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COMPLEX ANALYTIC COORDINATES IN ALMOST COMPLEX MANIFOLDS

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1. Formulation

A manifold is called a complex manifold if it can be covered by coordinate patches with complex coordinates in which the coordinates in overlapping patches are related by complex analytic transformations. On such a manifold scalar multiplication by i in the tangent space has an invariant meaning. An even dimensional $2n$ real manifold is called almost complex if there is a linear transformation J defined on the tangent space at every point (and varying differentiably with respect to local coordinates) whose square is minus the identity, i.e. if there is a real tensor field h_λ^μ satisfying

$$h_\lambda^\mu h_\mu^\nu = -\delta_\lambda^\nu, \tag{1.1}$$

where the summation convention is employed.[2] Greek and Latin indices run from 1 to $2n$ and 1 to n respectively. In a coordinate neighborhood on an even dimensional real manifold with coordinates $x^1, \cdots, x^{2n}$ one may introduce complex coordinates by setting, for example, $z^j = x^j + ix^{j+n}$, $j = 1, \cdots, n$. The almost complex structure given by J is called integrable if the manifold can be made into a complex manifold, with local coordinates $z^1, \cdots, z^n$, so that operating with J is equivalent to transforming dz^j and $d\overline{z^j}$ into idz^j and $-id\overline{z^j}$.

The problem we consider here is that of showing that an almost complex structure satisfying integrability conditions, to be formulated below, is integrable. It was observed by several people that for real analytic manifolds with real analytic h_λ^μ this may be derived from Frobenius' theorem; see, for instance, P. Libermann [10], and B. Eckmann and A. Frölicher [5] (the proof of [5] is also contained in [6]). We shall give a proof assuming the manifold to be of class $2n + 1$ and the h_λ^μ of class $2n$ (Theorem 1.1).

The problem of introducing analytic coordinates is purely local, for if z^j are are local coordinates with dz^j and $d\overline{z^j}$ transformed into idz^j and $-id\overline{z^j}$ under J, then the z^j will automatically be complex analytic functions of overlapping coordinates having the same transformation property with respect to J (see [6]).

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[2] For an exposition on complex and almost complex manifolds we refer the reader to the article by A. Frölicher [6], where an extensive bibliography is also to be found. See also Chapter 4 in A. Lichnerowicz [11].

In any coordinate patch we may choose complex valued coordinates $z^1, \cdots, z^{2n}$ with $z^{j+n} = \overline{z^j}$. (All complex coordinates to be used will have this property and in discussing them we shall usually refer simply to $z^1, \cdots, z^n$.) These coordinates may be chosen so that at the origin of the coordinate system the corresponding values (now complex) of the h_λ^μ are: $h_\lambda^\mu = 0$ if $\lambda \neq \mu$, $h_j^j = i$, $h_{j+n}^{j+n} = -i$, $j = 1, \cdots, n$. In order to formulate the integrability conditions suppose that we have complex analytic coordinates $\zeta^1, \cdots, \zeta^n$. Then $d\zeta^j = (\partial\zeta^j)/(\partial z^\mu)dz^\mu$ and $id\zeta^j = (\partial\zeta^j)/(\partial z^\mu)h_\lambda^\mu dz^\lambda$. Therefore $\zeta^j = w$ satisfies the system of equations

$$\frac{\partial w}{\partial z^\mu}(h_\lambda^\mu - i\delta_\lambda^\mu) = 0 \qquad \lambda = 1, \cdots, 2n. \tag{1.2}$$

In addition we have $2id\zeta^j = (\partial\zeta^j)/(\partial z^\mu)\,(h_\lambda^\mu + i\delta_\lambda^\mu)dz^\lambda$, and we see easily [6] that the system of forms $d\zeta^j, j = 1, \cdots, n$ is equivalent to the system $(h_\lambda^\mu + i\delta_\lambda^\mu)dz^\lambda$, $\mu = 1, \cdots, 2n$, from which follow, as necessary conditions, the *complete integrability conditions: the exterior differential of any form* $(h_\lambda^\mu + i\delta_\lambda^\mu)dz^\lambda$ *of the system may be expressed as a sum of exterior products of the forms of the system with first order forms.*

It is convenient to reformulate these conditions. For this purpose we set

$$z^{j+n} = \overline{z^j} = \bar z^j; \qquad \partial_j = \partial/\partial z^j, \qquad \bar\partial_j = \partial/\partial\bar z^j, \qquad \overline{\zeta^j} = \bar\zeta^j.$$

A complex valued function w satisfying (1.2) is called holomorphic with respect to the given almost complex structure. Because of the conditions (1.1) only n of these equations are independent; by our choice of coordinates, which led to the above values of the h_λ^μ at the origin, the last n of the equations are independent. We may solve these for the derivatives $\bar\partial_j w$ and rewrite these equations in the useful form

$$L_j w = \bar\partial_j w - a_j^k \partial_k w = 0, \quad a_j^k = 0 \text{ at } z^1 = \cdots = z^n = 0, \quad j = 1, \cdots, n. \tag{1.3}$$

In terms of the a_j^k the system of forms $d\zeta^j, j = 1, \cdots, n$ is equivalent to the system $dz^j + a_k^j\, d\bar z^k, j = 1, \cdots, n$ and the integrability conditions now have the form: *the operators L_j commute*, so that

$$\bar\partial_m a_j^k - a_m^p \partial_p a_j^k = \bar\partial_j a_m^k - a_j^p \partial_p a_m^k \quad j, k, m = 1, \cdots, n. \tag{1.4}$$

To solve our problem we seek n independent solutions $\zeta^1, \cdots, \zeta^n$ of 1.3. Any other solution can be seen to be a complex analytic function of these; conversely any analytic function of these is a solution of (1.3). Thus the problem admits the following formulation as a "complex Frobenius theorem", with which we shall work.

