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SOLUTIONS SHEET 7

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Exercise 1. We will abreviate $\mathbb{C}^- := \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.
(a) The set $\mathbb{C}^-$ is clearly a star shaped domain with possible centers on the ray $\mathbb{R}_{>0}$. Furtheremore, the function $1/z$ is holomorphic in $\mathbb{C}^-$ since it is a well-defined rational function there. By the Cauchy integral theorem for star shaped domains f has a primitive $F : \mathbb{C}^- \to \mathbb{C}$ which is explicitely given by

$$F(z) := \int_{[z_0,z]} \frac{\mathrm{d}\zeta}{\zeta} \tag{1}$$

for any $z_0 \in \mathbb{R}_{>0}$. The choice $z_0 = 1$ yields

$$F(1) = \int_{[1,1]} \frac{\mathrm{d}\zeta}{\zeta} = 0 \tag{2}$$

since the path $[1,1]\,(t) = 1$, $t \in [0,1]$, is clearly closed (we have that $[1,1]\,(0) = 1 = [1,1]\,(1)$) and thus again the Cauchy integral theorem implies that the integral over any closed path vanishes. Hence the primitive F of $1/z$ on $\mathbb{C}^-$ fulfilling $F(1) = 0$ is given by

$$\boxed{F(z) = \int_{[1,z]} \frac{\mathrm{d}\zeta}{\zeta} \qquad z \in \mathbb{C}^-.} \tag{3}$$

(b) Let $z_0 \in \mathbb{C}^-$. Let $B_r(z_0)$ denote the largest ball around z_0 contained in $\mathbb{C}^-$. By the Cauchy-Taylor expansion theorem we have that

$$F = \sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu \qquad a_\nu = \frac{F^{(\nu)}(z_0)}{\nu!} \tag{4}$$

in $B_r(z_0)$ since F is clearly holomorphic in $\mathbb{C}^-$ as a primitive. In order to calculate $F^{(\nu)}$, we have to compute $f^{(\nu)}$ since $F' = f$.

Lemma 0.1. *Consider the function $f : \mathbb{C}^\times \to \mathbb{C}$ defined by $f(z) := 1/z$. Then*

$$f^{(\nu)}(z_0) = (-1)^\nu \frac{\nu!}{z_0^{\nu+1}} \qquad \nu \in \mathbb{N}_0, z_0 \in \mathbb{C}^-. \tag{5}$$

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Proof. Proof by induction over $\nu \in \mathbb{N}_0$. For $\nu = 0$ the equation clearly holds. Assume it is true for some $\nu \in \mathbb{N}_0$. Then

$$f^{(\nu+1)}(z_0) = \left(f^{(\nu)}\right)'(z_0) = (-1)^\nu \nu!(-(\nu+1))\frac{1}{z_0^{\nu+2}} = (-1)^{\nu+1}\frac{(\nu+1)!}{z_0^{\nu+2}}. \tag{6}$$

□

Since $F^{(\nu)}(z_0)/\nu! = f^{(\nu-1)}(z_0)/\nu!$ for all $\nu \in \mathbb{N}$, lemma 0.1 implies that

$$\boxed{F = F(z_0) + \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{\nu}\frac{1}{z_0^\nu}(z-z_0)^\nu \qquad z \in B_r(z_0).} \tag{7}$$

By

$$\limsup_{\nu\to\infty}\left|\frac{(-1)^{\nu-1}}{\nu}\frac{1}{z_0^\nu}\right|^{1/\nu} = \frac{1}{|z_0|}\limsup_{\nu\to\infty}\frac{1}{\nu^{1/\nu}} = \frac{1}{|z_0|}\lim_{\nu\to\infty}\frac{1}{\nu^{1/\nu}} = \frac{1}{|z_0|} \tag{8}$$

we see that $R = |z_0|$ using the Cauchy-Hadamard formula.