Example. Evaluate the following integral by converting it into polar Coordinates.

$$I = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \cos(x^2+y^2)\, dy\, dx$$

$$\begin{cases} -1 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le 0 \end{cases}$$

It is the portion of the bottom of the disk of radius 1 centered at the origin.

$$\Rightarrow \begin{cases} \pi \le \theta \le 2\pi \\ 0 \le r \le 1 \end{cases}$$ and

$$\underbrace{dx\, dy = dA = r\, dr\, d\theta}$$

$$\Rightarrow I = \int_{\pi}^{2\pi} \int_0^1 \cos(r^2)\, r\, dr\, d\theta$$

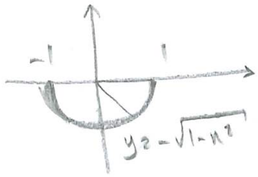
$$= \pi \left[\frac{\sin(r^2)}{2} \right]_0^1$$

$$= \frac{\pi}{2} \sin(1).$$

$$\sqrt{1-x^2} = y$$

$$1 - x^2 = y^2$$

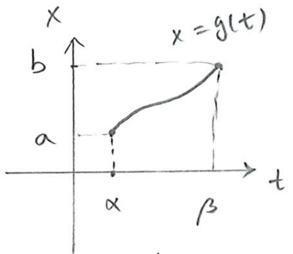
$$1 = x^2 + y^2$$



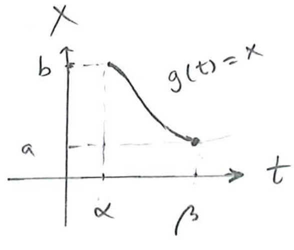
Change of variables in multiple integrals

Recall: Change of variable in single integrals.

Suppose $g \in C^1(\mathbb{R})$ is a monotonic function, $g : [\alpha, \beta] \longrightarrow [a, b]$.



Case 1



Case 2

$$\int_a^b f(x)\,dx = \left[\begin{array}{l} x = g(t) \\ dx = g'(t)\,dt \end{array}\right. \qquad \left\{\begin{array}{l} x = a \Longleftrightarrow t = \alpha \\ x = b \Longleftrightarrow t = \beta \\ \text{case } 1 \end{array}\right. \qquad \left\{\begin{array}{l} x = a \Longleftrightarrow t = \beta \\ x = b \Longleftrightarrow t = \alpha \\ \text{case } 2 \end{array}\right.$$

$$= \begin{cases} \displaystyle\int_\alpha^\beta f(g(t))\,\overset{>0}{g'(t)}\,dt & (\text{case } 1) \\ \displaystyle\int_\beta^\alpha f(g(t))\,\overset{<0}{g'(t)}\,dt = \int_\alpha^\beta f(g(t))\,\overset{>0}{(-g'(t))}\,dt & (\text{case } 2) \end{cases}$$

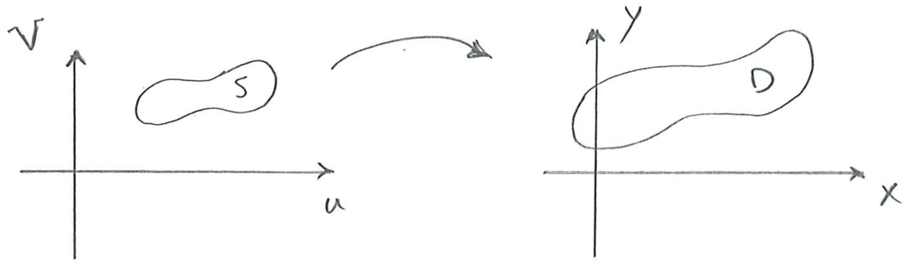
$$\Longrightarrow \int_a^b f(x)\,dx = \int_\alpha^\beta f(g(t))\,|g'(t)|\,dt$$

In general:

* let $g : [\alpha, \beta] \longrightarrow I$ be a differentiable function with a continuous derivative and $I \subseteq \mathbb{R}$ be an interval. Suppose $f : I \longrightarrow \mathbb{R}$ is a continuous function. Then,

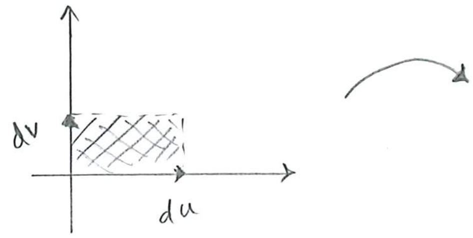
$$\int_\alpha^\beta f(g(t))\,g'(t)\,dt = \int_{g(\alpha)}^{g(\beta)} f(x)\,dx.$$

