# COMPLEX MANIFOLDS

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**Abstract**. Goal of this paper is to give an overview of the basic definitions of complex and Kähler manifolds together with the most important properties. The main theorem will be the *Newlander-Nirenberg Theorem* which gives a criterion under which an almost complex manifold is a complex one. The key role will be played by a certain tensor field, the so called *Nijenhuis tensor*.

## Contents

## 1. Functions of Several Complex Variables

This section summarizes the fundamental properties of functions of several complex variables needed later. The results are inspired by [FG10, pp. 14–30].

**Definition 1.1.** *Let $n \in \mathbb{Z}$, $n \geq 1$, $U \subseteq \mathbb{C}^n$ open and $a \in U$. A mapping $f : U \to \mathbb{C}$ is said to be **complex differentiable at $a$** if there exists $g : U \to \mathbb{C}^n$ such that $g$ is continuous at $a$ and*

$$f(z) = f(a) + \sum_{\nu=1}^{n} (z_\nu - a_\nu) g_\nu(z) \tag{1}$$

*holds for all $z \in D$. $f$ is said to be **holomorphic in $D$** if it is complex differentiable at every point $a \in D$. For $m \in \mathbb{Z}$, $m \geq 1$, a mapping $f : U \to \mathbb{C}^m$ is said to be*

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*holomorphic in $D$ if each component function $f_\nu$, $\nu = 1, \ldots, n$, is holomorphic in $D$.*

**Proposition 1.1.** *Let $n \in \mathbb{Z}$, $n \geq 1$, $D \subseteq \mathbb{C}^n$ open, $a \in U$ and $f : D \to \mathbb{C}$ real differentiable at $a$. Then*

$$\frac{\partial f}{\partial z_\nu}(a) = \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu}(a) - i\frac{\partial f}{\partial y_\nu}(a)\right) \tag{2}$$

*and*

$$\frac{\partial f}{\partial \overline{z}_\nu}(a) = \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu}(a) + i\frac{\partial f}{\partial y_\nu}(a)\right) \tag{3}$$

*holds for all $\nu = 1, \ldots, n$.*

**Theorem 1.1 (The Cauchy-Riemann Equations).** *Let $n \in \mathbb{Z}$, $n \geq 1$ and $D \subseteq \mathbb{C}^n$ open. A mapping $f : D \to \mathbb{C}$ is holomorphic in $D$ if and only if it is real differentiable at every $a \in D$ and the **Cauchy-Riemann equations***

$$\frac{\partial f}{\partial \overline{z}_\nu}(a) = 0 \tag{4}$$

*holds for all $a \in D$ and $\nu = 1, \ldots, n$.*

**Corollary 1.1.** *Let $m, n \in \mathbb{Z}$, $m, n \geq 1$, $D \subseteq \mathbb{C}^n$ open and $f : D \to \mathbb{C}^m$ holomorphic in $D$. If $f = g + ih$, $g, h : D \to \mathbb{R}^m$, then*

$$\boxed{\frac{\partial g_\mu}{\partial x_\nu}(a) = \frac{\partial h_\mu}{\partial y_\nu}(a) \qquad and \qquad \frac{\partial h_\mu}{\partial x_\nu}(a) = -\frac{\partial g_\mu}{\partial y_\nu}(a)} \tag{5}$$

*holds for any $a \in D$, $\nu = 1, \ldots, n$ and $\mu = 1, \ldots, m$.*

*Proof.* Fix $\mu = 1, \ldots, m$. By definition 1.1 $f_\mu$ is holomorphic in $D$. Hence $f_\mu$ is real differentiable in $D$ (see [FG10, p. 27]) and theorem 1.1 implies

$$\frac{\partial f_\mu}{\partial \overline{z}_\nu}(a) = 0$$

for all $a \in D$ and $\nu = 1, \ldots, n$. By proposition 1.1, this is equivalent to

$$\frac{\partial f}{\partial x_\nu}(a) + i\frac{\partial f}{\partial y_\nu}(a) = 0.$$

Using $f_\mu = g_\mu + ih_\mu$ and the $\mathbb{C}$-linearity of the operators $\frac{\partial}{\partial x_\nu}$ and $\frac{\partial}{\partial y_\nu}$ yields

$$\frac{\partial g_\mu}{\partial x_\nu}(a) - \frac{\partial h_\mu}{\partial y_\nu}(a) + i\left(\frac{\partial h_\mu}{\partial x_\nu}(a) + \frac{\partial g_\mu}{\partial y_\nu}(a)\right) = 0.$$

$\square$

## 2. Complex Structures on Vector Spaces

In what follows, let $n \in \mathbb{Z}$, $n \geq 1$. Consider an $n$-dimensional complex vector space and let $J \in \text{End}_{\mathbb{C}}(V)$ be defined by $J(v) := iv$. Then clearly $J \circ J = -\text{id}_V$. Since every $n$-dimensional complex vector space can be seen as a $2n$-dimensional real vector space in a natural way, i.e. if $(e_\nu)$ is a basis for the complex vector space $V$, then $(e_\nu, ie_\nu)$ is a basis for the real vector space $V$, the mapping $J$ induces an $\mathbb{R}$-endomorphism $J$ on the real vector space $V$ simply by $J(e_\nu) = ie_\nu$ and $J(ie_\nu) = -e_\nu$ for all $\nu = 1, \ldots, n$.

