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Maximum probability shortest path problem☆

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The maximum probability shortest path problem involves the constrained shortest path problem in a given graph where the arcs resources are independent normally distributed random variables. We maximize the probability that all resource constraints are jointly satisfied while the path cost does not exceed a given threshold. We use a second-order cone programming approximation for solving the continuous relaxation problem. In order to solve this stochastic combinatorial problem, a branch-and-bound algorithm is proposed, and numerical examples on randomly generated instances are given.

1. Introduction

Shortest path problem (SPP for short) is a fundamental problem in combinatorial optimization. Though interesting in its own right, algorithms for this problem are also used as building blocks in the design of algorithms for a large number of industrial complex problems. As a result, there has been an extensive literature on SPP and its various aspects [6,9,10]. SPP is solvable in polynomial time. However, it becomes NP-hard when one or more additional constraints are added [13]. In this paper, we focus on the resource constrained shortest path problem (RCSP hereafter), especially when a subset of resource constraints parameters is random variables. The deterministic RCSP has attracted considerable attention in the literature [3,14,16,21,22,28,34]. RCSP is NP-complete even for the case of one resource [13].

Handler and Zang [13] gave an exact algorithm based upon a Lagrangian relaxation, while Hassin [15] presented a pseudo-polynomial algorithm. Jensen and Berry [19] proposed an algorithm which combined dynamic programming and the use of dominance approach. Aneja et al. [2] presented a generalization of the Dijkstra algorithm to solve the problem. Beasley and Christofides [5] presented a branch-and-bound approach based on a Lagrangian relaxation. Avella et al. [3] proposed a heuristic algorithm to approximate the problem, which is based on the extension to the discrete case of an exponential penalty function.

In the deterministic SPP (or RCSP), all the parameters (distance, time or cost) are known. However in real world applications, the parameters are not known in advance due to different uncertainty factors, e.g., travel times between two cities. Therefore, it is natural to consider these parameters as random variables, which turn the underlying problem into a stochastic optimization problem [31]. Stochastic shortest path problem (SSPP) has been widely studied in the literature. Hutsona and Shierb [18], Mirchandani et al. [23] and Murthy et al. [24] considered the problem of selecting a path which maximizes utility functions or minimizes cost functions. Ohtsubo [26,27] selected a probability distribution over the set of successor nodes and formulated this problem as a Markov decision process. Provan [30], Polychronopoulos and Tsitsiklis [29]

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studied the expected shortest paths in the networks, where information on arc cost values is accumulated as the graph is traversed.

In this paper, we study a variant of stochastic resource constrained shortest path problem (called SRCSP for short), namely the maximum probability of resource constraints. We are given a simple and acyclic digraph $G = (V, A)$, a source node s, a sink node t, a threshold of the cost function C, and K resource limits d_1 to d_K. Each arc e has a deterministic cost $c(e)$ and resource consuming $\tilde{a}_k(e)$ units of resource k where $1 \le k \le K$. We assume that the resources consumed by traversing arcs are independently normally distributed, i.e., $\tilde{a}_j(e_1)$ and $\tilde{a}_i(e_2)$ are independent provided $i \ne j$. The objective is to maximize the probability of all resource constraints to be jointly satisfied while the path cost does not exceed a given threshold. As the deterministic RCSP is NP-hard [13], SRCSP is also NP-hard by choosing all the resource parameter means of the arcs equal to zero. To the best of our knowledge, there is only one paper where the probability that a resource constraint is satisfied, is maximized using a quasi-convex method [25].

We propose a branch and bound framework to solve the maximum probability RCSP. In branch-and-bound method, bounding plays a crucial role. So special interest is given to its relaxation problems. As the linear relaxation of this problem (where $x \in \{0, 1\}^{|A|}$ is replaced by $x \in [0, 1]^{|A|}$) is generally not convex, we propose an efficient convex approximation to come up with tight upper bounds. Moreover, we improve the convexity results of Henrion and Strugarek [17]. Furthermore, numerical experiments on a set of generated instances from the literature are conducted to illustrate the efficiency of our approach. The remainder of the paper is organized as follows. The mathematical formulation of SRCSP is given in Section 2. In Sections 3 and 4, the convex relaxations of SRCSP are presented for the case of individual chance constraint, i.e., $K = 1$ and for the case of joint chance constraints, i.e., $K > 1$ respectively. In Section 5, the standard approximation of the individual constraint maximum probability is introduced together with solving method. In Section 6, numerical results are given to compare the two relaxations either with individual or joint chance constraints. The conclusions are given in the last section.

