

COMPLEX MANIFOLDS

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Abstract. Goal of this paper is to give an overview of the basic definitions of complex and Kähler manifolds together with the most important properties. The main theorem will be the *Newlander-Nirenberg Theorem* which gives a criterion under which an almost complex manifold is a complex one. The key role will be played by a certain tensor field, the so called *Nijenhuis tensor*.

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1. Functions of Several Complex Variables

This section summarizes the fundamental properties of functions of several complex variables needed later. The results are inspired by [FG10, pp. 14–30].

Definition 1.1. Let $n \in \mathbb{Z}$, $n \geq 1$, $U \subseteq \mathbb{C}^n$ open and $a \in U$. A mapping $f : U \rightarrow \mathbb{C}$ is said to be **complex differentiable at a** if there exists $g : U \rightarrow \mathbb{C}^n$ such that g is continuous at a and

$$f(z) = f(a) + \sum_{\nu=1}^n (z_{\nu} - a_{\nu})g_{\nu}(z) \quad (1)$$

holds for all $z \in D$. f is said to be **holomorphic in D** if it is complex differentiable at every point $a \in D$. For $m \in \mathbb{Z}$, $m \geq 1$, a mapping $f : U \rightarrow \mathbb{C}^m$ is said to be

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holomorphic in D if each component function f_ν , $\nu = 1, \dots, n$, is holomorphic in D .

Proposition 1.1. *Let $n \in \mathbb{Z}$, $n \geq 1$, $D \subseteq \mathbb{C}^n$ open, $a \in U$ and $f : D \rightarrow \mathbb{C}$ real differentiable at a . Then*

$$\frac{\partial f}{\partial z_\nu}(a) = \frac{1}{2} \left(\frac{\partial f}{\partial x_\nu}(a) - i \frac{\partial f}{\partial y_\nu}(a) \right) \quad (2)$$

and

$$\frac{\partial f}{\partial \bar{z}_\nu}(a) = \frac{1}{2} \left(\frac{\partial f}{\partial x_\nu}(a) + i \frac{\partial f}{\partial y_\nu}(a) \right) \quad (3)$$

holds for all $\nu = 1, \dots, n$.

Theorem 1.1 (The Cauchy-Riemann Equations). *Let $n \in \mathbb{Z}$, $n \geq 1$ and $D \subseteq \mathbb{C}^n$ open. A mapping $f : D \rightarrow \mathbb{C}$ is holomorphic in D if and only if it is real differentiable at every $a \in D$ and the **Cauchy-Riemann equations***

$$\frac{\partial f}{\partial \bar{z}_\nu}(a) = 0 \quad (4)$$

holds for all $a \in D$ and $\nu = 1, \dots, n$.

Corollary 1.1. *Let $m, n \in \mathbb{Z}$, $m, n \geq 1$, $D \subseteq \mathbb{C}^n$ open and $f : D \rightarrow \mathbb{C}^m$ holomorphic in D . If $f = g + ih$, $g, h : D \rightarrow \mathbb{R}^m$, then*

$$\boxed{\frac{\partial g_\mu}{\partial x_\nu}(a) = \frac{\partial h_\mu}{\partial y_\nu}(a) \quad \text{and} \quad \frac{\partial h_\mu}{\partial x_\nu}(a) = -\frac{\partial g_\mu}{\partial y_\nu}(a)} \quad (5)$$

holds for any $a \in D$, $\nu = 1, \dots, n$ and $\mu = 1, \dots, m$.