Conversly, let $V$ be an $n$-dimensional real vector space with $J \in \text{End}_{\mathbb{R}}(V)$ such that $J \circ J = -\text{id}_V$. One can show, that

$$zv := xv + yJ(v) \tag{6}$$

for $z := x + iy \in \mathbb{C}$ and $v \in V$ makes $V$ into a complex vector space. This motivates the following definition.

**Definition 2.1.** *Let $V$ be an $n$-dimensional real vector space. A **complex structure on $V$** is a $\mathbb{R}$-linear mapping $J : V \to V$ such that $J \circ J = -\text{id}_V$. If $J$ is a complex structure on $V$, the tuple $(V, J)$ is called a **complex vector space**.*

**Lemma 2.1.** *Let $(V, J)$ be a complex vector space. Then $\dim V$ is even.*

*Proof.* That $\dim V$ must be even follows directly from

$$\left(\det(J)\right)^2 = \det(J \circ J) = \det(-\text{id}_V) = (-1)^{\dim V} \det(\text{id}_V) = (-1)^{\dim V}.$$

$\square$

## 3. Almost Complex Structures

If $M$ is a smooth manifold, then $T_pM$ is a finite dimensional real vector space. Hence we can generalize the definitions and results of the previous section to manifolds. The following definition is taken from [Sil08, p. 86].

**Definition 3.1.** *Let $M$ be a smooth manifold. An **almost complex structure on $M$** is a smooth tensor field $J \in \Gamma\left(T^{(1,1)}TM\right)$ such that $J_p \circ J_p = -\text{id}_{T_pM}$ holds for any $p \in M$. If $J$ is an almost complex structure on $M$, the tuple $(M, J)$ is called an **almost complex manifold**.*

**Proposition 3.1.** *Every almost complex manifold $(M, J)$ is of even dimension and orientable.*

*Proof.* Assume that $\dim M$ is odd. Let $p \in M$. Then by [Lee13, p. 57] we have that $\dim T_pM = \dim M$. Hence $\dim T_pM$ is odd. But by lemma 2.1, $\dim T_pM$ must be even since $(T_pM, J_p)$ is a complex vector space. Contradiction.

Since $M$ is a smooth manifold, there exists a Riemannian metric $g$ on $M$ (see [Lee13, p. 329]). Define

$$\widetilde{g}(X,Y) := g(X,Y) + g(JX,JY) \in \Gamma(T^{(0,2)}TM)$$

for all $X,Y \in \mathfrak{X}(M)$. Then

$$\widetilde{g}(JX,JY) = g(JX,JY) + g(-X,-Y) = g(JX,JY) + g(X,Y) = \widetilde{g}(X,Y)$$

by the bilinearity of $g$. Furthermore, clearly $\widetilde{g}$ is positive definite and symmetric, thus a Riemannian metric on $M$. Define

$$\omega(X,Y) := \widetilde{g}(X,JY).$$

Then by

$$\omega(Y,X) = \widetilde{g}(Y,JX) = \widetilde{g}(JX,Y) = \widetilde{g}(-X,JY) = -\omega(X,Y)$$

we see that $\omega$ is skew-symmetric. Hence $\omega \in \Omega^2(M)$. Let $p \in M$ and $u \in T_pM \setminus \{0\}$. Then also $-J_p(u) \neq 0$ since $J_p$ is invertible since $\det J_p = 1$. Furthermore, by [Lee13, p. 177], there exist $X,Y \in \mathfrak{X}(M)$, such that $X_p = u$ and $Y_p = -J_p(u)$. Hence

$$\begin{aligned}
\omega_p\big(u, -J_p(u)\big) &= \omega_p(X_p, Y_p) \\
&= \omega(X,Y)(p) \\
&= \widetilde{g}(X,JY)(p) \\
&= \widetilde{g}_p\big(X_p, (JY)_p\big) \\
&= \widetilde{g}_p\big(u, J_p(Y_p)\big) \\
&= \widetilde{g}_p\big(u, -(J_p \circ J_p)(u)\big) \\
&= \widetilde{g}_p(u,u) \\
&\neq 0
\end{aligned}$$

and by [Lee13, p. 565] we get that $\omega$ is nondegenrate. Let $\dim M = 2n$. By [Lee13, p. 567] this implies that $\omega_p \wedge \cdots \wedge \omega_p$ is nonzero for each $p \in M$. Hence $\omega \wedge \cdots \wedge \omega$ is a nonvanishing top form on $M$. Since any nonvanishing top form determines an orientation (see [Lee13, p. 381]), we have that $M$ is orientable. $\qquad\square$

