COMPLEX MANIFOLDS

YANNIS BÄHNI

Abstract. Goal of this paper is to give an overview of the basic definitions of complex and Kähler manifolds together with the most important properties. The main theorem will be the *Newlander-Nirenberg Theorem* which gives a criterion under which an almost complex manifold is a complex one. The key role will be played by a certain tensor field, the so called *Nijenhuis tensor*.

Contents

1 Functions of Several Complex Variables 1
2 Complex Structures on Vector Spaces 3
3 Almost Complex Structures 3
4 Complex Manifolds 5
5 The Nijenhuis Tensor and the Newlander-Nirenberg Theorem . 8
6 Kähler Manifolds 9
Appendix A Tensor Characterization Lemma 9
References 14

1. Functions of Several Complex Variables

This section summarizes the fundamental properties of functions of several complex variables needed later. The results are inspired by [FG10, pp. 14–30].

Definition 1.1. *Let $n \in \mathbb{Z}$, $n \geq 1$, $U \subseteq \mathbb{C}^n$ open and $a \in U$. A mapping $f : U \to \mathbb{C}$ is said to be* ***complex differentiable at a*** *if there exists $g : U \to \mathbb{C}^n$ such that g is continuous at a and*

$$f(z) = f(a) + \sum_{\nu=1}^{n} (z_\nu - a_\nu) g_\nu(z) \tag{1}$$

holds for all $z \in D$. f is said to be ***holomorphic in D*** *if it is complex differentiable at every point $a \in D$. For $m \in \mathbb{Z}$, $m \geq 1$, a mapping $f : U \to \mathbb{C}^m$ is said to be*

(Yannis Bähni) Universitt Zrich, Rämistrasse 71, 8006 Zurich
E-mail address: yannis.baehni@uzh.ch.

Yannis Bähni
yannis.baehni@uzh.ch

holomorphic in D if each component function f_ν, $\nu = 1, \ldots, n$, is holomorphic in D.

Proposition 1.1. *Let $n \in \mathbb{Z}$, $n \geq 1$, $D \subseteq \mathbb{C}^n$ open, $a \in U$ and $f : D \to \mathbb{C}$ real differentiable at a. Then*

$$\frac{\partial f}{\partial z_\nu}(a) = \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu}(a) - i\frac{\partial f}{\partial y_\nu}(a)\right) \tag{2}$$

and

$$\frac{\partial f}{\partial \overline{z}_\nu}(a) = \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu}(a) + i\frac{\partial f}{\partial y_\nu}(a)\right) \tag{3}$$

holds for all $\nu = 1, \ldots, n$.

Theorem 1.1 (The Cauchy-Riemann Equations). *Let $n \in \mathbb{Z}$, $n \geq 1$ and $D \subseteq \mathbb{C}^n$ open. A mapping $f : D \to \mathbb{C}$ is holomorphic in D if and only if it is real differentiable at every $a \in D$ and the **Cauchy-Riemann equations***

$$\frac{\partial f}{\partial \overline{z}_\nu}(a) = 0 \tag{4}$$

holds for all $a \in D$ and $\nu = 1, \ldots, n$.

Corollary 1.1. *Let $m, n \in \mathbb{Z}$, $m, n \geq 1$, $D \subseteq \mathbb{C}^n$ open and $f : D \to \mathbb{C}^m$ holomorphic in D. If $f = g + ih$, $g, h : D \to \mathbb{R}^m$, then*

$$\boxed{\frac{\partial g_\mu}{\partial x_\nu}(a) = \frac{\partial h_\mu}{\partial y_\nu}(a) \qquad \text{and} \qquad \frac{\partial h_\mu}{\partial x_\nu}(a) = -\frac{\partial g_\mu}{\partial y_\nu}(a)} \tag{5}$$

holds for any $a \in D$, $\nu = 1, \ldots, n$ and $\mu = 1, \ldots, m$.

Proof. Fix $\mu = 1, \ldots, m$. By definition 1.1 f_μ is holomorphic in D. Hence f_μ is real differentiable in D (see [FG10, p. 27]) and theorem 1.1 implies

$$\frac{\partial f_\mu}{\partial \overline{z}_\nu}(a) = 0$$

for all $a \in D$ and $\nu = 1, \ldots, n$. By proposition 1.1, this is equivalent to

$$\frac{\partial f}{\partial x_\nu}(a) + i\frac{\partial f}{\partial y_\nu}(a) = 0.$$

Using $f_\mu = g_\mu + ih_\mu$ and the $\mathbb{C}$-linearity of the operators $\frac{\partial}{\partial x_\nu}$ and $\frac{\partial}{\partial y_\nu}$ yields

$$\frac{\partial g_\mu}{\partial x_\nu}(a) - \frac{\partial h_\mu}{\partial y_\nu}(a) + i\left(\frac{\partial h_\mu}{\partial x_\nu}(a) + \frac{\partial g_\mu}{\partial y_\nu}(a)\right) = 0.$$

