COMPLEX MANIFOLDS

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Abstract.

Contents

1. Functions of Several Complex Variables

This section summarizes the fundamental properties of functions of several complex variables needed later. The results are inspired by [FG10, pp. 14–30].

Definition 1.1. *Let $n \in \mathbb{Z}$, $n \geq 1$, $U \subseteq \mathbb{C}^n$ open and $a \in U$. A mapping $f : U \to \mathbb{C}$ is said to be* ***complex differentiable at a*** *if there exists $g : U \to \mathbb{C}^n$ such that g is continuous at a and*

$$f(z) = f(a) + \sum_{\nu=1}^{n} (z_\nu - a_\nu) g_\nu(z) \tag{1}$$

holds for all $z \in D$. f is said to be ***holomorphic in D*** *if it is complex differentiable at every point $a \in D$. For $m \in \mathbb{Z}$, $m \geq 1$, a mapping $f : U \to \mathbb{C}^m$ is said to be holomorphic in D if each component function f_ν, $\nu = 1, \ldots, n$, is holomorphic in D.*

Proposition 1.1. *Let $n \in \mathbb{Z}$, $n \geq 1$, $D \subseteq \mathbb{C}^n$ open, $a \in U$ and $f : D \to \mathbb{C}$ real differentiable at a. Then*

$$\frac{\partial f}{\partial z_\nu}(a) = \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu}(a) - i\frac{\partial f}{\partial y_\nu}(a)\right) \tag{2}$$

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and

$$\frac{\partial f}{\partial \overline{z}_\nu}(a) = \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu}(a) + i\frac{\partial f}{\partial y_\nu}(a)\right) \tag{3}$$

holds for all $\nu = 1, \ldots, n$.

Theorem 1.1 (The Cauchy-Riemann Equations). *Let* $n \in \mathbb{Z}$*,* $n \geq 1$ *and* $D \subseteq \mathbb{C}^n$ *open. A mapping* $f : D \to \mathbb{C}$ *is holomorphic in* D *if and only if it is real differentiable at every* $a \in D$ *and the* ***Cauchy-Riemann equations***

$$\frac{\partial f}{\partial \overline{z}_\nu}(a) = 0 \tag{4}$$

holds for all $a \in D$ *and* $\nu = 1, \ldots, n$.

Corollary 1.1. *Let* $m, n \in \mathbb{Z}$*,* $m, n \geq 1$*,* $D \subseteq \mathbb{C}^n$ *open and* $f : D \to \mathbb{C}^m$ *holomorphic in* D*. If* $f = g + ih$*,* $g, h : D \to \mathbb{R}^m$*, then*

$$\boxed{\frac{\partial g_\mu}{\partial x_\nu}(a) = \frac{\partial h_\mu}{\partial y_\nu}(a) \qquad \textit{and} \qquad \frac{\partial h_\mu}{\partial x_\nu}(a) = -\frac{\partial g_\mu}{\partial y_\nu}(a)} \tag{5}$$

holds for any $a \in D$*,* $\nu = 1, \ldots, n$ *and* $\mu = 1, \ldots, m$.

Proof. Fix $\mu = 1, \ldots, m$. By definition 1.1 f_μ is holomorphic in D. Hence f_μ is real differentiable in D (see [FG10, p. 27]) and theorem 1.1 implies

$$\frac{\partial f_\mu}{\partial \overline{z}_\nu}(a) = 0$$

for all $a \in D$ and $\nu = 1, \ldots, n$. By proposition 1.1, this is equivalent to

$$\frac{\partial f}{\partial x_\nu}(a) + i\frac{\partial f}{\partial y_\nu}(a) = 0.$$

Using $f_\mu = g_\mu + ih_\mu$ and the $\mathbb{C}$-linearity of the operators $\frac{\partial}{\partial x_\nu}$ and $\frac{\partial}{\partial y_\nu}$ yields

$$\frac{\partial g_\mu}{\partial x_\nu}(a) - \frac{\partial h_\mu}{\partial y_\nu}(a) + i\left(\frac{\partial h_\mu}{\partial x_\nu}(a) + \frac{\partial g_\mu}{\partial y_\nu}(a)\right) = 0.$$

□

2. Almost Complex Structures

The following definition is taken from [Sil08, p. 86].

