

COMPLEX MANIFOLDS

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Abstract. Goal of this paper is to give an overview of the basic definitions of complex and Kähler manifolds together with the most important properties. The main theorem will be the *Newlander-Nirenberg Theorem* which gives a criterion under which an almost complex manifold is a complex one. The key role will be played by a certain tensor field, the so called *Nijenhuis tensor*.

Contents

1	Complex Structures on Vector Spaces	1
2	Almost Complex Structures	2
3	Complex Manifolds	3
4	The Nijenhuis Tensor and the Newlander-Nirenberg Theorem	6
5	Kähler Manifolds	7
Appendix A	Functions of Several Complex Variables	8
Appendix B	Tensor Characterization Lemma	9
References	14

1. Complex Structures on Vector Spaces

In what follows, let $n \in \mathbb{Z}$, $n \geq 1$. Consider an n -dimensional complex vector space and let $J \in \text{End}_{\mathbb{C}}(V)$ be defined by $J(v) := iv$. Then clearly $J \circ J = -\text{id}_V$. Since every n -dimensional complex vector space can be seen as a $2n$ -dimensional real vector space in a natural way, i.e. if (e_ν) is a basis for the complex vector space V , then (e_ν, ie_ν) is a basis for the real vector space V , the mapping J induces an \mathbb{R} -endomorphism \tilde{J} on the real vector space V simply by defining $\tilde{J}(e_\nu) := J(e_\nu) = ie_\nu$ and $\tilde{J}(ie_\nu) := J(ie_\nu) = -e_\nu$ for all $\nu = 1, \dots, n$.

Conversly, let V be an n -dimensional real vector space with $J \in \text{End}_{\mathbb{R}}(V)$ such that $J \circ J = -\text{id}_V$. One can show, that

$$zv := xv + yJ(v) \tag{1}$$

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for $z := x + iy \in \mathbb{C}$ and $v \in V$ makes V into a complex vector space. This motivates the following definition.

Definition 1.1. Let $n \in \mathbb{Z}$, $n \geq 1$ and V be an n -dimensional real vector space. A **complex structure on V** is a \mathbb{R} -linear mapping $J : V \rightarrow V$ such that $J \circ J = -\text{id}_V$. If J is a complex structure on V , the tuple (V, J) is called a **complex vector space**.

Lemma 1.1. Let (V, J) be a complex vector space. Then $\dim V$ is even.

Proof. That $\dim V$ must be even follows directly from

$$(\det(J))^2 = \det(J \circ J) = \det(-\text{id}_V) = (-1)^{\dim V} \det(\text{id}_V) = (-1)^{\dim V}$$

since $\det(J) \in \mathbb{R}$. □

2. Almost Complex Structures

If M is a smooth manifold, then $T_p M$ is a finite dimensional real vector space. Hence we can generalize the definitions and results of the previous section to manifolds. The following definition is taken from [Sil08, p. 86].

Definition 2.1. Let M be a smooth manifold. An **almost complex structure on M** is a smooth tensor field $J \in \Gamma(T^{(1,1)}TM)$ such that $J_p \circ J_p = -\text{id}_{T_p M}$ holds for every $p \in M$. If J is an almost complex structure on M , the tuple (M, J) is called an **almost complex manifold**.

Proposition 2.1. Every almost complex manifold (M, J) is of even dimension and orientable.

Proof. Assume that $\dim M$ is odd. Let $p \in M$. Then by [Lee13, p. 57] we have that $\dim T_p M = \dim M$. Hence $\dim T_p M$ is odd. But by lemma 1.1, $\dim T_p M$ must be even since $(T_p M, J_p)$ is a complex vector space. Contradiction.

Since M is a smooth manifold, there exists a Riemannian metric g on M (see [Lee13, p. 329]). Define

$$\tilde{g}(X, Y) := g(X, Y) + g(JX, JY) \in \Gamma(T^{(0,2)}TM)$$

for all $X, Y \in \mathfrak{X}(M)$. This is possible due to the tensor characterization lemma B.1. Then

$$\tilde{g}(JX, JY) = g(JX, JY) + g(-X, -Y) = g(JX, JY) + g(X, Y) = \tilde{g}(X, Y)$$

by the bilinearity of g . Furthermore, clearly \tilde{g} is positive definite and symmetric, thus a Riemannian metric on M . Define

$$\omega(X, Y) := \tilde{g}(X, JY).$$

Then by

$$\omega(Y, X) = \tilde{g}(Y, JX) = \tilde{g}(JX, Y) = \tilde{g}(-X, JY) = -\omega(X, Y)$$

we see that ω is skew-symmetric. Hence $\omega \in \Omega^2(M)$. Let $p \in M$ and $u \in T_p M \setminus \{0\}$. Then also $-J_p(u) \neq 0$ since J_p is invertible by $\det J_p = 1$. Furthermore, by [Lee13, p. 177], there exist $X, Y \in \mathfrak{X}(M)$, such that $X_p = u$ and $Y_p = -J_p(u)$. Hence