**Remark 3.1.** The converse of proposition 3.1 is not true in general. One can show using results on fibre bundles and Chern classes, that if $\mathbb{S}^n$ admits an almost complex structure, then $n = 2^k - 2$ for $k \in \mathbb{Z}$, $k \geq 1$ (see [Ste51, p. 219]). So for example $\mathbb{S}^4$ does not admit an almost complex structure. Actually, it can be shown that $\mathbb{S}^2$ and $\mathbb{S}^6$ are the only spheres which admit an almost complex structure (see [BS53, p. 434]).

## 4. Complex Manifolds

The definition of smooth manifolds adapts smoothly to the complex case.

**Definition 4.1.** *Let $n \in \mathbb{Z}$, $n \geq 1$. An **n-dimensional complex manifold** is a second countable Hausdorff space $M$ equipped with a holomorphic structure, that is a maximal holomorphic atlas $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ of complex charts $(U_\alpha, \varphi_\alpha)$, such that all the transition maps are holomorphic.*

**Examples 4.1 (Complex Manifolds).**

1. The complex $n$-space $\mathbb{C}^n$ is an $n$-dimensional complex manifold.
2. Let $\{\omega_1, \ldots, \omega_{2n}\}$ be a real basis of $\mathbb{C}^n$ and define

$$G := \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_{2n}. \tag{7}$$

   Then the discrete group $G$ acts freely and properly discontinuously on $\mathbb{C}^n$ by translation. Thus $\mathbb{T}^n := \mathbb{C}^n/G$ is an $n$-dimensional complex manifold, called a **complex torus** (see [FG10, pp. 206–207]).
3. The quotient $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$ is an $n$-dimensional complex manifold, called the **complex projective space** (see [FG10, pp. 208–210]).

**Lemma 4.1.** *Let $n, k \in \mathbb{Z}$, $n, k \geq 1$. Let $V$ be an $n$-dimensional real vector space. Then*

$$V \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{k} \cong \mathrm{L}(\underbrace{V, \ldots, V}_{k}; V) \tag{8}$$

*canonically. If $(e_\nu)$ is a basis of $V$ and $(e_\nu^*)$ the corresponding basis of $V^*$, then $f \in \mathrm{End}(V)$ corresponds to*

$$\sum_{\nu=1}^{n} f(e_\nu) \otimes e_\nu^*. \tag{9}$$

*Proof.* It is easily checked that

$$\Phi : \begin{cases} V \times V^* \times \cdots \times V^* \to \mathrm{L}(V, \ldots, V; V) \\ (v, f_1, \ldots, f_k) \mapsto \big((v_1, \ldots, v_k) \mapsto f_1(v_1) \cdots f_k(v_k)v\big) \end{cases}$$

is multilinear. Thus by the universal property of the tensor product there exists a unique mapping $\widehat{\Phi} \in \mathrm{Hom}\big(V \times V^* \times \cdots \times V^*; \mathrm{L}(V, \ldots, V; V)\big)$ such that $\Phi = \widehat{\Phi} \circ \otimes$. It is also easily checked that $\widehat{\Phi}$ is an isomorphism. Let $f \in \mathrm{End}(V)$. Then for any $v \in V$ we have

$$\widehat{\Phi}\left(\sum_{\nu=1}^{n} f(e_\nu) \otimes e_\nu^*\right)(v) = \sum_{\nu=1}^{n} \widehat{\Phi}\big(f(e_\nu) \otimes e_\nu^*\big)(v)$$

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$$= \sum_{\nu=1}^{n} e_{\nu}^{*}(v)f(e_{\nu})$$

$$= f\left(\sum_{\nu=1}^{n} e_{\nu}^{*}(v)e_{\nu}\right)$$

$$= f(v).$$

$\square$

**Proposition 4.1.** *Any complex manifold admits a canonical almost complex structure.*

*Proof.* Fix a complex manifold $M$. We define $J$ in terms of local coordinates. Let $\left(U, (x^{\nu}, y^{\nu})\right)$ be a chart. By lemma 4.1 it is also enough to construct an endomorphism $J_p$ for every $p \in U$. We define

$$J_p\left(\left.\frac{\partial}{\partial x^{\nu}}\right|_p\right) := \left.\frac{\partial}{\partial y^{\nu}}\right|_p \qquad \text{and} \qquad J_p\left(\left.\frac{\partial}{\partial y^{\nu}}\right|_p\right) := -\left.\frac{\partial}{\partial x^{\nu}}\right|_p$$

