SOLUTIONS SHEET 8

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Exercise 1. Let $U := \mathbb{C} \setminus \{\pm e^{\pm i\pi/4}\}$ and define $F : U \to \mathbb{C}$ by

$$F(z) := \frac{1}{1+z^4}. \tag{1}$$

Clearly $F \in \mathcal{O}(U)$ as a well defined rational function, U is open in $\mathbb{C}$ and $\mathbb{R} \subseteq U$. Furthermore $F|_{\mathbb{R}} = f$. Hence F is a holomorphic continuation of f. Since having an analytic continuation is equivalent to be real-analytic (see [FL03, p. 100]), we have that f is real-analytic.

Let $x_0 \in \mathbb{R}$. The Taylor series expansion of f is completely determined by the one of F. This is due to the fact, that $F^{(\nu)}(x_0) = f^{(\nu)}(x_0)$ for all $\nu \in \mathbb{N}_0$. By Cauchy-Taylor, F is expandable into a power series in the largest ball around x_0 contained in U, i.e.

$$F(z) = \sum_{\nu=0}^{\infty} a_\nu (z - x_0)^\nu \tag{2}$$

and the convergence is normal there. Thus the radius of convergence R of the expansion (2) is at least $|x_0 - e^{i\pi/4}|$ if $x_0 \geq 0$ and $|x_0 + e^{i\pi/4}|$ if $x_0 \leq 0$. Let $r := |x_0 - e^{i\pi/4}|$ and assume $x_0 \geq 0$ (the case $x_0 \leq 0$ is similar). Furthermore assume that $R > r$. Hence the series expansion (2) converges in $B_R(x_0)$ and therefore defines a function $G : B_R(x_0) \to \mathbb{C}$ by

$$G(z) := \sum_{\nu=0}^{\infty} a_\nu (z - x_0)^\nu \tag{3}$$

with $G|_{B_r}(x_0) = F$. Since G is expandable in a power series, we have $G \in \mathcal{O}(B_R(x_0))$ by [RS02, p. 187]. Since any holomorphic function is continuous, we have $G \in \mathscr{C}(B_R(x_0))$. Let $(z_\nu)_{\nu\in\mathbb{N}}$ be a sequence in $B_r(x_0)$ such that $\lim_{\nu\to\infty} z_\nu = e^{i\pi/4}$. Clearly

$$\lim_{\nu\to\infty} F(z_\nu) = \infty \tag{4}$$

and since $G|_{B_r(x_0)} = F$ we have

$$\lim_{\nu\to\infty} G(z_\nu) = \infty. \tag{5}$$

But since $R > r$, G is continuous at $e^{i\pi/4}$ and so we must have

$$G(e^{i\pi/4}) = \lim_{\nu\to\infty} G(z_\nu) = \infty. \tag{6}$$

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Thus the series G diverges at $e^{i\pi/4}$, contradicting that $e^{i\pi/4} \in B_R(x_0)$. In conclusion

$$\boxed{R(x_0) = \min\left(\left|x_0 - e^{i\pi/4}\right|, \left|x_0 + e^{i\pi/4}\right|\right) \qquad x_0 \in \mathbb{R}.} \tag{7}$$

Exercise 2. Since $f \in \mathcal{O}(\mathbb{E})$ we have that $f \in \mathscr{C}(\mathbb{E})$. Thus since $\partial B_r(0)$, $0 \leq r < 1$, is compact we have that $|f|$ attains its supremum on $\partial B_r(0)$. Hence we have

$$M(r) = \max_{|z|=r} |f(z)| \tag{8}$$

and actually $M : [0,1) \to \mathbb{R}_{\geq 0}$.
(i) Let $0 < R < 1$. Since $\overline{B_R}(0)$ is compact and $f \in \mathscr{C}(\mathbb{E})$, we have that f is uniformly continuous on $\overline{B_R}(0)$ (see for example [Alt16, p. 138] or [Lee11, p. 215]). By the reversed triangle inequality, also $|f|$ is uniformly continuous on $\overline{B_R}(0)$.