2. SRCSP formulation

SRCSP can be formulated as a stochastic combinatorial optimization problem in the following way: let $x \in \{0, 1\}^{|A|}$ such that each component x_a of x represents an arc $a \in A$. For a directed path P, we define the corresponding $x = x(P)$ such that $x_a = 1$ if and only if $a \in P$. Then, SRCSP can be mathematically formulated as follows:

$$\begin{aligned}
\max \quad & \boldsymbol{Pr}\{\tilde{a}_k^T x \le d_k, k = 1, \ldots, K\} \\
& c^T x \le C && (1) \\
\text{s.t.} \quad & Mx = b && (2) \\
& x \in \{0, 1\}^n && (3)
\end{aligned}$$

where $c \in R^n,\ C \in R,\ M \in \mathbb{R}^{m \times n}$ is the *node–arc incidence matrix* [1], and $b \in \mathbb{R}^m$ is a vector where all elements are 0 except the s-th and the t-th components which are equal to 1 and -1 respectively. $\tilde{a}_k \in R^n,\ k = 1, \ldots, K$ are independent random vectors in R^n with a multivariate normal distribution with a known mean μ_k and a known covariance matrix V_k; $d_k \in R,\ k = 1, \ldots, K$.

This problem can be reformulated as:

$$\begin{aligned}
& \max \quad p \\
& \text{s.t.} \quad p \le \boldsymbol{Pr}\{\tilde{a}_k^T x \le d_k,\ k = 1, \ldots, K\} && (4) \\
\text{SRCSP} \quad & c^T x \le C && (5) \\
& Mx = b \\
& x \in \{0, 1\}^n.
\end{aligned}$$

Assumptions. Let x^* be the optimal solution of SRCSP, we assume $\frac{1}{2} \le \boldsymbol{Pr}\{\tilde{a}_k^T x^* \le d_k\},$ for any $k \in \{1, \ldots, K\}$. In other words, the resource threshold is at least as large as the expectation of the resources consumed by traversing the path.

We consider two different variants of SRCSP: the case where $K = 1$, i.e., individual probabilistic constraint; and the case of joint chance constraints where $K > 1$.

3. SRCSP with individual chance constraint

In this section, we consider the individually probabilistic SRCSP, i.e., $K = 1$. Before giving the mathematical formulation of the problem, we introduce a small instance to illustrate SRCSP problem as in Fig. 1. In each arc, there is not only one deterministic cost but also a random resource consumption which is assumed to be normally distributed. For instance, for arc (1, 2), the cost is 1 and the resource consumption is normally distributed with mean 3 and variance 1. For the sake of simplicity, all the resource consumptions of the arcs are assumed to be independent. The threshold of path costs C is set to

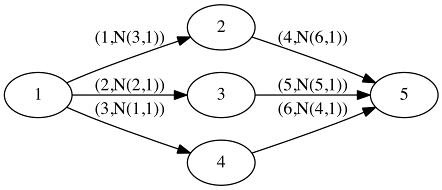

Fig. 1. SRCSP with one resource constraint.

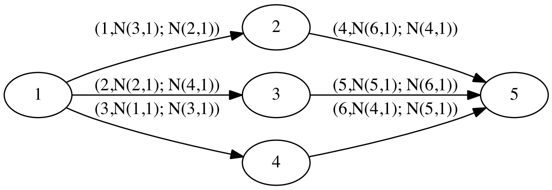

Fig. 2. SRCSP with two resource constraints.

8 and the resource threshold d_1 is set to be 10. Without the resource constraint, the shortest path is $1 \to 2 \to 5$ whose cost is 5 while its probability of satisfying the resource constraint is 0.76. However, for SRCSP problem, the optimal path is $1 \to 3 \to 5$ whose probability is 0.98 and the cost is 7.

