COMPLEX MANIFOLDS

YANNIS BÄHNI

Abstract.

Contents

1. Functions of Several Complex Variables

This section summarizes the fundamental properties of functions of several complex variables needed later. The results are inspired by [FG10, pp. 14–30].

Definition 1.1. *Let $n \in \mathbb{Z}$, $n \geq 1$, $U \subseteq \mathbb{C}^n$ open and $a \in U$. A mapping $f : U \to \mathbb{C}$ is said to be* ***complex differentiable at a*** *if there exists $g : U \to \mathbb{C}^n$ such that g is continuous at a and*

$$f(z) = f(a) + \sum_{\nu=1}^{n} (z_\nu - a_\nu) g_\nu(z) \tag{1}$$

holds for all $z \in D$. f is said to be ***holomorphic in D*** *if it is complex differentiable at every point $a \in D$. For $m \in \mathbb{Z}$, $m \geq 1$, a mapping $f : U \to \mathbb{C}^m$ is said to be holomorphic in D if each component function f_ν, $\nu = 1, \ldots, n$, is holomorphic in D.*

Proposition 1.1. *Let $n \in \mathbb{Z}$, $n \geq 1$, $D \subseteq \mathbb{C}^n$ open, $a \in U$ and $f : D \to \mathbb{C}$ real differentiable at a. Then*

$$\frac{\partial f}{\partial z_\nu}(a) = \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu}(a) - i\frac{\partial f}{\partial y_\nu}(a)\right) \quad \text{and} \quad \frac{\partial f}{\partial \bar{z}_\nu}(a) = \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu}(a) + i\frac{\partial f}{\partial y_\nu}(a)\right) \tag{2}$$

holds for all $\nu = 1, \ldots, n$.

(Yannis Bähni) Universitt Zrich, Rämistrasse 71, 8006 Zurich
E-mail address: yannis.baehni@uzh.ch.

Yannis Bähni
yannis.baehni@uzh.ch

Theorem 1.1 (The Cauchy-Riemann Equations). *Let $n \in \mathbb{Z}$, $n \geq 1$ and $D \subseteq \mathbb{C}^n$ open. A mapping $f : D \to \mathbb{C}$ is holomorphic in D if and only if it is real differentiable at every $a \in D$ and the* ***Cauchy-Riemann equations***

$$\frac{\partial f}{\partial \bar{z}_\nu}(a) = 0 \tag{3}$$

holds for all $a \in D$ and $\nu = 1, \ldots, n$.

Corollary 1.1. *Let $m, n \in \mathbb{Z}$, $m, n \geq 1$, $D \subseteq \mathbb{C}^n$ open and $f : D \to \mathbb{C}^m$ holomorphic in D. If $f = g + ih$, $g, h : D \to \mathbb{R}^m$, then*

$$\boxed{\frac{\partial g_\mu}{\partial x_\nu}(a) = \frac{\partial h_\mu}{\partial y_\nu}(a) \qquad \text{and} \qquad \frac{\partial h_\mu}{\partial x_\nu}(a) = -\frac{\partial g_\mu}{\partial y_\nu}(a)} \tag{4}$$

holds for any $a \in D$, $\nu = 1, \ldots, n$ and $\mu = 1, \ldots, m$.

Proof. Fix $\mu = 1, \ldots, m$. By definition 1.1 f_μ is holomorphic in D. Hence f_μ is real differentiable in D (see [FG10, p. 27]) and theorem 1.1 implies

$$\frac{\partial f_\mu}{\partial \bar{z}_\nu}(a) = 0$$

for all $a \in D$ and $\nu = 1, \ldots, n$. By proposition 1.1, this is equivalent to

$$\frac{\partial f}{\partial x_\nu}(a) + i\frac{\partial f}{\partial y_\nu}(a) = 0.$$

Using $f_\mu = g_\mu + ih_\mu$ and the $\mathbb{C}$-linearity of the operators $\frac{\partial}{\partial x_\nu}$ and $\frac{\partial}{\partial y_\nu}$ yields

$$\frac{\partial g_\mu}{\partial x_\nu}(a) - \frac{\partial h_\mu}{\partial y_\nu}(a) + i\left(\frac{\partial h_\mu}{\partial x_\nu}(a) + \frac{\partial g_\mu}{\partial y_\nu}(a)\right) = 0.$$

□

2. Almost Complex Structures

The following definition is taken from [Sil08, p. 86].

