COMPLEX MANIFOLDS

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Abstract. Goal of this paper is to give an overview of the basic definitions of complex and Kähler manifolds together with the most important properties. The main theorem will be the *Newlander-Nirenberg Theorem* which gives a criterion under which an almost complex manifold is a complex one. The key role will be played by a certain tensor field, the so called *Nijenhuis tensor*.

Contents

1. Functions of Several Complex Variables

This section summarizes the fundamental properties of functions of several complex variables needed later. The results are inspired by [FG10, pp. 14–30].

Definition 1.1. *Let $n \in \mathbb{Z}$, $n \geq 1$, $U \subseteq \mathbb{C}^n$ open and $a \in U$. A mapping $f : U \to \mathbb{C}$ is said to be* ***complex differentiable at a*** *if there exists $g : U \to \mathbb{C}^n$ such that g is continuous at a and*

$$f(z) = f(a) + \sum_{\nu=1}^{n} (z_\nu - a_\nu) g_\nu(z) \tag{1}$$

holds for all $z \in D$. f is said to be ***holomorphic in D*** *if it is complex differentiable at every point $a \in D$. For $m \in \mathbb{Z}$, $m \geq 1$, a mapping $f : U \to \mathbb{C}^m$ is said to be*

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holomorphic in D if each component function f_ν, $\nu = 1, \ldots, n$, is holomorphic in D.

Proposition 1.1. *Let $n \in \mathbb{Z}$, $n \geq 1$, $D \subseteq \mathbb{C}^n$ open, $a \in U$ and $f : D \to \mathbb{C}$ real differentiable at a. Then*

$$\frac{\partial f}{\partial z_\nu}(a) = \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu}(a) - i\frac{\partial f}{\partial y_\nu}(a)\right) \tag{2}$$

and

$$\frac{\partial f}{\partial \overline{z}_\nu}(a) = \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu}(a) + i\frac{\partial f}{\partial y_\nu}(a)\right) \tag{3}$$

holds for all $\nu = 1, \ldots, n$.

Theorem 1.1 (The Cauchy-Riemann Equations). *Let $n \in \mathbb{Z}$, $n \geq 1$ and $D \subseteq \mathbb{C}^n$ open. A mapping $f : D \to \mathbb{C}$ is holomorphic in D if and only if it is real differentiable at every $a \in D$ and the* ***Cauchy-Riemann equations***

$$\frac{\partial f}{\partial \overline{z}_\nu}(a) = 0 \tag{4}$$

holds for all $a \in D$ and $\nu = 1, \ldots, n$.

Corollary 1.1. *Let $m, n \in \mathbb{Z}$, $m, n \geq 1$, $D \subseteq \mathbb{C}^n$ open and $f : D \to \mathbb{C}^m$ holomorphic in D. If $f = g + ih$, $g, h : D \to \mathbb{R}^m$, then*

$$\boxed{\frac{\partial g_\mu}{\partial x_\nu}(a) = \frac{\partial h_\mu}{\partial y_\nu}(a) \qquad \text{and} \qquad \frac{\partial h_\mu}{\partial x_\nu}(a) = -\frac{\partial g_\mu}{\partial y_\nu}(a)} \tag{5}$$

holds for any $a \in D$, $\nu = 1, \ldots, n$ and $\mu = 1, \ldots, m$.

Proof. Fix $\mu = 1, \ldots, m$. By definition 1.1 f_μ is holomorphic in D. Hence f_μ is real differentiable in D (see [FG10, p. 27]) and theorem 1.1 implies

$$\frac{\partial f_\mu}{\partial \overline{z}_\nu}(a) = 0$$

for all $a \in D$ and $\nu = 1, \ldots, n$. By proposition 1.1, this is equivalent to

$$\frac{\partial f}{\partial x_\nu}(a) + i\frac{\partial f}{\partial y_\nu}(a) = 0.$$

Using $f_\mu = g_\mu + ih_\mu$ and the $\mathbb{C}$-linearity of the operators $\frac{\partial}{\partial x_\nu}$ and $\frac{\partial}{\partial y_\nu}$ yields

$$\frac{\partial g_\mu}{\partial x_\nu}(a) - \frac{\partial h_\mu}{\partial y_\nu}(a) + i\left(\frac{\partial h_\mu}{\partial x_\nu}(a) + \frac{\partial g_\mu}{\partial y_\nu}(a)\right) = 0.$$

□

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2. Complex Structures on Vector Spaces