□

Yannis Bähni
yannis.baehni@uzh.ch

2. Complex Structures on Vector Spaces

In what follows, let $n \in \mathbb{Z}$, $n \geq 1$. Consider an n-dimensional complex vector space and let $J \in \operatorname{End}_{\mathbb{C}}(V)$ be defined by $J(v) := iv$. Then clearly $J \circ J = -\operatorname{id}_V$. Since every n-dimensional complex vector space can be seen as a $2n$-dimensional real vector space in a natural way, i.e. if (e_ν) is a basis for the complex vector space V, then (e_ν, ie_ν) is a basis for the real vector space V, the mapping J induces an $\mathbb{R}$-endomorphism J on the real vector space V simply by $J(e_\nu) = ie_\nu$ and $J(ie_\nu) = -e_\nu$ for all $\nu = 1, \dots, n$.
Conversly, let V be an n-dimensional real vector space with $J \in \operatorname{End}_{\mathbb{R}}(V)$ such that $J \circ J = -\operatorname{id}_V$. One can show, that

$$zv := xv + yJ(v) \tag{6}$$

for $z := x + iy \in \mathbb{C}$ and $v \in V$ makes V into a complex vector space. This motivates the following definition.

Definition 2.1. *Let V be an n-dimensional real vector space. A **complex structure on V** is a $\mathbb{R}$-linear mapping $J : V \to V$ such that $J \circ J = -\operatorname{id}_V$. If J is a complex structure on V, the tuple (V, J) is called a **complex vector space**.*

Lemma 2.1. *Let (V, J) be a complex vector space. Then $\dim V$ is even.*

Proof. That $\dim V$ must be even follows directly from

$$\big(\det(J)\big)^2 = \det(J \circ J) = \det(-\operatorname{id}_V) = (-1)^{\dim V} \det(\operatorname{id}_V) = (-1)^{\dim V}.$$

□

3. Almost Complex Structures

If M is a smooth manifold, then T_pM is a finite dimensional real vector space. Hence we can generalize the definitions and results of the previous section to manifolds. The following definition is taken from [Sil08, p. 86].

Definition 3.1. *Let M be a smooth manifold. An **almost complex structure on M** is a smooth tensor field $J \in \Gamma\left(T^{(1,1)}TM\right)$ such that $J_p \circ J_p = -\operatorname{id}_{T_pM}$ holds for any $p \in M$. If J is an almost complex structure on M, the tuple (M, J) is called an **almost complex manifold**.*

Proposition 3.1. *Every almost complex manifold (M, J) is of even dimension and orientable.*

Proof. Assume that $\dim M$ is odd. Let $p \in M$. Then by [Lee13, p. 57] we have that $\dim T_pM = \dim M$. Hence $\dim T_pM$ is odd. But by lemma 2.1, $\dim T_pM$ must be even since (T_pM, J_p) is a complex vector space. Contradiction.

Yannis Bähni
yannis.baehni@uzh.ch

Since M is a smooth manifold, there exists a Riemannian metric g on M (see [Lee13, p. 329]). Define

$$\widetilde{g}(X,Y) := g(X,Y) + g(JX,JY) \in \Gamma(T^{(0,2)}TM)$$

for all $X, Y \in \mathfrak{X}(M)$. This is possible due to the tensor characterization lemma A.1. Then

$$\widetilde{g}(JX,JY) = g(JX,JY) + g(-X,-Y) = g(JX,JY) + g(X,Y) = \widetilde{g}(X,Y)$$

by the bilinearity of g. Furthermore, clearly $\widetilde{g}$ is positive definite and symmetric, thus a Riemannian metric on M. Define

$$\omega(X,Y) := \widetilde{g}(X,JY).$$

Then by

$$\omega(Y,X) = \widetilde{g}(Y,JX) = \widetilde{g}(JX,Y) = \widetilde{g}(-X,JY) = -\omega(X,Y)$$

we see that ω is skew-symmetric. Hence $\omega \in \Omega^2(M)$. Let $p \in M$ and $u \in T_pM \setminus \{0\}$. Then also $-J_p(u) \neq 0$ since J_p is invertible since $\det J_p = 1$. Furthermore, by [Lee13, p. 177], there exist $X, Y \in \mathfrak{X}(M)$, such that $X_p = u$ and $Y_p = -J_p(u)$. Hence

$$\begin{aligned}
\omega_p\big(u, -J_p(u)\big) &= \omega_p(X_p, Y_p) \\
&= \omega(X,Y)(p) \\
&= \widetilde{g}(X,JY)(p) \\
&= \widetilde{g}_p\big(X_p, (JY)_p\big) \\
&= \widetilde{g}_p\big(u, J_p(Y_p)\big) \\
&= \widetilde{g}_p\big(u, -(J_p \circ J_p)(u)\big) \\
&= \widetilde{g}_p(u,u) \\
&\neq 0
\end{aligned}$$

and by [Lee13, p. 565] we get that ω is nondegenrate. Let $\dim M = 2n$. By [Lee13, p. 567] this implies that $\omega_p \wedge \cdots \wedge \omega_p$ is nonzero for each $p \in M$. Hence $\omega \wedge \cdots \wedge \omega$ is a nonvanishing top form on M. Since any nonvanishing top form determines an orientation (see [Lee13, p. 381]), we have that M is orientable. □

