COMPLEX MANIFOLDS

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Abstract. Goal of this paper is to give an overview of the basic definitions of complex and Kähler manifolds together with the most important properties. The main theorem will be the *Newlander-Nirenberg Theorem* which gives a criterion under which an almost complex manifold is a complex one. The key role will be played by a certain tensor field, the so called *Nijenhuis tensor*.

Contents

1. Complex Structures on Vector Spaces

In what follows, let $n \in \mathbb{Z}$, $n \geq 1$. Consider an n-dimensional complex vector space and let $J \in \operatorname{End}_{\mathbb{C}}(V)$ be defined by $J(v) := iv$. Then clearly $J \circ J = -\operatorname{id}_V$. Since every n-dimensional complex vector space can be seen as a $2n$-dimensional real vector space in a natural way, i.e. if (e_ν) is a basis for the complex vector space V, then (e_ν, ie_ν) is a basis for the real vector space V, the mapping J induces an $\mathbb{R}$-endomorphism $\widetilde{J}$ on the real vector space V simply by defining $\widetilde{J}(e_\nu) := J(e_\nu) = ie_\nu$ and $\widetilde{J}(ie_\nu) := J(ie_\nu) = -e_\nu$ for all $\nu = 1, \ldots, n$.
Conversly, let V be an n-dimensional real vector space with $J \in \operatorname{End}_{\mathbb{R}}(V)$ such that $J \circ J = -\operatorname{id}_V$. One can show, that

$$zv := xv + yJ(v) \tag{1}$$

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for $z := x + iy \in \mathbb{C}$ and $v \in V$ makes V into a complex vector space. This motivates the following definition.

Definition 1.1. *Let $n \in \mathbb{Z}$, $n \geq 1$ and V be an n-dimensional real vector space. A **complex structure on V** is a $\mathbb{R}$-linear mapping $J : V \to V$ such that $J \circ J = -\operatorname{id}_V$. If J is a complex structure on V, the tuple (V, J) is called a **complex vector space**.*

Lemma 1.1. *Let (V, J) be a complex vector space. Then $\dim V$ is even.*

Proof. That $\dim V$ must be even follows directly from

$$\big(\det(J)\big)^2 = \det(J \circ J) = \det(-\operatorname{id}_V) = (-1)^{\dim V} \det(\operatorname{id}_V) = (-1)^{\dim V}$$

since $\det(J) \in \mathbb{R}$. $\square$

2. Almost Complex Structures

If M is a smooth manifold, then T_pM is a finite dimensional real vector space. Hence we can generalize the definitions and results of the previous section to manifolds. The following definition is taken from [Sil08, p. 86].

Definition 2.1. *Let M be a smooth manifold. An **almost complex structure on M** is a smooth tensor field $J \in \Gamma\left(T^{(1,1)}TM\right)$ such that $J_p \circ J_p = -\operatorname{id}_{T_pM}$ holds for every $p \in M$. If J is an almost complex structure on M, the tuple (M, J) is called an **almost complex manifold**.*

Proposition 2.1. *Every almost complex manifold (M, J) is of even dimension and orientable.*

Proof. Assume that $\dim M$ is odd. Let $p \in M$. Then by [Lee13, p. 57] we have that $\dim T_pM = \dim M$. Hence $\dim T_pM$ is odd. But by lemma 1.1, $\dim T_pM$ must be even since (T_pM, J_p) is a complex vector space. Contradiction.
Since M is a smooth manifold, there exists a Riemannian metric g on M (see [Lee13, p. 329]). Define

$$\widetilde{g}(X, Y) := g(X, Y) + g(JX, JY) \in \Gamma(T^{(0,2)}TM)$$

for all $X, Y \in \mathfrak{X}(M)$. This is possible due to the tensor characterization lemma B.1. Then

$$\widetilde{g}(JX, JY) = g(JX, JY) + g(-X, -Y) = g(JX, JY) + g(X, Y) = \widetilde{g}(X, Y)$$

by the bilinearity of g. Furthermore, clearly $\widetilde{g}$ is positive definite and symmetric, thus a Riemannian metric on M. Define

$$\omega(X, Y) := \widetilde{g}(X, JY).$$

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Then by

$$\omega(Y,X) = \widetilde{g}(Y,JX) = \widetilde{g}(JX,Y) = \widetilde{g}(-X,JY) = -\omega(X,Y)$$

we see that ω is skew-symmetric. Hence $\omega \in \Omega^2(M)$. Let $p \in M$ and $u \in T_pM \setminus \{0\}$. Then also $-J_p(u) \neq 0$ since J_p is invertible by $\det J_p = 1$. Furthermore, by [Lee13, p. 177], there exist $X, Y \in \mathfrak{X}(M)$, such that $X_p = u$ and $Y_p = -J_p(u)$. Hence

