Part III. The Cohomology Theory of Bundles

§29. THE STEPWISE EXTENSION OF A CROSS-SECTION

29.1. Extendability when Y is q-connected. We turn now to the application of cohomology theory to the problem of constructing a cross-section of a bundle. It will be assumed throughout that the base space of the bundle $\mathfrak{B}$ is a finite complex K as defined in §19.1. The problem of constructing a cross-section is a special case of that of extending a cross-section f already given over a subspace of K. We assume that the subspace is a subcomplex L.

If L does not contain all of the 0-dimensional skeleton K^0 of K, then f can be extended continuously over $L \cup K^0$ by defining $f(x)$ in Y_x arbitrarily for each vertex x not in L.

Assuming that f is given on $L \cup K^0$, consider the problem of extending over $L \cup K^1$. If f can be extended over $L \cup K^1$ then, for each 1-cell σ of K not in L, the cross-section $f|\dot{\sigma}$ ($\dot{\sigma}$ = boundary of σ) can be extended over σ. Conversely, a set of extensions over the individual 1-cells form an extension over $L \cup K^1$; for the interiors of the 1-cells are disjoint open sets of $L \cup K^1$.

Let σ be a 1-cell not in L, and $\mathfrak{B}_\sigma$ the part of $\mathfrak{B}$ over σ. Since σ is contractible on itself to a point, $\mathfrak{B}_\sigma$ is a product bundle (§11.6). Therefore we have a bundle map

$$\phi_\sigma\colon\quad \sigma \times Y \to \mathfrak{B}_\sigma \tag{1}$$

and a map

$$p_\sigma\colon\quad B_\sigma \to Y \tag{2}$$

such that

$$\phi_\sigma(p(b),p_\sigma(b)) = b \qquad \text{for } b \in B_\sigma. \tag{3}$$

The composition of $f|\dot{\sigma}$ and p_σ is a map $f_{\dot{\sigma}}\colon\ \dot{\sigma} \to Y$. If f extends over σ, then $p_\sigma f|\sigma$ extends $f_{\dot{\sigma}}$ over σ. Conversely, if $f_{\dot{\sigma}}$ extends over σ to a map f_σ, then $f(x) = \phi_\sigma(x,f_\sigma(x))$ extends f over σ. Since σ is a 1-cell, the extendability of $f_{\dot{\sigma}}$ is just the question of whether the images of the two vertices of σ can be joined by a curve in Y. Thus, *if Y is arcwise connected, each $f_{\dot{\sigma}}$ can be extended, and the cross-section f on $L \cup K^0$ can be extended over $L \cup K^1$.*

Suppose, in general, a cross-section f is given on $L \cup K^q$, and we consider the problem of extending it over $L \cup K^{q+1}$. Again the problem reduces to extending, over each $(q+1)$-cell σ of K not in L, the cross-section $f|\dot{\sigma}$. Since σ is contractible, we may choose a product representation as in (1) and a projection (2) satisfying (3). Setting $f_{\dot{\sigma}} = p_\sigma[f|\dot{\sigma}]$, we reduce the problem to extending $f_{\dot{\sigma}}\colon\ \dot{\sigma} \to Y$ to a map $f_\sigma'\colon\ \sigma \to Y$. Again there is a blanket assumption which permits the extension, namely, the homotopy group $\pi_q(Y,y_0) = 0$ for each base point y_0 (if Y is arcwise connected, it suffices to impose this for a single base point, and we write $\pi_q(Y) = 0$).

We say that a space Y is *q-connected* ($q \geqq 0$) if it is arcwise connected and $\pi_i(Y) = 0$ for $i = 1, \cdots, q$.

With this definition, we may summarize the preceding argument in

29.2. Theorem. *If f is a cross-section of the part of $\mathfrak{B}$ over a subcomplex L of K, then f may be extended over $L \cup K^0$. If Y is q-connected, then f may be extended over $L \cup K^{q+1}$.*

29.3. Corollary. *If Y is q-connected, and $K - L$ has dimension $\leqq q + 1$, then any cross-section over L may be extended over K.*

Note that a solid space (§12.1) is q-connected for every q. The above corollary is therefore a sharpening of §12.2.

29.4. The obstruction cocycle. The stepwise extension of a cross-section does not lead to an interesting situation until a dimension q is reached for which $\pi_q(Y)$ is not zero. Suppose then that f is given on $L \cup K^q$. For any $(q+1)$-cell σ the preceding construction yields a map $f_{\dot{\sigma}}\colon\ \dot{\sigma} \to Y$ whose extendability over σ is equivalent to that of f. Assuming $\pi_q(Y) \neq 0$, we meet with an *obstruction* to extending f. We propose to measure this obstruction. The next few articles are devoted to the measuring procedure. Roughly it runs as follows. An orientation of σ induces one of $\dot{\sigma}$; the latter and $f_{\dot{\sigma}}$ determine an element of $\pi_q(Y)$ denoted by $c(f,\sigma)$. This function of oriented $(q+1)$-cells proves to be a cocycle. If f is altered on the q-cells of $K - L$, the cocycle varies by a coboundary. We thus arrive at a cohomology class of K mod L which gives a precise measure of the obstruction. Its vanishing is necessary and sufficient for $f|L \cup K^{q-1}$ to be extendable over $L \cup K^{q+1}$.

There are several difficulties. The first of these concerns the base point of $\pi_q(Y)$. This is eliminated by assuming that Y is q-simple (§16.5), so that any map of an oriented q-sphere in Y determines a unique element of $\pi_q(Y,y_0)$ for any y_0. That this assumption is not too restrictive is shown by §16.9 and §16.11. The first asserts that any group space is q-simple for all q. The second asserts the same for a coset space of a Lie group by a connected subgroup. Also $\pi_1 = 0$

implies q-simple for each q. Thus, the hypothesis is satisfied if we are dealing with a principal bundle, or with a sphere bundle, or with most of the bundles associated with a sphere bundle. It is probable that the hypothesis can be avoided since it has been avoided in a similar situation.

In the case $q = 1$, simplicity means that $\pi_1(Y)$ is abelian (§16.4). Thus, in every case, $\pi_q(Y)$ is abelian. The case $q = 0$ will be included by defining $\pi_0(Y)$ to be the reduced 0-dimensional singular homology group $H_0(Y)$, i.e. only such 0-cycles are used as have a coefficient sum of zero. Then a map of the boundary of an oriented 1-cell σ into Y determines an element of $\pi_0(Y)$ whose vanishing is necessary and sufficient for the extendability of the map over σ. In the case $Y = G$, this convention supersedes the convention $\pi_0(G) = G/G_e$ of §16.10.

There is a second and more serious difficulty. The map $f_{\dot\sigma}$ depends on the choice of the product representation (1). A different choice may lead to an entirely different element of $\pi_q(Y)$. As will be shown there are several different assumptions which eliminate this difficulty, e.g. G is connected, or $\pi_1(K) = 0$, or that the characteristic class $\chi\colon\ \pi_1(K) \rightarrow G/G_e$ is zero. But this would also eliminate important cases, e.g. tensor bundles over non-orientable manifolds. Moreover, it is possible to circumvent the difficulty without restrictions. This is accomplished by an altered procedure, and an elaboration of the cohomology theory to be used. We discuss now the altered procedure.

29.5. *The cross-section $f|\dot\sigma$ can be extended to a cross-section over σ if and only if it can be extended to a map of σ into B_σ.*

Half of the assertion is trivial. Suppose $f'\colon\ \sigma \rightarrow B_\sigma$ is an extension of $f|\dot\sigma$, but is not necessarily a cross-section. Choose a product representation of B_σ as in (1), (2) and (3) of §29.1. Define

$$f''(x) = \phi_\sigma(x, p_\sigma f'(x)), \qquad x \in \sigma.$$

Then $f''|\dot\sigma = f'|\dot\sigma = f|\dot\sigma$, and $pf''(x) = x$. Hence f'' is a cross-section extending $f|\dot\sigma$.

29.6. *If Y_x is the fibre over a point x of σ, then the inclusion map of Y_x in B_σ induces an isomorphism $\pi_q(Y_x) \approx \pi_q(B_\sigma)$. Therefore $f|\dot\sigma$ is homotopic in B_σ to a map $f_{\dot\sigma}\colon\ \dot\sigma \rightarrow Y_x$. The homotopy class of $f_{\dot\sigma}$ in Y_x depends only on that of $f|\dot\sigma$ in B_σ; and $f_{\dot\sigma}$ is extendable to $f_\sigma\colon\ \sigma \rightarrow Y_x$ if and only if $f|\dot\sigma$ is extendable to a cross-section over σ.*

This is a trivial consequence of (i) the existence of a product representation $\sigma \times Y$ for B_σ, (ii) the vanishing of all homotopy groups of σ, and (iii) the result §17.8 on the homotopy groups of a product space.

A direct visualization is provided by choosing a contraction of σ on itself to the point x and picturing a covering homotopy which contracts B_σ into the fibre Y_σ. The latter deforms $f|\dot\sigma$ into $f_{\dot\sigma}$. If the contraction of $\dot\sigma$ over σ into x is chosen to sweep out each point of $\sigma - x$ just once (e.g. a radial contraction), then the covering homotopy of $f|\dot\sigma$ sweeps out an extension of $f|\dot\sigma$ to a single-valued continuous cross-section over $\sigma - x$. We thus obtain an extension with a singularity at x; and $f_{\dot\sigma}$ is clearly a measure of this singularity.

29.7. We choose now, in each $(q + 1)$-cell σ, a reference point x_σ, and denote by Y_σ the fibre over x_σ. Choose, for each $(q + 1)$-cell σ, an orientation; and denote by σ the oriented cell, and by $\dot\sigma$ the oriented boundary. If f is a cross-section over $L \cup K^q$, we define $c(f,\sigma)$ to be the element of $\pi_q(Y_\sigma)$ given by 29.6 applied to $f|\dot\sigma$. (We are assuming that Y is q-simple.) Then we have

29.8. *A cross-section f over $L \cup K^q$ is extendable over $L \cup K^{q+1}$ if and only if $c(f,\sigma) = 0$ for each $(q + 1)$-cell σ.*

Denote by $c(f)$ the function of σ given by $c(f)(\sigma) = c(f,\sigma)$. We call $c(f)$ *the obstruction cocycle of f.* The terminology anticipates showing that $c(f)$ is a cochain, in some sense, and proving that its coboundary is zero. A cochain in the usual sense is a function assigning to oriented cells elements of an abelian group—the same group for each cell. In the present case, the values of $c(f)$ lie in different groups. It is true that they are all isomorphic, but there is no natural unique isomorphism between any two of them. Thus we meet, in a different form, the second of the difficulties described in §29.4. As promised there, we circumvent the difficulty by broadening the notion of cochain so as to allow functions such as $c(f)$. This requires a broadening of the related concepts of cohomology theory to which the next two articles are devoted.

§30. Bundles of Coefficients

30.1. Definitions. By a *bundle of groups* is meant a bundle with a fibre Y which is a group, and the group G acts as automorphisms of Y. As observed in §6.6, we can define, in each fibre Y_x, a group structure so that each admissible map $Y \to Y_x$ is an isomorphism.

By a *bundle of coefficients* (for homology or cohomology groups) is meant a bundle of groups where the fibre is an abelian group, written additively, and the group of the bundle is totally disconnected. The fibre will be denoted by π, and the group of the bundle (acting as automorphisms of π) by Γ.

Since Γ is totally disconnected, the results of §13 may be applied. In particular, any curve C from x_0 to x_1 in X determines an isomorphism $C^\#$: $\pi_1 \approx \pi_0$ which depends only on the homotopy class of C. If C' is a curve from x_1 to x_2, then $(CC')^\# = C^\# C'^\#$. The bundle is determined up to an equivalence by its characteristic homomorphism $\chi\colon \pi_1(X) \to \Gamma$ (X connected and locally connected).

The bundle of coefficients is called *simple* if it is a product bundle. According to §13.7, this happens if and only if χ is zero, i.e. each closed path operates as the identity.

Examples of bundles of coefficients are easily obtained. According to §13.8 we need only choose a group π and a homomorphism χ of $\pi_1(X)$ into the automorphism group of π to obtain one such. But certain bundles of coefficients arise naturally from other bundles.

30.2. The associated bundle $\mathcal{B}(\pi_q)$. The example of most importance to us is the following. Let $\mathcal{B} = \{B,p,X,Y,G\}$ be a bundle such that Y is q-simple. Then each fibre Y_x is also q-simple and the groups $\pi = \pi_q(Y)$ and $\pi_x = \pi_q(Y_x)$ are defined without reference to a base point in Y_x (e.g. as the group of homotopy classes of maps of a q-sphere into Y_x). Let Π be the union of the sets π_x for all x in X. Define ρ: $\Pi \to X$ so that ρ maps π_x into x for each x. We proceed to define a topology and bundle structure in Π.

For each coordinate neighborhood V_j of $\mathcal{B}$ define

$$\psi_j\colon \quad V_j \times \pi \to \rho^{-1}(V_j) \tag{1}$$

by

$$\psi_j(x,\alpha) = (\phi_{j,x})_*\alpha, \tag{2}$$

i.e. $\psi_j(x,\alpha)$ is the image of α under the isomorphism $\pi \to \pi_x$ induced by $\phi_{j,x}\colon \quad Y \to Y_x$. Define

$$\rho_j\colon \quad \rho^{-1}(V_j) \to \pi \tag{3}$$

by

$$\rho_j(\beta) = (p_j|Y_x)_*(\beta) \qquad \text{for } \beta \in \pi_x. \tag{4}$$

It is easily seen that $\rho\psi_j(x,\alpha) = x$, $\rho_j\psi_j(x,\alpha) = \alpha$ and $\psi_j(\rho(\beta),\rho_j(\beta)) = \beta$. It follows that (1) is a 1-1 map. If we set $\gamma_{ji}(x) = \rho_j\psi_{i,x}$, we have

$$\gamma_{ji}(x) = g_{ji}(x)_*. \tag{5}$$

Thus we have most of the elements of a bundle structure. As yet we do not have a topology in Π, a topology in π, and a group of transformations of π. The last is readily obtained. Each g in G induces an automorphism g_* of π. Let H be the subgroup which acts as the identity in π. Set $\Gamma = G/H$ and let $\eta\colon \quad G \to \Gamma$ be given by $\eta(g) = g_*$. Then Γ is an automorphism group of π, and by (5) the coordinate transformations of the projected bundle are in Γ.

It remains to assign topologies to π, Γ, and Π. We give to π and Γ

the discrete topologies. This choice leads to difficulties in the most general situation since we must show that (5) is continuous in x. We therefore make the restriction: *let X be locally arcwise connected.* Then, if $x_1 \varepsilon V_i \cap V_j$, there exists a neighborhood N of x_1 such that, for each x_2 in N there is a curve C in $V_i \cap V_j$ from x_1 to x_2. Since g_{ji} is continuous, it maps C into a curve in G from $g_{ji}(x_1)$ to $g_{ji}(x_2)$. Then $g_{ji}(x_1)$ and $g_{ji}(x_2)$ are homotopic maps of Y on itself. Therefore they induce the same homomorphism of π. This shows that γ_{ji} is constant over N, hence continuous over $V_i \cap V_j$.

There is an alternative restriction which is equally effective: *let the component G_e of e in G be arcwise connected and open in G.* It follows quickly that $G_e \subset H$, and G/G_e is discrete with the coset space topology. Since G/H is a coset space of G/G_e, it too has the discrete topology as a coset space of G. This implies that η: $G \to G/H$ is continuous when G/H is discrete. Then the continuity of g_{ji} implies that of γ_{ji}.

Finally we topologize Π so that the 1-1 maps ψ_j are homeomorphisms. There is just one way of doing this. For each α in Π we select a j such that $\rho(\alpha) \varepsilon V_j$, and define neighborhoods of α to be the images of neighborhoods of $(\rho(\alpha),\rho_j(\alpha))$ under ψ_j. That this yields a topology, under which ρ and each ψ_j,ρ_j are continuous, follows quickly from the continuity of γ_{ji}.

The bundle of coefficients so constructed is denoted by $\mathfrak{B}(\pi_q)$. It is defined whenever Y is q-simple, and X is locally arcwise connected. Since π is discrete, the bundle is a covering space of X (§14).

30.3. Lemma. *If C is a curve from x_1 to x_2 in X, and Y_{x_2} is translated along C^{-1} into Y_{x_1} in $\mathfrak{B}$, then the induced isomorphism of π_{x_2} into π_{x_1} coincides with the unique isomorphism $C^\#$ obtained by translating π_{x_2} along C^{-1} into π_{x_1} in the bundle $\mathfrak{B}(\pi_q)$.*

The proof is obtained quickly by first observing that it holds when C lies in some V_j, and then noting that any curve is a composition of a finite number of such curves.

30.4. Theorem. *Let $\mathfrak{B}$ be a bundle whose fibre Y is q-simple, and whose base space X is arcwise connected and arcwise locally connected. Suppose also that the component G_e of e in G is arcwise connected. If the characteristic class χ: $\pi_1(X) \to G/G_e$ of B is trivial, then the bundle $\mathfrak{B}(\pi_q)$ is a product bundle.*

The hypothesis on χ means that, if a fibre Y_0 is translated around a closed curve C into itself, the resulting self map is homotopic to the identity. By §30.3, this implies that $C^\#$ is the identity map of $\pi_q(Y_0)$. Hence the characteristic class χ: $\pi_1(X) \to \Gamma$ of $\mathfrak{B}(\pi_q)$ is also trivial. By §13.7, it must be a product bundle.

It is to be noted that formula (5) implies that $\mathfrak{B}(\pi_q)$ is the bundle weakly associated with $\mathfrak{B}$ relative to η: $G \to \Gamma$.

30.5. LEMMA. *Let h: $\mathfrak{B} \to \mathfrak{B}'$ be a bundle map where the common fibre Y is q-simple, and the base spaces X,X' are locally arcwise connected. If $x \varepsilon X$ and $x' = \bar{h}(x)$, let h_{x*} be the isomorphism $\pi_q(Y_x) \approx \pi_q(Y_{x'})$ induced by h_x. Then the maps h_{x*}, for all x in X, define a bundle map κ: $\mathfrak{B}(\pi_q) \to \mathfrak{B}'(\pi_q)$.*

If $x \varepsilon V_j \cap \bar{h}^{-1}(V_k')$, and $x' = \bar{h}(x)$, we have

$$(\phi'^{-1}_{k,x'})_* h_{x*} \phi_{j,x*} = (\phi'^{-1}_{k,x'} h_x \phi_{j,x})_* = \bar{g}_{kj}(x)_*.$$

The continuity of $\bar{g}_{kj}(x)_*$ in x is proved precisely as in the case of (5) above. By §2.6, there is a unique bundle map κ: $\mathfrak{B}(\pi_q) \to \mathfrak{B}'(\pi_q)$ corresponding to $\bar{h}$ and the $\bar{g}_{kj}(x)_*$. A brief glance at the construction of κ in §2.6 reveals that $\kappa_x = h_{x*}$ for each x; and the lemma is proved.

30.6. It is to be noted that the preceding construction of $\mathfrak{B}(\pi_q)$ can be carried through with the homology group $H_q(Y)$ replacing $\pi_q(Y)$. The only properties of $\pi_q(Y)$ used in the discussion are (i) it is an abelian group, (ii) any g in G induces an automorphism g_* of π, (iii) $(gg')_* = g_* g'_*$, and (iv) g_* depends only on the homotopy class of g. Since $H_q(Y)$ has the same four properties, the weakly associated bundle $\mathfrak{B}(H_q)$ can be defined in the same fashion.

The cohomology group $H^q(Y)$ satisfies all save (iii) which is replaced by $(gg')^* = g'^* g^*$. If we set $g^\# = (g^{-1})^*$, then $g \to g^\#$ is a homomorphism of G onto a group Γ of automorphisms of H^q. With $g^\#$ in place of g^*, the four properties hold, and we may define $\mathfrak{B}(H^q)$.

30.7. The bundle of homotopy groups. Another important example of a bundle of coefficients is provided by the homotopy groups $\pi_q(X,x)$ where X is arcwise connected, arcwise locally connected, and semi-locally 1-connected (§13.8).

We set $\pi_x = \pi_q(X,x)$, and let Π denote the union of the groups π_x for all x in X. Define ρ: $\Pi \to X$ by $\rho(\pi_x) = x$.

Let x_0 be a reference point, and let the fibre π be π_{x_0}. As shown in §16.4, $\pi_1(X,x_0)$ acts as a group of automorphisms of π. We define Γ to be the factor group of $\pi_1(X,x_0)$ by the subgroup which acts as the identity automorphism of π. We give to π and Γ the discrete topology.

For each x in X there exists a neighborhood V of x which is arcwise connected and such that any closed curve in V is homotopic to a point in X. For such a neighborhood V and any $\alpha \varepsilon \pi_x$, we define the *neighborhood* $V(\alpha)$ in Π to be the set of elements obtained by deforming α along curves in V, i.e. if C is a curve from x' to x in V, then $C^\#\alpha$ is in $V(\alpha)$.

Clearly ρ maps $V(\alpha)$ onto V. In fact, under the topology defined in Π by these neighborhoods, *ρ maps $V(\alpha)$ topologically onto V.* Suppose C_1,C_2 are two curves in V from x to x'. Then $C_1C_2^{-1}$ is contractible in X, and this implies that $C_1^\# = C_2^\#$. It follows that $\rho|V(\alpha)$ is 1-1. Since, for each V and α, ρ maps $V(\alpha)$ into V, ρ is continuous. If β is in $V(\alpha)$, and $V'(\beta) \subset V(\alpha)$, it is easily seen that $V' \subset V$. Hence $(\rho|V(\alpha))^{-1}$ maps V' into $V'(\beta)$. This proves that $\rho|V(\alpha)$ is topological.

We have thereby shown that ρ: $\Pi \to X$ is a covering in the sense of §14.1. By §14.3, it may be given a bundle structure. One checks readily that the characteristic class of the bundle is the natural map $\pi_1(X,x_0) \to \Gamma$.

§31. Cohomology Groups Based on a Bundle of Coefficients

31.1. Introductory remarks. This generalization of cohomology theory was given first by Reidemeister [81]. He called the bundle of coefficients an Überdeckung. Subsequently, an extensive survey was made by the author [87]. In the latter, the bundle of coefficients was called a *system of local coefficients*. Although their definitions differ, it is easily proved that, in a connected and semi-locally 1-connected space (e.g. a complex), a system of local coefficients is a bundle of coefficients.

The following treatment is restricted to cohomology. One may also treat homology theory with coefficients in a bundle. A reader, familiar with the parallelism between cohomology and homology, will be able to state and prove the corresponding facts about the latter. But these will not be used in the sequel.

We will assume that the reader is familiar with certain basic material concerning ordinary homology theory with integer coefficients in a complex. All such can be found in the book of Lefschetz [64]. We could avoid this and achieve greater simplicity if we dealt only with simplicial complexes and simplicial maps. But the needs of subsequent articles demand the use of the cell complex (of §19.1) and arbitrary continuous maps.

It is not generally realized that the satisfactory use of cell complexes in homology theory presupposes the theorem on the invariance of the homology groups. For example, if the cell complex K consists of a single n-cell and its faces, then, for $q > 0$, $H_q(K) = 0$ is a consequence of the invariance theorem; but, to my knowledge, is not provable in any other way. If K is a simplex, the fact is directly deducible from the definition of H_q in terms of cycles and boundaries. This difference accounts for the preferred treatment accorded simplicial complexes.

But one rarely computes the homology groups of spaces from simplicial decompositions. The number of simplexes required can be impractically large. For example a simplicial division of a torus requires a minimum of 42 elements. A cell decomposition with 16 elements can be given. In higher dimensions the discrepancy is greater. An n-simplex has $2^n - 1$ elements. A cellular decomposition of an n-cell need have only $2n + 1$ elements.

Still fewer cells are needed if one allows cell complexes in which identifications occur on the boundaries of the cells. These are frequently used to compute ordinary homology groups. For the generalization to be given, it seems to be necessary that the closed cells be at least simply connected. However we shall adhere to the definition of §19.1.

31.2. Cochains, cocycles and cohomology. Let K be a finite cell complex and let $X = |K|$ be the space of K. Let $\mathfrak{B} = \{\Pi,\rho,X,\pi,\Gamma\}$ be a bundle of coefficients over X (§30.1).

For each $q \geqq 0$ and each q-cell σ of K, we choose a reference point x_σ in σ, and denote by π_σ the fibre of $\mathfrak{B}$ over x_σ. We call π_σ *the coefficient group of* σ.

A *q-cochain of K with coefficients in* $\mathfrak{B}$ is a function c which attaches to each oriented q-cell σ an element $c(\sigma)$ of π_σ and satisfies $c(-\sigma) = -c(\sigma)$ where $-\sigma$ denotes the orientation opposite to that of σ. The q-cochain c is said to be zero on a subcomplex L if $c(\sigma) = 0$ for each σ in L. We add cochains by adding functional values:

$$(c_1 + c_2)(\sigma) = c_1(\sigma) + c_2(\sigma). \tag{1}$$

It follows that the q-cochains form an additive abelian group denoted by $C^q(K;\mathfrak{B})$. Those which are zero on L form a subgroup $C^q(K,L;\mathfrak{B})$.

If we choose a fixed orientation of each cell, then a choice of one element from π_σ for each q-cell σ determines a unique q-cochain which on each oriented σ has the prescribed value in π_σ. It follows that $C^q(K,L;\mathfrak{B})$ is isomorphic to the direct sum $\Sigma\pi_\sigma$ for q-cells σ in $K - L$.

