

## SOLUTIONS SHEET 8

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**Exercise 1.** Let  $U := \mathbb{C} \setminus \{\pm e^{\pm i\pi/4}\}$  and define  $F : U \rightarrow \mathbb{C}$  by

$$F(z) := \frac{1}{1 + z^4}. \quad (1)$$

Clearly  $F \in \mathcal{O}(U)$  as a well defined rational function,  $U$  is open in  $\mathbb{C}$  and  $\mathbb{R} \subseteq U$ . Furthermore  $F|_{\mathbb{R}} = f$ . Hence  $F$  is a holomorphic continuation of  $f$ . Since having an analytic continuation is equivalent to be real-analytic (see [FL03, p. 100]), we have that  $f$  is real-analytic.

Let  $x_0 \in \mathbb{R}$ . The Taylor series expansion of  $f$  is completely determined by the one of  $F$ . This is due to the fact, that  $F^{(\nu)}(x_0) = f^{(\nu)}(x_0)$  for all  $\nu \in \mathbb{N}_0$ . By Cauchy-Taylor,  $F$  is expandable into a power series in the largest ball around  $x_0$  contained in  $U$ , i.e.

$$F(z) = \sum_{\nu=0}^{\infty} a_{\nu}(z - x_0)^{\nu} \quad (2)$$

and the convergence is normal there. Thus the radius of convergence  $R$  of the expansion (2) is at least  $|x_0 - e^{i\pi/4}|$  if  $x_0 \geq 0$  and  $|x_0 + e^{i\pi/4}|$  if  $x_0 \leq 0$ . Let  $r := |x_0 - e^{i\pi/4}|$  and assume  $x_0 \geq 0$  (the case  $x_0 \leq 0$  is similar). Furthermore assume that  $R > r$ . Hence the series expansion (2) converges in  $B_R(x_0)$  and therefore defines a function  $G : B_R(x_0) \rightarrow \mathbb{C}$  by

$$G(z) := \sum_{\nu=0}^{\infty} a_{\nu}(z - x_0)^{\nu} \quad (3)$$

with  $G|_{B_r(x_0)} = F$ . Since  $G$  is expandable in a power series, we have  $G \in \mathcal{O}(B_R(x_0))$  by [RS02, p. 187]. Since any holomorphic function is continuous, we have  $G \in \mathcal{C}(B_R(x_0))$ . Let  $(z_{\nu})_{\nu \in \mathbb{N}}$  be a sequence in  $B_r(x_0)$  such that  $\lim_{\nu \rightarrow \infty} z_{\nu} = e^{i\pi/4}$ . Clearly

$$\lim_{\nu \rightarrow \infty} F(z_{\nu}) = \infty \quad (4)$$

and since  $G|_{B_r(x_0)} = F$  we have

$$\lim_{\nu \rightarrow \infty} G(z_{\nu}) = \infty. \quad (5)$$

But since  $R > r$ ,  $G$  is continuous at  $e^{i\pi/4}$  and so we must have

$$G(e^{i\pi/4}) = \lim_{\nu \rightarrow \infty} G(z_{\nu}) = \infty. \quad (6)$$

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Thus the series  $G$  diverges at  $e^{i\pi/4}$ , contradicting that  $e^{i\pi/4} \in B_R(x_0)$ . In conclusion

$$\boxed{R(x_0) = \min(|x_0 - e^{i\pi/4}|, |x_0 + e^{i\pi/4}|) \quad x_0 \in \mathbb{R}.} \quad (7)$$

**Exercise 2.** Since  $f \in \mathcal{O}(\mathbb{E})$  we have that  $f \in \mathcal{C}(\mathbb{E})$ . Thus since  $\partial B_r(0)$ ,  $0 \leq r < 1$ , is compact we have that  $|f|$  attains its supremum on  $\partial B_r(0)$ . Hence we have

$$M(r) = \max_{|z|=r} |f(z)|. \quad (8)$$

(i) Let  $0 < R < 1$ . Since  $\overline{B_R}(0)$  is compact and  $f \in \mathcal{C}(\mathbb{E})$ , we have that  $f$  is uniformly continuous on  $\overline{B_R}(0)$  (see for example [Alt16, p. 138]). By the reversed triangle inequality also  $|f|$  is uniformly continuous on  $\overline{B_R}(0)$ . Let  $r_1, r_2 \in [0, R]$  and  $\varepsilon > 0$ . Assume first that  $M(r_2) \geq M(r_1)$ . We find  $\varphi \in \mathbb{R}$  such that

$$M(r_2) = |f(r_2 e^{i\varphi})|. \quad (9)$$

Now

$$M(r_2) - M(r_1) \leq M(r_2) - |f(r_1 e^{i\varphi})| = |f(r_2 e^{i\varphi})| - |f(r_1 e^{i\varphi})| < \varepsilon$$