Proposition 0.1. *M is uniformly continuous on $[0,R]$, $0 < R < 1$.*

Proof. Fix $0 < R < 1$ and let $\varepsilon > 0$. Since $|f|$ is uniformly continuous on $\overline{B_R}(0)$, we find $\delta > 0$ such that for all $z_1, z_2 \in \overline{B_R}(0)$, $|z_1 - z_2| < \delta$ implies $||f(z_1)| - |f(z_2)|| < \varepsilon$. Let $r_1, r_2 \in [0,R]$. There are two cases to distinguish. First assume $M(r_1) \geq M(r_2)$. We find $\varphi_1 \in \mathbb{R}$ such that $M(r_1) = |f(r_1 e^{i\varphi_1})|$. If $|r_1 - r_2| = |r_1 e^{i\varphi_1} - r_2 e^{i\varphi_1}| < \delta$, we have that

$$M(r_1) - M(r_2) \leq M(r_1) - |f(r_2 e^{i\varphi_1})| = |f(r_1 e^{i\varphi_1})| - |f(r_2 e^{i\varphi_1})| < \varepsilon \tag{9}$$

since $r_1 e^{i\varphi_1}, r_2 e^{i\varphi_1} \in \overline{B_R}(0)$. Now assume $M(r_1) \leq M(r_2)$. Again, we find $\varphi_2 \in \mathbb{R}$ such that $M(r_2) = |f(r_2 e^{i\varphi_2})|$ and analogously if $|r_1 - r_2| = |r_1 e^{i\varphi_2} - r_2 e^{i\varphi_2}| < \delta$, we have that

$$M(r_2) - M(r_1) \leq M(r_2) - |f(r_1 e^{i\varphi_2})| = |f(r_2 e^{i\varphi_2})| - |f(r_1 e^{i\varphi_2})| < \varepsilon$$

since $r_1 e^{i\varphi_2}, r_2 e^{i\varphi_2} \in \overline{B_R}(0)$. In conclusion $|M(r_1) - M(r_2)| < \varepsilon$ whenever $|r_1 - r_2| < \delta$. Hence M is uniformly continuous on $[0,R]$. □

Corollary 0.1. *$M \in \mathscr{C}([0,1))$.*

Proof. By proposition 0.1, $M \in \mathscr{C}([0,R])$ for any $0 < R < 1$ since uniform continuity implies continuity. Let $r_0 \in [0,1)$. We find $0 \leq r_0 < R < 1$ and since continuity is a local property, we have that M is continuous at r_0. Since the choice of r_0 was arbitray, we conclude that $M \in \mathscr{C}([0,1))$. □

Lemma 0.1. *M is nondecreasing.*

Proof. Let $0 \leq r_1 < r_2 < 1$. We have $\overline{B_{r_2}}(0) \subseteq \mathbb{E}$. Thus f is holomorphic in the bounded domain $B_{r_2}(0)$ and continuous on $\overline{B_{r_2}}(0)$. The maximum modulus principle [FL03, p. 91] implies

$$|f(z)| \leq \max_{\zeta \in \partial B_{r_2}(0)} |f(\zeta)| = M(r_2) \tag{10}$$

for all $z \in \overline{B_{r_2}}(0)$. In particular

$$M(r_1) = \max_{\zeta \in \partial B_{r_1}(0)} |f(\zeta)| \leq \max_{\zeta \in \partial B_{r_2}(0)} |f(\zeta)| = M(r_2). \tag{11}$$