FORMULATION. *Under a transformation of variables $(\zeta^1, \cdots, \zeta^n) \to (z^1, \cdots, z^n)$ the Cauchy-Riemann equations $\partial w/\partial\bar\zeta^j = 0$ are transformed into a system* (1.3) *satisfying the integrability relations* (1.4). (*The a_j^k vanishing at the origin means that z^j behaves like ζ^j to first order.*) *Show, conversely, that a given system* (1.3) *satisfying* (1.4) *may be transformed to the Cauchy-Riemann equations by a nonsingular transformation of variables.*

For $n = 1$ the compatibility conditions are vacuous and the problem becomes that of introducing isothermal coordinates with respect to the Riemannian metric $ds^2 = | dz + ad\bar{z} |^2$.

In order to formulate our main result we recall the notion of Hölder continuity. A function f defined in a domain in euclidean N space is said to be Hölder continuous (with exponent α, $0 < \alpha < 1$) if in every compact subset A it satisfies a Hölder condition (exponent α), i.e. if

$$\text{l.u.b.} \, | f(P) - f(Q) | \, | P - Q |^{-\alpha} < \infty$$

for all points P, Q in A; here $| P - Q |$ represents the distance from P to Q. The function is of class $C^{k+\alpha}$, k a nonnegative integer, if it is of class C^k and if its derivatives of k^{th} order are Hölder continuous. (In the following theorem the functions considered are defined in a neighborhood of the origin in euclidean $2n$-space, with Re z^j and Im z^j, $j = 1, \cdots, n$ as real coordinates.).

We shall prove

THEOREM 1.1. *If the coefficients a_j^k in* (1.3) *are of class C^{2n} in a neighborhood of the origin, and satisfy* (1.4), *then, in some neighborhood of the origin, there exist n solutions $\zeta^1, \cdots, \zeta^n$ of* (1.3) *such that the Jacobian of $\zeta^1, \cdots, \zeta^n, \bar{\zeta}^1, \cdots \bar{\zeta}^n$ with respect to $z^1, \cdots, \bar{z}^n$ is different from zero, so that the equations* (1.3) *reduce to $\partial w/\partial\bar{\zeta}^j = 0$. Each function ζ^j is of class $C^{2n+\beta}$ for any positive $\beta < 1$. If in addition the coefficients a_j^i are of class $C^{k+\alpha}$, for integral $k \geqq 2n$, and $0 < \alpha < 1$, then each ζ^j is of class $C^{k+1+\alpha}$.*

Thus on an almost complex manifold of class C^{2n+1}, with h_λ^μ of class C^{2n} satisfying the complete integrability conditions, it is possible to introduce complex analytic coordinates, these are however of class C^{2n}. In order for these to be of class C^{2n+1} we require that there exist suitable coordinates in which the h_λ^μ are of class $C^{2n+\alpha}$, for some $\alpha > 0$.

Our method of proof, which is outlined in the next section, is based entirely on known potential theoretic inequalities. Other potential theoretic problems on almost complex manifolds have been treated by J. J. Kohn and D. C. Spencer [8]; further references may be found there.

We hope to treat other problems associated with overdetermined systems of differential equations at some later time. In passing we mention the problem of transforming a *single* equation

$$Lw = \bar{\partial}_1 w - a^j \partial_j w = 0, \qquad a^j(0, \cdots, 0) = 0 \text{ for } j = 1, \cdots n,$$

into the equation $\partial w/\partial\bar{\zeta}^1 = 0$ by a suitable change of independent variables $z^1, \cdots, z^n \to \zeta^1, \cdots, \zeta^n$; there is no integrability condition like (1.4). In general no such transformation exists. By considerations similar to those of the next section, in which the z coordinates are considered as functions of the ζ coordinates, one finds, setting $a^1 = a$, that the following conditions are necessary and sufficient:

$$(1 - | a |^2) L\bar{a}^j = \bar{a}^j(\overline{L\bar{a}} - a \, L\bar{a}), \qquad j = 2, \cdots, n.$$

We wish to thank Professor S. S. Chern for bringing the problem to our attention and for pointing out the formulation above.

2. Outline of proof

We recall first the treatment of the classical case of one complex dimension (Korn [9], Lichtenstein [12], see also Bers [1], Chern [2], and Courant-Hilbert [3]). It is required to find a solution of

$$\bar{\partial}_z w = a(z)\partial_z w, \qquad a(0) = 0,[3] \tag{2.1}$$

in $|z| = |x + iy| < r$, with $\partial_z w \neq 0$; here $\partial_z = \partial/\partial z$, $\bar{\partial}_z = \partial/\partial\bar{z}$. The treatment of (2.1) makes the use of an inverse operator T of the Cauchy-Riemann operator $\bar{\partial}_z$:

$$w(z) = Tf = \frac{1}{2\pi i}\iint_{|\tau|<r} \frac{f(\tau)}{z - \tau}\, d\bar{\tau}d\tau.$$

If f satisfies a Hölder condition in $|z| < r$ then $w(z) = Tf$ is a solution of the equation $\bar{\partial}_z w = f(z)$.