$$\begin{aligned} \omega_p(u, -J_p(u)) &= \omega_p(X_p, Y_p) \\ &= \omega(X, Y)(p) \\ &= \tilde{g}(X, JY)(p) \\ &= \tilde{g}_p(X_p, (JY)_p) \\ &= \tilde{g}_p(u, J_p(Y_p)) \\ &= \tilde{g}_p(u, -(J_p \circ J_p)(u)) \\ &= \tilde{g}_p(u, u) \\ &\neq 0 \end{aligned}$$

and by [Lee13, p. 565] we get that ω is nondegenerate. Let $\dim M = 2n$. By [Lee13, p. 567] this implies that $\omega_p \wedge \cdots \wedge \omega_p$ is nonzero for each $p \in M$. Hence $\omega \wedge \cdots \wedge \omega$ is a nonvanishing top form on M . Since any nonvanishing top form determines an orientation (see [Lee13, p. 381]), we have that M is orientable. \square

Remark 2.1. The converse of proposition 2.1 is not true in general. One can show using results on fibre bundles and Chern classes, that if \mathbb{S}^n admits an almost complex structure, then $n = 2^k - 2$ for $k \in \mathbb{Z}$, $k \geq 1$ (see [Ste51, p. 219]). So for example \mathbb{S}^4 does not admit an almost complex structure. Actually, it can be shown that \mathbb{S}^2 and \mathbb{S}^6 are the only spheres which admit an almost complex structure (see [BS53, p. 434]).

3. Complex Manifolds

The definition of smooth manifolds adapts smoothly to the complex case.

Definition 3.1. Let $n \in \mathbb{Z}$, $n \geq 1$. An **n -dimensional complex manifold** is a second countable Hausdorff space M equipped with a holomorphic structure, that is a maximal holomorphic atlas of complex charts, such that all the transition maps are holomorphic.

Examples 3.1 (Complex Manifolds).

1. The complex n -space \mathbb{C}^n is an n -dimensional complex manifold.
2. Let $\{\omega_1, \dots, \omega_{2n}\}$ be a real basis of \mathbb{C}^n and define

$$G := \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_{2n}. \tag{2}$$

Then the discrete group G acts freely and properly discontinuously on \mathbb{C}^n by translation. Thus $\mathbb{T}^n := \mathbb{C}^n/G$ is an n -dimensional complex manifold, called a **complex torus** (see [FG10, pp. 206–207]).

3. The quotient $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$ is an n -dimensional complex manifold, called the **complex projective space** (see [FG10, pp. 208–210]).

Lemma 3.1. *Let $n, k \in \mathbb{Z}$, $n, k \geq 1$. Let V be an n -dimensional real vector space. Then*

$$V \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_k \cong \mathbf{L}(\underbrace{V, \dots, V}_k; V) \quad (3)$$

canonically. If (e_ν) is a basis of V and (e_ν^) the corresponding basis of V^* , then $f \in \text{End}(V)$ corresponds to*

$$\sum_{\nu=1}^n f(e_\nu) \otimes e_\nu^*. \quad (4)$$

Proof. It is easily checked that

$$\Psi : \begin{cases} V \times V^* \times \cdots \times V^* \rightarrow \mathbf{L}(V, \dots, V; V) \\ (v, f_1, \dots, f_k) \mapsto ((v_1, \dots, v_k) \mapsto f_1(v_1) \cdots f_k(v_k)v) \end{cases}$$

is multilinear. Thus by the universal property of the tensor product there exists a unique linear mapping $\tilde{\Psi} \in \text{Hom}_{\mathbb{R}}(T^{(1,k)}(V); \mathbf{L}(V, \dots, V; V))$ such that $\Psi = \tilde{\Psi} \circ \otimes$. It is also easily checked that $\tilde{\Psi}$ is an isomorphism. Let $f \in \text{End}(V)$. Then for any $v \in V$ we have

$$\begin{aligned} \tilde{\Psi} \left(\sum_{\nu=1}^n f(e_\nu) \otimes e_\nu^* \right) (v) &= \sum_{\nu=1}^n \tilde{\Psi} (f(e_\nu) \otimes e_\nu^*) (v) \\ &= \sum_{\nu=1}^n e_\nu^*(v) f(e_\nu) \\ &= f \left(\sum_{\nu=1}^n e_\nu^*(v) e_\nu \right) \\ &= f(v). \end{aligned}$$

□

Proposition 3.1. *Any complex manifold admits a canonical almost complex structure.*