Proof. Fix $\mu = 1, \dots, m$. By definition 1.1 f_μ is holomorphic in D . Hence f_μ is real differentiable in D (see [FG10, p. 27]) and theorem 1.1 implies

$$\frac{\partial f_\mu}{\partial \bar{z}_\nu}(a) = 0$$

for all $a \in D$ and $\nu = 1, \dots, n$. By proposition 1.1, this is equivalent to

$$\frac{\partial f}{\partial x_\nu}(a) + i \frac{\partial f}{\partial y_\nu}(a) = 0.$$

Using $f_\mu = g_\mu + ih_\mu$ and the \mathbb{C} -linearity of the operators $\frac{\partial}{\partial x_\nu}$ and $\frac{\partial}{\partial y_\nu}$ yields

$$\frac{\partial g_\mu}{\partial x_\nu}(a) - \frac{\partial h_\mu}{\partial y_\nu}(a) + i \left(\frac{\partial h_\mu}{\partial x_\nu}(a) + \frac{\partial g_\mu}{\partial y_\nu}(a) \right) = 0.$$

□

2. Complex Structures on Vector Spaces

In what follows, let $n \in \mathbb{Z}$, $n \geq 1$. Consider an n -dimensional complex vector space and let $J \in \text{End}_{\mathbb{C}}(V)$ be defined by $J(v) := iv$. Then clearly $J \circ J = -\text{id}_V$. Since every n -dimensional complex vector space can be seen as a $2n$ -dimensional real vector space in a natural way, i.e. if (e_ν) is a basis for the complex vector space V , then (e_ν, ie_ν) is a basis for the real vector space V , the mapping J induces an \mathbb{R} -endomorphism J on the real vector space V simply by $J(e_\nu) = ie_\nu$ and $J(ie_\nu) = -e_\nu$ for all $\nu = 1, \dots, n$.

Conversely, let V be an n -dimensional real vector space with $J \in \text{End}_{\mathbb{R}}(V)$ such that $J \circ J = -\text{id}_V$. One can show, that

$$zv := xv + yJ(v) \quad (6)$$

for $z := x + iy \in \mathbb{C}$ and $v \in V$ makes V into a complex vector space. This motivates the following definition.

Definition 2.1. *Let V be an n -dimensional real vector space. A **complex structure on V** is a \mathbb{R} -linear mapping $J : V \rightarrow V$ such that $J \circ J = -\text{id}_V$. If J is a complex structure on V , the tuple (V, J) is called a **complex vector space**.*

Lemma 2.1. *Let (V, J) be a complex vector space. Then $\dim V$ is even.*

Proof. That $\dim V$ must be even follows directly from

$$(\det(J))^2 = \det(J \circ J) = \det(-\text{id}_V) = (-1)^{\dim V} \det(\text{id}_V) = (-1)^{\dim V}.$$

□

3. Almost Complex Structures

If M is a smooth manifold, then $T_p M$ is a finite dimensional real vector space. Hence we can generalize the definitions and results of the previous section to manifolds. The following definition is taken from [Sil08, p. 86].

Definition 3.1. *Let M be a smooth manifold. An **almost complex structure on M** is a smooth tensor field $J \in \Gamma(T^{(1,1)}TM)$ such that $J_p \circ J_p = -\text{id}_{T_p M}$ holds for any $p \in M$. If J is an almost complex structure on M , the tuple (M, J) is called an **almost complex manifold**.*

Proposition 3.1. *Every almost complex manifold (M, J) is of even dimension and orientable.*

Proof. Assume that $\dim M$ is odd. Let $p \in M$. Then by [Lee13, p. 57] we have that $\dim T_p M = \dim M$. Hence $\dim T_p M$ is odd. But by lemma 2.1, $\dim T_p M$ must be even since $(T_p M, J_p)$ is a complex vector space. Contradiction.