Definition 2.1. *Let* M *be a smooth manifold. An* ***almost complex structure on*** $\boldsymbol{M}$ *is a smooth tensor field* $J \in \Gamma\left(T^{(1,1)}TM\right)$ *such that* $J_p \circ J_p = -\operatorname{id}_{T_pM}$ *holds for any* $p \in M$*. If* J *is an almost complex structure on* M*, the tuple* (M, J) *is called an* ***almost complex manifold****.*

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Proposition 2.1. *Every almost complex manifold is of even dimension and orientable.*

Proof. Assume that $n := \dim M$ is odd. Let $p \in M$. Then by [Lee13, p. 57] we have that $\dim T_pM = n$. Hence $\dim T_pM$ is odd. But by

$$\big(\det(J_p)\big)^2 = \det(J_p \circ J_p) = \det(-\mathrm{id}_{T_pM}) = (-1)^n \det(\mathrm{id}_{T_pM}) = (-1)^n$$

we see that n must be even since $\det(J_p) \in \mathbb{R}$ and hence $\big(\det(J_p)\big)^2 > 0$. Contradiction.

$\square$

Remark 2.1. The converse of proposition 2.1 is not true in general. One can show using results on fibre bundles and Chern classes, that if $\mathbb{S}^n$ admits an almost complex structure, then $n = 2^k - 2$ for $k \in \mathbb{Z}$, $k \geq 1$ (see [Ste51, p. 219]). So for example $\mathbb{S}^4$ does not admit an almost complex structure. Actually, it can be shown that $\mathbb{S}^2$ and $\mathbb{S}^6$ are the only spheres which admit an almost complex structure (see [BS53, p. 434]).

3. Complex Manifolds

The definition of smooth manifolds adapts smoothly to the complex case.

Definition 3.1. *Let $n \in \mathbb{Z}$, $n \geq 1$. An **n-dimensional complex manifold** is a second countable Hausdorff space M equipped with a holomorphic structure, that is a holomorphic atlas $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ of complex charts $(U_\alpha, \varphi_\alpha)$, such that all the transition maps are holomorphically compatible.*

Lemma 3.1. *Let V be a real vector space of dimension $n \in \mathbb{Z}$, $n \geq 1$. Then*

$$V \otimes V^* \cong \mathrm{End}(V) \tag{6}$$

canonically. If (e_ν) is a basis of V and (e_ν^*) the corresponding basis of V^*, then $f \in \mathrm{End}(V)$ corresponds to

$$\sum_{\nu=1}^{n} f(e_\nu) \otimes e_\nu^*. \tag{7}$$

Proof. It is easily checked that

$$\Phi : \begin{cases} V \times V^* \to \mathrm{End}(V) \\ (v, f) \mapsto \big(u \mapsto f(u)v\big) \end{cases}$$

is bilinear. Thus by the universal property of the tensor product there exists a unique mapping $\widehat{\Phi} \in \mathrm{Hom}(V \otimes V^*; \mathrm{End}(V))$ such that $\Phi = \widehat{\Phi} \circ \otimes$. It is also easily

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checked that $\widehat{\Phi}$ is an isomorphism.
Let $f \in \mathrm{End}(V)$. Then for any $v \in V$ we have

$$\begin{aligned}\widehat{\Phi}\left(\sum_{\nu=1}^{n} f(e_\nu) \otimes e_\nu^*\right)(v) &= \sum_{\nu=1}^{n} \widehat{\Phi}\left(f(e_\nu) \otimes e_\nu^*\right)(v) \\ &= \sum_{\nu=1}^{n} e_\nu^*(v) f(e_\nu) \\ &= f\left(\sum_{\nu=1}^{n} e_\nu^*(v) e_\nu\right) \\ &= f(v).\end{aligned}$$

$\square$

Proposition 3.1. *Any complex manifold admits a canonical almost complex structure.*

