Yannis Bähni
yannis.baehni@uzh.ch

# SOLUTIONS SHEET 8

YANNIS BÄHNI

**Exercise 1.** We follow [IL03, pp. 99–101]. Let $U := \mathbb{C} \setminus \{\pm e^{\pm i\pi/4}\}$ and define $F : U \to \mathbb{C}$ by

$$F(z) := \frac{1}{1+z^4}. \tag{1}$$

Clearly $F \in \mathcal{O}(U)$ as a well-defined rational function, $U$ is open in $\mathbb{C}$ and $\mathbb{R} \subseteq U$. Furthermore $F|_{\mathbb{R}} = f$. Hence $F$ is a holomorphic continuation of $f$. Since having an analytic continuation is equivalent to be real-analytic (see [IL03, p. 100]), we have that $f$ is real-analytic.
Let $x_0 \in \mathbb{R}$. The Taylor series expansion of $f$ is completely determined by the one of $F$. So the only thing which restricts the radius of convergence of the Taylor series expansions are the singularities of $F$. I will again formalize why this is the case. Let

$$F(z) = \sum_{\nu=0}^{\infty} a_\nu (z - x_0)^\nu \tag{2}$$

be the Taylor expansion of $F$ around $x_0$. By Cauchy-Taylor the radius of convergence of the expansion (2) is at least $|x_0 - e^{i\pi/2}|$ if $x_0 \geq 0$ and $|x_0 + e^{i\pi/4}|$ if $x_0 \leq 0$. Let $r := |x_0 - e^{i\pi/4}|$ and assume $x_0 \geq 0$. (the case $x_0 \leq 0$ is similar) and $R > r$. Hence the series in (2) converges in $B_R(x_0)$. Hence it defines a function $G : B_R(x_0) \to \mathbb{C}$ by

$$G(z) := \sum_{\nu=0}^{\infty} a_\nu (z - x_0)^\nu \tag{3}$$

with $G|_{B_r}(x_0) = F$. Since $G$ is expandable in a power series, we have $G \in \mathcal{O}(B_R(x_0))$ by [RS02, p. 187]. Since any holomorphic functionis continuous, we have $G \in \mathscr{C}(B_R(x_0))$. Let $(z_\nu)_{\nu\in\mathbb{N}}$ be a sequence in $B_r(x_0)$ such that $\lim_{\nu\to\infty} z_\nu = e^{i\pi/4}$. Clearly

$$\lim_{\nu\to\infty} F(z_\nu) = \infty \tag{4}$$

and since $G|_{B_r(x_0)} = F$ we have

$$\lim_{\nu\to\infty} G(z_\nu) = \infty. \tag{5}$$

But since $R > r$, $G$ is continuous at $e^{i\pi/4}$ and so we must have

$$G(e^{i\pi/4}) = \lim_{\nu\to\infty} G(z_\nu) = \infty. \tag{6}$$

(Yannis Bähni) UNIVERSITY OF ZURICH, RÄMISTRASSE 71, 8006 ZURICH
*E-mail address*: yannis.baehni@uzh.ch.

Yannis Bähni
yannis.baehni@uzh.ch

Thus the series $G$ diverges at $e^{i\pi/4}$, contradicting that $e^{i\pi/4} \in B_R(x_0)$. Now for general $x_0 \in \mathbb{R}$, the radius of convergence $R$ of the Taylor series expansion of $f$ in $x_0$ is the radius of convergence of the restriction of the Taylor series expansion of $F$ in $x_0$ on $\mathbb{R}$, hence

$$R = \begin{cases} |x_0 - e^{i\pi/4}| & x_0 \geq 0, \\ |x_0 + e^{i\pi/4}| & x_0 \leq 0. \end{cases}$$

**Exercise 2.**
(i) Since $f \in \mathcal{O}(\mathbb{E})$ we have that $f \in \mathscr{C}(\mathbb{E})$. Thus since $\partial B_r(0)$, $0 \leq r < 1$, is compact we have that $|f|$ attains its supremum on $\partial B_r(0)$. Hence we have

$$M(r) = \max_{|z|=r} |f(z)|. \tag{7}$$

First we show monotonicity. Let $0 \leq r_1 < r_2 < 1$. We have $\overline{B_{r_2}}(0) \subseteq \mathbb{E}$. Thus $f$ is holomorphic in the bounded domain $B_{r_2}(0)$ and continuous on $\overline{B_{r_2}}(0)$. Then the maximum principle implies that

$$|f(z)| \leq \max_{\zeta \in \partial B_{r_2}(0)} |f(\zeta)| = M(r_2) \tag{8}$$

for all $z \in \overline{B_{r_2}}(0)$. In particular

$$M(r_1) = \max_{\zeta \in \partial B_{r_1}(0)} |f(\zeta)| \leq \max_{\zeta \in \partial B_{r_2}(0)} |f(\zeta)| = M(r_2). \tag{9}$$

Thus $M$ is monotonically increasing.

