Appendix

This appendix has been added (November 1956) to call attention to some of the important advances in the theory of fibre bundles since 1951, and to show how they answer, wholly or in part, questions raised in the text. The order of the following material approximates that of the related subjects in the text.

1. Local cross-sections of a subgroup. A generalization of the conjecture made at the end of §7.5, p. 33 has been proved by P. S. Mostert: Local cross sections in locally compact groups, *Proc. Amer. Math. Soc.* *4* (1953), 645–649. He shows that, if B is a locally-compact and finite-dimensional group, and G is a closed subgroup, then G has a local cross-section in B.

2. The covering homotopy theorem. The hypotheses of the covering homotopy theorems, §11.3, p. 50, and §11.7, p. 54, can be weakened without affecting the conclusions by replacing the condition "X is a normal, locally-compact C_σ-space" by "X is normal and paracompact." See W. Huebsch, On the covering homotopy theorem, *Annals of Math.* *61* (1955), 555–563.

3. The existence of cross-sections. The hypotheses of the existence theorem §12.2, p. 55, may be relaxed. The conclusion still holds if the base space X is normal and paracompact, and the fibre Y is solid. This improvement is of the same nature as that made in the covering homotopy theorem; and the modifications in the proof are similar.

4. Homotopy groups. At the ends of §15.10, p. 80, and §21.7, p. 114, it is stated that very few homotopy groups have been successfully computed. The situation in 1956 is entirely different. Major advances in the theory have been made. It has been shown that the homotopy groups of finite, simply-connected complexes are finitely generated, and are effectively computable. The computations have been made in numerous special cases. It is notable that the concept of *fibre space* (a somewhat broader notion than *fibre bundle*) played a vital role in this development. A complete review of these results and their implications for fibre bundles would be too long. The following references give a substantial indication of the progress.

J.-P. Serre, Homologie singulière des espaces fibrés, *Annals of Math.* *54* (1951), 425–505.

——, Groupes d'homotopie et classes de groupes abéliens, *ibid.* *58* (1953), 258–294.

H. Cartan, *Algèbres d'Eilenberg-MacLane et homotopie*, Seminar notes 1954/1955, Paris.

H. Toda, Calcul des groupes d'homotopie des spheres, *C. R. Acad. Sci. Paris 240* (1955), 147–149.

——, Le produit de Whitehead et l'invariant de Hopf, *ibid. 241* (1955), 849–850.

In this last note Toda announces that there is no mapping $S^{31} \to S^{16}$ of Hopf invariant 1. Hence there is no real division algebra of dimension 16 (see §20.7, p. 110).

5. Homotopy groups of Lie groups. Articles 22 to 25, pp. 114–134, of the text are devoted to the computation of a few of the homotopy groups of the classical Lie groups and their coset spaces. Far more extensive results have been obtained by better methods. For a survey of these see A. Borel, Topology of Lie groups and characteristic classes, *Bull. Amer. Math. Soc. 61* (1955), 397–432.

6. Sphere bundles over spheres. In §26.6 to 26.10 various sphere bundles over spheres are exhibited which are not equivalent to product bundles but have the homotopy groups and homology structure of products. I. M. James and J. H. C. Whitehead have devised a homotopy invariant of such bundles which enables them to distinguish many of these spaces from products and one another. See: The homotopy theory of sphere bundles over spheres I and II, *Proc. London Math. Soc. 4* (1954), 196–218, and *5* (1955), 148–166.

7. The tangent bundle of S^n. In a paper by J. H. C. Whitehead and the author (Vector fields on the n-sphere, *Proc. Nat. Acad. Sci. 37* (1951), 58–63), many of the results of §§27, 40, and 41 are generalized and proved by easier methods. For example, Theorems 27.8 and 27.9 are special cases of the following: If n and k are related by $n + 1 = 2^k(2r + 1)$, then any set of 2^k continuous vector fields tangent to S^n are somewhere dependent. Theorems 27.18 and 40.12 are included in: *If n and k are as above, and $2^k \leqq q \leqq n - 2^k$, then S^n does not admit a continuous field of tangent q-planes nor a continuous quadratic form which is nonsingular of signature q.*

Theorem 41.20 becomes: *If S^n admits an almost (= quasi) complex structure, then n must be of the form $2^k - 2$.* This last result has been greatly improved by A. Borel and J.-P. Serre (Groupes de Lie et puissances réduites de Steenrod, *Amer. Jour. Math. 75* (1953), 409–448) as follows: *The only spheres which admit an almost complex structure are S^2 and S^6.*

The result 27.16 has been improved by I. M. James: Note on factor spaces, *Jour. London Math. Soc. 28* (1953), 278–285.

8. The fibering of spheres by spheres. The results of §28 have been considerably improved. In the paper by Whitehead and the author, referred to above, it is shown that, if S^{n+r} is an r-sphere bundle over S^n, then $n = 2^k$ and $r = 2^k - 1$ for some k. The impossibility of fibering a sphere by spheres has been shown in many other cases by J. Adem: Relations on iterated reduced powers, *Proc. Nat. Acad. Sci.* *39* (1953), 636–638.