for all $\nu = 1, \ldots, n$. As standard linear algebra shows, there is a unique linear mapping associated with $J_p$ (see [HK71, p. 69]). Let $v := a^{\nu} \left.\frac{\partial}{\partial x^{\nu}}\right|_p + b^{\nu} \left.\frac{\partial}{\partial y^{\nu}}\right|_p \in T_pM$. Then

$$(J_p \circ J_p)(v) = J_p\left(a^{\nu} J_p\left(\left.\frac{\partial}{\partial x^{\nu}}\right|_p\right) + b^{\nu} J_p\left(\left.\frac{\partial}{\partial y^{\nu}}\right|_p\right)\right)$$

$$= J_p\left(a^{\nu} \left.\frac{\partial}{\partial y^{\nu}}\right|_p - b^{\nu} \left.\frac{\partial}{\partial x^{\nu}}\right|_p\right)$$

$$= -a^{\nu} \left.\frac{\partial}{\partial x^{\nu}}\right|_p - b^{\nu} \left.\frac{\partial}{\partial y^{\nu}}\right|_p$$

$$= -v$$

and thus $J_p \circ J_p = -\operatorname{id}_{T_pM}$.

Next we have to show that above locally defined mapping is well-defined, i.e. does not depend on the choice of coordinates. Assume that $p \in U \cap V$ for another chart $\left(V, (u^i, v^i)\right)$. By the change of coordinates formula [Lee13, p. 64] we get that

$$\left.\frac{\partial}{\partial x^{\nu}}\right|_p = \frac{\partial u^{\mu}}{\partial x^{\nu}}(\widehat{p}) \left.\frac{\partial}{\partial u^{\mu}}\right|_p + \frac{\partial v^{\mu}}{\partial x^{\nu}}(\widehat{p}) \left.\frac{\partial}{\partial v^{\mu}}\right|_p$$

and

$$\left.\frac{\partial}{\partial y^{\nu}}\right|_p = \frac{\partial u^{\mu}}{\partial y^{\nu}}(\widehat{p}) \left.\frac{\partial}{\partial u^{\mu}}\right|_p + \frac{\partial v^{\mu}}{\partial y^{\nu}}(\widehat{p}) \left.\frac{\partial}{\partial v^{\mu}}\right|_p$$

where $\widehat{p}$ denotes the coordinate representation of $p$ with respect to the coordinates $(x^\nu, y^\nu)$. Corollary 1.1 implies

$$
\begin{aligned}
J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) &= \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p}) J_p\left(\left.\frac{\partial}{\partial u^\mu}\right|_p\right) + \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p}) J_p\left(\left.\frac{\partial}{\partial v^\mu}\right|_p\right) \\
&= \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p}) \left.\frac{\partial}{\partial v^\mu}\right|_p - \frac{\partial v^\mu}{\partial x^\nu}(\widehat{p}) \left.\frac{\partial}{\partial u^\mu}\right|_p \\
&= \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p}) \left.\frac{\partial}{\partial v^\mu}\right|_p + \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p}) \left.\frac{\partial}{\partial u^\mu}\right|_p \\
&= \left.\frac{\partial}{\partial y^\nu}\right|_p
\end{aligned}
$$

and

$$
\begin{aligned}
J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right) &= \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p}) J_p\left(\left.\frac{\partial}{\partial u^\mu}\right|_p\right) + \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p}) J_p\left(\left.\frac{\partial}{\partial v^\mu}\right|_p\right) \\
&= \frac{\partial u^\mu}{\partial y^\nu}(\widehat{p}) \left.\frac{\partial}{\partial v^\mu}\right|_p - \frac{\partial v^\mu}{\partial y^\nu}(\widehat{p}) \left.\frac{\partial}{\partial u^\mu}\right|_p \\
&= -\frac{\partial v^\mu}{\partial x^\nu}(\widehat{p}) \left.\frac{\partial}{\partial v^\mu}\right|_p - \frac{\partial u^\mu}{\partial x^\nu}(\widehat{p}) \left.\frac{\partial}{\partial u^\mu}\right|_p \\
&= -\left.\frac{\partial}{\partial x^\nu}\right|_p.
\end{aligned}
$$

Left to check is smoothness. According to lemma 4.1 the corresponding rough tensor field is given by

$$
J_p\left(\left.\frac{\partial}{\partial x^\nu}\right|_p\right) \otimes dx^\nu|_p + J_p\left(\left.\frac{\partial}{\partial y^\nu}\right|_p\right) \otimes dy^\nu|_p = \left.\frac{\partial}{\partial y^\nu}\right|_p \otimes dx^\nu|_p - \left.\frac{\partial}{\partial x^\nu}\right|_p \otimes dy^\nu|_p
$$

for any $p \in U$. Thus the smoothness criteria for tensor fields [Lee13, pp. 317–318] together with [Lee13, p. 36] yields that $J \in \Gamma\left(T^{(1,1)}TM\right)$. $\qquad \square$