When $K = 1$, we have the following deterministic reformulation of SRCSP:

$$\max \quad p$$

$$\text{s.t.} \quad F^{-1}(p)(x^T V_1 x)^{\frac{1}{2}} \le d_1 - \mu_1^T x \tag{6}$$

$$c^T x \le C \tag{7}$$

$$Mx = b \tag{8}$$

$$x \in \{0, 1\}^n \tag{9}$$

where $F^{-1}(\cdot)$ is the inverse of the standard normal cumulative distribution function. We solve the linear relaxation of this problem by using a binary search algorithm.

Consider the following optimization model with $p \ge \frac{1}{2}$:

$$\begin{aligned} & \max \quad 0 \\ & \text{s.t.} \quad F^{-1}(p)(x^T V_1 x)^{\frac{1}{2}} \le d_1 - \mu_1^T x \\ \text{(SRCSPI)} \quad & c^T x \le C \\ & Mx = b \\ & 0 \le x \le 1. \end{aligned}$$

Solving SRCSP consists in finding a feasible solution of SRCSPI. If the problem has a feasible solution, then it means that the optimal value of SRCSPI is greater than or equal to p. Then, we increase the value of p and compute a new feasible solution.

Algorithm 1 (*Binary Search Procedure*)**.**

- *Initialization:* Let $p_1 \ge \frac{1}{2}$ be a feasible solution of SRCSPI, i.e., lower bound of SRCSP. p_l and p_u are lower and upper bounds of SRCSPI respectively, e.g., $p_l = \frac{1}{2}$ and $p_u = 1$. Set the iteration counter to $t \leftarrow 1$.
- *Update:* Solve SRCSPI with $p = p_t$. If SRCSPI has an optimal solution, set $p_l = p_t$. Otherwise, set $p_u = p_t$.
- *Stopping criteria:* If $\frac{p_u - p_l}{p_u} \le \epsilon$ where ϵ is a given tolerance, return p_t and stop. Otherwise, set $t \leftarrow t + 1$ and go to step *Update* with $p_t = \frac{p_l + p_u}{2}$.

4. SRCSP with joint probabilistic constraints

In this section, we consider the joint probabilistic SRCSP, i.e., $K > 1$, and give an approximation approach and a convexity result to the corresponding linear relaxation problem. Like the individual probabilistic SRCSP, we also introduce a small instance to illustrate SRCSP problem as follows (see Fig. 2):

Each arc has two normally distributed resources consumptions and a deterministic cost. For instance, arc (1, 2) has cost equal to 1, the first resource consumption has mean 3 and variance 1, and the second resource consumption has mean 6 and variance 1. For the sake of simplicity, we also assume that all the resource consumptions of the arcs are independent.

The threshold of path costs C and the first resource threshold d_1 are set to be the same as the instance of Fig. 1. Moreover, the second resource threshold d_2 is set to be the same as d_1. For the joint probabilistic SRCSP problem, the optimal path is $1 \to 2 \to 5$ whose joint probability is 0.76, whereas the path $1 \to 3 \to 5$, whose joint probability is 0.49, is the optimal solution for the individual probabilistic SRCSP in Fig. 1.

When $K > 1$, the joint probabilistic SRCSP can be formulated as follows:

$$\begin{aligned}
&\max && p \\
&\text{s.t.} && p \le \boldsymbol{Pr}\{\tilde{a}_k^T x \le d_k,\ k = 1, \ldots, K\} && (10)\\
&\text{SRCSPJ} && c^T x \le C && (11)\\
& && Mx = b \\
& && 0 \le x_i \le 1, \quad i = 1, \ldots, n. && (12)
\end{aligned}$$

Henrion and Strugarek [17] studied the convexity of chance constrained problem with normally distributed coefficients together with the independence assumption on the coefficient matrix row vectors. Let $M(p)$ be the feasible set of the probabilistic constraints.

$$M(p) := \{x : \boldsymbol{Pr}\{\tilde{a}_k^T x \le d_k,\ k = 1, \ldots, K\} \ge p\}.$$

Theorem 4.0.1 *([17]). If $p > F(\max\{\sqrt{3}, u^*\})$, then $M(p)$ is convex where $u^* = \max_{k=1,\ldots,K} 4\lambda_{\max}^{(k)}[\lambda_{\min}^{(k)}]^{-\frac{3}{2}}\|\mu_k\|$ and $\lambda_{\max}^{(k)}$ and $\lambda_{\min}^{(k)}$ are the largest and smallest eigenvalues of Σ_k.*

From this theorem, we note that when the mean vector $\|\mu_k\|$ is equal to zero, then $M(p)$ is convex for $p > F(\sqrt{3})$. However, we can prove that $M(p)$ is still convex for smaller values of p.