Change of variables in $\mathbb{R}^2$

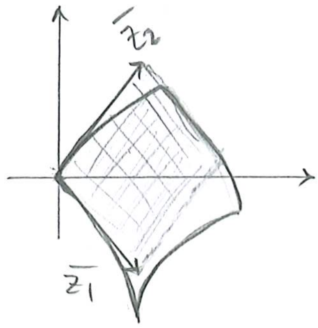


$x = x(u,v)$

$y = y(u,v)$



area element = du dv



$$\bar{z}_1 = \left(\frac{\partial x}{\partial u}\,du,\ \frac{\partial y}{\partial u}\,du\right)$$

$$\bar{z}_2 = \left(\frac{\partial x}{\partial v}\,dv,\ \frac{\partial y}{\partial v}\,dv\right)$$

$$\text{area element}: \left|\bar{z}_1 \times \bar{z}_2\right| = \left|\det\begin{bmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial x}{\partial u}du & \frac{\partial y}{\partial u}du & 0 \\ \frac{\partial x}{\partial v}dv & \frac{\partial y}{\partial v}dv & 0 \end{bmatrix}\right|$$

$$\Rightarrow \underline{\text{area element}} = \left|\det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}\right| du\,dv$$

$$= du\,dv\left|\det \underbrace{\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}}_{=\frac{\partial(x,y)}{\partial(u,v)}\ \text{Jacobian}}\right|$$

$$= du\,dv\left|\det\frac{\partial(x,y)}{\partial(u,v)}\right| = \underline{dx\,dy}$$

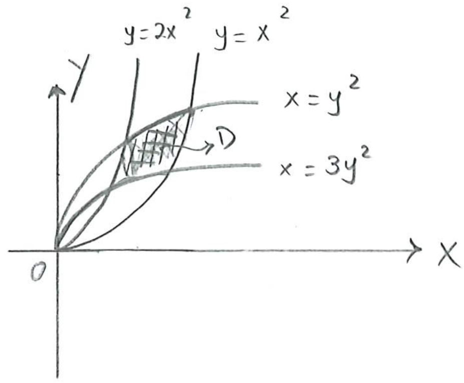
$$\iint_D f(x,y)\,dx\,dy = \iint_S f(x(u,v),y(u,v))\left|\det\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Example: Cartesian to polar transformation

$$\left| \det \frac{\partial(x,y)}{\partial(r,\theta)} \right| = \left| \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \right| = \left| r(\cos^2\theta + \sin^2\theta) \right| = r.$$

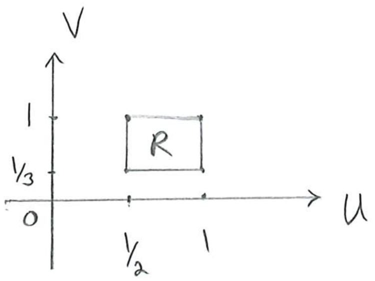
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \Rightarrow dx\,dy = dA = r\,dr\,d\theta$$

Example: Compute area of D.



We define: $\begin{cases} u = \dfrac{x^2}{y} \\ v = \dfrac{y^2}{x} \end{cases}$

$(x,y) \in D \iff (u,v) \in R$

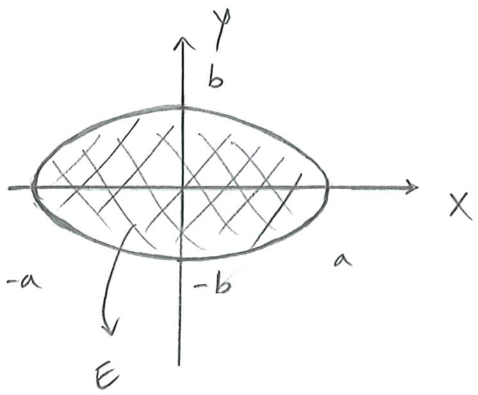


$$\det \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{2x}{y} & -\frac{x^2}{y^2} \\ -\frac{y^2}{x^2} & \frac{2y}{x} \end{vmatrix} = 4-1 = 3$$

$$\Rightarrow A(D) = \iint_D dx\,dy = \iint_R \left| \det\left(\frac{\partial(x,y)}{\partial(u,v)}\right) \right| du\,dv = \iint_R \frac{1}{3}\,du\,dv = \frac{1}{3}\iint_R du\,dv$$

$$= \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{2}{3} = \underline{\underline{\frac{1}{9}}}$$