Proof. Fix a complex manifold M . We define J in terms of local coordinates. Let $(U, (x^\nu, y^\nu))$ be a chart. By lemma 3.1 it is also enough to construct an endomorphism J_p for every $p \in U$. We define

$$J_p \left(\frac{\partial}{\partial x^\nu} \Big|_p \right) := \frac{\partial}{\partial y^\nu} \Big|_p \quad \text{and} \quad J_p \left(\frac{\partial}{\partial y^\nu} \Big|_p \right) := -\frac{\partial}{\partial x^\nu} \Big|_p$$

for all $\nu = 1, \dots, n$. As standard linear algebra shows, there is a unique linear mapping associated with J_p (see [HK71, p. 69]). Let $v := a^\nu \frac{\partial}{\partial x^\nu} \Big|_p + b^\nu \frac{\partial}{\partial y^\nu} \Big|_p \in T_p M$. Then

$$\begin{aligned} (J_p \circ J_p)(v) &= J_p \left(a^\nu J_p \left(\frac{\partial}{\partial x^\nu} \Big|_p \right) + b^\nu J_p \left(\frac{\partial}{\partial y^\nu} \Big|_p \right) \right) \\ &= J_p \left(a^\nu \frac{\partial}{\partial y^\nu} \Big|_p - b^\nu \frac{\partial}{\partial x^\nu} \Big|_p \right) \\ &= -a^\nu \frac{\partial}{\partial x^\nu} \Big|_p - b^\nu \frac{\partial}{\partial y^\nu} \Big|_p \\ &= -v \end{aligned}$$

and thus $J_p \circ J_p = -\text{id}_{T_p M}$.

Next we have to show that above locally defined mapping is well-defined, i.e. does not depend on the choice of coordinates. Assume that $p \in U \cap V$ for another chart $(V, (u^i, v^i))$. By the change of coordinates formula [Lee13, p. 64] we get that

$$\frac{\partial}{\partial x^\nu} \Big|_p = \frac{\partial u^\mu}{\partial x^\nu}(\hat{p}) \frac{\partial}{\partial u^\mu} \Big|_p + \frac{\partial v^\mu}{\partial x^\nu}(\hat{p}) \frac{\partial}{\partial v^\mu} \Big|_p$$

and

$$\frac{\partial}{\partial y^\nu} \Big|_p = \frac{\partial u^\mu}{\partial y^\nu}(\hat{p}) \frac{\partial}{\partial u^\mu} \Big|_p + \frac{\partial v^\mu}{\partial y^\nu}(\hat{p}) \frac{\partial}{\partial v^\mu} \Big|_p$$

where \hat{p} denotes the coordinate representation of p with respect to the coordinates (x^ν, y^ν) . Corollary A.1 implies

$$\begin{aligned} J_p \left(\frac{\partial}{\partial x^\nu} \Big|_p \right) &= \frac{\partial u^\mu}{\partial x^\nu}(\hat{p}) J_p \left(\frac{\partial}{\partial u^\mu} \Big|_p \right) + \frac{\partial v^\mu}{\partial x^\nu}(\hat{p}) J_p \left(\frac{\partial}{\partial v^\mu} \Big|_p \right) \\ &= \frac{\partial u^\mu}{\partial x^\nu}(\hat{p}) \frac{\partial}{\partial v^\mu} \Big|_p - \frac{\partial v^\mu}{\partial x^\nu}(\hat{p}) \frac{\partial}{\partial u^\mu} \Big|_p \\ &= \frac{\partial v^\mu}{\partial y^\nu}(\hat{p}) \frac{\partial}{\partial v^\mu} \Big|_p + \frac{\partial u^\mu}{\partial y^\nu}(\hat{p}) \frac{\partial}{\partial u^\mu} \Big|_p \\ &= \frac{\partial}{\partial y^\nu} \Big|_p \end{aligned}$$

and

$$\begin{aligned}
 J_p \left(\frac{\partial}{\partial y^\nu} \Big|_p \right) &= \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p}) J_p \left(\frac{\partial}{\partial u^\mu} \Big|_p \right) + \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p}) J_p \left(\frac{\partial}{\partial v^\mu} \Big|_p \right) \\
 &= \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p}) \frac{\partial}{\partial v^\mu} \Big|_p - \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p}) \frac{\partial}{\partial u^\mu} \Big|_p \\
 &= -\frac{\partial v^\mu}{\partial x^\nu}(\widehat{p}) \frac{\partial}{\partial v^\mu} \Big|_p - \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p}) \frac{\partial}{\partial u^\mu} \Big|_p \\
 &= -\frac{\partial}{\partial x^\nu} \Big|_p.
 \end{aligned}$$

Left to check is smoothness. According to lemma 3.1 the corresponding rough tensor field is given by

$$J_p \left(\frac{\partial}{\partial x^\nu} \Big|_p \right) \otimes dx^\nu|_p + J_p \left(\frac{\partial}{\partial y^\nu} \Big|_p \right) \otimes dy^\nu|_p = \frac{\partial}{\partial y^\nu} \Big|_p \otimes dx^\nu|_p - \frac{\partial}{\partial x^\nu} \Big|_p \otimes dy^\nu|_p$$

for any $p \in U$. Thus the smoothness criteria for tensor fields B.2 together with [Lee13, p. 36] yields that $J \in \Gamma(T^{(1,1)}TM)$. \square

A question which naturally arises by considering proposition 3.1 is, if the converse is also true, i.e. if every almost complex manifold is a complex manifold. This is in general not the case. Let \mathbb{P} denote the naturally oriented underlying smooth manifold of the complex projective plane. Again using results about Chern numbers it can be shown that

$$\mathbb{P} \# (\mathbb{S}^1 \times \mathbb{S}^3) \# (\mathbb{S}^1 \times \mathbb{S}^3) \quad \text{and} \quad (\mathbb{S}^1 \times \mathbb{S}^3) \# (\mathbb{S}^1 \times \mathbb{S}^3) \# (\mathbb{S}^2 \times \mathbb{S}^2) \quad (5)$$

have almost complex structures but no complex structure (see [Ven66, p. 1627]).

