L3 Introductory examples

1MA901/1MA406 Linear algebra

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Engelsk-svensk ordlista

English	Swedish
linear system of equations	linjärt ekvationssystem
solution set	lösningsmängd
matrix	matris
back substitution	bakåtsubstitution

Defining systems of linear equations

Let $n \geq 1$ be an integer, and $a_1, \ldots, a_n, b \in \mathbb{R}$. A *linear equation in n unknowns* is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

A *linear system* of m equations in n unknowns is then a system of the form

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}\right. .$$

The system is said to be a $(m \times n)$ linear system.

Examples of linear systems and the solution

The following are examples of linear systems

$$\textbf{(a)} \quad \begin{cases} x_1 + x_2 = 4 \\ 4x_1 + x_2 = 1 \end{cases}$$

$$\textbf{(b)} \quad \begin{cases} x + y + z = 6 \\ 7x - y - z = 3 \end{cases}$$

$$\textbf{(c)} \quad \begin{cases} x_1 + x_2 = 6 \\ 2x_1 - x_2 = 1 \\ 3x_1 + 2x_2 = -2 \end{cases}$$

(a) is a 2×2 system, **(b)** is a 2×3 system, and **(c)** is a 3×2 system.

Definition
A *solution* to a $m \times n$ linear system is an n-tuple satisfying all the m equations.

Example
The pair $(x_1, x_2) = (-1, 5)$ is a solution for **(a)** as

$$\textbf{(a)} \quad \begin{cases} 1(-1) + 1(5) = 4 \\ 4(-1) + 1(5) = 1 \end{cases}$$

Geometric interpretation of a solution

A linear equation in 2 unknowns can be described as a line in $\mathbb{R}^2$. Hence, the solution to a linear 2×2 system is the point of intersection of the two lines in the system.

Example

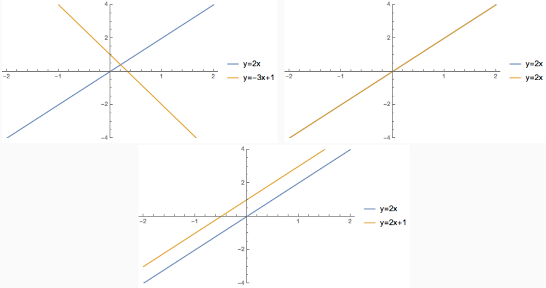

Figure 1: Three systems of linear equations with a unique, infinitely many and no solution(s) respectively.

Geometric interpretation of a solution

The corresponding systems are defined by the equations:

$$\left\{\begin{array}{rcrcr} 2x_1 & - & x_2 & = & 0 \\ 3x_1 & + & x_2 & = & 1, \end{array}\right. \quad \left\{\begin{array}{rcrcr} 2x_1 & - & x_2 & = & 0 \\ 2x_1 & - & x_2 & = & 0 \end{array}\right.$$

$$\text{and} \quad \left\{\begin{array}{rcrcr} 2x_1 & - & x_2 & = & 0 \\ 2x_1 & - & x_2 & = & -1. \end{array}\right.$$

Equivalent systems

Consider the following two systems of equations

$$\begin{cases} x_1 & + & x_2 & = & 4 \\ 4x_1 & + & x_2 & = & 1 \end{cases} \quad \text{and} \quad \begin{cases} 2x_1 & + & 2x_2 & = & 8 \\ -3x_1 & + & & = & 3. \end{cases}$$

By insertion we see that $(x_1, x_2) = (-1, 5)$ is a solution for both systems.

Definition
Two system of equations involving the same variables are said to be *equivalent* if they have the same solution set, and we use $\sim$ to denote that two systems are equivalent, *i.e.*

$$\begin{cases} x_1 & + & x_2 & = & 4 \\ 4x_1 & + & x_2 & = & 1 \end{cases} \sim \begin{cases} 2x_1 & + & 2x_2 & = & 8 \\ -3x_1 & + & & = & 3. \end{cases}$$

Remark
Let A, B and C be three systems of linear equations. It is clear that this is an equivalence relation, since

- *$A \sim A$, i.e. A has the same solution set as A (reflexive)*
- *$A \sim B \implies B \sim A$ (symmetric)*
- *if $A \sim B$ and $B \sim C$ then $A \sim C$ (transitive).*

Operations to obtain equivalent systems

There are three operations which can be applied to a system of equation without changing its solution set.