Since M is a smooth manifold, there exists a Riemannian metric g on M (see [Lee13, p. 329]). Define

$$\tilde{g}(X, Y) := g(X, Y) + g(JX, JY) \in \Gamma(T^{(0,2)}TM)$$

for all $X, Y \in \mathfrak{X}(M)$. Then

$$\tilde{g}(JX, JY) = g(JX, JY) + g(-X, -Y) = g(JX, JY) + g(X, Y) = \tilde{g}(X, Y)$$

by the bilinearity of g . Furthermore, clearly \tilde{g} is positive definite and symmetric, thus a Riemannian metric on M . Define

$$\omega(X, Y) := \tilde{g}(X, JY).$$

Then by

$$\omega(Y, X) = \tilde{g}(Y, JX) = \tilde{g}(JX, Y) = \tilde{g}(-X, JY) = -\omega(X, Y)$$

we see that ω is skew-symmetric. Hence $\omega \in \Omega^2(M)$. Let $p \in M$ and $u \in T_p M \setminus \{0\}$. Then also $-J_p(u) \neq 0$ since J_p is invertible since $\det J_p = 1$. Furthermore, by [Lee13, p. 177], there exist $X, Y \in \mathfrak{X}(M)$, such that $X_p = u$ and $Y_p = -J_p(u)$. Hence

$$\begin{aligned} \omega_p(u, -J_p(u)) &= \omega_p(X_p, Y_p) \\ &= \omega(X, Y)(p) \\ &= \tilde{g}(X, JY)(p) \\ &= \tilde{g}_p(X_p, (JY)_p) \\ &= \tilde{g}_p(u, J_p(Y_p)) \\ &= \tilde{g}_p(u, -(J_p \circ J_p)(u)) \\ &= \tilde{g}_p(u, u) \\ &\neq 0 \end{aligned}$$

and by [Lee13, p. 565] we get that ω is nondegenerate. Let $\dim M = 2n$. By [Lee13, p. 567] this implies that $\omega_p \wedge \cdots \wedge \omega_p$ is nonzero for each $p \in M$. Hence $\omega \wedge \cdots \wedge \omega$ is a nonvanishing top form on M . Since any nonvanishing top form determines an orientation (see [Lee13, p. 381]), we have that M is orientable. \square

Remark 3.1. The converse of proposition 3.1 is not true in general. One can show using results on fibre bundles and Chern classes, that if \mathbb{S}^n admits an almost complex structure, then $n = 2^k - 2$ for $k \in \mathbb{Z}$, $k \geq 1$ (see [Ste51, p. 219]). So for example \mathbb{S}^4 does not admit an almost complex structure. Actually, it can be shown that \mathbb{S}^2 and \mathbb{S}^6 are the only spheres which admit an almost complex structure (see [BS53, p. 434]).

4. Complex Manifolds

The definition of smooth manifolds adapts smoothly to the complex case.

Definition 4.1. Let $n \in \mathbb{Z}$, $n \geq 1$. An **n -dimensional complex manifold** is a second countable Hausdorff space M equipped with a holomorphic structure, that is a maximal holomorphic atlas $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ of complex charts $(U_\alpha, \varphi_\alpha)$, such that all the transition maps are holomorphically compatible.

Examples 4.1 (Complex Manifolds).

1. The complex n -space \mathbb{C}^n is an n -dimensional complex manifold.
2. Let $\{\omega_1, \dots, \omega_{2n}\}$ be a real basis of \mathbb{C}^n and define

$$G := \mathbb{Z}\omega_1 + \dots + \mathbb{Z}\omega_{2n}. \quad (7)$$

Then the discrete group G acts freely and properly discontinuously on \mathbb{C}^n by translation. Thus $\mathbb{T}^n := \mathbb{C}^n/G$ is an n -dimensional complex manifold, called a **complex torus** (see [FG10, pp. 206–207]).

3. The quotient $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$ is an n -dimensional complex manifold, called the **complex projective space** (see [FG10, pp. 208–210]).