A question which naturally arises by considering proposition 4.1 is, if the converse is also true, i.e. if every almost complex manifold is a complex manifold. This is in general not the case. Let $\mathbb{P}$ denote the naturally oriented underlying smooth manifold of the complex projective plane. Again using results about Chern numbers it can be shown that

$$
\mathbb{P}\#(\mathbb{S}^1 \times \mathbb{S}^3)\#(\mathbb{S}^1 \times \mathbb{S}^3) \qquad \text{and} \qquad (\mathbb{S}^1 \times \mathbb{S}^3)\#(\mathbb{S}^1 \times \mathbb{S}^3)\#(\mathbb{S}^2 \times \mathbb{S}^2) \tag{10}
$$

have almost complex structures but no complex structure (see [Ven66, p. 1627]).

### 5. The Nijenhuis Tensor and the Newlander-Nirenberg Theorem

As we have seen in the last section, not every almost complex manifold is a complex manifold. Under which condition is this possible?

**Definition 5.1.** *Let $(M, J)$ be an almost complex manifold. For $X, Y \in \mathfrak{X}(M)$ we define the **Nijenhuis tensor $N$** as*

$$N(X, Y) := [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] \tag{11}$$

*where $[X, Y]$ denotes the usual Lie-bracket of vector fields.*

**Proposition 5.1.** *Let $(M, J)$ be an almost complex manifold and $N$ be the Nijenhuis tensor. Then $N \in \Gamma\left(T^{(1,2)}TM\right)$.*

*Proof.* First of all, $N(X, Y) \in \mathfrak{X}(M)$ for all $X, Y \in \mathfrak{X}(M)$. This follows immediately by considering $J$ as a mapping $J : \mathfrak{X}(M) \to \mathfrak{X}(M)$ (see [KN96a, p. 26]), the fact that the Lie Bracket of two smooth vector fields is again a smooth vector field (see [Lee13, p. 186]) and that $\mathfrak{X}(M)$ is a $\mathscr{C}^\infty(M)$-module (see [Lee13, p. 177]). Let $f \in \mathscr{C}^\infty(M)$ and $X, Y, Z \in \mathfrak{X}(M)$. Then

$$
\begin{aligned}
N(fX + Y, Z) =\ & [J(fX + Y), JZ] - J[fX + Y, JZ] - J[J(fX + Y), Z] \\
& - [fX + Y, Z] \\
=\ & [fJX + JY, JZ] - J[fX + Y, JZ] - J[fJX + JY, Z] \\
& - [fX + Y, Z] \\
=\ & [fJX, JZ] + [JY, JZ] - J[fX, JZ] - J[Y, JZ] - J[fJX, Z] \\
& - J[JY, Z] - [fX, Z] - [Y, Z] \\
=\ & f[JX, JZ] - (JZf)JX + [JY, JZ] - fJ[X, JZ] + (JZf)JX \\
& - [Y, JZ] - fJ[JX, Z] + (Zf)JJX - J[JY, Z] - f[X, Z] \\
& + (Zf)X - [Y, Z] \\
=\ & fN(X, Z) + N(Y, Z).
\end{aligned}
$$

by [Lee13, pp. 187–188]. Linearity in the second argument is shown similarly. Hence $N : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is bilinear over $\mathscr{C}^\infty(M)$. So by [KN96a, p. 26] we have that $N \in \Gamma\left(T^{(1,2)}TM\right)$. $\qquad\square$

**Theorem 5.1 (Newlander-Nirenberg).** *Let $(M, J)$ be an almost complex manifold. Then $M$ is a complex manifold, where the complex structure is so that the canonically induced almost complex structure is $J$, if and only if the Nijenhuis tensor $N$ vanishes identically.*

*Proof.* Assume $M$ is a complex manifold. Let $(U, (x^\nu, y^\nu))$ be a chart. From proposition 5.1 it is enough to consider the coordinate vector fields $\frac{\partial}{\partial x^\nu}$ and $\frac{\partial}{\partial y^\nu}$. But from

the explicit definition of $J$ in proposition 4.1 and the property, that the Lie-Bracket of coordinate vector fields vanishes, together with the $\mathscr{C}^\infty(M)$-linearity of $J$ we get that $N$ vanishes identically on each chart, and thus on $M$.

The other direction however is far more technical and uses results on partial differential equations. A complete proof can either be found in the original paper [NN57] or in [Sil08, p. 106], where references to more recent proofs are given. $\qquad\square$

## 6. Kähler Manifolds

The following is inspired by [KN96b, pp. 146–149] and introduces the concepts from a complex viewpoint. This is in contrast to the symplectic approach provided for example in [Sil08].