$$\begin{aligned}
\omega_p\left(u, -J_p(u)\right) &= \omega_p(X_p, Y_p) \\
&= \omega(X,Y)(p) \\
&= \widetilde{g}(X,JY)(p) \\
&= \widetilde{g}_p\left(X_p, (JY)_p\right) \\
&= \widetilde{g}_p\left(u, J_p(Y_p)\right) \\
&= \widetilde{g}_p\left(u, -(J_p \circ J_p)(u)\right) \\
&= \widetilde{g}_p(u,u) \\
&\neq 0
\end{aligned}$$

and by [Lee13, p. 565] we get that ω is nondegenrate. Let $\dim M = 2n$. By [Lee13, p. 567] this implies that $\omega_p \wedge \cdots \wedge \omega_p$ is nonzero for each $p \in M$. Hence $\omega \wedge \cdots \wedge \omega$ is a nonvanishing top form on M. Since any nonvanishing top form determines an orientation (see [Lee13, p. 381]), we have that M is orientable. □

Remark 2.1. The converse of proposition 2.1 is not true in general. One can show using results on fibre bundles and Chern classes, that if $\mathbb{S}^n$ admits an almost complex structure, then $n = 2^k - 2$ for $k \in \mathbb{Z}$, $k \geq 1$ (see [Ste51, p. 219]). So for example $\mathbb{S}^4$ does not admit an almost complex structure. Actually, it can be shown that $\mathbb{S}^2$ and $\mathbb{S}^6$ are the only spheres which admit an almost complex structure (see [BS53, p. 434]).

3. Complex Manifolds

The definition of smooth manifolds adapts smoothly to the complex case.

Definition 3.1. *Let $n \in \mathbb{Z}$, $n \geq 1$. An **n-dimensional complex manifold** is a second countable Hausdorff space M equipped with a holomorphic structure, that is a maximal holomorphic atlas of complex charts, such that all the transition maps are holomorphic.*

Examples 3.1 (Complex Manifolds).

1. The complex n-space $\mathbb{C}^n$ is an n-dimensional complex manifold.
2. Let $\{\omega_1, \ldots, \omega_{2n}\}$ be a real basis of $\mathbb{C}^n$ and define

$$G := \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_{2n}. \tag{2}$$

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Then the discrete group G acts freely and properly discontinuously on $\mathbb{C}^n$ by translation. Thus $\mathbb{T}^n := \mathbb{C}^n/G$ is an n-dimensional complex manifold, called a ***complex torus*** (see [FG10, pp. 206–207]).

3. The quotient $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$ is an n-dimensional complex manifold, called the ***complex projective space*** (see [FG10, pp. 208–210]).

Lemma 3.1. *Let $n, k \in \mathbb{Z}$, $n, k \geq 1$. Let V be an n-dimensional real vector space. Then*

$$V \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{k} \cong \mathrm{L}(\underbrace{V, \ldots, V}_{k}; V) \tag{3}$$

canonically. If (e_ν) is a basis of V and (e_ν^*) the corresponding basis of V^*, then $f \in \mathrm{End}(V)$ corresponds to

$$\sum_{\nu=1}^{n} f(e_\nu) \otimes e_\nu^*. \tag{4}$$

Proof. It is easily checked that

$$\Psi : \begin{cases} V \times V^* \times \cdots \times V^* \to \mathrm{L}(V, \ldots, V; V) \\ (v, f_1, \ldots, f_k) \mapsto \big((v_1, \ldots, v_k) \mapsto f_1(v_1) \cdots f_k(v_k) v\big) \end{cases}$$

is multilinear. Thus by the universal property of the tensor product there exists a unique linear mapping $\widetilde{\Psi} \in \mathrm{Hom}_{\mathbb{R}}\left(T^{(1,k)}(V); \mathrm{L}(V, \ldots, V; V)\right)$ such that $\Psi = \widetilde{\Psi} \circ \otimes$. It is also easily checked that $\widetilde{\Psi}$ is an isomorphism. Let $f \in \mathrm{End}(V)$. Then for any $v \in V$ we have

$$\begin{aligned} \widetilde{\Psi}\left(\sum_{\nu=1}^{n} f(e_\nu) \otimes e_\nu^*\right)(v) &= \sum_{\nu=1}^{n} \widetilde{\Psi}\left(f(e_\nu) \otimes e_\nu^*\right)(v) \\ &= \sum_{\nu=1}^{n} e_\nu^*(v) f(e_\nu) \\ &= f\left(\sum_{\nu=1}^{n} e_\nu^*(v) e_\nu\right) \\ &= f(v). \end{aligned}$$

□

Proposition 3.1. *Any complex manifold admits a canonical almost complex structure.*

