}$ gives the rate of change in the y-direction, for x fixed.



Alternative notations: $\frac{\partial f}{\partial x} = f'_x = f_1 = D_1 f = f_x$,

$$\frac{\partial f}{\partial y} = f'_y = f_2 = D_2 f = f_y.$$

Example. ① $f(x,y) = xy^2$

$f'_x(x,y) = y^2$ $\qquad$ $f'_y(x,y) = 2xy$.

② $f(x,y) = \sin^2(2x^3y)$

$f'_x(x,y) = 2\sin(2x^3y)\cdot\cos(2x^3y)\cdot 6x^2y = \sin(4x^3y).6x^2y.$

$f'_y(x,y) = 2\sin(2x^3y)\cos(2x^3y)\cdot 2x^3 = \sin(4x^3y)\cdot 2x^3.$

③ $f(x,y) = \sqrt{\ln(1+x^2+3y^2)}$

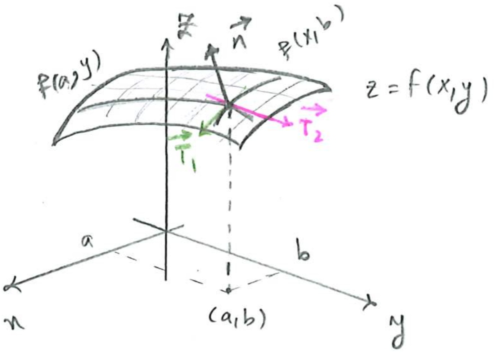
$$f_1(x,y) = f'_x(x,y) = \frac{1}{2}\left(\ln(1+x^2+3y^2)\right)^{-\frac{1}{2}} \cdot \frac{2x}{1+x^2+3y^2}$$

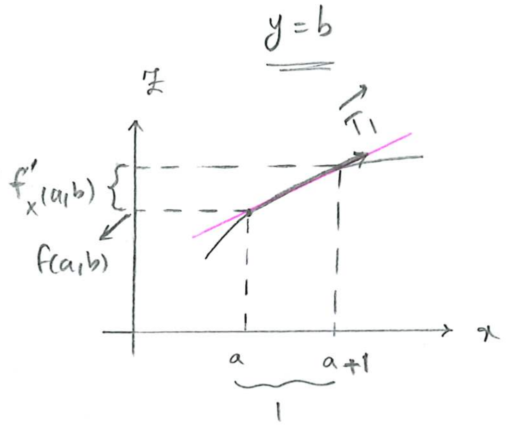
$$= \frac{x}{(1+x^2+3y^2)\left(\ln(1+x^2+3y^2)\right)^{\frac{1}{2}}}.$$

$$f_2(x,y) = f'_y(x,y) = \frac{1}{2}\left(\ln(1+x^2+3y^2)\right)^{-\frac{1}{2}} \frac{6y}{1+x^2+3y^2}$$

$$= \frac{3y}{(1+x^2+3y^2)\left(\ln(1+x^2+3y^2)\right)^{\frac{1}{2}}}.$$

Tangent plane:





$$\vec{T_1} = (a+1-a,\ 0,\ f(a,b) + f'_x(a,b)\cdot 1 - f(a,b))$$

$$= (1, 0, f'_x(a,b)) = \vec{i} + \vec{k}\, f'_x(a,b) \quad \text{and}$$

$$\vec{T_2} = (0, 1, f'_y(a,b))$$

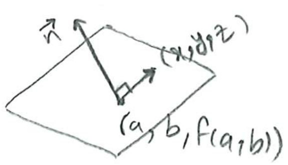
Normal vector to the function surface: $\vec{n} = \vec{T_1} \times \vec{T_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & f'_x(a,b) \\ 0 & 1 & f'_y(a,b) \end{vmatrix} = (-f'_x(a,b), -f'_y(a,b), 1)$