A solution of (2.1) is obtained by solving the integral equation

$$w(z) = T(a\partial_z w) + z.$$

For r sufficiently small, and a satisfying a Hölder condition with exponent α, this integral equation is solved by iterations, setting $w_{n+1}(z) = T(a\partial_z w_n) + z$, $w_0 = 0$. The convergence of this iteration scheme is based on the following potential theoretic property of the operator T. Setting

$$H_\alpha[f] = \text{l.u.b.}\, |f(z) - f(\tilde{z})| \cdot |z - \tilde{z}|^{-\alpha}, \quad \text{for} \quad |z|,\ |\tilde{z}| < \text{r},$$

the property is expressed by the inequality

$$\begin{aligned} \| Tf \| \equiv \text{l.u.b.}\, |Tf| + r\ \text{l.u.b.}\, |DTf| + r^{1+\alpha}H_\alpha[DTf] \\ \leqq \text{constant}\ r(\text{l.u.b.}\, |f| + r^\alpha H_\alpha[f]) \end{aligned} \tag{2.2}$$

where the constant depends only on α; here D represents either of the differential operators ∂_z, $\bar{\partial}_z$. The convergence of the iteration is proved using the norm $\| \ \|$ in (2.2) (see [2], [3]); for r sufficiently small $\partial_z w \neq 0$.

Consider now the general case $n > 1$, and denote by T^j the integral operator

$$T^j f = \frac{1}{2\pi i}\iint_{|\tau|<r} f(z^1, \cdots z^{j-1}, \tau, z^{j+1}, \cdots, z^n)\, \frac{d\bar{\tau}d\tau}{z^j - \tau}, \tag{2.3}$$

with r fixed. Clearly the T^j commute with each other; furthermore T^j commutes with ∂_k, $\bar{\partial}_k$ for $j \neq k$ and, formally the operator $\bar{\partial}_j T^j$(not summed) equals the identity.

Using the operators T^j we may write down a solution of the inhomogeneous Cauchy-Riemann equations

$$\bar{\partial}_j w = f_j, \qquad j = 1, \cdots, n, \tag{2.4}$$

[3] Although only the dependence of a on z is indicated, its dependence on $\bar{z}$ is also to be understood. This notation will be used throughout.

assuming that $F = \{f_1, \cdots, f_n\}$ satisfies the compatibility relations

$$\bar{\partial}_k f_j = \bar{\partial}_j f_k \qquad j, k = 1, \cdots, n,$$

and that the f_j have suitable differentiability properties. Such a solution is given by

$$w = \sum_{s=0}^{n-1} \frac{(-1)^s}{(s+1)!} \sum' T^{j_1}\bar{\partial}_{j_1} \cdots T^{j_s}\bar{\partial}_{j_s} \cdot T^k f_k \equiv \mathbf{T}F \tag{2.5}$$

where $\sum'$ *denotes summation extended over all* $(s + 1)$*-tuples with* $j_1, \cdots j_s, k$ *distinct*. Differentiating, one finds, namely, after some manipulation, that

$$\bar{\partial}_j w - f_j = \sum_{s=0}^{n-2} \frac{(-1)^s}{(s+2)!} \sum^j T^{j_1}\bar{\partial}_{j_1} \cdots T^{j_s}\bar{\partial}_{j_s} T^k(\bar{\partial}_j f_k - \bar{\partial}_k f_j), \tag{2.6}$$

where $\sum^j$ *denotes summation over all* $(s + 1)$*-tuples with* $j_1, \cdots j_s, k$ *distinct and different from* j, and the right side vanishes if the compatibility relations hold.

One might now expect that, analogous to the case $n = 1$, one would obtain solutions of the system (1.3) by solving the integral equation

$$w = \mathbf{T}F[w] + \text{complex analytic function of } (z^1, \cdots, z^n)$$

where $F[w]$ is the vector function whose j^{th} component is $\sum a_j^k \partial_k w$. Since, however, each component of F involves derivatives of w with respect to all the z^k, while in T^j integration over only z^j occurs, one would not expect the usual convergence proof for iterations to work.

We proceed differently. Instead of considering the new coordinates $\zeta^1, \cdots, \zeta^n$ as solutions of (1.3) we consider $z^1, \cdots, z^n$ as functions of the $\zeta^1, \cdots, \zeta^n$ (we shall not display explicitly their dependence on the conjugates $\bar{\zeta}^1 \cdots, \bar{\zeta}^n$, see footnote 3). Setting

$$d_j = \partial/\partial\zeta^j, \qquad \bar{d}_j = \partial/\partial\bar{\zeta}^j,$$

we have

$$\bar{d}_j w = \partial_k w \cdot \bar{d}_j z^k + \bar{\partial}_k w \cdot \bar{d}_j \bar{z}^k = \partial_k w \cdot (\bar{d}_j z^k + a_m^k \bar{d}_j \bar{z}^m)$$

if w satisfies (1.3); thus $\bar{d}_j w = 0$ if the variables z satisfy the non-linear system

$$\bar{d}_j z^k + a_m^k \bar{d}_j \bar{z}^m = 0, \qquad j, k = 1, \cdots, n. \tag{2.7}$$

On the other hand if the variables z satisfy (2.7) then

$$\bar{d}_j w = \bar{d}_j \bar{z}^k \cdot (\bar{\partial}_k w - a_k^i \partial_i w).$$

Thus the Cauchy-Riemann equations $\bar{d}_j w = 0$ are equivalent to the system (1.3) if the z^i satisfy (2.7) with the matrix $(\bar{d}_j \bar{z}^k)$ non-singular. (For a solution of (2.7), $\bar{d}_j z^k$ vanishes at the origin so that the matrix $\bar{d}_j \bar{z}^k$ is non-singular there provided the transformation from ζ to z variable is non-singular there.)

Our procedure is to find solutions of the system (2.7). This system, though non-linear, is more convenient than (1.3) because of the property that *differentiation* *occurs with respect to only one of the independent variables in each equation.* For fixed j the equations (2.7) represent the "characteristic" equations of the surface ζ^k = constant for all $k \neq j$.