Choose now for each cell σ a fixed reference orientation and let σ also denote the oriented cell. If σ is a q-cell and is a face of the $(q + 1)$-cell τ (written: $\sigma < \tau$), let $[\sigma:\tau] = \pm 1$ denote the incidence number of σ and τ. If σ is a q-face of the $(q + 2)$-cell ξ, and τ, τ' are the two $(q + 1)$-cells such that $\sigma < \tau < \xi$, $\sigma < \tau' < \xi$, then we have the usual relation

$$[\sigma:\tau][\tau:\xi] + [\sigma:\tau'][\tau':\xi] = 0. \tag{2}$$

For each relation $\sigma < \tau$, we choose a curve C in τ from x_τ to x_σ and denote by $w_{\sigma\tau}$ the isomorphism $C^\#$ of π_σ onto π_τ. Note that w is independent of the choice of C since τ is simply-connected. It follows that $\sigma < \tau < \xi$ implies

$$w_{\tau\xi}w_{\sigma\tau} = w_{\sigma\xi}. \tag{3}$$

For any q-cochain c we define its *coboundary* δc in $C^{q+1}(K;\mathcal{B})$ by

$$\delta c(\tau) = \Sigma[\sigma:\tau]w_{\sigma\tau}(c(\sigma)) \tag{4}$$

where τ is a $(q+1)$-cell and the sum is extended over all q-faces σ of τ. Since each w is a homomorphism, it follows from (1) that δ is a homomorphism. If c is zero on L, it is clear that δc is zero on L. Hence

$$\delta:\quad C^q(K,L;\mathcal{B}) \rightarrow C^{q+1}(K,L;\mathcal{B}). \tag{5}$$

If the orientation of σ is reversed, both $[\sigma:\tau]$ and $c(\sigma)$ change sign and $\delta c(\tau)$ remains unchanged. If we reverse that of τ, both sides of (4) change sign. It follows that δ is independent of the choice of the reference orientations.

If we calculate $\delta\delta c$ from (4), apply (3) and then (2), we arrive at the basic relation

$$\delta\delta = 0. \tag{6}$$

The kernel of (5), denoted by $Z^q(K,L;\mathcal{B})$, is called *the group of q-cocycles of K mod L with coefficients in* $\mathcal{B}$. The image of (5), denoted by $B^{q+1}(K,L;\mathcal{B})$ is called *the group of (q + 1)-coboundaries of K mod L*. By (6), we have

$$Z^q(K,L;\mathcal{B}) \supset B^q(K,L;\mathcal{B}).$$

We define *the qth cohomology group of K mod L with coefficients in* $\mathcal{B}$ by

$$H^q(K,L;\mathcal{B}) = Z^q(K,L;\mathcal{B})/B^q(K,L;\mathcal{B}).$$

To have a proper definition for all $q \geqq 0$, we define $C^q(K,L;\mathcal{B}) = 0$ when $K - L$ has no q-cells, and we set $B^0(K,L;\mathcal{B}) = 0$.

The cohomology groups are independent of the choice of the base points x_σ. For suppose x'_σ is a second set of choices. Choose a path in σ from x'_σ to x_σ and use it to define an isomorphism u_σ: $\pi_\sigma \rightarrow \pi'_\sigma$. Use these to map each old cochain into a new one. This gives an isomorphism of the old group of cochains onto the new one. If $\sigma < \tau$, then $w'_{\sigma\tau}u_\sigma$ and $u_\tau w_{\sigma\tau}$ are both isomorphisms of π_σ onto π'_τ induced by traversing curves in τ; so they must be equal. From this it follows that the isomorphisms of the old cochains onto the new ones commute with δ. Hence they induce isomorphisms of the respective cohomology groups.

31.3. Simple coefficients. Whenever the bundle $\mathcal{B}$ of coefficients is a product bundle, the cohomology groups $H^q(K,L;\mathcal{B})$ reduce in a natural way to the ordinary cohomology groups $H^q(K,L;\pi)$. This is proved as follows.

Choose a bundle map μ: $\mathfrak{B} \to \pi$ which exists since $\mathfrak{B}$ is a product. Then μ maps each π_σ isomorphically onto π, and in such a way that $\mu w_{\sigma\tau}|\pi_\sigma = \mu|\pi_\sigma$ for each $\sigma < \tau$. Using μ we obtain, quickly, isomorphisms $C^q(K,L;\mathfrak{B}) \approx C^q(K,L;\pi)$ for each q which commute with δ. They induce therefore isomorphisms of the cohomology groups.

31.4. The Kronecker index. In the case of ordinary cochains with coefficients in π, one has the notion of a Kronecker index. Let c be a p-cochain with coefficients in π, and let $z = \Sigma_{i=1}^r a_i\sigma_i$ be a p-chain with integer coefficients. Then the *Kronecker index* $c\cdot z$ in π is defined by

$$c\cdot z = \sum_{i=1}^{r} a_i c(\sigma_i). \tag{7}$$

Clearly, $c\cdot z$ is bilinear. Furthermore it is readily shown that

$$(\delta c)\cdot z = c\cdot(\partial z) \tag{8}$$

when c is a $(p-1)$-cochain and z is a p-chain.

Passing to the case of a bundle $\mathfrak{B}$ of coefficients, let $c \in C^p(K;\mathfrak{B})$, and let z be a p-chain with ordinary integer coefficients. We note that (7) has no meaning because the various $c(\sigma_i)$ lie in different groups. It is possible to bring them together into the same group by translating along curves of X into a single fibre. If $\mathfrak{B}$ is not a product, the result will depend on the choice of the curves. Thus, we must abandon a Kronecker index in the usual sense.

However, if the chain z lies on a subcomplex E of K such that $\mathfrak{B}|E$ is a product bundle, then the terms of (7) can be accumulated in a single fibre, in just one way, by using curves in E. Adding them in this fibre determines a unique value of $c\cdot z$. It is easily seen that the values of $c\cdot z$ obtained in the various fibres of $\mathfrak{B}|E$ correspond to one another under translation along curves in E. We are thus led to a bilinear operation, called the Kronecker index, which pairs $C^p(K;\mathfrak{B})$ and $C_p(E)$ to the cross-sections of $\mathfrak{B}|E$.

Since $\mathfrak{B}|E$ is a product, a cochain c' on E is an ordinary cochain, and $c'\cdot z$ is definable as usual. A cochain c of K with coefficients in $\mathfrak{B}$ determines a cochain c' on E by restricting c to cells of E. It is clear that $c\cdot z = c'\cdot z$ under this correspondence. It follows that (8) holds for the extended Kronecker index.

Any closed cell σ of K and its faces form a subcomplex E such that $\mathfrak{B}|E$ is a product. In this sense, we have

$$c(\sigma) = c\cdot\sigma. \tag{9}$$

And (8) gives

$$\delta c(\sigma) = c\cdot\partial\sigma. \tag{10}$$

Thus, we always have a "local" Kronecker index with the usual properties.

31.5. Carrier of a mapping. A *carrier* for a continuous function h: $K \to K'$ of one cell complex into another is a function assigning to each cell σ of K a closed subcomplex E_σ of K' such that $h(\sigma) \subset E_\sigma$ and $\sigma < \tau$ implies $E_\sigma \subset E_\tau$. A carrier for a map h: $(K,L) \to (K',L')$ is required to satisfy the additional condition that $E_\sigma \subset L'$ when σ is in L.

The intersection of two carriers of h is again a carrier. If, for each σ, E_σ is the smallest closed subcomplex containing $h(\sigma)$, then $\{E_\sigma\}$ is called the minimal carrier. It is contained in every carrier.

A carrier $\{E_\sigma\}$ of h is called *solid* if, for each σ, E_σ is contractible to a point (see §12.1). This implies that E_σ is connected, simply-connected, and the homology groups $H_q(E_\sigma)$, with integer coefficients, are zero for $q = 1, 2, \cdots$. The last means that each q-cycle on E_σ is a boundary. Since E_σ is connected, a 0-cycle of the form $v_1 - v_2$, where v_1,v_2 are vertices of E_σ, bounds a 1-chain on E_σ.

31.6. Chain homomorphisms and homotopies. Let $\{E_\sigma\}$ be a solid carrier for h. Then *there exist chain homomorphisms (integer coefficients)*

(11) $$h_\#\colon \quad C_q(K) \to C_q(K'), \qquad q = 0, 1, \cdots,$$

such that $h_\#$ carries a vertex into a vertex,

(12) $$h_\#\sigma \subset E_\sigma, \qquad \text{and} \qquad \partial h_\# = h_\#\partial.$$

The proof proceeds by induction. To each vertex v of K we assign a vertex $h_\# v$ of E_v. We then extend $h_\#$ to all 0-chains by the requirement of linearity. Assuming that (11) is defined for $q < p$, we choose a base for $C_p(K)$ consisting of one orientation of each p-cell. If σ is a base element, then (12) implies that $h_\#\partial\sigma$ is a $(p-1)$-cycle on E_σ. Since E_σ is solid, we can choose a p-chain on E_σ whose boundary is $h_\#\partial\sigma$ and denote it by $h_\#\sigma$. We then extend $h_\#$ to all p-chains by the requirement of linearity. This completes the general step of the induction.

Any two chain homomorphisms $h_\#,h'_\#$, having the same solid carrier $\{E_\sigma\}$, are chain homotopic, i.e. there exist homomorphisms

(13) $$D\colon \quad C_q(K) \to C_{q+1}(K'), \qquad q = 0, 1, \cdots,$$

such that

(14) $$D\sigma \subset E_\sigma, \qquad \text{and} \qquad \partial Dz = h'_\# z - h_\# z - D\partial z$$

for any chain z.

This again is proved by induction. We define Dv, for a vertex v, to be a 1-chain of E_v whose boundary is $h'_\# v - h_\# v$. Assuming D

defined for $q < p$, then, for any base element σ of $C_p(K)$, by applying (14) with $z = \partial\sigma$, we find that $h'_{\#}\sigma - h_{\#}\sigma - D\partial\sigma$ is a p-cycle on E_σ. We let $D\sigma$ be a $(p+1)$-chain of E_σ which has it for boundary.

31.7. Induced homomorphisms. Let $\mathfrak{B},\mathfrak{B}'$ be bundles of coefficients over K,K', and let h: $\mathfrak{B} \to \mathfrak{B}'$ be a bundle map such that the induced map $\bar{h}$: $(K,L) \to (K',L')$ has a solid carrier $\{E_\sigma\}$. Select a chain homomorphism $h_{\#}$ as above.

Let c' be an element of $C^q(K',L';\mathfrak{B}')$, and let σ be an oriented q-cell. Since E_σ is simply connected, $\mathfrak{B}'|E_\sigma$ is a product bundle; hence, we can form the Kronecker index $c'\cdot h_{\#}\sigma$, and it is a cross-section of $\mathfrak{B}'|E_\sigma$ (see §31.4). Let h_σ denote the isomorphism of the coefficient group π_σ onto the fibre of $\mathfrak{B}'$ over $\bar{h}(x_\sigma)$. Then we define $h^{\#}c'$ by

(15) $$h^{\#}c'(\sigma) = h_\sigma^{-1}(c'\cdot h_{\#}\sigma).$$

It is readily checked that $h^{\#}$ is a homomorphism

(16) $$h^{\#}\colon\quad C^q(K',L';\mathfrak{B}') \to C^q(K,L;\mathfrak{B}).$$

If c' is a $(q-1)$-cochain of K', and σ is a q-cell of K, then (12) and (8) yield

$$\begin{aligned}(h^{\#}\delta c')(\sigma) &= h_\sigma^{-1}(\delta c'\cdot h_{\#}\sigma) = h_\sigma^{-1}(c'\cdot\partial h_{\#}\sigma)\\ &= h_\sigma^{-1}(c'\cdot h_{\#}\partial\sigma) = h^{\#}c'\cdot\partial\sigma = (\delta h^{\#}c')(\sigma).\end{aligned}$$

This proves

(17) $$\delta h^{\#} = h^{\#}\delta.$$

It follows that $h^{\#}$ carries cocycles into cocycles and coboundaries into coboundaries, thereby inducing a homomorphism

(18) $$h^*\colon\quad H^q(K',L';\mathfrak{B}') \to H^q(K,L;\mathfrak{B}).$$

31.8. The uniqueness of h^* for proper maps. The definition of h^* depends on the choice of $\{E_\sigma\}$, and on the choice of $h_{\#}$. Consider first the latter choice. Let $h_{\#},h'_{\#}$ be any two such choices; then a chain homotopy (13) may be chosen. Define a corresponding *cochain homotopy*

(19) $$D\colon\quad C^q(K',L';\mathfrak{B}') \to C^{q-1}(K,L;\mathfrak{B})$$

by using the local Kronecker index:

(20) $$Dc'(\sigma) = h_\sigma^{-1}(c'\cdot D\sigma).$$

Using (14) we obtain

(21) $$\delta Dc' = h'^{\#}c' - h^{\#}c' - D\delta c'.$$

This implies, for a cocycle c', that $h'^{\#}c'$ and $h^{\#}c'$ belong to the same cohomology class. Thus we have proved that h^* is independent of the choice of $h_{\#}$.

If we had started with two maps h and h' having the common solid carrier $\{E_\sigma\}$, the foregoing argument proves that $h^* = h'^*$

In general, h^* depends on the choice of the solid carrier. We shall say that h is a *proper* map if the minimal carrier is solid. Then we can define h^* using the minimal carrier. Since any solid carrier contains the minimal carrier, any h^* coincides with the one assigned to the minimal carrier. *Thus, for proper maps, h^* is unique and can be constructed from any solid carrier.*

It is to be noted that any inclusion map $(K,L) \to (K',L')$ is proper where K is a subcomplex of K' and L is a subcomplex of L'. The minimal carrier of σ consists of σ and its faces.

If K,K' are simplicial complexes, and $\bar{h}$ is simplicial, then $\bar{h}(\sigma)$ is a simplex and is therefore solid. Thus simplicial maps are proper.

There is a wide class of proper maps for which the $h_{\#}$ assigned to the minimal carrier is unique. The map $\bar{h}$: $K \to K'$ is called *cellular* if, for each q, $\bar{h}$ maps the q-skeleton K^q of K into K'^q. Suppose that $\bar{h}$ is both proper and cellular. For any q-cell σ of K, the minimal carrier E_σ is a subcomplex of K'^q, and is thereby q-dimensional. Then $H_q(E_\sigma) = 0$ means $Z_q(E_\sigma) = 0$; for, the absence of $(q+1)$-cells in E_σ means $B_q(E_\sigma) = 0$. But $Z_q(E_\sigma)$ is the kernel of ∂: $C_q(E_\sigma) \to C_{q-1}(E_\sigma)$. Since this kernel is zero, there can be at most one q-chain of E_σ whose boundary is $h_{\#}\partial\sigma$. Thus, at each stage of the inductive construction of $h_{\#}$, the choice is unique:

If $\bar{h}$: $K \to K'$ is both proper and cellular, then there is just one $h_{\#}$ associated with the minimal carrier.

31.9. Subdivision. Let (K',L') be a subdivision of (K,L), i.e. their spaces coincide, and each cell of K (L) is the union of the cells of K' (L') which it contains. We assert that *the inclusion maps*

$$h:\quad (K,L) \to (K',L'), \qquad h':\quad (K',L') \to (K,L) \tag{22}$$

are proper. In the case of h, the minimal carrier of σ is the subdivision in K' of the complex composed of σ and its faces. Denote this by $Sd\ \sigma$. Since any subdivision of a cell is solid, h is proper. If σ' is in K', its interior lies in the interior of just one cell σ of K; then its minimal carrier $E_{\sigma'}$ consists of σ and its faces. Thus h' is proper.

Using these carriers, choose $h_{\#}$ and $h'_{\#}$ and let $h^{\#},h'^{\#}$, as in (16), be defined accordingly. Let i be the identity map of (K,L) and let $i_{\#},i^{\#}$ be identity maps of chains and cochains. Now $i_{\#}$ and $h'_{\#}h_{\#}$ have in common the minimal solid carrier $\{E_\sigma\}$ where E_σ consists of σ and its faces. Since i is cellular, the uniqueness statement of 31.8 yields $i_{\#} = h'_{\#}h_{\#}$. It follows that $i^* = h^*h'^*$.

Letting i' denote the identity map of (K',L'), we find that $i'_{\#}$ and $h_{\#}h'_{\#}$ have the common solid carrier $\{Sd\ E_{\sigma'}\}$. Hence they are chain homotopic. As in §31.8, this implies $i'^* = h'^*h^*$. Thus we have proved

INVARIANCE UNDER SUBDIVISION: *If (K',L') is a subdivision of (K,L), then the inclusion maps of (22) are proper, and they induce isomorphisms*

$$H^q(K,L;\mathcal{B}) \approx H^q(K',L';\mathcal{B}).$$

31.10. The h^* of a general map. It is important to define h^* when h is not proper. The procedure is to factor h into the composition of three proper maps

$$(23) \qquad (K,L) \xrightarrow{i} (K_1,L_1) \xrightarrow{h'} (K'_1,L'_1) \xrightarrow{i'} (K',L')$$

where (K_1,L_1) is a simplicial subdivision of (K,L), (K'_1,L'_1) is a simplicial subdivision of (K',L'), i and i' are inclusion maps and $h'(x) = h(x)$ for each x.

The first regular subdivision of K is simplicial. It is constructed, inductively, by introducing one new vertex on each σ and subdividing σ into the join of this vertex with the subdivision of the boundary of σ. Thus, simplicial subdivisions can be found. Arbitrarily fine subdivisions can be found by repeated barycentric subdivisions of the regular subdivision.

The existence of (23) is proved as follows. Let (K'_1,L'_1) be any simplicial subdivision of (K',L'). The open stars of vertices of K'_1 cover K'. Choose a simplicial subdivision of (K_1,L_1) of (K,L) so fine that the image of each simplex of K_1 lies in the open star of a vertex of K'_1. If $h(\sigma)$ lies in the star of v, then each closed simplex of K'_1 which meets $h(\sigma)$ has v as a vertex. Therefore their union is contractible to v. But this union is the minimal carrier of $h(\sigma)$. It follows that h' is proper. By §31.9, i and i' are proper.

Assuming now that h: $\mathcal{B} \to \mathcal{B}'$ is a map of bundles of coefficients, we choose a factorization (23), and define h^* by

$$(24) \qquad h^* = i^*h'^*i'^*.$$

Various facts must now be proved to insure that this definition of h^* is satisfactory. We list these without proofs since the proofs are simple applications of the cochain homotopy construction of §31.8.

31.11. *The h^* of (24) is independent of the choice of the factorization (23).*

31.12. *If h is a proper map, the h^* defined in §31.7 coincides with the h^* of (24).*

31.13. *If h: $\mathfrak{B} \to \mathfrak{B}'$ and h': $\mathfrak{B}' \to \mathfrak{B}''$, then $(h'h)^* = h^*h'^*$.*

31.14. *The identity map $\mathfrak{B} \to \mathfrak{B}$ induces the identity map of $H^q(K,L;\mathfrak{B})$.*

If h: $\mathfrak{B} \to \mathfrak{B}'$ induces a homeomorphism $\bar{h}$: $(K,L) \to (K',L')$, the last two propositions imply that h^* is an isomorphism. This is a precise formulation of the statement: *$H^q(K,L;\mathfrak{B})$ is a topological invariant.*

31.15. *If h_0,h_1: $\mathfrak{B} \to \mathfrak{B}'$ are homotopic maps, then $h_0^* = h_1^*$.*

Let h: $\mathfrak{B} \times I \to \mathfrak{B}'$ be the homotopy. Let h'_0,h'_1: $\mathfrak{B} \to \mathfrak{B} \times I$ be defined by $h'_0(x) = (x,0)$ and $h'_1(x) = (x,1)$. Then $h_i = hh'_i$ $(i = 0,1)$. By §31.13, it suffices to prove that $h'_0{}^* = h'_1{}^*$. But, relative to the product complex $K \times I$ (see §19.1), h'_0 and h'_1 have the common solid carrier $E_\sigma = \sigma \times I$. We may therefore choose $h'_{0\#} = h'_{1\#}$, and the desired result follows.

31.16. In the factorization (23), the map i is cellular as well as proper while h' and i' need not be cellular. Suppose the subdivision K_1 of K is chosen so fine that, for any vertex v of K_1, the image of the star of v lies in the star of some vertex v' of K'_1. As is well known (the simplicial approximation theorem), setting $k(v) = v'$ determines a unique simplicial map k: $(K_1,L_1) \to (K'_1,L'_1)$ such that $k(x)$ lies in the closure of the smallest simplex containing $h(x)$. Then $k \simeq h$, and the minimal carrier of k is contained in that of h'. The latter implies that $h'_\#$ can be chosen to be the unique $k_\#$ (k is cellular and proper). Then $h'^* = k^*$.

If K'_1 is not K', then i' is not cellular; however its inverse i'' is cellular, and i''^* is the inverse of i'^*. Then (24) becomes

$$h^* = i^*k^*(i''^*)^{-1}. \tag{24'}$$

We have proved:

Any h^* can be factored into the form (24′) where i,k and i'' are proper cellular maps.

This result enables us to extend to the general h^* those of its properties proved when h is proper and cellular.

31.17. The coboundary operator. Let $\mathfrak{B}$ be a bundle of coefficients over (K,L) and let i and j be the inclusion maps

$$L \xrightarrow{i} K \xrightarrow{j} (K,L).$$

These maps are proper and cellular, so $i_\#$ and $j_\#$ are unique. These determine homomorphisms

$$0 \to C^q(K,L;\mathcal{B}) \xrightarrow{j^\#} C^q(K;\mathcal{B}) \xrightarrow{i^\#} C^q(L;\mathcal{B}) \to 0. \tag{25}$$

Direct interpretations are as follows. $C^q(K,L;\mathcal{B})$ is the subgroup of $C^q(K;\mathcal{B})$ consisting of cochains which are zero on L; and $j^\#$ is the inclusion map. For any c in $C^q(K;\mathcal{B})$, $i^\# c$ is the cochain of L obtained by restricting c to cells of L. The image of $i^\#$ is $C^q(L;\mathcal{B})$ since a function of cells of L can be extended to a function of cells of K by assigning values to the cells of $K - L$. It follows that the sequence (25) is exact.

Consider now the diagram

$$\begin{array}{ccccccccc} 0 & \to & C^q(K,L;\mathcal{B}) & \xrightarrow{j^\#} & C^q(K;\mathcal{B}) & \xrightarrow{i^\#} & C^q(L;\mathcal{B}) & \to & 0 \\ & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\ 0 & \to & C^{q+1}(K,L;\mathcal{B}) & \xrightarrow{j^\#} & C^{q+1}(K;\mathcal{B}) & \xrightarrow{i^\#} & C^{q+1}(L;\mathcal{B}) & \to & 0. \end{array} \tag{26}$$

If c is in $Z^q(L;\mathcal{B})$, i.e. $\delta c = 0$, we choose an extension c' of c in $C^q(K;\mathcal{B})$. Then

$$i^\#\delta c' = \delta i^\# c' = \delta c = 0.$$

Therefore $\delta c'$ lies in $C^{q+1}(K,L;\mathcal{B})$. Since $\delta\delta c' = 0$, $\delta c'$ is a cocycle. If, also, $i^\# c'' = c$. Then $\delta(c' - c'') = \delta c' - \delta c''$, and $c' - c''$ is zero on L. Thus the cohomology class in $H^{q+1}(K,L;\mathcal{B})$ of $\delta c'$ is independent of the extension c'. We obtain thus a unique homomorphism $Z^q(L;\mathcal{B}) \to H^{q+1}(K,L;\mathcal{B})$. It is easily proved that it carries $B^{q+1}(L;\mathcal{B})$ into zero. It thereby induces a homomorphism

$$\delta\colon\ H^q(L;\mathcal{B}) \to H^{q+1}(K,L;\mathcal{B}), \qquad q = 0, 1, \cdots. \tag{27}$$

31.18. The cohomology sequence. Associated with the bundle $\mathcal{B}$ of coefficients over (K,L) is the infinite sequence of groups and homomorphisms

$$\cdots \to H^{q-1}(L;\mathcal{B}) \xrightarrow{\delta} H^q(K,L;\mathcal{B}) \xrightarrow{j^*} H^q(K;\mathcal{B}) \xrightarrow{i^*} H^q(L;\mathcal{B}) \to \cdots \tag{28}$$

It is called *the cohomology sequence of* (K,L) *with coefficients in* $\mathcal{B}$.

The cohomology sequence is exact.

As an example we shall prove exactness at the terms of the form $H^q(K,L;\mathcal{B})$; the proofs for the other two cases are left to the reader. Let c be a $(q-1)$-cocycle of L representing $\bar{c}$ in $H^{q-1}(L;\mathcal{B})$. To find $\delta\bar{c}$, we choose an extension c' of c in $C^{q-1}(K;\mathcal{B})$. By definition, $\delta c'$ in $Z^q(K,L;\mathcal{B})$ represents $\delta\bar{c}$. Since $j^\#$ is an inclusion, $j^\#\delta c' = \delta c'$

regarded as an element of $Z^q(K;\mathfrak{B})$. But $\delta c'$ is a coboundary. Hence $\delta c'$ represents zero in $H^q(K;\mathfrak{B})$. Thus $j^*\delta\bar{c} = 0$, and we have proved that the kernel of j^* contains the image of δ.

Now, let $\bar{c}$ in $H^q(K,L;\mathfrak{B})$ be such that $j^*\bar{c} = 0$. Let c in $Z^q(K,L;\mathfrak{B})$ represent $\bar{c}$. Then $j^\# c = c$ in $Z^q(K;\mathfrak{B})$ represents $j^*\bar{c} = 0$. Hence $c = \delta c'$ where $c' \varepsilon C^{q-1}(K;\mathfrak{B})$. Let $c_1 = i^\# c'$. Then $\delta c_1 = \delta i^\# c' = i^\# \delta c' = i^\# j^\# c = 0$. Therefore c_1 is a $(q-1)$-cocycle of L. If $\bar{c}_1$ is its class in $H^{q-1}(L;\mathfrak{B})$, we have $\delta\bar{c}_1 = \bar{c}$. This proves that the kernel of j^* is contained in the image of δ; and completes the proof of the exactness of (28) at the term $H^q(K,L;\mathfrak{B})$.

31.19. Commutativity. Let $\mathfrak{B},\mathfrak{B}'$ be bundles of coefficients over (K,L), (K',L') respectively, and let h: $\mathfrak{B} \to \mathfrak{B}'$. Then h restricted to $\mathfrak{B}|L$ is a bundle map h_1: $\mathfrak{B}|L \to \mathfrak{B}'|L'$. We obtain the diagram

$$(29)\qquad \begin{array}{ccc} H^{q+1}(K',L';\mathfrak{B}') & \xrightarrow{h^*} & H^{q+1}(K,L;\mathfrak{B}) \\ \uparrow\delta & & \uparrow\delta \\ H^q(L';\mathfrak{B}') & \xrightarrow{h_1^*} & H^q(L;\mathfrak{B}). \end{array}$$

We assert that

$$(30)\qquad h^*\delta = \delta h_1^*.$$

In view of the definition (24), it suffices to prove (30) when h is a proper map. Let h_2 denote h regarded as a map of $\mathfrak{B}$ over K into $\mathfrak{B}'$ over K'. Using the minimal carrier, we select an $h_\#$ satisfying (11) and (12). Then we set $h_{2\#} = h_\#$ and $h_{1\#} = h_\#|C_q(L)$. Passing to the associated cochain maps, as in (15) and (16), we obtain the commutativity relations

$$h_2^\# j'^\# = j^\# h^\#, \qquad h_1^\# i'^\# = i^\# h_2^\#.$$

Let c be a representative cocycle of $\bar{c}$ in $H^q(L';\mathfrak{B}')$. Extend c to c' in $C^q(K';\mathfrak{B}')$. Then, by definition, $h^\#\delta c'$ represents $h^*\delta\bar{c}$. Using (17), $h^\#\delta c' = \delta h_2^\# c'$. But $i^\# h_2^\# c' = h_1^\# i'^\# c' = h_1^\# c$ represents $h_1^*\bar{c}$. Therefore $\delta h_2^\# c'$ represents $\delta h_1^* \bar{c}$, and the proof is complete.