1. We may interchange the order of the equations
2. Both sides of an equation may be multiplied by the same nonzero real number
3. A multiple of one equation may be added to another.

By (1) we have

$$\left\{\begin{array}{cccccc} x_1 & + & x_2 & = & 4 \\ 4x_1 & + & x_2 & = & 1 \end{array}\right. \sim \left\{\begin{array}{cccccc} 4x_1 & + & x_2 & = & 1 \\ x_1 & + & x_2 & = & 4 \end{array}\right. .$$

Further, by (2) we have

$$\left\{\begin{array}{cccccc} x_1 & + & x_2 & = & 4 \\ 4x_1 & + & x_2 & = & 1 \end{array}\right. \sim \left\{\begin{array}{cccccc} x_1 & + & x_2 & = & 4 \\ 12x_1 & + & 3x_2 & = & 3 \end{array}\right. .$$

Finally, (3) implies for instance that

$$\left\{\begin{array}{cccccc} x_1 & + & x_2 & = & 4 \\ 4x_1 & + & x_2 & = & 1 \end{array}\right. \sim \left\{\begin{array}{cccccc} x_1 & + & x_2 & = & 4 \\ & + & -3x_2 & = & -15 \end{array}\right. ,$$

where the first equation is multiplied by (-4) and then added to the second.

Proof that systems under these row operations are equivalent

Theorem
The n-tuple x *is a solution to the equations* A_1 *and* A_2 *if and only if it is a solution to* A_1 *and* $A_1 + A_2$*.*

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Proof.
We start by proving the implication $\Rightarrow$. By our assumption x is a solution to the equations A_1 and A_2. Let A_1 and A_2 be of the form

$$a_1x_1 + a_2x_2 + \ldots a_nx_n = b,$$

and

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = d.$$

However, then x must also satisfy $A_1 + A_2$ as we have

$$(a_1 + c_1)x_1 + (a_2 + c_2)x_2 + \cdots + (a_n + c_n)x_n = b + d.$$

($\Leftarrow$): One the other hand, if x is a solution to A_1 and $A_1 + A_2$ then it is also a solution to

$$(A_1 + A_2) - A_1 = A_2.$$

□

Change of equations but not solution set

The two system of equations

$$\left\{\begin{array}{rcrcl} 2x_1 & - & x_2 & = & 0 \\ 3x_1 & + & x_2 & = & 1, \end{array}\right. \qquad \left\{\begin{array}{rcrcl} & & -5/3x_2 & = & -2/3 \\ 6x_1 & + & 2x_2 & = & 2 \end{array}\right.$$

are equivalent. The following plots in Figure 2 of these linear equations illustrates that the solutions set is left the same even though the equations are not.

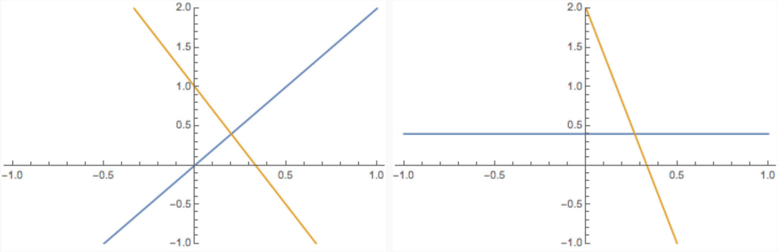

Figure 2: Two systems of linear equations with a the same solution set.