4. The Nijenhuis Tensor and the Newlander-Nirenberg Theorem

As we have seen in the last section, not every almost complex manifold is a complex manifold. Under which condition is this possible?

Definition 4.1. Let (M, J) be an almost complex manifold. For $X, Y \in \mathfrak{X}(M)$ we define the **Nijenhuis tensor** N as

$$N(X, Y) := [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] \quad (6)$$

where $[X, Y]$ denotes the usual Lie-bracket of vector fields.

Proposition 4.1. Let (M, J) be an almost complex manifold and N be the Nijenhuis tensor. Then $N \in \Gamma(T^{(1,2)}TM)$.

Proof. First of all, $N(X, Y) \in \mathfrak{X}(M)$ for all $X, Y \in \mathfrak{X}(M)$. This follows immediately by considering J as a mapping $J : \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ using the tensor characterization lemma B.1, the fact that the Lie Bracket of two smooth vector fields is again a smooth vector field (see [Lee13, p. 186]) and that $\mathfrak{X}(M)$ is a $\mathcal{C}^\infty(M)$ -module (see [Lee13, p. 177]). Let $f \in \mathcal{C}^\infty(M)$ and $X, Y, Z \in \mathfrak{X}(M)$. Then

$$\begin{aligned}
 N(fX + Y, Z) &= [J(fX + Y), JZ] - J[fX + Y, JZ] - J[J(fX + Y), Z] \\
 &\quad - [fX + Y, Z] \\
 &= [fJX + JY, JZ] - J[fX + Y, JZ] - J[fJX + JY, Z] \\
 &\quad - [fX + Y, Z] \\
 &= [fJX, JZ] + [JY, JZ] - J[fX, JZ] - J[Y, JZ] - J[fJX, Z] \\
 &\quad - J[JY, Z] - [fX, Z] - [Y, Z] \\
 &= f[JX, JZ] - (JZf)JX + [JY, JZ] - fJ[X, JZ] + (JZf)JX \\
 &\quad - [Y, JZ] - fJ[JX, Z] + (Zf)JJX - J[JY, Z] - f[X, Z] \\
 &\quad + (Zf)X - [Y, Z] \\
 &= fN(X, Z) + N(Y, Z).
 \end{aligned}$$

by [Lee13, pp. 187–188]. Linearity in the second argument is shown similarly. Hence $N : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ is bilinear over $\mathcal{C}^\infty(M)$. Again by the tensor characterization lemma B.1 we have that $N \in \Gamma(T^{(1,2)}TM)$. \square

Theorem 4.1 (Newlander-Nirenberg). *Let (M, J) be an almost complex manifold. Then M is a complex manifold, where the complex structure is so that the canonically induced almost complex structure is J , if and only if the Nijenhuis tensor N vanishes identically.*

Proof. Assume M is a complex manifold. Let $(U, (x^\nu, y^\nu))$ be a chart. From proposition 4.1 it is enough to consider the coordinate vector fields $\frac{\partial}{\partial x^\nu}$ and $\frac{\partial}{\partial y^\nu}$. But from the explicit definition of J in proposition 3.1 and the property, that the Lie-Bracket of coordinate vector fields vanishes, together with the $\mathcal{C}^\infty(M)$ -linearity of J we get that N vanishes identically on each chart, and thus on M .

The other direction however is far more technical and uses results on partial differential equations. A complete proof can either be found in the original paper [NN57] or in [Sil08, p. 106], where references to more recent proofs are given. \square

5. Kähler Manifolds

The following is inspired by [KN96, pp. 146–149] and introduces the concepts from a complex viewpoint. This is in contrast to the symplectic approach provided for example in [Sil08].

Definition 5.1. Let (M, J) be an almost complex manifold. A **Hermitian metric on M** is a Riemannian metric g such that

$$g(JX, JY) = g(X, Y) \quad (7)$$

holds for all $X, Y \in \mathfrak{X}(M)$. If g is a Hermitian metric on M , the triple (M, J, g) is called an **almost Hermitian manifold**.

Lemma 5.1. Every almost complex manifold admits a Hermitian metric.

Proof. The existence was shown in the proof of proposition 2.1. \square

Definition 5.2. Let (M, J, g) be an almost Hermitian manifold. The **fundamental 2-form Ω** is defined to be

$$\Omega(X, Y) := g(X, JY) \quad (8)$$

for all $X, Y \in \mathfrak{X}(M)$.