31.20. The 0-dimensional group. When K is a connected complex, the ordinary cohomology group $H^0(K;\pi)$ is isomorphic to π. This is not the case for $H^0(K;\mathfrak{B})$ when $\mathfrak{B}$ is not a product bundle. The structure of this group is obtained as follows. Since $B^0(K;\mathfrak{B}) = 0$, we have

$$H^0(K;\mathfrak{B}) = Z^0(K;\mathfrak{B}).$$

We must interpret the condition for a 0-cochain c to be a cocycle. Let σ be an edge with vertices A and B so that $\partial\sigma = B - A$. We can suppose that $x_\sigma = A$. Then

$$\delta c(\sigma) = w_{B\sigma}c(B) - c(A).$$

Thus, $\delta c(\sigma) = 0$ is equivalent to the statement: $c(B)$ translates along σ into $c(A)$. Then $\delta c = 0$ is equivalent to the statement: for any two vertices A and B, translation along a curve in K^1 from B to A carries $c(B)$ into $c(A)$. Since any curve in K from A to B is homotopic to one in K^1, we may replace K^1 by K in the statement. Then, translating $c(A)$, say, to the various points of K provides a uniquely defined cross-section of $\mathfrak{B}$. It follows that the 0-cocycles are in 1-1 correspondence with the cross-sections of $\mathfrak{B}$. Since the fibre π is discrete, the cross-sections are in 1-1 correspondence with the elements of π pointwise invariant under the operations of $\chi(\pi_1(K))$ on π. *Thus, $H^0(K;\mathfrak{B})$ is isomorphic to the subgroup of π pointwise invariant under $\chi(\pi_1(K))$.*

31.21. The ordinary homology and cohomology theory of complexes has been extended to spaces other than complexes by two distinct methods: the Čech method based on coverings and their nerves, and the method based on singular simplexes. One would expect to find corresponding generalizations for homology and cohomology theory with coefficients in bundles. These do exist. We shall have no need of them in the sequel since our work is restricted to complexes. For a discussion of these matters see [87].

§32. THE OBSTRUCTION COCYCLE

32.1. The proof of $\delta c(f) = 0$. Let $\mathfrak{B}$ be a bundle over the cell complex K with a fibre Y which is q-simple, let L be a subcomplex of K, and let f be a cross-section of $\mathfrak{B}|L \cup K^q$. According to §30.2, the groups $\pi_q(Y_x)$ form a bundle $\mathfrak{B}(\pi_q)$ of coefficients over K. Let $c(f)$ be the obstruction cocycle of f as defined in §§29.7–29.8. By §31.2, *$c(f)$ is a $(q + 1)$-cochain of K mod L with coefficients in $\mathfrak{B}(\pi_q)$.* To prove that it is a cocycle requires the use of a "homotopy addition theorem." The latter relates the addition of spherical cycles to the addition of the corresponding elements of homotopy groups. It is used in the proof of the Hurewicz isomorphism (§15.10), and is a consequence of it. We avoid the addition theorem by assuming §15.10 and deriving the following consequence.

32.2. LEMMA. *If K is $(q - 1)$-connected (§29.1), then the natural homomorphism $\pi_q(K^q) \rightarrow H_q(K^q)$ is an isomorphism, and $H_q(K^q)$ coincides with the group of q-cycles $Z_q(K)$; hence*

$$\pi_q(K^q) \approx Z_q(K).$$

The relation $\pi_i(K^q) \approx \pi_i(K)$, $i < q$, holds in any complex. (This follows quickly from the well-known homotopy approximation theorem:

If a complex of dimension $\leqq q$ is mapped into K, the map is homotopic to a map into K^q, leaving fixed any points already mapped into K^q.)

Since K is $(q-1)$-connected, it follows that K^q is $(q-1)$-connected. Then the Hurewicz theorem, §15.10, gives $\pi_q(K^q) \approx H_q(K^q)$. Since $C_{q+1}(K^q) = 0$, we have $B_q(K^q) = 0$; hence $H_q(K^q) = Z_q(K^q) = Z_q(K)$.

32.3. Lemma. *If K is $(q-1)$-connected, and the bundle $\mathfrak{B}$ over K is equivalent to $K \times Y$, and if f is a cross-section defined over K^q, then $c(f)$ is a coboundary in K.*

Let p' be the projection $K \times Y \to Y$, and let $f' = p'f$. In this case $\mathfrak{B}(\pi_q)$ is a product bundle, and we may identify each $\pi_q(Y_x)$ with $\pi_q(Y)$. Then $c(f,\sigma)$ is the element of $\pi_q(Y)$ represented by $f'|\dot{\sigma}$. Letting $C_p(K)$ denote the group of p-chains of K with integer coefficients, we have the diagram

$$C_{q+1}(K) \xrightarrow{\partial} Z_q(K) \xleftarrow{\psi} \pi_q(K^q) \xrightarrow{f'_*} \pi_q(Y)$$

where ψ is the isomorphism of §32.2. Then, for any oriented $(q+1)$-cell σ,

$$c(f,\sigma) = f'_*\psi^{-1}\partial\sigma.$$

Since $C_{q-1}(K)$ is a free abelian group, the kernel of ∂: $C_q(K) \to C_{q-1}(K)$, namely $Z_q(K)$, is a direct summand of $C_q(K)$. Therefore the homomorphism $f'_*\psi^{-1}$ extends to a homomorphism

$$d\colon \quad C_q(K) \to \pi_q(Y).$$

As a function on the oriented q-cells of K, we have $d \in C^q(K;\pi_q(Y))$. Then the relation $c(f,\sigma) = d(\partial\sigma)$, and the general relation $d(\partial\sigma) = (\delta d)(\sigma)$ imply $\delta d = c(f)$.

32.4. Theorem. *The obstruction cochain $c(f)$ is a cocycle.*

We must show that $\delta c(f)$ is zero on any $(q+2)$-cell ζ. Let K' be the subcomplex consisting of ζ and its faces. Let $c' = c(f)|K'$. Then $\delta c'$ has the same value on ζ as does $\delta c(f)$. If $\mathfrak{B}'$ is $\mathfrak{B}|K'$, and if $f' = f|K'^q$, then it is clear that $c' = c(f')$. Now K' is a cell; hence it is $(q-1)$-connected, and $\mathfrak{B}' = K' \times Y$. Then, by §32.3, $c' = \delta d$ is a coboundary. Since $\delta\delta d = 0$ for any cochain, we have $\delta c' = 0$, and the theorem is proved.

32.5. Homotopies of cross-sections. If f is a cross-section of the bundle $\mathfrak{B} = \{B,p,X,Y,G\}$, a *homotopy* of f is a map F: $X \times I \to B$ such that $pF(x,t) = x$ for all t and $F(x,0) = f(x)$. If we define f_t: $X \to B$ by $f_t(x) = F(x,t)$, then f_t is a cross-section. We call F a *homotopy* of f_0 into f_1; and f_0 and f_1 are said to be *homotopic*.

32.6. LEMMA. *Let* $\mathfrak{B}$ *be a bundle over* K, *and let* f_0, f_1 *be homotopic cross-sections of* $\mathfrak{B}|L \cup K^q$. *Then* $c(f_0) = c(f_1)$.

For any $(q+1)$-cell σ, the homotopy $f_0 \simeq f_1$ induces a homotopy $f_0|\dot{\sigma} \simeq f_1|\dot{\sigma}$ in B_σ. Therefore they determine the same element of $\pi_q(B_\sigma)$; hence, by §29.6, the same element of π_σ.

32.7. Invariance of $c(f)$ under mappings.

LEMMA. *Let* $\mathfrak{B}, \mathfrak{B}'$ *be bundles over complexes* (K,L), (K',L') *respectively, and let* h: $\mathfrak{B} \to \mathfrak{B}'$ *be a bundle map which induces a proper cellular map* $\bar{h}$: $(K,L) \to (K',L')$. *Let* k: $\mathfrak{B}(\pi_q) \to \mathfrak{B}'(\pi_q)$ *be the induced map of the coefficient bundles* (§30.5), *and let*

$$h^{\#}: \quad C^{q+1}(K',L';\mathfrak{B}'(\pi_q)) \to C^{q+1}(K,L;\mathfrak{B}(\pi_q))$$

denote the unique cochain homomorphism induced by k (§31.8). *Let* f' *be a cross-section of* $\mathfrak{B}'|L' \cup K'^q$, *and let* f *be the cross-section of* $\mathfrak{B}|L \cup K^q$ *induced by* f' *and* h (§2.11). *Then*

$$c(f) = h^{\#}c(f').$$

We must prove that the two sides have the same value on any $(q+1)$-cell σ of K. Since f is induced by h, the statement to be proved is a commutativity relation. It is a consequence of a rather large number of trivial commutativity relations. Let E_σ be the minimal carrier of σ, E_σ^q its q-skeleton, and τ a $(q+1)$-cell of E_σ. We obtain the diagram

$$\begin{tikzcd}
C_{q+1}(\sigma) \arrow[r, "\partial"] \arrow[d, "h_\#"] & Z_q(\dot{\sigma}) \arrow[r, phantom, "="] \arrow[d, "h_\#"] & H_q(\dot{\sigma}) \arrow[d, "\bar{h}_*"] & \pi_q(\dot{\sigma}) \arrow[l, "\psi"'] \arrow[r, "f_{\dot{\sigma}*}"] \arrow[d, "\bar{h}_*"] & \pi_q(Y_\sigma) \arrow[d, "h_{1*}"] \arrow[dr, "h_\sigma"] & \\
C_{q+1}(E_\sigma) \arrow[r, "\partial"] & Z_q(E_\sigma^q) \arrow[r, phantom, "="] & H_q(E_\sigma^q) & \pi_q(E_\sigma^q) \arrow[l, "\psi'"'] \arrow[r, "f'_*"] & \pi_q(B'_\sigma) & \pi_q(Y_0) \arrow[l, "i_*"'] \\
C_{q+1}(\tau) \arrow[r, "\partial"] \arrow[u, "k_\#"] & Z_q(\dot{\tau}) \arrow[r, phantom, "="] \arrow[u, "k_\#"] & H_q(\dot{\tau}) \arrow[u, "k_*"] & \pi_q(\dot{\tau}) \arrow[l, "\psi''"'] \arrow[r, "f'_{\dot{\tau}*}"] \arrow[u, "k_*"] & \pi_q(Y_\tau) \arrow[u, "j_*"] \arrow[ur, "w_\tau"'] &
\end{tikzcd}$$

We have denoted by B'_σ the part of B' over E_σ. The fibre of $\mathfrak{B}$ over x_σ is Y_σ, Y_0 is the fibre of $\mathfrak{B}'$ over $\bar{h}(x_\sigma)$, and Y_τ is the fibre over x'_τ. Restricting h to the domain Y_σ and range B'_σ gives h_1. The maps i, j and k are inclusions, h_σ is induced by h_{x_σ}: $Y_\sigma \to Y_0$, and w_τ is induced by translating Y_τ along a curve in E_σ to Y_0.

Observe first that commutativity holds in each square and triangle of the diagram. This is trivial for the six squares on the left and the upper right triangle. Since $f_{\dot{\sigma}} \simeq f|\dot{\sigma}$, and $hf = f'\bar{h}$, we obtain $h_1 f_{\dot{\sigma}} \simeq f'\bar{h}|\dot{\sigma}$. This implies commutativity in the upper right square. The same for the lower right square follows from $f'_{\dot\tau} \simeq f'|\dot\tau$. Since translation of Y_τ along a curve of E_σ keeps Y_τ in B'_σ, commutativity holds in the lower right triangle.

The fact that ψ and ψ'' are isomorphisms has already been used in defining $c(f)$ and $c(f')$. Perhaps the only non-trivial point of this proof is that ψ' is an isomorphism onto. This follows from §32.2 and the assumption that E_σ is solid. Thus, we may reverse the arrows of ψ, ψ', ψ'' and commutativity still holds.

In the two triangles on the right, all maps are isomorphisms onto. This follows since the contractibility of E_σ implies that B'_σ can be contracted into Y_0.

By definition,

$$c(f',\tau) = f'_{\dot\tau *}\psi''^{-1}\partial\tau.$$

Applying the definition (§31.4) of the Kronecker index in E_σ,

$$c(f')\cdot\tau = w_\tau f'_{\dot\tau *}\psi''^{-1}\partial\tau.$$

Identifying $C_{q+1}(\tau)$ with a subgroup of $C_{q+1}(E_\sigma)$ under $k_\#$, and using commutativity in the lower half of the diagram, we obtain

$$c(f')\cdot\tau = i_*^{-1}f'_*\psi'^{-1}\partial\tau.$$

Since both sides are additive, the last relation holds with τ replaced by any chain in $C_{q+1}(E_\sigma)$, in particular, by $h_\#\sigma$. It follows, from §31.7, that

$$h^\# c(f')(\sigma) = h_\sigma^{-1}(c(f')\cdot h_\#\sigma) = h_\sigma^{-1}i_*^{-1}f'_*\psi'^{-1}\partial h_\#\sigma.$$

Using commutativity in the upper half of the diagram, the right side of the last equation reduces to $f_{\dot\sigma *}\psi^{-1}\partial\sigma$. But this is the definition of $c(f,\sigma)$; and the proof is complete.

§33. The Difference Cochain

33.1. Motivation. Recall that the obstruction cocycle $c(f)$ is met after a stepwise extension to $L \cup K^q$ of a cross-section given on L. We will show that an alteration of the extension over the q-cells alters $c(f)$ by a coboundary. For this purpose we must introduce the difference cochain associated with two different extensions over the q-cells. Now two maps f_0,f_1 of a q-cell τ which agree on $\dot\tau$ determine, in a natural way, a map of a q-sphere, and this, in turn, determines an element $d(f_0,f_1,\tau)$ of $\pi_q(Y_\tau)$. In this way we obtain a cochain $d(f_0,f_1)$ in $C^q(K,L;\mathfrak{B}(\pi_q))$. If we were to adopt this direct and intuitive definition of the difference cochain, we would be required to give it a formal treatment as extensive as that of $c(f)$. Fortunately $d(f_0,f_1)$ is essentially an obstruction cochain on the product complex $K \times I$. We shall use this fact to define d, and then its properties will follow quickly from those of $c(f)$. The cost of this procedure is that we must digress to consider the product complex and cross-products of cochains.

33.2. Products of chains and cochains. As in §19.1, the product $K \times K_1$ of two cell complexes is a cell complex whose cells are the products $\sigma \times \tau$ of cells of K and K_1. Using ordinary chains with integer coefficients, we have the well-known result that, for all p and q, there exists a bilinear pairing of $C_p(K)$ and $C_q(K_1)$ to $C_{p+q}(K \times K_1)$, denoted by $\times$, with the following properties: if σ and τ are oriented cells, then $\sigma \times \tau$ is an orientation of their product cell, and

$$(1) \qquad \partial(\sigma \times \tau) = \partial\sigma \times \tau + (-1)^p \sigma \times \partial\tau, \qquad p = \dim \sigma.$$

We wish to extend this result to cochains with coefficients in bundles. Because of the limited application we will not consider the most general situation. We shall suppose that the bundle $\mathfrak{B}'$ of coefficients over $K \times K_1$ is the one induced by the projection $K \times K_1 \to K$ and a bundle $\mathfrak{B}$ of coefficients over K. The coefficients for K_1 will be ordinary integers, denoted by J. The reference point for $\sigma \times \tau$ will be the point (x_σ, y_τ). Then the coefficient group $\pi_{\sigma\times\tau}$ is naturally isomorphic to π_σ under the projection $\mathfrak{B}' \to \mathfrak{B}$. The pairing of π_σ and J to $\pi_{\sigma\times\tau}$ is the ordinary multiplication of a group element by an integer followed by the inverse of the isomorphism $\pi_{\sigma\times\tau} \approx \pi_\sigma$. Then, if

$$u \in C^p(K;\mathfrak{B}), \qquad v \in C^q(K_1;J),$$

we define $u \times v \in C^{p+q}(K \times K_1;\mathfrak{B}')$ by setting $u \times v = 0$ on all cells $\sigma \times \tau$ unless $\dim \sigma = p$ and $\dim \tau = q$, in which case

$$(2) \qquad u \times v(\sigma \times \tau) = u(\sigma)v(\tau).$$

It is easily seen that $u \times v$ is bilinear. If we agree that $u(\sigma) = 0$ whenever $\dim \sigma \neq \dim u$, then (2) defines $u \times v$ on any product cell.

It is important to prove the analog of (1):

$$(3) \qquad \delta(u \times v) = \delta u \times v + (-1)^p u \times \delta v.$$

We must evaluate both sides of (3) on a product cell $\sigma \times \tau$. This cell and its faces form a simply-connected subcomplex of $K \times K_1$ which is the product of the analogous subcomplexes for σ and τ. This reduces the proof to the case of simple coefficients. For any cochain w and cell ζ, we have the basic relation $(\delta w)(\zeta) = w(\partial\zeta)$. Using this repeatedly, we have

$$\begin{aligned}(\delta(u \times v))(\sigma \times \tau) &= u \times v(\partial(\sigma \times \tau)) \\ &= u \times v(\partial\sigma \times \tau + (-1)^r \sigma \times \partial\tau), \qquad r = \dim \sigma \\ &= u(\partial\sigma)v(\tau) + (-1)^r u(\sigma)v(\partial\tau) \\ &= \delta u(\sigma)v(\tau) + (-1)^r u(\sigma)\delta v(\tau) \\ &= (\delta u \times v)(\sigma \times \tau) + (-1)^r(u \times \delta v)(\sigma \times \tau).\end{aligned}$$

The last term is zero unless $\dim u = \dim \sigma$; hence replacing $(-1)^r$ by $(-1)^p$ leaves it unaltered. Then (3) follows.

33.3. The complex $K \times I$. Let $\mathfrak{B}$ be a bundle over (K,L). For convenience of notation, let

$$\mathfrak{B}^\square = \mathfrak{B} \times I, \qquad K^\square = K \times I,$$
$$L^\square = (K \times 0) \cup (L \times I) \cup (K \times 1).$$

We regard I as a complex composed of two 0-cells $\bar{0}$ and $\bar{1}$ and the 1-cell $\bar{I}$. We also let $\bar{0},\bar{1}$ stand for the generating 0-cochains of $C^0(I)$ (integer coefficients); and $\bar{I}$ will denote a generator of $C^1(I)$ chosen so that

$$\delta\bar{0} = -\bar{I}, \qquad \delta\bar{1} = \bar{I}. \tag{4}$$

If $d \in C^p(K,L;\mathfrak{B}(\pi_q))$, it is readily checked that $d \times \bar{I}$ is zero on $L^\square$. Hence

$$d \times \bar{I} \in C^{p+1}(K^\square,L^\square;\, \mathfrak{B}^\square(\pi_q)).$$

Since $\tau \to \tau \times I$ is a 1-1 correspondence between the p-cells of $K - L$ and the $(p+1)$-cells of $K^\square - L^\square$, it follows that $d \to d \times \bar{I}$ is an isomorphism

$$C^p(K,L;\mathfrak{B}(\pi_q)) \approx C^{p+1}(K^\square,L^\square;\mathfrak{B}^\square(\pi_q)). \tag{5}$$

Since $\delta\bar{I} = 0$, (3) implies that the isomorphisms (5) commute with δ:

$$\delta(d \times \bar{I}) = \delta d \times \bar{I}. \tag{6}$$

Therefore (5) induces

$$H^p(K,L;\mathfrak{B}(\pi_q)) \approx H^{p+1}(K^\square,L^\square;\mathfrak{B}^\square(\pi_q)). \tag{7}$$

33.4. Definition. Let f_0,f_1 be cross-sections of the part of $\mathfrak{B}$ over $L \cup K^q$, let $f_0 = f_1$ on L, and let k be a homotopy (as in §32.5)

$$f_0|L \cup K^{q-1} \simeq f_1|L \cup K^{q-1} \qquad \text{relative to } L.$$

The *associated* cross-section F of the part of $\mathfrak{B}^\square$ $(= \mathfrak{B} \times I)$ over $L^\square \cup K^{\square q}$ is defined by

$$\begin{aligned} F(x,0) &= (f_0(x),0), \qquad F(x,1) = (f_1(x),1), \\ F(x,t) &= (k(x,t),t) \qquad \text{for } x \in L \cup K^{q-1},\ t \in I.\end{aligned} \tag{8}$$

Then an obstruction cocycle

$$c(F) \in C^{q+1}(K^{\square},L \times I;\mathcal{B}^{\square}(\pi_q))$$

is defined. It coincides with $c(f_0) \times \bar{0}$ on $K \times 0$ and with $c(f_1) \times \bar{1}$ on $K \times 1$. Hence

$$c(F) - c(f_0) \times \bar{0} - c(f_1) \times \bar{1} \in C^{q+1}(K^{\square},L^{\square};\mathcal{B}^{\square}(\pi_q)).$$

Using the isomorphism (5), we define the *deformation cochain*

$$d(f_0,k,f_1) \in C^q(K,L;\mathcal{B}(\pi_q))$$

by

(9) $$d(f_0,k,f_1) \times \bar{I} = (-1)^{q+1}\{c(F) - c(f_0) \times \bar{0} - c(f_1) \times \bar{1}\}.$$

Whenever $f_0 = f_1$ on $L \cup K^{q-1}$ and $k(x,t) = f_0(x)$ for all t, we abbreviate $d(f_0,k,f_1)$ by $d(f_0,f_1)$ and call it the *difference cochain*.

33.5. The coboundary formula.

THEOREM. *Under the hypotheses of §33.4,*

$$\delta d(f_0,k,f_1) = c(f_0) - c(f_1).$$

If we apply δ to both sides of (9), use (6), (3), (4), and the fact that $c(F)$, $c(f_0)$, $c(f_1)$ are cocycles (§32.7), we obtain

$$(\delta d(f_0,k,f_1)) \times \bar{I} = c(f_0) \times \bar{I} - c(f_1) \times \bar{I}.$$

Since the operation $u \to u \times \bar{I}$ is an isomorphism of cochains (see (5)), the theorem follows.

33.6. Invariance under mappings.

THEOREM. *Let f_0,k,f_1 be as in §33.4. Let h: $\mathcal{B}' \to \mathcal{B}$ induce a proper cellular map $\bar{h}$: $(K',L') \to (K,L)$, and let f_0',f_1',k' be the cross-sections and homotopy induced by h and f_0,f_1,k. Then*

$$h^{\#}d(f_0,k,f_1) = d(f_0',k',f_1').$$

The proof is entirely mechanical so we only sketch its outline. Define $h^{\square}$: $\mathcal{B}'^{\square} \to \mathcal{B}^{\square}$ by $h^{\square}(b',t) = (h(b'),t)$. Then verify that (i) $h^{\square\#}(d \times \bar{I}) = (h^{\#}d) \times \bar{I}$, and (ii) the cross-section F' associated with f_0',k',f_1' is induced by $h^{\square}$ and F. Now apply $h^{\square\#}$ to both sides of (9) and use §32.7 to provide $h^{\square\#}c(F) = c(F')$, and

$$h^{\square\#}(c(f_i) \times \bar{\imath}) = (h^{\#}c(f_i)) \times \bar{\imath} = c(f_i') \times \bar{\imath}, \qquad i = 0,1.$$

This gives $(h^{\#}d(f_0,k,f_1)) \times \bar{I} = d(f_0',k',f_1') \times \bar{I}$, and this, in view of (5), proves the theorem.

33.7. The addition formula.

THEOREM. *Let f_0,f_1,f_2 be cross-sections defined on $L \cup K^q$ which coincide on L. Let $f_i' = f_i|L \cup K^{q-1}$ ($i = 0,1,2$). Let k,k' be homotopies*

$$k:\quad f_0' \simeq f_1', \qquad k':\quad f_1' \simeq f_2', \qquad \text{relative to } L,$$

and let k'': $f_0' \simeq f_2'$ rel. L be their composition (i.e. $k''(x,t) = k(x,2t)$ *for* $0 \leqq t \leqq 1/2$, $k''(x,t) = k'(x,2t - 1)$ *for* $1/2 \leqq t \leqq 1$). *Then*

$$d(f_0,k'',f_2) = d(f_0,k,f_1) + d(f_1,k',f_2).$$

To prove that the two sides coincide on a q-cell τ, it suffices to restrict attention to the subcomplex consisting of τ and its faces. If §33.6 is applied to the inclusion map of this subcomplex, we thereby reduce the proof to the case where K consists of a q-cell τ and its faces. Then $\mathcal{B} = K \times Y$, $\mathcal{B}^\square = K \times I \times Y$, and we have projections p': $\mathcal{B} \to Y$ and $p'^\square$: $\mathcal{B}^\square \to Y$. The coefficient bundles are product bundles, and all coefficient groups may be identified with $\pi_q(Y)$ under these projections.