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Proof. Fix a complex manifold M. We define J in terms of local coordinates. Let $\big(U,(x^\nu,y^\nu)\big)$ be a chart. By lemma 3.1 it is also enough to construct an endomorphism J_p for every $p \in U$. We define

$$J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) := \left.\frac{\partial}{\partial y^\nu}\right|_p \qquad \text{and} \qquad J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right) := -\left.\frac{\partial}{\partial x^\nu}\right|_p$$

for all $\nu = 1,\dots,n$. As standard linear algebra shows, there is a unique linear mapping associated with J_p (see [HK71, p. 69]). Let $v := a^\nu \frac{\partial}{\partial x^\nu}\big|_p + b^\nu \frac{\partial}{\partial y^\nu}\big|_p \in T_pM$. Then

$$\begin{aligned}
(J_p\circ J_p)(v) &= J_p\left(a^\nu J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) + b^\nu J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right)\right)\\
&= J_p\left(a^\nu \left.\frac{\partial}{\partial y^\nu}\right|_p - b^\nu \left.\frac{\partial}{\partial x^\nu}\right|_p\right)\\
&= -a^\nu \left.\frac{\partial}{\partial x^\nu}\right|_p - b^\nu \left.\frac{\partial}{\partial y^\nu}\right|_p\\
&= -v
\end{aligned}$$

and thus $J_p \circ J_p = -\operatorname{id}_{T_pM}$.

Next we have to show that above locally defined mapping is well-defined, i.e. does not depend on the choice of coordinates. Assume that $p \in U \cap V$ for another chart $\big(V,(u^i,v^i)\big)$. By the change of coordinates formula [Lee13, p. 64] we get that

$$\left.\frac{\partial}{\partial x^\nu}\right|_p = \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})\left.\frac{\partial}{\partial u^\mu}\right|_p + \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})\left.\frac{\partial}{\partial v^\mu}\right|_p$$

and

$$\left.\frac{\partial}{\partial y^\nu}\right|_p = \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})\left.\frac{\partial}{\partial u^\mu}\right|_p + \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})\left.\frac{\partial}{\partial v^\mu}\right|_p$$

where $\widehat{p}$ denotes the coordinate representation of p with respect to the coordinates (x^ν,y^ν). Corollary A.1 implies

$$\begin{aligned}
J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) &= \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p}) J_p\left(\left.\frac{\partial}{\partial u^\mu}\right|_p\right) + \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p}) J_p\left(\left.\frac{\partial}{\partial v^\mu}\right|_p\right)\\
&= \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})\left.\frac{\partial}{\partial v^\mu}\right|_p - \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})\left.\frac{\partial}{\partial u^\mu}\right|_p\\
&= \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})\left.\frac{\partial}{\partial v^\mu}\right|_p + \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})\left.\frac{\partial}{\partial u^\mu}\right|_p\\
&= \left.\frac{\partial}{\partial y^\nu}\right|_p
\end{aligned}$$

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and

$$\begin{aligned}J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right) &= \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})J_p\left(\left.\frac{\partial}{\partial u^\mu}\right|_p\right) + \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})J_p\left(\left.\frac{\partial}{\partial v^\mu}\right|_p\right) \\ &= \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})\left.\frac{\partial}{\partial v^\mu}\right|_p - \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})\left.\frac{\partial}{\partial u^\mu}\right|_p \\ &= -\frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})\left.\frac{\partial}{\partial v^\mu}\right|_p - \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})\left.\frac{\partial}{\partial u^\mu}\right|_p \\ &= -\left.\frac{\partial}{\partial x^\nu}\right|_p.\end{aligned}$$

Left to check is smoothness. According to lemma 3.1 the corresponding rough tensor field is given by

$$J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right)\otimes \mathrm{d}x^\nu|_p + J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right)\otimes \mathrm{d}y^\nu|_p = \left.\frac{\partial}{\partial y^\nu}\right|_p \otimes \mathrm{d}x^\nu|_p - \left.\frac{\partial}{\partial x^\nu}\right|_p \otimes \mathrm{d}y^\nu|_p$$

for any $p \in U$. Thus the smoothness criteria for tensor fields B.2 together with [Lee13, p. 36] yields that $J \in \Gamma\left(T^{(1,1)}TM\right)$. □

A question which naturally arises by considering proposition 3.1 is, if the converse is also true, i.e. if every almost complex manifold is a complex manifold. This is in general not the case. Let $\mathbb{P}$ denote the naturally oriented underlying smooth manifold of the complex projective plane. Again using results about Chern numbers it can be shown that

$$\mathbb{P}\#(\mathbb{S}^1\times\mathbb{S}^3)\#(\mathbb{S}^1\times\mathbb{S}^3) \qquad \text{and} \qquad (\mathbb{S}^1\times\mathbb{S}^3)\#(\mathbb{S}^1\times\mathbb{S}^3)\#(\mathbb{S}^2\times\mathbb{S}^2) \tag{5}$$

have almost complex structures but no complex structure (see [Ven66, p. 1627]).