Theorem 4.0.2. *If $u_k = 0$, $k = 1, \ldots, K$ and $p > F(\sqrt{2})$, then $M(p)$ is convex.*

Proof. The proof is twofold: first we prove $G(z) := \frac{1}{F^{-1}(p^z)}$ is concave on the interval $(0, 1]$. Then we prove that $\sqrt{x^T \Sigma_k x}$ is convex.

$$M(p) = \left\{x : F^{-1}(p^{z_k})\|\Sigma_k^{1/2}x\| \le D_k, \sum_{k=1}^{K} z_k = 1, z_k \ge 0, k = 1, \ldots, K\right\}.$$

The second derivative of $G(z)$ is

$$G''(z) = \frac{-(\ln p)^2 p^z F^{-1}(p^z)[F^{-1}(p^z) f(F^{-1}(p^z)) - 2p^z + p^z(F^{-1}(p^z))^2]}{(F^{-1}(p^z))^4 (f(F^{-1}(p^z)))^2}.$$

When $p > 0.5$, $F^{-1}(p^z)$ is positive. In order to verify the positivity, it is sufficient to show this property for the term

$$-2 + (F^{-1}(p^z))^2.$$

As $F^{-1}(p^z) \ge \sqrt{2}$, i.e., $p^z \ge F(\sqrt{2})$, $G''(z)$ is non-positive. Thus, if $p \ge F(\sqrt{2})$, then $\frac{1}{F^{-1}(p^z)}$ is concave on the interval $(0, 1]$.

The Hessian matrix of $\sqrt{x^T \Sigma_k x}$ is given by

$$H(\sqrt{x^T \Sigma_k x}) = \frac{x^T \Sigma_k x \Sigma_k - \Sigma_k x x^T \Sigma_k}{(x^T \Sigma_k x)^{\frac{3}{2}}}.$$

For any given $v \in R^n$,

$$v^T H(\sqrt{x^T \Sigma_k x}) v = \frac{(x^T \Sigma_k x)(v^T \Sigma_k v) - (v^T \Sigma_k x)^2}{(x^T \Sigma_k x)^{\frac{3}{2}}}.$$

By the Cauchy–Schwarz inequality, we have

$$v^T H(\sqrt{x^T \Sigma_k x}) v \ge 0.$$

Therefore, $\sqrt{x^T \Sigma_k x}$ is convex. In this case $\sqrt{x^T \Sigma_k x} - \frac{D_k}{F^{-1}(p^{z_k})}$ is convex, and $M(p)$ is a convex set. $\square$

4.1. Convex relaxed SRCSPJ

We have the equivalent deterministic reformulation of SRCSPJ as following:

$$\begin{aligned} \max \quad & p \\ \text{s.t.} \quad & F^{-1}(p^{y_k})(x^T V_k x)^{\frac{1}{2}} \le d_k - \mu_k^T x, \quad k = 1, \dots, K \\ & \sum_{k=1}^{K} y_k = 1, \quad y_k \ge 0 \quad k = 1, \dots, K \\ & c^T x \le C \\ & Mx = b, \quad x \in \{0, 1\}^n. \end{aligned} \tag{13}$$

Theorem 4.1.1. *The optimal value of SRCSPJ is equal to the optimal value of the following problem:*

$$\begin{aligned} \max \quad & p_0^{\sum_{k=1}^{K} y_k} \\ s.t. \quad & F^{-1}(p_0^{y_k})(x^T V_k x)^{\frac{1}{2}} \le d_k - \mu_k^T x, \quad k = 1, \dots, K \\ & 0 \le y_k \le -\log_{p_0}(2), \quad k = 1, \dots, K \\ & c^T x \le C \\ & Mx = b, \quad x \in \{0, 1\}^n \end{aligned} \tag{14}$$

where $p_0 \ge \frac{1}{2}$ is a fixed lower bound of $\boldsymbol{Pr}\{\tilde{a}_k^T x^* \le d_k\}$, $k \in \{1, \dots, K\}$, where x^* is an optimal solution of SRCSPJ.