whenever  $|r_2 - r_1| = |r_2 e^{i\varphi} - r_1 e^{i\varphi}| < \delta$  by the uniform continuity of  $|f|$  on  $[0, R]$ . Now assume  $M(r_2) \leq M(r_1)$ . Again we find  $\psi \in \mathbb{R}$  such that  $M(r_1) = |f(r_1 e^{i\psi})|$ . Thus

$$M(r_1) - M(r_2) \leq M(r_1) - |f(r_2 e^{i\psi})| = |f(r_1 e^{i\psi})| - |f(r_2 e^{i\psi})| < \varepsilon \quad (10)$$

whenever  $|r_2 - r_1| < \delta$ . Thus we have

$$|M(r_1) - M(r_2)| < \varepsilon \quad (11)$$

whenever  $|r_1 - r_2| < \delta$ . Hence  $M$  is uniformly continuous on  $[0, R]$ . Now  $M$  is clearly continuous on  $[0, 1)$ , since if not, there would exist a point  $r_0 \in [0, 1)$  where  $M$  is not continuous, but  $r_0 < R < 1$  for some suitable choice of  $R$ , and since uniform continuity implies continuity,  $M$  would be continuous at  $r_0$ . Thus we conclude  $M \in \mathcal{C}([0, 1))$ .

Next we show monotonicity. Let  $0 \leq r_1 < r_2 < 1$ . We have  $\overline{B_{r_2}}(0) \subseteq \mathbb{E}$ . Thus  $f$  is holomorphic in the bounded domain  $B_{r_2}(0)$  and continuous on  $\overline{B_{r_2}}(0)$ . Then the maximum principle implies that

$$|f(z)| \leq \max_{\zeta \in \partial B_{r_2}(0)} |f(\zeta)| = M(r_2) \quad (12)$$

for all  $z \in \overline{B_{r_2}}(0)$ . In particular

$$M(r_1) = \max_{\zeta \in \partial B_{r_1}(0)} |f(\zeta)| \leq \max_{\zeta \in \partial B_{r_2}(0)} |f(\zeta)| = M(r_2). \quad (13)$$

Thus  $M$  is monotonically increasing.

(ii) Proof by contradiction. Assume that  $f$  is not constant and that  $M$  is not strictly increasing. Hence we find  $0 \leq r_1 < r_2 < 1$  such that  $M(r_1) = M(r_2)$  since by part (i) we already know that  $M$  is monotone increasing. Thus we find  $z_0 \in B_{r_1}(0)$  such that  $M(r_1) = |f(z_0)|$ . An application of the maximum principle similar to part (i) yields

$$|f(z)| \leq \max_{\zeta \in \partial B_{r_2}(0)} |f(\zeta)| = M(r_2) = M(r_1) = |f(z_0)| \quad (14)$$

for all  $z \in \overline{B_{r_2}}(0)$ . But  $z_0 \in B_{r_1}(0)$  and  $r_1 < r_2$ , thus  $B_{r_2-r_1}(z_0) \neq \{z_0\}$ . Hence  $|f|$  has a local maximum in  $B_{r_2}(0)$  and thus by the maximum principle,  $f$  is constant in  $B_{r_2}(0)$ . Since  $0 < r_2$ ,  $B_{r_2}(0)$  is not discrete in  $\mathbb{E}$ , hence if we define  $g : \mathbb{E} \rightarrow \mathbb{C}$  by  $g(z) := f(z_0)$ , clearly  $g \in \mathcal{O}(\mathbb{E})$  and  $f = g$  on  $B_{r_2}(0)$ . Hence by the second version of the identity principle we have  $f = g$  on  $\mathbb{E}$  which implies that  $f$  is constant on  $\mathbb{E}$ . Contradiction.

**Exercise 3.** Proof by contradiction. Since no point  $\zeta \in \partial B_R(z_0)$  is singular, we find a neighbourhood  $U_\zeta$  of  $\zeta$  and a function  $f_\zeta \in \mathcal{O}(U_\zeta)$ , such that

$$f_\zeta(z) = \sum_{\nu=0}^{\infty} a_\nu(z - z_0)^\nu \quad (15)$$

for all  $z \in U_\zeta \cap B_R(z_0)$ . Since each  $U_\zeta$  contains an open set containing  $\zeta$ , we find  $r_\zeta$ , such that  $B_{r_\zeta}(\zeta) \subseteq U_\zeta$ . Therefore we have that

$$\partial B_R(z_0) \subseteq \bigcup_{\zeta \in \partial B_R(z_0)} B_{r_\zeta}(\zeta) \quad (16)$$

is an open cover of  $\partial B_R(z_0)$ . Since  $\partial B_R(z_0)$  is compact, we find  $\zeta_1, \dots, \zeta_n$  such that  $B_{r_{\zeta_1}}(\zeta_1), \dots, B_{r_{\zeta_n}}(\zeta_n)$  still covers  $\partial B_R(z_0)$ . The next step is conceptually easy, but notationally ugly. We will explain it in a quite informal way. Now the intersection  $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu)$  is open and thus if  $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu) \neq \emptyset$ , we find an open ball contained in the intersection  $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu) \neq \emptyset$ . Taking the minimum of all radii of those balls lying in the intersection (this is possible since there are only finitely many ones), we find  $\hat{R} > R$  such that

$$\partial B_{\hat{R}}(z_0) \subseteq \bigcup_{k=1}^n B_{r_{\zeta_k}}(\zeta_k). \quad (17)$$