4. The Nijenhuis Tensor and the Newlander-Nirenberg Theorem

As we have seen in the last section, not every almost complex manifold is a complex manifold. Under which condition is this possible?

Definition 4.1. *Let (M, J) be an almost complex manifold. For $X, Y \in \mathfrak{X}(M)$ we define the* ***Nijenhuis tensor N*** *as*

$$N(X,Y) := [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] \tag{6}$$

where $[X, Y]$ denotes the usual Lie-bracket of vector fields.

Proposition 4.1. *Let (M, J) be an almost complex manifold and N be the Nijenhuis tensor. Then $N \in \Gamma\left(T^{(1,2)}TM\right)$.*

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Proof. First of all, $N(X,Y) \in \mathfrak{X}(M)$ for all $X, Y \in \mathfrak{X}(M)$. This follows immediately by considering J as a mapping $J : \mathfrak{X}(M) \to \mathfrak{X}(M)$ using the tensor characterization lemma B.1, the fact that the Lie Bracket of two smooth vector fields is again a smooth vector field (see [Lee13, p. 186]) and that $\mathfrak{X}(M)$ is a $\mathscr{C}^\infty(M)$-module (see [Lee13, p. 177]). Let $f \in \mathscr{C}^\infty(M)$ and $X, Y, Z \in \mathfrak{X}(M)$. Then

$$\begin{aligned}
N(fX+Y,Z) = &[J(fX+Y),JZ] - J[fX+Y,JZ] - J[J(fX+Y),Z] \\
&- [fX+Y,Z] \\
= &[fJX+JY,JZ] - J[fX+Y,JZ] - J[fJX+JY,Z] \\
&- [fX+Y,Z] \\
= &[fJX,JZ] + [JY,JZ] - J[fX,JZ] - J[Y,JZ] - J[fJX,Z] \\
&- J[JY,Z] - [fX,Z] - [Y,Z] \\
= &f[JX,JZ] - (JZf)JX + [JY,JZ] - fJ[X,JZ] + (JZf)JX \\
&- [Y,JZ] - fJ[JX,Z] + (Zf)JJX - J[JY,Z] - f[X,Z] \\
&+ (Zf)X - [Y,Z] \\
= &fN(X,Z) + N(Y,Z).
\end{aligned}$$

by [Lee13, pp. 187–188]. Linearity in the second argument is shown similarly. Hence $N : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is bilinear over $\mathscr{C}^\infty(M)$. Again by the tensor characterization lemma B.1 we have that $N \in \Gamma\left(T^{(1,2)}TM\right)$. □

Theorem 4.1 (Newlander-Nirenberg). *Let (M, J) be an almost complex manifold. Then M is a complex manifold, where the complex structure is so that the canonically induced almost complex structure is J, if and only if the Nijenhuis tensor N vanishes identically.*

Proof. Assume M is a complex manifold. Let $(U, (x^\nu, y^\nu))$ be a chart. From proposition 4.1 it is enough to consider the coordinate vector fields $\frac{\partial}{\partial x^\nu}$ and $\frac{\partial}{\partial y^\nu}$. But from the explicit definition of J in proposition 3.1 and the property, that the Lie-Bracket of coordinate vector fields vanishes, together with the $\mathscr{C}^\infty(M)$-linearity of J we get that N vanishes identically on each chart, and thus on M.
The other direction however is far more technical and uses results on partial differential equations. A complete proof can either be found in the original paper [NN57] or in [Sil08, p. 106], where references to more recent proofs are given. □

5. Kähler Manifolds

The following is inspired by [KN96, pp. 146–149] and introduces the concepts from a complex viewpoint. This is in contrast to the symplectic approach provided for example in [Sil08].

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Definition 5.1. *Let (M, J) be an almost complex manifold. A **Hermitian metric on M** is a Riemannian metric g such that*

$$g(JX, JY) = g(X, Y) \tag{7}$$

*holds for all $X, Y \in \mathfrak{X}(M)$. If g is a Hermitian metric on M, the triple (M, J, g) is called an **almost Hermitian manifold**.*

Lemma 5.1. *Every almost complex manifold admits a Hermitian metric.*

Proof. The existence was shown in the proof of proposition 2.1. □

Definition 5.2. *Let (M, J, g) be an almost Hermitian manifold. The **fundamental 2-form Ω** is defined to be*

$$\Omega(X, Y) := g(X, JY) \tag{8}$$

for all $X, Y \in \mathfrak{X}(M)$.

Definition 5.3. *Let (M, J, g) be an almost Hermitian manifold with fundamental 2-form Ω. The Hermitian metric is said to be a **Kähler metric**, if $\mathrm{d}\Omega = 0$. An almost complex manifold with a Kähler metric is called an **almost Kähler manifold** and a complex manifold with a Kähler metric is called a **Kähler manifold**.*

Appendix A. Functions of Several Complex Variables

This section summarizes the fundamental properties of functions of several complex variables needed later. The results are inspired by [FG10, pp. 14–30].