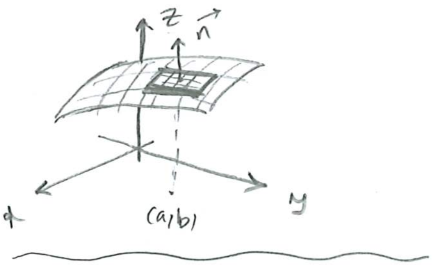
$$= -f'_x(a,b)\vec{i} - f'_y(a,b)\vec{j} + \vec{k}$$

Tangent plane:

$$(x-a,\ y-b,\ z-f(a,b)) \cdot \vec{n} = 0 \iff -f'_x(a,b)(x-a) - f'_y(a,b)(y-b) + z - f(a,b) = 0.$$

$$\iff \underline{z = f(a,b) + f'_x(a,b)(x-a) + f'_y(a,b)(y-b)}$$





* The normal vector is normal to the function surface at the certain point, and is normal to the tangent plane of the surface at that point.

Example (a) Find the equation of tangent plane of $f(x,y) = \dfrac{2xy}{x^2+y^2}$ at $(\overset{a}{1}, \overset{b}{2}, \underbrace{4/5}_{f(1,2)})$.

Tangent plane's equation: $z = f(a,b) + f'_x(a,b)(x-a) + f'_y(a,b)(y-b)$

$$f'_x(x,y) = \frac{2y(x^2+y^2) - 2xy \cdot 2x}{(x^2+y^2)^2} = \frac{2y^3 - 2x^2y}{(x^2+y^2)^2}$$

$$\Rightarrow f'_x(1,2) = \frac{12}{25}$$

$$f'_y(x,y) = \frac{2x^3 - 2xy^2}{(x^2+y^2)^2} \Rightarrow f'_y(1,2) = \frac{-6}{25}$$

$$\Rightarrow \text{Tangent plane: } \quad z = \frac{4}{5} + \frac{12}{5}(x-1) - \frac{6}{25}(y-2).$$

(b) At which point is the tangent plane of $f(x,y) = \dfrac{2xy}{x^2+y^2}$ horizontal?

The tangent plane is horizontal $\Leftrightarrow$ The normal vector, $\vec{n}$, is vertical.

$$\Leftrightarrow (-f'_x, -f'_y, 1) = \vec{n} = (0,0,1)$$

$$\Leftrightarrow f'_x(x,y) = f'_y(x,y) = 0$$

$$\Leftrightarrow \begin{cases} 2y(y^2-x^2) = 0 \\ 2x(x^2-y^2) = 0 \end{cases}$$

$$\Leftrightarrow (x+y)(x-y) = 0 \quad \text{or}$$

$$y = 0 \quad \text{or} \quad x = 0$$

$$\Leftrightarrow y = -x \quad \text{or} \quad y = x$$

except $(x,y) = (0,0)$.

Higher order partial derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f''_{xx}(x,y)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f''_{yy}(x,y)$$

"pure"

Other notations:

$$f_{xx}(x,y) = \frac{\partial^2 z}{\partial x^2} = f_{11}(x,y)$$

$$f_{yy}(x,y) = \frac{\partial^2 z}{\partial y^2} = f_{22}(x,y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f''_{yx}(x,y)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f''_{xy}(x,y)$$

"mixed"

$$f_{xy}(x,y) = f_{12}(x,y) = \frac{\partial^2 z}{\partial y \partial x}$$

$$f_{yx}(x,y) = f_{21}(x,y) = \frac{\partial^2 z}{\partial x \partial y}$$

Theorem. If $f, f'_x, f'_y, f''_{xy}, f''_{yx}$ are continuous, then:

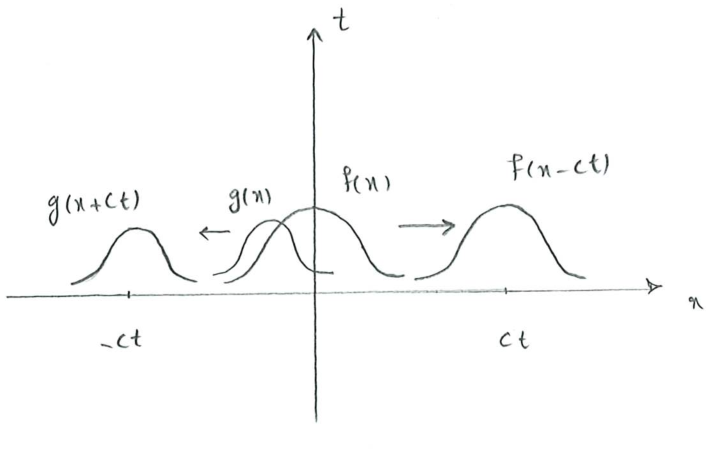
$$f''_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = f''_{xy}$$

Examples. (Applications)

- Solutions to the Laplace equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ are called harmonic functions. They appear in physics (e.g. electromagnetics)

- The wave equation $\frac{\partial^2 h}{\partial t^2} = c^2 \frac{\partial^2 h}{\partial x^2}$ has solution:

$$h(t,x) = f(x - ct) + g(x + ct).$$



Differentiability:

For $f: \mathbb{R} \to \mathbb{R}$, the existence of the derivative at $x=a$ means

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \iff \lim_{h\to 0} \underbrace{\left| \frac{f(a+h)-f(a)-f'(a)h}{h} \right|}_{:=\alpha(h)} = 0 \iff$$

$$(*)\quad f(a+h)-f(a) = f'(a)h + h\,\alpha(h), \quad \alpha(h) \to 0 \text{ as } h\to 0.$$

Geometric interpretation: The linear approximation improves as $h \to 0$.

The formulation of differentiability, $(*)$, can be generalized to $\mathbb{R}^n$,

Definition.

$f: \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\bar{x} = (x_1, \dots, x_n)$ if there exist $A_1, \dots, A_n \in \mathbb{R}$ such that $f(\bar{x}+\bar{h}) - f(\bar{x}) = A_1 h_1 + \dots + A_n h_n + |\bar{h}|\,\alpha(\bar{h})$

where $\alpha(\bar{h}) \to 0$ as $\bar{h} \to 0$. Then $A_j = f'_j(\bar{x})$, $j = 1, \dots, n$.

** Differentiability implies the existence of partial derivatives.

If $n=2$: $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a,b) if there exist

$A, B \in \mathbb{R}$ such that $f(a+h, b+k) - f(a,b) = Ah + Bk + \sqrt{h^2+k^2}\,\alpha(h,k)$

where $\alpha(h,k) \to 0$ as $(h,k) \to 0$.

$$f(a+h, b+k) = \underbrace{f(a,b) + f'_x(a,b)\,h + f'_y(a,b)\,k}_{\text{tangent plane approximation}} + \sqrt{h^2+k^2}\,\alpha(h,k)$$

The approximation improves as $(h,k) \to 0$.

* Differentiability is <u>stronger</u> than existence of partial derivatives $f'_x(x,y)$ and $f'_y(x,y)$.

<u>But</u> if $f \in C^1(\mathbb{R}^2)$, that is $f'_x, f'_y \in C(\mathbb{R}^2)$, then f is differentiable for all $(x,y) \in \mathbb{R}^2$.

<u>Theorem</u>. If $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ is differentiable at (a,b), then f is continuous at (a,b).

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* $g \in C(\mathbb{R}^2)$ means g is continuous.