**Definition 6.1.** *Let $(M, J)$ be an almost complex manifold. A **Hermitian metric on $M$** is a Riemannian metric $g$ such that*

$$g(JX, JY) = g(X, Y) \tag{12}$$

*holds for all $X, Y \in \mathfrak{X}(M)$. If $g$ is a Hermitian metric on $M$, the triple $(M, J, g)$ is called an **almost Hermitian manifold**.*

**Lemma 6.1.** *Every almost complex manifold admits a Hermitian metric.*

*Proof.* The existence was shown in the proof of proposition 3.1. $\qquad\square$

**Definition 6.2.** *Let $(M, J, g)$ be an almost Hermitian manifold. The **fundamental 2-form** $\Omega$ is defined to be*

$$\Omega(X, Y) := g(X, JY) \tag{13}$$

*for all $X, Y \in \mathfrak{X}(M)$.*

**Definition 6.3.** *Let $(M, J, g)$ be an almost Hermitian manifold with fundamental 2-form $\Omega$. The Hermitian metric is said to be a **Kähler metric**, if $d\Omega = 0$. An almost complex manifold with a Kähler metric is called an **almost Kähler manifold** and a complex manifold with a Kähler metric is called a **Kähler manifold**.*

## Appendix A. Tensor Characterization Lemma

**Definition A.1.** *Let $k, l \in \mathbb{Z}$, $k, l \geq 0$ and $M$ a smooth manifold. Then the **bundle of mixed tensors of type $(k, l)$** is defined by*

$$T^{(k,l)}TM := \coprod_{p \in M} T^{(k,l)}(T_pM). \tag{14}$$

**Proposition A.1.** *The bundle of mixed tensors of type $(k, l)$ has an unique natural structure as a smooth vector bundle of rank $n^{k+l}$ over $M$.*

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*Proof.* For each $p \in M$ let $E_p := T^{(k,l)}(T_p M)$. By [Lee13, p. 57] and [Lee13, p. 313] $\dim E_p = n^{k+l}$. Furthermore, let $E := T^{(k,l)}TM$ and $\pi : E \to M$ be defined by $\pi(p, A) := p$. Let $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ be an atlas for $M$. For each $\alpha \in A$ define

$$\Phi_\alpha : \begin{cases} \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{n^{k+l}} \\ (p, A) \mapsto \left(p, (A^{i_1 \ldots i_k}_{j_1 \ldots j_l})\right) \end{cases}$$

Clearly, the inverse is given by

$$\Phi_\alpha^{-1} : \begin{cases} U_\alpha \times \mathbb{R}^{n^{k+l}} \to \pi^{-1}(U_\alpha) \\ \left(p, (A^{i_1 \ldots i_k}_{j_1 \ldots j_l})\right) \mapsto (p, A) \end{cases} \quad .$$

Hence each $\Phi_\alpha$ is bijective. Now we have to check, that $\Phi_\alpha|_{E_p}$ is an isomorphism. So let $\lambda \in \mathbb{R}$ and $B \in E_p$. Then

$$\begin{aligned} \Phi_\alpha|_{E_p}(p, \lambda A + B) &= \left(p, (\lambda A + B)^{i_1 \ldots i_k}_{j_1 \ldots j_l}\right) \\ &= \left(p, \lambda(A^{i_1 \ldots i_k}_{j_1 \ldots j_l}) + (B^{i_1 \ldots i_k}_{j_1 \ldots j_l})\right) \\ &= \lambda \Phi_\alpha|_{E_p}(p, A) + \Phi_\alpha|_{E_p}(p, B). \end{aligned}$$

Now let $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$. We consider the mapping

$$\Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^{n^{k+l}} \to (U_\alpha \cap U_\beta) \times \mathbb{R}^{n^{k+l}}.$$

Define $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n^{k+l}, \mathbb{R})$ by

$$\tau_{\alpha\beta} := (\delta^i_j).$$

Then we have that

$$\left(\Phi_\alpha \circ \Phi_\beta^{-1}\right) \left(p, (A^{i_1 \ldots i_k}_{j_1 \ldots j_l})\right) = \left(p, (A^{i_1 \ldots i_k}_{j_1 \ldots j_l})\right) = \left(p, \tau_{\alpha\beta}(p)(A^{i_1 \ldots i_k}_{j_1 \ldots j_l})\right).$$

Since $\tau_{\alpha\beta}$ is constant, it is smooth (see [Lee13, p. 36]). Hence we can apply the vector bundle chart lemma [Lee13, p. 253] and the result follows. $\qquad\square$

What follows is a reformulation of the smoothness criteria for tensor fields ([Lee13, pp. 317–318]) for tensor fields of type $(1, k)$.