Remark 3.1. The converse of proposition 3.1 is not true in general. One can show using results on fibre bundles and Chern classes, that if $\mathbb{S}^n$ admits an almost complex structure, then $n = 2^k - 2$ for $k \in \mathbb{Z}$, $k \geq 1$ (see [Ste51, p. 219]). So for example $\mathbb{S}^4$ does not admit an almost complex structure. Actually, it can be shown that $\mathbb{S}^2$ and $\mathbb{S}^6$ are the only spheres which admit an almost complex structure (see [BS53, p. 434]).

Yannis Bähni
yannis.baehni@uzh.ch

4. Complex Manifolds

The definition of smooth manifolds adapts smoothly to the complex case.

Definition 4.1. *Let $n \in \mathbb{Z}$, $n \geq 1$. An* ***n-dimensional complex manifold*** *is a second countable Hausdorff space M equipped with a holomorphic structure, that is a maximal holomorphic atlas $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ of complex charts $(U_\alpha, \varphi_\alpha)$, such that all the transition maps are holomorphic.*

Examples 4.1 (Complex Manifolds).

1. The complex n-space $\mathbb{C}^n$ is an n-dimensional complex manifold.
2. Let $\{\omega_1, \ldots, \omega_{2n}\}$ be a real basis of $\mathbb{C}^n$ and define
$$G := \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_{2n}. \tag{7}$$
Then the discrete group G acts freely and properly discontinuously on $\mathbb{C}^n$ by translation. Thus $\mathbb{T}^n := \mathbb{C}^n/G$ is an n-dimensional complex manifold, called a ***complex torus*** (see [FG10, pp. 206–207]).
3. The quotient $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$ is an n-dimensional complex manifold, called the ***complex projective space*** (see [FG10, pp. 208–210]).

Lemma 4.1. *Let $n, k \in \mathbb{Z}$, $n, k \geq 1$. Let V be an n-dimensional real vector space. Then*
$$V \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{k} \cong \mathrm{L}(\underbrace{V, \ldots, V}_{k}; V) \tag{8}$$
canonically. If (e_ν) is a basis of V and (e_ν^*) the corresponding basis of V^*, then $f \in \mathrm{End}(V)$ corresponds to
$$\sum_{\nu=1}^{n} f(e_\nu) \otimes e_\nu^*. \tag{9}$$

Proof. It is easily checked that
$$\Phi : \begin{cases} V \times V^* \times \cdots \times V^* \to \mathrm{L}(V, \ldots, V; V) \\ (v, f_1, \ldots, f_k) \mapsto \big((v_1, \ldots, v_k) \mapsto f_1(v_1) \cdots f_k(v_k) v\big) \end{cases}$$
is multilinear. Thus by the universal property of the tensor product there exists a unique mapping $\widehat{\Phi} \in \mathrm{Hom}\left(V \times V^* \times \cdots \times V^*; \mathrm{L}(V, \ldots, V; V)\right)$ such that $\Phi = \widehat{\Phi} \circ \otimes$. It is also easily checked that $\widehat{\Phi}$ is an isomorphism. Let $f \in \mathrm{End}(V)$. Then for any $v \in V$ we have
$$\widehat{\Phi}\left(\sum_{\nu=1}^{n} f(e_\nu) \otimes e_\nu^*\right)(v) = \sum_{\nu=1}^{n} \widehat{\Phi}\left(f(e_\nu) \otimes e_\nu^*\right)(v)$$

Yannis Bähni
yannis.baehni@uzh.ch

$$
\begin{aligned}
&= \sum_{\nu=1}^{n} e_\nu^*(v) f(e_\nu) \\
&= f\left(\sum_{\nu=1}^{n} e_\nu^*(v) e_\nu\right) \\
&= f(v).
\end{aligned}
$$

□

Proposition 4.1. *Any complex manifold admits a canonical almost complex structure.*

Proof. Fix a complex manifold M. We define J in terms of local coordinates. Let $\big(U, (x^\nu, y^\nu)\big)$ be a chart. By lemma 4.1 it is also enough to construct an endomorphism J_p for every $p \in U$. We define

$$
J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) := \left.\frac{\partial}{\partial y^\nu}\right|_p \qquad \text{and} \qquad J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right) := -\left.\frac{\partial}{\partial x^\nu}\right|_p
$$

for all $\nu = 1, \ldots, n$. As standard linear algebra shows, there is a unique linear mapping associated with J_p (see [HK71, p. 69]). Let $v := a^\nu \frac{\partial}{\partial x^\nu}\big|_p + b^\nu \frac{\partial}{\partial y^\nu}\big|_p \in T_pM$. Then