Proof. First we define J_p in terms of local coordinates. By lemma 3.1 it is also enough to construct an endomorphism. Let $p \in M$. Given a chart $\big(U, (x^\nu, y^\nu)\big)$ with $p \in U$, we define

$$J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) := \left.\frac{\partial}{\partial y^\nu}\right|_p \quad \text{and} \quad J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right) := -\left.\frac{\partial}{\partial x^\nu}\right|_p$$

for all $\nu = 1, \ldots, n$. As standard linear algebra shows, there is a unique linear mapping associated with J_p (see [HK71, p. 69]). Let $v := a^\nu \frac{\partial}{\partial x^\nu}\big|_p + b^\nu \frac{\partial}{\partial y^\nu}\big|_p \in T_pM$. Then

$$\begin{aligned}(J_p \circ J_p)(v) &= J_p\left(a^\nu J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) + b^\nu J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right)\right) \\ &= J_p\left(a^\nu \left.\frac{\partial}{\partial y^\nu}\right|_p - b^\nu \left.\frac{\partial}{\partial x^\nu}\right|_p\right) \\ &= -a^\nu \left.\frac{\partial}{\partial x^\nu}\right|_p - b^\nu \left.\frac{\partial}{\partial y^\nu}\right|_p \\ &= -v\end{aligned}$$

and thus $J_p \circ J_p = -\mathrm{id}_{T_pM}$.
Next we have to show that above locally defined mapping is well-defined, i.e. does not depend on the choice of coordinates. Assume that p lies also in the domain of the chart $\big(V, (u^i, v^i)\big)$. By the change of coordinates formula [Lee13, p. 64] we get

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that

$$\left.\frac{\partial}{\partial x^\nu}\right|_p = \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})\left.\frac{\partial}{\partial u^\mu}\right|_p + \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})\left.\frac{\partial}{\partial v^\mu}\right|_p$$

and

$$\left.\frac{\partial}{\partial y^\nu}\right|_p = \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})\left.\frac{\partial}{\partial u^\mu}\right|_p + \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})\left.\frac{\partial}{\partial v^\mu}\right|_p$$

where $\widehat{p}$ denotes the coordinate representation of p with respect to the coordinates (x^ν, y^ν). Corollary 1.1 implies

$$\begin{aligned}
J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) &= \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})J_p\left(\left.\frac{\partial}{\partial u^\mu}\right|_p\right) + \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})J_p\left(\left.\frac{\partial}{\partial v^\mu}\right|_p\right)\\
&= \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})\left.\frac{\partial}{\partial v^\mu}\right|_p - \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})\left.\frac{\partial}{\partial u^\mu}\right|_p\\
&= \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})\left.\frac{\partial}{\partial v^\mu}\right|_p + \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})\left.\frac{\partial}{\partial u^\mu}\right|_p\\
&= \left.\frac{\partial}{\partial y^\nu}\right|_p
\end{aligned}$$

and

$$\begin{aligned}
J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right) &= \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})J_p\left(\left.\frac{\partial}{\partial u^\mu}\right|_p\right) + \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})J_p\left(\left.\frac{\partial}{\partial v^\mu}\right|_p\right)\\
&= \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})\left.\frac{\partial}{\partial v^\mu}\right|_p - \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})\left.\frac{\partial}{\partial u^\mu}\right|_p\\
&= -\frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})\left.\frac{\partial}{\partial v^\mu}\right|_p - \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})\left.\frac{\partial}{\partial u^\mu}\right|_p\\
&= -\left.\frac{\partial}{\partial x^\nu}\right|_p.
\end{aligned}$$

Left to check is smoothness. According to lemma 3.1 the corresponding rough tensor field is given by

$$J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right)\otimes \mathrm{d}x^\nu|_p + J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right)\otimes \mathrm{d}y^\nu|_p = \left.\frac{\partial}{\partial y^\nu}\right|_p \otimes \mathrm{d}x^\nu|_p - \left.\frac{\partial}{\partial x^\nu}\right|_p\otimes \mathrm{d}y^\nu|_p$$

for any $p \in U$. Thus the smoothness criteria for tensor fields [Lee13, pp. 317–318] together with [Lee13, p. 36] yields that $J \in \Gamma\left(T^{(1,1)}TM\right)$. $\square$

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