Lemma 4.1. Let V be a real vector space of dimension $n \in \mathbb{Z}$, $n \geq 1$. Then

$$V \otimes V^* \cong \text{End}(V) \quad (8)$$

canonically. If (e_ν) is a basis of V and (e_ν^*) the corresponding basis of V^* , then $f \in \text{End}(V)$ corresponds to

$$\sum_{\nu=1}^n f(e_\nu) \otimes e_\nu^*. \quad (9)$$

Proof. It is easily checked that

$$\Phi : \begin{cases} V \times V^* \rightarrow \text{End}(V) \\ (v, f) \mapsto (u \mapsto f(u)v) \end{cases}$$

is bilinear. Thus by the universal property of the tensor product there exists a unique mapping $\widehat{\Phi} \in \text{Hom}(V \otimes V^*; \text{End}(V))$ such that $\Phi = \widehat{\Phi} \circ \otimes$. It is also easily checked that $\widehat{\Phi}$ is an isomorphism.

Let $f \in \text{End}(V)$. Then for any $v \in V$ we have

$$\begin{aligned} \widehat{\Phi} \left(\sum_{\nu=1}^n f(e_\nu) \otimes e_\nu^* \right) (v) &= \sum_{\nu=1}^n \widehat{\Phi} (f(e_\nu) \otimes e_\nu^*) (v) \\ &= \sum_{\nu=1}^n e_\nu^*(v) f(e_\nu) \end{aligned}$$

$$\begin{aligned} &= f \left(\sum_{\nu=1}^n e_{\nu}^*(v) e_{\nu} \right) \\ &= f(v). \end{aligned}$$

□

Proposition 4.1. *Any complex manifold admits a canonical almost complex structure.*

Proof. Fix a complex manifold M . We define J in terms of local coordinates. Let $(U, (x^{\nu}, y^{\nu}))$ be a chart. By lemma 4.1 it is also enough to construct an endomorphism J_p for every $p \in U$. We define

$$J_p \left(\frac{\partial}{\partial x^{\nu}} \Big|_p \right) := \frac{\partial}{\partial y^{\nu}} \Big|_p \quad \text{and} \quad J_p \left(\frac{\partial}{\partial y^{\nu}} \Big|_p \right) := -\frac{\partial}{\partial x^{\nu}} \Big|_p$$

for all $\nu = 1, \dots, n$. As standard linear algebra shows, there is a unique linear mapping associated with J_p (see [HK71, p. 69]). Let $v := a^{\nu} \frac{\partial}{\partial x^{\nu}} \Big|_p + b^{\nu} \frac{\partial}{\partial y^{\nu}} \Big|_p \in T_p M$. Then

$$\begin{aligned} (J_p \circ J_p)(v) &= J_p \left(a^{\nu} J_p \left(\frac{\partial}{\partial x^{\nu}} \Big|_p \right) + b^{\nu} J_p \left(\frac{\partial}{\partial y^{\nu}} \Big|_p \right) \right) \\ &= J_p \left(a^{\nu} \frac{\partial}{\partial y^{\nu}} \Big|_p - b^{\nu} \frac{\partial}{\partial x^{\nu}} \Big|_p \right) \\ &= -a^{\nu} \frac{\partial}{\partial x^{\nu}} \Big|_p - b^{\nu} \frac{\partial}{\partial y^{\nu}} \Big|_p \\ &= -v \end{aligned}$$

and thus $J_p \circ J_p = -\text{id}_{T_p M}$.

Next we have to show that above locally defined mapping is well-defined, i.e. does not depend on the choice of coordinates. Assume that $p \in U \cap V$ for another chart $(V, (u^i, v^i))$. By the change of coordinates formula [Lee13, p. 64] we get that

$$\frac{\partial}{\partial x^{\nu}} \Big|_p = \frac{\partial u^{\mu}}{\partial x^{\nu}}(\widehat{p}) \frac{\partial}{\partial u^{\mu}} \Big|_p + \frac{\partial v^{\mu}}{\partial x^{\nu}}(\widehat{p}) \frac{\partial}{\partial v^{\mu}} \Big|_p$$

and

$$\frac{\partial}{\partial y^{\nu}} \Big|_p = \frac{\partial u^{\mu}}{\partial y^{\nu}}(\widehat{p}) \frac{\partial}{\partial u^{\mu}} \Big|_p + \frac{\partial v^{\mu}}{\partial y^{\nu}}(\widehat{p}) \frac{\partial}{\partial v^{\mu}} \Big|_p$$