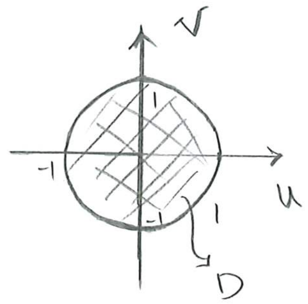
Example. Compute the area of the ellipse.



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \le 1 \quad \text{(ellipse area)}$$

$a, b > 0$

We define (introduce): $\begin{cases} x = au \\ y = bv \end{cases} \Rightarrow \begin{cases} \frac{x}{a} = u \\ \frac{y}{b} = v \end{cases}$

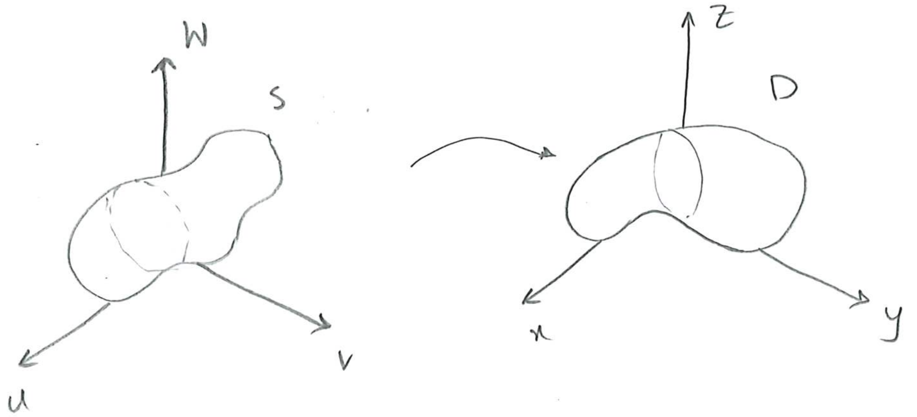


$$(x, y) \in E \iff u^2 + v^2 \le 1 \iff (u, v) \in D$$

$$\text{Area of ellipse} = \iint_E dx\, dy = \iint_D \left| \det \left(\frac{\partial(x,y)}{\partial(u,v)} \right) \right| du\, dv$$

$$= \iint_D \left| \det \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \right| du\, dv = ab \iint_D du\, dv = ab\pi .$$

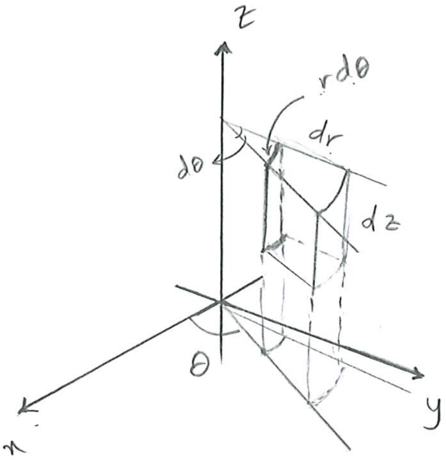
Change of variables in triple integrals



$$\iiint_D f(x,y,z)\, dx\, dy\, dz = \iiint_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \det \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\, dv\, dw$$

Cylindrical coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



$$\det \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\underbrace{dV}_{\text{volume element}} = r\,dr\,d\theta\,dz$$

Spherical coordinates:

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

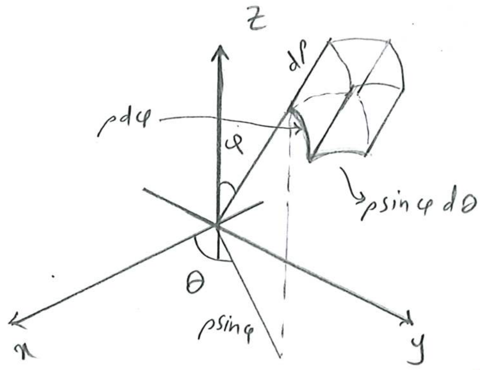
$$\det\left|\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)}\right| = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$$

$$= \cos\varphi\,\rho^2 \begin{vmatrix} \cos\varphi\cos\theta & -\sin\varphi\sin\theta \\ \cos\varphi\sin\theta & \sin\varphi\cos\theta \end{vmatrix} + \rho^2\sin\varphi \begin{vmatrix} \sin\varphi\cos\theta & -\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \sin\varphi\cos\theta \end{vmatrix}$$

$$= \rho^2\cos\varphi\,(\cos\varphi\sin\varphi) + \rho^2\sin\varphi\,(\sin^2\varphi)$$

$$= \rho^2\sin\varphi\,(\underbrace{\cos^2\varphi + \sin^2\varphi}_{1}) = \rho^2\sin\varphi$$