We shall find solutions of (2.7) by solving a corresponding integral equation to which iterations can be applied successfully. For r fixed let T^j and $\mathbf{T}$ be the operators defined by (2.3) and (2.5) where now $\zeta^1, \cdots, \zeta^n$ are the independent variables, so that ∂_j, $\bar\partial_j$ are replaced by d_j, $\bar d_j$. Set $Z = \{z^1, \cdots, z^n\}$, and denote the coefficients a^i_m which are functions of $(z^1, \cdots, z^n)$ by $a^i_m(Z)$. Setting, furthermore,

$$(2.8) \qquad -a^i_m \bar d_j \bar z^m = f^i_j[Z], \qquad F^i = \{f^i_1, \cdots, f^i_n\},$$

we form the non-linear integral equation system corresponding to (2.7)

$$(2.9) \qquad z^i = \zeta^i + \mathbf{T}F^i[Z] - z^i_0[Z], \quad \text{or} \quad Z = \mathfrak{J}[Z]$$

in abbreviation, where $z^i_0[Z]$ is the value of $\mathbf{T}F^i[Z]$ at $\zeta^1 = , \cdots = \zeta^n = 0$. For r sufficiently small we shall solve this in the polycylinder $|\zeta^j| < r, j = 1, \cdots, n$, by iterations.

So far the integrability relations (1.4) have not been used. They enter now in showing that the solution of (2.9) satisfies also (2.7). To show that the functions

$$(2.10) \qquad g^i_j = \bar d_j z^i + a^i_m \bar d_j \bar z^m$$

vanish we differentiate (2.9) with respect to $\bar\zeta^j$ and find, by (2.6) that

$$(2.11) \qquad \begin{aligned} g^i_j &= \bar d_j z^i + a^i_m \bar d_j \bar z^m \\ &= \textstyle\sum_{s=0}^{n-2} \frac{(-1)^s}{(s+2)!} \sum^j T^{j_1}\bar d_{j_1} \cdots T^{j_s}\bar d_{j_s} \cdot T^k(\bar d_j f^i_k - \bar d_k f^i_j). \end{aligned}$$

Now

$$\bar d_j f^i_k - \bar d_k f^i_j = \bar d_k a^i_m \cdot \bar d_j \bar z^m - \bar d_j a^i_m \bar d_k \bar z^m$$

since the mixed second derivatives of the $\bar z^m$ cancel,

$$\begin{aligned} &= \partial_p a^i_m \cdot \bar d_k z^p \cdot \bar d_j \bar z^m + \bar\partial_p a^i_m \cdot \bar d_k \bar z^p \cdot \bar d_j \bar z^m \\ &\quad - \partial_p a^i_m \cdot \bar d_j z^p \bar d_k \bar z^m - \bar\partial_p a^i_m \cdot \bar d_j \bar z^p \cdot \bar d_k \bar z^m \\ &= \partial_p a^i_m \cdot (\bar d_j \bar z^m \cdot g^p_k - \bar d_k \bar z^m \cdot g^p_j) \end{aligned}$$

using the integrability relations (1.4) and (2.10). Inserting this expression into (2.11) we obtain

$$(2.12) \qquad \begin{aligned} g^i_j = \textstyle\sum_{s=0}^{n-2} \frac{(-1)^s}{(s+2)!} \sum^j T^{j_1}\bar d_{j_1} \cdots T^{j_s}\bar d_{j_s} \\ \cdot\, T^k[\partial_p a^i_m \cdot (\bar d_j \bar z^m \cdot g^p_k - \bar d^k \bar z^m \cdot g^p_j)], \end{aligned}$$

i.e. a system of linear integral equations satisfied by the g^i_j, *which, as we show, admits only the null solution, for r small.*

In proving the existence of solutions of (2.9) and (2.7) having first order derivatives (Theorem 5.1) we found it necessary, because of the special nature of the operator **T**, to work with a special class of functions z^i—those admitting only "mixed" higher order derivatives (as described in the next section). For this we require that the a_j^i have derivatives up to order $2n$. (Having an inverse operator of the Cauchy-Riemann equations (2.4) with better "smoothing" properties than **T** would simplify the proof.)

We remark finally that we have an alternative proof of the existence of holomorphic coordinates in case $n = 2$, which involves several transformations of coordinates. Each transformation simplifies the system (1.3) by eliminating coefficients a_j^i, and requires solving a nonlinear integral equation involving only *one* complex independent variable.

In Sections 3–5 we prove the existence of solutions of (2.7) having continuous first derivatives (and "mixed" higher order derivatives). In Section 6 we complete the proof of Theorem (1.1) by obtaining the full differentiability properties of the solutions.

3. Normed function spaces

We consider complex valued functions of $\zeta^1, \cdots, \zeta^{n3}$ in the polycylinder $|\zeta^j| < r \leqq \frac{1}{4}, j = 1, \cdots, n$.

NOTATION. For brevity we shall use the symbol D_j to denote either of the differentiation operators d_j, $\bar{d}_j$. D^k will denote a general k^{th} order derivative $D^k = D_{i_1} \cdots D_{i_k}$ *with the* $i_1, \cdots i_k$ *distinct.* (Thus we shall consider only "mixed" derivatives.) $D^{k,j} = D_{i_1} \cdots D_{i_k}$ will denote such a derivative *with the* $i_1, \cdots, i_k$ *distinct and different from* j. For fixed positive $\alpha < 1$ we introduce the difference quotient operators

$$\delta_j f = |\delta\zeta^j|^{-\alpha}\{f(\zeta^1, \zeta^2, \cdots, \zeta^j + \delta\zeta^j, \cdots, \zeta^n) - f(\zeta^1, \cdots, \zeta^j, \cdots, \zeta^n)\}, \qquad |\zeta^j + \delta\zeta^j| < r,$$

(the resulting function clearly depends also on $\delta\zeta^j$), and denote a typical product of such operators by

$$\delta^m = \delta_{j_1} \cdots \delta_{j_m}$$

for $0 \leqq m \leqq n$ *and* $j_1, \cdots, j_m$ *distinct*; δ^0 will denote the identity operator. Clearly δ_j and D_k commute for $j \neq k$.