Let F_0 be the cross-section associated with f_0,k,f_1; F_1 with f_1,k',f_2; and F with f_0,k'',f_2. Let I' denote the complex obtained by dividing I into two subintervals $I_0 = [0,1/2]$ and $I_1 = [1/2,1]$. Extend F to a cross-section F' over $(K \times I')^q$ by setting $F'(x,1/2) = (f_1(x),1/2)$. Let

$$\sigma = \tau \times I, \qquad \sigma_0 = \tau \times I_0, \qquad \sigma_1 = \tau \times I_1.$$

Then the chain $\sigma_0 + \sigma_1$ is the subdivision of σ. Hence $\partial\sigma_0 + \partial\sigma_1$ is the subdivision of $\partial\sigma$. Applying §32.2, it follows that $\partial\sigma$ represents in $\pi_q((K \times I')^q)$ the sum of the elements represented by $\partial\sigma_0$ and $\partial\sigma_1$. Taking images under $p'^\square F'$, we obtain

$$c(F,\sigma) = c(F',\sigma_0) + c(F',\sigma_1).$$

Defining g: $\mathcal{B} \times I_0 \to \mathcal{B} \times I$ by $g(b,t) = (b,2t)$, then $gF' = F_0\bar{g}$, and this implies $c(F',\sigma_0) = c(F_0,\sigma)$. Similarly $c(F',\sigma_1) = c(F_1,\sigma)$. Therefore

$$c(F,\sigma) = c(F_0,\sigma) + c(F_1,\sigma). \tag{10}$$

Since f_i ($i = 0,1,2$) is defined on all of K, $c(f_i) = 0$. The theorem follows now from (10), (9) and the isomorphism (5).

33.8. LEMMA. *We have $d(f_0,k,f_1) = 0$ if and only if k can be extended to a homotopy*

$$f_0|L \cup K^q \simeq f_1|L \cup K^q.$$

Now $d = 0$ if and only if $c(F)$ is zero on each $\tau \times I$; and this occurs if and only if F is extendable over each $\tau \times I$. But the latter is equivalent to the extendability of k.

33.9. LEMMA. *If f_0 is a cross-section of the part of $\mathfrak{B}$ over $L \cup K^q$, and*

$$d \; \varepsilon \; C^q(K,L;\mathfrak{B}(\pi_q)),$$

then $f_0|L \cup K^{q-1}$ may be extended to a cross-section f_1 defined on $L \cup K^q$ such that

$$d(f_0,f_1) = d.$$

For each q-cell τ, we shall extend $f_0|\dot{\tau}$ to a cross-section f_1 over τ so that the value of $d(f_0,f_1)$ on τ is $d(\tau)$. This reduces the proof to the case where K consists of a q-cell τ and its faces. We may therefore suppose that $\mathfrak{B} = K \times Y$, and $\mathfrak{B}^\square = K \times I \times Y$. Let p': $\mathfrak{B}^\square \to Y$ be the natural projection. Define

$$F(x,0) = (f_0(x),0) \text{ for } x \; \varepsilon \; \tau; \qquad F(x,t) = (f_0(x),t) \text{ for } x \; \varepsilon \; \dot{\tau}.$$

Then F is defined on $E = \tau \times 0 \cup \dot{\tau} \times I$. Let g: $(\tau \times I)^\cdot \to Y$ represent $(-1)^{q+1} d(\tau) \; \varepsilon \; \pi_q(Y)$. Since E is a q-cell, there exists a homotopy k of $g|E$ into $p'F$ (shrinking E to a point deforms both maps into constant maps, and the two resulting image points can be connected by a curve in Y, for Y is q-simple). By §16.2, the homotopy $k|(\tau \times 1)^\cdot$ can be extended to a homotopy of $\tau \times 1$. Then g is homotopic to a map g_1 such that $g_1|E = p'F$; and g_1 represents $(-1)^{q+1} d(\tau)$. Using the representation $\mathfrak{B} = K \times Y$, define $f_1(x) = (x,g_1(x,1))$. It follows immediately that $d(f_0,f_1)(\tau) = d(\tau)$.

§34. EXTENSION AND DEFORMATION THEOREMS

34.1. Extensions of cross-sections. We put together now the results of the preceding articles. It is assumed that $\mathfrak{B}$ is a bundle over the cell complex K, L is a subcomplex, and Y is q-simple.

34.2. THEOREM. *Let f be a cross-section of $\mathfrak{B}|L \cup K^{q-1}$, and let f be extendable over $L \cup K^q$. Then the set $\{c(f')\}$ of $(q+1)$-dimensional obstruction cocycles of all such extensions f' of f forms a single cohomology class*

$$\bar{c}(f) \; \varepsilon \; H^{q+1}(K,L;\mathfrak{B}(\pi_q));$$

and f is extendable over $L \cup K^{q+1}$ if and only if $\bar{c}(f) = 0$.

Let f_0, f_1 be two extensions of f over $L \cup K^q$. Then §33.5 gives $\delta d(f_0,f_1) = c(f_0) - c(f_1)$; hence $c(f_0)$ and $c(f_1)$ belong to the same cohomology class.

Let f_0 be an extension of f over $L \cup K^q$, and c a cocycle in the cohomology class of $c(f_0)$. Then there is a q-cochain d such that $\delta d = c(f_0) - c$. By §33.9, there is an extension f_1 of f over $L \cup K^q$ such that $d(f_0,f_1) = d$. It follows from §33.5, that $c(f_1) = c$.

If f is extendable over $L \cup K^{q+1}$ and f' is such an extension, then $f_0 = f'|L \cup K^q$ is extendable over $L \cup K^{q+1}$, so $c(f_0) = 0$, and $\bar{c}(f) = 0$.

If $\bar{c}(f)$ is zero, the part already proved provides an extension f' over $L \cup K^q$ such that $c(f') = 0$. Then f' is extendable over $L \cup K^{q+1}$. This completes the proof.

34.3. Corollary. *If f is a cross-section of $\mathfrak{B}|L \cup K^q$, then $f|L \cup K^{q-1}$ is extendable over $L \cup K^{q+1}$ if and only if $c(f)$ is a coboundary in $K - L$.*

34.4. Corollary. *If $\mathfrak{B}$ is a bundle over (K,L) and, for each $q = 1, 2, \cdots$, dim $(K - L)$, Y is $(q-1)$-simple and $H^q(K,L;\mathfrak{B}(\pi_{q-1})) = 0$, then any cross-section f of $\mathfrak{B}|L$ can be extended to a full cross-section of $\mathfrak{B}$. In particular, if $H^q(K;\mathfrak{B}(\pi_{q-1})) = 0$ for $q = 1, 2, \cdots$, dim K, then $\mathfrak{B}$ has a cross-section.*

If f is extendable to $L \cup K^q$, the assumption $H^{q+1} = 0$ and the preceding corollary imply that f is extendable to $L \cup K^{q+1}$. The result follows by induction.

34.5. Homotopies of cross-sections. Suppose now that f_0,f_1 are two cross-sections of $\mathfrak{B}$, and $f_0|L = f_1|L$. And let the problem be to construct a homotopy

$$k\colon\ f_0 \simeq f_1, \qquad \text{relative } L \text{ (see §32.5).}$$

Defining $\mathfrak{B}^\square$, $K^\square$ and $L^\square$ as in §33.3, and setting

$$f(x,0) = (f_0(x),0), \qquad f(x,1) = (f_1(x),0),$$
$$f(x,t) = (f_0(x),t) \qquad \text{for } x \in L,\ t \in I,$$

we obtain a cross-section of $\mathfrak{B}^\square|L^\square$. If the homotopy k exists, then $f'(x,t) = (k(x,t),t)$ is an extension of f to a full cross-section. Conversely if f' is such an extension of f, the x-coordinate, $k(x,t)$, of $f'(x,t)$ is the required homotopy. Thus *the homotopy problem is equivalent to an extension problem.* Using this equivalence, the preceding results of this article yield the following three propositions concerning the homotopy problem. The proofs are omitted since they are obvious formal translations. We note that, if

$$k\colon\ f_0|L \cup K^{q-1} \simeq f_1|L \cup K^{q-1} \qquad \text{relative } L,$$

then $d(f_0,k,f_1)$ is a cocycle; for, by §33.5, $\delta d = c(f_0) - c(f_1)$ and both obstructions are zero since f_0,f_1 are full cross-sections.

34.6. Theorem. *Let f_0,f_1 be two cross-sections of $\mathfrak{B}$ which coincide on L, and let k be a homotopy*

$$k\colon\ f_0|L \cup K^{q-2} \simeq f_1|L \cup K^{q-2} \qquad \textit{relative } L$$

which is extendable to a homotopy

$$k'\colon\ f_0|L \cup K^{q-1} \simeq f_1|L \cup K^{q-1} \qquad \textit{relative } L.$$

Then the set $\{d(f_0,k',f_1)\}$ *of deformation cocycles of all such extensions* k' *forms a single cohomology class*

$$\bar{d}(f_0,k,f_1) \in H^q(K,L;\mathfrak{B}(\pi_q)),$$

and k *is extendable to a homotopy*

$$f_0|L \cup K^q \simeq f_1|L \cup K^q \qquad \textit{relative } L$$

if and only if $\bar{d}(f_0,k,f_1) = 0$.

34.7. COROLLARY. *If* k *is a homotopy*

$$k\colon\ f_0|L \cup K^{q-1} \simeq f_1|L \cup K^{q-1} \qquad \textit{relative } L,$$

then $k|(L \cup K^{q-2}) \times I$ *is extendable to a homotopy* $f_0|L \cup K^q \simeq f_1|L \cup K^q$ *if and only if* $d(f_0,k,f_1)$ *is a coboundary in* $K - L$.

34.8. THEOREM. *If* $\mathfrak{B}$ *is a bundle over* (K,L), *and, for each* $q = 0, 1, \cdots, \dim (K - L)$, Y *is* q*-simple and* $H^q(K,L;\mathfrak{B}(\pi_q)) = 0$, *then any two cross-sections of* $\mathfrak{B}$, *equal on* L, *are homotopic relative to* L.

34.9. Extension of a homotopy. We prove now a homotopy extension theorem which will provide a reinterpretation of these results.

THEOREM. *Let* $\mathfrak{B}$ *be a bundle over* (K,L), f *a cross-section of* $\mathfrak{B}$, *and* $F'\colon\ L \times I \to B$ *a homotopy of* $f' = f|L$. *Then* F' *can be extended to a homotopy* $F\colon\ K \times I \to B$ *of* f.

We order the cells of $K - L$ in a finite sequence so that no cell precedes any of its faces. The extension of F' to F is carried out a cell at a time in the prescribed order. For the extension over a particular cell σ, we need use only the part of $\mathfrak{B}$ over σ. In this way we reduce the proof to the case where K is a q-cell σ and its faces, and L is the collection of proper faces of σ. Choose, then, a product representation

$$\phi\colon\ \sigma \times Y \to \mathfrak{B}, \qquad p'\colon\ B \to Y$$

with the usual properties. Define

$$h\colon\ (\sigma \times 0) \cup (\dot{\sigma} \times I) \to Y$$

by

$$h(x,t) = \begin{cases} p'F'(x,t), & x \in \dot{\sigma}, \\ p'f(x), & x \in \sigma,\ t = 0. \end{cases}$$

According to §16.2, there is a retraction r of $\sigma \times I$ into $\sigma \times 0 \cup \dot{\sigma} \times I$. Define

$$F(x,t) = \phi(x,hr(x,t)).$$

Then F is the desired extension of F'.

34.10. THEOREM. *Let f_0, f_1 be two cross-sections of $\mathfrak{B}$ which coincide on $L \cup K^{q-1}$, then there exists a homotopy*

$$F: \quad f_0 \simeq f_1', \qquad \textit{relative } L \cup K^{q-2},$$

such that

$$f_1' = f_1 \qquad \textit{on} \qquad L \cup K^q$$

if and only if the difference cocycle $d(f_0,f_1)$ is a coboundary in $K - L$.

Taking $k(x,t) = f_0(x)$ for $x \in L \cup K^{q-1}$, then §34.7 states that there exists a homotopy

$$k': \quad f_0|L \cup K^q \simeq f_1|L \cup K^q \quad \text{relative } L \cup K^{q-2}$$

if and only if $d(f_0,f_1)$ is a coboundary. If k' exists, then §34.9 provides the extension F. If F exists, then $F|(L \cup K^q) \times I$ is a homotopy k'. Thus k' exists if and only if F exists.

§35. THE PRIMARY OBSTRUCTION AND THE CHARACTERISTIC COHOMOLOGY CLASS

35.1. Assumption on the dimension q. The result of §34 on the obstruction to an extension can be summarized as follows. If, in the stepwise process of extending a cross-section, we meet with a non-zero obstruction $c(f)$, then it is a cocycle, and it may be varied within its cohomology class by altering the choice of the extension at the last step. If the class of $c(f)$ is zero, the alteration of the last step can be chosen so that the next step of the extension is possible.

The weakness of this result is only too apparent if one asks the question: Suppose the class of $c(f)$ is not zero, can one alter the choice of the extension over the last *two* steps so as to make the next step of the extension possible? If not, what can be accomplished by redefining over three steps, etc.? A few special results have been achieved in this direction. A redefinition over two stages can alter the cohomology class of $c(f)$, usually by some kind of "product" of lower dimensional classes (see [89]). The general problem is highly interesting and much research remains to be done. (See App. sect. 11.)

There is a special case however where the results of article 34 are fully satisfactory. We turn to this now.

Throughout this article we shall let q denote the least integer such that $\pi_q(Y) \neq 0$.

We continue the convention that $\pi_0(Y)$ is the reduced 0th homology group with integer coefficients (in the singular sense). Thus $q = 0$ is possible. If $q > 0$, then Y is arcwise connected, and the condition $\pi_q \neq 0$ is independent of the base point. If $q = 1$, we assume that π_1 is abelian. If $q > 1$, then $\pi_1 = 0$. Thus, in all cases, Y is q-simple.

It is to be noted that q and π_q are effectively computable, at least for triangulable spaces Y; for, by the Hurewicz theorem 15.10, q is the dimension of the first non-vanishing homology group, and $\pi_q(Y) \approx H_q(Y)$.

35.2. LEMMA. *Any cross-section f of $\mathfrak{B}|L$ is extendable to a cross-section of $\mathfrak{B}|L \cup K^q$. If f_1,f_2 are any two such extensions then $c(f_1) - c(f_2)$ is a coboundary in $K - L$.*

Since Y is $(q - 1)$-connected, the first statement follows from §29.2. If we apply §34.8 to $\mathfrak{B}|L \cup K^{q-1}$, we obtain a homotopy

$$k\colon\ f_1|L \cup K^{q-1} \simeq f_2|L \cup K^{q-1} \qquad \text{relative } L.$$

Then $d(f_1,k,f_2)$ is defined and by §33.5 its coboundary is $c(f_1) - c(f_2)$.

In view of the lemma, we can state:

35.3. Definition. If $\mathfrak{B}$ is a bundle over (K,L), q is the least integer such that $\pi_q(Y) \neq 0$, and f is a cross-section of $\mathfrak{B}|L$, then the cohomology class of the obstruction $c(f')$, where f' is any extension of f over $L \cup K^q$, is called the *primary obstruction* to the extension of f. It is denoted by $\bar{c}(f)$ and is an element of $H^{q+1}(K,L;\mathfrak{B}(\pi_q))$. In the special case that L is vacuous, the cohomology class of $c(f')$ is denoted by $\bar{c}(\mathfrak{B})$ and is called *the characteristic cohomology class of* $\mathfrak{B}$. It is the primary obstruction to the construction of a cross-section.

35.4. The vanishing of $\bar{c}(f)$.

THEOREM. *The primary obstruction $\bar{c}(f)$ is an invariant of the homotopy class of f. Its vanishing is a necessary and sufficient condition for f to be extendable over $L \cup K^{q+1}$.*

Let f' be an extension of f to $L \cup K^q$, and let a homotopy $f \simeq f_1$ be given. By §34.9, the homotopy is extendable to $f' \simeq f_1'$. Then, by §32.4, $c(f') = c(f_1')$. Hence $\bar{c}(f) = \bar{c}(f_1)$, and the first assertion is proved.

If f is extendable over $L \cup K^{q+1}$ and f' is such an extension, then, by §29.8, $c(f'|L \cup K^q) = 0$. Hence $\bar{c}(f) = 0$. Conversely, if $\bar{c}(f) = 0$, and f' is an extension of f to $L \cup K^q$, then $c(f')$ is a coboundary in $K - L$, and, by §34.3, $f'|L \cup K^{q-1}$ is extendable over $L \cup K^{q+1}$.

35.5. COROLLARY. *The vanishing of $\bar{c}(\mathfrak{B})$ is a necessary and sufficient condition for the existence of a cross-section over K^{q+1}.*

35.6. Invariance under mappings.

LEMMA. *Let $\mathfrak{B},\mathfrak{B}'$ be bundles over X,X', respectively, let h_0, h_1 be homotopic maps $\mathfrak{B} \to \mathfrak{B}'$, and let f' be a cross-section of $\mathfrak{B}'$. Then the cross-sections f_0,f_1 of $\mathfrak{B}$, induced by h_0,f' and h_1,f' respectively, are homotopic.*

Let $h\colon\ \mathfrak{B} \times I \to \mathfrak{B}'$ be a homotopy of h_0 into h_1. Let f be the cross-section of $\mathfrak{B} \times I$ induced by h,f'. Then f has the form $f(x,t) = (k(x,t),t)$, and k is the required homotopy.

35.7. Theorem. *Let $\mathfrak{B},\mathfrak{B}'$ be bundles over (K,L), (K',L') respectively, let h: $\mathfrak{B} \to \mathfrak{B}'$ induce a map $\bar{h}$: $(K,L) \to (K',L')$, and let*

$$h^*: \quad H^{q+1}(K',L';\mathfrak{B}'(\pi_q)) \to H^{q+1}(K,L;\mathfrak{B}(\pi_q))$$

be the induced homomorphism of cohomology groups. Let f' be a cross-section of $\mathfrak{B}'|L'$, and f the induced cross-section of $\mathfrak{B}|L$. Then

$$h^*\bar{c}(f') = \bar{c}(f).$$

By §31.16, h is homotopic to a map which can be factored into the form (24′) of §31.16. By §35.6, this alters f by a homotopy, and, by §35.4, $\bar{c}(f)$ is unchanged. Since h^* is unchanged by a homotopy (§31.15), we can suppose that h itself factors into $i''^{-1}ki$. Now i''^* is an isomorphism (§31.9). If we can show that i''^* carries the primary obstruction into itself, the same will hold for $(i''^*)^{-1}$. Thus, it suffices to prove the theorem in the special case that $\bar{h}$ is a proper cellular map.

Choose an extension of f' to a cross-section f'_1 of $\mathfrak{B}'|L' \cup K^q$. Since $\bar{h}$ is cellular, f'_1 and $\bar{h}$ induce a cross-section f_1 of $\mathfrak{B}|L \cup K^q$ which extends f. Then $c(f'_1)$, $c(f_1)$ are cocycles representing $\bar{c}(f'),\bar{c}(f)$ respectively. By §32.7, $h^\# c(f'_1) = c(f_1)$; and the theorem is proved.

35.8. Corollary. *If, in §35.7, h induces a homeomorphism of (K,L) onto (K',L'), then the conclusion asserts the topological invariance of the primary obstruction. In particular, if (K,L), (K',L') are two cellular decompositions of the same space and subspace, $\mathfrak{B} = \mathfrak{B}'$ and $h =$ the identity, it follows that $\bar{c}(f)$ is independent of the choice of the cell complex used to compute it.*

35.9. The generalized $\bar{c}(f)$. There is a useful generalization of the primary obstruction. Let us replace the assumption $\pi_i(Y) = 0$ for $i < q$ by the following weaker conditions on Y and (K,L) jointly:

(1) $\qquad Y$ is i-simple $\qquad$ for $i = 1, \cdots, q-1$.

(2) $\qquad H^{i+1}(K,L;\mathfrak{B}(\pi_i)) = 0 \quad$ for $i = 0, 1, \cdots, q-1$.

(3) $\qquad H^i(K,L;\mathfrak{B}(\pi_i)) = 0 \quad$ for $i = 0, 1, \cdots, q-1$.

(It is understood that H^0 is the *reduced* cohomology group.)

In the development of the primary obstruction, the only place, where the assumption $\pi_i(Y) = 0$ for $i < q$ was used, occurred in the proof of §35.2. It was used once to obtain an extension of the cross section f over L to $L \cup K^q$. But §34.4 states that (1) and (2) insure this. The assumption was used again to obtain a homotopy connecting two such extensions restricted to $L \cup K^{q-1}$. The homotopy was provided by §34.8. But (1) and (3) are the hypotheses of §34.8. The assumption was not used again except, possibly, for a tacit use in §35.6 where h: $\mathcal{B} \to \mathcal{B}'$ implies $h^*\bar{c}(f') = \bar{c}(f)$. But here we would require that (1), (2) and (3) hold for both (K,L) and (K',L'). Thus we have

35.10. THEOREM. *All of the preceding theorems of §35 hold if the restriction $\pi_i(Y) = 0$ for $i < q$ is replaced by the weaker condition (1) above, and only such bundles $\mathcal{B}$ over (K,L) are considered as satisfy conditions (2) and (3) above.*

There are important special cases where the weaker conditions hold. Thus (1) holds if Y is a group (e.g. for principal bundles), or if $\pi_1(Y) = 0$. Conditions (2) and (3) will hold if K is a $(q + 1)$-sphere and $L = 0$.

35.11. Bundles over spheres. Let $\mathcal{B}$ be a bundle over the $(q + 1)$-sphere S, and let Y be q-simple. Then $\bar{c}(\mathcal{B})$ in $H^{q+1}(S;\pi_q(Y))$ is defined. We suppose that $q > 0$ so that $\mathcal{B}(\pi_q)$ is a product bundle. Then, also, we can form the Kronecker index of $\bar{c}(\mathcal{B})$ with each homology class (integer coefficients, see §31.4) and obtain a homomorphism $H_{q+1}(S) \to \pi_q(Y_0)$ where Y_0 is the fibre over $x_0 \in S$. Defining Δ as in §17.3 and letting ψ be the Hurewicz isomorphism (§15.10) we have the diagram

$$\begin{array}{ccc} & H_{q+1}(S) & \\ \psi \nearrow & & \searrow \bar{c}(\mathcal{B}) \\ & \Delta & \\ \pi_{q+1}(S) & \longrightarrow & \pi_q(Y_0). \end{array} \tag{4}$$

35.12. THEOREM. *Under the above hypotheses, we have*

$$\bar{c}(\mathcal{B})\cdot\psi(\alpha) = -\Delta\alpha, \qquad \alpha \in \pi_{q+1}(S).$$

Let K be cellular decomposition of S whose $(q + 1)$-cells consist of the two hemispheres E_1,E_2 into which S is divided by a great q-sphere S'. We suppose $x_0 \in S'$. Orient E_1,E_2 so that

$$\partial E_1 = -\partial E_2. \tag{5}$$

Then $E_1 + E_2$ is a cycle representing a generator u of $H_{q+1}(S)$. It suffices to prove the theorem for the generator $\alpha = \psi^{-1}u$. Let f be a cross-section of $\mathcal{B}|E_2$, and let $f' = f|S'$. Then $c(f')$ is a cocycle representing $\bar{c}(\mathcal{B})$, and

$$c(f',E_2) = 0 \tag{6}$$

since f' is extendable over E_2.

Choose a homotopy k shrinking E_1 over itself into x_0, and then extend k to a homotopy of S (see §16.2). Then k deforms the identity into a map g_1 which maps (E_2,S') on (S,x_0) with degree 1. Hence $g_1|(E_2,S')$ represents α. Cover k by a homotopy k' of f into a map f_1: $(E_2,S') \to (B,Y_0)$. Since $pf_1 = g_1$, it follows that $f_1|S'$ represents $\Delta\alpha$ when S' is oriented so as to be positively incident to E_2. On the other hand $k'|S'$ deforms f' over $B|E_1$ into $f_1|S'$. Then, by definition, $f_1|S'$ represents $c(f',E_1)$ when S' is oriented so as to be positively incident to E_1. So, by (5),

$$c(f',E_1) = -\Delta\alpha. \tag{7}$$

Combining (6) and (7) gives

$$c(f')\cdot(E_1 + E_2) = -\Delta\alpha,$$

and the theorem is proved.

§36. The Primary Difference of Two Cross-sections

36.1. Assumption on the dimension q. We consider again the problem of the homotopy classification of cross-sections. Let $\mathfrak{B}$ be a bundle over (K,L), and let f_0,f_1 be cross-sections of $\mathfrak{B}$ which coincide on L. The problem is to construct a homotopy $f_0 \simeq f_1$ relative to L. As shown in §34.5, this is equivalent to an extension problem in $\mathfrak{B}^\square = \mathfrak{B} \times I$. We make the same assumption as in §35, namely, *q is the least integer such that $\pi_q(Y) \neq 0$.* Then the results of §35 on the extension problem yield corresponding results for the homotopy problem. We state these now. The proofs of the first few propositions are omitted since they are entirely mechanical.

36.2. Lemma. *If f_0,f_1 are two cross-sections of $\mathfrak{B}$ which coincide on L, then there exists a homotopy*

$$k:\quad f_0|L \cup K^{q-1} \simeq f_1|L \cup K^{q-1} \qquad \textit{relative } L.$$

If k,k' are two such homotopies, then $d(f_0,k,f_1) - d(f_0,k',f_1)$ is a coboundary in $K - L$.

36.3. Definition. The conclusion of the preceding lemma asserts that the cohomology class of $d(f_0,k,f_1)$ depends only on f_0,f_1. This class, denoted by

$$\bar{d}(f_0,f_1) \in H^q(K,L;\mathfrak{B}(\pi_q))$$

is called the *primary difference* of f_0 and f_1.

36.4. The vanishing of $\bar{d}(f_0,f_1)$.

Theorem. *The primary difference $\bar{d}(f_0,f_1)$ is an invariant of the homotopy classes relative to L of f_0 and f_1. Its vanishing is a necessary and sufficient condition for*

$$f_0|L \cup K^q \simeq f_1|L \cup K^q \qquad \textit{relative } L.$$

Using §34.10, we can restate the last proposition:

36.5 COROLLARY. *The vanishing of $\bar{d}(f_0,f_1)$ is a necessary and sufficient condition for the existence of a homotopy $f_0 \simeq f_1'$ relative L such that $f_1' = f_1$ on $L \cup K^q$.*

36.6. The addition formula.

THEOREM. *Let f_0,f_1,f_2 be three cross-sections of $\mathfrak{B}$ which coincide on L, then*

$$\bar{d}(f_0,f_2) = \bar{d}(f_0,f_1) + \bar{d}(f_1,f_2).$$

This follows directly from §33.7.