Definition 5.3. Let (M, J, g) be an almost Hermitian manifold with fundamental 2-form Ω . The Hermitian metric g is said to be a **Kähler metric**, if $d\Omega = 0$. An almost complex manifold with a Kähler metric is called an **almost Kähler manifold** and a complex manifold with a Kähler metric is called a **Kähler manifold**.

Appendix A. Functions of Several Complex Variables

This section summarizes the fundamental properties of functions of several complex variables needed later. The results are inspired by [FG10, pp. 14–30].

Definition A.1. Let $n \in \mathbb{Z}$, $n \geq 1$, $U \subseteq \mathbb{C}^n$ open and $a \in U$. A mapping $f : U \rightarrow \mathbb{C}$ is said to be **complex differentiable at a** if there exists $g : U \rightarrow \mathbb{C}^n$ such that g is continuous at a and

$$f(z) = f(a) + \sum_{\nu=1}^n (z_\nu - a_\nu) g_\nu(z) \quad (9)$$

holds for all $z \in D$. f is said to be **holomorphic in D** if it is complex differentiable at every point $a \in D$. For $m \in \mathbb{Z}$, $m \geq 1$, a mapping $f : U \rightarrow \mathbb{C}^m$ is said to be **holomorphic in D** if each component function f_ν , $\nu = 1, \dots, m$, is holomorphic in D .

Proposition A.1. Let $n \in \mathbb{Z}$, $n \geq 1$, $D \subseteq \mathbb{C}^n$ open, $a \in D$ and $f : D \rightarrow \mathbb{C}$ real differentiable at a . Then

$$\frac{\partial f}{\partial z_\nu}(a) = \frac{1}{2} \left(\frac{\partial f}{\partial x_\nu}(a) - i \frac{\partial f}{\partial y_\nu}(a) \right) \quad (10)$$

and

$$\frac{\partial f}{\partial \bar{z}_\nu}(a) = \frac{1}{2} \left(\frac{\partial f}{\partial x_\nu}(a) + i \frac{\partial f}{\partial y_\nu}(a) \right) \quad (11)$$

holds for all $\nu = 1, \dots, n$.

Theorem A.1 (The Cauchy-Riemann Equations). *Let $n \in \mathbb{Z}$, $n \geq 1$ and $D \subseteq \mathbb{C}^n$ open. A mapping $f : D \rightarrow \mathbb{C}$ is holomorphic in D if and only if it is real differentiable at every $a \in D$ and the **Cauchy-Riemann equations***

$$\frac{\partial f}{\partial \bar{z}_\nu}(a) = 0 \quad (12)$$

holds for all $a \in D$ and $\nu = 1, \dots, n$.

Corollary A.1. *Let $m, n \in \mathbb{Z}$, $m, n \geq 1$, $D \subseteq \mathbb{C}^n$ open and $f : D \rightarrow \mathbb{C}^m$ holomorphic in D . If $f = g + ih$, $g, h : D \rightarrow \mathbb{R}^m$, then*

$$\boxed{\frac{\partial g_\mu}{\partial x_\nu}(a) = \frac{\partial h_\mu}{\partial y_\nu}(a) \quad \text{and} \quad \frac{\partial h_\mu}{\partial x_\nu}(a) = -\frac{\partial g_\mu}{\partial y_\nu}(a)} \quad (13)$$

holds for any $a \in D$, $\nu = 1, \dots, n$ and $\mu = 1, \dots, m$.

Proof. Fix $\mu = 1, \dots, m$. By definition A.1 f_μ is holomorphic in D . Hence f_μ is real differentiable in D (see [FG10, p. 27]) and theorem A.1 implies

$$\frac{\partial f_\mu}{\partial \bar{z}_\nu}(a) = 0$$

for all $a \in D$ and $\nu = 1, \dots, n$. By proposition A.1, this is equivalent to

$$\frac{\partial f}{\partial x_\nu}(a) + i \frac{\partial f}{\partial y_\nu}(a) = 0.$$

Using $f_\mu = g_\mu + ih_\mu$ and the \mathbb{C} -linearity of the operators $\frac{\partial}{\partial x_\nu}$ and $\frac{\partial}{\partial y_\nu}$ yields

$$\frac{\partial g_\mu}{\partial x_\nu}(a) - \frac{\partial h_\mu}{\partial y_\nu}(a) + i \left(\frac{\partial h_\mu}{\partial x_\nu}(a) + \frac{\partial g_\mu}{\partial y_\nu}(a) \right) = 0.$$

□

Appendix B. Tensor Characterization Lemma

Definition B.1. *Let $k, l \in \mathbb{Z}$, $k, l \geq 0$ and M a smooth manifold. Then the **bundle of mixed tensors of type (k, l)** is defined by*

$$T^{(k,l)}TM := \coprod_{p \in M} T^{(k,l)}(T_p M). \quad (14)$$

Proposition B.1. *The bundle of mixed tensors of type (k, l) has a unique natural structure as a smooth vector bundle of rank n^{k+l} over M .*

