L4 Systems of linear equations and row echelon form

1MA406/1MA901 Linear algebra

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Outline of the course

1. Mathematical thinking (2 lectures)
   - Logic and mathematical proofs
   - Sets and functions
2. Matrices and systems of linear equations (4 lectures)
   - Introductory examples and outline
   - **Systems of linear equations and row echelon form**
   - Matrix arithmetic and matrix algebra
   - Elementary matrices and the matrix inverse
3. Determinants (1 lectures)
   - Determinants
4. Vector spaces (2 lectures)
   - Vector spaces and its subspaces
   - Basis and dimensions
5. Linear transformations (1 lectures)
   - Linear transformations
6. Orthogonality (2 lectures)
   - The scalar product and orthogonal subspaces
   - Least squares problems
7. Eigenvalues (3 lectures)
   - Eigenvalues and eigenvectors
   - Diagonalization
   - Applications

Engelsk-svensk ordlista

English	Swedish
Homogeneous system	Homogent system
Nontrivial	Icke-trivial
Augmented matrix	utökad matris
row operations	radoperationer
Over- and underdetermined	Över- och underbestämt
Inconsistent system	olösbart system

Matrix representation of linear systems

For a linear system of equations of the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

we say that the m by n array (matrix) A of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

is the *coefficient matrix* of the linear system.

By appending the column of values b_i we obtain the *augmented matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix},$$

Row operations on augmented matrices

We define three *elementary row operations* to use on augmented matrices

1. Interchange two rows
2. Multiply a row by a nonzero real number
3. Replace a row by its sum with a multiple of another row.

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Definition

We say that two augmented matrices are equivalent if they differ by a finite number of elementary row operations.

From an earlier example we saw that

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 7 & -1 & -1 & 3 \end{pmatrix} \sim \begin{pmatrix} 8 & 0 & 0 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$

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Solve the system of equations

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The solution is

$$(x_1, x_2, x_3) = (3, -2, 4).$$

Row echelon form and Gaussian elimination

Definition

A matrix is said to be in *row echelon form* if the following are satisfied

1. The first nonzero entry in each nonzero row is 1
2. If row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k
3. If there are rows whose entries are all zero, they are all below the rows having nonzero entries

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Example

The following matrices are in row echelon form

$$\begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and these are *not*

$$\begin{pmatrix} 2 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The process of using row operations to obtain row echelon form is known as *Gaussian elimination*.

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$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then the system is inconsistent.

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Example

The linear systems with augmented matrices A and B given below

$$A = \begin{pmatrix} 1 & 1 & 6 \\ 2 & -1 & 1 \\ 3 & 2 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 1 & 6 \\ 7 & -1 & -1 & 3 \end{pmatrix}$$

are overdetermined and underdetermined respectively. In particular, the system corresponding to A is inconsistent.

The solution to an underdetermined system

A consistent underdetermined system always have an infinite solution set.

If we write an $m \times n$ underdetermined system in row echelon form, and assume that the system is consistent. Suppose that there are $r < n$ nonzero rows then there are r leading variables. These are determined by the r equations and in terms of the $n - r$ *free variables* which can be assigned any value.

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Example

Find all solutions to the following system of equations

$$\left\{ \begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & = & 0 \\ 2x_1 & - & 2x_2 & + & x_3 & = & 2 \\ 3x_1 & - & x_2 & + & 2x_3 & = & 2 \end{array} \right. .$$

The solution is presented on the next slide.

Cont. example

We apply row reduction on the augmented matrix, and obtain

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & -2 & 1 & 2 \\ 3 & -1 & 2 & 2 \end{pmatrix} \begin{array}{l} -2 \quad -3 \\ + \\ \quad\;\; + \end{array} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -4 & -1 & 2 \\ 0 & -4 & -1 & 2 \end{pmatrix} \begin{array}{l} \\ -1 \\ + \end{array}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -4 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/4 & -1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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x_1 and x_2 are the leading variables in the row echelon form. Thus, this is a $3 - 2 = 1$ parametric solution. Pick any unknown and set an arbitrary value for it, e.g. let $x_3 = t$. Then we can solve for x_2 and obtain

$$-4x_2 - t = 2 \iff x_2 = -\frac{2+t}{4}.$$

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Continuing for x_1 we obtain

$$x_1 + x_2 + x_3 = 0 \iff x_1 = \frac{2+t}{4} - t = \frac{2-3t}{4}.$$

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$$x_1 + x_2 + x_3 = 0 \iff x_1 = \frac{2+t}{4} - t = \frac{2-3t}{4}.$$

Hence, we get a parameter solution of one parameter t of the form

$$(x_1, x_2, x_3) = \left(\frac{2-3t}{4}, -\frac{2+t}{4}, t \right).$$

This means that we get a unique solution for each fixed value of $t \in \mathbb{R}$, and thus we have infinitely many solutions.

Reduced row echelon form

Definition

A matrix is said to be in *reduced row echelon form* if

1. The matrix is in row echelon form
2. The first nonzero entry in each row is the only nonzero entry in its column.

Finding the reduced row echelon form of a matrix is known as *Gauss-Jordan reduction*

Example

Use Gauss-Jordan reduction to solve the system

$$\begin{array}{ccccccccc} -x_1 & + & x_2 & - & x_3 & + & 3x_4 & = & 0 \\ 3x_1 & - & x_2 & - & x_3 & - & x_4 & = & 0 \\ 2x_1 & - & x_2 & - & 2x_3 & - & x_4 & = & 0 \end{array}.$$

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The solution is

$$(x_1, x_2, x_3, x_4) = t(-1/2, -3, 1/2, 1),$$

for all $t \in \mathbb{R}$.