Definition 2.1. *Let M be a smooth manifold. An* ***almost complex structure on M*** *is a smooth tensor field $J \in \Gamma\left(T^{(1,1)}TM\right)$ such that $J_p \circ J_p = -\operatorname{id}_{T_pM}$ holds for any $p \in M$. If J is an almost complex structure on M, the tuple (M, J) is called an* ***almost complex manifold****.*

Proposition 2.1. *Every almost complex manifold is of even dimension and orientable.*

Proof. Assume that $n := \dim M$ is odd. Let $p \in M$. Then by [Lee13, p. 57] we have that $\dim T_pM = n$. Hence $\dim T_pM$ is odd. But by

$$\left(\det(J_p)\right)^2 = \det(J_p \circ J_p) = \det(-\operatorname{id}_{T_pM}) = (-1)^n \det(\operatorname{id}_{T_pM}) = (-1)^n$$

we see that n must be even since $\det(J_p) \in \mathbb{R}$ and hence $\left(\det(J_p)\right)^2 > 0$. Contradiction.

□

Yannis Bähni
yannis.baehni@uzh.ch

3. Complex Manifolds

The definition of smooth manifolds adapts smoothly to the complex case.

Definition 3.1. *Let $n \in \mathbb{Z}$, $n \geq 1$. An **n-dimensional complex manifold** is a second countable Hausdorff space M equipped with a holomorphic structure, that is a holomorphic atlas $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ of complex charts $(U_\alpha, \varphi_\alpha)$, such that all the transition maps are holomorphically compatible.*

Lemma 3.1. *Let V be a real vector space of dimension $n \in \mathbb{Z}$, $n \geq 1$. Then*

$$V \otimes V^* \cong \operatorname{End}(V) \tag{5}$$

canonically. If (e_ν) is a basis of V and (e_ν^*) the corresponding basis of V^*, then $f \in \operatorname{End}(V)$ corresponds to

$$\sum_{\nu=1}^{n} f(e_\nu) \otimes e_\nu^*. \tag{6}$$

Proof. It is easily checked that

$$\Phi : \begin{cases} V \times V^* \to \operatorname{End}(V) \\ (v, f) \mapsto \big(u \mapsto f(u)v\big) \end{cases}$$

is bilinear. Thus by the universal property of the tensor product there exists a unique mapping $\widehat{\Phi} \in \operatorname{Hom}(V \otimes V^*; \operatorname{End}(V))$ such that $\Phi = \widehat{\Phi} \circ \otimes$. It is also easily checked that $\widehat{\Phi}$ is an isomorphism.
Let $f \in \operatorname{End}(V)$. Then for any $v \in V$ we have

$$\widehat{\Phi}\left(\sum_{\nu=1}^{n} f(e_\nu) \otimes e_\nu^*\right)(v) = \sum_{\nu=1}^{n} \widehat{\Phi}\left(f(e_\nu) \otimes e_\nu^*\right)(v) = \sum_{\nu=1}^{n} e_\nu^*(v) f(e_\nu) = f\left(\sum_{\nu=1}^{n} e_\nu^*(v) e_\nu\right) = f(v).$$

□

Proposition 3.1. *Any complex manifold admits a canonical almost complex structure.*

Proof. First we define J_p in terms of local coordinates. By lemma 3.1 it is also enough to construct an endomorphism. Let $p \in M$. Given a chart $\big(U, (x^\nu, y^\nu)\big)$ with $p \in U$, we define

$$J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) := \left.\frac{\partial}{\partial y^\nu}\right|_p \qquad \text{and} \qquad J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right) := -\left.\frac{\partial}{\partial x^\nu}\right|_p$$

for all $\nu = 1, \dots, n$. As standard linear algebra shows, there is a unique linear mapping associated with J_p (see [HK71, p. 69]). Let $v := a^\nu \frac{\partial}{\partial x^\nu}\big|_p + b^\nu \frac{\partial}{\partial y^\nu}\big|_p \in T_pM$. Then

$$\begin{aligned}(J_p \circ J_p)(v) &= J_p\left(a^\nu J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) + b^\nu J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right)\right) \\ &= J_p\left(a^\nu \left.\frac{\partial}{\partial y^\nu}\right|_p - b^\nu \left.\frac{\partial}{\partial x^\nu}\right|_p\right)\end{aligned}$$

