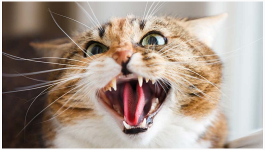
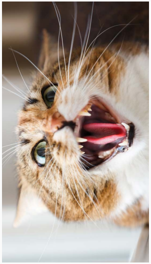
L13 Linear transformations

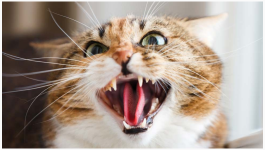
1ma901/1ma406 Linear algebra

Jonas Nordqvist

Computer graphics







Computer graphics – not in this course!

Defining a linear transformation

A function or a map from a vector space V to a vector space W, assigns to each vector $\mathsf{v} \in V$ a vector $\mathsf{w} \in W$.

Definition
A mapping L from a vector space V to a vector space W is said to be a *linear transformation* if it satisfies

$$L(\alpha \mathsf{v}_1 + \beta \mathsf{v}_2) = \alpha L(\mathsf{v}_1) + \beta L(\mathsf{v}_2),$$

for all $\mathsf{v}_1, \mathsf{v}_2 \in V$ and all scalars α, β.

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To prove that a map is linear it is sufficient to prove

1. $L(\mathrm{v}_1 + \mathrm{v}_2) = L(\mathrm{v}_1) + L(\mathrm{v}_2)$
2. $L(\alpha \mathrm{v}) = \alpha L(\mathrm{v})$

Examples

Example
Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = 5x \quad \text{and} \quad g(x) = x^2.$$

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Then f is linear, and g is not , since

$$f(x+y) = 5(x+y) = 5x + 5y = f(x) + f(y)$$

and

$$\alpha f(x) = \alpha 5x = 5(\alpha x) = f(\alpha x).$$

However,

$$g(x+y) = (x+y)^2 = x^2 + 2xy + y^2 \neq g(x) + g(y).$$

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Let L_1 and L_2 be maps from $\mathbb{R}^3$ to itself defined by

$$L_1(\mathrm{v}) = 5\mathrm{v} \quad \text{and} \quad L_2(\mathrm{v}) = 5\mathrm{v} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

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$$L(\mathsf{v}_1 + \mathsf{v}_2) = 5(\mathsf{v}_1 + \mathsf{v}_2) = 5\mathsf{v}_1 + 5\mathsf{v}_2 = L(\mathsf{v}_1) + L(\mathsf{v}_2),$$

and

$$\alpha L_1(\mathsf{v}_1) = \alpha(5\mathsf{v}_1) = 5(\alpha\mathsf{v}_1) = L(\alpha\mathsf{v}_1).$$

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and

$$L_2(\mathsf{v}_1) + L_2(\mathsf{v}_2) = 5\mathsf{v}_1 + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 5\mathsf{v}_2 + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = L_2(\mathsf{v}_1 + \mathsf{v}_2) + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \neq L_2(\mathsf{v}_1 + \mathsf{v}_2).$$

Example
For each $\mathrm{x} = (x_1, x_2)^T \in \mathbb{R}^2$ define the map $L(\mathrm{x}) = (-x_1, x_1 + x_2)$. Is this linear?

Remark
Assume that L is as in the previous example. Then

$$L(\mathrm{x}) = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_1 + x_2 \end{pmatrix}.$$

Hence, we have

$$L(\mathrm{x}) = A\mathrm{x},$$

where

$$A = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Matrices are linear transformations

Given an $m \times n$ matrix A we can define a linear transformation L_A from $\mathbb{R}^n$ to $\mathbb{R}^m$ (or a subspace of these) by

$$L_A(\mathsf{x}) := A\mathsf{x}.$$

It is clearly linear as

$$\begin{aligned} L_A(\alpha \mathsf{x} + \beta \mathsf{y}) &= A(\alpha \mathsf{x} + \beta \mathsf{y}) \\ &= \alpha A\mathsf{x} + \beta A\mathsf{y} \\ &= \alpha L_A(\mathsf{x}) + \beta L_A(\mathsf{y}) \end{aligned}$$

Properties of linear transformations

A linear transformation L from V to W has the following properties

1. $L(0_V) = 0_W$
2. if $v_1, \dots, v_n$ are elements of V and $\alpha_1, \dots, \alpha_n$ are scalars, then

$$L(\alpha_1 v_1 + \dots \alpha_n v_n) = \alpha_1 L(v_1) + \dots \alpha_n L(v_n).$$

3. $L(-v) = -L(v)$ for all $v \in V$.

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Proof.
(1) follows from $\alpha L(\mathrm{v}) = L(\alpha \mathrm{v})$ with $\alpha = 0$, and (3) by $\alpha = -1$. $\square$

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Example
An important linear opeartor is the so called *identity operator* defined by $I(\mathrm{v}) = \mathrm{v}$ for all $\mathrm{v} \in V$.

Kernel

Definition
Let $L : V \to W$ be a linear transformation. The *kernel* of L denoted $\ker(L)$ is defined by

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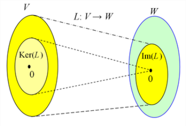

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Example
Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L(\mathrm{x}) = (-x_1, x_2)^T$, and put $S = \{\mathrm{x} \in \mathbb{R}^2 \mid x_2 = 0\}$. Then $L(S)$ consists of all vectors of the form $(a, 0)^T$ for some real value a.

If $L : V \to W$ is a linear transformation and S is a subsapce of V, then

1. $\ker(L)$ is a subspace of V
2. $L(S)$ is a subspace of W.

Example
Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \mathrm{x} = (x_1, x_2)^T.$$

We define $L_A(\mathrm{x}) = A\mathrm{x}$. The range is equal to the column space of A, and its kernel is the null space of A.

Matrices are linear transformations between $\mathbb{R}^n$ and $\mathbb{R}^m$

Theorem
If L is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ then there exists an $m \times n$ matrix A such that

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In particular the jth column of A is given by

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Proof.

Put $\mathrm{a}_j = L(\mathrm{e}_j)$, then we have

$$A = (\mathrm{a}_1, \ldots, \mathrm{a}_n).$$

If

$$\mathrm{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1\mathrm{e}_1 + \cdots + x_n\mathrm{e}_n$$

then

$$\begin{aligned} L(\mathrm{x}) &= x_1 L(\mathrm{e}_1) + \cdots + x_n L(\mathrm{e}_n) \\ &= x_1\mathrm{a}_1 + \cdots + x_n\mathrm{a}_n \\ &= A\mathrm{x}. \end{aligned}$$

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Find the matrix representations of the linear transformation

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Example
Find the matrix representation of the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ which reflects all vectors in the x-axis.

Dilations and contractions

Definition
A linear transformation from a vector space V to itself is also called *operator*.

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The matrix is given by cI, where I is the identity matrix.