$$
\begin{aligned}
(J_p \circ J_p)(v) &= J_p\left(a^\nu J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) + b^\nu J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right)\right) \\
&= J_p\left(a^\nu \left.\frac{\partial}{\partial y^\nu}\right|_p - b^\nu \left.\frac{\partial}{\partial x^\nu}\right|_p\right) \\
&= -a^\nu \left.\frac{\partial}{\partial x^\nu}\right|_p - b^\nu \left.\frac{\partial}{\partial y^\nu}\right|_p \\
&= -v
\end{aligned}
$$

and thus $J_p \circ J_p = -\mathrm{id}_{T_pM}$.
Next we have to show that above locally defined mapping is well-defined, i.e. does not depend on the choice of coordinates. Assume that $p \in U \cap V$ for another chart $\big(V, (u^i, v^i)\big)$. By the change of coordinates formula [Lee13, p. 64] we get that

$$
\left.\frac{\partial}{\partial x^\nu}\right|_p = \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p}) \left.\frac{\partial}{\partial u^\mu}\right|_p + \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p}) \left.\frac{\partial}{\partial v^\mu}\right|_p
$$

and

$$
\left.\frac{\partial}{\partial y^\nu}\right|_p = \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p}) \left.\frac{\partial}{\partial u^\mu}\right|_p + \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p}) \left.\frac{\partial}{\partial v^\mu}\right|_p
$$

Yannis Bähni
yannis.baehni@uzh.ch

where $\widehat{p}$ denotes the coordinate representation of p with respect to the coordinates (x^ν, y^ν). Corollary 1.1 implies

$$\begin{aligned} J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) &= \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})J_p\left(\left.\frac{\partial}{\partial u^\mu}\right|_p\right) + \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})J_p\left(\left.\frac{\partial}{\partial v^\mu}\right|_p\right) \\ &= \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})\left.\frac{\partial}{\partial v^\mu}\right|_p - \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})\left.\frac{\partial}{\partial u^\mu}\right|_p \\ &= \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})\left.\frac{\partial}{\partial v^\mu}\right|_p + \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})\left.\frac{\partial}{\partial u^\mu}\right|_p \\ &= \left.\frac{\partial}{\partial y^\nu}\right|_p \end{aligned}$$

and

$$\begin{aligned} J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right) &= \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})J_p\left(\left.\frac{\partial}{\partial u^\mu}\right|_p\right) + \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})J_p\left(\left.\frac{\partial}{\partial v^\mu}\right|_p\right) \\ &= \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})\left.\frac{\partial}{\partial v^\mu}\right|_p - \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})\left.\frac{\partial}{\partial u^\mu}\right|_p \\ &= -\frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})\left.\frac{\partial}{\partial v^\mu}\right|_p - \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})\left.\frac{\partial}{\partial u^\mu}\right|_p \\ &= -\left.\frac{\partial}{\partial x^\nu}\right|_p. \end{aligned}$$

Left to check is smoothness. According to lemma 4.1 the corresponding rough tensor field is given by

$$J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) \otimes \mathrm{d}x^\nu|_p + J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right) \otimes \mathrm{d}y^\nu|_p = \left.\frac{\partial}{\partial y^\nu}\right|_p \otimes \mathrm{d}x^\nu|_p - \left.\frac{\partial}{\partial x^\nu}\right|_p \otimes \mathrm{d}y^\nu|_p$$

for any $p \in U$. Thus the smoothness criteria for tensor fields **??** together with [Lee13, p. 36] yields that $J \in \Gamma\left(T^{(1,1)}TM\right)$. ☐

A question which naturally arises by considering proposition 4.1 is, if the converse is also true, i.e. if every almost complex manifold is a complex manifold. This is in general not the case. Let $\mathbb{P}$ denote the naturally oriented underlying smooth manifold of the complex projective plane. Again using results about Chern numbers it can be shown that

$$\mathbb{P}\#(\mathbb{S}^1 \times \mathbb{S}^3)\#(\mathbb{S}^1 \times \mathbb{S}^3) \qquad \text{and} \qquad (\mathbb{S}^1 \times \mathbb{S}^3)\#(\mathbb{S}^1 \times \mathbb{S}^3)\#(\mathbb{S}^2 \times \mathbb{S}^2) \tag{10}$$

have almost complex structures but no complex structure (see [Ven66, p. 1627]).

Yannis Bähni
yannis.baehni@uzh.ch

5. The Nijenhuis Tensor and the Newlander-Nirenberg Theorem

As we have seen in the last section, not every almost complex manifold is a complex manifold. Under which condition is this possible?

Definition 5.1. *Let (M, J) be an almost complex manifold. For $X, Y \in \mathfrak{X}(M)$ we define the **Nijenhuis tensor N** as*

$$N(X,Y) := [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] \tag{11}$$

where $[X, Y]$ denotes the usual Lie-bracket of vector fields.

