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SOLUTIONS SHEET 1

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Exercise 1.
a) It is well known, that the set $M_2(\mathbb{R})$ is a ring with identity. It is also clear, that the given set together with the usual operations constitutes a subring with identity of $M_2(\mathbb{R})$. Therefore it is enough to show commutativity and the existence of inverse elements regarding multiplication. Let $x, y, x', y' \in \mathbb{R}$. Then we have

$$\begin{aligned}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\begin{pmatrix} x' & y' \\ -y' & x' \end{pmatrix} &= \begin{pmatrix} xx' - yy' & xy' + yx' \\ -yx' - xy' & -yy' + xx' \end{pmatrix} \\ &= \begin{pmatrix} x'x - y'y & x'y + y'x \\ -y'x - x'y & -y'y + x'x \end{pmatrix} \\ &= \begin{pmatrix} x' & y' \\ -y' & x' \end{pmatrix}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\end{aligned}$$

Since $\mathbb{R}$ is a field. Furthermore we have

$$\det \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = x^2 + y^2 \tag{1}$$

Hence $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ is invertible if and only if $(x, y) \neq (0, 0)$, which means that every non-zero element is invertible. Thus the set constitutes a field under the given operations. Now define a mapping ι by

$$\iota(x + iy) := \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \tag{2}$$

We have

$$\iota((x + iy) + (u + iv)) = \begin{pmatrix} x + u & y + v \\ -y - v & x + u \end{pmatrix} = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} + \begin{pmatrix} u & v \\ -v & u \end{pmatrix} = \iota(x + iy) + \iota(u + iv)$$

and

$$\iota((x + iy)(u + iv)) = \begin{pmatrix} xu - yv & xv + uy \\ -xv - uy & xu - yv \end{pmatrix} = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}\begin{pmatrix} u & v \\ -v & u \end{pmatrix} = \iota(x + iy)\iota(u + iv).$$

Obviously $\ker(\iota) = \{0\}$ and ι is surjective, hence ι is an isomorphism of fields.

b) Consider the abelian group $(\mathbb{C}, \cdot)$. We show $(S^1, \cdot) \leq (\mathbb{C}, \cdot)$, where

$$S^1 := \partial\mathbb{E} = \{z \in \mathbb{C} : |z| = 1\}.$$

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Clearly $1 \in S^1$. If $z, z' \in S^1$, we have $|z| = |z'| = 1$ and therefore $|zz'| = |z|\,|z'| = 1$ which implies $zz' \in S^1$. Also we have $|1/z| = 1/|z| = 1$ for $z \in S^1$ which implies $1/z \in S^1$. With the terminology established in (2) we consider the restriction

$$\iota|_{S^1} : S^1 \to \iota(S^1) \tag{3}$$

which is an isomorphism of groups. We will show that

$$\iota(S^1) = \mathrm{SO}(2). \tag{4}$$

Let $x + iy \in S^1$. Then $1 = |x + iy|^2 = x^2 + y^2$ and so

$$\iota(x+iy)(\iota(x+iy))^t = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} x^2+y^2 & -xy+yx \\ -yx+xy & x^2+y^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $\iota(S^1)$ is a subgroup of the abelian group in a), we have also that $(\iota(x+iy))^t\iota(x+iy)$ is the identity matrix. Also $\det(\iota(x+iy)) = x^2 + y^2 = 1$ so $\iota(S^1) \subseteq \mathrm{SO}(2)$. Now we have

$$\mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} : \varphi \in \mathbb{R} \right\}$$

By

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos(-\varphi) & \sin(-\varphi) \\ -\sin(-\varphi) & \cos(-\varphi) \end{pmatrix}$$

we have

$$\iota(\cos(-\varphi) + i\sin(-\varphi)) = \begin{pmatrix} \cos(-\varphi) & \sin(-\varphi) \\ -\sin(-\varphi) & \cos(-\varphi) \end{pmatrix}$$

and with

$$|\cos(-\varphi) + i\sin(-\varphi)|^2 = \cos^2(-\varphi) + \sin^2(-\varphi) = 1$$

this implies $\mathrm{SO}(2) \subseteq \iota(S^1)$.

Exercise 2. See separate sheet.