Proof. Let $p^*, \bar{x}$ and $\bar{y}_k$, $k = 1, \dots, K$ be the optimal solutions of (13), while $\hat{x}$ and $\hat{y}_k$, $k = 1, \dots, K$ be the optimal solutions of (14).

First, we prove $p^* \le p_0^{\sum_{k=1}^{K} \hat{y}_k}$. Let $\bar{y}_k^* = \bar{y}_k \log_{p_0} p^*$. Thus $p_0^{\bar{y}_k^*} = (p^*)^{\bar{y}_k}$. Since $\frac{1}{2}$ is a lower bound of $\boldsymbol{Pr}\{\tilde{a}_k^T \bar{x} \le d_k\}$, $k \in \{1, \dots, K\}$, then we have $(p^*)^{\bar{y}_k} \ge \frac{1}{2}$. Furthermore, as $p_0 \le 1$, thus $\bar{y}_k^* = \bar{y}_k \log_{p_0} p^* \le \log_{p_0} \frac{1}{2} = -\log_{p_0}(2)$. Therefore, $\bar{x}$ and $\bar{y}_k^*$, $k = 1, \dots, K$ are feasible solutions of (14). Then $p^* = p_0^{\sum_{k=1}^{K} \bar{y}_k^*} \le p_0^{\sum_{k=1}^{K} \hat{y}_k}$, the latter is the optimal value of (14).

Second, we prove $p^* \ge p_0^{\sum_{k=1}^{K} \hat{y}_k}$. Let $\hat{p} = p_0^{\sum_{k=1}^{K} \hat{y}_k}$ and $\hat{y}_k^* = \hat{y}_k \log_{\hat{p}} p_0$. We have $\sum_{k=1}^{K} \hat{y}_k^* = 1$. Furthermore, as $\hat{p}$ and p_0 are less than one, then $\hat{y}_k^* \ge 0$. So $\hat{p}$, $\hat{y}_k^*$ and $\hat{x}$ are feasible solutions of (13). Thus $\hat{p} \le p^*$, i.e., $p^* \ge p_0^{\sum_{k=1}^{K} \hat{y}_k}$. □

4.1.1. Relaxation of SRCSPJ

The idea to approximate the problem SRCSPJ is the following: firstly, we approximate $F^{-1}(p_0^{y_k})$ with a piecewise tangent approximation of y_k [8]. Secondly, by applying the linearization method introduced in [11], we get an approximation, which is a second-order cone programming (SOCP for short) problem. Furthermore, the optimal value of the SOCP problem is an upper bound of SRCSPJ, and it converges to the optimal value of SRCSPJ as the number of segments N goes to infinity.

Theorem 4.1.2. *By applying the tangent approximation of $F^{-1}(p_0^{y_k})$ and the standard linearization method, we have the approximation of SRCSPJ, correspondingly:*

$$\begin{aligned} \min \quad & \sum_{k=1}^{K} y_k \\ s.t. \quad & (\tilde{z}_k^T V_k \tilde{z}_k)^{\frac{1}{2}} \le d_k - \mu_k^T x, \quad k = 1, \dots, K \\ & \tilde{z}_{ki} \ge a_j x_i + b_j y_{ki}, \quad j = 0, 1, \dots, N, \; i = 1, \dots, n \\ & 0 \le y_k \le -\log_{p_0}(2), \quad k = 1, \dots, K \\ & 0 \le y_{ki} \le y_k, \quad y_{ki} \ge y_k + x_i - 1, \; i = 1, \dots, n, k = 1, \dots, K \\ & c^T x \le C, \qquad Mx = b, \quad x \in \{0, 1\}^n \end{aligned} \tag{15}$$

where $a_0 = 0$, $b_0 = 0$. Let y^* and p^* be the optimal values of the approximation and SRCSPJ, respectively. Then p_0^{y*} is an upper bound of SRCSPJ, i.e., $p_0^{y*} \ge p^*$. Furthermore, $\lim_{N \to \inf} p_0^{y*} = p^*$.