In what follows, let $n \in \mathbb{Z}$, $n \geq 1$. Consider an n-dimensional complex vector space and let $J \in \operatorname{End}_{\mathbb{C}}(V)$ be defined by $J(v) := iv$. Then clearly $J \circ J = -\operatorname{id}_V$. Since every n-dimensional complex vector space can be seen as a $2n$-dimensional real vector space in a natural way, i.e. if (e_ν) is a basis for the complex vector space V, then (e_ν, ie_ν) is a basis for the real vector space V, the mapping J induces an $\mathbb{R}$-endomorphism J on the real vector space V simply by $J(e_\nu) = ie_\nu$ and $J(ie_\nu) = -e_\nu$ for all $\nu = 1, \ldots, n$.
Conversly, let V be an n-dimensional real vector space with $J \in \operatorname{End}_{\mathbb{R}}(V)$ such that $J \circ J = -\operatorname{id}_V$. One can show, that

$$zv := xv + yJ(v) \tag{6}$$

for $z := x + iy \in \mathbb{C}$ and $v \in V$ makes V into a complex vector space. This motivates the following definition.

Definition 2.1. *Let V be an n-dimensional real vector space. A* ***complex structure on V*** *is a $\mathbb{R}$-linear mapping $J : V \to V$ such that $J \circ J = -\operatorname{id}_V$. If J is a complex structure on V, the tuple (V, J) is called a* ***complex vector space****.*

Lemma 2.1. *Let (V, J) be a complex vector space. Then $\dim V$ is even.*

Proof. That $\dim V$ must be even follows directly from

$$\big(\det(J)\big)^2 = \det(J \circ J) = \det(-\operatorname{id}_V) = (-1)^{\dim V} \det(\operatorname{id}_V) = (-1)^{\dim V}.$$

□

3. Almost Complex Structures

If M is a smooth manifold, then T_pM is a finite dimensional real vector space. Hence we can generalize the definitions and results of the previous section to manifolds. The following definition is taken from [Sil08, p. 86].

Definition 3.1. *Let M be a smooth manifold. An* ***almost complex structure on M*** *is a smooth tensor field $J \in \Gamma\left(T^{(1,1)}TM\right)$ such that $J_p \circ J_p = -\operatorname{id}_{T_pM}$ holds for any $p \in M$. If J is an almost complex structure on M, the tuple (M, J) is called an* ***almost complex manifold****.*

Proposition 3.1. *Every almost complex manifold is of even dimension and orientable.*

Proof. Assume that $\dim M$ is odd. Let $p \in M$. Then by [Lee13, p. 57] we have that $\dim T_pM = \dim M$. Hence $\dim T_pM$ is odd. But by lemma 2.1, $\dim T_pM$ must be even since (T_pM, J_p) is a complex vector space. Contradiction.

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Assume that J is an almost complex structure on M. Since M is a smooth manifold, there exists a Riemannian metric g on M (see [Lee13, p. 329]). Define

$$\widetilde{g}(X,Y) := g(X,Y) + g(JX,JY) \in \Gamma(T^{(0,2)}TM)$$

for all $X, Y \in \mathfrak{X}(M)$. Then

$$\widetilde{g}(JX,JY) = g(JX,JY) + g(-X,-Y) = g(JX,JY) + g(X,Y) = \widetilde{g}(X,Y)$$

by the bilinearity of g. Furthermore, clearly $\widetilde{g}$ is positive definite and symmetric, thus a Riemannian metric on M. Define

$$\omega(X,Y) := \widetilde{g}(X,JY).$$

Then by

$$\omega(Y,X) = \widetilde{g}(Y,JX) = \widetilde{g}(JX,Y) = \widetilde{g}(-X,JY) = -\omega(X,Y)$$

we see that ω is skew-symmetric. Hence $\omega \in \Omega^2(M)$. Let $p \in M$ and $u \in T_pM \setminus \{0\}$. Then also $-J_p(u) \neq 0$ since J_p is invertible since $\det J_p = 1$. Hence

$$\omega\left(u, -J_p(u)\right) = \widetilde{g}_p\left(u, -(J_p \circ J_p)(u)\right) = \widetilde{g}(u,u) \neq 0$$

and by [Lee13, p. 565] we get that ω is nondegenrate. Let $\dim M = 2n$. By [Lee13, p. 567] this implies that $\omega_p \wedge \cdots \wedge \omega_p$ is nonzero for each $p \in M$. Hence $\omega \wedge \cdots \wedge \omega$ is a nonvanishing top form on M. Since any nonvanishing top form determines an orientation (see [Lee13, p. 381]), we have that M is orientable. $\square$

Remark 3.1. The converse of proposition 3.1 is not true in general. One can show using results on fibre bundles and Chern classes, that if $\mathbb{S}^n$ admits an almost complex structure, then $n = 2^k - 2$ for $k \in \mathbb{Z}$, $k \geq 1$ (see [Ste51, p. 219]). So for example $\mathbb{S}^4$ does not admit an almost complex structure. Actually, it can be shown that $\mathbb{S}^2$ and $\mathbb{S}^6$ are the only spheres which admit an almost complex structure (see [BS53, p. 434]).