36.7. The coboundary formula.

THEOREM. *Let f_0,f_1 be two cross-sections of $\mathfrak{B}|L$. Then $\bar{d}(f_0,f_1)$ is defined and is in $H^q(L;\mathfrak{B}(\pi_q))$. Under the coboundary operator*

$$\delta\colon\ H^q(L;\mathfrak{B}(\pi_q)) \to H^{q+1}(K,L;\mathfrak{B}(\pi_q)) \qquad \text{(see §31.6)},$$

we have

$$\delta\bar{d}(f_0,f_1) = \bar{c}(f_0) - \bar{c}(f_1).$$

By §35.2, we can choose extensions f_0',f_1' of f_0,f_1 over $L \cup K^q$. Applying §36.2 (with $L = 0$) we obtain a homotopy

$$k'\colon\ f_0'|K^{q-1} \simeq f_1'|K^{q-1}.$$

Now apply §33.5 (with $L = 0$) and we obtain

$$\delta d(f_0',k',f_1') = c(f_0') - c(f_1'). \tag{1}$$

Let k denote the homotopy k' restricted to L^{q-1}. By definition, $d(f_0,k,f_1)$ is a cocycle in the class $\bar{d}(f_0,f_1)$. Furthermore, $d(f_0',k',f_1')$ is an extension of $d(f_0,k,f_1)$ to a cochain of K. Hence $\delta d(f_0',k',f_1')$ is a cocycle in the class $\delta\bar{d}(f_0,f_1)$. Since $c(f_i')$ $(i = 0,1)$ is a cocycle in the class $\bar{c}(f_i)$, the theorem follows from (1) above.

36.8. Invariance under mappings.

LEMMA. *Let f_0,f_1 be cross-sections of $\mathfrak{B}$ which coincide on L. Define $\mathfrak{B}^\square(= \mathfrak{B} \times I)$, $K^\square$ and $L^\square$ as in §33.3. Let the cross-section F of the part of $\mathfrak{B}^\square$ over $L^\square$ be given by*

$$F(x,0) = (f_0(x),0), \quad F(x,1) = (f_1(x),1), \quad F(x,t) = (f_0(x),t) \qquad \text{for } x \in L.$$

Then, under the isomorphism (7) of §33.3, we have

$$\bar{d}(f_0,f_1) \times \bar{I} = (-1)^{q+1}\bar{c}(F).$$

Using a homotopy k given by §36.2, and noting that $c(f_0) = c(f_1) = 0$, the result follows from (9) of §33.4.

36.9. THEOREM. *Let $h\colon\ \mathfrak{B} \to \mathfrak{B}'$ induce a map $h\colon\ (K,L) \to (K',L')$. Let f_0',f_1' be cross-sections of $\mathfrak{B}'$ which agree on L'; and let f_0,f_1*

be the cross-sections of $\mathfrak{B}$ *induced by h. Then*

$$h^*\bar{d}(f_0',f_1') = \bar{d}(f_0,f_1).$$

Construct F as in §36.8. In a similar way, construct the cross-section F' of $\mathfrak{B}'^{\square}|L'^{\square}$. Define $h^{\square}$: $\mathfrak{B}^{\square} \to \mathfrak{B}'^{\square}$ by $h^{\square}(b,t) = (h(b),t)$. It is obvious that F is induced by F' and $h^{\square}$. By §35.7, we have $h^{\square *}\bar{c}(F') = \bar{c}(F)$. The conclusion of §36.8 provides

$$h^{\square *}(\bar{d}(f_0',f_1') \times \bar{I}) = \bar{d}(f_0,f_1) \times \bar{I}. \tag{1}$$

If we can show that

$$h^{\square *}(\bar{d} \times \bar{I}) = (h^*\bar{d}) \times \bar{I}, \tag{2}$$

the theorem will follow from (1) and the fact that $d \to d \times \bar{I}$ is an isomorphism. If $\bar{h}$ is a proper cellular map, we have noted in the proof of §33.6 that $h^{\square \#}(d \times \bar{I}) = (h^{\#}d) \times I$; and this implies (2) for this case. For the general case, we apply the factorization (24′) of §31.16. Since i, k, and i'' are proper cellular maps, the general case of (2) follows from the special case.

36.10. Corollary. *When* $\bar{h}$ *is a homeomorphism, the conclusion of §36.9 asserts the topological invariance of the primary difference. When* (K,L), (K',L') *are two cellular decompositions of the same space and subspace,* $\mathfrak{B} = \mathfrak{B}'$ *and* $h =$ *the identity, it asserts that* $\bar{d}(f_0,f_1)$ *is independent of the cellular decomposition used to compute it.*

36.11. The generalized primary difference. In §35.9 it is noted that the assumption on Y and q can be replaced by weaker conditions in defining the primary obstruction. The same obtains for the primary difference. Since the latter is just a primary obstruction in $(K^{\square},L^{\square})$ (see §36.8), it is enough for the weakened conditions of §35.9 to hold in the bundle $\mathfrak{B}^{\square}$. Using the isomorphisms (7) of §33.3, these conditions translate into

(1) Y is i-simple for $i = 1, \cdots, q-1$.
(2) $H^i(K,L;\mathfrak{B}(\pi_i)) = 0$ for $i = 0, 1, \cdots, q-1$.
(3) $H^{i-1}(K,L;\mathfrak{B}(\pi_i)) = 0$ for $i = 1, \cdots, q-1$.

These conditions suffice for all the results of this article except §36.7. Here the primary difference lies in L; so we impose (1), (2), and (3) with L in place of (K,L). In addition the theorem involves primary obstructions in (K,L). Hence we impose conditions (1), (2), and (3) of §35.9. The two sets of conditions suffice to prove §36.7.

§37. EXTENSIONS OF FUNCTIONS, AND THE HOMOTOPY CLASSIFICATIONS OF MAPS

37.1. Assumption on the dimension q. We continue with the assumption that q is the least integer such that $\pi_q(Y) \neq 0$, and we will apply the primary difference to the problems of extending a cross-section, and of the homotopy classification of cross-sections.

37.2. First extension theorem. *Let* $\dim (K - L) \leqq q + 1$. *Let* $\mathfrak{B}$ *be a bundle over* (K,L) *which has a cross-section* f_0. *Then, for each* $d \in H^q(K,L;\mathfrak{B}(\pi_q))$, *there exists an extension* f_1 *of* $f_0|L$ *to a full cross-section such that*

$$\bar{d}(f_0,f_1) = d.$$

Let d' be a cocycle in the class d. According to §33.9, $f_0|L \cup K^{q-1}$ extends to a cross-section f_1 over $L \cup K^q$ such that $d(f_0,f_1) = d'$. By §33.5,

$$c(f_0|L \cup K^q) - c(f_1) = \delta d(f_0,f_1) = \delta d' = 0,$$

for d' is a cocycle. Since $f_0|L \cup K^q$ is extendable, $c(f_0|L \cup K^q) = 0$. Hence $c(f_1) = 0$, and f_1 is extendable over $L \cup K^{q+1} = K$.

37.3. THEOREM. *Let* f_0,f *be cross-sections of* $\mathfrak{B}|L$, *and let* f_0 *be extendable over* $L \cup K^{q+1}$. *Then*

$$\bar{c}(f) = \delta\bar{d}(f,f_0).$$

Therefore, f *is extendable over* $L \cup K^{q+1}$ *if and only if* $\delta\bar{d}(f,f_0) = 0$.

Since f_0 is extendable over $L \cup K^{q+1}$, by §35.4, we have $\bar{c}(f_0) = 0$. Then $\delta\bar{d}(f,f_0) = \bar{c}(f)$ follows from §36.7. The last statement follows now from §35.4.

37.4. Second extension theorem. *Let* $\dim (K - L) \leqq q + 1$, *and suppose the bundle* $\mathfrak{B}$ *over* (K,L) *has a cross-section* f_0. *Let* i: $\mathfrak{B}|L \to \mathfrak{B}$ *be the inclusion map so that*

$$i^*: \quad H^q(K;\mathfrak{B}(\pi_q)) \to H^q(L;\mathfrak{B}(\pi_q)).$$

Then a cross-section f *of* $\mathfrak{B}|L$ *is extendable to a cross-section of* $\mathfrak{B}$ *if and only if there exists an element* d' *in* $H^q(K;\mathfrak{B}(\pi_q))$ *such that*

$$i^*d' = \bar{d}(f,f_0|L).$$

Furthermore, for each such d', *there exists an extension* f' *of* f *such that* $\bar{d}(f',f_0) = d'$.

Suppose f is extendable to a cross-section f' of $\mathfrak{B}$. If we observe that f and $f_0|L$ are the cross-sections induced by f',f_0, and the inclusion map i, then $i^*\bar{d}(f',f_0) = \bar{d}(f,f_0|L)$ follows from §36.9; and $d' = \bar{d}(f',f_0)$ is the required element.

Conversely, suppose d' given and $i^*d' = \bar{d}(f,f_0|L)$. By §36.2, there exists a homotopy

$$k:\quad f|L^{q-1} \simeq f_0|L^{q-1}.$$

By §34.9, k extends to a homotopy k': $f \simeq f_1$ and $f_1 = f_0$ on L^{q-1}. According to §36.4, $\bar{d}(f,f_0|L) = \bar{d}(f_1,f_0|L)$. Therefore $d(f_1,f_0|L)$ is a cocycle in the class $\bar{d}(f,f_0|L)$. Let d_1 be a cocycle in the class d'. Then $i^\# d_1$ represents $i^*d' = \bar{d}(f,f_0|L)$. It follows that there is a $(q-1)$-cochain c of L such that

$$\delta c = i^\# d_1 - d(f_1,f_0|L).$$

Extend c to a cochain c' of K by defining it arbitrarily on the $(q-1)$-cells of $K - L$. Let $d_2 = d_1 - \delta c'$. Then

$$\begin{aligned} i^\# d_2 &= i^\# d_1 - i^\# \delta c' = i^\# d_1 - \delta i^\# c' \\ &= i^\# d_1 - \delta c = d(f_1,f_0|L). \end{aligned}$$

Thus d_2 *is an extension of* $d(f_1,f_0|L)$ *to a cocycle of* K *belonging to the class* d'.

Define d_3 in $C^q(K,L;\mathfrak{B}(\pi_q))$ by

$$d_3 = d_2 \text{ on } K - L, \qquad d_3 = 0 \text{ on } L.$$

Then §33.9 provides an extension f_0' of $f_0|L \cup K^{q-1}$ to $L \cup K^q$ such that

$$d(f_0',f_0) = d_3.$$

Define f_1' on $L \cup K^q$ by

$$f_1'|L = f_1, \qquad f_1'|K^q - L^q = f_0'|K^q - L^q.$$

Since L intersects the closure of $K^q - L^q$ in a subset of L^{q-1}, and $f_1 = f_0 = f_0'$ on L^{q-1}, it follows that f_1' is a continuous extension of f_1. By its definition,

$$d(f_1',f_0) = \begin{cases} d(f_0',f_0) = d_3 & \text{in } K - L, \\ d(f_1,f_0|L) & \text{in } L. \end{cases}$$

Therefore $d(f_1',f_0) = d_2$. Applying §33.5, we have

$$\delta d(f_1',f_0) = c(f_1') - c(f_0).$$

Since d_2 is a cocycle, $\delta d(f_1',f_0) = 0$. Since f_0 is defined over all $(q+1)$-cells, $c(f_0) = 0$. Hence $c(f_1') = 0$. Thus f_1' extends to a cross-section f_1'' over $L \cup K^{q+1} = K$. Then $d(f_1'',f_0) = d(f_1',f_0) = d_2$, and $\bar{d}(f_1'',f_0) = d'$. By §34.9, the reverse of the homotopy k': $f \simeq f_1$ extends to a homotopy $f_1'' \simeq f'$. Then f' is an extension of f to all of K; and, by §36.4, $\bar{d}(f',f_0) = \bar{d}(f_1'',f_0) = d'$. This completes the proof.

37.5. Classification theorem. *Let* dim $(K - L) = q$, *and suppose the bundle* $\mathfrak{B}$ *over* (K,L) *has a cross-section* f_0. *If we restrict attention to cross-sections* f *of* $\mathfrak{B}$ *which coincide with* f_0 *on* L, *and to homotopies relative to* L, *then the assignment of* $\bar{d}(f,f_0)$ *to each such* f *sets up a 1-1 correspondence between homotopy classes of cross-sections and elements of* $H^q(K,L;\mathfrak{B}(\pi_q))$.

If $f \simeq f'$ rel. L, §36.4 asserts that $\bar{d}(f,f_0) = \bar{d}(f',f_0)$. Thus each homotopy class corresponds to a single cohomology class.

Suppose $\bar{d}(f,f_0) = \bar{d}(f',f_0)$. By the addition formula §36.6, we have $\bar{d}(f,f') = 0$. Since $K = L \cup K^q$, §36.5 provides a homotopy $f \simeq f'$ rel. L. Thus, distinct homotopy classes correspond to distinct cohomology classes.

Now let $d \in H^q(K,L;\mathfrak{B}(\pi_q))$ be given. By §37.2, there exists an extension f of $f_0|L$ to all of K such that $\bar{d}(f_0,f) = -d$. By §36.6, we have $\bar{d}(f,f_0) = d$. This completes the proof.

37.6. Specialization of results to $K \times Y$. If we specialize $\mathfrak{B}$ to be the product bundle $K \times Y$ the preceding results may be given a slightly revised and simpler form. Note first that the coefficient bundle $\mathfrak{B}(\pi_q)$ is likewise a product, hence we may deal with ordinary cohomology groups with coefficients in $\pi_q = \pi_q(Y)$.

Any cross-section of $K \times Y$ is the graph of a map $K \to Y$, and any map provides a cross-section. A homotopy k of a map $K \to Y$ provides a homotopy $k'(x,t) = (x,k(x,t))$ of the graph, and conversely. The relation "graph" is a 1-1 correspondence which preserves relations such as equality on L, one function is an extension of another, and homotopic relative to L.

If f: $L \to Y$, we define the primary obstruction $\bar{c}(f)$ (to the extension of f to K) to be the primary obstruction to extending its graph. If f_0,f_1: $K \to Y$ and $f_0|L = f_1|L$, we define the primary difference $\bar{d}(f_0,f_1)$ to be the primary difference of their graphs.

With these conventions, all of the preceding work, beginning with §32, may be divested of the bundle language, and restated in terms of maps of complexes into Y, *and their homotopies. We shall assume any such restatement without further comment.*

37.7. Interpretation for the generalized $\bar{c}$ and $\bar{d}$. In keeping with the remarks of §35.9 and §36.11, the hypothesis on Y and q in the preceding theorems can be weakened. In §37.2, the conditions of §36.11 are adequate. For §37.3 we require that the conditions of §36.11 hold with L in place of (K,L), and the conditions of §35.9 hold for (K,L). In §37.4, it suffices for the conditions of §36.11 to hold with L in place of (K,L) and with K in place of (K,L). For §37.5, the conditions of §36.11 suffice.

37.8. The primary obstruction to contracting Y. We continue with the assumption that q is the least integer such that $\pi_q(Y) \neq 0$. We assume, moreover, that *Y is a complex.*

Let y_0 be a point of Y and let g_0: $Y \to Y$ be the constant map $g_0(y) = y_0$. Let g_1: $Y \to Y$ be the identity map. Define

$$\bar{d}(Y) \in H^q(Y;\pi_q), \qquad \pi_q = \pi_q(Y),$$

by

$$\bar{d}(Y) = \bar{d}(g_0,g_1) \qquad \text{see §37.6.}$$

We call it *the primary obstruction to contracting Y into y_0.*

If $q > 0$, then Y is arcwise connected, and any two constant maps are homotopic. Then §36.4 asserts that $\bar{d}(Y)$ is independent of the choice of y_0. In any case we can assume that y_0 is a vertex of Y.

37.9. Lemma. *Let f_0: $K \to Y$ be the constant map $f_0(x) = y_0$, and let f: $K \to Y$ be any map. Then $\bar{d}(f_0,f) = f^*\bar{d}(Y)$.*

We prove first an elementary fact about the graph relation between maps and cross-sections. Let g: $Y \to Y$ and let g' be its graph: $g'(y) = (y,g(y))$. Let $\bar{f}$: $K \times Y \to Y \times Y$ be defined by $\bar{f}(x,y) = (f(x),y)$. Then $\bar{f}$ is a bundle map inducing the map f of the base space K into the base space Y. Let ϕ' be the cross-section induced by g' and $\bar{f}$, and let ϕ' be the graph of ϕ. By definition of the induced cross-section, $\bar{f}\phi' = g'f$. Using this, we find that $\phi = gf$. This may be restated: *Under the graph relationship between cross-sections and maps, induced cross-sections under $\bar{f}$ correspond to compositions with f.*

Since $g_0(y) = y_0$, we have $g_0f(x) = y_0$. Therefore f_0 is induced by f and g_0. Since $g_1(y) = y$, we have $g_1f(x) = f(x)$. Therefore f is induced by f and g_1. Then §36.4 states

$$\bar{d}(f_0,f) = f^*\bar{d}(g_0,g_1) = f^*\bar{d}(Y).$$

In §§37.3–37.5, we take f_0 to be the constant map; and, using the lemma, we obtain the following three results.

37.10. Extension theorems.

Theorem. *Let L be a subcomplex of K, and f: $L \to Y$, then $\bar{c}(f) = \delta f^*\bar{d}(Y)$. Thus f is extendable to a map $L \cup K^{q+1} \to Y$ if and only if $\delta f^*\bar{d}(Y) = 0$.*

37.11. Theorem. *If $\dim (K - L) \leqq q + 1$, i: $L \to K$ is the inclusion map, and f is a map $L \to Y$, then f extends to a map $K \to Y$ if and only if there exists a $d' \in H^q(K;\pi_q)$ such that $i^*d' = f^*\bar{d}(Y)$. For each such d', there exists an extension f' of f such that $f'^*\bar{d}(Y) = d'$.*

37.12. Homotopy classification theorem. *Let $\dim K = q$. Then the assignment of $f^*\bar{d}(Y)$ to each map f: $K \to Y$ sets up a 1-1 correspondence between homotopy classes of maps $K \to Y$ and elements of $H^q(K;\pi_q)$.*

In the last theorem, we have applied §37.5 with $L = 0$. This restriction on L can be lifted by using the notion of the difference homomorphism $(f - f_0)^*$: $H^q(Y) \to H^q(K,L)$ where $f = f_0$ on L. This is described in [89].

37.13. The case $Y = S^q$. Assume now that Y is a q-sphere S^q. Then $\pi_q(S^q)$ is infinite cyclic. Thus the cohomology groups appearing in the preceding three theorems may be treated as the ordinary groups with integer coefficients. Furthermore, $H^q(S^q;\pi_q)$ is infinite cyclic and

(1) $$\bar{d}(S^q) \text{ generates } H^q(S^q;\pi_q).$$

To prove this, we represent S^q as a complex consisting of two q-cells E_+,E_- with S^{q-1} as a common boundary. We take an arbitrary cellular decomposition on S^{q-1}. We can suppose $y_0 \in S^{q-1}$. To compute a representative cocycle of $\bar{d}(S^q)$, we must select a homotopy k of the identity map $g_1|S^{q-1}$ into the constant map g_0. Let k shrink S^{q-1} over E_- into y_0. Constructing F as in §33.4, we find that F maps the boundary of $E_+ \times I$ on S^q with degree 1, and the boundary of $E_- \times I$ on S^q with degree 0. Hence $d(g_1,k,g_0)$ is zero on E_- and is a generator of π_q on E_+; and (1) follows.

It is worth noting in this case that the primary obstructions and differences appearing in §§37.10–37.12 are effectively computable, at least for cellular maps. With a little more effort, using the Hurewicz isomorphism $H_q(Y) \approx \pi_q(Y)$, one can prove the same without restricting Y to be S^q. However we have no general rules for effectively computing the obstructions and differences appearing in the theorems 37.3, 37.4, and 37.5. This is one of the chief problems of the theory.

37.14. The relation $p^*\bar{c}(\mathfrak{B}) = 0$.

THEOREM. *If* $\mathfrak{B} = \{B,p,X,Y\}$ *and both* B *and* X *are triangulable (i.e. admit cellular decompositions), then*

$$p^*\bar{c}(\mathfrak{B}) = 0.$$

Let $\mathfrak{B}^2 = \{B^2,p',B,Y\}$ be the bundle induced over B by p: $B \to X$ and the bundle $\mathfrak{B}$ over X, and let $\tilde{p}$: $\mathfrak{B}^2 \to \mathfrak{B}$ be the natural map. By §35.7, we have

$$\tilde{p}^*\bar{c}(\mathfrak{B}) = \bar{c}(\mathfrak{B}^2).$$

In §10.4 we showed that $\mathfrak{B}^2$ has a cross-section. Therefore $\bar{c}(\mathfrak{B}^2) = 0$. This gives

$$\tilde{p}^*\bar{c}(\mathfrak{B}) = 0$$

which is the form the conclusion of the theorem should have.

37.15. We give now an unpublished result due to G. W. Whitehead which adds a measure of precision to the preceding result. Let $\mathfrak{B} =$

$\{B,p,X,Y\}$. Let $y_0 \in B$, $x_0 = p(y_0)$, and $Y_0 = p^{-1}(x_0)$. We assume that X is triangulable with x_0 as a vertex, and that B is triangulable with Y_0 as a subcomplex. Define f by $f(x_0) = y_0$ so that f is a cross-section of the part of $\mathcal{B}$ over x_0. We obtain the diagram

$$\begin{array}{ccccc}
\bar{d}(Y_0) & \delta & p_1^*\bar{c}(f) & k^* & \\
H^q(Y_0;\pi_q) & \rightarrow & H^{q+1}(B,Y_0;\mathcal{B}^2(\pi_q)) & \rightarrow & H^{q+1}(B;\mathcal{B}^2(\pi_q)) \\
\uparrow p_0^* & & \uparrow p_1^* & & \uparrow p^* \\
 & \delta & & j^* & \\
H^q(x_0;\pi_q) & \rightarrow & H^{q+1}(X,x_0;\mathcal{B}(\pi_q)) & \rightarrow & H^{q+1}(X;\mathcal{B}(\pi_q)) \\
 & & \bar{c}(f) & & \bar{c}(\mathcal{B})
\end{array}$$

where k,j are inclusion maps, and p_0,p_1 are maps induced by p. Since the part of $\mathcal{B}$ over x_0 is a product, the coefficient groups $\mathcal{B}^2(\pi_q),\mathcal{B}(\pi_q)$, on the left, reduce to π_q. Then, we have

37.16. Theorem. *Under the above hypotheses,*

$$\delta\bar{d}(Y_0) = p_1^*\bar{c}(f).$$

Treating X as a pair $(X,0)$, then f and j induce the vacuous cross-section. Hence

$$j^*\bar{c}(f) = \bar{c}(\mathcal{B})$$

follows from §35.7. Applying §37.14, we obtain $p^*j^*\bar{c}(f) = 0$. Commutativity in the right square of the diagram gives

$$k^*p_1^*\bar{c}(f) = 0.$$

Exactness of the cohomology sequence assures us that there is a $d \in H^q(Y_0;\pi_q)$ such that $\delta d = p_1^*\bar{c}(f)$.

Let g_0 be the cross-section of the part of $\mathcal{B}^2$ over Y_0 induced by f. Recall (§10.2) that

$$B^2 \subset B \times B$$

consists of pairs (b,b') in B such that $p(b) = p(b')$. Thus the part of B^2 over Y_0 is just $Y_0 \times Y_0$. In this representation g_0 is the graph of the constant map $Y_0 \rightarrow y_0$. The cross-section g_1 of $\mathcal{B}^2$ is given by $g_1(b) = (b,b)$. Then $g_1|Y_0$ is the graph of the identity map. Hence, by §37.8,

$$\bar{d}(g_0,g_1|Y_0) = \bar{d}(Y_0).$$

By §36.7, we have

$$\delta\bar{d}(Y_0) = \bar{c}(g_0) - \bar{c}(g_1|Y_0).$$

The last term is zero since $g_1|Y_0$ is extendable to the cross-section g_1. Since g_0 is induced by f_0, we have $p_1^*\bar{c}(f) = \bar{c}(g_0)$, and the theorem follows.

37.17. Historical remarks. The theorems 37.11 and 37.12 with $Y = S^q$ are the extension and classification theorems due to Hopf. He stated them in the language of homology. The simpler formulation in terms of cohomology is due to Whitney. An excellent treatment is given in the book of Hurewicz and Wallman [56].

The notion of the characteristic class of a bundle is due to Whitney [103]. Independently, Stiefel treated the characteristic classes of the tangent sphere bundle of a manifold [91].

The primary obstructions and differences for maps $K \to Y$ are due to Eilenberg [31]. The development which we have given of these ideas (in §§29–37) follows closely the treatment given by Eilenberg. The formulation for bundles requires only the complication of a bundle of coefficients.

§38.* THE WHITNEY CHARACTERISTIC CLASSES OF A SPHERE BUNDLE

38.1. Conventions. Throughout this article $\mathfrak{B}$ will denote an $(n - 1)$-sphere bundle over a complex K. For notational convenience, we denote by Y^q the Stiefel manifold

$$Y^q = V_{n,n-q} = O_n/O_q, \quad q = 0, 1, \cdots, n - 1.$$

And we define $\mathfrak{B}^q$ to be the associated bundle of $\mathfrak{B}$ with the fibre Y^q. Then $\mathfrak{B}^0$ is the principal bundle of $\mathfrak{B}$, and $\mathfrak{B}^{n-1} = \mathfrak{B}$. Since $O_{q-1} \subset O_q$, we have natural projections

$$(1) \qquad O_n = Y^0 \to Y^1 \to \cdots \to Y^{n-1} = S^{n-1}$$

and Y^{q-1} is a $(q - 1)$-sphere bundle over Y^q (see §7.8). By §9.6, these projections induce projections

$$(2) \qquad B^0 \to B^1 \to \cdots \to B^{n-1} \to K,$$

and any composition of them is the projection of a bundle structure with a suitable Stiefel manifold as fibre. In particular, B^{q-1} is a $(q - 1)$-sphere bundle over B^q. The composition of $B^q \to B^{q+1} \to \cdots \to K$ is the projection of $\mathfrak{B}^q$.

38.2. Definition. The qth *characteristic class of* $\mathfrak{B}$ $(q = 1, \cdots, n)$, in the sense of Whitney [103], is defined to be the characteristic class of $\mathfrak{B}^{q-1}$. We denote it by $c^q(\mathfrak{B})$; thus

$$(3) \qquad c^{q+1}(\mathfrak{B}) = \bar{c}(\mathfrak{B}^q).$$

According to §25.6, π_q is the first non-zero homotopy group of Y^q. Therefore

$$(4) \qquad c^{q+1}(\mathfrak{B}) \in H^{q+1}(K;\mathfrak{B}^q(\pi_q)).$$

* See App. sect. 9 and 10.