Proposition 5.1. *Let (M, J) be an almost complex manifold and N be the Nijenhuis tensor. Then $N \in \Gamma\left(T^{(1,2)}TM\right)$.*

Proof. First of all, $N(X, Y) \in \mathfrak{X}(M)$ for all $X, Y \in \mathfrak{X}(M)$. This follows immediately by considering J as a mapping $J : \mathfrak{X}(M) \to \mathfrak{X}(M)$ using the tensor characterization lemma A.1, the fact that the Lie Bracket of two smooth vector fields is again a smooth vector field (see [Lee13, p. 186]) and that $\mathfrak{X}(M)$ is a $\mathscr{C}^\infty(M)$-module (see [Lee13, p. 177]). Let $f \in \mathscr{C}^\infty(M)$ and $X, Y, Z \in \mathfrak{X}(M)$. Then

$$\begin{aligned}
N(fX + Y, Z) =& [J(fX + Y), JZ] - J[fX + Y, JZ] - J[J(fX + Y), Z] \\
&- [fX + Y, Z] \\
=& [fJX + JY, JZ] - J[fX + Y, JZ] - J[fJX + JY, Z] \\
&- [fX + Y, Z] \\
=& [fJX, JZ] + [JY, JZ] - J[fX, JZ] - J[Y, JZ] - J[fJX, Z] \\
&- J[JY, Z] - [fX, Z] - [Y, Z] \\
=& f[JX, JZ] - (JZf)JX + [JY, JZ] - fJ[X, JZ] + (JZf)JX \\
&- [Y, JZ] - fJ[JX, Z] + (Zf)JJX - J[JY, Z] - f[X, Z] \\
&+ (Zf)X - [Y, Z] \\
=& fN(X, Z) + N(Y, Z).
\end{aligned}$$

by [Lee13, pp. 187–188]. Linearity in the second argument is shown similarly. Hence $N : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is bilinear over $\mathscr{C}^\infty(M)$. Again by the tensor characterization lemma A.1 we have that $N \in \Gamma\left(T^{(1,2)}TM\right)$. □

Theorem 5.1 (Newlander-Nirenberg). *Let (M, J) be an almost complex manifold. Then M is a complex manifold, where the complex structure is so that the canonically induced almost complex structure is J, if and only if the Nijenhuis tensor N vanishes identically.*

Proof. Assume M is a complex manifold. Let $(U, (x^\nu, y^\nu))$ be a chart. From proposition 5.1 it is enough to consider the coordinate vector fields $\frac{\partial}{\partial x^\nu}$ and $\frac{\partial}{\partial y^\nu}$. But from

Yannis Bähni
yannis.baehni@uzh.ch

the explicit definition of J in proposition 4.1 and the property, that the Lie-Bracket of coordinate vector fields vanishes, together with the $\mathscr{C}^\infty(M)$-linearity of J we get that N vanishes identically on each chart, and thus on M.
The other direction however is far more technical and uses results on partial differential equations. A complete proof can either be found in the original paper [NN57] or in [Sil08, p. 106], where references to more recent proofs are given. $\square$

6. Kähler Manifolds

The following is inspired by [KN96, pp. 146–149] and introduces the concepts from a complex viewpoint. This is in contrast to the symplectic approach provided for example in [Sil08].

Definition 6.1. *Let (M, J) be an almost complex manifold. A **Hermitian metric on M** is a Riemannian metric g such that*

$$g(JX, JY) = g(X, Y) \tag{12}$$

*holds for all $X, Y \in \mathfrak{X}(M)$. If g is a Hermitian metric on M, the triple (M, J, g) is called an **almost Hermitian manifold**.*

Lemma 6.1. *Every almost complex manifold admits a Hermitian metric.*

Proof. The existence was shown in the proof of proposition 3.1. $\square$

Definition 6.2. *Let (M, J, g) be an almost Hermitian manifold. The **fundamental 2-form Ω** is defined to be*

$$\Omega(X, Y) := g(X, JY) \tag{13}$$

for all $X, Y \in \mathfrak{X}(M)$.

Definition 6.3. *Let (M, J, g) be an almost Hermitian manifold with fundamental 2-form Ω. The Hermitian metric is said to be a **Kähler metric**, if $\mathrm{d}\Omega = 0$. An almost complex manifold with a Kähler metric is called an **almost Kähler manifold** and a complex manifold with a Kähler metric is called a **Kähler manifold**.*

Appendix A. Tensor Characterization Lemma

Definition A.1. *Let $k, l \in \mathbb{Z}$, $k, l \geq 0$ and M a smooth manifold. Then the **bundle of mixed tensors of type (k, l)** is defined by*

$$T^{(k,l)}TM := \coprod_{p \in M} T^{(k,l)}(T_pM). \tag{14}$$

Proposition A.1. *The bundle of mixed tensors of type (k, l) has an unique natural structure as a smooth vector bundle of rank n^{k+l} over M.*