Definition A.1. *Let $n \in \mathbb{Z}$, $n \geq 1$, $U \subseteq \mathbb{C}^n$ open and $a \in U$. A mapping $f : U \to \mathbb{C}$ is said to be **complex differentiable at a** if there exists $g : U \to \mathbb{C}^n$ such that g is continuous at a and*

$$f(z) = f(a) + \sum_{\nu=1}^{n} (z_\nu - a_\nu) g_\nu(z) \tag{9}$$

*holds for all $z \in D$. f is said to be **holomorphic in D** if it is complex differentiable at every point $a \in D$. For $m \in \mathbb{Z}$, $m \geq 1$, a mapping $f : U \to \mathbb{C}^m$ is said to be holomorphic in D if each component function f_ν, $\nu = 1, \ldots, n$, is holomorphic in D.*

Proposition A.1. *Let $n \in \mathbb{Z}$, $n \geq 1$, $D \subseteq \mathbb{C}^n$ open, $a \in U$ and $f : D \to \mathbb{C}$ real differentiable at a. Then*

$$\frac{\partial f}{\partial z_\nu}(a) = \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu}(a) - i\frac{\partial f}{\partial y_\nu}(a)\right) \tag{10}$$

and

$$\frac{\partial f}{\partial \bar{z}_\nu}(a) = \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu}(a) + i\frac{\partial f}{\partial y_\nu}(a)\right) \tag{11}$$

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holds for all $\nu = 1, \dots, n$.

Theorem A.1 (The Cauchy-Riemann Equations). *Let* $n \in \mathbb{Z}$, $n \geq 1$ *and* $D \subseteq \mathbb{C}^n$ *open. A mapping* $f : D \to \mathbb{C}$ *is holomorphic in* D *if and only if it is real differentiable at every* $a \in D$ *and the* ***Cauchy-Riemann equations***

$$\frac{\partial f}{\partial \overline{z}_\nu}(a) = 0 \tag{12}$$

holds for all $a \in D$ *and* $\nu = 1, \dots, n$.

Corollary A.1. *Let* $m, n \in \mathbb{Z}$, $m, n \geq 1$, $D \subseteq \mathbb{C}^n$ *open and* $f : D \to \mathbb{C}^m$ *holomorphic in* D. *If* $f = g + ih$, $g, h : D \to \mathbb{R}^m$, *then*

$$\boxed{\frac{\partial g_\mu}{\partial x_\nu}(a) = \frac{\partial h_\mu}{\partial y_\nu}(a) \qquad \text{and} \qquad \frac{\partial h_\mu}{\partial x_\nu}(a) = -\frac{\partial g_\mu}{\partial y_\nu}(a)} \tag{13}$$

holds for any $a \in D$, $\nu = 1, \dots, n$ *and* $\mu = 1, \dots, m$.

Proof. Fix $\mu = 1, \dots, m$. By definition A.1 f_μ is holomorphic in D. Hence f_μ is real differentiable in D (see [FG10, p. 27]) and theorem A.1 implies

$$\frac{\partial f_\mu}{\partial \overline{z}_\nu}(a) = 0$$

for all $a \in D$ and $\nu = 1, \dots, n$. By proposition A.1, this is equivalent to

$$\frac{\partial f}{\partial x_\nu}(a) + i\frac{\partial f}{\partial y_\nu}(a) = 0.$$

Using $f_\mu = g_\mu + ih_\mu$ and the $\mathbb{C}$-linearity of the operators $\frac{\partial}{\partial x_\nu}$ and $\frac{\partial}{\partial y_\nu}$ yields

$$\frac{\partial g_\mu}{\partial x_\nu}(a) - \frac{\partial h_\mu}{\partial y_\nu}(a) + i\left(\frac{\partial h_\mu}{\partial x_\nu}(a) + \frac{\partial g_\mu}{\partial y_\nu}(a)\right) = 0.$$

□

Appendix B. Tensor Characterization Lemma

Definition B.1. *Let* $k, l \in \mathbb{Z}$, $k, l \geq 0$ *and* M *a smooth manifold. Then the* ***bundle of mixed tensors of type*** (k, l) *is defined by*

$$T^{(k,l)}TM := \coprod_{p \in M} T^{(k,l)}(T_pM). \tag{14}$$

Proposition B.1. *The bundle of mixed tensors of type* (k, l) *has an unique natural structure as a smooth vector bundle of rank* n^{k+l} *over* M.