From now on we shall use c and C with or without super and sub-scripts to denote constants depending only on n and α.

For functions z and f in the polycylinder we introduce various

NORMS.

$$(3.1) \qquad \tilde{H}_\alpha[z] = \sum_{m=0}^{n} \frac{r^{m\alpha}}{m!} \text{l.u.b.} |\delta^m z|,$$

where the l.u.b. is taken with respect to all δ^m, i.e. with respect to all ζ^j and $\zeta^j + \delta\zeta^j$ having absolute value less than r.

$$(3.2) \qquad \begin{aligned} |z|_n &= \sum_k \frac{r^k}{k!} \text{l.u.b.} |D^k z| \\ |z|_{n+\alpha} &= \sum_k \frac{r^k}{k!} \text{l.u.b.} \tilde{H}_\alpha[D^k z] \leqq \sum_{k,m=0}^n \frac{r^{k+m\alpha}}{k!m!} \text{l.u.b.} |\delta^m D^k z| . \end{aligned}$$

Here the l.u.b. extend over all derivatives D^k of order k, all δ^m, and all points in the polycylinder. Because of our requirement on the D^k, k is at most n, and we sum over k from 0 to n. Finally we introduce the norm

$$(3.3) \qquad |f|_{n-1+\alpha}^j = \sum_k \frac{r^k}{k!} \text{l.u.b.} \tilde{H}_\alpha[D^{k,j} f],$$

and observe that

$$(3.4) \qquad |D_j f|_{n-1+\alpha}^j \leqq \frac{c_0}{r} |f|_{n+\alpha}, \qquad j = 1, \cdots, n.$$

It is easily verified that each of the four norms above has the multiplicative property: *the norm of a product of two functions is not greater than the product of their norms*. Furthermore, with the aid of the theorem of the mean, we may prove

$$(3.5) \qquad \tilde{H}_\alpha[z] \leqq c_0' \, |z|_n .$$

We shall denote the set of functions z having derivatives $D^k z$ and with $|z|_{n+\alpha} < \infty$ by $\tilde{C}^{n+\alpha}$. It is easily seen that $\tilde{C}^{n+\alpha}$ is a complete linear normed space, with norm $|\ |_{n+\alpha}$, and, in view of the multiplicative property of the norm, that *$\tilde{C}^{n+\alpha}$ is in fact a Banach algebra*. (It is clear that we can define Banach algebras corresponding to the other norms above.) Functions in $\tilde{C}^{n+\alpha}$ admit all "mixed" derivatives up to order n.

For n-tuples of functions $Z = \{z^1, \cdots, z^n\}$, $F = \{f_1, \cdots, f_n\}$ we define the norms

$$(3.6) \qquad \begin{aligned} |Z|_{n+\alpha} &= \max_j |z^j|_{n+\alpha} \\ |F|_{n-1+\alpha} &= \max_j |f_j|_{n-1+\alpha}^j . \end{aligned}$$

We shall denote by **B** the Banach space of n-tuples Z with components z^j in $\tilde{C}^{n+\alpha}$, the norm in **B** being $|Z|_{n+\alpha}$.

We shall need to estimate a function $a(Z) = a(z^1, \cdots, \bar{z}^n)$ of Z in **B**. We shall denote any k^{th} order derivative of a with respect to its arguments by $D_Z^k a$, $k \geqq 0$.

LEMMA 3.1. *Suppose that $|Z|_{n+\alpha} \leqq 1$ and that a has continuous derivatives with respect to its arguments up to order $2n - 1$ which are bounded in absolute value by K. Then*

$$(3.7) \qquad |a(Z)|_{n-1+\alpha}^m \leqq cK;$$

if, in addition, a vanishes for $Z = 0$ then

$$(3.8) \qquad |a(Z)|_{n-1+\alpha}^m \leqq cK \, |Z|_{n+\alpha}, \qquad m = 1, \cdots, n.$$

We shall sketch the

PROOF. Observe first that

$$| a(Z) | \leqq \begin{cases} K \\ c'K \mid Z \mid_{n+\alpha} & \text{if} \quad a(0) = 0. \end{cases} \tag{3.9}$$

For $j > 0$ any derivative $D^j a$ may be expressed as a linear combination of terms t of the form

$$t = (D_Z^k a)(D^{j_1} z^{i_1}) \cdots (D^{j_r} z^{i_r})(D^{j_{r+1}} \bar{z}^{i_{r+1}}) \cdots D^{j_k} \bar{z}^{i_k}), \quad 1 \leqq k \leqq j,$$

with $D^{j_1} \cdots D^{j_k} = D^j$. It follows that $r^j \mid t \mid \leqq K \mid Z \mid_{n+\alpha}$ and hence $r^j \mid D^j a \mid \leqq c''K \mid Z \mid_{n+\alpha}$ for $j \geqq 1$. Combining this result with (3.9) we find

$$| a(Z) |_n \leqq \begin{cases} c'''K \\ c'''K \mid Z \mid_{n+\alpha} & \text{if} \quad a(0) = 0. \end{cases} \tag{3.10}$$

We wish now to estimate $r^j \tilde{H}_\alpha[D^j a]$ for $j < n$. If $j = 0$ we find, by (3.5) and (3.10)

$$\tilde{H}_\alpha[a] \leqq c_0' \mid a(Z) \mid_n \leqq \begin{cases} c_1 K \\ c_1 K \mid Z \mid_{n+\alpha} & \text{if} \quad a(0) = 0. \end{cases} \tag{3.11}$$