where \widehat{p} denotes the coordinate representation of p with respect to the coordinates (x^{ν}, y^{ν}) . Corollary 1.1 implies

$$J_p \left(\frac{\partial}{\partial x^{\nu}} \Big|_p \right) = \frac{\partial u^{\mu}}{\partial x^{\nu}}(\widehat{p}) J_p \left(\frac{\partial}{\partial u^{\mu}} \Big|_p \right) + \frac{\partial v^{\mu}}{\partial x^{\nu}}(\widehat{p}) J_p \left(\frac{\partial}{\partial v^{\mu}} \Big|_p \right)$$

$$\begin{aligned}
&= \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p}) \frac{\partial}{\partial v^\mu} \Big|_p - \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p}) \frac{\partial}{\partial u^\mu} \Big|_p \\
&= \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p}) \frac{\partial}{\partial v^\mu} \Big|_p + \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p}) \frac{\partial}{\partial u^\mu} \Big|_p \\
&= \frac{\partial}{\partial y^\nu} \Big|_p
\end{aligned}$$

and

$$\begin{aligned}
J_p \left(\frac{\partial}{\partial y^\nu} \Big|_p \right) &= \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p}) J_p \left(\frac{\partial}{\partial u^\mu} \Big|_p \right) + \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p}) J_p \left(\frac{\partial}{\partial v^\mu} \Big|_p \right) \\
&= \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p}) \frac{\partial}{\partial v^\mu} \Big|_p - \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p}) \frac{\partial}{\partial u^\mu} \Big|_p \\
&= -\frac{\partial v^\mu}{\partial x^\nu}(\widehat{p}) \frac{\partial}{\partial v^\mu} \Big|_p - \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p}) \frac{\partial}{\partial u^\mu} \Big|_p \\
&= -\frac{\partial}{\partial x^\nu} \Big|_p.
\end{aligned}$$

Left to check is smoothness. According to lemma 4.1 the corresponding rough tensor field is given by

$$J_p \left(\frac{\partial}{\partial x^\nu} \Big|_p \right) \otimes dx^\nu|_p + J_p \left(\frac{\partial}{\partial y^\nu} \Big|_p \right) \otimes dy^\nu|_p = \frac{\partial}{\partial y^\nu} \Big|_p \otimes dx^\nu|_p - \frac{\partial}{\partial x^\nu} \Big|_p \otimes dy^\nu|_p$$

for any $p \in U$. Thus the smoothness criteria for tensor fields [Lee13, pp. 317–318] together with [Lee13, p. 36] yields that $J \in \Gamma(T^{(1,1)}TM)$. \square

A question which naturally arises by considering proposition 4.1 is, if the converse is also true, i.e. if every almost complex manifold is a complex manifold. This is in general not the case. Let \mathbb{P} denote the naturally oriented underlying smooth manifold of the complex projective plane. Again using results about Chern numbers it can be shown that

$$\mathbb{P} \# (\mathbb{S}^1 \times \mathbb{S}^3) \# (\mathbb{S}^1 \times \mathbb{S}^3) \quad \text{and} \quad (\mathbb{S}^1 \times \mathbb{S}^3) \# (\mathbb{S}^1 \times \mathbb{S}^3) \# (\mathbb{S}^2 \times \mathbb{S}^2) \quad (10)$$

have almost complex structures but no complex structure (see [Ven66, p. 1627]).

5. The Nijenhuis Tensor and the Newlander-Nirenberg Theorem

As we have seen in the last section, not every almost complex manifold is a complex manifold. Under which condition is this possible?

Definition 5.1. Let (M, J) be an almost complex manifold. For $X, Y \in \mathfrak{X}(M)$ we define the **Nijenhuis tensor** N as

$$N(X, Y) := [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] \quad (11)$$

where $[X, Y]$ denotes the usual Lie-bracket of vector fields.

