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SOLUTIONS SHEET 7

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Exercise 1. We will abreviate $\mathbb{C}^- := \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.
(a) The set $\mathbb{C}^-$ is clearly a star shaped domain with possible centers on the ray $\mathbb{R}_{>0}$. Furtheremore, the function $1/z$ is holomorphic in $\mathbb{C}^-$ since it is a well-defined rational function there. By the Cauchy integral theorem for star shaped domains f has a primitive $F : \mathbb{C}^- \to \mathbb{C}$ which is explicitely given by

$$F(z) := \int_{[z_0,z]} \frac{\mathrm{d}\zeta}{\zeta} \tag{1}$$

for any $z_0 \in \mathbb{R}_{>0}$. The choice $z_0 = 1$ yields

$$F(1) = \int_{[1,1]} \frac{\mathrm{d}\zeta}{\zeta} = 0 \tag{2}$$

since the path $[1,1]\,(t) = 1$, $t \in [0,1]$, is clearly closed (we have that $[1,1]\,(0) = 1 = [1,1]\,(1)$) and thus again the Cauchy integral theorem implies that the integral over any closed path vanishes. Hence the primitive F of $1/z$ on $\mathbb{C}^-$ fulfilling $F(1) = 0$ is given by

$$\boxed{F(z) = \int_{[1,z]} \frac{\mathrm{d}\zeta}{\zeta} \qquad z \in \mathbb{C}^-.} \tag{3}$$

(b) Let $z_0 \in \mathbb{C}^-$. Let $B_r(z_0)$ denote the largest ball around z_0 contained in $\mathbb{C}^-$. By the Cauchy-Taylor expansion theorem we have that

$$F = \sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu \qquad a_\nu = \frac{F^{(\nu)}(z_0)}{\nu!} \tag{4}$$

in $B_r(z_0)$ since F is clearly holomorphic in $\mathbb{C}^-$ as a primitive. In order to calculate $F^{(\nu)}$, we have to compute $f^{(\nu)}$ since $F' = f$.

Lemma 0.1. *Consider the function $f : \mathbb{C}^\times \to \mathbb{C}$ defined by $f(z) := 1/z$. Then*

$$f^{(\nu)}(z_0) = (-1)^\nu \frac{\nu!}{z_0^{\nu+1}} \qquad \nu \in \mathbb{N}_0, z_0 \in \mathbb{C}^-. \tag{5}$$

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Proof. Proof by induction over $\nu \in \mathbb{N}_0$. For $\nu = 0$ the equation clearly holds. Assume it is true for some $\nu \in \mathbb{N}_0$. Then

$$f^{(\nu+1)}(z_0) = \left(f^{(\nu)}\right)'(z_0) = (-1)^\nu \nu!(-(\nu+1))\frac{1}{z_0^{\nu+2}} = (-1)^{\nu+1}\frac{(\nu+1)!}{z_0^{\nu+2}}. \tag{6}$$

□

Since $F^{(\nu)}(z_0)/\nu! = f^{(\nu-1)}(z_0)/\nu!$ for all $\nu \in \mathbb{N}$, lemma 0.1 implies that

$$\boxed{F = F(z_0) + \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{\nu}\frac{1}{z_0^\nu}(z - z_0)^\nu \qquad z \in B_r(z_0).} \tag{7}$$

By

$$\limsup_{\nu\to\infty}\left|\frac{(-1)^{\nu-1}}{\nu}\frac{1}{z_0^\nu}\right|^{1/\nu} = \frac{1}{|z_0|}\limsup_{\nu\to\infty}\frac{1}{\nu^{1/\nu}} = \frac{1}{|z_0|}\lim_{\nu\to\infty}\frac{1}{\nu^{1/\nu}} = \frac{1}{|z_0|} \tag{8}$$

we see that $R = |z_0|$ using the Cauchy-Hadamard formula.

Exercise 2. For all $z \in B_{2\pi}^\times(0)$ we have that

$$\frac{e^z - 1}{z} = \sum_{\nu=1}^{\infty}\frac{z^{\nu-1}}{\nu!} = \sum_{\mu=0}^{\infty}\frac{z^\mu}{(\mu+1)!}. \tag{9}$$

From (9) it is immediate that

$$\lim_{z\to 0}\frac{e^z-1}{z} = \lim_{z\to 0}\sum_{\mu=0}^{\infty}\frac{z^\mu}{(\mu+1)!} = 1 \tag{10}$$

since the radius of convergence of the right side in (9) is ∞ and thus $\sum_{\mu=0}^\infty \frac{z^\mu}{(\mu+1)!}$ is clearly continuous at 0. Hence $f \in \mathscr{C}(B_{2\pi}(0))$ and $f \in \mathcal{O}(B_{2\pi}^\times(0))$. Since by continuity f is bounded on any compactum $\overline{B_\varepsilon}(0)$, $0 < \varepsilon < 2\pi$, Riemann's theorem on removable singularities implies that $f \in \mathcal{O}(B_{2\pi}(0))$. Now the largest disc $B_r(0)$ contained in $B_{2\pi}(0)$ is $B_{2\pi}(0)$ itself, and thus by the Cauchy-Taylor expansion theorem f can be Taylor expanded around 0. The expansion is of the form

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu = \sum_{\nu=0}^{\infty}\frac{f^{(\nu)}(0)}{\nu!}z^\nu = \sum_{\nu=0}^{\infty}\frac{B_\nu}{\nu!}z^\nu \tag{11}$$

where $B_\nu := f^{(\nu)}(0)$ for all $\nu \in \mathbb{N}_0$.