For $j \geqq 1$ consider $r^j \tilde{H}_\alpha[t]$. Using the multiplicative property of $\tilde{H}_\alpha[\ ]$, and the fact that $\mid Z \mid_{n+\alpha} \leqq 1$, we easily find that

$$\begin{aligned} r^j \tilde{H}_\alpha[t] &\leqq c_1' \tilde{H}_\alpha[D_Z^k a] \mid Z \mid_{n+\alpha} \\ &\leqq c_1'' \mid D_Z^k a \mid_n \mid Z \mid_{n+\alpha} \qquad \text{by} \quad (3.5). \end{aligned}$$

By (3.10) applied to the function $D_Z^k a$ we have

$$r^j \tilde{H}_\alpha[t] \leqq c_1'' c''' K \mid Z \mid_{n+\alpha} ,$$

and consequently

$$r^j \tilde{H}_\alpha[D^j a] \leqq c_1''' K \mid Z \mid_{n+\alpha} , \qquad 1 \leqq j < n.$$

This combined with (3.11) yields the desired inequalities (3.7), (3.8).

4. Potential theoretic lemmas

In this section we state some useful properties of the integral operators T^j. These follow from property (2.2) of the operators T^j which may be expressed in the form: *for a function y defined in the polycylinder*

$$\begin{aligned} &\text{l.u.b.} \mid T^j y \mid + r^\alpha \, \text{l.u.b.} \mid \delta_j T^j y \mid + r \, \text{l.u.b.} \mid D_j T^j y \mid \\ &\quad + r^{1+\alpha} \, \text{l.u.b.} \mid \delta_j D_j T^j y \mid \leqq \tilde{c} r \, [\text{l.u.b.} \mid y \mid + r^\alpha \, \text{l.u.b.} \mid \delta_j y \mid] \end{aligned} \tag{4.1}$$

where each l.u.b. is taken with respect to the ζ^j variable only, the others being fixed.

LEMMA 4.1. $\quad \mid T^j D_j f \mid_{n-1+\alpha}^{l} \leqq c_2 \mid f \mid_{n-1+\alpha}^{l} , \qquad j, l = 1, \cdots, n, j \neq l.$

We sketch the

PROOF. It suffices to derive such a bound for the functions $r^{k+m\alpha} \delta^m D^{k,l} T^j D_j f$,

$0 \leqq k \leqq n-1$; $0 \leqq m \leqq n$. If, for $j_1, \cdots, j_m$ distinct and different from j, and $i_1, \cdots, i_k$ distinct and different from j, l we consider functions

$$y = \delta_{j_1} \cdots \delta_{j_m} D_{i_1} \cdots D_{i_k} D_j f$$

we see that it suffices to derive such a bound for the functions

$$\eta = r^{k+m\alpha} \delta_{j_1} \cdots \delta_{j_m} D_{i_1} \cdots D_{i_k} T^j D_j f = r^{k+m\alpha} T^j y,$$

and

$$r^{\alpha} \delta_j \eta, \qquad r D_j \eta \quad \text{and} \quad r^{1+\alpha} \delta_j D_j \eta.$$

From (4.1) however, it follows that these four functions are bounded in absolute value by

$$\tilde{c} r^{k+1+m\alpha} [\text{l.u.b.} \mid y \mid + r^{\alpha} \text{ l.u.b.} \mid \delta_j y \mid] \leqq c_2' \mid f \mid_{n-1+\alpha}^{l},$$

proving the lemma.

By similar considerations we can prove

LEMMA 4.2.

$$\mid T^j(gh) \mid_{n-1+\alpha}^{l} \leqq c_3 r \mid g \mid_{n-1+\alpha}^{j} \mid h \mid_{n-1+\alpha}^{l}, \qquad j, l = 1, \cdots, n.$$

COROLLARY.

$$\mid T^j g \mid_{n-1+\alpha}^{l} \leqq \begin{cases} c_3 r \mid g \mid_{n-1+\alpha}^{j}, \\ c_3 r \mid g \mid_{n-1+\alpha}^{l}, \end{cases} \qquad j, l = 1, \cdots, n.$$

LEMMA 4.3.

$$\mid T^j f \mid_{n+\alpha} \leqq c_4 r \mid f \mid_{n-1+\alpha}^{j} \qquad j = 1, \cdots, n.$$

PROOF. By the corollary to Lemma 4.2 we have

$$\mid T^j f \mid_{n-1+\alpha}^{l} \leqq c_3 r \mid f \mid_{n-1+\alpha}^{j} \qquad l = 1, \cdots, n.$$

Therefore we need only show that the functions

$$r^n D^n T^j f, \qquad r^{n+l\alpha} \delta^l D^n T^j f$$

are bounded in absolute value by $c_5 r \mid f \mid_{n-1+\alpha}^{j}$, and this follows, as in Lemma 4.1, with the aid of (4.1).

From Lemmas 4.1 and 4.3 follows immediately

THEOREM 4.1.

$$\mid \mathbf{T} F \mid_{n+\alpha} \leqq C r \mid F \mid_{n-1+\alpha}.$$

5. An existence theorem

In this section we prove

THEOREM 5.1. *If the coefficients* a_j^i *possess continuous derivatives up to order* $2n$ *then for* r *sufficiently small the system of integral equations* (2.9) *admits a unique solution* Z *in* $\mathbf{B}$ *satisfying also* (2.7) *and such that the transformation from the* ζ *coordinates to* z *coordinates has non-vanishing Jacobian.*

We first show that the integral equation system (2.9)

$$Z = \mathfrak{J}[Z]$$

has, for r sufficiently small, a unique solution Z satisfying $|Z|_{n+\alpha} \leqq 4r$. To this end it suffices to show that if $|Z|_{n+\alpha}, |\tilde{Z}|_{n+\alpha} \leqq 4r$ then

$$(5.1) \qquad \begin{aligned} |\mathfrak{J}[Z]|_{n+\alpha} &\leqq 4r \\ |\mathfrak{J}[Z] - \mathfrak{J}[\tilde{Z}]|_{n+\alpha} &\leqq \tfrac{1}{2}|Z - \tilde{Z}|_{n+\alpha}, \end{aligned}$$

for then the usual iteration scheme yields a unique "fixed point" of $\mathfrak{J}$ in $|Z|_{n+\alpha} \leqq 4r$.