Proposition 5.1. Let (M, J) be an almost complex manifold and N be the associated Nijenhuis tensor. Then $N \in \Gamma(T^{(1,2)}TM)$.

Proof. First of all, $N(X, Y) \in \mathfrak{X}(M)$ for all $X, Y \in \mathfrak{X}(M)$. This follows immediately by considering J as a mapping $J : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ (see [KN96a, p. 26]), the fact that the Lie Bracket of two smooth vector fields is again a smooth vector field (see [Lee13, p. 186]) and that $\mathfrak{X}(M)$ is a $\mathcal{C}^\infty(M)$ -module (see [Lee13, p. 177]). Let $f \in \mathcal{C}^\infty(M)$ and $X, Y, Z \in \mathfrak{X}(M)$. Then

$$\begin{aligned} N(fX + Y, Z) &= [J(fX + Y), JZ] - J[fX + Y, JZ] - J[J(fX + Y), Z] \\ &\quad - [fX + Y, Z] \\ &= [fJX + JY, JZ] - J[fX + Y, JZ] - J[fJX + JY, Z] \\ &\quad - [fX + Y, Z] \\ &= [fJX, JZ] + [JY, JZ] - J[fX, JZ] - J[Y, JZ] - J[fJX, Z] \\ &\quad - J[JY, Z] - [fX, Z] - [Y, Z] \\ &= f[JX, JZ] - (JZf)JX + [JY, JZ] - fJ[X, JZ] + (JZf)JX \\ &\quad - [Y, JZ] - fJ[JX, Z] + (Zf)JJX - J[JY, Z] - f[X, Z] \\ &\quad + (Zf)X - [Y, Z] \\ &= fN(X, Z) + N(Y, Z). \end{aligned}$$

by [Lee13, pp. 187–188]. Linearity in the second argument is shown similarly. Hence $N : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ is bilinear over $\mathcal{C}^\infty(M)$. So by [KN96a, p. 26] we have that $N \in \Gamma(T^{(1,2)}TM)$. \square

Theorem 5.1 (Newlander-Nirenberg). Let (M, J) be an almost complex manifold. Then M is a complex manifold, where the complex structure is so that the canonically induced almost complex structure is J , if and only if the Nijenhuis tensor N vanishes identically.

Proof. Assume M is a complex manifold. Let $(U, (x^\nu, y^\nu))$ be a chart. From proposition 5.1 it is enough to consider the coordinate vector fields $\frac{\partial}{\partial x^\nu}$ and $\frac{\partial}{\partial y^\nu}$. But from the explicit definition of J in proposition 4.1 and the property, that the Lie-Bracket of coordinate vector fields vanishes, together with the $\mathcal{C}^\infty(M)$ -linearity of J we get that N vanishes identically on each chart, and thus on M .

The other direction however is far more technical and uses results on partial differential equations. A complete proof can either be found in the original paper [NN57] or in [Sil08, p. 106], where references to more recent proofs are given. \square

6. Kähler Manifolds

The following is inspired by [KN96b, pp. 146–149] and introduces the concepts from a complex viewpoint. This is in contrast to the symplectic approach provided for example in [Sil08].

Definition 6.1. *Let (M, J) be an almost complex manifold. A **Hermitian metric on M** is a Riemannian metric g such that*

$$g(JX, JY) = g(X, Y) \quad (12)$$

*holds for all $X, Y \in \mathfrak{X}(M)$. If g is a Hermitian metric on M , the triple (M, J, g) is called an **almost Hermitian manifold**.*

Lemma 6.1. *Every almost complex manifold admits a Hermitian metric.*

Proof. The existence was shown in the proof of proposition 3.1. \square

Definition 6.2. *Let (M, J, g) be an almost Hermitian manifold. The **fundamental 2-form Ω** is defined to be*

$$\Omega(X, Y) := g(X, JY) \quad (13)$$

for all $X, Y \in \mathfrak{X}(M)$.