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Proof. For each $p \in M$ let $E_p := T^{(k,l)}(T_pM)$. By [Lee13, p. 57] and [Lee13, p. 313] $\dim E_p = n^{k+l}$. Furthermore, let $E := T^{(k,l)}TM$ and $\pi : E \to M$ be defined by $\pi(p, A) := p$. Let $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ be an atlas for M. For each $\alpha \in A$ define

$$\Phi_\alpha : \begin{cases} \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{n^{k+l}} \\ (p, A) \mapsto \left(p, (A^{i_1 \dots i_k}_{j_1 \dots j_l})\right) \end{cases}$$

Clearly, the inverse is given by

$$\Phi_\alpha^{-1} : \begin{cases} U_\alpha \times \mathbb{R}^{n^{k+l}} \to \pi^{-1}(U_\alpha) \\ \left(p, (A^{i_1 \dots i_k}_{j_1 \dots j_l})\right) \mapsto (p, A) \end{cases}.$$

Hence each Φ_α is bijective. Now we have to check, that $\Phi_\alpha|_{E_p}$ is an isomorphism. So let $\lambda \in \mathbb{R}$ and $B \in E_p$. Then

$$\begin{aligned} \Phi_\alpha|_{E_p}(p, \lambda A + B) &= \left(p, (\lambda A + B)^{i_1 \dots i_k}_{j_1 \dots j_l})\right) \\ &= \left(p, \lambda(A^{i_1 \dots i_k}_{j_1 \dots j_l}) + (B^{i_1 \dots i_k}_{j_1 \dots j_l})\right) \\ &= \lambda \Phi_\alpha|_{E_p}(p, A) + \Phi_\alpha|_{E_p}(p, B). \end{aligned}$$

Now let $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$. We consider the mapping

$$\Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^{n^{k+l}} \to (U_\alpha \cap U_\beta) \times \mathbb{R}^{n^{k+l}}.$$

Define $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n^{k+l}, \mathbb{R})$ by

$$\tau_{\alpha\beta} := (\delta^i_j).$$

Then we have that

$$\left(\Phi_\alpha \circ \Phi_\beta^{-1}\right)\left(p, (A^{i_1 \dots i_k}_{j_1 \dots j_l})\right) = \left(p, (A^{i_1 \dots i_k}_{j_1 \dots j_l})\right) = \left(p, \tau_{\alpha\beta}(p)(A^{i_1 \dots i_k}_{j_1 \dots j_l})\right).$$

Since $\tau_{\alpha\beta}$ is constant, it is smooth (see [Lee13, p. 36]). Hence we can apply the vector bundle chart lemma [Lee13, p. 253] and the result follows. □

What follows is a reformulation of the smoothness criteria for tensor fields ([Lee13, pp. 317–318]) for tensor fields of type $(1, k)$.

Proposition B.2 (Smoothness Criteria for Tensor Fields). *Let M be smooth manifold and let $A : M \to T^{(1,k)}TM$ be a rough section. Then the following are equivalent:*

(a) $A \in \Gamma(T^{(1,k)}TM)$.
(b) In every smooth coordinate chart, the component functions of A are smooth.
(c) For all $X_1, \dots, X_k \in \mathfrak{X}(M)$, the rough section $A(X_1, \dots, X_k) : M \to TM$ defined by

$$A(X_1, \dots, X_k)(p) := A_p(X_1|_p, \dots, X_k|_p) \tag{15}$$

is a smooth vector field.

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(d) *If $X_1, \dots, X_k$ are smooth vector fields on some open subset $U \subseteq M$, then also $A(X_1, \dots, X_k)$ is a smooth vector field on U.*

Proof. We prove (a)⇔(b) and (b)⇒(c)⇒(d)⇒(b).
To prove (a)⇔(b), let $(U, (x^i))$ be a smooth chart. Actually, we can prove this for general tensor fields of type (k, l). Proposition B.1 together with the proof of the vector bundle chart lemma [Lee13, pp. 253–254] implies, that the corresponding chart on $T^{(k,l)}TM$ is given by $\left(\pi^{-1}(U), \widetilde{\varphi}\right)$, where $\widetilde{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^{n^{k+l}}$ is defined by

$$\widetilde{\varphi} := (\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}}) \circ \Phi$$

where $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^{n^{k+l}}$ is given as in the proof of proposition B.1. Now we consider the coordinate representation $\widehat{A}$ in the given charts (see [Lee13, p. 35]). Since A is a rough section, we have that

$$A^{-1}\left(\pi^{-1}(U)\right) = (\pi \circ A)^{-1}(U) = \mathrm{id}_M^{-1}(U) = U.$$

Hence $\varphi\left(U \cap A^{-1}(\pi^{-1}(U))\right) = \varphi(U)$, which is open, and $\widehat{A} : \varphi(U) \to \widetilde{\varphi}\left(\pi^{-1}(U)\right)$ is given by

$$\begin{aligned}
\widehat{A}(x) &= (\widetilde{\varphi} \circ A \circ \varphi^{-1})(x) \\
&= (\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}})\left(\Phi(\varphi^{-1}(x), A_{\varphi^{-1}(x)})\right) \\
&= (\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}})\left(\varphi^{-1}(x), \left(A^{i_1 \dots i_k}_{j_1 \dots j_l}(\varphi^{-1}(x))\right)\right) \\
&= \left(x, \left(\widehat{A}^{i_1 \dots i_k}_{j_1 \dots j_l}(x)\right)\right).
\end{aligned}$$