Yannis Bähni
yannis.baehni@uzh.ch

Proof. For each $p \in M$ let $E_p := T^{(k,l)}(T_pM)$. By [Lee13, p. 57] and [Lee13, p. 313] $\dim E_p = n^{k+l}$. Furthermore, let $E := T^{(k,l)}TM$ and $\pi : E \to M$ be defined by $\pi(p, A) := p$. Let $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ be an atlas for M. For each $\alpha \in A$ define

$$\Phi_\alpha : \begin{cases} \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{n^{k+l}} \\ (p, A) \mapsto \big(p, (A^{i_1 \ldots i_k}_{j_1 \ldots j_l})\big) \end{cases}$$

Clearly, the inverse is given by

$$\Phi_\alpha^{-1} : \begin{cases} U_\alpha \times \mathbb{R}^{n^{k+l}} \to \pi^{-1}(U_\alpha) \\ \big(p, (A^{i_1 \ldots i_k}_{j_1 \ldots j_l})\big) \mapsto (p, A) \end{cases}.$$

Hence each Φ_α is bijective. Now we have to check, that $\Phi_\alpha|_{E_p}$ is an isomorphism. So let $\lambda \in \mathbb{R}$ and $B \in E_p$. Then

$$\begin{aligned} \Phi_\alpha|_{E_p}(p, \lambda A + B) &= \big(p, (\lambda A + B)^{i_1 \ldots i_k}_{j_1 \ldots j_l})\big) \\ &= \big(p, \lambda(A^{i_1 \ldots i_k}_{j_1 \ldots j_l}) + (B^{i_1 \ldots i_k}_{j_1 \ldots j_l})\big) \\ &= \lambda \Phi_\alpha|_{E_p}(p, A) + \Phi_\alpha|_{E_p}(p, B). \end{aligned}$$

Now let $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$. We consider the mapping

$$\Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^{n^{k+l}} \to (U_\alpha \cap U_\beta) \times \mathbb{R}^{n^{k+l}}.$$

Define $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n^{k+l}, \mathbb{R})$ by

$$\tau_{\alpha\beta} := (\delta^i_j).$$

Then we have that

$$\left(\Phi_\alpha \circ \Phi_\beta^{-1}\right)\big(p, (A^{i_1 \ldots i_k}_{j_1 \ldots j_l})\big) = \big(p, (A^{i_1 \ldots i_k}_{j_1 \ldots j_l})\big) = \big(p, \tau_{\alpha\beta}(p)(A^{i_1 \ldots i_k}_{j_1 \ldots j_l})\big).$$

Since $\tau_{\alpha\beta}$ is constant, it is smooth (see [Lee13, p. 36]). Hence we can apply the vector bundle chart lemma [Lee13, p. 253] and the result follows. □

What follows is a reformulation of the smoothness criteria for tensor fields ([Lee13, pp. 317–318]) for tensor fields of type $(1, k)$.

Proposition A.2 (Smoothness Criteria for Tensor Fields). *Let M be smooth manifold and let $A : M \to T^{(1,k)}TM$ be a rough section. Then the following are equivalent:*

(a) $A \in \Gamma(T^{(1,k)}TM)$.
(b) *In every smooth coordinate chart, the component functions of A are smooth.*
(c) *For all $X_1, \ldots, X_k \in \mathfrak{X}(M)$, the rough section $A(X_1, \ldots, X_k) : M \to TM$ defined by*

$$A(X_1, \ldots, X_k)(p) := A_p(X_1|_p, \ldots, X_k|_p) \tag{15}$$

is a smooth vector field.

Yannis Bähni
yannis.baehni@uzh.ch

(*d*) *If* $X_1, \ldots, X_k$ *are smooth vector fields on some open subset* $U \subseteq M$*, then also* $A(X_1, \ldots, X_k)$ *is a smooth vector field on* U*.*

Proof. We prove (a)⇔(b) and (b)⇒(c)⇒(d)⇒(b).
To prove (a)⇔(b), let $(U, (x^i))$ be a smooth chart. Proposition A.1 together with the proof of the vector bundle chart lemma [Lee13, pp. 253–254] implies, that the corresponding chart on $T^{(k,l)}TM$ is given by $\left(\pi^{-1}(U), \widetilde{\varphi}\right)$, where $\widetilde{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^{n^{k+l}}$ is defined by

$$\widetilde{\varphi} := (\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}}) \circ \Phi$$

where $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^{n^{k+l}}$ is given as in the proof of proposition A.1. Now we consider the coordinate representation $\widehat{A}$ in the given charts (see [Lee13, p. 35]). Since A is a rough section, we have that

$$A^{-1}\left(\pi^{-1}(U)\right) = (\pi \circ A)^{-1}(U) = \mathrm{id}_M^{-1}(U) = U.$$