4. Complex Manifolds

The definition of smooth manifolds adapts smoothly to the complex case.

Definition 4.1. *Let $n \in \mathbb{Z}$, $n \geq 1$. An **n-dimensional complex manifold** is a second countable Hausdorff space M equipped with a holomorphic structure, that is a holomorphic atlas $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ of complex charts $(U_\alpha, \varphi_\alpha)$, such that all the transition maps are holomorphically compatible.*

Lemma 4.1. *Let V be a real vector space of dimension $n \in \mathbb{Z}$, $n \geq 1$. Then*

$$V \otimes V^* \cong \mathrm{End}(V) \tag{7}$$

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canonically. If (e_ν) is a basis of V and (e_ν^*) the corresponding basis of V^*, then $f \in \mathrm{End}(V)$ corresponds to

$$\sum_{\nu=1}^n f(e_\nu) \otimes e_\nu^*. \tag{8}$$

Proof. It is easily checked that

$$\Phi : \begin{cases} V \times V^* \to \mathrm{End}(V) \\ (v, f) \mapsto \big(u \mapsto f(u)v\big) \end{cases}$$

is bilinear. Thus by the universal property of the tensor product there exists a unique mapping $\widehat{\Phi} \in \mathrm{Hom}(V \otimes V^*; \mathrm{End}(V))$ such that $\Phi = \widehat{\Phi} \circ \otimes$. It is also easily checked that $\widehat{\Phi}$ is an isomorphism.
Let $f \in \mathrm{End}(V)$. Then for any $v \in V$ we have

$$\begin{aligned} \widehat{\Phi}\left(\sum_{\nu=1}^n f(e_\nu) \otimes e_\nu^*\right)(v) &= \sum_{\nu=1}^n \widehat{\Phi}\left(f(e_\nu) \otimes e_\nu^*\right)(v) \\ &= \sum_{\nu=1}^n e_\nu^*(v) f(e_\nu) \\ &= f\left(\sum_{\nu=1}^n e_\nu^*(v) e_\nu\right) \\ &= f(v). \end{aligned}$$

□

Proposition 4.1. *Any complex manifold admits a canonical almost complex structure.*

Proof. Fix a complex manifold M. We define J in terms of local coordinates. Let $\big(U, (x^\nu, y^\nu)\big)$ be a chart. By lemma 4.1 it is also enough to construct an endomorphism J_p for every $p \in U$. We define

$$J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) := \left.\frac{\partial}{\partial y^\nu}\right|_p \qquad \text{and} \qquad J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right) := -\left.\frac{\partial}{\partial x^\nu}\right|_p$$

for all $\nu = 1, \ldots, n$. As standard linear algebra shows, there is a unique linear mapping associated with J_p (see [HK71, p. 69]). Let $v := a^\nu \frac{\partial}{\partial x^\nu}\big|_p + b^\nu \frac{\partial}{\partial y^\nu}\big|_p \in T_pM$. Then

$$(J_p \circ J_p)(v) = J_p\left(a^\nu J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) + b^\nu J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right)\right)$$

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$$
\begin{aligned}
&= J_p\left(a^\nu \frac{\partial}{\partial y^\nu}\bigg|_p - b^\nu \frac{\partial}{\partial x^\nu}\bigg|_p\right)\\
&= -a^\nu \frac{\partial}{\partial x^\nu}\bigg|_p - b^\nu \frac{\partial}{\partial y^\nu}\bigg|_p\\
&= -v
\end{aligned}
$$

and thus $J_p \circ J_p = -\operatorname{id}_{T_pM}$.
Next we have to show that above locally defined mapping is well-defined, i.e. does not depend on the choice of coordinates. Assume that $p \in U \cap V$ for another chart $\big(V, (u^i, v^i)\big)$. By the change of coordinates formula [Lee13, p. 64] we get that

$$
\frac{\partial}{\partial x^\nu}\bigg|_p = \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})\frac{\partial}{\partial u^\mu}\bigg|_p + \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})\frac{\partial}{\partial v^\mu}\bigg|_p
$$