9. Characteristic classes of sphere bundles. Knowledge of the Stiefel and Whitney characteristic classes (§38, 39) has been greatly extended by the work of R. Thom: Espaces fibrés en sphères et carrés de Steenrod, *Ann. Sci. École Norm. Sup. 69* (1952), 109–182. His first main result is a formula which characterizes the Whitney classes W^i $(i = 0, 1, \cdots, n)$ of an $(n-1)$-sphere bundle $B \to X$, namely:

$$W^i = \phi^{-1}\mathrm{Sq}^i\phi W^0$$

In this formula ϕ is an isomorphism $H^q(X) \approx H^{q+n}(A,B)$ where $A \to X$ is the associated n-cell bundle; and Sq^i: $H^n \to H^{n+i}$ is the squaring operation on cohomology mod 2 which this author defined for other purposes (Cyclic reduced powers of cohomology classes, *Proc. Nat. Acad. Sci. 39* (1953), 213–223).

Using this formula, Thom obtains simple proofs of the various properties of the W^i. In particular the Whitney duality theorem, stated but not proved in §38.13, is an easy consequence of the Cartan identity

$$\mathrm{Sq}^i(u \smile v) = \Sigma_{j=0}^{i}\mathrm{Sq}^j u \smile \mathrm{Sq}^{i-j} v.$$

Next Thom considers a differentiable imbedding of the differentiable r-manifold X in a differentiable $(r + n)$-manifold M. Letting B be the normal bundle of X in M, he realizes the associated bundle A as a tubular neighborhood of X in M, and obtains a new formulation of the isomorphism ϕ which is purely topological in character. This provides him with a definition of normal classes for *any* topological imbedding of X in M, and it gives the usual normal classes when the imbedding is differentiable. He considers next the diagonal imbedding of X in $X \times X$. When X is differentiable, he shows that the tangent bundle of X is isomorphic to its normal bundle in $X \times X$. When X is an arbitrary manifold, he defines its *tangent* classes to be its normal classes in $X \times X$. This extends the definition of the Stiefel characteristic classes to arbitrary manifolds; and at the same time it proves that the Stiefel classes, defined in terms of a differential structure, are independent of that structure.

Wu Wen-Tsun has improved on this result (Classes caracteristiques et i-carrés d'une variété, *C. R. Acad. Sci. Paris 230* (1950), 508). Basing his work on that of Thom, he derives formulas for the Stiefel classes of a manifold X which involve only the cohomology ring of X, and the squaring operations. These formulas enable one to compute easily the Stiefel classes in special cases. In particular they give a quick proof that $c^2(X) = 0$ when X is an orientable 3-manifold (see §39.9).

10. The theory of characteristic classes. Recent developments have produced a change in the point of view on characteristic classes. They are no longer regarded primarily as obstructions to cross-sectioning suitable bundles. The new attitude is based on the theorem of §19 that a bundle over a complex K with group G is uniquely determined by the universal bundle $B_G \to X_G$ and a mapping $f\colon\ K \to X_G$. Letting H^* denote the cohomology ring, the image of $H^*(X_G)$ in $H^*(K)$ under f^* is called the characteristic ring of the bundle. A set of generators of $H^*(X_G)$ are called *universal* characteristic classes, and their images in $H^*(K)$ are called the characteristic classes of the bundle.

This procedure presupposes the ability to compute successfully $H^*(X_G)$. The work of A. Borel in this direction has been very important (Sur la cohomologie des espaces fibrés principaux et des espaces homogenes de groupes de Lie compacts, *Annals of Math. 57* (1953), 115–207). He has extended greatly the results of Hopf concerning the structure of $H^*(G)$, and has applied the spectral sequence technique of Leray to obtain theorems on the structure of $H^*(X_G)$. In case G is an orthogonal group, then $H^*(X_G)$ with coefficients mod 2 is a polynomial ring whose generators are the universal Whitney classes. If G is a unitary group, then $H^*(X_G)$ with integer coefficients is a polynomial ring generated by the universal Chern classes. Other special cases have led to new characteristic classes such as the Pontrjagin classes associated with the special orthogonal groups.

This new approach has been fruitful in the applications of fibre bundle theory to differential geometry, complex manifolds and algebraic varieties. For a survey of these see the paper of A. Borel referred to in §5 above, and the monograph of F. Hirzebruch: Neue topologische Methoden in der algebraischen Geometrie, *Ergeb. der Math.*, Springer (1956), Berlin.

11. Secondary obstructions. In case the primary obstruction to finding a cross-section of a bundle is zero, the secondary obstruction is defined and is a set of cohomology classes. For sphere bundles, the secondary obstruction has been analysed successfully by S. D. Liao: On the theory of obstructions of fiber bundles, *Annals of Math. 60* (1954), 146–191. Another special case has been treated by E. G. Kundert: Über Schnittflächen in speziellen Faserungen und Felder reeller und komplexer Linienelemente, *Annals of Math. 54* (1951), 215–246. A general treatment of primary and secondary obstructions is given in the monograph of V. A. Boltyanskii: Homology theory of mappings and vector fields, (in Russian), *Trudy Mat. Inst. Steklov*, no. 47 (1955).

12. Fields of line elements. The subject of tangent fields of line elements has been thoroughly analysed by L. Marcus: Line element fields and Lorentz structures on differentiable manifolds, *Annals of Math. 62* (1955), 411–417. In particular, the statement on p. 207, lines 12b and 13b, must be corrected by deleting the word "twice." See also H. Samelson, A theorem on differentiable manifolds, *Portugaliae Math. 10* (1951), 129–133.