Hence $\varphi\left(U \cap A^{-1}(\pi^{-1}(U))\right) = \varphi(U)$, which is open, and $\widehat{A} : \varphi(U) \to \widetilde{\varphi}\left(\pi^{-1}(U)\right)$ is given by

$$\begin{aligned}
\widehat{A}(x) &= (\widetilde{\varphi} \circ A \circ \varphi^{-1})(x) \\
&= (\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}})\left(\Phi(\varphi^{-1}(x), A_{\varphi^{-1}(x)})\right) \\
&= (\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}})\left(\varphi^{-1}(x), \left(A^{i_1 \ldots i_k}_{j_1 \ldots j_l}(\varphi^{-1}(x))\right)\right) \\
&= \left(x, \left(\widehat{A}^{i_1 \ldots i_k}_{j_1 \ldots j_l}(x)\right)\right).
\end{aligned}$$

By [Lee13, p. 35] A is smooth if and only if in any chart $\widehat{A}$ is smooth. This is furthermore equivalent to that each $\widehat{A}^{i_1 \ldots i_k}_{j_1 \ldots j_l}$ is smooth and thus equivalent to that $A^{i_1 \ldots i_k}_{j_1 \ldots j_l}$ is smooth (see [Lee13, p. 33]).
To prove (b)⇒(c), let $(U, (x^i))$ be a smooth chart. Then write $X_1, \ldots, X_k \in \mathfrak{X}(M)$ as

$$X_\nu = X_\nu^{\mu_\nu} \frac{\partial}{\partial x^{\mu_\nu}}.$$

for $\nu = 1, \ldots, k$. For $p \in U$ lemma 4.1 implies

$$\begin{aligned}
A(X_1, \ldots, X_n)(p) &= A_p(X_1|_p, \ldots, X_k|_p) \\
&= A_p\left(X_1^{\mu_1}(p) \left.\frac{\partial}{\partial x^{\mu_1}}\right|_p, \ldots, X_1^{\mu_k}(p) \left.\frac{\partial}{\partial x^{\mu_k}}\right|_p\right) \\
&= X_1^{\mu_1}(p) \cdots X_k^{\mu_k}(p) A_p\left(\left.\frac{\partial}{\partial x^{\mu_1}}\right|_p, \ldots, \left.\frac{\partial}{\partial x^{\mu_k}}\right|_p\right) \\
&= X_1^{\mu_1}(p) \cdots X_k^{\mu_k}(p) A^i_{\mu_1 \ldots \mu_k}(p) \left.\frac{\partial}{\partial x^i}\right|_p.
\end{aligned}$$

By the smoothness criterion for vector fields [Lee13, p. 175] we have that each component function $X^{\mu_n}_\nu$ is smooth. Thus if A is smooth, we have by that each $A^i_{j_1\dots j_k}$ is smooth and since $\mathscr{C}^\infty(M)$ is an $\mathbb{R}$-algebra (see [Lee13, p. 33]), we have that

$$X^{\mu_1}_1 \cdots X^{\mu_k}_k A^i_{\mu_1\dots\mu_k}$$

is smooth for $i = 1,\dots,n$. Thus again by the smoothness criterion together with the localness of smoothness [Lee13, p. 35] we get that $A(X_1,\dots,X_k) \in \mathfrak{X}(M)$.
To prove (c)⇒(d), we use that smoothness is a local property (see [Lee13, p. 35]). Let $p \in U$. Then by [Cat17, p. 14] we find a smooth bump function ψ supported in U and identically equal to 1 on some neighbourhood V of p. Set

$$\widetilde{X}_i|_p := \begin{cases} \psi(p)X_i|_p & p \in \operatorname{supp}\psi \\ 0 & p \in M \setminus \operatorname{supp}\psi \end{cases}.$$

Then the gluing lemma for smooth maps [Lee13, p. 35] implies $X_1,\dots,X_k \in \mathfrak{X}(M)$. Hence by (c) we get that $A(\widetilde{X}_1,\dots,\widetilde{X}_k) \in \mathfrak{X}(M)$ and so the restriction $A(\widetilde{X}_1,\dots,\widetilde{X}_k)|_V$ is smooth. But $A(\widetilde{X}_1,\dots,\widetilde{X}_k)|_V = A(X_1,\dots,X_k)$ and so we are done.
Lasty to prove (d)⇒(b), each vector field locally defined by

$$X_{j_\nu} = \delta^{\mu_\nu}_{j_\nu}\frac{\partial}{\partial x^{\mu_\nu}}.$$

is smooth. Thus by

$$A(X_1,\dots,X_n)(p) = \delta^{\mu_1}_{j_1}\cdots\delta^{\mu_k}_{j_k}A^i_{\mu_1\dots\mu_k}(p)\frac{\partial}{\partial x^i}\bigg|_p = A^i_{j_1\dots j_k}(p)\frac{\partial}{\partial x^i}\bigg|_p$$

we get that $A^i_{j_1\dots j_k}$ is smooth and hence by (b) also A. □

Theorem A.1 (Tensor Characterization Lemma). *A mapping*

$$\underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{k} \to \mathscr{C}^\infty(M) \qquad \text{or} \qquad \underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{k} \to \mathfrak{X}(M)$$

is induced by an element of $\Gamma(T^{(0,k)}TM)$ or $\Gamma(T^{(1,k)}TM)$, respectively, if and only if they are multilinear over $\mathscr{C}^\infty(M)$.