Definition 6.3. *Let (M, J, g) be an almost Hermitian manifold with fundamental 2-form Ω . The Hermitian metric is said to be a **Kähler metric**, if $d\Omega = 0$. An almost complex manifold with a Kähler metric is called an **almost Kähler manifold** and a complex manifold with a Kähler metric is called a **Kähler manifold**.*

Appendix A. Tensor Characterization Lemma

Theorem A.1 (Tensor Characterization Lemma). *A mapping*

$$\underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_k \rightarrow \mathcal{C}^\infty(M) \quad \text{or} \quad \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_k \rightarrow \mathfrak{X}(M)$$

is induced by an element of $\Gamma(T^{(0,k)}TM)$ or $\Gamma(T^{(1,k)}TM)$, respectively, if and only if they are multilinear over $\mathcal{C}^\infty(M)$.

Proof. Let $A \in \Gamma(T^{(1,k)}TM)$. Since $T^{(1,k)}T_pM \cong L((T_pM)^k; T_pM)$ (see [KN96a, p. 23]), we define a mapping $\mathcal{A} : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ by

$$\mathcal{A}(X_1, \dots, X_k)_p := A_p(X_1|_p, \dots, X_k|_p) \in T_pM$$

where the smoothness follows from an adapted version of [Lee13, pp. 317–318]. Next we show that \mathcal{A} is multilinear over $\mathcal{C}^\infty(M)$. Let $f \in \mathcal{C}^\infty(M)$ and $X_\nu, \tilde{X}_\nu \in \mathfrak{X}(M)$, $\nu = 1, \dots, k$. Then for any $p \in M$ we have that

$$\begin{aligned} \mathcal{A}(X_1, \dots, fX_\nu + \tilde{X}_\nu, \dots, X_k)_p &= A_p(X_1|_p, \dots, (fX_\nu + \tilde{X}_\nu)|_p, \dots, X_k|_p) \\ &= A_p(X_1|_p, \dots, f(p)X_\nu|_p + \tilde{X}_\nu|_p, \dots, X_k|_p) \\ &= f(p)A_p(X_1|_p, \dots, X_\nu|_p, \dots, X_k|_p) \\ &\quad + A_p(X_1|_p, \dots, \tilde{X}_\nu|_p, \dots, X_k|_p) \\ &= f(p)\mathcal{A}(X_1, \dots, X_\nu, \dots, X_k)_p \\ &\quad + \mathcal{A}(X_1, \dots, \tilde{X}_\nu, \dots, X_k)_p \\ &= (f\mathcal{A}(X_1, \dots, X_\nu, \dots, X_k))_p \\ &\quad + \mathcal{A}(X_1, \dots, \tilde{X}_\nu, \dots, X_k)_p. \end{aligned}$$

Conversely, suppose that $\mathcal{A} : \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ is multilinear over $\mathcal{C}^\infty(M)$. Let $p \in M$. First we show that \mathcal{A} acts locally, i.e. if $X_\nu = \tilde{X}_\nu$ in some neighbourhood of p implies that also

$$\mathcal{A}(X_1, \dots, X_\nu, \dots, X_k) = \mathcal{A}(X_1, \dots, \tilde{X}_\nu, \dots, X_k)$$

on U . By the multilinearity of \mathcal{A} it is enough to show that if X_ν vanishes on U then so does \mathcal{A} . There exists a smooth bump function ψ for $\{p\}$ supported in U (see [Lee13, p. 44]). Hence $\psi X_\nu = 0$ on M and $\psi(p) = 1$. Thus

$$0 = \mathcal{A}(X_1, \dots, \psi X_\nu, \dots, X_k)_p = \psi(p)\mathcal{A}(X_1, \dots, X_\nu, \dots, X_k)_p.$$

and since $\psi(p) = 1$ we have that

$$\mathcal{A}(X_1, \dots, X_\nu, \dots, X_k)_p = 0$$

for any $p \in U$. □

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