We derive first a preliminary result. Consider, for $|Z|_{n+\alpha} \leqq 4r$, the functions (2.8)

$$f_j^i[Z] = -a_k^i \bar{d}_j \bar{z}^k$$

and assume that the a_k^i vanish at $Z = 0$ and have derivatives up to order $2n$ bounded in absolute value by K.

Lemma 5.1. *If* $|Z|_{n+\alpha}, |\tilde{Z}|_{n+\alpha} \leqq 4r$ *then*

$$|f_j^i[Z]|^j_{n-1+\alpha} \leqq c_6 K r,$$

$$|f_j^i[Z] - f_j^i[\tilde{Z}]|^j_{n-1+\alpha} \leqq c_6 K |Z - \tilde{Z}|_{n+\alpha}, \quad i, j = 1, \cdots, n.$$

Proof. By the multiplicative property of our norms we have

$$(5.2) \qquad \begin{aligned} |f_j^i[Z]|^j_{n-1+\alpha} &\leqq \textstyle\sum_k |a_k^i|^j_{n-1+\alpha} |d_j z^k|^j_{n-1+\alpha} \\ &\leqq \frac{c_0}{r} \textstyle\sum_k |a_k^i|^j_{n-1+\alpha} |z^k|_{n+\alpha}, & \text{by (3.4)}, \\ &\leqq 4c_0 \textstyle\sum_k |a_k^i|^j_{n-1+\alpha} \\ &\leqq 4nc_0 cK4r & \text{by (3.8)}. \end{aligned}$$

Similarly, we find with the aid of Lemma 3.1 and (3.4) that

$$\begin{aligned} |f_j^i[Z] - f_j^i[\tilde{Z}]|^j_{n-1+\alpha} &\leqq \textstyle\sum_k |a_k^i(\tilde{Z})\, \bar{d}_j(\bar{z}^k - \bar{\tilde{z}}^k)|^j_{n-1+\alpha} \\ &\quad + \textstyle\sum_k |\bar{d}_j \bar{z}^k (a_k^i(Z) - a_k^i(\tilde{Z}))|^j_{n-1+\alpha} \\ &\leqq \textstyle\sum |a_k^i(\tilde{Z})|^j_{n-1+\alpha} \cdot |d_j(z^k - \tilde{z}^k)|^j_{n-1+\alpha} \\ &\quad + \textstyle\sum |d_j z^k|^j_{n-1+\alpha} \cdot |a_k^i(Z) - a_k^i(\tilde{Z})|^j_{n-1+\alpha} \\ &\leqq ncK4c_0 |Z - \tilde{Z}|_{n+\alpha} + 4c_0 \textstyle\sum_k |a_k^i(Z) - a_k^i(\tilde{Z})|^j_{n-1+\alpha}. \end{aligned}$$

By the theorem of the mean we have

$$\begin{aligned} &|a_k^i(Z) - a_k^i(\tilde{Z})|^j_{n-1+\alpha} \\ &\qquad \leqq [\textstyle\sum_m |\partial_m a_k^i(Z')|^j_{n-1+\alpha} + \sum_m |\bar{\partial}_m a_k^i(Z')|^j_{n-1+\alpha}]\, |Z - \tilde{Z}|^j_{n-1+\alpha} \end{aligned}$$

for some Z' satisfying $|Z'|_{n+\alpha} \leqq 4r$, so that by Lemma 3.1

$$|a_k^i(Z) - a_k^i(\tilde{Z})|^j_{n-1+\alpha} \leqq c_6' K |Z - \tilde{Z}|_{n+\alpha}.$$

Inserting this into the inequality above we obtain the inequality

$$|f_j^i(Z) - f_j^i(\tilde{Z})|_{n-1+\alpha}^{j} \leqq c_6'' K|Z - \tilde{Z}|_{n+\alpha}$$

which, together with (5.2), yields the desired result.

We now prove (5.1). Setting $F^i[Z] = \{f_1^i[Z], \cdots, f_n^i[Z]\}$, we denote the i^{th} component of $\mathfrak{Z}[Z]$ by

$$y^i[Z] = \zeta^i + \mathbf{T}F^i[Z] - z_0^i[Z] \qquad i = 1, \cdots, n,$$

and observe that

$$(5.3)\qquad \begin{aligned} |y^i[Z]|_{n+\alpha} &\leqq |\zeta^i|_{n+\alpha} + 2|\mathbf{T}F^i[Z]|_{n+\alpha} \\ &\leqq (2 + 2^{1-\alpha})r + 2Cr|F^i[Z]|_{n-1+\alpha}, \quad \text{by Theorem 4.1,} \\ &\leqq (2 + 2^{1-\alpha})r + 2Crc_6Kr, \qquad \text{by Lemma 5.1,} \\ &\leqq 4r \end{aligned}$$

for r sufficiently small. Thus the first part of (5.1) is verified. To prove the second we see that

$$\begin{aligned} |y^i[Z] - y^i[\tilde{Z}]|_{n+\alpha} &\leqq 2|\mathbf{T}(F^i[Z] - F^i[\tilde{Z}])|_{n+\alpha} \\ &\leqq 2Cr|F^i[Z] - F^i[\tilde{Z}]|_{n-1+\alpha}, \quad \text{by Theorem 4.1,} \\ &\leqq 2Crc_6K|Z - \tilde{Z}|_{n+\alpha}, \qquad \text{by Lemma 5.1,} \\ &\leqq \tfrac{1}{2}|Z - \tilde{Z}|_{n+\alpha} \end{aligned}$$

for r sufficiently small, completing the proof of (5.1).