Yannis Bähni
yannis.baehni@uzh.ch

$$
\begin{aligned}
&= -a^\nu \frac{\partial}{\partial x^\nu}\bigg|_p - b^\nu \frac{\partial}{\partial y^\nu}\bigg|_p \\
&= -v
\end{aligned}
$$

and thus $J_p \circ J_p = -\operatorname{id}_{T_pM}$.
Next we have to show that above locally defined mapping is well-defined, i.e. does not depend on the choice of coordinates. Assume that p lies also in the domain of the chart $\big(V, (u^i, v^i)\big)$. By the change of coordinates formula [Lee13, p. 64] we get that

$$
\frac{\partial}{\partial x^\nu}\bigg|_p = \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})\frac{\partial}{\partial u^\mu}\bigg|_p + \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})\frac{\partial}{\partial v^\mu}\bigg|_p \qquad \text{and} \qquad \frac{\partial}{\partial y^\nu}\bigg|_p = \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})\frac{\partial}{\partial u^\mu}\bigg|_p + \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})\frac{\partial}{\partial v^\mu}\bigg|_p.
$$

where $\widehat{p}$ denotes the coordinate representation of p with respect to the coordinates (x^ν, y^ν). Corollary 1.1 implies

$$
\begin{aligned}
J_p\left(\frac{\partial}{\partial x^\nu}\bigg|_p\right) &= \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p}) J_p\left(\frac{\partial}{\partial u^\mu}\bigg|_p\right) + \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p}) J_p\left(\frac{\partial}{\partial v^\mu}\bigg|_p\right) \\
&= \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})\frac{\partial}{\partial v^\mu}\bigg|_p - \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})\frac{\partial}{\partial u^\mu}\bigg|_p \\
&= \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})\frac{\partial}{\partial v^\mu}\bigg|_p + \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})\frac{\partial}{\partial u^\mu}\bigg|_p \\
&= \frac{\partial}{\partial y^\nu}\bigg|_p
\end{aligned}
$$

and

$$
\begin{aligned}
J_p\left(\frac{\partial}{\partial y^\nu}\bigg|_p\right) &= \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p}) J_p\left(\frac{\partial}{\partial u^\mu}\bigg|_p\right) + \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p}) J_p\left(\frac{\partial}{\partial v^\mu}\bigg|_p\right) \\
&= \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})\frac{\partial}{\partial v^\mu}\bigg|_p - \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})\frac{\partial}{\partial u^\mu}\bigg|_p \\
&= -\frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})\frac{\partial}{\partial v^\mu}\bigg|_p - \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})\frac{\partial}{\partial u^\mu}\bigg|_p \\
&= -\frac{\partial}{\partial x^\nu}\bigg|_p.
\end{aligned}
$$

Left to check is smoothness. According to lemma 3.1 the corresponding rough tensor field is given by

$$
J_p\left(\frac{\partial}{\partial x^\nu}\bigg|_p\right) \otimes \mathrm{d}x^\nu|_p + J_p\left(\frac{\partial}{\partial y^\nu}\bigg|_p\right) \otimes \mathrm{d}y^\nu|_p = \frac{\partial}{\partial y^\nu}\bigg|_p \otimes \mathrm{d}x^\nu|_p - \frac{\partial}{\partial x^\nu}\bigg|_p \otimes \mathrm{d}y^\nu|_p
$$

for any $p \in U$. Thus the smoothness criteria for tensor fields [Lee13, pp. 317–318] together with [Lee13, p. 36] yields that $J \in \Gamma\left(T^{(1,1)}TM\right)$. □

Yannis Bähni
yannis.baehni@uzh.ch

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