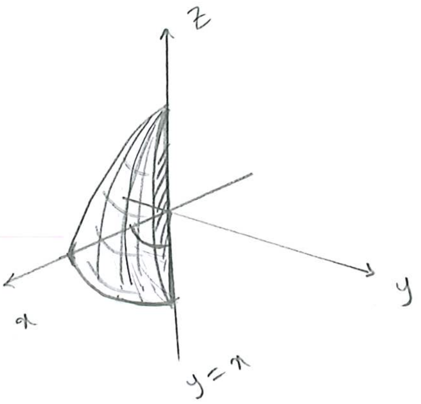
$$dV = \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$



Example. Compute the volume of D.

$$D = \left\{ (x,y,z) \in \mathbb{R}^3 : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1,\ z \geq 0,\ x \geq 0,\ 0 \leq y \leq x \right\}$$

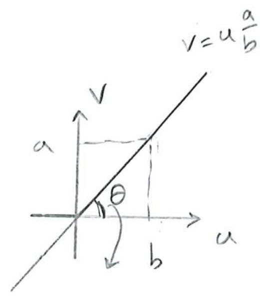
$a, b, c > 0$



$$V = \iiint_D dx\, dy\, dz = \iiint_{\substack{u^2+v^2+w^2 \leq 1 \\ w \geq 0,\ u \geq 0,\ 0 \leq v \leq \frac{au}{b}}} \left| \det \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\, dv\, dw$$

We define : $\begin{cases} x = au \\ y = bv \\ z = cw \end{cases}$

$$\Rightarrow V = \iiint_{\substack{u^2+v^2+w^2 \leq 1 \\ w \geq 0,\ u \geq 0,\ 0 \leq v \leq \frac{a}{b}u}} \left| \det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \right| du\, dv\, dw = \iiint_{\substack{u^2+v^2+w^2 \leq 1 \\ w \geq 0,\ u \geq 0,\ 0 \leq v \leq u\left(\frac{a}{b}\right)}} abc\ du\, dv\, dw$$



$\theta = \arctan\left(\frac{a}{b}\right)$

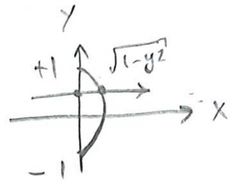
$$= abc \int_{\rho=0}^{1} \int_{\varphi=0}^{\pi/2} \int_{\theta=0}^{\arctan\left(\frac{a}{b}\right)} \rho^2 \sin\varphi \, d\theta\, d\varphi\, d\rho$$

Spherical Coord.

$$= abc \left[\frac{\rho^3}{3}\right]_0^1 \left(\arctan\left(\frac{a}{b}\right)\right) \left[-\cos\varphi\right]_0^{\pi/2} = \frac{abc}{3} \arctan\left(\frac{a}{b}\right)$$

Example. Evaluate the integral below by converting it into an integral in cylindrical coordinates.

$$I = \int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz \, dz \, dx \, dy$$



$-1 \leq y \leq 1$

$0 \leq x \leq \sqrt{1-y^2}$ } The right half of the circle of radius 1 centered at the origin.