Proof. We are proving only the second statement. Any element in $\Gamma(T^{(1,k)}TM)$ induces a mapping $\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M) \to \mathfrak{X}(M)$ by part (c) of the smoothness criteria for tensor fields A.2. Thus we have to show that $\mathscr{A}$ is multilinear over $\mathscr{C}^\infty(M)$. Let $f \in \mathscr{C}^\infty(M)$ and $X_\nu, \widetilde{X}_\nu \in \mathfrak{X}(M)$, $\nu = 1,\dots,k$. Then for any $p \in M$ we have that

$$\begin{aligned}\mathscr{A}(X_1,\dots,fX_\nu+\widetilde{X}_\nu,\dots,X_k)_p =& A_p(X_1|_p,\dots,(fX_\nu+\widetilde{X}_\nu)_p,\dots,X_k|_p) \\ =& A_p(X_1|_p,\dots,f(p)X_\nu|_p+\widetilde{X}_\nu|_p,\dots,X_k|_p)\end{aligned}$$

Yannis Bähni
yannis.baehni@uzh.ch

$$
\begin{aligned}
&= f(p)A_p(X_1|_p, \ldots, X_\nu|_p, \ldots, X_k|_p) \\
&\quad + A_p(X_1|_p, \ldots, \widetilde{X}_\nu|_p, \ldots, X_k|_p) \\
&= f(p)\mathscr{A}(X_1, \ldots, X_\nu, \ldots, X_k)_p \\
&\quad + \mathscr{A}(X_1, \ldots, \widetilde{X}_\nu, \ldots, X_k)_p \\
&= \big(f\mathscr{A}(X_1, \ldots, X_\nu, \ldots, X_k)\big)_p \\
&\quad + \mathscr{A}(X_1, \ldots, \widetilde{X}_\nu, \ldots, X_k)_p.
\end{aligned}
$$

Conversly, suppose that $\mathscr{A} : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is multilinear over $\mathscr{C}^\infty(M)$. Let $p \in M$. First we show that $\mathscr{A}$ acts locally, i.e. if $X_\nu = \widetilde{X}_\nu$ in some neighbourhood U of p implies that also

$$\mathscr{A}(X_1, \ldots, X_\nu, \ldots, X_k) = \mathscr{A}(X_1, \ldots, \widetilde{X}_\nu, \ldots, X_k)$$

on U. By the multilinearity of $\mathscr{A}$ it is enough to show that if X_ν vanishes on U then so does $\mathscr{A}$. There exists a smooth bump function ψ for $\{p\}$ supported in U (see [Lee13, p. 44]). Hence $\psi X_\nu = 0$ on M and $\psi(p) = 1$. Thus

$$0 = \mathscr{A}(X_1, \ldots, \psi X_\nu, \ldots, X_k)_p = \psi(p)\mathscr{A}(X_1, \ldots, X_\nu, \ldots, X_k)_p.$$

and since $\psi(p) = 1$ we have that

$$\mathscr{A}(X_1, \ldots, X_\nu, \ldots, X_k)_p = 0$$

for any $p \in U$.
Next we show that $\mathscr{A}$ actually acts pointwise, i.e. if $X_\nu|_p$ vanishes so does $\mathscr{A}$. Let $(U, (x^i))$ be a chart containing p and $X_\nu = X_\nu^i \frac{\partial}{\partial x^i}$ on U. The same construction as used showing the implication (c)$\Rightarrow$(d) in the proof of proposition A.2 yields the existence of $f^1, \ldots, f^n \in \mathscr{C}^\infty(M)$ and $\widetilde{X}_1, \ldots, \widetilde{X}_n \in \mathfrak{X}(M)$ such that $f^i = X_\nu^i$ and $\widetilde{X}_i = \frac{\partial}{\partial x^i}$ on a neighbourhood $V \subseteq U$ of p. Thus by the previous localization, we get that

$$\mathscr{A}(X_1, \ldots, X_\nu, \ldots, X_k) = \mathscr{A}(X_1, \ldots, f^i\widetilde{X}_i, \ldots, X_k) = f^i\mathscr{A}(X_1, \ldots, \widetilde{X}_i, \ldots, X_k)$$

in U. Since $0 = X_\nu^i(p) = f^i(p)$, $\mathscr{A}$ vanishes at p. Hence $\mathscr{A}$ depends only on the value of X_ν at p. Thus define a rough section $A : M \to T^{(1,k)}TM$ by

$$A_p(v_1, \ldots, v_k) := \mathscr{A}(V_1, \ldots, V_k)(p)$$

where $V_1, \ldots, V_k \in \mathfrak{X}(M)$ are any extensions of $v_1, \ldots, v_k \in T_pM$ (see [Lee13, p. 177]). By the above, the choice of the extensions does not matter and the resulting rough section is smooth by proposition A.2 part (c), hence $A \in \Gamma(T^{(1,k)}TM)$. □

Yannis Bähni
yannis.baehni@uzh.ch

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