Proof. For each $p \in M$ let $E_p := T^{(k,l)}(T_p M)$. By [Lee13, p. 57] and [Lee13, p. 313] $\dim E_p = n^{k+l}$. Furthermore, let $E := T^{(k,l)}TM$ and $\pi : E \rightarrow M$ be defined by $\pi(p, A) := p$. Let $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ be an atlas for M . For each $\alpha \in A$ define

$$\Phi_\alpha : \begin{cases} \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^{n^{k+l}} \\ (p, A) \mapsto (p, (A_{j_1 \dots j_l}^{i_1 \dots i_k})) \end{cases}$$

Clearly, the inverse is given by

$$\Phi_\alpha^{-1} : \begin{cases} U_\alpha \times \mathbb{R}^{n^{k+l}} \rightarrow \pi^{-1}(U_\alpha) \\ (p, (A_{j_1 \dots j_l}^{i_1 \dots i_k})) \mapsto (p, A) \end{cases}.$$

Hence each Φ_α is bijective. Now we have to check, that $\Phi_\alpha|_{E_p}$ is an isomorphism. So let $\lambda \in \mathbb{R}$ and $B \in E_p$. Then

$$\begin{aligned} \Phi_\alpha|_{E_p}(p, \lambda A + B) &= (p, (\lambda A + B)_{j_1 \dots j_l}^{i_1 \dots i_k}) \\ &= (p, \lambda (A_{j_1 \dots j_l}^{i_1 \dots i_k}) + (B_{j_1 \dots j_l}^{i_1 \dots i_k})) \\ &= \lambda \Phi_\alpha|_{E_p}(p, A) + \Phi_\alpha|_{E_p}(p, B). \end{aligned}$$

Now let $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$. We consider the mapping

$$\Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^{n^{k+l}} \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^{n^{k+l}}.$$

Define $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(n^{k+l}, \mathbb{R})$ by

$$\tau_{\alpha\beta} := (\delta_j^i).$$

Then we have that

$$(\Phi_\alpha \circ \Phi_\beta^{-1})(p, (A_{j_1 \dots j_l}^{i_1 \dots i_k})) = (p, (A_{j_1 \dots j_l}^{i_1 \dots i_k})) = (p, \tau_{\alpha\beta}(p)(A_{j_1 \dots j_l}^{i_1 \dots i_k})).$$

Since $\tau_{\alpha\beta}$ is constant, it is smooth (see [Lee13, p. 36]). Hence we can apply the vector bundle chart lemma [Lee13, p. 253] and the result follows. \square

What follows is a reformulation of the smoothness criteria for tensor fields ([Lee13, pp. 317–318]) for tensor fields of type $(1, k)$.

Proposition B.2 (Smoothness Criteria for Tensor Fields). *Let M be smooth manifold and let $A : M \rightarrow T^{(1,k)}TM$ be a rough section. Then the following are equivalent:*

- (a) $A \in \Gamma(T^{(1,k)}TM)$.
- (b) In every smooth coordinate chart, the component functions of A are smooth.
- (c) For all $X_1, \dots, X_k \in \mathfrak{X}(M)$, the rough section $A(X_1, \dots, X_k) : M \rightarrow TM$ defined by

$$A(X_1, \dots, X_k)(p) := A_p(X_1|_p, \dots, X_k|_p) \tag{15}$$

is a smooth vector field.

(d) If X_1, \dots, X_k are smooth vector fields on some open subset $U \subseteq M$, then also $A(X_1, \dots, X_k)$ is a smooth vector field on U .

Proof. We prove (a) \Leftrightarrow (b) and (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (b).

To prove (a) \Leftrightarrow (b), let $(U, (x^i))$ be a smooth chart. Actually, we can prove this for general tensor fields of type (k, l) . Proposition B.1 together with the proof of the vector bundle chart lemma [Lee13, pp. 253–254] implies, that the corresponding chart on $T^{(k,l)}TM$ is given by $(\pi^{-1}(U), \tilde{\varphi})$, where $\tilde{\varphi} : \pi^{-1}(U) \rightarrow \varphi(U) \times \mathbb{R}^{n^{k+l}}$ is defined by

$$\tilde{\varphi} := (\varphi \times \text{id}_{\mathbb{R}^{n^{k+l}}}) \circ \Phi$$

where $\Phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^{n^{k+l}}$ is given as in the proof of proposition B.1. Now we consider the coordinate representation \hat{A} in the given charts (see [Lee13, p. 35]). Since A is a rough section, we have that

$$A^{-1}(\pi^{-1}(U)) = (\pi \circ A)^{-1}(U) = \text{id}_M^{-1}(U) = U.$$

Hence $\varphi(U \cap A^{-1}(\pi^{-1}(U))) = \varphi(U)$, which is open, and $\hat{A} : \varphi(U) \rightarrow \tilde{\varphi}(\pi^{-1}(U))$ is given by

