Linnaeus University
Department of Mathematics
Karl-Olof Lindahl

Written Exam on Basic Numerical Methods, 1DV519, 5 ECTS
Saturday the 25th of November 2017, 12.00-17.00

The solutions should be complete, logically correct, well structured and easy to follow.
If you prefer you may write the answers in Swedish.
Aids: Calculator (you may use a scientific calculator but *not* with internet connection)
Grades: 15p-16p⇒E; 17p-18p⇒D; 19p-23p⇒C; 24p-27p⇒B; 28p-30p⇒A.

1. a) Given the following set of points $(x, y) : (1.0, 1.0), (2.0, 3.0), (3.0, 4.0), (4.0, 6.0)$; fit the data with a polynomial of degree *two* using the least square method. (4p)

   Selected part of the solution: Model $y = c_1 + c_2 x$. In the least square sense the function $y = -0.5 + 1.6x$ is the best approximation. The parameters of the model, c_1 and c_2, are obtained solving the system $A^T A\binom{c_1}{c_2} = A^T B$ where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}, \quad \text{and } B = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 6 \end{pmatrix}.$$

   b) How accurately do we need to know the natural number e to calculate e^2 with 4 correct decimals? (1p)

   Solution: Put $f(x) = x^2$ so that $f'(x) = 2x$.

   Want: $|\Delta f| \leq 0.5 \cdot 10^{-4}$.

   By the error propagation formula

$$\Delta x \approx \frac{\Delta f}{f'(x)}.$$

   For $x = e$ we then obtain

$$|\Delta x| \approx \left|\frac{\Delta f}{2x}\right| \leq \frac{0.5 \cdot 10^{-4}}{e} \leq \frac{0.5 \cdot 10^{-4}}{2 \cdot 2.718} \approx 0.92 \cdot 10^{-5}.$$

2. Use the Newton-Raphson method to find approximations of all solutions of the equation $f(x) = sin(2x) + xe^x + 1$ with 4 correct decimals.

   Selected part of solution: Crude localization of the zeros of $f(x) = sin(2x) + xe^x$ gives that f has two zeros $\alpha_1 \approx 2$ and $\alpha_2 \approx 4.5$. Using a few iteration with Newton Raphson gives $\alpha_1 \approx 2.0811$. Moreover $f(2.0811) > 0$ and $f(2.08105) < 0$ so $\alpha_1 \approx 2.0811$ with four correct decimals. Similarily $\alpha_2 \approx 4.5036$. Moreover $f(4.50365) > 0$ and $f(4.5036) < 0$ so $\alpha_2 \approx 4.5036$ with four correct decimals.

   (5p)

3. a) Use the trapetzoidal method to calculate approximate values of the integral

$$I = \int_1^2 \ln(x^3)dx,$$

for 3 different step lengths: $h = 1, 0.5, 0.25$. Use 6 correct decimals of function values. (2p)

Answer: $T(1) = 0.693147$; $T(0.5) = 0.752039$; $T(0.25) = 0.767399$.

b) Use Richardson extrapolation on the values of I obtained in a) to find an improved approximation of I. Estimate all the errors involved including the truncation error. (3p)

Answer: Using Richardsson extrapolation we obtain $I = 0.772576 \pm 0.6 \cdot 10^{-4}$. The truncation error is estimated by $|E_{tr}| \approx 0.57 \cdot 10^{-4}$ and the tabular error $E_{tab} \approx 0.5 \cdot 10^{-6}$.

4. Let $y(x)$ be the solution of $y'(x) = \frac{y}{2+\sin(x)}$ for which $y(2) = 1$.

a) Find an approximate value of $y(2.6)$ using Euler forward with step length $h = 0.1$. Use 6 correctly rounded decimals of function values in the written presentation. (2p)

Answer: $y(2.6; 0.1) \approx 1.526614$.

b) Calculate an iterative improvement of the approximate value of $y(2.6)$ obtained in a) using Richardson extrapolation. Also estimate the truncation error. (3p)

Selected part pf solution: With the double step length $h = 0.2$ using Euler Forward we obtain $y(2.6; 0.2) \approx 1.489543$. Using Richardson extrapolation and the fact that the truncation error in Euler Forward is of order 1 ($\mathcal{O}(h)$), we get

$$\tilde{y}(2.6) = y(2.6; 0.1) + \frac{y(2.6; 0.1) - y(2.6; 0.2)}{2^1 - 1} = 1.563685.$$

The truncation error is approximately $\frac{y(2.6;0.1)-y(2.6;0.2)}{2^1-1} \approx 0.37071$.

5. a) Find the error term and the order for the approximation formula

$$f(x) \approx \frac{1}{12h}\left(f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)\right).$$

(2p)

b) Assume that we have the following, correctly rounded function values for f: $f(0.4) = 0.010582$, $f(0.450) = 0.015034$, $f(0.475) = 0.017662$, $f(0.5) = 0.020574$, $f(0.525) = 0.023787$, $f(0.550) = 0.027313$, $f(0.6) = 0.035358$. Use the above given approximation of $f'(x)$, and Richardson extrapolation to approximate $f'(0.5)$, and estimate the error in the approximation. (2p)

c) Find the roots of the equation $x^2 + 9^{12}x = 3$ with four correct significant digits. (2p)

d) Assume that r is a zero of $f(x)$ and that $r + \Delta r$ is a zero of $f(x) + \epsilon \cdot g(x)$. Then

$$\Delta r \approx -\frac{\epsilon g(r)}{f'(r)}, \quad \text{for } \epsilon \text{ sufficiently smaller than } f'(r). \qquad (1)$$

Use this result to estimate the largest root of the equation

$$P(x) = (x-1)(x-2)(x-3)(x-4)(x-5)(x-6) - 10^{-6}x^7 = 0.$$

(2p)

e) Prove the above given formula (1). (2p)

Good Luck!