Also, by §25.6,

$$(5) \qquad \pi_q(Y^q) = \begin{cases} \infty & \text{if } q \text{ is even, or } q = n - 1, \\ 2 & \text{if } q \text{ is odd and } < n - 1. \end{cases}$$

In the second case, $\mathfrak{B}^q(\pi_q)$ is a product bundle; for, a cyclic group of order 2 has no non-trivial automorphisms. Thus,

$$(4') \qquad c^{2q}(\mathfrak{B}) \in H^{2q}(K; \text{mod } 2), \qquad 1 < 2q < n.$$

In the first case, the group of $\mathfrak{B}^q(\pi_q)$ is O_n/R_n which is cyclic of order 2. It operates effectively on $\pi_q(Y^q)$. To see this, choose a generator f: $S^q \to Y^q$ of the group as described in §25.6. Let $r \in O_n$ have determinant -1 and act as the identity in the space orthogonal to S^q. Then left translation of Y^q by r maps $f(S^q)$ on itself with degree -1; so r reverses sign in $\pi_q(Y^q)$. Thus, by §13.7,

If q is even or $q = n - 1$, $\mathfrak{B}^q(\pi_q)$ is a product bundle if and only if $\chi(\mathfrak{B})$: $\pi_1(K) \to O_n/R_n$ is trivial.

For example, let K be a manifold and $\mathfrak{B}$ the tangent sphere bundle. If K is orientable, then $\mathfrak{B}$ is equivalent to a bundle in R_n, so $\chi(\mathfrak{B}) = 0$, and each $\mathfrak{B}^q(\pi_q)$ is a product bundle. If K is non-orientable, $\mathfrak{B}$ is not reducible to R_n; therefore the weakly associated bundle with fibre O_n/R_n is not a product bundle (see §9.5), hence $\chi(\mathfrak{B})$ is non-trivial and $\mathfrak{B}^q(\pi_q)$ is not a product bundle (q even or $q = n - 1$).

38.3. Interpretation. By its definition, $c^{q+1}(\mathfrak{B})$ is the primary obstruction to forming a cross-section of $\mathfrak{B}^q$. Since π_q is the first non-zero homotopy group of Y^q, $\mathfrak{B}^q|K^q$ has a cross-section, and $\mathfrak{B}^q|K^{q+1}$ has a cross-section if and only if $c^{q+1}(\mathfrak{B}) = 0$. Now, for any integer p, §9.5 states that $\mathfrak{B}^q|K^p$ has a cross-section if and only if $\mathfrak{B}|K^p$ is equivalent in O_n to a bundle with group O_q. But O_q operating in S^{n-1} leaves a great $(n - q - 1)$-sphere S' pointwise fixed. Then the subbundle of $\mathfrak{B}|K^p$ corresponding to S' is a product bundle. Thus, we have proved

38.4. THEOREM. *For each $q = 0, 1, \cdots, n - 1$, there exists a map*

$$\psi: \quad K^q \times S^{n-q-1} \to B$$

which, for each x in K^q, maps the fibre $x \times S^{n-q-1}$ orthogonally into the fibre over x in B. And there exists a similar map of $K^{q+1} \times S^{n-q-1}$ if and only if $c^{q+1}(\mathfrak{B}) = 0$.

38.5. The cohomology sequence of a coefficient sequence. The characteristic classes of $\mathfrak{B}$ are not independent. To state the relations requires the use of a little-known operation on cohomology groups which we describe first. Let

$$(6) \qquad 0 \to L \xrightarrow{\lambda} M \xrightarrow{\mu} N \to 0$$

be an *exact* sequence of abelian groups and homomorphisms, i.e. kernel $\lambda = 0$, image λ = kernel μ, and image $\mu = N$. If we pass to cochains in K with coefficients in L, M and N, then λ,μ induce homomorphisms of the cochain groups, and we have the diagram

$$
\begin{array}{ccccccccc}
0 & \rightarrow & C^p(K;L) & \xrightarrow{\lambda} & C^p(K;M) & \xrightarrow{\mu} & C^p(K;N) & \rightarrow & 0 \\
 & & \downarrow\delta & & \downarrow\delta & & \downarrow\delta & & \\
0 & \rightarrow & C^{p+1}(K;L) & \xrightarrow{\lambda} & C^{p+1}(K;M) & \xrightarrow{\mu} & C^{p+1}(K;N) & \rightarrow & 0
\end{array}
\tag{7}
$$

where each line is an exact sequence. Clearly we have commutativity in each square: $\lambda\delta = \delta\lambda$ and $\mu\delta = \delta\mu$.

Let z in $C^p(K;N)$ be a cocycle. Choose u in $C^p(K;M)$ such that $\mu u = z$. It follows that $\mu\delta u = 0$. Exactness of the lower line provides a w in $C^{p+1}(K;L)$ such that $\lambda w = \delta u$. Furthermore w is a cocycle; for λ: $C^{p+2}(K;L) \rightarrow C^{p+2}(K;M)$ has kernel $= 0$, and $\lambda\delta w = \delta\lambda w = \delta\delta u = 0$. Thus, we have proved

For each cocycle z in $C^p(K;N)$, there exists a cocycle w in $C^{p+1}(K;L)$ and a cochain u in $C^p(K;M)$ such that

$$\lambda w = \delta u, \qquad \mu u = z. \tag{8}$$

Suppose now that z,u',w' is a second triple satisfying analogous relations. Since $\mu u = \mu u' = z$, we have $\mu(u - u') = 0$. Exactness provides a v in $C^p(K;L)$ such that $\lambda v = u - u'$. Then

$$\lambda\delta v = \delta\lambda v = \delta u - \delta u' = \lambda w - \lambda w'.$$

Since λ has kernel $= 0$, we have

$$\delta v = w - w'.$$

Thus the assignment of w to z defines a unique map

$$Z^p(K;N) \rightarrow H^{p+1}(K;L). \tag{9}$$

It is easily proved that (9) is a homomorphism. Suppose now that $z = \delta c$ is a coboundary. Choose d in $C^{p-1}(K;M)$ such that $\mu d = c$. Then $\mu\delta d = z$. Taking $u = \delta d$, we have $\delta u = 0$. Hence we may choose $w = 0$. Thus, under (9), coboundaries are mapped into zero. It follows that (9) induces a homomorphism

$$\delta^*: \quad H^p(K;N) \rightarrow H^{p+1}(K;L). \tag{10}$$

It is to be noted that the diagram of (7) is algebraically identical to the diagram (26) of §31.17 obtained from a complex K, a subcomplex L and just one coefficient group. Here $C^p(K;L)$ corresponds to $C^p(K,L)$, $C^p(K;M)$ to $C^p(K)$, $C^p(K;N)$ to $C^p(L)$, etc. A comparison of the definitions shows that δ^* in (10) corresponds to δ: $H^p(L) \to H^{p+1}(K,L)$ defined in §31.17. In §31.18, the exactness of the cohomology sequence of (K,L) was stated and proved in part. This proof was based entirely on the algebraic properties of the diagram (26) of §31.18. Since the diagram (7) above enjoys these properties, it follows that the same argument proves that *the sequence*

$$\cdots \to H^{p-1}(K;N) \xrightarrow{\delta^*} H^p(K;L) \xrightarrow{\lambda^*} H^p(K;M) \xrightarrow{\mu^*} H^p(K;N) \to \cdots \tag{11}$$

is also exact. We refer to the sequence (11) as *the cohomology sequence of K and the coefficient sequence (6).*

For the application we have in mind the preceding must be generalized to cochains with coefficients in bundles. We replace (6) by

$$0 \to \mathfrak{B}(L) \xrightarrow{\lambda} \mathfrak{B}(M) \xrightarrow{\mu} \mathfrak{B}(N) \to 0 \tag{12}$$

where $\mathfrak{B}(L)$, etc., are coefficient bundles over K, and λ,μ are bundle homomorphisms in the sense that, for each x in K, the sequence

$$0 \to L_x \xrightarrow{\lambda_x} M_x \xrightarrow{\mu_x} N_x \to 0 \tag{13}$$

is exact, and the homomorphisms λ_x,μ_x commute with the translations of L_x, M_x, N_x along curves in K. We obtain then the diagram (7) with L, M, and N replaced by $\mathfrak{B}(L)$, $\mathfrak{B}(M)$, and $\mathfrak{B}(N)$ respectively. The generalization of the definition of δ^* and the proof of exactness of the generalized sequence (11) is entirely mechanical.

38.6. The coefficient sequence for the Whitney classes. We intend to show, for a sphere bundle $\mathfrak{B}$, that $\delta^* c^{2q}(\mathfrak{B}) = c^{2q+1}(\mathfrak{B})$. This requires $\mathfrak{B}(N)$ to be the coefficient bundle of $c^{2q}(\mathfrak{B})$, and $\mathfrak{B}(L)$ to be the same of $c^{2q+1}(\mathfrak{B})$. Before the relation makes sense, we must define $\mathfrak{B}(M)$, λ and μ, and prove exactness.

Lemma. *Let the integer q satisfy $2 \leqq 2q < n$, and let $\mathfrak{B}'$ be the $(2q-1)$-sphere bundle*

$$p': \quad Y^{2q-1} \to Y^{2q} \qquad \text{(see §38.1).}$$

Then, in the section

$$\pi_{2q}(Y^{2q}) \xrightarrow{\Delta} \pi_{2q-1}(S_0^{2q-1}) \xrightarrow{i_*} \pi_{2q-1}(Y^{2q-1}) \tag{14}$$

of the homotopy sequence of $\mathfrak{B}'$, the kernel of Δ is zero and i_* is a homomorphism onto.

By §25.6, a generator of $\pi_{2q-1}(Y^{2q-1})$ is represented by the fibre over the point v_0 of Y^{2q}. Therefore i_* is a homomorphism onto. Since the image of i_* is cyclic of order 2, the kernel of i_* consists of the "even" elements of $\pi_{2q-1}(S_0^{2q-1})$. By the exactness of the homotopy sequence, the latter is the image of Δ. But $\pi_{2q}(Y^{2q}) = \infty$. Hence the kernel of Δ is zero, and the lemma is proved.

38.7. It follows that we may adjoin zero groups on the left and right ends of (14) and have an exact sequence as in (6). But we need a sequence of *bundles* as in (12), i.e. for each $x \varepsilon K$, an exact sequence of groups as in (13). Now, for each $x \varepsilon K$, the bundle projection p': $B^{2q-1} \to B^{2q}$ reduces to a bundle projection

$$(15) \qquad p'_x: \quad Y_x^{2q-1} \to Y_x^{2q}$$

of the fibres over x. Denote this bundle by $\mathfrak{B}'_x$. Now $\mathfrak{B}'_x$ is a replica of the bundle $\mathfrak{B}'$ of the lemma. Hence the terms of its homotopy sequence corresponding to (14) form the desired sequence (13).

Before this choice is properly defined we must clarify the situation with respect to the base point of the homotopy groups. Since $Y^p = R_n/R_p$, we have, by §16.11, that Y^p is simple in all dimensions. Hence its homotopy groups can be defined, without reference to a base point, as homotopy classes of maps of a sphere. Thus, no base point is needed for the end terms of (14). For the middle term no base point is needed, however, a particular fibre is assumed. We eliminate this choice of a fibre as follows.

Since Y^{2q} is simply-connected, the bundle of groups $\mathfrak{B}'(\pi_{2q-1}(S^{2q-1}))$ over Y^{2q} is a product bundle. Thus we have unique isomorphisms connecting the homotopy groups of the various fibres of $\mathfrak{B}'$. Using these isomorphisms an element of a homotopy group of one fibre determines a class of equivalent elements—one on each fibre. These equivalence classes form a group isomorphic to $\pi_{2q-1}(S^{2q-1})$. It is this group which we take as the middle term of (14). Then the sequence (14) is assigned to $\mathfrak{B}'$ without any choices being necessary.

We now define the sequence

$$L_x \xrightarrow{\lambda_x} M_x \xrightarrow{\mu_x} N_x$$

to be the section of the homotopy sequence of $\mathfrak{B}'_x$ corresponding to the section (14) of $\mathfrak{B}'$. Using the local product representations for the bundles $\mathfrak{B}^{2q-1}, \mathfrak{B}^{2q}$ over K, it is easy to see that the λ_x, μ_x commute with translations of the groups along a curve lying in a coordinate neighborhood, and then, by composition, along any curve. It follows that this choice of the sequence (13) defines a sequence (12) of coefficient bundles.

It is to be noted that $\mathfrak{B}(M)$ is just the weakly associated bundle of $\mathfrak{B}$ with fibre $\pi_{2q-1}(S^{2q-1})$ and group O_n/R_n operating in such a way that the non-trivial element reverses sign.

38.8. The coboundary relation.

THEOREM. *Let $\mathfrak{B}$ be an $(n-1)$-sphere bundle over K, let q be an integer with $2 \leqq 2q < n$. Then, with respect to the preceding choice of the sequence (12) of coefficient bundles, we have*

$$\delta^* c^{2q}(\mathfrak{B}) = c^{2q+1}(\mathfrak{B}).$$

As shown in §38.3, we may choose a cross-section f of $\mathfrak{B}^{2q-1}|K^{2q-1}$. Then $c(f) \varepsilon C^{2q}(K;\mathfrak{B}(N))$ is defined, and is a cocycle representing $c^{2q}(\mathfrak{B})$. Composing f with

$$p': \quad B^{2q-1} \to B^{2q}, \tag{16}$$

we obtain a cross-section $p'f$ of $\mathfrak{B}^{2q}|K^{2q-1}$. Since the fibre of $\mathfrak{B}^{2q}$ is $(2q-1)$-connected, we may extend $p'f$ to a cross-section g of $B^{2q}|K^{2q}$. Then $c(g) \varepsilon C^{2q+1}(K;\mathfrak{B}(L))$ is a cocycle representing $c^{2q+1}(\mathfrak{B})$, and

$$p'f = g|K^{2q-1}. \tag{17}$$

Now g imbeds K^{2q} topologically in B^{2q}. Let $\mathfrak{B}'_g$ denote the bundle of (16) restricted to $g(K^{2q})$. Then we may regard $\mathfrak{B}'_g$ as a $(2q-1)$-sphere bundle over K^{2q}. Then (17) states that f is a cross-section of $\mathfrak{B}'_g|K^{2q-1}$. Let $c(f,g)$ denote the obstruction to extending f to a cross-section of $\mathfrak{B}'_g$. Then the coefficients of $c(f,g)$ are elements of the $(2q-1)$st homotopy groups of fibres of the bundle (16), i.e.

$$c(f,g) \varepsilon C^{2q}(K;\mathfrak{B}(M)).$$

It should be kept in mind that $c(f,g)$ need not be a cocycle of K since $\mathfrak{B}'_g$ is only defined on K^{2q}.

Recalling the definition of δ^*, the theorem will follow once we have proved

$$\mu(-c(f,g)) = c(f), \tag{18}$$
$$\lambda c(g) = \delta(-c(f,g)). \tag{19}$$

To prove (18), let σ be a $2q$-cell and x its reference point. Shrinking σ to x, deforms $g|\sigma$ into a constant map g_0 with $g_0(\sigma) = y \varepsilon Y^{2q}_x$. A covering homotopy deforms $f|\dot{\sigma}$ into a map f_0 of $\dot{\sigma}$ into the fibre S^{2q-1}_0 of (16) over y. Then f_0 represents simultaneously

$$c(f,\sigma) \varepsilon \pi_{2q-1}(Y^{2q-1}_x), \quad \text{and} \quad c(f,g,\sigma) \varepsilon \pi_{2q-1}(S^{2q-1}_0).$$

Since

$$\mu_x: \quad \pi_{2q-1}(S_0^{2q-1}) \to \pi_{2q-1}(Y_x^{2q-1})$$

is induced by the inclusion map, we have proved that $\mu_x c(f,g,\sigma) = c(f,\sigma)$. Since $\pi_{2q-1}(Y_x^{2q-1})$ is cyclic of order 2, $c(f,\sigma) = -c(f,\sigma)$. This proves (18).

For (19), let σ denote a $(2q+1)$-cell of K. If we apply §35.12 to the bundle $\mathfrak{B}_g'|g(\dot{\sigma})$, we obtain

$$(\delta c(f,g))\cdot\sigma = c(f,g)\cdot\partial\sigma = -\Delta\alpha \tag{20}$$

where $\alpha \,\varepsilon\, \pi_{2q}(g(\dot{\sigma}))$ is represented by $g|\partial\sigma$. Shrinking $\dot{\sigma}$ over σ into the reference point x of σ, a covering homotopy deforms $g|\partial\sigma$ into a map representing $c(g,\sigma)$. A second covering homotopy deforms $\mathfrak{B}'|g(\dot{\sigma})$ into a bundle map into the bundle Y_x^{2q-1} over Y_x^{2q}. Since Δ commutes with bundle maps, it follows that $\Delta\alpha = \Delta_x c(g,\sigma)$. But Δ_x is λ_x. Combining with (20), we have

$$\delta c(f,g)\cdot\sigma = -\lambda_x c(g,\sigma).$$

This implies (19), and the proof is complete.

38.9. The 0 and 1-dimensional classes. The last theorem states nothing about $c^1(\mathfrak{B})$. We shall remedy this by defining $c^0(\mathfrak{B})$ suitably and proving a similar result.

Let $H = H_0(Y^0)$ with integer coefficients, and let $\tilde{H}$ be the reduced group, i.e. the subgroup of H generated by 0-cycles having a coefficient sum of zero. The operation of forming the coefficient sum of a 0-cycle defines a homomorphism μ of H into the group J of integers. Then

$$0 \to \tilde{H} \xrightarrow{\lambda} H \xrightarrow{\mu} J \to 0$$

is an exact sequence (λ = the inclusion).

For each x in K, let

$$L_x = \tilde{H}_0(Y_x^0), \quad M_x = H_0(Y_x^0), \quad N_x = J.$$

Let λ_x be the inclusion, and μ_x the coefficient sum. We obtain thus an exact sequence of coefficient bundles, as in (12), and $\mathfrak{B}(N)$ is a product bundle. Thus $H^0(K;\mathfrak{B}(N)) = H^0(K;J)$. Define the "unit" 0-cocycle $c \,\varepsilon\, C^0(K;J)$ by $c(v) = 1$ for each vertex v. Then $\delta c = 0$, and its cohomology class is denoted by c^0. *We set $c^0(\mathfrak{B}) = c^0$ for any sphere bundle $\mathfrak{B}$.*

38.10. THEOREM. *With respect to the above sequence of coefficient bundles, we have $\delta^* c^0 = c^1(\mathfrak{B})$.*

Let f be a cross-section of $\mathfrak{B}^0|K^0$. For each vertex v of K, let $c^0(f,v)$ be the element of M_v represented by the point $f(v)$ with coefficient 1. Then $c^0(f)$ is in $C^0(K;\mathcal{B}(M))$. Then $\mu_v c^0(f,v) = 1$ which proves that $\mu c^0(f)$ is the unit 0-cocycle.

Referring to the definition of δ^*, the theorem will follow once we have proved

$$\mu c(f) = \delta c^0(f). \tag{21}$$

Let σ be an oriented edge, and $\partial\sigma = v - v'$. Let v be the reference point of σ, and w the isomorphism $M_{v'}$ onto M_v obtained by translation along σ. Then

$$\delta c^0(f)\cdot\sigma = c^0(f)\cdot\partial\sigma = c^0(f,v) - wc^0(f,v').$$

The translation w can be achieved by a homotopy of $Y^0_{v'}$ along σ into Y^0_v. This carries $f(v')$ along σ into $f'(v')$. Then $wc^0(f,v')$ is represented by the cycle $1f'(v')$. Hence $\delta c^0(f)\cdot\sigma$ is represented by the cycle $1f(v) - 1f'(v')$. By definition of the obstruction, it also represents $c(f,\sigma)$. Since μ is an inclusion map, (21) follows, and the proof is complete.

38.11. THEOREM. *Every odd dimensional characteristic class of a sphere bundle has order 2.*

For $q > 0$ and $2q < n$, $c^{2q}(\mathcal{B})$ has coefficients mod 2, hence it is of order 2. Since δ^* is a homomorphism, $c^{2q+1}(\mathcal{B}) = \delta^* c^{2q}(\mathcal{B})$ is also of order 2.

For the case $q = 0$, let f_0 be a cross-section of $\mathcal{B}^0|K^0$. For each vertex v, let $f_1(v)$ be a point in the component of Y^0_v not containing $f_0(v)$. ($Y^0 = O_n$ has two components.) Then f_1 is also a cross-section of $\mathcal{B}^0|K^0$. It is clear that $c(f_1,\sigma) = -c(f_0,\sigma)$ for each edge σ. Then §33.5 gives

$$\delta d(f_0,f_1) = c(f_0) - c(f_1) = 2c(f_0)$$

which implies $2c^1(\mathcal{B}) = 0$.

REMARK. The above result shows that every characteristic class of an n-sphere bundle $\mathcal{B}$ has order 2 except $c^0(\mathcal{B})$ and, possibly, $c^n(\mathcal{B})$ when n is even (note the exception in (4) and (4')).

38.12. THEOREM. *The following conditions on an n-sphere bundle $\mathcal{B}$ are equivalent:*

(i) $c^1(\mathcal{B}) = 0,$

(ii) $\chi(\mathcal{B})\colon\ \pi_1(K) \to O_n/R_n$ *is trivial,*

(iii) $\mathcal{B}$ *is equivalent to a bundle with group* R_n,

(iv) *for each q, the coefficient bundle of* $c^q(\mathcal{B})$ *is a product.*

The equivalence of (ii) and (iv) was noted in §38.2. By definition, $\chi(\mathcal{B}) = \chi(\mathcal{B}')$ where $\mathcal{B}'$ is the weakly associated bundle with fibre

O_n/R_n. Since the group of $\mathcal{B}'$ is discrete, $\chi(\mathcal{B}') = 0$ if and only if $\mathcal{B}'$ is a product bundle. Since $\mathcal{B}'$ is a principle bundle, it is a product bundle if and only if it has a cross-section. By §9.5 this last condition is equivalent to (iii). Thus (ii) and (iii) are equivalent.

Since $c^1(\mathcal{B})$ is the primary obstruction of $\mathcal{B}^0$, condition (i) implies that $\mathcal{B}^0|K^1$ has a cross-section. Since $\mathcal{B}^0$ is a principal bundle, this implies that $\mathcal{B}^0|K^1$ is a product bundle. Then the associated bundle $\mathcal{B}'|K^1$ is also a product; hence $\chi(\mathcal{B}'|K^1) = 0$. But $\chi(\mathcal{B}'|K^1)$ is the composition of

$$\pi_1(K^1) \xrightarrow{f_*} \pi_1(K) \xrightarrow{\chi(\mathcal{B}')} O_n/R_n$$

where f is the inclusion map. Since f_* is onto, it follows that $\chi(\mathcal{B}') = 0$. Thus (i) implies (ii).

Suppose (iii) holds and $\mathcal{B}$ is represented as a bundle with group R_n. Then the same holds for $\mathcal{B}^0$. Since the left translations of O_n by R_n map R_n on itself, R_n determines a subbundle of $\mathcal{B}^0$ with fibre R_n. Since R_n is arcwise connected, the portion of this subbundle over K^1 has a cross-section. This provides a cross-section of $\mathcal{B}^0|K^1$; so (i) holds.

38.13. Remarks. The preceding results are due to Whitney [106]. We have recast the results somewhat and taken full account of the fact that the coefficient bundles may not be product bundles. For example, Whitney states the relation $c^{2q+1} = \delta^* c^{2q}$ in the form $c^{2q+1} = \frac{1}{2}\,\delta\omega c^{2q}$. He regards c^{2q+1} as having integer coefficients, c^{2q} as having coefficients mod 2, and ω as the inverse of reduction mod 2. Thus ω corresponds to i_*^{-1} in §38.6 and 1/2 to Δ^{-1}.

Whitney has announced [106] a "duality theorem" for sphere bundles. Let $\mathcal{B},\mathcal{B}'$ be sphere bundles over K with groups O_m, O'_n respectively. We may regard $O_m \times O'_n$ as a subgroup of O''_{m+n}. If we set $g''_{ji}(x) = g_{ji}(x)g'_{ji}(x)$, we obtain coordinate transformations for a sphere bundle $\mathcal{B}''$ over K with group O''_{m+n}. Then the duality theorem reads

$$(22) \qquad c^r(\mathcal{B}'') = \sum_{p+q=r} c^p(\mathcal{B}) \smile c^q(\mathcal{B}').$$

As Whitney has shown, it has numerous important applications.

No proof of (22) in full generality has been published.* The proposition is somewhat ambiguous. The use of the cup product presupposes that the coefficient groups $\pi_{p-1}(Y^{p-1})$ and $\pi_{q-1}(Y'^{q-1})$ are paired to $\pi_{r-1}(Y''^{r-1})$. A clarification of (22) would present such a pairing in a natural geometric fashion. W. T. Wu [108] has proved the special case obtained by reducing everything mod 2. Reduction mod 2

* See App. sect. 9.

eliminates the ambiguity. It also eliminates difficulty with the coefficient bundles: all such become product bundles when reduced mod 2. A clarification and proof of (22), in full generality, is needed.

§39.* THE STIEFEL CHARACTERISTIC CLASSES OF DIFFERENTIABLE MANIFOLDS

39.1. Definitions and interpretation. Let M denote a compact, connected, n-dimensional, differentiable manifold. Using the result that M is triangulable, the concepts of §38 can be applied to the tangent sphere bundle $\mathfrak{B}$ of M (defined in §12.10). The *Stiefel characteristic classes of M* are defined to be the Whitney classes of $\mathfrak{B}$.

$$c^q(M) = c^q(\mathfrak{B}), \qquad q = 0, 1, \cdots, n.$$

The result of §38.4 translates as follows:

For each q, there exists a continuous field of tangent, orthogonal $(n - q)$-frames (see §7.7) defined over the q-dimensional skeleton M^q of M. There exists such a field over M^{q+1} if and only if $c^{q+1}(M) = 0$.

The translation is effected by selecting a fixed orthogonal $(n - q)$-frame spanning S^{n-q-1}.

39.2. THEOREM. *$c^1(M) = 0$ if and only if M is orientable.*

This follows from §38.12; for the condition (iii) of §38.12 is equivalent to the orientability of M.