Thus M is nondecreasing. □

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(ii) Proof by contradiction. Assume that f is not constant and that M is not increasing. Hence we find $0 \leq r_1 < r_2 < 1$ such that $M(r_1) = M(r_2)$, since by part (i) we already know that M is nondecreasing. We find $z_0 \in \partial B_{r_1}(0)$ such that $M(r_1) = |f(z_0)|$. An application of the maximum principle similar to part (i) yields

$$|f(z)| \leq \max_{\zeta \in \partial B_{r_2}(0)} |f(\zeta)| = M(r_2) = M(r_1) = |f(z_0)| \tag{12}$$

for all $z \in \overline{B_{r_2}}(0)$. Define $r := \min(r_1, r_2 - r_1)$. Since $B_r(z_0) \neq \{z_0\}$, $B_r(z_0) \subseteq B_{r_2}(0)$ and by (12) we have that $|f(z)| \leq |f(z_0)|$ for all $z \in B_r(0)$, $|f|$ has a local maximum at $z_0 \in B_{r_2}(0)$. Thus by the maximum modulus principle, f is constant in $B_{r_2}(0)$. Since $0 < r_2$, $B_{r_2}(0)$ is not discrete in $\mathbb{E}$, hence if we define $g : \mathbb{E} \to \mathbb{C}$ by $g(z) := f(z_0)$, clearly $g \in \mathcal{O}(\mathbb{E})$ and $f = g$ on $B_{r_2}(0)$. Hence by the second version of the identity principle [FL03, p. 85] we have $f = g$ on $\mathbb{E}$ which implies that f is constant on $\mathbb{E}$. Contradiction.

Exercise 3. Proof by contradiction. Assume that $\sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu$ has radius of convergence $0 < R < \infty$ and that $\partial B_R(z_0)$ contains no singular points. Thus for any $\zeta \in \partial B_R(z_0)$, we find an open neighbourhood U_ζ of ζ and a function $f_\zeta \in \mathcal{O}(U_\zeta)$, such that

$$f_\zeta(z) = \sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu \tag{13}$$

for all $z \in U_\zeta \cap B_R(z_0)$. Since each U_ζ is an open neighbourhood of ζ, we find $r_\zeta > 0$, such that $B_{r_\zeta}(\zeta) \subseteq U_\zeta$. Since clearly

$$\partial B_R(z_0) \subseteq \bigcup_{\zeta \in \partial B_R(z_0)} B_{r_\zeta}(\zeta) \tag{14}$$

$(B_{r_\zeta}(\zeta))_{\zeta \in \partial B_R(z_0)}$ is an open cover of $\partial B_R(z_0)$. Since $\partial B_R(z_0)$ is compact, we find $\zeta_1, \ldots, \zeta_n$ such that $B_{r_{\zeta_1}}(\zeta_1), \ldots, B_{r_{\zeta_n}}(\zeta_n)$ still covers $\partial B_R(z_0)$. The next step is conceptually easy, but notationally ugly. We will explain it in a quite informal way. Now the intersection $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu)$ is open and thus if $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu) \neq \varnothing$ and $\nu \neq \mu$, we find an open ball centered at some point $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu) \cap \partial B_R(z_0)$ contained in the intersection $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu)$. Taking the minimum of all radii of those balls lying in the intersection (this is possible since there are only finitely many ones), say $\widehat{R} > R$, we have that

$$B_{\widehat{R}}(z_0) \setminus B_R(z_0) \subseteq \bigcup_{k=1}^{n} B_{r_{\zeta_k}}(\zeta_k). \tag{15}$$

We construct a function $g : B_{\widehat{R}}(z_0) \to \mathbb{C}$. Define $g(z) := \sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu$ if $z \in B_R(z_0)$. If $z \in B_{\widehat{R}}(z_0) \setminus B_R(z_0)$, we find $\nu \in \{1, \ldots, n\}$, such that $z \in B_{r_{\zeta_\nu}}(\zeta_\nu)$ by (15). Define $g(z) := f_{\zeta_\nu}(z)$. This definition is well defined. Indeed, if $z \in B_{r_{\zeta_\mu}}(\zeta_\mu)$ for some $\mu \neq \nu$, we have that $f_{\zeta_\nu} = f_{\zeta_\mu}$ in $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu) \cap B_R(z_0)$, which is open and therefore not discrete in $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu)$. Thus by the second version of the identity principle we have $f_{\zeta_\nu} = f_{\zeta_\mu}$ on $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu)$ and thus $f_{\zeta_\nu}(z) = f_{\zeta_\mu}(z)$. Since $f_{\zeta_\nu} \in \mathcal{O}(U_{\zeta_\nu})$ and by the theorem on interchangeability of differentiation and summation we have that any