Proof. By applying the standard linearization technique introduced in [11] and the theory presented in [8], we show that the optimal value of (15) is an upper bound of SRCSPJ. Finally, one can apply the results of the piecewise linear approximation presented in [33]. □

Hence, we get a convex relaxation of SRCSP as follows:

$$\begin{aligned}
\min \quad & \sum_{k=1}^{K} y_k \\
\text{s.t.} \quad & (\tilde{z}_k^T V_k \tilde{z}_k)^{\frac{1}{2}} \le d_k - \mu_k^T x, \quad k = 1, \dots, K \\
& \tilde{z}_{ki} \ge a_j x_i + b_j y_{ki}, \quad j = 0, 1, \dots, N, i = 1, \dots, n \\
& 0 \le y_k \le -\log_{p_0}(2), \quad k = 1, \dots, K \qquad (16)\\
& 0 \le y_{ki} \le y_k, \quad y_{ki} \ge y_k + x_i - 1, \ i = 1, \dots, n, k = 1, \dots, K \\
& c^T x \le C, \quad Mx = b, \ 0 \le x_i \le 1, \ i = 1, \dots, n \\
& M\tilde{y}_k = y_k b, \quad k = 1, \dots, K \qquad (17)
\end{aligned}$$

where $\tilde{y}_k = (y_{k1}, \dots, y_{kn})$. The constraint (17) is redundant in problem (15) because of binary constraints, but it plays an important role in the relaxation. The relaxation is called hereafter the convex relaxation.

5. Individually relaxed SRCSPJ

In this section, we give another stochastic shortest path problem, Maximum Probability on the Individual Constraints (SRCSPJI for short):

$$\text{SRCSPJI} \quad \begin{aligned}
\max \quad & p \\
\text{s.t.} \quad & p \le \boldsymbol{Pr}\{\tilde{a}_k^T x \le d_k\}, \quad k = 1, \dots, K \\
& c^T x \le C \\
& Mx = b \\
& x \in \{0, 1\}^n.
\end{aligned} \qquad (18)$$

It is easy to show that Maximum Probability on the Individual Constraints is a relaxation of SRCSPJ. The Maximum Probability on the Individual Constraints is called hereafter the individual approximation.

A major shortcoming of the individual approximation is that the approximation quality depends heavily on how close to 1 is $\boldsymbol{Pr}\{\tilde{a}_k^T x \le d_k\}, k = 1, \dots, K$. For instance, we assume $K = 5, x^*$ is an optimal solution of SRCSPJ and $p_k = \boldsymbol{Pr}\{\tilde{a}_k^T x^* \le d_k\}$. If $p_k = 1, \ k = 1, 2, 3, 4$ and $p_5 = 0.8$, then 0.8 is the optimal value of both the individual approximation and the SRCSPJ. In other words, the solution of individual approximation is the exact solution of SRCSPJ. However, if $p_k = 0.8, \ k = 1, 2, 3, 4, 5$, then 0.8 is the optimal value of the individual approximation whilst the optimal value of SRCSPJ is $0.8^5 = 0.328$ far from 0.8.

The individual relaxation has an equivalent deterministic reformulation given by:

$$\begin{aligned}
\max \quad & p \\
\text{s.t.} \quad & F^{-1}(p)(x^T V_k x)^{\frac{1}{2}} \le d_k - \mu_k^T x, \quad k = 1, \dots, K \\
& c^T x \le C \\
& Mx = b, \quad x \in \{0, 1\}^n.
\end{aligned} \qquad (19)$$

Correspondingly, we have the linear relaxation of SRCSPJI as follows:

$$\begin{aligned}
\max \quad & p \\
\text{s.t.} \quad & F^{-1}(p)(x^T V_k x)^{\frac{1}{2}} \le d_k - \mu_k^T x, \quad k = 1, \dots, K \\
& c^T x \le C \\
& Mx = b, \quad 0 \le x_i \le 1, \ i = 1, \dots, n.
\end{aligned} \qquad (20)$$

Naturally, the optimal value of (20) is a upper bound of SRCSPJ, i.e., Maximum Probability on the Joint Constraints.