39.3. If M is orientable, then §38.12 asserts that the Stiefel classes belong to cohomology groups of M with ordinary coefficients (either infinite cyclic or cyclic of order 2). In particular, the coefficients of $c^n(M)$ lie in the infinite cyclic group $\pi_{n-1}(V_{n,1})$ $(V_{n,1} = S^{n-1})$. As is well known, the group $H^n(M;\pi_{n-1})$ is also infinite cyclic.

If M is non-orientable, the ordinary group $H^n(M)$, with integer coefficients, is cyclic of order 2. However §38.12 asserts that $\mathfrak{B}(\pi_{n-1})$ is not a product bundle. We will show that $H^n(M;\mathfrak{B}(\pi_{n-1}))$ is infinite cyclic. This requires a mild digression.

39.4. Homology with coefficients in a bundle. If π is an infinite cyclic group, we define a *pairing* of π with itself to the group J of integers by choosing an isomorphism ψ: $\pi \approx J$ and defining the *product* $\alpha\beta$, for α,β in π, by

$$(1) \qquad \alpha\beta = \psi(\alpha)\psi(\beta).$$

Then $\alpha\beta$ is an integer, the product is bilinear, and $\alpha^2 = 1$ if α generates π. There are just two possible choices for ψ and they differ in sign; hence the form of (1) shows that the product is independent of ψ. From this it follows that, if π' is also infinite cyclic and w: $\pi \approx \pi'$, then

$$(2) \qquad \alpha\beta = w(\alpha)w(\beta).$$

* See App. sect. 9 and 10.

Let $\mathcal{B}(\pi)$ be a bundle of coefficients over a complex K. We shall deal with both chains and cochains in $\mathcal{B}(\pi)$. The former we have not defined. The group $C_q(K;\mathcal{B}(\pi))$ of q-chains is defined to be the group $C^q(K;\mathcal{B}(\pi))$ of q-cochains. If $u \varepsilon C_q(K;\mathcal{B}(\pi))$, we define $\partial u \varepsilon C_{q-1}(K;\mathcal{B}(\pi))$ by

$$(\partial u)(\sigma) = \sum_{\tau} [\sigma:\tau] w_{\sigma\tau}^{-1}(u(\tau)) \qquad \text{(see §31.2).} \tag{3}$$

The sum is taken over q-cells τ having σ as a face. One proves $\partial\partial = 0$, and defines the homology group $H_q(K;\mathcal{B}(\pi))$ in the usual way.

Assuming π to be infinite cyclic, we define the *Kronecker index* of a q-cochain c and a q-chain u to be the integer

$$c \cdot u = \sum_{\sigma} c(\sigma)u(\sigma). \tag{4}$$

The sum is taken over all q-cells—one term for each cell. It is clear that $c(\sigma)u(\sigma)$ is independent of the orientation of σ. Using (2) with $w = w_{\sigma\tau}$, one proves easily that

$$c \cdot \partial v = \delta c \cdot v \tag{5}$$

for any q-cochain c and $(q + 1)$-chain v. Then the Kronecker index of a cocycle and a cycle depends only on their respective homology and cohomology classes. This yields a pairing of $H^q(K;\mathcal{B}(\pi))$ with $H_q(K;\mathcal{B}(\pi))$ to J which is also called the Kronecker index. It is clearly bilinear.

39.5. The fundamental n-cycle of a manifold. Now let $K = M$ be a differentiable n-manifold (orientable or non-orientable), let $\mathcal{B}$ be the tangent sphere bundle, $\pi = \pi_{n-1}(S^{n-1})$, and $\mathcal{B}(\pi)$ the associated coefficient bundle. We suppose the subdivision K of M is so fine that any two adjacent cells are contained in a coordinate neighborhood of M. Define $z \varepsilon C_n(K;\mathcal{B}(\pi))$ as follows. An orientation of an n-cell σ determines a concordant orientation of the tangent plane at the reference point x_σ. The latter determines a concordant orientation of the unit $(n - 1)$-sphere Y_σ (the fibre of $\mathcal{B}$ over x_σ). This in turn determines a generator $z(\sigma)$ of $\pi_\sigma = \pi_{n-1}(Y_\sigma)$. Clearly $z(-\sigma) = -z(\sigma)$. An $(n - 1)$-cell τ is a face of just two n-cells, σ, σ' say. An orientation of a neighborhood V of $\sigma \cup \sigma'$ determines concordant orientations of σ and σ', i.e. $[\tau:\sigma] = -[\tau:\sigma']$, and translation of $z(\sigma)$ along a path in V carries $z(\sigma)$ into $z(\sigma')$. It follows that $\partial z = 0$. We call z the *fundamental n-cycle* of M.

One proves now in the standard manner that any n-cycle of $C_n(M;\mathcal{B}(\pi))$ is a multiple of z. It follows that *$H_n(M;\mathcal{B}(\pi))$ is cyclic infinite.*

A cochain on the cell τ is one which is zero on all cells except possibly τ; it is called an *elementary* cochain. An n-cochain is always a cocycle due to the absence of $(n + 1)$-cells. If c is an n-cocycle on σ, σ' is an adjacent n-cell, and τ is the common $(n - 1)$-face, then there is an $(n - 1)$-cochain d on τ such that $c - \delta d$ is an n-cocycle on σ' (one uses here the fact that σ, σ' are the only n-cells having τ as a face). Any n-cell σ can be connected to a reference n-cell σ_0 by a "path" of successively adjacent n-cells. Using a succession of steps as above, one shows that any n-cocycle on σ is cohomologous to one on σ_0. Since any n-cocycle is a sum of elementary cocycles, it follows that *any n-cocycle is cohomologous to a cocycle on σ_0.*

As π_{σ_0} is infinite cyclic, it follows that $H^n(M;\mathfrak{B}(\pi))$ is a factor group of the infinite cyclic group of n-cocycles on σ_0. Hence H^n is a cyclic group. Define $c(\sigma) = 0$ for $\sigma \neq \sigma_0$, and $c(\sigma_0) = z(\sigma_0)$. Then c generates the group of n-cocycles on σ_0; and the cohomology class of c generates $H^n(M;\mathfrak{B}(\pi))$. Since

$$c\cdot z = c(\sigma_0)z(\sigma_0) = 1$$

we have $(mc)\cdot z = m$ for any integer m. But $mc = \delta d$ implies, by (5),

$$mc\cdot z = \delta d\cdot z = d\cdot\partial z = 0,$$

so $m = 0$. Therefore *$H^n(M;\mathfrak{B}(\pi))$ is infinite cyclic.*

39.6. The n-dimensional class of an n-manifold.

THEOREM. *If $n = \dim M$ is odd, then $c^n(M) = 0$. Therefore the tangent sphere bundle of M has a cross-section, i.e. M has a continuous field of non-zero tangent vectors.*

Since n is odd, $c^n(M)$ has order 2 (see §38.11). But $H^n(M;\mathfrak{B}(\pi))$ is infinite cyclic. Hence $c^n(M) = 0$.

There is a more general result which holds for manifolds of arbitrary dimension:

39.7 THEOREM. *If z is the fundamental n-cycle of M (see §39.5), then*

$$c^n(M)\cdot z = \text{the Euler number of } M.$$

It is known that the Euler number e of M is zero for any manifold of odd dimension. By the results of §39.5, $c^n(M)\cdot z = 0$ implies $c^n(M) = 0$. Thus, the above theorem generalizes the preceding one.

We shall omit the proof. It can be found in the book of Alexandroff and Hopf [1, p. 549]. It is a long proof and we have nothing to add. We shall give, however, a brief intuitive discussion which suggests the truth of the result.

Let K be a simplicial triangulation of M. Let K' and K'' denote.

respectively, the first and second barycentric subdivisions of K. A vertex v of K'' lies in the interior of just one simplex of K, let $\phi(v)$ denote the barycenter of that simplex. This vertex assignment determines a unique simplicial map ϕ: $K'' \to K'$. It is easily shown that the fixed points of ϕ are the barycenters of the simplexes of K.

Now x and $\phi(x)$ lie on a single simplex of K' and are joined by a unique line segment of the simplex. We assume that the triangulation is differentiable. Then the segment has a tangent direction at each point, and we can define $f(x)$ to be the unit tangent vector at x. It follows that f is defined and continuous except at the barycenters of the simplexes of K.

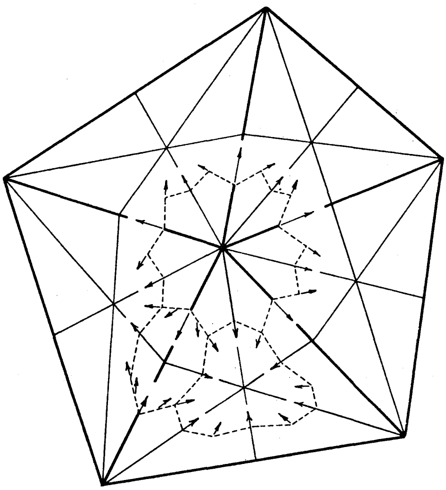

FIG. 8.

Let K'^* be the cellular decomposition of M dual to K' so that, for each q-simplex τ of K', there is a dual (transverse) $(n - q)$-cell of K'^* which is the union of those simplexes of K'' having the barycenter of τ as vertex of least order. Then the singularities of f occur at the centers of the n-cells of K'^*; so f provides a cross-section of $\mathfrak{B}|K'^{*^{n-1}}$, and $c(f)$ in $Z^n(K'^*;\mathfrak{B}(\pi_{n-1}))$ is defined.

Figure 8 illustrates the case of a 2-manifold. The heavy lines are the edges of K, these and the light lines are the edges of K'. Three of the 2-cells of K'^* are outlined by dotted lines. One is dual to a vertex of K' which is also a vertex of K; a second is dual to the barycenter of an edge of K; and the third is dual to the barycenter of a 2-simplex of K.

In general, K'^* has one n-cell for each simplex σ of K, namely, the dual, σ^*, of the barycenter of σ. Let z be the fundamental n-cycle of

K'^* (§39.5). Then

$$c(f,\sigma^*)\cdot z(\sigma^*) = (-1)^{\dim \sigma}. \tag{6}$$

To prove this we take the barycenter x_σ of σ as the reference point of σ^*. Let S_σ be the fibre of $\mathfrak{B}$ over x_σ. For $v \in S_\sigma$, let $g(v)$ be the point in which the line segment from x_σ in the direction v meets $\partial\sigma^*$. If the triangulation is sufficiently fine and *smooth*, g will provide a topological map of S_σ on $\partial\sigma^*$. Furthermore, g^{-1} will represent the generator $z(\sigma^*)$ of $\pi_{n-1}(S_\sigma)$. If f': $\partial\sigma^* \to S_\sigma$ is homotopic to $f|\partial\sigma^*$, it follows that (6) is just the degree of the map gf' of $\partial\sigma^*$ on itself (= the index of the singularity). Let $q = \dim \sigma$. Now σ^* is the product of the q-cell $\sigma \cap \sigma^*$ and the $(n - q)$-cell $\tilde{\sigma} \cap \sigma^*$ where $\tilde{\sigma}$ is the dual cell of σ. All vectors on $\partial(\tilde{\sigma} \cap \sigma^*)$ head "outward" so gf' is the identity on it. All vectors on $\partial(\sigma \cap \sigma^*)$ head "inward," so gf' is the antipodal transformation on it. The antipodal transformation has degree $(-1)^q$. Then the composite map gf' on the boundary of the product cell has degree $(-1)^q$.

If α_q denotes the number of q-simplexes of K, (6) implies that

$$c(f)\cdot z = \sum_{q=0}^{n} (-1)^q \alpha_q.$$

The right side is sometimes taken as the definition of the Euler number e of M. The alternative definition is $e = \sum_{q=0}^{n}(-1)^q R_q$ where R_q is the qth Betti number of M (i.e. the rank of $H_q(M)$). In any case, the equality of the two sums is a standard theorem. This completes our intuitive proof.

39.8. Corollary. *A differentiable manifold admits a continuous field of non-zero tangent vectors if and only if its Euler number is zero.*

39.9. Remarks. The preceding results provide satisfactory "computations" of $c^1(M)$ and $c^n(M)$. Corresponding results for the other Stiefel classes have not been obtained. Whitney [105] has announced that a representative cocycle for $c^q(M)$ mod 2 is obtained by assigning the value 1 to each q-cell of the subdivision K'^* (§39.7) of M. No proof of this has appeared.

Stiefel [91] has given a proof that $c^2(M) = 0$ if M is an orientable 3-manifold. We will not attempt to reproduce it here. His argument is sketchy in a major detail. He asserts that any mod 2 homology class of M is representable by a 2-manifold M' differentiably imbedded in M without singularities, and the structure of the "normal" bundle of M' in M is independent of the imbedding. This seems to be highly likely; but a full proof would be quite awkward. (See App. sect. 9.)

Granting that $c^2(M) = 0$, Stiefel goes on to prove that any orientable 3-manifold is parallelizable (i.e. its tangent bundle is a product bundle). The vanishing of $c^2(M)$ means that a 2-field f can be constructed on the 2-skeleton of M. Then the 3-cocycle $c(f)$ is defined with coefficients in $\pi_2(V_{3,2}) = \pi_2(R_3) = 0$; i.e. the secondary obstruction vanishes identically. Hence f extends to a 2-field f' over all of M. The 2-field f' is extended to a 3-field over M by adjoining, at each point x, a unit vector perpendicular to the pair of vectors $f'(x)$ so that the three vectors give a prescribed orientation of M.

Recently Thom [92, 93] and Wu [112, 113] have announced interesting results relating the Stiefel classes to products of other basic cohomology classes of the manifold. These offer methods for computing the Stiefel classes. They also show that the Stiefel classes are topological invariants of the manifold, i.e. they are independent of the differential structure. It is not known whether or not there exists a topological manifold having two differential structures with inequivalent tangent bundles. (See App. sect. 7.)

§40. QUADRATIC FORMS ON MANIFOLDS

40.1. Formulation of the problem. In §12.12 we proved that any differentiable manifold M admits a Riemannian metric. We consider now the problem of constructing *indefinite* quadratic forms, precisely, a covariant, second order, symmetric, tensor function which, at each point, has a non-zero determinant. The *signature* k of the quadratic form at a point is the number of negative characteristic values of the matrix. A simple continuity argument shows that k is independent of the point.

The problem is to construct a cross-section of a suitable tensor bundle over M. Just as the vector field problem led to a study of the Stiefel manifolds $V_{n,k}$, the present problem requires a preliminary study of the fibres involved.

As before, L_n denotes the group of non-singular, real matrices of order n, O_n the orthogonal group, and R_n the rotation group. Let S_n be the subset of L_n of symmetric matrices, and let $S_{n,k}$ be the subset of S_n of matrices of signature k. It is easily seen that $S_{n,k}$ is an open set in S_n, and $S_n = \bigcup_{k=0}^{n} S_{n,k}$.

If the problem is to construct a quadratic form of signature k over a differentiable n-manifold, then we are seeking a cross-section of the bundle, associated with the tangent bundle, with fibre $S_{n,k}$ and group O_n where σ in O_n operates on τ in $S_{n,k}$ by the similarity $\tau \rightarrow \sigma\tau\sigma'$ ($\sigma' =$ transpose of σ). (We are assuming that the reduction of the tangent bundle to the group O_n has already been carried out, then the covariant, contravariant and mixed variance problems become equivalent (see

§12.11).) We proceed to study the structure of $S_{n,k}$ under these operations of O_n.

40.2. The imbedding of $M_{n,k}$ in O_n. Choose as a reference point of $S_{n,k}$ the matrix

$$\sigma_k = \begin{vmatrix} -I_k & 0 \\ 0 & I_{n-k} \end{vmatrix} \tag{1}$$

where I_k denotes the unit diagonal matrix of order k. Define ϕ: $O_n \to S_{n,k}$ by

$$\phi(\sigma) = \sigma\sigma_k\sigma' \qquad (\sigma' = \text{transpose of } \sigma). \tag{2}$$

The subgroup of O_n carried into σ_k by ϕ is $O_k \times O'_{n-k}$. We may therefore identify $\phi(O_n)$ with the Grassmann manifold $M_{n,k}$ (see §7.9). Since any orthogonal element of $S_{n,k}$ is orthogonally equivalent to σ_k, we have

$$M_{n,k} = \phi(O_n) = O_n \cap S_{n,k}. \tag{3}$$

It is to be observed that, under this identification of the coset space $M_{n,k}$ with a set of matrices, the left translation of $M_{n,k}$ by σ in O_n corresponds to the similarity $\tau \to \sigma\tau\sigma'$ of the matrices.

40.3. The deformation retraction of $S_{n,k}$ into $M_{n,k}$. Any complex non-singular matrix τ can be factored in one and only one way into a product $\tau = \sigma\alpha$ where σ is unitary and α is positive definite Hermitian (see Chevalley [12; p. 14]). This decomposition is continuous and sets up a homeomorphism of the complex linear group with the product space of U_n and the space of positive definite Hermitian matrices.

An examination of the factorization shows that, if τ is real, so also are σ and α. Then $\sigma \varepsilon O_n$ and $\alpha \varepsilon S_{n,0}$. Therefore the function

$$\psi(\sigma,\alpha) = \sigma\alpha \tag{4}$$

defines a homeomorphism ψ: $O_n \times S_{n,0} \to L_n$.

40.4. Lemma. *If $\sigma \varepsilon O_n$, and $\alpha \varepsilon S_{n,0}$, then $\sigma\alpha \varepsilon S_{n,k}$ if and only if*

$$\sigma \varepsilon M_{n,k} \qquad \textit{and} \qquad \sigma\alpha = \alpha\sigma. \tag{5}$$

If $\sigma\alpha \varepsilon S_{n,k}$, then $(\sigma\alpha)' = \sigma\alpha$ implies $\alpha = \sigma^2(\sigma'\alpha\sigma)$. By the uniqueness of the factorization ψ^{-1}, we have $\sigma^2 = I_n$ and $\alpha = \sigma'\alpha\sigma$. Therefore $\sigma = \sigma'$ and $\sigma\alpha = \alpha\sigma$. Regarding σ as a linear transformation, let C (C') be the space of negative (positive) characteristic vectors of σ. Since $\sigma\alpha = \alpha\sigma$, α transforms C into C and C' into C'. Thus we may choose characteristic vectors for α, each lying in C or in C'. Then each will be a characteristic vector for $\sigma\alpha$. Since the characteristic values of α are positive it follows that σ and $\sigma\alpha$ have the same signature. Hence $\sigma \varepsilon M_{n,k}$.

Conversely, if (5) holds, then $(\sigma\alpha)' = \alpha'\sigma' = \alpha\sigma = \sigma\alpha$. So $\sigma\alpha$ is symmetric. The argument above shows that σ and $\sigma\alpha$ have the same signature; hence $\sigma\alpha \,\varepsilon\, S_{n,k}$.

40.5. Define $S'_{n,k}$ to be the subset of pairs (σ,α) in $M_{n,k} \times S_{n,0}$ such that $\sigma\alpha = \alpha\sigma$. Let μ in O_n operate on $M_{n,k} \times S_{n,0}$ by

$$\mu\cdot(\sigma,\alpha) = (\mu\sigma\mu',\mu\alpha\mu').$$

It is clear that μ transforms $S'_{n,k}$ into itself. Then the preceding lemma can be restated:

40.6. THEOREM. *The map ψ of (4) is a homeomorphism of $S'_{n,k}$ with $S_{n,k}$, and it commutes with the operations of O_n.*

Since ψ is an equivalence of the pair $(S'_{n,k},O_n)$ with the pair $(S_{n,k},O_n)$, we shall identify them under ψ and omit the ψ: $(\sigma,\tau) = \sigma\tau$.

40.7. Define p: $S_{n,k} \to M_{n,k}$ by

$$p(\sigma\alpha) = \sigma. \tag{6}$$

Then p is a continuous retraction of $S_{n,k}$ into $M_{n,k}$. Define a homotopy k by

$$k(\sigma\alpha,t) = \sigma[tI_n + (1-t)\alpha]. \tag{7}$$

Since α commutes with σ, so also does $tI_n + (1-t)\alpha$. Hence k deforms $S_{n,k}$ over itself. It is easy to verify the relations

$$k(\sigma\alpha,0) = \sigma\alpha, \qquad k(\sigma\alpha,1) = \sigma, \tag{8}$$

$$pk(\sigma\alpha,t) = \sigma, \tag{9}$$

$$k(\sigma,t) = \sigma, \tag{10}$$

$$\mu k(\sigma\alpha,t)\mu' = k(\mu\sigma\alpha\mu',t), \qquad \mu \,\varepsilon\, O_n. \tag{11}$$

In words:

40.8. THEOREM. *The homotopy k is a deformation retraction of $S_{n,k}$ into $M_{n,k}$ which commutes with the operations of O_n. For each σ in $M_{n,k}$, k contracts $p^{-1}(\sigma)$ over itself into σ.*

40.9. Reduction of the problem. Let M be a differentiable n-manifold, and let $\mathfrak{B}$ be the bundle over M with fibre $Y = S_{n,k}$ and group O_n associated with its tangent sphere bundle (i.e. $\mathfrak{B}$ is the bundle of quadratic forms of signature k at the various points of M). Let $\mathfrak{B}'$ be the subbundle of $\mathfrak{B}$ corresponding to the subspace $Y' = M_{n,k}$ of Y. (It must be noted that each σ in O_n maps $M_{n,k}$ on itself.)

For each x in M, choose an admissible ξ: $Y \to Y_x$, and define k' by

$$k'(b,t) = \xi k(\xi^{-1}b,t), \qquad b \,\varepsilon\, Y_x. \tag{12}$$

From (11) it follows that k' is independent of the choice of ξ. Setting $\xi = \phi_{i,x}$, where ϕ_i is a coordinate function of $\mathfrak{B}$, we find that k' is continuous. Thus:

40.10. THEOREM. *The homotopy k' is a deformation retraction of B into B', and for each x it contracts Y_x over itself into Y'_x. Then, for any cross-section f of $\mathfrak{B}$, k' provides a homotopy of f into a cross-section of $\mathfrak{B}'$ Thus $\mathfrak{B}$ has a cross-section if and only if $\mathfrak{B}'$ has a cross-section.*

If we return to the interpretation of $M_{n,k}$ as the manifold of k-planes in n-space, we may restate the result in the form

40.11. THEOREM. *A compact differentiable manifold admits an everywhere defined, continuous, non-singular, quadratic form of signature k if and only if it admits a continuous field of tangent k-planes.*

40.12. Applications. (See App. sect. 7.)

COROLLARY. *The results of Theorems 27.14 through 27.18 concerning fields of tangent k-planes over spheres are equally valid if "tangent k-plane" is replaced by "quadratic form of signature k."*

40.13. THEOREM. *If M is compact and dim M is odd, then M admits a quadratic form of signature 1. If dim M is even, then this holds if the Euler number of M is zero.*

In either case M has a tangent vector field (§39.6 and §39.7) which provides a field of tangent line elements (1-planes).

For 2-manifolds, the theorem may be completed by saying "if and only if the Euler number is zero." In this case, the vector field problem and the line element problem have 1-spheres as fibres and the first fibre is a 2-fold covering of the second. Choose a 1-field f over the 1-skeleton of M, and let f' be the induced field of line elements. For any 2-cell σ, the double covering maps $c(f,\sigma)$ onto $c(f',\sigma)$. It follows that the sum of the indices of $c(f')$ is twice that of $c(f)$, i.e. twice the Euler number. Thus, we have

The only compact 2-manifolds which admit a quadratic form of signature 1 are the torus and Klein bottle.

The above argument can be extended to arbitrary manifolds of even dimensions. One obtains an obstruction $c(f')$ having a sum of indices equal to twice the Euler number. However it is not the *primary* obstruction. The primary obstruction has dimension 2 and is zero. Instead $c(f')$ is a *secondary* obstruction. It may be possible to alter its cohomology class by an alteration of f' by a 1-cocycle on the 1-skeleton. So far as the author knows, this problem is unsolved. It may be worth noting that, if $H^1(M; \text{mod } 2) = 0$, then no alteration of f' on the 1-skeleton can affect the class of $c(f')$. (See App. sect. 11 and 12.)

If an n-manifold admits a quadratic form of signature k, a change of sign provides one of signature $n - k$. Therefore, a compact 3-manifold admits a quadratic form of any possible signature.

Further results along these lines should be obtainable without too great an effort, at least for 4-manifolds. The lower dimensional homotopy groups of the fibre $M_{n,k}$ are known (§25.8). An interesting special case is the problem of constructing a form of signature 2 on the complex projective plane.

40.14. The bundle $S_{n,k} \to M_{n,k}$. It may be of interest to show that the map p: $S_{n,k} \to M_{n,k}$ given in §40.7 admits a bundle structure.

The fibre Y is defined to be $p^{-1}(\sigma_k)$. These are the symmetric matrices which commute with σ_k. They have the form $\begin{vmatrix} A & 0 \\ 0 & B \end{vmatrix}$ where A is a negative definite $k \times k$ matrix and B is positive definite. Since the space of definite matrices of a fixed order is a cell, it follows that Y is a product space of two cells; hence Y is a cell.

The elements of O_n which map Y on itself are those which commute with σ_k. They form the subgroup denoted by $O_k \times O'_{n-k}$. The subgroup H of the latter which commutes with all elements of Y is the 4-group of matrices of the form $\begin{vmatrix} \pm I_k & 0 \\ 0 & \pm I_{n-k} \end{vmatrix}$. The group of Y is the factor group $G = O_k \times O'_{n-k}/H$.