$$\begin{aligned} \hat{A}(x) &= (\tilde{\varphi} \circ A \circ \varphi^{-1})(x) \\ &= (\varphi \times \text{id}_{\mathbb{R}^{n^{k+l}}})(\Phi(\varphi^{-1}(x), A_{\varphi^{-1}(x)})) \\ &= (\varphi \times \text{id}_{\mathbb{R}^{n^{k+l}}})(\varphi^{-1}(x), (A_{j_1 \dots j_l}^{i_1 \dots i_k}(\varphi^{-1}(x)))) \\ &= (x, (\hat{A}_{j_1 \dots j_l}^{i_1 \dots i_k}(x))). \end{aligned}$$

By [Lee13, p. 35] A is smooth if and only if in any chart \hat{A} is smooth. This is furthermore equivalent to that each $\hat{A}_{j_1 \dots j_l}^{i_1 \dots i_k}$ is smooth and thus equivalent to that $A_{j_1 \dots j_l}^{i_1 \dots i_k}$ is smooth (see [Lee13, p. 33]).

To prove (b) \Rightarrow (c), let $(U, (x^i))$ be a smooth chart. Then write $X_1, \dots, X_k \in \mathfrak{X}(M)$ as

$$X_\nu = X_\nu^{\mu_\nu} \frac{\partial}{\partial x^{\mu_\nu}}.$$

for $\nu = 1, \dots, k$. For $p \in U$ lemma 3.1 implies

$$\begin{aligned} A(X_1, \dots, X_k)(p) &= A_p(X_1|_p, \dots, X_k|_p) \\ &= A_p \left(X_1^{\mu_1}(p) \frac{\partial}{\partial x^{\mu_1}} \Big|_p, \dots, X_k^{\mu_k}(p) \frac{\partial}{\partial x^{\mu_k}} \Big|_p \right) \\ &= X_1^{\mu_1}(p) \cdots X_k^{\mu_k}(p) A_p \left(\frac{\partial}{\partial x^{\mu_1}} \Big|_p, \dots, \frac{\partial}{\partial x^{\mu_k}} \Big|_p \right) \\ &= X_1^{\mu_1}(p) \cdots X_k^{\mu_k}(p) A_{\mu_1 \dots \mu_k}^i(p) \frac{\partial}{\partial x^i} \Big|_p. \end{aligned}$$

By the smoothness criterion for vector fields [Lee13, p. 175] we have that each component function $X_\nu^{\mu_n}$ is smooth. Thus if A is smooth, we have by that each $A_{j_1 \dots j_k}^i$ is smooth and since $\mathcal{C}^\infty(M)$ is an \mathbb{R} -algebra (see [Lee13, p. 33]), we have that

$$X_1^{\mu_1} \dots X_k^{\mu_k} A_{\mu_1 \dots \mu_k}^i$$

is smooth for $i = 1, \dots, n$. Thus again by the smoothness criterion together with the localness of smoothness [Lee13, p. 35] we get that $A(X_1, \dots, X_k) \in \mathfrak{X}(M)$.

To prove (c) \Rightarrow (d), we use that smoothness is a local property (see [Lee13, p. 35]). Let $p \in U$. Then by [Cat17, p. 14] we find a smooth bump function ψ supported in U and identically equal to 1 on some neighbourhood V of p . Set

$$\tilde{X}_\nu|_p := \begin{cases} \psi(p)X_\nu|_p & p \in \text{supp } \psi \\ 0 & p \in M \setminus \text{supp } \psi \end{cases}.$$

Then the gluing lemma for smooth maps [Lee13, p. 35] implies $\tilde{X}_1, \dots, \tilde{X}_k \in \mathfrak{X}(M)$. Hence by (c) we get that $A(\tilde{X}_1, \dots, \tilde{X}_k) \in \mathfrak{X}(M)$ and so the restriction $A(\tilde{X}_1, \dots, \tilde{X}_k)|_V$ is smooth. But $A(\tilde{X}_1, \dots, \tilde{X}_k)|_V = A(X_1, \dots, X_k)$ and so we are done. Lastly to prove (d) \Rightarrow (b), each vector field locally defined by

$$X_{j_\nu} = \delta_{j_\nu}^{\mu_\nu} \frac{\partial}{\partial x^{\mu_\nu}}.$$

is smooth. Thus by

$$A(X_1, \dots, X_n)(p) = \delta_{j_1}^{\mu_1} \dots \delta_{j_k}^{\mu_k} A_{\mu_1 \dots \mu_k}^i(p) \frac{\partial}{\partial x^i} \Big|_p = A_{j_1 \dots j_k}^i(p) \frac{\partial}{\partial x^i} \Big|_p$$

we get that $A_{j_1 \dots j_k}^i$ is smooth and hence by (b) also A . \square

Theorem B.1 (Tensor Characterization Lemma). *A mapping*

$$\underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_k \rightarrow \mathcal{C}^\infty(M) \quad \text{or} \quad \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_k \rightarrow \mathfrak{X}(M)$$

is induced by an element of $\Gamma(T^{(0,k)}TM)$ or $\Gamma(T^{(1,k)}TM)$, respectively, if and only if they are multilinear over $\mathcal{C}^\infty(M)$.