By [Lee13, p. 35] A is smooth if and only if in any chart $\widehat{A}$ is smooth. This is furthermore equivalent to that each $\widehat{A}^{i_1 \dots i_k}_{j_1 \dots j_l}$ is smooth and thus equivalent to that $A^{i_1 \dots i_k}_{j_1 \dots j_l}$ is smooth (see [Lee13, p. 33]).
To prove (b)⇒(c), let $(U, (x^i))$ be a smooth chart. Then write $X_1, \dots, X_k \in \mathfrak{X}(M)$ as

$$X_\nu = X_\nu^{\mu_\nu} \frac{\partial}{\partial x^{\mu_\nu}}.$$

for $\nu = 1, \dots, k$. For $p \in U$ lemma 3.1 implies

$$\begin{aligned}
A(X_1, \dots, X_n)(p) &= A_p(X_1|_p, \dots, X_k|_p) \\
&= A_p\left(X_1^{\mu_1}(p) \left.\frac{\partial}{\partial x^{\mu_1}}\right|_p, \dots, X_1^{\mu_k}(p) \left.\frac{\partial}{\partial x^{\mu_k}}\right|_p\right) \\
&= X_1^{\mu_1}(p) \cdots X_k^{\mu_k}(p) A_p\left(\left.\frac{\partial}{\partial x^{\mu_1}}\right|_p, \dots, \left.\frac{\partial}{\partial x^{\mu_k}}\right|_p\right) \\
&= X_1^{\mu_1}(p) \cdots X_k^{\mu_k}(p) A^i_{\mu_1 \dots \mu_k}(p) \left.\frac{\partial}{\partial x^i}\right|_p.
\end{aligned}$$

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By the smoothness criterion for vector fields [Lee13, p. 175] we have that each component function $X_\nu^{\mu_n}$ is smooth. Thus if A is smooth, we have by that each $A^i_{j_1\dots j_k}$ is smooth and since $\mathscr{C}^\infty(M)$ is an $\mathbb{R}$-algebra (see [Lee13, p. 33]), we have that

$$X_1^{\mu_1}\cdots X_k^{\mu_k}A^i_{\mu_1\dots\mu_k}$$

is smooth for $i = 1,\dots,n$. Thus again by the smoothness criterion together with the localness of smoothness [Lee13, p. 35] we get that $A(X_1,\dots,X_k)\in\mathfrak{X}(M)$.
To prove (c)⇒(d), we use that smoothness is a local property (see [Lee13, p. 35]). Let $p\in U$. Then by [Cat17, p. 14] we find a smooth bump function ψ supported in U and identically equal to 1 on some neighbourhood V of p. Set

$$\widetilde{X}_i|_p := \begin{cases}\psi(p)X_i|_p & p\in\operatorname{supp}\psi\\ 0 & p\in M\setminus\operatorname{supp}\psi\end{cases}.$$

Then the gluing lemma for smooth maps [Lee13, p. 35] implies $X_1,\dots,X_k\in\mathfrak{X}(M)$. Hence by (c) we get that $A(\widetilde{X}_1,\dots,\widetilde{X}_k)\in\mathfrak{X}(M)$ and so the restriction $A(\widetilde{X}_1,\dots,\widetilde{X}_k)|_V$ is smooth. But $A(\widetilde{X}_1,\dots,\widetilde{X}_k)|_V = A(X_1,\dots,X_k)$ and so we are done.
Lasty to prove (d)⇒(b), each vector field locally defined by

$$X_{j_\nu} = \delta^{\mu_\nu}_{j_\nu}\frac{\partial}{\partial x^{\mu_\nu}}.$$

is smooth. Thus by

$$A(X_1,\dots,X_n)(p) = \delta^{\mu_1}_{j_1}\cdots\delta^{\mu_k}_{j_k}A^i_{\mu_1\dots\mu_k}(p)\frac{\partial}{\partial x^i}\bigg|_p = A^i_{j_1\dots j_k}(p)\frac{\partial}{\partial x^i}\bigg|_p$$

we get that $A^i_{j_1\dots j_k}$ is smooth and hence by (b) also A. □

Theorem B.1 (Tensor Characterization Lemma). *A mapping*

$$\underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{k}\to\mathscr{C}^\infty(M)\qquad\textit{or}\qquad\underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{k}\to\mathfrak{X}(M)$$

is induced by an element of $\Gamma(T^{(0,k)}TM)$ *or* $\Gamma(T^{(1,k)}TM)$*, respectively, if and only if they are multilinear over* $\mathscr{C}^\infty(M)$*.*