and

$$
\frac{\partial}{\partial y^\nu}\bigg|_p = \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})\frac{\partial}{\partial u^\mu}\bigg|_p + \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})\frac{\partial}{\partial v^\mu}\bigg|_p
$$

where $\widehat{p}$ denotes the coordinate representation of p with respect to the coordinates (x^ν, y^ν). Corollary 1.1 implies

$$
\begin{aligned}
J_p\left(\frac{\partial}{\partial x^\nu}\bigg|_p\right) &= \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})J_p\left(\frac{\partial}{\partial u^\mu}\bigg|_p\right) + \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})J_p\left(\frac{\partial}{\partial v^\mu}\bigg|_p\right)\\
&= \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})\frac{\partial}{\partial v^\mu}\bigg|_p - \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})\frac{\partial}{\partial u^\mu}\bigg|_p\\
&= \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})\frac{\partial}{\partial v^\mu}\bigg|_p + \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})\frac{\partial}{\partial u^\mu}\bigg|_p\\
&= \frac{\partial}{\partial y^\nu}\bigg|_p
\end{aligned}
$$

and

$$
\begin{aligned}
J_p\left(\frac{\partial}{\partial y^\nu}\bigg|_p\right) &= \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})J_p\left(\frac{\partial}{\partial u^\mu}\bigg|_p\right) + \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})J_p\left(\frac{\partial}{\partial v^\mu}\bigg|_p\right)\\
&= \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p})\frac{\partial}{\partial v^\mu}\bigg|_p - \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p})\frac{\partial}{\partial u^\mu}\bigg|_p\\
&= -\frac{\partial v^\mu}{\partial x^\nu}(\widehat{p})\frac{\partial}{\partial v^\mu}\bigg|_p - \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p})\frac{\partial}{\partial u^\mu}\bigg|_p\\
&= -\frac{\partial}{\partial x^\nu}\bigg|_p.
\end{aligned}
$$

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Left to check is smoothness. According to lemma 4.1 the corresponding rough tensor field is given by

$$J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right)\otimes \mathrm{d}x^\nu|_p + J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right)\otimes \mathrm{d}y^\nu|_p = \left.\frac{\partial}{\partial y^\nu}\right|_p \otimes \mathrm{d}x^\nu|_p - \left.\frac{\partial}{\partial x^\nu}\right|_p \otimes \mathrm{d}y^\nu|_p$$

for any $p \in U$. Thus the smoothness criteria for tensor fields [Lee13, pp. 317–318] together with [Lee13, p. 36] yields that $J \in \Gamma\left(T^{(1,1)}TM\right)$. □

A question which naturally arises by considering proposition 4.1 is, if the converse is also true, i.e. if every almost complex manifold is a complex manifold. This is in general not the case. Let $\mathbb{P}$ denote the naturally oriented underlying smooth manifold of the complex projective plane. Again using results about Chern numbers it can be shown that

$$\mathbb{P}\#(\mathbb{S}^1\times\mathbb{S}^3)\#(\mathbb{S}^1\times\mathbb{S}^3) \qquad \text{and} \qquad (\mathbb{S}^1\times\mathbb{S}^3)\#(\mathbb{S}^1\times\mathbb{S}^3)\#(\mathbb{S}^2\times\mathbb{S}^2) \tag{9}$$

have almost complex structures but no complex structure (see [Ven66, p. 1627]).

5. The Nijenhuis Tensor and the Newlander-Nirenberg Theorem

As we have seen in the last section, not every almost complex manifold is a complex manifold. Under which condition is this possible?

Definition 5.1. *Let (M, J) be an almost complex manifold. For $X, Y \in \mathfrak{X}(M)$ we define the **Nijenhuis tensor N** as*

$$N(X,Y) := [JX, JY] - J\,[X, JY] - J\,[JX, Y] - [X, Y] \tag{10}$$

where $[X, Y]$ denotes the usual Lie-bracket of vector fields.