Exercise 3. Clearly $\varnothing \in \mathcal{Z}$ and $\mathbb{C} \in \mathcal{Z}$ since $\mathbb{C}^c = \varnothing$ which is finite. Let $U, V \in \mathcal{Z}$. Then U^c and V^c are both finite and so is

$$(U \cap V)^c = U^c \cup V^c.$$

Hence $U \cap V \in \mathcal{Z}$. Let $(U_\alpha)_{\alpha \in A}$ be a sequence in $\mathcal{Z}$. Then we have that U_α^c is finite for any $\alpha \in A$. Therefore by

$$\left(\bigcup_{\alpha \in A} U_\alpha \right)^c = \bigcap_{\alpha \in A} U_\alpha^c \subseteq U_\beta^c$$

for any $\beta \in A$ we have that $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{Z}$. Hence $(\mathbb{C}, \mathcal{Z})$ is a topological space. We show the following result:

Lemma 0.1. *Let X be an infinite set. Then $(X, \mathcal{Z})$ is not Hausdorff.*

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Proof. Towards a contradiction assume that $(X, \mathcal{Z})$ is Hausdorff. Hence for any $p, q \in X$ we find (open) neighbourhoods U of p and V of q such that $U \cap V = \varnothing$. Thus U^c is finite and since $U \cap V = \varnothing$ we have that $U \subseteq V^c$ and thus U is finite. But this would imply that

$$X = U \cup U^c$$

is the union of finite sets which would mean that X itself is finite. Contradiction. □

Exercise 4. We use the terminology established in [FL05, pp. 8–16] which results in considering a function $f : M \to \mathbb{C}$. We show the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).
Assume f is continuous in $z^* \in M$ and fix $\varepsilon > 0$. Now consider the set $B_\varepsilon(f(z^*))$ which is a neighbourhood of $f(z^*)$. By (i) there exists a neighbourhood U of z^* such that $f(U \cap M) \subseteq B_\varepsilon(f(z^*))$. Since U is a neighbourhood of z^* it contains a δ-neighbourhood $B_\delta(z^*)$ of z^*.
Assume that (ii) holds. Let $(z_\nu)_{\nu \in \mathbb{N}}$ be a sequence in M such that $z_\nu \to z^*$ and U be any neighbourhood of $f(z^*)$. Since U is a neighbourhood of $f(z^*)$ it contains a ε-neighbourhood $B_\varepsilon(f(z^*))$. By (ii) we find $\delta > 0$ such that $z \in B_\delta(z^*)$ implies $f(z) \in B_\varepsilon(f(z^*))$. Since $z_\nu \to z^*$ we also have $z_\nu \in B_\delta(z^*)$ for almost all z_ν. In conclusion, $f(z_\nu) \in U$ for almost all $f(z_\nu)$.
Assume that (iii) holds. Towards a contradiction assume that (i) does not hold. Hence there exists a neighbourhood V of $f(z^*)$ such that for any neighbourhood U of z^* we have $f(U \cap M) \not\subseteq V$. The latter means that $z \in f(U \cap M)$ but $z \notin V$ for some z. Consider the sets $B_{1/\nu}(z^*)$ for $\nu \in \mathbb{N}_{>0}$. They are clearly neighbourhoods of z^*. Now by assumption, for any $\nu \in \mathbb{N}_{>0}$ there is some $f(z_\nu) \in f(B_{1/\nu}(z^*) \cap M)$ which is not in V. This defines a sequence $(z_\nu)_{n \in \mathbb{N}_{>0}}$ in M. We claim that $z_\nu \to z^*$. Indeed, if we have any neighbourhood U of z^* we find by definition an ε-neighbourhood $B_\varepsilon(z^*) \subseteq U$. But by the archimidean principle we have $1/\nu < \varepsilon$ for ν small enough, thus $z_\nu \in B_\varepsilon(z^*)$ for almost all ν. Now $z_\nu \to z^*$ but clearly $f(z_\nu) \not\to f(z^*)$ since none of the z_ν is in V. Contradiction.

References

[FL05] W. Fischer and I. Lieb. *Funktionentheorie: Komplexe Analysis in einer Veränderlichen.* vieweg studium; Aufbaukurs Mathematik. Vieweg+Teubner Verlag, 2005. ISBN: 9783834800138.