Example. Show that $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$ is not differentiable, but $f'_x(x,y)$ and $f'_y(x,y)$ exist, for all $(x,y) \in \mathbb{R}^2$.

$$\lim_{\substack{(x,y)\to(0,0) \\ (x,y)\in\Gamma_1}} f(x,y) = \lim_{x\to 0} \frac{x \cdot 0}{x^2+0} = 0$$

$$\boxed{\Gamma_1 : y=0}$$

$$\lim_{\substack{(x,y)\to(0,0) \\ (x,y)\in\Gamma_2}} f(x,y) = \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

$$\boxed{\Gamma_2 : y=x}$$

$\Rightarrow$ f is not continuous $\Rightarrow$ f is not differentiable.

$$\underset{(a,b)\neq(0,0)}{f'_x(a,b)} = \left.\frac{y(x^2+y^2)-2x^2y}{(x^2+y^2)^2}\right|_{(a,b)} = \frac{b^3-a^2b}{(a^2+b^2)^2}$$

$$\underset{(a,b)\neq(0,0)}{f'_y(a,b)} = \left.\frac{x(x^2+y^2)-2xy^2}{(x^2+y^2)^2}\right|_{(a,b)} = \frac{a^3-ab^2}{(a^2+b^2)^2}$$

$$f'_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$$f'_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

$\Rightarrow f'_x(a,b)$, $f'_y(a,b)$ exist, for all $(a,b)\in\mathbb{R}^2$.

The Chain Rule

Recall. $f, g : \mathbb{R} \to \mathbb{R}$ $\quad \frac{d}{dt}(f(g(t))) = f'(g(t)) \cdot \underbrace{g'(t)}_{\text{inner derivative}}$

Example. $\left(\sin(x^2)\right)' = \cos(x^2) \cdot 2x$

Theorem. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, and suppose $g_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \dots, n$ are differentiable.

Set $g(t) = (g_1(t), g_2(t), \dots, g_n(t))$. Then

$f \circ g(t) = f(g_1(t), g_2(t), \dots, g_n(t))$ is differentiable,

and
$$\frac{d}{dt}(f \circ g(t)) = f_1'(g(t))\, g_1'(t) + f_2'(g(t)) \cdot g_2'(t) + \dots + f_n'(g(t))\, g_n'(t).$$
$$= \sum_{j=1}^{n} f_j'(g(t))\, g_j'(t).$$

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* If z is a function of x and y with continuous first partial derivatives and if x and y are differentiable functions of t, then:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\begin{cases} z = f(x, y) & f : \mathbb{R} \to \mathbb{R} \\ x = x(t) & x : \mathbb{R} \to \mathbb{R} \\ y = y(t) & y : \mathbb{R} \to \mathbb{R} \end{cases}$$

$$z = t^2 e^{t^2/t} = t^2 e^t \leftarrow \frac{dz}{dt}$$

Example:

(a) $z = xe^{xy}$, $\underbrace{x = t^2, \quad y = t^{-1}}$. $\frac{dz}{dt} = ?$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (e^{xy} + xye^{xy})(2t) + x^2 e^{xy}\left(\frac{-1}{t^2}\right)$$

$$= (e^t + te^t)(2t) - t^4 e^t t^{-2}$$

$$= 2te^t + 2t^2e^t - t^2e^t = \boxed{2te^t + t^2e^t}$$

(b) $z = x^2y^3 + y\cos x$, $x = \ln(t^2)$, $y = \sin(4t)$. $\frac{dz}{dt} = ?$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2xy^3 - y\sin x)\left(\frac{2}{t}\right) + (3x^2y^2 + \cos x)(4\cos(4t))$$

$$= \frac{4\cdot \ln(t^2)\sin^3(4t) - 2\sin(4t)\sin(\ln(t^2))}{t} +$$

$$(4\cos(4t))(3\ln^2(t^2)\sin^2(4t) + \cos(\ln(t^2)))$$

System of Functions :

We have $f_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ $\qquad j = 1, 2, \dots, m$.