(i) Clearly we have $B_0 = f^{(0)}(0) = f(0) = 1$. Using the formula from (ii) with $n = 2$ we get $1 + 2B_1 = 0$ which is equivalent to $B_1 = -1/2$. Next we consider the function $f(z) + z/2$. We claim that this function is even. Indeed, if $z \neq 0$ we have

$$\begin{aligned} f(-z) - \frac{z}{2} &= \frac{-z}{e^{-z}-1} - \frac{z}{2} = -\frac{z(1+e^{-z})}{2(e^{-z}-1)} = -\frac{z(e^z+1)}{2(1-e^z)} = \frac{z(e^z+1)}{2(e^z-1)} \\ &= \frac{z(e^z-1+2)}{2(e^z-1)} = \frac{z}{e^z-1} + \frac{z}{2} = f(z) + \frac{z}{2}. \end{aligned}$$

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Now we have

$$f(z) + \frac{z}{2} = 1 + \sum_{\nu=2}^{\infty} \frac{B_\nu}{\nu!} z^\nu \qquad z \in B_{z\pi}(0). \tag{12}$$

Since $f(z) + z/2$ is even, we get

$$0 = \sum_{\nu=2}^{\infty} \frac{B_\nu}{\nu!} z^\nu - \sum_{\nu=2}^{\infty} \frac{B_\nu}{\nu!}(-1)^\nu z^\nu = \sum_{\nu=2}^{\infty} \frac{B_\nu}{\nu!}(1 + (-1)^{\nu+1}) z^\nu = 2\sum_{\nu=1}^{\infty} \frac{B_{2\nu+1}}{(2\nu+1)!} z^{2\nu+1} \tag{13}$$

The uniqueness of Taylor coefficients therefore implies that $B_{2\nu+1} = 0$ for all $\nu \in \mathbb{N}$.

(ii) Since both $e^z - 1 = \sum_{\nu=1}^{\infty} \frac{z^\nu}{\nu!}$ and $\sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} z^\nu$ have radius of convergence 2π around 0, the product theorem for power series [RS02, p. 195] implies that

$$z = (e^z - 1)\sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} z^\nu = \left(\sum_{\nu=1}^{\infty} \frac{z^\nu}{\nu!}\right)\left(\sum_{\nu=0}^{\infty} \frac{B_\nu}{\nu!} z^\nu\right) = \sum_{\lambda=0}^{\infty}\left(\sum_{\mu+\nu=\lambda} a_\mu \frac{B_\nu}{\nu!}\right) z^\lambda \tag{14}$$

where $a_\mu := 1/\mu!$ for $\mu \in \mathbb{N}$ and $a_0 := 0$. By the uniqueness of the Taylor coefficients we get

$$0 = \sum_{\mu+\nu=\lambda} a_\mu \frac{B_\nu}{\nu!} = \sum_{\nu=0}^{\lambda} a_{\lambda-\nu} \frac{B_\nu}{\nu!} = \sum_{\nu=0}^{\lambda-1} \frac{B_\nu}{\nu!(\lambda-\nu)!} = \frac{1}{\lambda!}\sum_{\nu=0}^{\lambda-1} \binom{\lambda}{\nu} B_\nu \tag{15}$$

for all $\lambda \in \mathbb{N}_{>1}$ or equivalently

$$\boxed{\sum_{\nu=0}^{\lambda-1} \binom{\lambda}{\nu} B_\nu = 0 \qquad \lambda \in \mathbb{N}_{>1}.} \tag{16}$$

(iii) Showing that $B_\nu \in \mathbb{Q}$ for all $\nu \in \mathbb{N}_0$ is a simple proof by induction using (ii) and the fact that the binomial coefficients are integers. The case $\nu = 0, 1$ is clear since $B_0 = 1 \in \mathbb{Q}$ and $B_1 = -1/2 \in \mathbb{Q}$. Hence assume that the statement holds for some $\lambda \in \mathbb{N}_{>1}$. Then (ii) yields

$$B_{\lambda+1} = -\frac{1}{\binom{\lambda+2}{\lambda+1}} \sum_{\nu=0}^{\lambda} \binom{\lambda+2}{\nu} B_\nu \tag{17}$$

but this is a sum of integers and rational numbers, hence rational. Therefore we conclude by the principle of mathematical induction.
Towards a contradiction, assume that the sequence $(B_\nu)_{\nu\in\mathbb{N}_0}$ is bounded, i.e. $|B_\nu| \leq M$ for some $M > 0$. By

$$0 \leq \limsup_{\nu\to\infty} \left|\frac{B_\nu}{\nu!}\right|^{1/\nu} = \limsup_{\nu\to\infty} \frac{|B_\nu|^{1/\nu}}{(\nu!)^{1/\nu}} \leq \limsup_{\nu\to\infty} \frac{M^{1/\nu}}{(\nu!)^{1/\nu}} = 0 \tag{18}$$

we get $R = \infty$ by Cauchy-Hadamard. This contradicts the finite radius of convergence of 2π. Hence $(B_\nu)_{\nu\in\mathbb{N}_0}$ is unbounded.

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References

[RS02] R. Remmert and G. Schumacher. *Funktionentheorie 1*. Springer-Lehrbuch. Springer Berlin Heidelberg, 2002. ISBN: 9783540418559.