Example
Show that the two systems A and B are equivalent

$$\mathbf{A}:\begin{cases} x_1 + x_2 + x_3 = 1 \\ 7x_1 - x_2 - x_3 = 3 \end{cases} \quad \mathbf{B}:\begin{cases} 8x_1 = 4 \\ 2x_1 + 2x_2 + 2x_3 = 2 \end{cases}.$$

Solution.
We will prove this using a direct proof by writing A as B by using the operations defined in the previous slide. If we add the first equation to the second in A we see that

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 7x_1 - x_2 - x_3 = 3 \end{cases} \sim \begin{cases} x_1 + x_2 + x_3 = 1 \\ 8x_1 = 4 \end{cases}.$$

In addition, if we multiply the first equation by the nonzero number 2, and then interchange the rows we obtain

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 8x_1 = 4 \end{cases} \sim \begin{cases} 8x_1 = 4 \\ 2x_1 + 2x_2 + 2x_3 = 2 \end{cases},$$

which proves that $A \sim B$. $\square$

Triangle form

Definition
A system is said to be in *strict triangular form* if for all integers $k \in \{1, 2 \ldots, n\}$ we have that for the kth equation the coefficients of the first $k-1$ variables are all zero and the coefficient of x_k is nonzero.

Example
The system

$$\begin{cases} x_1 & + & x_2 & + & x_3 & = & 1 \\ & - & x_2 & - & x_3 & = & 2 \\ & & & & 3x_3 & = & 5 \end{cases}$$

is in strict triangle form.

Observation
Systems which are in strict triangular form are easy to solve.

We immediately obtain from the example that $x_3 = 5/3$. Substituting this value into the second equation we obtain

$$-x_2 - 5/3 = 2 \iff x_2 = -11/3.$$

Again, utilizing $(x_2, x_3) = (5/3, -11/3)$ we can easily solve the first equation, where we obtain

$$x_1 - 11/3 + 5/3 = 1 \iff x_1 = 3.$$

Hence, the solution to the system in the example is $(x_1, x_2, x_3) = (3, -11/3, 5/3)$.

The technique we just used to solve the system is known as *back substitution*.

Example
Solve the system of equations

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 3x_1 - x_2 - x_3 = 2 \\ 2x_1 + 2x_2 + 3x_3 = -2 \end{cases}.$$

Example: Solve the system of equations

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Solve the system of equations

$$\left\{\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 1 \\ 3x_1 & - & x_2 & - & x_3 & = & 2 \\ 2x_1 & + & 2x_2 & & +3x_3 & = & -2 \end{array}\right. .$$

The solution is

$$(x_1, x_2, x_3) = (3/4, 17/4, -4).$$

Matrix representation of linear systems

For a linear system of equations of the form

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \quad\vdots & & & & & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}$$

we say that the m by n array (matrix) A of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

is the *coefficient matrix* of the linear system.

By appending the column of values b_i we obtain the *augmented matrix*

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix} \sim \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right).$$

Row operations on augmented matrices

We define three *elementary row operations* to use on augmented matrices

1. Interchange two rows
2. Multiply a row by a nonzero real number
3. Replace a row by its sum with a multiple of another row.

Definition
We say that two augmented matrices are equivalent if they differ by a finite number of elementary row operations.

From an earlier example we saw that

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 7 & -1 & -1 & 3 \end{pmatrix} \sim \begin{pmatrix} 8 & 0 & 0 & 4 \\ 2 & 2 & 2 & 2 \end{pmatrix}.$$

Example
Solve the system of equations

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{pmatrix}.$$

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Example
Solve the system of equations

$$\begin{pmatrix} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{pmatrix}.$$

The solution is

$$(x_1, x_2, x_3) = (3, -2, 4).$$

Row echelon form and Gaussian elimination

Definition
A matrix is said to be in *row echelon form* if the following are satisfied

1. The first nonzero entry in each nonzero row is 1
2. If row k does not consist entirely of zeros, the number of leading zero entries in row $k+1$ is greater than the number of leading zero entries in row k
3. If there are rows whose entries are all zero, they are all below the rows having nonzero entries

Example
The following matrices are in row echelon form

$$\begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and these are *not*

$$\begin{pmatrix} 2 & 4 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The process to obtain row echelon form is known as *Gaussian elimination*.