5.1. Solving the relaxed SRCSPJI

For the linear relaxed SRCSPJI, we can still use a binary search algorithm to solve it. Consider the following optimization model when $p \ge \frac{1}{2}$ is fixed:

$$\text{(RSRCSPJI)} \quad \begin{aligned}
\max \quad & 0 \\
\text{s.t.} \quad & F^{-1}(p)(x^T V_k x)^{\frac{1}{2}} \le d_k - \mu_k^T x, \quad k = 1, \dots, K \\
& c^T x \le C \\
& Mx = b, \quad 0 \le x_i \le 1, \ i = 1, \dots, n.
\end{aligned}$$

Table 1
Computational results of SRCSP as $(n, m) = (23, 40)$.

DATA	Convex relaxation + B&B					Individual relaxation + B&B				
Name	UB_1	CPU (s)	Optimum	CPU (s)	Nodes	UB_2	CPU (s)	Optimum	CPU (s)	Nodes
Inst1	0.85	6.0	0.57	79.0	13	0.92	4.4	0.57	174.2	49
Inst2	0.84	6.3	0.54	79.3	13	0.92	4.0	0.54	114.3	32
Inst3	0.60	7.2	0.58	35.2	6	0.80	4.1	0.58	47.5	14
Inst4	0.73	7.0	0.43	113.0	21	0.85	4.7	0.43	122.3	36
Inst5	0.70	7.1	0.36	77.2	15	0.81	4.0	0.36	97.6	29

The main idea of the approach is the same as aforementioned when $K = 1$, i.e., finding a feasible solution of RSRCSPJI. If the problem has a feasible solution, it means that the optimal value of RSRCSPJI is greater than or equal to p. Then, we increase the value of p and compute a new feasible solution.

Algorithm 2 (*Binary Search Procedure*).

- *Initialization:* Let $p_1 \geq \frac{1}{2}$ be a feasible solution of RSRCSPJI, i.e., lower bound of the relaxed SRCSPJI. p_l and p_u are lower and upper bounds of RSRCSPJI respectively, e.g., $p_l = \frac{1}{2}$ and $p_u = 1$. Set the iteration counter to $t \leftarrow 1$.
- *Update:* Solve RSRCSPJI with $p = p_t$. If RSRCSPJI has an optimal solution, set $p_l = p_t$. Otherwise, set $p_u = p_t$.
- *Stopping criteria:* if $\frac{p_u - p_l}{p_u} \leq \epsilon$ where ϵ is a tolerance parameter. Return p_t and stop. Otherwise, set $t \leftarrow t + 1$ and go to step *Update* with $p_t = \frac{p_l + p_u}{2}$.

6. Numerical results

Both the convex relaxation and the individual relaxation problems as well as a branch-and-bound algorithm [7] were implemented in Matlab and solved by Sedumi [32] on an Intel(R)D @ 2.00 GHz with 4.0 GB RAM. For Sedumi, we use default parameters except the accuracy parameter, where the value is set to 10^{-4}. The accuracy parameter ϵ is also set to 10^{-4}. Our algorithm can perform with smaller values than 10^{-4} for the numerical precision. We choose 10^{-4} as a threshold for our experiments as this value is far enough to come up with good solutions from numerical point of view. Smaller values of ϵ do not bring any significant improvement.

We considered both deterministic graphs and randomly generated graphs for our tests. We used two deterministic directed graphs with $(|V|, |A|)$ equal to $(23, 40)$ and $(50, 413)$ respectively, where the first graph is from Ji [20], while the other is a modified graph of the OR-library (see [4]). The random graphs are generated with the same rules as Goel et al. [12], where there are two parameters: node size n and connectivity p_c which is the probability of a link existence between nodes i and j. Moreover, we add the link between i and $i+1$ to guarantee that there exists a path from node 1 to node n in the random generated graph. For the sake of simplicity, the random variables in the same row vector are assumed to be independent, i.e., the covariance matrix V_k is a diagonal matrix. The number of resources K is set to 10, while the input data for the models is randomly generated as follows. The cost c is uniformly generated on the interval $[0, 5]$. The mean μ_k is uniformly generated on the interval $[4, 6]$ while the entries of the covariance matrix V_k are uniformly generated on the interval $[0, 1]$. Resource threshold parameter d_k is uniformly generated on the interval $[15, 20]$. The threshold of path costs C is set to the double of the cost of the shortest path. Finally, for the piecewise tangent approximation, we choose three tangent points, i.e., $z_1 = 0.01$, $z_2 = 0.15$ and $z_3 = 0.45$. We observed that for some generated instances, the generated resource threshold parameter d_k is too large which leads the probability of the problem to be close to 1. To avoid this situation, we choose the instances accordingly.