It means we have: $-\pi/2 \leq \theta \leq \pi/2$

$0 \leq r \leq 1$

$x^2+y^2 \leq z \leq \sqrt{x^2+y^2}$

and we know: $r^2 \leq z \leq r$.

$$\Rightarrow I = \int_{-\pi/2}^{\pi/2} \int_{0}^{1} \int_{r^2}^{r} (r\cos\theta)(r\sin\theta) z \; r \, dz \, dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_{0}^{1} r^3 \cos\theta \sin\theta \left[\frac{z^2}{2}\right]_{r^2}^{r} dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_{0}^{1} r^3 \cos\theta \sin\theta \left(\frac{r^2}{2} - \frac{r^4}{2}\right) dr \, d\theta = \int_{-\pi/2}^{\pi/2} \frac{\cos\theta \sin\theta}{2} \left[\frac{r^6}{6} - \frac{r^8}{8}\right]_0^1 d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{\cos\theta \sin\theta}{2} \cdot \frac{1}{24}\right) d\theta = \frac{1}{48} \int_{-\pi/2}^{\pi/2} \cos\theta \sin\theta \, d\theta$$

$$= \frac{1}{48} \left[\frac{1}{2} \sin^2\theta\right]_{-\pi/2}^{\pi/2} = \frac{1}{96} \left(\sin^2(\pi/2) - \sin^2(-\pi/2)\right) = 0$$

Example. Evaluate $\iiint_E 16z \, dv$ by converting it into an integral in spherical coordinates. E is the upper half of the sphere $x^2+y^2+z^2=1$.

According to the upper half of the sphere, we get:

$$0 \le \rho \le 1$$
$$0 \le \theta \le 2\pi$$
$$0 \le \varphi \le \frac{\pi}{2}$$

$$\Rightarrow \iiint_E 16z \, dv = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho^2 \sin\varphi \, (16\rho\cos\varphi) \, d\rho \, d\theta \, d\varphi$$

$$= 16 \int_0^{\pi/2} \int_0^{2\pi} \left[\frac{\rho^4}{4}\right]_0^1 \sin\varphi \cos\varphi \, d\theta \, d\varphi$$

$$= 4 \cdot (2\pi) \int_0^{\frac{\pi}{2}} \sin\varphi \cos\varphi \, d\varphi$$

$$= 8\pi \left[\frac{\sin^2\varphi}{2}\right]_0^{\pi/2} = 4\pi \left(\sin^2(\pi/2) - \sin^2(0) \right)$$

$$= 4\pi$$

Exercise 14.4. 24 (9th edition)

Find the volume of the region lying above the xy-plane, inside the cylinder $x^2+y^2=4$, and below the plane $z=x+y+4$.

$$\text{Volume} = \iint_D F(x,y)\,dA = \iiint_D \int_0^{x+y+4} dV = \iint_D (x+y+4)\,dx\,dy$$

$D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 4\}$

$$\int_0^{2\pi}\int_0^{2} (r\cos\theta + r\sin\theta + 4)\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[2r^2 + \frac{r^3}{3}(\cos\theta+\sin\theta)\right]_0^2 d\theta$$

$$= \int_0^{2\pi} \left(8 + \frac{8}{3}(\cos\theta+\sin\theta)\right) d\theta =$$

$$\left[8\theta + \frac{8}{3}(\sin\theta - \cos\theta)\right]_0^{2\pi} = \underline{16\pi} \ (\text{unit})^3$$

Another solution:

$$V = \iint_D (x+y+4)\,dx\,dy = \iint_D x\,dA + \iint_D y\,dA + 4\iint_D dA$$

$D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 4\}$

$$= 0 + 0 + 4\,\text{Area}(D) = 4\cdot\pi(2)^2 = \underline{16\pi} \ (\text{unit})^3$$

Example. Compute $\iiint_K (x^2+y^2)\,dx\,dy\,dz$, where K is the region bounded by $z=x^2+y^2$ and $z=2-x^2-y^2$.

$$\begin{cases} z = x^2+y^2 \\ z = 2-x^2-y^2 \end{cases} \Rightarrow x^2+y^2 = 2-x^2-y^2 \Rightarrow 2x^2+2y^2 = 2$$

$$\Rightarrow x^2+y^2 = 1.$$

$D:\ x^2+y^2 \le 1$

$$\Rightarrow \iiint_K (x^2+y^2)\,dx\,dy\,dz = \iint_D \left(\int_{z=x^2+y^2}^{z=2-x^2-y^2} (x^2+y^2)\,dz \right) dx\,dy$$

$$= \iint_D (x^2+y^2)(2-x^2-y^2-x^2-y^2)\,dx\,dy$$

$$= \int_0^{2\pi} \int_0^1 r^2(2-2r^2)r\,dr\,d\theta = (2\pi)\int_0^1 (2r^3-2r^5)\,dr$$

$$= (2\pi)\left[\frac{r^4}{2} - \frac{r^6}{3}\right]_0^1 = \boxed{\frac{\pi}{3}}$$