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power series is holomorphic within its radius of convergence, we have that $g \in \mathcal{O}(B_{\widehat{R}}(z_0))$. An application of Cauchy-Taylor yields

$$g(z) = \sum_{\nu=0}^{\infty} \frac{g^{(\nu)}(z_0)}{\nu!}(z-z_0)^{\nu} = \sum_{\nu=0}^{\infty} a_{\nu}(z-z_0)^{\nu} \tag{16}$$

for all $z \in B_{\widehat{R}}(z_0)$ since $g(z) = \sum_{\nu=0}^{\infty} a_{\nu}(z-z_0)^{\nu}$ in $B_R(z_0)$ and thus by the theorem on interchangeability of differentiation and summation of power series we have that $g^{(\nu)}(z_0)/\nu! = a_{\nu}$ for all $\nu \in \mathbb{N}_0$. By $\widehat{R} > R$ we have that $\sum_{\nu=0}^{\infty} a_{\nu}(z-z_0)^{\nu}$ is convergent in $B_{\widehat{R}}(z_0) \setminus \overline{B_R}(z_0)$, contradicting that $\sum_{\nu=0}^{\infty} a_{\nu}(z-z_0)^{\nu}$ is divergent there by the definition of the radius of convergence.

Exercise 4. Central is Weierstrass' differentiation theorem for compact convergent series. For each $\nu \in \mathbb{N}_0$ let

$$f_{\nu}(z) := \sum_{\mu=0}^{\infty} c_{\nu\mu}(z-z_0)^{\mu} \tag{17}$$

be convergent in $B_r(z_0)$, $r > 0$, $z_0 \in \mathbb{C}$. Furthermore, assume that

$$f(z) := \sum_{\nu=0}^{\infty} f_{\nu}(z) = \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} c_{\nu\mu}(z-z_0)^{\mu} \tag{18}$$

is normally convergent in $B_r(z_0)$. Since $r > 0$, the theorem on interchangeability of differentiation and summation of power series implies that $f_{\nu} \in \mathcal{O}(B_r(z_0))$ for all $\nu \in \mathbb{N}_0$. Since $\sum_{\nu=0}^{\infty} f_{\nu}$ is normally convergent in $B_r(z_0)$, we have that $\sum_{\nu=0}^{\infty} f_{\nu}$ is locally uniformly convergent in $B_r(z_0)$ (see [RS02, p. 92]) and thus compactly convergent in $B_r(z_0)$ (see [RS02, p. 85]). Hence Weierstrass' theorem implies that the limit function f is holomorphic in $B_r(z_0)$. Thus by the expansion theorem of Cauchy-Taylor, for any $z \in B_r(z_0)$ we find a disc centered at z where f is expandable in a Taylor series. This implies that f is analytic in $B_r(z_0)$ (analytic in the sense of Weierstrass, see [RS02, p. 210]). Furthermore, Weierstrass' theorem also implies that for any $k \in \mathbb{N}_0$ we have

$$f^{(k)}(z) = \sum_{\nu=0}^{\infty} f_{\nu}^{(k)}(z) = \sum_{\nu=0}^{\infty}\sum_{\mu=k}^{\infty} k!\binom{\mu}{k} c_{\nu\mu}(z-z_0)^{\mu-k} \tag{19}$$

for all $z \in B_r(z_0)$ by the theorem on interchangeability of differentiation and summation of power series. Since $f \in \mathcal{O}(B_r(z_0))$, the expansion theorem of Cauchy-Taylor implies that

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k = \sum_{k=0}^{\infty}\left(\sum_{\nu=0}^{\infty} c_{\nu k}\right)(z-z_0)^k \tag{20}$$

for all $z \in B_r(z_0)$.

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