We solve SRCSP by using branch-and-bound algorithm [7], where the upper bound is calculated by solving the convex relaxation (16) and the individual linear relaxation of (20), respectively. We denote the optimal objective values of the upper bounds by UB_1 and UB_2, while we denote the optimal value of branch-and-bound by *Optimum*.

Numerical results are given by Tables 1–3, where column one gives the name of the instances in the first two tables and the connectivity probability p_c in the last one. The columns two and three present the optimal value of the convex relaxation and the corresponding CPU time respectively. Columns four, five and six give the optimal value of the branch-and-bound, the relative CPU time and the number of nodes respectively. The last five columns give the upper bound, CPU time, optimal values, CPU time and the number of nodes for the individual relaxation and its branch-and-bound respectively. All the CPU times are given in seconds.

From Tables 1 to 3, we observe that for the continuous SRCSP, the upper bound of the convex relaxation is less than the one for the individual approximation for all instances, and the largest gap between the two upper bounds is 0.24. Admittedly, it takes more CPU time to get the upper bound for the convex relaxation than for the individual approximation. For the combinatorial SRCSP, when using the convex relaxation, a smaller number of nodes has been considered in the branch-and-bound algorithm compared to the individual approximation. This can be explained by the smaller upper bounds and therefore a smaller number of non-rejected subtrees. Accordingly, it takes less CPU time for the convex relaxation than for solving the individual approximation for all the instances of the graphs. Moreover, in the worst case for all the graphs, the CPU time of the branch-and bound of the individual approximation is more than double of the one of the convex relaxation. Finally, our numerical results show that our approaches are robust, and can be easily extended to solve larger size instances.

Table 2
Computational results of SRCSP as $(n, m) = (50, 413)$.

DATA	Convex relaxation + B&B					Individual relaxation + B&B				
Name	UB_1	CPU (s)	Optimum	CPU (s)	Nodes	UB_2	CPU (s)	Optimum	CPU (s)	Nodes
Inst1	0.76	48.4	0.33	950.7	23	0.87	41.3	0.33	1400.9	39
Inst2	0.62	44.9	0.11	1756.5	46	0.86	37.8	0.11	2843.3	87
Inst3	0.78	51.0	0.42	1021.9	24	0.86	39.8	0.42	1835.6	52
Inst4	0.85	46.7	0.14	1673.4	42	0.92	37.0	0.14	3396.3	102
Inst5	0.78	53.0	0.21	2835.1	67	0.88	39.8	0.21	3679.9	103

Table 3
Computational results of SRCSP with random graphs as $n = 50$.

DATA	Convex relaxation + B&B					Individual relaxation + B&B				
Name	UB_1	CPU (s)	Optimum	CPU (s)	Nodes	UB_2	CPU (s)	Optimum	CPU (s)	Nodes
$p_c = 0.3$	0.83	48.7	0.35	598.6	17	0.88	42.1	0.35	2706.1	74
$p_c = 0.3$	0.85	45.8	0.31	1380.0	40	0.89	42.3	0.31	1806.0	50
$p_c = 0.4$	0.81	62.3	0.29	2523.2	49	0.90	60.7	0.29	5221.3	95
$p_c = 0.4$	0.82	64.4	0.29	1236.3	24	0.87	63.2	0.29	2789.5	51
$p_c = 0.5$	0.73	90.8	0.20	4010.1	56	0.83	89.0	0.20	5521.2	67

7. Conclusions

In this paper, we study and solve a stochastic version of the resource constrained shortest path problem, i.e., the maximum probability problem. To solve this problem, we propose to use a branch-and-bound framework to come up with the optimal solution. Firstly, we improve the convexity results of its corresponding linear relaxations when $\mu = 0$. Then, as its linear relaxation is generally not convex, we put forward a convex relaxation whose optimal value provides an upper bound. To measure the quality of the convex relaxation, we implement the Maximum Probability on the Individual Constraints. Finally, our numerical results on the instances from the literature substantiate that the convex relaxation outperforms the individual approximation. As further research, we plan to generalize the assumptions to the case where the random variables are dependent, and show that our approach could be extended to other stochastic combinatorial optimization problems.

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