$f_j(\bar{x}) = f_j(x_1, x_2, \dots, x_n)$

Set $\bar{F} = (f_1, f_2, \dots, f_m) \implies \bar{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ (vector-valued function of several variables)

$$\bar{y} = \bar{F}(\bar{x}) \longleftrightarrow \begin{cases} y_1 = f_1(x_1, \dots, x_n) \\ y_2 = f_2(x_1, \dots, x_n) \\ \vdots \\ y_m = f_m(x_1, \dots, x_n) \end{cases}$$

Jacobian matrix : $D\bar{F}(\bar{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}_{m \times n}$ component

$(m \times n)$

$\longrightarrow$ variable

$$= \frac{\partial(f_1, \dots, f_m)}{\partial(x_1, \dots, x_n)}$$

Example. Find the Jacobian matrix of $\bar{F}(a,b) = (x(a,b), y(a,b))$ with:

$x = 2a + 3b^2$, $y = 5a^2 + 7b^2 + 11ab$.

$n = m = 2$

$\bar{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\bar{F}(a,b) = (f_1(a,b), f_2(a,b)) = (x(a,b), y(a,b))$

$$D\bar{F}((a,b)) = \begin{bmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} \end{bmatrix} = \begin{bmatrix} 2 & 6b \\ 10a + 11b & 14b + 11a \end{bmatrix}.$$

General case of the chain rule:

Suppose $\bar{F}: \mathbb{R}^n \to \mathbb{R}^m$ and $\bar{g}: \mathbb{R}^k \to \mathbb{R}^n$ are differentiable and $\bar{x} \in \mathbb{R}^k$.

Then:
$$D(\bar{F} \circ \bar{g})(\bar{x}) = \underbrace{\underbrace{D\bar{F}(\bar{g}(\bar{x}))}_{m \times n} \cdot \underbrace{D\bar{g}(\bar{x})}_{n \times k}}_{m \times k}$$

matrix multiplication

(In the previous theorem, we had $k = m = 1$, which is a particular case of the formula above.)

Example. If $z = \sin(x^2 y)$, where $x = st^2$ and $y = s^2 + \frac{1}{t}$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ by using the chain rule.

Set: $\bar{F}: \mathbb{R}^2 \to \mathbb{R}$ and $\bar{g}: \mathbb{R}^2 \to \mathbb{R}^2$

$\bar{F}(x,y) = z = \sin(x^2 y)$

$$\bar{g}(s,t) = (x(s,t), y(s,t)) = \left(st^2, s^2 + \frac{1}{t}\right)$$

$$\Rightarrow D(\bar{F} \circ \bar{g})((s,t)) = D\bar{F}(\bar{g}((s,t))) \cdot D\bar{g}((s,t)) = D\bar{F}((x,y)) \cdot D\bar{g}((s,t))$$

$$= \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{bmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} & \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \end{pmatrix}$$

$$\Rightarrow \begin{cases} \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \end{cases}$$

$$\Rightarrow \frac{\partial z}{\partial s} = \left(2xy \cos(x^2y)\right) t^2 + \left(x^2 \cos(x^2y)\right) 2s$$

$$= \left(2st^2 \left(s^2+\frac{1}{t}\right)t^2 + 2s^3t^4\right) \cos\left(s^4t^4 + s^2t^3\right)$$

$$= \left(4s^3t^4 + 2st^3\right) \cos\left(s^4t^4 + s^2t^3\right)$$

$$\Rightarrow \frac{\partial z}{\partial t} = \left(2xy \cos(x^2y)\right)(2st) + x^2 \cos(x^2y)\left(\frac{-1}{t^2}\right)$$

$$= 4s^2t^3\left(s^2+\frac{1}{t}\right)\cos\left(s^2t^4\left(s^2+\frac{1}{t}\right)\right) - s^2t^2\cos\left(s^2t^4\left(s^2+\frac{1}{t}\right)\right)$$

$$= \left(4s^4t^3 + 4s^2t^2 - s^2t^2\right)\cos\left(s^2t^4\left(s^2+\frac{1}{t}\right)\right)$$

$$= \left(4s^4t^3 + 3s^2t^2\right)\cos\left(s^2t^4 + s^2t^3\right).$$