Proof. We are proving only the second statement. Any element in $\Gamma(T^{(1,k)}TM)$ induces a mapping $\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ by part (c) of the smoothness criteria for tensor fields B.2. Thus we have to show that $\mathscr{A}$ is multilinear over $\mathscr{C}^\infty(M)$. Let $f\in\mathscr{C}^\infty(M)$ and $X_\nu,\widetilde{X}_\nu\in\mathfrak{X}(M)$, $\nu=1,\dots,k$. Then for any $p\in M$ we have that

$$\begin{aligned}\mathscr{A}(X_1,\dots,fX_\nu+\widetilde{X}_\nu,\dots,X_k)_p =& A_p(X_1|_p,\dots,(fX_\nu+\widetilde{X}_\nu)_p,\dots,X_k|_p)\\ =& A_p(X_1|_p,\dots,f(p)X_\nu|_p+\widetilde{X}_\nu|_p,\dots,X_k|_p)\end{aligned}$$

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$$
\begin{aligned}
&= f(p)A_p(X_1|_p, \dots, X_\nu|_p, \dots, X_k|_p) \\
&\quad + A_p(X_1|_p, \dots, \widetilde{X}_\nu|_p, \dots, X_k|_p) \\
&= f(p)\mathscr{A}(X_1, \dots, X_\nu, \dots, X_k)_p \\
&\quad + \mathscr{A}(X_1, \dots, \widetilde{X}_\nu, \dots, X_k)_p \\
&= \big(f\mathscr{A}(X_1, \dots, X_\nu, \dots, X_k)\big)_p \\
&\quad + \mathscr{A}(X_1, \dots, \widetilde{X}_\nu, \dots, X_k)_p.
\end{aligned}
$$

Conversly, suppose that $\mathscr{A} : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is multilinear over $\mathscr{C}^\infty(M)$. Let $p \in M$. First we show that $\mathscr{A}$ acts locally, i.e. if $X_\nu = \widetilde{X}_\nu$ in some neighbourhood U of p implies that also

$$\mathscr{A}(X_1, \dots, X_\nu, \dots, X_k) = \mathscr{A}(X_1, \dots, \widetilde{X}_\nu, \dots, X_k)$$

on U. By the multilinearity of $\mathscr{A}$ it is enough to show that if X_ν vanishes on U then so does $\mathscr{A}$. There exists a smooth bump function ψ for $\{p\}$ supported in U (see [Lee13, p. 44]). Hence $\psi X_\nu = 0$ on M and $\psi(p) = 1$. Thus

$$0 = \mathscr{A}(X_1, \dots, \psi X_\nu, \dots, X_k)_p = \psi(p)\mathscr{A}(X_1, \dots, X_\nu, \dots, X_k)_p.$$

and since $\psi(p) = 1$ we have that

$$\mathscr{A}(X_1, \dots, X_\nu, \dots, X_k)_p = 0$$

for any $p \in U$.
Next we show that $\mathscr{A}$ actually acts pointwise, i.e. if $X_\nu|_p$ vanishes so does $\mathscr{A}$. Let $(U, (x^i))$ be a chart containing p and $X_\nu = X_\nu^i \frac{\partial}{\partial x^i}$ on U. The same construction as used showing the implication (c)⇒(d) in the proof of proposition B.2 yields the existence of $f^1, \dots, f^n \in \mathscr{C}^\infty(M)$ and $\widetilde{X}_1, \dots, \widetilde{X}_n \in \mathfrak{X}(M)$ such that $f^i = X_\nu^i$ and $\widetilde{X}_i = \frac{\partial}{\partial x^i}$ on a neighbourhood $V \subseteq U$ of p. Thus by the previous localization, we get that

$$\mathscr{A}(X_1, \dots, X_\nu, \dots, X_k) = \mathscr{A}(X_1, \dots, f^i\widetilde{X}_i, \dots, X_k) = f^i\mathscr{A}(X_1, \dots, \widetilde{X}_i, \dots, X_k)$$

in U. Since $0 = X_\nu^i(p) = f^i(p)$, $\mathscr{A}$ vanishes at p. Hence $\mathscr{A}$ depends only on the value of X_ν at p. Thus define a rough section $A : M \to T^{(1,k)}TM$ by

$$A_p(v_1, \dots, v_k) := \mathscr{A}(V_1, \dots, V_k)(p)$$

where $V_1, \dots, V_k \in \mathfrak{X}(M)$ are any extensions of $v_1, \dots, v_k \in T_pM$ (see [Lee13, p. 177]). By the above, the choice of the extensions does not matter and the resulting rough section is smooth by proposition B.2 part (c), hence $A \in \Gamma(T^{(1,k)}TM)$. □

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