If $b \in O_n$ and $\tau \in S_{n,k}$, we will adopt the notation $b \cdot \tau$ for $b\tau b'$. Then we may use the symbolism of §7.4. Let f be a local cross-section of $O_k \times O'_{n-k}$ in O_n. Then f is defined in a neighborhood V of σ_k in $M_{n,k}$, and $f(\sigma)\cdot\sigma_k = \sigma$. For b in O_n set $V_b = b \cdot V$, and

$$f_b(\sigma) = bf(b^{-1}\cdot\sigma), \qquad (\sigma \in V_b).$$

Define

$$\phi_b\colon \quad V_b \times Y \to p^{-1}(V_b)$$

by

$$\phi_b(\sigma,\alpha) = f_b(\sigma)\cdot\alpha$$

and define p_b: $p^{-1}(V_b) \to Y$ by

$$p_b(\tau) = [f_b p(\tau)]^{-1}\cdot\tau.$$

Now $\alpha \in Y$ implies $\alpha = \sigma_k\beta$ where β is positive definite. If $\mu = f_b(\sigma)$, then

$$\begin{aligned} p\phi_b(\sigma,\alpha) &= p(\mu\sigma_k\beta\mu') = p(\mu\sigma_k\mu'\mu\beta\mu') \\ &= \mu\sigma_k\mu' \end{aligned}$$

since $\mu\beta\mu'$ is positive definite. Then

$$\begin{aligned} p\phi_b(\sigma,\alpha) &= f_b(\sigma)\cdot\sigma_k = [bf(b^{-1}\cdot\sigma)]\cdot\sigma_k \\ &= b\cdot[f(b^{-1}\cdot\sigma)\cdot\sigma_k] = b\cdot(b^{-1}\cdot\sigma) = \sigma. \end{aligned}$$

Likewise

$$p_b\phi_b(\sigma,\alpha) = f_b(\sigma)^{-1}\cdot\phi_b(\sigma,\alpha) = f_b(\sigma)^{-1}\cdot[f_b(\sigma)\cdot\alpha] = \alpha.$$

Therefore ϕ_b is a homeomorphism. Computing coordinate transformations, we obtain

$$g_{cb}(\sigma)\cdot\alpha = p_c\phi_b(\sigma,\alpha) = f_c(\sigma)^{-1}\cdot[f_b(\sigma)\cdot\alpha].$$

Therefore $g_{cb}(\sigma)$ is the image in G of $f_c(\sigma)^{-1}f_b(\sigma)$ in $O_k \times O'_{n-k}$.

If the foregoing is compared with the proof of §7.4, one observes that $\{f_c(\sigma)^{-1}f_b(\sigma)\}$ are the coordinate transformations of the bundle $O_n \to M_{n,k}$. It follows that the bundle $S_{n,k}$ over $M_{n,k}$ is weakly associated with the bundle O_n over $M_{n,k}$ under the natural homomorphism of $O_k \times O'_{n-k}$ into G. It is easily shown that the principal bundle of $S_{n,k} \to M_{n,k}$ is the bundle $O_n/H \to M_{n,k}$. Since the latter is not a product for $0 < k < n$, neither is the former.

§41. Complex Analytic Manifolds and Exterior Forms of Degree 2

41.1. Quasi-complex manifolds. If M is a complex analytic manifold of n complex dimensions, its tangent bundle has, for fibre Y, the n-dimensional complex vector space, and, for group, the complex linear group CL_n. Passage to real and imaginary parts of the coordinates in M represents M as a real analytic manifold of $2n$ dimensions, Y becomes a real $2n$-space of variables $(x_1, \cdots, x_n, y_1, \cdots, y_n)$, and CL_n is imbedded in L_{2n}. If γ is a complex matrix in CL_n, and $\gamma = \alpha + i\beta$ where α,β are real $n \times n$ matrices, then, as an element of L_{2n}, γ is represented by the $2n \times 2n$ real matrix $\begin{vmatrix} \alpha & -\beta \\ \beta & \alpha \end{vmatrix}$. Conversely any matrix of L_{2n} of the latter form belongs to CL_n. Thus the tangent bundle of the real manifold is represented as a bundle in the subgroup CL_n of L_{2n}.

Let M be a real, differentiable $2n$-manifold. The foregoing shows that a necessary condition for M to be differentiably equivalent to the real form of a complex analytic manifold is that the tangent bundle of M be equivalent, in its group L_{2n}, to a bundle in the subgroup CL_n. A manifold satisfying this necessary condition will be called *a quasi-complex manifold.*

We shall restrict our attention to real manifolds, and derive conditions for such to be quasi-complex. Our results, so far as complex analytic manifolds are concerned, will be of the negative form: A particular real manifold is not quasi-complex, so it does not admit a complex analytic structure. It seems highly unlikely that every quasi-complex manifold has a complex analytic structure.

41.2. Unitary sphere bundles. Let M be a real differentiable $2n$-manifold, and $\mathfrak{B}$ its tangent bundle. We have seen in §12.9 that $\mathfrak{B}$ is equivalent in L_{2n} to a bundle $\mathfrak{B}'$ with group O_{2n}. If M is quasi-complex, $\mathcal{B}$ is equivalent in L_{2n} to a bundle $\mathcal{B}_1$ with group CL_n. By the analog of §12.9 for the complex case, $\mathcal{B}_1$ is equivalent in CL_n to a bundle $\mathcal{B}_1'$ in the unitary group $U_n = (CL_n) \cap O_{2n}$. It follows that $\mathcal{B}'$ and $\mathcal{B}_1'$ are equivalent in L_{2n}. According to §12.9, this equivalence holds also in O_{2n}. This proves: *M is quasi-complex if and only if its tangent sphere bundle is equivalent in O_{2n} to a bundle in the unitary group U_n.*

In general a sphere bundle will be called a *unitary sphere bundle* if its fibre is an odd dimensional sphere S^{2n-1} and its group is U_n.

41.3. The Chern characteristic classes. Let $\mathcal{B}$ denote a unitary $(2n-1)$-sphere bundle over a cell complex K. For notational convenience let Y'^q denote the "complex" Stiefel manifold (§25.7)

$$Y'^q = W_{n,n-q} = U_n/U_q, \quad q = 0, 1, \cdots, n-1.$$

Define $\mathcal{B}'^q$ to be the associated bundle of $\mathcal{B}$ with fibre Y'^q. Then $\mathcal{B}'^0$ is the principal bundle of $\mathcal{B}$, and $\mathcal{B}'^{n-1} = \mathcal{B}$. Since $U_{q-1} \subset U_q$, we have natural projections

$$(1) \qquad U_n = Y'^0 \to Y'^1 \to \cdots \to Y'^{n-1} = S^{2n-1},$$

and Y'^{q-1} is a unitary $(2q-1)$-sphere bundle over Y'^q. By §9.6, these projections induce projections

$$(2) \qquad B'^0 \to B'^1 \to \cdots \to B'^{n-1} \to K,$$

and any composition of them is the projection of a bundle structure. In particular B'^{q-1} is a unitary $(2q-1)$-sphere bundle over B'^q. The composition of $B'^q \to B'^{q+1} \to \cdots \to K$ is the projection of $\mathcal{B}'^q$.

41.4. With $\mathcal{B}$ as above, we define the $2q$th *characteristic class of* $\mathcal{B}$ $(q = 1, \cdots, n)$, in the sense of Chern [7], to be the characteristic class of $\mathcal{B}'^{q-1}$. We denote it by $c'^{2q}(\mathcal{B})$. Thus

$$(3) \qquad c'^{2q}(\mathcal{B}) = \bar{c}(\mathcal{B}'^{q-1}).$$

Since U_n is connected, $\pi_0(U_n) = 0$; hence all coefficient bundles are products, and we may use ordinary coefficients. According to §25.7, the first non-zero homotopy group of Y'^{q-1} is π_{2q-1} and it is infinite cyclic, thus

$$(4) \qquad c'^{2q}(\mathcal{B}) \in H^{2q}(K; \pi_{2q-1}).$$

We state the analog of §38.4; the proof is similar.

41.5. THEOREM. *For each $q = 1, \cdots, n$, there exists a map*

$$\psi\colon \quad K^{2q-1} \times S^{2n-2q+1} \to B$$

which, for each x, maps the fibre $x \times S^{2n-2q+1}$ by a unitary transfor-

mation into the fibre over x in B. And there exists a similar map of $K^{2q} \times S^{2n-2q+1}$ if and only if $c'^{2q}(\mathfrak{B}) = 0$.

The form this theorem takes in the complex analytic case is of interest:

41.6. Theorem. *If K is a complex analytic manifold of n complex dimensions, then, for each $q = 1, \cdots, n$, there exist $n - q + 1$ fields of tangent complex vectors which are independent at each point of K^{2q-1}; and there exist $n - q + 1$ such fields independent at each point of K^{2q} if and only if c'^{2q}(tangent bundle) $= 0$.*

41.7. Relations between the Chern and Whitney classes. The Chern classes are related to the Whitney classes. To exhibit the relationship, we adopt several conventions. The fibre of $\mathfrak{B}$ is the unit sphere in the space of n complex variables $(z_1, \cdots, z_n)$. The subgroup U_q of U_n operates trivially in the subspace $z_1 = \cdots = z_q = 0$. We pass to real coordinates by setting

$$x_{2i-1} = \mathfrak{R}z_i, \qquad x_{2i} = \mathfrak{I}z_i$$

Let O_q be the subgroup of O_{2n} operating trivially in the subspace $x_1 = \cdots = x_q = 0$. Then we have the obvious relations

$$U_n \cap O_{2q} = U_n \cap O_{2q+1} = U_q. \tag{5}$$

Thus, distinct cosets of U_q in U_n are contained in distinct cosets of O_{2q} and O_{2q+1} in O_{2n}. These coset inclusions induce natural imbeddings of the coset spaces:

$$U_n/U_q \subset O_{2n}/O_{2q}, \qquad U_n/U_q \subset O_{2n}/O_{2q+1}.$$

In the notations of §41.3 and §38.1, these become

$$Y'^q \subset Y^{2q}, \qquad Y'^q \subset Y^{2q+1}. \tag{6}$$

It is important to note that these imbeddings conform with the left translations of $U_n \subset O_{2n}$.

Let $\mathfrak{B}'$ be a unitary $(2n - 1)$-sphere bundle over K, and let $\mathfrak{B}$ denote the corresponding orthogonal bundle under the imbedding $U_n \subset O_{2n}$. Let $\mathfrak{B}^q$ be defined as in §38.1, and $\mathfrak{B}'^q$ as in §41.3. Then the inclusions (6) induce the relations in the following diagram

$$\begin{array}{ccccccccccccccc} B'^0 & = & B'^0 & \to & B'^1 & = & B'^1 & \to & B'^2 & \cdots & B'^{n-1} & = & B'^{n-1} & \to & K \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ B^0 & \to & B^1 & \to & B^2 & \to & B^3 & \to & B^4 & \cdots & B^{2n-2} & \to & B^{2n-1} & \to & K \end{array} \tag{7}$$

Each vertical arrow is an inclusion. The last two (on the right) are equalities.

The fibre of $B'^{q-1} \to B'^q$ is

$$U_q/U_{q-1} = S^{2q-1} = O_{2q}/O_{2q-1}$$

and it therefore coincides with the fibre of $B^{2q-1} \to B^{2q}$. It follows that $B'^{q-1} \to B'^q$ is the bundle obtained by restricting $B^{2q-1} \to B^{2q}$ to the subspace B'^q. The fibre S^{2q-1} generates both $\pi_{2q-1}(Y^{2q-1}) = 2$ and $\pi_{2q-1}(Y'^{q-1}) = \infty$. Thus the homomorphism

$$\lambda\colon\ \pi_{2q-1}(Y'^{q-1}) \to \pi_{2q-1}(Y^{2q-1}) \tag{8}$$

induced by the inclusion is onto. When $q = n$, λ is an equality.

Let f' be a cross-section of $\mathfrak{B}'^{q-1}|K^{2q-1}$. Then f' is also a cross-section f of $\mathfrak{B}^{2q-1}|K^{2q-1}$. It follows that $\lambda c(f') = c(f)$. We note also that f' is a cross-section of $\mathfrak{B}^{2q-2}|K^{2q-1}$. Since the first non-zero homotopy group of Y^{2q-2} is π_{2q-2}, it follows that the primary obstruction of $\mathfrak{B}^{2q-2}$ is zero. Thus we have proved

41.8. THEOREM. *If $\mathfrak{B}'$ is a unitary $(2n-1)$-sphere bundle over K and $\mathfrak{B}$ the corresponding orthogonal sphere bundle, then, for $q = 0, 1, \cdots, n-1$,*

$$c^{2q+1}(\mathfrak{B}) = 0, \qquad c^{2q}(\mathfrak{B}) = \lambda c^{2q}(\mathfrak{B}').$$

and

$$c^{2n}(\mathfrak{B}) = c^{2n}(\mathfrak{B}').$$

Since λ is reduction mod 2, the relation $\delta^* c^{2q}(\mathfrak{B}) = c^{2q+1}(\mathfrak{B})$, and the exactness of the sequence (11) of §38.5 show that the two sets of relations in the above theorem are not independent. Either set implies the other.

41.9. COROLLARY. *In order that a real, differentiable, compact $2n$-manifold be quasi-complex, it is necessary that each odd dimensional Stiefel class be zero and each even ($<2n$) dimensional class be the mod 2 image of a cohomology class with integer coefficients. In particular the manifold must be orientable (see §39.2).*

These appear to be rather strong conditions. Just how strong is not known. Does there exist an orientable differentiable manifold with a non-zero odd dimensional Stiefel class? Any guess is useless. We need effective methods for computing Stiefel classes, and applications to many examples.

41.10. Skew matrices. We approach the present problem from a new angle. Let $\mathfrak{B}$ denote a $(2n-1)$-sphere bundle with group O_{2n}. According to §9.5, $\mathfrak{B}$ *is equivalent to a bundle in U_n if and only if the weakly associated bundle with fibre O_{2n}/U_n has a cross-section.* We proceed to a study of this fibre.

Let W'_n be the set of real, non-singular, $2n \times 2n$, skew symmetric matrices, and let $W_n = O_{2n} \cap W'_n$. Let

(9) $$\sigma_0 = \begin{vmatrix} 0 & -I_n \\ I_n & 0 \end{vmatrix}$$

be a reference point of W_n. Define ϕ: $O_{2n} \to W_n$ by

(10) $$\phi(\sigma) = \sigma\sigma_0\sigma' \qquad (\sigma' = \text{transpose of } \sigma).$$

Now $\phi(\sigma) = \sigma_0$ if and only if σ has the form

(11) $$\sigma = \begin{vmatrix} \alpha & -\beta \\ \beta & \alpha \end{vmatrix}.$$

If we pass from complex coordinates $(z_1, \cdots, z_n)$ to real coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ where $z_i = x_i + iy_i$, then orthogonal matrices of the form (11) correspond exactly to unitary matrices. This imbedding of U_n in O_{2n} is equivalent to the one in §41.7 under conjugation by the orthogonal transformation carrying $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ into $(x_1, y_1, x_2, y_2, \cdots, x_n, y_n)$.

It follows that ϕ induces an identification

(12) $$O_{2n}/U_n = W_n.$$

Under this identification a left translation of the coset space by σ in O_{2n} corresponds to conjugation of W_n by σ (i.e. $\tau \to \sigma\tau\sigma'$).

Consider now the effect of the factorization $\tau = \sigma\alpha$ of §40.3 when τ is in W'_n. The analog of §40.4 is

41.11. Lemma. *If $\sigma \in O_{2n}$ and $\alpha \in S_{2n,0}$, then $\sigma\alpha \in W'_n$ if and only if*

(13) $$\sigma \in W_n \quad \text{and} \quad \sigma\alpha = \alpha\sigma.$$

Suppose $\sigma\alpha \in W'_n$. Then $(\sigma\alpha)' = -\sigma\alpha$ implies $\alpha = -\sigma^2(\sigma'\alpha\sigma)$. Since σ^2 is orthogonal and $\sigma'\alpha\sigma$ is positive definite, the uniqueness of the factorization ψ of §40.3 yields

$$-\sigma^2 = I_{2n}, \qquad \alpha = \sigma'\alpha\sigma.$$

But these conditions are equivalent to (13). The converse argument is trivial.

In analogy with §40.5, we shall identify W'_n with the subspace of $W_n \times S_{2n,0}$ consisting of pairs (σ,α) satisfying (13). Define the projection p: $W'_n \to W_n$ by

(14) $$p(\sigma\alpha) = \sigma.$$

Then p is a continuous retraction of W'_n into W_n. Define the homotopy k by the formula (7) of §40.7. Then the formulas (8), (9), (10) and (11) of §40.7 continue to hold, and we have

41.12. THEOREM. *There is a deformation retraction k of the manifold W'_n of skew matrices into the submanifold W_n of orthogonal skew matrices. The homotopy k commutes with the operations of O_{2n} on W'_n* (i.e. $\tau \rightarrow \sigma\tau\sigma'$). *For each σ, k contracts $p^{-1}(\sigma)$ over itself into σ.*

One may prove more here, namely: p: $W'_n \rightarrow W_n$ admits a bundle structure with fibre $S_{2n,0}$ and group U_n/H where H is the group of two elements $\pm I_{2n}$. The proof is similar to that of §40.14 for the analogous case $S_{n,k} \rightarrow M_{n,k}$.

41.13. Reduction of the problem. Let M be a real, differentiable $2n$-manifold. Let $\mathfrak{B}$ denote the tangent sphere bundle of M, and let $\mathfrak{B}'_s$ be the weakly associated bundle with fibre W'_n. Then $\mathfrak{B}'_s$ is the bundle of 2nd order skew symmetric tensors over M having non-zero determinants. (Since the group of $\mathfrak{B}$ is O_{2n}, the variance of the tensors is irrelevant; see §12.11.) Let $\mathfrak{B}_s$ denote the subbundle corresponding to W_n.

For each x in M, choose an admissible ξ mapping W'_n onto the fibre over x, and define the homotopy k' by (12) of §40.9. In analogy with §40.10 we have

41.14. THEOREM. *The homotopy k' is a deformation retraction of B'_s into B_s and contracts each fibre over itself. Then k' deforms any cross-section of $\mathfrak{B}'_s$ into one of $\mathfrak{B}_s$. Thus $\mathfrak{B}'_s$ has a cross-section if and only if $\mathfrak{B}_s$ has a cross-section.*

Referring to the first paragraph of §41.10, we have

41.15. COROLLARY. *The real, differentiable 2n-manifold M is quasi complex if and only if it admits a 2nd order, skew symmetric, tensor field which is non-singular at each point.*

A tensor field of the type prescribed in the corollary is otherwise known as a non-singular exterior form of degree 2.

41.16. Applications. Various facts about the topology of the compact manifolds W_n are readily available. Since O_{2n} has two components and U_n is connected, W_n has two components. They are homeomorphic since O_{2n} operates transitively. This means that the primary obstruction to finding a cross-section of a bundle with fibre W_n is 1-dimensional. In the tensor problem of §41.15, the vanishing of this obstruction is equivalent to orientability. In general, the vanishing is equivalent to the reducibility of the bundle to the group R_{2n}. When this happens, the bundle is the union of two disjoint isomorphic subbundles; and the problem reduces to finding a cross-section of one of them, say, the one with fibre R_{2n}/U_n which we will denote by Z_n.

Since $U_1 = R_2$, Z_1 is a point. This gives:

A 2-manifold is quasi-complex if and only if it is orientable.

It is well known, of course, that any orientable 2-manifold admits a complex analytic structure.

In the decomposition of R_4 into the product space of the symplectic group Sp_1 and R_3, we have $Sp_1 \subset U_2$ and $U_2 \cap R_3 = R_2 = U_1$. Then U_2 is the product space of Sp_1 and U_1. This implies that

$$R_4/U_2 = R_3/R_2 = S^2.$$

Therefore Z_2 is a 2-sphere. Hence $\pi_2(Z_2)$ and $\pi_3(Z_2)$ are infinite cyclic and $\pi_4(Z_2)$ is cyclic of order 2. If M is an orientable 4-manifold, the skew-tensor problem of §41.15 leads to a 3-dimensional primary obstruction with coefficients in an infinite cyclic group.

For a general n, the lower homotopy groups of Z_n can be deduced from the homotopy sequence of the bundle $R_{2n} \to Z_n$:

$$\pi_4(R_{2n}) \to \pi_4(Z_n) \xrightarrow{\lambda} \pi_3(U_n) \to \pi_3(R_{2n}) \to \pi_3(Z_n) \to \pi_2(U_n)$$
$$\to \pi_2(R_{2n}) \to \pi_2(Z_n) \xrightarrow{\mu} \pi_1(U_n) \to \pi_1(R_{2n}) \to \pi_1(Z_n).$$

When $n > 2$, $\pi_4(R_{2n}) = 0$ and λ is an isomorphism onto (see §§24.6, 25.1, 25.4). Exactness of the sequence implies $\pi_4(Z_n) = 0$. Since λ is onto, and $\pi_2(U_2) = 0$, we have $\pi_3(Z_n) = 0$. Since μ is onto, $\pi_1(U_n)$ is infinite cyclic, $\pi_1(R_{2n})$ is cyclic of order 2, and $\pi_2(R_{2n}) = 0$, it follows that $\pi_2(Z_2)$ is infinite cyclic. Since U_n is connected and μ is onto, we have $\pi_1(Z_n) = 0$. Thus, for $n > 2$, we have

$$\pi_i(Z_n) = \begin{cases} 0 & i = 1, \\ \infty & i = 2, \\ 0 & i = 3, \\ 0 & i = 4. \end{cases}$$

41.17. Quasi-complex spheres. We turn to the problem of determining the dimensions of spheres which are quasi-complex. In the euclidean space C^{2n} of coordinates $(x_1, \cdots, x_{2n})$, let S be the $(2n - 2)$-sphere defined by $x_{2n} = 0$ and $\Sigma_1^{2n-1} x_i^2 = 1$. Let $\mathcal{B}_s$ be the bundle of orthogonal skew-tensors of order 2 over S. If $b \in B_s$ lies over $x \in S$ and T_x is the tangent $(2n - 2)$-plane to S at x, then b is an orthogonal transformation of T_x on itself which carries each vector v of T_x into a vector perpendicular to v (this follows from the skew-symmetry of the matrix representation). We assign to b a linear transformation $\psi(b)$ of C^{2n} as follows: it carries the vector $x_0 = (0, \cdots, 0, 1)$ into x, it carries x into $-x_0$, and, in the $(2n - 2)$-space L_x parallel to T_x, the operation $\psi(b)$ is obtained by parallel translation of the operation b in T_x. Then $\psi(b)$ carries each vector into an orthogonal vector;

hence $\psi(b)$ is in $W_n \subset R_{2n}$. If $\sigma \in W_n$, then $x = \sigma x_0$ is in S and $\sigma|L_x$ is skew-symmetric. It follows that ψ maps B_s topologically onto W_n. If p: $R_{2n} \to S^{2n-1}$ is defined by $p(\sigma) = \sigma x_0$, then p maps W_n onto S, and, for each x in S, ψ maps the fibre of $\mathfrak{B}_s$ over x onto $p^{-1}(x) \cap W_n$. We have proved:

41.18. THEOREM. *With respect to the projection p: $W_n \to S^{2n-2}$ given by $p(\sigma) = \sigma(x_0)$, W_n is a bundle over S^{2n-2} with fibre W_{n-1}. It is equivalent to the bundle of orthogonal skew-tensors of order 2 over S^{2n-2}.*

If the bundle $R_{2n} \to S^{2n-1}$ is restricted to the hemisphere $x_{2n} \geqq 0$, we obtain a product bundle. Hence the same is true of the bundle restricted to S^{2n-2}. Thus ψ imbeds B_s in $S^{2n-2} \times R_{2n-1}$. We cannot conclude that $\mathfrak{B}_s$ is a product bundle since ψ is, in no sense, a bundle mapping.

It is to be noted that the relation "W_n is a bundle over S^{2n-2} with fibre W_{n-1}" is analogous to the relation "R_n is a bundle over S^{n-1} with fibre R_{n-1}." The latter was used to compute homotopy groups of R_n. One may do likewise for the homotopy groups of W_n and obtain the results already given in §41.16.

As to the problem of which spheres are quasi-complex; the main results are embodied in the following theorem of Kirchhoff [61].

41.19. THEOREM. *If S^{2n-2} is quasi-complex, then the bundle $R_{2n} \to S^{2n-1}$ admits a cross-section; and is therefore equivalent to a product bundle.*

By assumption, there is a cross-section f of the bundle $W_n \to S^{2n-2}$ of §41.18. Any vector x in S^{2n-1} is uniquely expressible in the form

$$x = \lambda x_0 + \mu y, \quad y \in S^{2n-2}, \quad \mu \geqq 0, \quad \lambda^2 + \mu^2 = 1.$$

Set

$$\sigma(x) = \lambda I_{2n} + \mu f(y).$$

Since $f(y)$ is skew-orthogonal, $f(y)^2 = -I_{2n}$.
Hence

$$\begin{aligned}\sigma(x)\sigma(x)' = [\lambda I_{2n} + \mu f(y)][\lambda I_{2n} - \mu f(y)] &= \lambda^2 I_{2n} - \mu^2 f(y)^2 \\ &= (\lambda^2 + \mu^2) I_{2n} = I_{2n}.\end{aligned}$$

Therefore $\sigma(x) \in R_{2n}$. Also

$$\sigma(x)\cdot x_0 = \lambda I_{2n}\cdot x_0 + \mu f(y)\cdot x_0 = \lambda x_0 + \mu y = x.$$

It follows that $\sigma(x)$ is a cross-section of $R_{2n} \to S^{2n-1}$, and the theorem is proved.

In §24.8, we have shown that the characteristic map T_{4m+2} of the bundle $R_{4m+2} \to S^{4m+1}$ is not homotopic to a constant; hence the bundle is not a product. Thus, we have

41.20. Corollary. *For $m = 1, 2, \cdots$, the sphere of dimension $4m$ is not a quasi-complex manifold.* (See App. sect. 7.)

It should be kept in mind that this asserts only that S^{4m}, *with its usual differential structure,* is not differentiably equivalent to the real form of a complex analytic manifold. The question here is a special case of a general one: If M and M' are two differentiable manifolds on the same space, are their tangent bundles equivalent?

41.21. Since S^2 admits a complex analytic structure it is quasi-complex. We have also that: *S^6 is quasi-complex.* To prove this, let S^7 be the set of Cayley numbers of norm 1 (see §20.5). Let S^6 be the equator of points c of S^7 which are orthogonal to the Cayley unit 1. Since right multiplication by $b \varepsilon S^7$ is orthogonal, c orthogonal to 1 implies cb orthogonal to b. Hence left multiplication of S^7 by c in S^6 carries each point into an orthogonal point. Then the matrix $f(c)$ of this left multiplication lies in W_4. Since $c1 = c$, f is a cross-section of the bundle $W_4 \rightarrow S^6$. Then §41.18 asserts that S^6 is quasi-complex.

41.22. It is interesting to note that *W_n is contractible to a point in R_{2n}.* This is obtained by modifying the proof of §41.19. Set

$$k(\sigma,t) = tI_{2n} + (1 - t^2)^{1/2}\sigma, \quad \sigma \varepsilon W_n, \ 0 \leqq t \leqq 1.$$

As before it follows that $k(\sigma,t)$ is orthogonal; then k is the required homotopy.