**Proposition A.2.** *Let $M$ be a smooth manifold and let $A : M \to T^{(k,l)}TM$ be a rough section. Then $A \in \Gamma(T^{(k,l)}TM)$ if and only if in every smooth coordinate chart, the component functions of $A$ are smooth.*

*Proof.* Let $(U, (x^i))$ be a smooth chart. Proposition A.1 together with the proof of the vector bundle chart lemma [Lee13, pp. 253–254] implies, that the corresponding chart on $T^{(k,l)}TM$ is given by $\left(\pi^{-1}(U), \widetilde{\varphi}\right)$, where $\widetilde{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^{n^{k+l}}$ is defined by

$$\widetilde{\varphi} := (\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}}) \circ \Phi$$

where $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^{n^{k+l}}$ is given as in the proof of proposition A.1. Now we consider the coordinate representation $\widehat{A}$ in the given charts (see [Lee13, p. 35]). Since $A$ is a rough section, we have that

$$A^{-1}\left(\pi^{-1}(U)\right) = (\pi \circ A)^{-1}(U) = \mathrm{id}_M^{-1}(U) = U.$$

Hence $\varphi\left(U \cap A^{-1}(\pi^{-1}(U))\right) = \varphi(U)$, which is open, and $\widehat{A} : \varphi(U) \to \widetilde{\varphi}\left(\pi^{-1}(U)\right)$ is given by

$$\begin{aligned}
\widehat{A}(x) &= (\widetilde{\varphi} \circ A \circ \varphi^{-1})(x) \\
&= (\varphi \times \mathrm{id}_{\mathbb{R}^{n+l}}) \left(\Phi(\varphi^{-1}(x), A_{\varphi^{-1}(x)})\right) \\
&= (\varphi \times \mathrm{id}_{\mathbb{R}^{n+l}}) \left(\varphi^{-1}(x), \left(A_{j_1\ldots j_l}^{i_1\ldots i_k}(\varphi^{-1}(x))\right)\right) \\
&= \left(x, \left(\widehat{A}_{j_1\ldots j_l}^{i_1\ldots i_k}(x)\right)\right).
\end{aligned}$$

By [Lee13, p. 35] $A$ is smooth if and only if in any chart $\widehat{A}$ is smooth. This is furthermore equivalent to that each $\widehat{A}_{j_1\ldots j_l}^{i_1\ldots i_k}$ is smooth and thus equivalent to that $A_{j_1\ldots j_l}^{i_1\ldots i_k}$ is smooth (see [Lee13, p. 33]). $\qquad\square$

**Proposition A.3.** *Let $M$ be smooth manifold and let $A : M \to T^{(1,k)}TM$ be a rough section. Then $A \in \Gamma(T^{(1,k)}TM)$ if and only if for all $X_1, \ldots, X_k \in \mathfrak{X}(M)$, the rough section $A(X_1, \ldots, X_k) : M \to TM$ defined by*

$$A(X_1, \ldots, X_k)(p) := A_p(X_1|_p, \ldots, X_k|_p) \tag{15}$$

*is a smooth vector field.*

*Proof.* Let $(U, (x^i))$ be a smooth chart. Then write $X_1, \ldots, X_k \in \mathfrak{X}(M)$ as

$$X_\nu = X_\nu^{\mu_\nu} \frac{\partial}{\partial x^{\mu_\nu}}.$$

for $\nu = 1, \ldots, k$. For $p \in U$ lemma 4.1 implies

$$\begin{aligned}
A(X_1, \ldots, X_n)(p) &= A_p(X_1|_p, \ldots, X_k|_p) \\
&= A_p\left(X_1^{\mu_1}(p)\frac{\partial}{\partial x^{\mu_1}}\bigg|_p, \ldots, X_1^{\mu_k}(p)\frac{\partial}{\partial x^{\mu_k}}\bigg|_p\right) \\
&= X_1^{\mu_1}(p)\cdots X_k^{\mu_k}(p) A_p\left(\frac{\partial}{\partial x^{\mu_1}}\bigg|_p, \ldots, \frac{\partial}{\partial x^{\mu_k}}\bigg|_p\right) \\
&= X_1^{\mu_1}(p)\cdots X_k^{\mu_k}(p) A_{\mu_1\ldots\mu_k}^i(p)\frac{\partial}{\partial x^i}\bigg|_p.
\end{aligned}$$

By the smoothness criterion for vector fields [Lee13, p. 175] we have that each component function $X_\nu^{\mu_n}$ is smooth. Thus if $A$ is smooth, we have by that each

