SOLUTIONS SHEET 9

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Exercise 1. Let $\Gamma := [z_1, z_2] + [z_2, z_3] + [z_3, z_1]$. Γ is a cycle since it is the positively oriented boundary chain of the non-empty domain Δ° (example 1 in Fischer/Lieb). Now by Satz 1.3 we have that $n(\Gamma, z)$ is locally constant for $z \in \mathbb{C} \setminus |\Gamma|$ and 0 for z in the unbounded pathcomponent. Thus we get

$$n(\Gamma, z) = 0 \qquad z \in \mathbb{C} \setminus \Delta.$$

By the wall-crossing lemma (Satz 3.1) we get furthermore

$$n(\Gamma, z) = 1 \qquad z \in \Delta^\circ$$

since by assumption every line segment is not a singleton, hence we find a ball around a point in the line segment and we can apply Satz 3.1 which just yields

$$n(\Gamma, z_1) = n(\Gamma, z_2) + 1 = 1 \qquad z_1 \in \Delta^\circ, z_2 \in \mathbb{C} \setminus \Delta.$$

Exercise 2. See separate sheet.

Exercise 3.

(i) Let $f \in \mathcal{O}(G_1 \cup G_2)$ (clearly $G_1 \cup G_2$ is open and connected since it is path-connected by the non-empty intersection). This implies $f \in \mathcal{O}(G_1)$ and $f \in \mathcal{O}(G_2)$. Since G_1 and G_2 are elementary domains, there exist primitives $F_1 : G_1 \to \mathbb{C}$ and $F_2 : G_2 \to \mathbb{C}$ of f. Consider the auxiliary function $\varphi : G_1 \cap G_2 \to \mathbb{C}$ defined by $\varphi(z) := F_1(z) - F_2(z)$. This is clearly not the empty function since $G_1 \cap G_2 \neq \varnothing$ by assumption. Now

$$\varphi'(z) = F_1'(z) - F_2'(z) = f(z) - f(z) = 0$$

for all $z \in G_1 \cap G_2$ implies that φ is locally constant on $G_1 \cap G_2$ and thus by connectedness, φ is constant on $G_1 \cap G_2$. Hence there exists $\lambda \in \mathbb{C}$ such that $F_1(z) = F_2(z) + \lambda$ for all $z \in G_1 \cap G_2$. Thus

$$F(z) := \begin{cases} F_1(z) & z \in G_1 \\ F_2(z) + \lambda & z \in G_2 \end{cases}$$

is a well defined primitive of f in $G_1 \cup G_2$.

(ii) Consider $G_1 := \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and $G_2 := \mathbb{C} \setminus \mathbb{R}_{\geq 0}$. Clearly, G_1 and G_2 are elementary domains, since they are star-shaped domains. Now $G_1 \cap G_2 = \mathbb{C} \setminus \mathbb{R} \neq \varnothing$ which is not connected since

$$\mathbb{C} \setminus \mathbb{R} = \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\} \cup \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

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Furthermore, $G_1 \cup G_2 = \mathbb{C} \setminus \{0\}$. But $\mathbb{C} \setminus \{0\}$ is clearly not an elementary domain since $f(z) := \frac{1}{z}$ does not have a primitive there since

$$\int_{\partial \mathbb{E}} f(\zeta) \, \mathrm{d}\zeta = 2\pi i.$$

Hence the assumption that $G_1 \cap G_2 \neq \varnothing$ is connected is necessary.

(iii)

Exercise 4.

Definition 0.1. *Let $G \subseteq \mathbb{C}^\times$ be a domain. A function $\varphi \in \mathscr{C}(G; \mathbb{R})$ is said to be a* ***branch of the argument****, if $z = |z| \, e^{i\varphi(z)}$ for all $z \in G$.*

Proposition 0.1. *Let $G \subseteq \mathbb{C}^\times$ be a domain. There exists a branch of the argument on G if and only if there exists a branch of the logarithm on G.*

Proof. Assume that there exists a branch of the argument φ. Thus for all $z \in G$ we have

$$z = |z| \, e^{i\varphi(z)} = e^{\log|z| + i\varphi(z)}$$

and by the continuity of φ, $f(z) := \log|z| + i\varphi(z)$ is a branch of the logarithm. Conversly, by

$$z = e^{f(z)} = e^{\operatorname{Re} f} e^{i \operatorname{Im} f} = |z| \, e^{i \operatorname{Im} f}$$

for all $z \in G$ we have that $\operatorname{Im} f$ is a branch of the argument on G, since f is continuous and so is $\operatorname{Im} f$. $\square$