We therefore conclude that there is a solution Z of (2.9)

$$z^i = \zeta^i + \mathbf{T}F^i[Z] - z_0^i[Z]$$

for r sufficiently small. Since, as we showed in (5.3), $|\mathbf{T}F^i[Z]|_{n+\alpha} \leqq c_6CKr^2$ it follows easily that for r sufficiently small the Jacobian of the z^i variables with respect to the ζ^i variables is different from zero, and that the matrix $(\bar{d}_j\bar{z}^k)$ is nonsingular, being close to the identity.

We have finally to show that the solution Z thus obtained is also a solution of the differential equations (2.7)

$$g_j^i = \bar{d}_j z^i + a_m^i \bar{d}_j \bar{z}^m = 0$$

(for r possibly restricted still further). To this end we employ the integral equation system (2.12) satisfied by the g_j^i,

$$g_j^i = \sum_{s=0}^{n-2} \frac{(-1)^s}{(s+2)!} \sum^j T^{j_1}\bar{d}_{j_1} \cdots T^{j_s}\bar{d}_{j_s} \cdot T^k[\partial_p a_m^i \cdot (\bar{d}_j \bar{z}^m \cdot g_k^p - \bar{d}_k \bar{z}^m \cdot g_j^p)]$$

where the inner sum extends over all distinct $(s + 1)$-tuples with $j_1, \cdots, j_s, k$ different from j. (We remark that in the derivation of this system the functions

z^i were assumed to have "mixed" second order continuous derivatives, which they do since they belong to $\tilde{C}^{n+\alpha}$.) By Lemma 4.1 it follows that

$$\begin{aligned}\sum_{i,j} | g_j^i |_{n-1+\alpha}^j \leqq c_7 \sum_{i,j} \sum_{k\neq j} | T^k[\partial_p a_m^i \cdot (\bar{d}_j \bar{z}^m \cdot g_k^p - \bar{d}_k \bar{z}^m \cdot g_j^p)] |_{n-1+\alpha}^j \\ \leqq c_7 c_3 r \sum_{i,j,m} \sum_{k\neq j} [\, | \partial_p a_m^i \cdot \bar{d}_j \bar{z}^m |_{n-1+\alpha}^j \, | g_k^p |_{n-1+\alpha}^k \\ + | \partial_p a_m^i \cdot \bar{d}_k \bar{z}^m |_{n-1+\alpha}^k \, | g_j^p |_{n-1+\alpha}^j]\end{aligned}$$

by Lemma 4.2,

$$\begin{aligned}&\leqq c_8 r \max_{i,j,m,p} | \partial_p a_m^i |_{n-1+\alpha}^j \max_{m,k} | d_k z^m |_{n-1+\alpha}^k \cdot [\sum_{p,j} | g_j^p |_{n-1+\alpha}^j] \\ &\leqq c_9 K r \sum_{p,j} | g_j^p |_{n-1+\alpha}^j ,\end{aligned}$$

by (3.7) (applied to $\partial_p a_m^i$) and (3.4). It follows that the $g_j^i = 0$ if r is so small that $c_9 K r < 1$. This completes the proof of Theorem 5.1.

6. Proof of Theorem 1.1

We have constructed a solution $z^1, \cdots, z^n$ of the system (2.7) in a polycylinder $| \zeta^j | < r, j = 1, \cdots, n$ for r sufficiently small; this solution vanishes at the origin and has nonvanishing Jacobian with respect to the ζ variables. The functions $z^1, \cdots, z^n$ therefore map the polycylinder homeomorphically onto a neighborhood U of the origin in the z space. In U therefore the coordinates $\zeta^1, \cdots, \zeta^n$ are solutions of (1.3). In order to complete the proof of Theorem 1.1 it remains only to establish the differentiability properties of the functions ζ^j. For this purpose we rely on known differentiability theorems for elliptic partial differential equations.

Consider the equations (2.7) with j fixed which the z^k satisfy. For r sufficiently small, so that the a_m^k are small, these equations form a nonlinear elliptic system of partial differential equations for the functions $z^1, \cdots, z^n$ of the variables $\zeta^j, \bar{\zeta}^j$. These functions are of class $C^{1+\alpha}$ with respect to the variables $\zeta^j, \bar{\zeta}^j$. We may therefore apply differentiability theorems for elliptic equations (see for example [4]) and infer that the functions z^k have continuous second partial derivatives with respect to the variables $\zeta^j, \bar{\zeta}^j$, since the coefficients are $2n$ times differentiable. Since the z^k already have "mixed" second derivatives with respect to the various ζ^j it follows that the z^k are of class C^2. Therefore the inverse functions ζ^j are also of class C^2 in U. By differentiating and combining the equations (1.3) we infer that each ζ satisfies the equation

$$\partial_j \bar{\partial}_j w - \partial_j a_j^k \partial_k w = 0$$

which is elliptic for r sufficiently small. The desired differentiability properties of $\zeta^1, \cdots, \zeta^n$, i.e. the remainder of Theorem 1.1, follow from well known differentiability theorems for linear elliptic equations of second order, as derived for instance by E. Hopf [7] (see also [4]).

Under the latter conditions of Theorem 1.1, that the coefficients are of class $C^{k+\alpha}$ it follows also from the theory of elliptic equations that the z^i as functions of the ζ coordinates in the polycylinder are of class $C^{k+1+\alpha}$ there.

We remark that if we are given a family of almost complex structures depending differentiably on one or more parameters, and satisfying (1.4) then the complex analytic ζ coordinates which we have constructed will be differentiable in these parameters.

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