Proof. We are proving only the second statement. Any element in $\Gamma(T^{(1,k)}TM)$ induces a mapping $\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ by part (c) of the smoothness criteria for tensor fields B.2. Thus we have to show that \mathcal{A} is multilinear over $\mathcal{C}^\infty(M)$. Let $f \in \mathcal{C}^\infty(M)$ and $X_\nu, \tilde{X}_\nu \in \mathfrak{X}(M)$, $\nu = 1, \dots, k$. Then for any $p \in M$ we have that

$$\begin{aligned} \mathcal{A}(X_1, \dots, fX_\nu + \tilde{X}_\nu, \dots, X_k)_p &= A_p(X_1|_p, \dots, (fX_\nu + \tilde{X}_\nu)|_p, \dots, X_k|_p) \\ &= A_p(X_1|_p, \dots, f(p)X_\nu|_p + \tilde{X}_\nu|_p, \dots, X_k|_p) \end{aligned}$$

$$\begin{aligned}
&= f(p)A_p(X_1|_p, \dots, X_\nu|_p, \dots, X_k|_p) \\
&\quad + A_p(X_1|_p, \dots, \tilde{X}_\nu|_p, \dots, X_k|_p) \\
&= f(p)\mathcal{A}(X_1, \dots, X_\nu, \dots, X_k)_p \\
&\quad + \mathcal{A}(X_1, \dots, \tilde{X}_\nu, \dots, X_k)_p \\
&= (f\mathcal{A}(X_1, \dots, X_\nu, \dots, X_k))_p \\
&\quad + \mathcal{A}(X_1, \dots, \tilde{X}_\nu, \dots, X_k)_p.
\end{aligned}$$

Conversely, suppose that $\mathcal{A} : \mathfrak{X}(M) \times \dots \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$ is multilinear over $\mathcal{C}^\infty(M)$. Let $p \in M$. First we show that \mathcal{A} acts locally, i.e. if $X_\nu = \tilde{X}_\nu$ in some neighbourhood U of p implies that also

$$\mathcal{A}(X_1, \dots, X_\nu, \dots, X_k) = \mathcal{A}(X_1, \dots, \tilde{X}_\nu, \dots, X_k)$$

on U . By the multilinearity of \mathcal{A} it is enough to show that if X_ν vanishes on U then so does \mathcal{A} . There exists a smooth bump function ψ for $\{p\}$ supported in U (see [Lee13, p. 44]). Hence $\psi X_\nu = 0$ on M and $\psi(p) = 1$. Thus

$$0 = \mathcal{A}(X_1, \dots, \psi X_\nu, \dots, X_k)_p = \psi(p)\mathcal{A}(X_1, \dots, X_\nu, \dots, X_k)_p.$$

and since $\psi(p) = 1$ we have that

$$\mathcal{A}(X_1, \dots, X_\nu, \dots, X_k)_p = 0$$

for any $p \in U$.

Next we show that \mathcal{A} actually acts pointwise, i.e. if $X_\nu|_p$ vanishes so does \mathcal{A} . Let $(U, (x^i))$ be a chart containing p and $X_\nu = X_\nu^i \frac{\partial}{\partial x^i}$ on U . The same construction as used showing the implication (c) \Rightarrow (d) in the proof of proposition B.2 yields the existence of $f^1, \dots, f^n \in \mathcal{C}^\infty(M)$ and $\tilde{X}_1, \dots, \tilde{X}_n \in \mathfrak{X}(M)$ such that $f^i = X_\nu^i$ and $\tilde{X}_i = \frac{\partial}{\partial x^i}$ on a neighbourhood $V \subseteq U$ of p . Thus by the previous localization, we get that

$$\mathcal{A}(X_1, \dots, X_\nu, \dots, X_k) = \mathcal{A}(X_1, \dots, f^i \tilde{X}_i, \dots, X_k) = f^i \mathcal{A}(X_1, \dots, \tilde{X}_i, \dots, X_k)$$

in U . Since $0 = X_\nu^i(p) = f^i(p)$, \mathcal{A} vanishes at p . Hence \mathcal{A} depends only on the value of X_ν at p . Thus define a rough section $A : M \rightarrow T^{(1,k)}TM$ by

$$A_p(v_1, \dots, v_k) := \mathcal{A}(V_1, \dots, V_k)(p)$$

where $V_1, \dots, V_k \in \mathfrak{X}(M)$ are any extensions of $v_1, \dots, v_k \in T_p M$ (see [Lee13, p. 177]). By the above, the choice of the extensions does not matter and the resulting rough section is smooth by proposition B.2 part (c), hence $A \in \Gamma(T^{(1,k)}TM)$. \square

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