$A^i_{j_1\ldots j_k}$ is smooth and since $\mathscr{C}^\infty(M)$ is an $\mathbb{R}$-algebra (see [Lee13, p. 33]), we have that

$$X_1^{\mu_1}\cdots X_k^{\mu_k}A^i_{\mu_1\ldots\mu_k}$$

is smooth for $i = 1,\ldots,n$. Thus again by the smoothness criterion together with the localness of smoothness [Lee13, p. 35] we get that $A(X_1,\ldots,X_k) \in \mathfrak{X}(M)$. Conversely, each vector field locally defined by

$$X_{j_\nu} = \delta^{\mu_\nu}_{j_\nu}\frac{\partial}{\partial x^{\mu_\nu}}.$$

is smooth. Thus by

$$A(X_1,\ldots,X_n)(p) = \delta^{\mu_1}_{j_1}\cdots\delta^{\mu_k}_{j_k}A^i_{\mu_1\ldots\mu_k}(p)\frac{\partial}{\partial x^i}\bigg|_p = A^i_{j_1\ldots j_k}(p)\frac{\partial}{\partial x^i}\bigg|_p$$

we get that $A^i_{j_1\ldots j_k}$ is smooth and hence $A$. $\qquad\square$

**Theorem A.1 (Tensor Characterization Lemma).** *A mapping*

$$\underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{k}\to\mathscr{C}^\infty(M) \qquad or \qquad \underbrace{\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)}_{k}\to\mathfrak{X}(M)$$

*is induced by an element of $\Gamma(T^{(0,k)}TM)$ or $\Gamma(T^{(1,k)}TM)$, respectively, if and only if they are multilinear over $\mathscr{C}^\infty(M)$.*

*Proof.* Let $A \in \Gamma(T^{(1,k)}TM)$. Since $T^{(1,k)}T_pM \cong L\left((T_pM)^k;T_pM\right)$ (see [KN96a, p. 23]), we define a mapping $\mathscr{A} : \mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ by

$$\mathscr{A}(X_1,\ldots,X_k)_p := A_p(X_1|_p,\ldots,X_k|_p) \in T_pM$$

where the smoothness follows from an adapted version of [Lee13, pp. 317–318]. Next we show that $\mathscr{A}$ is multilinear over $\mathscr{C}^\infty(M)$. Let $f \in \mathscr{C}^\infty(M)$ and $X_\nu, \widetilde{X}_\nu \in \mathfrak{X}(M)$, $\nu = 1,\ldots,k$. Then for any $p \in M$ we have that

$$\begin{aligned}
\mathscr{A}(X_1,\ldots,fX_\nu + \widetilde{X}_\nu,\ldots,X_k)_p &= A_p(X_1|_p,\ldots,(fX_\nu + \widetilde{X}_\nu)_p,\ldots,X_k|_p)\\
&= A_p(X_1|_p,\ldots,f(p)X_\nu|_p + \widetilde{X}_\nu|_p,\ldots,X_k|_p)\\
&= f(p)A_p(X_1|_p,\ldots,X_\nu|_p,\ldots,X_k|_p)\\
&\quad + A_p(X_1|_p,\ldots,\widetilde{X}_\nu|_p,\ldots,X_k|_p)\\
&= f(p)\mathscr{A}(X_1,\ldots,X_\nu,\ldots,X_k)_p\\
&\quad + \mathscr{A}(X_1,\ldots,\widetilde{X}_\nu,\ldots,X_k)_p\\
&= \left(f\mathscr{A}(X_1,\ldots,X_\nu,\ldots,X_k)\right)_p\\
&\quad + \mathscr{A}(X_1,\ldots,\widetilde{X}_\nu,\ldots,X_k)_p.
\end{aligned}$$

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Conversly, suppose that $\mathscr{A} : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is multilinear over $\mathscr{C}^\infty(M)$. Let $p \in M$. First we show that $\mathscr{A}$ acts locally, i.e. if $X_\nu = \widetilde{X}_\nu$ in some neighbourhood of $p$ implies that also

$$\mathscr{A}(X_1, \ldots, X_\nu, \ldots, X_k) = \mathscr{A}(X_1, \ldots, \widetilde{X}_\nu, \ldots, X_k)$$

on $U$. By the multilinearity of $\mathscr{A}$ it is enough to show that if $X_\nu$ vanishes on $U$ then so does $\mathscr{A}$. There exists a smooth bump function $\psi$ for $\{p\}$ supported in $U$ (see [Lee13, p. 44]). Hence $\psi X_\nu = 0$ on $M$ and $\psi(p) = 1$. Thus

$$0 = \mathscr{A}(X_1, \ldots, \psi X_\nu, \ldots, X_k)_p = \psi(p)\mathscr{A}(X_1, \ldots, X_\nu, \ldots, X_k)_p.$$

and since $\psi(p) = 1$ we have that

$$\mathscr{A}(X_1, \ldots, X_\nu, \ldots, X_k)_p = 0$$

for any $p \in U$. $\qquad\qquad\square$

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