Next we construct a function  $g : B_{\hat{R}}(z_0) \rightarrow \mathbb{C}$ . Define  $g(z) := \sum_{\nu=0}^{\infty} a_\nu(z - z_0)^\nu$  if  $z \in B_R(z_0)$ . If  $z \in (B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu)) \setminus B_R(z_0)$ , we have that  $f_{\zeta_\nu} = f_{\zeta_\mu}$  in  $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu) \cap B_R(z_0)$ , which is open and therefore not discrete in  $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu)$ . Thus by the second version of the identity principle we have  $f_{\zeta_\nu} = f_{\zeta_\mu}$  on  $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu)$ . Therefore  $g(z) := f_{\zeta_\nu}(z) = f_{\zeta_\mu}(z)$  is well defined. In the remaining cases, define  $g(z) := f_{\zeta_\nu}(z)$  if  $z \in B_{r_{\zeta_\nu}}(\zeta_\nu)$ . Since  $f_{\zeta_\nu} \in \mathcal{O}(U_{\zeta_\nu})$  and by the theorem on interchangeability of differentiation and summation we have that any power series is holomorphic within its radius of convergence, we have that  $g \in \mathcal{O}(B_{\hat{R}}(z_0))$ . An application of Cauchy-Taylor yields

$$g(z) = \sum_{\nu=0}^{\infty} \frac{g^{(\nu)}(z_0)}{\nu!} (z - z_0)^\nu = \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(z_0)}{\nu!} (z - z_0)^\nu \quad (18)$$

for all  $z \in B_{\hat{R}}(z_0)$  since  $g = f$  on  $B_R(z_0)$ . Furthermore, since  $\hat{R} > R$  we have that  $\sum_{\nu=0}^{\infty} a_\nu(z - z_0)^\nu$  is convergent in  $B_{\hat{R}}(z_0) \setminus \overline{B_R}(z_0)$ , contradicting that  $\sum_{\nu=0}^{\infty} a_\nu(z - z_0)^\nu$  is divergent there by the definition of the radius of convergence. Contradiction.

**Exercise 4.** Central is Weierstrass' differentiation theorem for compact convergent series. For each  $\nu \in \mathbb{N}_0$  let

$$f_\nu(z) := \sum_{\mu=0}^{\infty} c_{\nu\mu}(z - z_0)^\mu \quad (19)$$

be convergent in  $B_r(z_0)$ ,  $r > 0$ ,  $z_0 \in \mathbb{C}$ . Furthermore, assume that

$$f(z) := \sum_{\nu=0}^{\infty} f_\nu(z) = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} c_{\nu\mu}(z - z_0)^\mu \quad (20)$$

is normally convergent in  $B_r(z_0)$ . Since  $r > 0$ , the theorem on interchangeability of differentiation and summation of power series implies that  $f_\nu \in \mathcal{O}(B_r(z_0))$  for all  $\nu \in \mathbb{N}_0$ . Since  $\sum_{\nu=0}^{\infty} f_\nu$  is normally convergent in  $B_r(z_0)$ , we have that  $\sum_{\nu=0}^{\infty} f_\nu$  is locally uniformly convergent in  $B_r(z_0)$  and thus compactly convergent in  $B_r(z_0)$ . Hence Weierstrass' theorem implies that the limit function  $f$  is holomorphic in  $B_r(z_0)$ . Thus by the expansion theorem of Cauchy-Taylor, for any  $z \in B_r(z_0)$  we find a disc centered at  $z$  where  $f$  is expandable in a Taylor series. This implies that  $f$  is analytic in  $B_r(z_0)$ . Furthermore, the same theorem implies that for any  $k \in \mathbb{N}_0$  we have

$$f^{(k)}(z) = \sum_{\nu=0}^{\infty} f_\nu^{(k)}(z) = \sum_{\nu=0}^{\infty} \sum_{\mu=k}^{\infty} k! \binom{\mu}{k} c_{\nu\mu}(z - z_0)^{\mu-k} \quad (21)$$

for all  $z \in B_r(z_0)$  by the theorem on interchangeability of differentiation and summation of power series. Since  $f \in \mathcal{O}(B_r(z_0))$ , the expansion theorem of Cauchy-Taylor implies that

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k = \sum_{k=0}^{\infty} \left( \sum_{\nu=0}^{\infty} c_{\nu k} \right) (z - z_0)^k \quad (22)$$

for all  $z \in B_r(z_0)$ .

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