Proposition 5.1. *Let (M, J) be an almost complex manifold and N be the associated Nijenhuis tensor. Then $N \in \Gamma\left(T^{(1,2)}TM\right)$.*

Proof. First of all, $N(X,Y) \in \mathfrak{X}(M)$ for all $X, Y \in \mathfrak{X}(M)$. This follows immediately by considering J as a mapping $J : \mathfrak{X}(M) \to \mathfrak{X}(M)$ (see [KN96, p. 26]), the fact that the Lie Bracket of two smooth vector fields is again a smooth vector field (see [Lee13, p. 186]) and that $\mathfrak{X}(M)$ is a $\mathscr{C}^\infty(M)$-module (see [Lee13, p. 177]). Let $f \in \mathscr{C}^\infty(M)$ and $X, Y, Z \in \mathfrak{X}(M)$. Then

$$\begin{aligned}
N(fX+Y, Z) &= [J(fX+Y), JZ] - J\,[fX+Y, JZ] - J\,[J(fX+Y), Z] \\
&\quad - [fX+Y, Z] \\
&= [fJX+JY, JZ] - J\,[fX+Y, JZ] - J\,[fJX+JY, Z] \\
&\quad - [fX+Y, Z] \\
&= [fJX, JZ] + [JY, JZ] - J\,[fX, JZ] - J\,[Y, JZ] - J\,[fJX, Z] \\
&\quad - J\,[JY, Z] - [fX, Z] - [Y, Z]
\end{aligned}$$

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$$\begin{aligned}
=& f\left[JX,JZ\right]-(JZf)JX+\left[JY,JZ\right]-fJ\left[X,JZ\right]+(JZf)JX\\
&-\left[Y,JZ\right]-fJ\left[JX,Z\right]+(Zf)JJX-J\left[JY,Z\right]-f\left[X,Z\right]\\
&+(Zf)X-\left[Y,Z\right]\\
=& fN(X,Z)+N(Y,Z).
\end{aligned}$$

by [Lee13, pp. 187–188]. Linearity in the second argument is shown similarly. Hence $N:\mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ is bilinear over $\mathscr{C}^\infty(M)$. So by [KN96, p. 26] we have that $N\in\Gamma\left(T^{(1,2)}TM\right)$. □

Theorem 5.1 (Newlander-Nirenberg). *Let (M,J) be an almost complex manifold. Then M is a complex manifold, where the complex structure is so that the canonically induced almost complex structure is J, if and only if the Nijenhuis tensor N vanishes identically.*

Proof. Assume M is a complex manifold. Let $(U,(x^\nu,y^\nu))$ be a chart. From proposition 5.1 it is enough to consider the coordinate vector fields $\frac{\partial}{\partial x^\nu}$ and $\frac{\partial}{\partial y^\nu}$. But from the explicit definition of J in proposition 4.1 and the property, that the Lie-Bracket of coordinate vector fields vanishes, together with the $\mathscr{C}^\infty(M)$-linearity of J we get that N vanishes identically on each chart, and thus on M.
The other direction however is far more technical and uses results on partial differential equations. A complete proof can either be found in the original paper [NN57] or in [Sil08, p. 106], where references to more recent proofs are given. □

6. Kähler Manifolds

References

[BS53] A. Borel and J.-P. Serre. "Groupes de Lie et Puissances Reduites de Steenrod". In: *American Journal of Mathematics* 75.3 (1953), pp. 409–448. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/2372495.

[FG10] Klaus Fritzsche and Hans Grauert. *From Holomorphic Functions to Complex Manifolds*. Graduate Texts in Mathematics. Springer-Verlag New York, Inc., 2010.

[Gri07] Pierre Antoine Grillet. *Abstract Algebra*. Graduate Texts in Mathematics. Springer Science + Business Media, LLC, 2007.

[HK71] Kenneth Hoffman and Ray Kunze. *Linear Algebra*. Second Edition. Prentice Hall, 1971.

[KN96] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of Differential Geometry*. John Wiley & Sons, Inc., 1996.

[Lee13] John M. Lee. *Introduction to Smooth Manifolds*. Second Edition. Graduate Texts in Mathematics. Springer, 2013.

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[NN57] A. Newlander and L. Nirenberg. “Complex Analytic Coordinates in Almost Complex Manifolds”. In: *Annals of Mathematics* 65.3 (1957), pp. 391–404. ISSN: 0003486X. URL: http://www.jstor.org/stable/1970051.

[Sil08] Ana Cannas da Silva. *Lectures on Symplectic Geometry*. Corrected 2nd printing. Lecture Notes in Mathematics 1764. Springer Verlag Berlin Heidelberg, 2008.

[Ste51] Norman Steenrod. *The Topology of Fibre Bundles*. Princeton University Press, 1951.

[Ven66] A. Van De Ven. “On the Chern Numbers of Certain Complex and Almost Complex Manifolds”. In: *Proceedings of the National Academy of Sciences of the United States of America* 55.6 (1966), pp. 1624–1627. ISSN: 00278424. URL: http://www.jstor.org/stable/57245.