An extensive example

Example

Find all solutions to the following system of equations

$$\left\{\begin{array}{lllllll} x_1 & + & 2x_2 & + & 3x_3 & = & 2 \\ x_1 & - & 2x_2 & - & x_3 & = & -2 \\ 4x_1 & + & x_2 & + & ax_3 & = & b \end{array}\right.$$

for all values of the constants a and b.

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Example

Find all solutions to the following system of equations

$$
\begin{cases}
x_1 + 2x_2 + 3x_3 = 2 \\
x_1 - 2x_2 - x_3 = -2 \\
4x_1 + x_2 + ax_3 = b
\end{cases}
$$

for all values of the constants a and b.

Solution.

We proceed as previously and apply row reduction on the augmented matrix

$$
\begin{pmatrix} 1 & 2 & 3 & 2 \\ 1 & -2 & -1 & -2 \\ 4 & 1 & a & b \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -4 & -4 & -4 \\ 0 & -7 & -12+a & -8+b \end{pmatrix}
$$

$$
\sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -4 & -4 & -4 \\ 0 & 0 & -5+a & -1+b \end{pmatrix}
$$

Depending on the choice of a and b we could get either unique solution, infinitely many or none. We start by the case of no solutions. We continue on the next slide.

An extensive example (cont.)

Case I: No solutions

This scenario will occur if $a = 5$ and $b \neq 1$, because in that case we will have $0 = b - 1$ in the last equation, and since we then require b to be different from one we get no solutions.

An extensive example (cont.)

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Case II: Parametric solution

This will occur if the last equation reads $0 = 0$, hence if $a = 5$ and $b = 1$. Then we put $x_3 = t$, and can solve for x_1 and x_2 in the first and second equation. We get

$$-4x_2 - 4t = -4 \iff x_2 + t = 1 \iff x_2 = 1 - t.$$

And for x_1 we get

$$x_1 + 2(1 - t) + 3t = 2 \iff x_1 = -t.$$

Hence, our solution is of the form $(x_1, x_2, x_3) = (-t, 1 - t, t)$.

An extensive example (cont.)

Case III: Unique solution

The last and final case occurs if $a \neq 5$. Then we obtain a unique solution of the form

$$x_3 = \frac{b-1}{a-5}.$$

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And finally

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Our solution is then

$$(x_1, x_2, x_3) = \left(-\frac{b-1}{a-5}, \frac{a-b-4}{a-5}, \frac{b-1}{a-5}\right),$$

and we found all solutions for all values of a and b.

Homogeneous systems

A system of linear equations of the form

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is said to be *homogeneous* if $b_1 = \cdots = b_m = 0$.

Homogeneous systems are always consistent as $(x_1, \ldots, x_n) = (0, \ldots, 0)$ is always a solution. This is called its *trivial solution*.

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Theorem

An $m \times n$ homogeneous system of linear equations has a nontrivial solution if $n > m$.

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Theorem

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Proof.

First, we know that the system is consistent since it is homogeneous. Each equation can fix at most one variable, and since $n > m$ we have that there are more variables than equations, and thus at least one variable is free. Since the free variable may be anything there exists nontrivial solutions. $\square$

Application: Traffic flow

In a city, four intersections are distributed as in Figure 1. Compute the amount of traffic in each intersection.

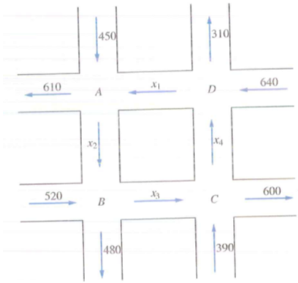


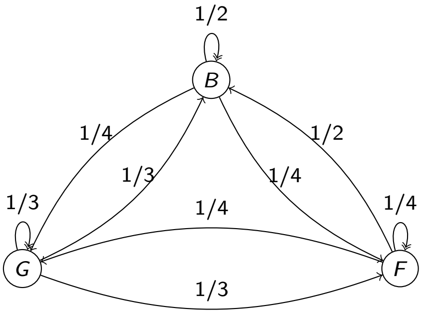
Figure: Traffic flow in four intersections.

Application: Leontief input-output model

Nobel prize laurete Wassily Leontief gave some early results on so-called input-output models, which are economic models that represents the dependencies between different sectors of an economy.

Suppose we have a small autonomous economy consisting of three sectors farming, beer production and royal guard. Each sector depends on each other, although the question is this relation is given. Suppose that the beer production goes up, what amount of increase of farming and the royal guard would this require?

The following graph describes the dependencies between the sectors.



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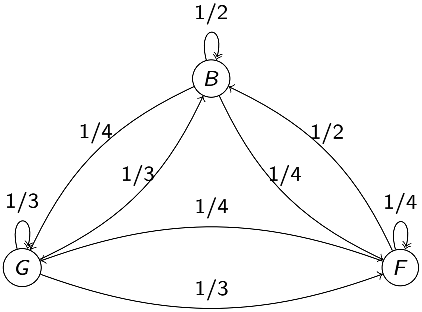

Let x_1 denote one unit of beer production, x_2 denotes one unit of farming, and x_3 one unit of guarding. In order to produce one unit of beer production we need

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{3}x_3 = x_1.$$

We can set up similar equations for the other sectors and thus receive a vector $x = (x_1, x_2, x_3)$ which would for example tell us a reasonable pricing of each good.

Preparation for next exercise session: Read and do the recommended exercises.

Next lecture we consider the arithmetic and algebra of matrices.

Thank you for today!