(ii) Proof by contradiction. Assume that $f$ is not constant and that $M$ is not strictly increasing. Hence we find $0 \leq r_1 < r_2 < 1$ such that $M(r_1) = M(r_2)$ since by part (i) we already know that $M$ is monotone increasing. Thus we find $z_0 \in B_{r_1}(0)$ such that $M(r_1) = |f(z_0)|$. An application of the maximum principle similar to part (i) yields

$$|f(z)| \leq \max_{\zeta \in \partial B_{r_2}(0)} |f(\zeta)| = M(r_2) = M(r_1) = |f(z_0)| \tag{10}$$

for all $z \in \overline{B_{r_2}}(0)$. But $z_0 \in B_{r_1}(0)$ and $r_1 < r_2$, thus $B_{r_2 - r_1}(z_0) \neq \{z_0\}$. Hence $|f|$ has a local maximum in $B_{r_2}(0)$ and thus by the maximum principle, $f$ is constant in $B_{r_2}(0)$. Since $0 < r_2$, $B_{r_2}(0)$ is not discrete in $\mathbb{E}$, hence if we define $g : \mathbb{E} \to \mathbb{C}$ by $g(z) := f(z_0)$, clearly $g \in \mathcal{O}(\mathbb{R})$ and $f = g$ on $B_{r_2}(0)$. Hence by the second version of the identity principle we have $f = g$ on $\mathbb{E}$ which implies that $f$ is constant on $\mathbb{E}$. Contradiction.

**Exercise 3.** Proof by contradiction. Since no point $\zeta \in \partial B_R(z_0)$ is singular, we find a neighbourhood $U_\zeta$ of $\zeta$ and a function $f_\zeta \in \mathcal{O}(U_\zeta)$, such that

$$f_\zeta(z) = \sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu \tag{11}$$

for all $z \in U_\zeta \cap B_R(z_0)$. Since each $U_\zeta$ contains an open set containing $\zeta$, we find $r_\zeta$, such that $B_{r_\zeta}(\zeta) \subseteq U_\zeta$. Therefore we have that

$$\partial B_R(z_0) \subseteq \bigcup_{\zeta \in \partial B_R(z_0)} B_{r_\zeta}(\zeta) \tag{12}$$

Yannis Bähni
yannis.baehni@uzh.ch

is an open cover of $\partial B_R(z_0)$. Since $\partial B_R(z_0)$ is compact, we find $\zeta_1, \dots, \zeta_n$ such that $B_{r_{\zeta_1}}(\zeta_1), \dots, B_{r_{\zeta_n}}(\zeta_n)$ still covers $\partial B_R(z_0)$. The next step is conceptually easy, but notationally ugly. We will explain it in a quite informal way. Now the intersection $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu)$ is open and thus if $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu) \neq \varnothing$, we find an open ball contained in the intersection $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu) \neq \varnothing$. Taking the minimum of all radii of those balls lying in the intersection (this is possible since there are only finitely many ones), we find $\hat{R} > R$ such that

$$\partial B_{\hat{R}}(z_0) \subseteq \bigcup_{k=1}^{n} B_{r_{\zeta_k}}(\zeta_k). \tag{13}$$

Next we construct a function $g : B_{\hat{R}}(z_0) \to \mathbb{C}$. Define $g(z) := \sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu$ if $z \in B_R(z_0)$. If $z \in (B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu)) \setminus B_R(z_0)$, we have that $f_{\zeta_\nu} = f_{\zeta_\mu}$ in $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu) \cap B_R(z_0)$, which is open and therefore not discrete in $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu)$. Thus by the second version of the identity principle we have $f_{\zeta_\nu} = f_{\zeta_\mu}$ on $B_{r_{\zeta_\nu}}(\zeta_\nu) \cap B_{r_{\zeta_\mu}}(\zeta_\mu)$. Therefore $g(z) := f_{\zeta_\nu}(z) = f_{\zeta_\mu}(z)$ is well defined. In the remaining cases, define $g(z) := f_{\zeta_\nu}(z)$ if $z \in B_{r_{\zeta_\nu}}(\zeta_\nu)$. Since $f_{\zeta_\nu} \in \mathcal{O}(\zeta_\nu)$ and by the theorem on interchangeability of differentiation and summation we have that any power series is holomorphic within its radius of convergence, we have that $g \in \mathcal{O}(B_{\hat{R}}(z_0))$. An application of Cauchy-Taylor yields

$$g(z) = \sum_{\nu=0}^{\infty} \frac{g^{(\nu)}(z_0)}{\nu!}(z - z_0)^\nu = \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(z_0)}{\nu!}(z - z_0)^\nu \tag{14}$$

for all $z \in B_{\hat{R}}(z_0)$ since $g = f$ on $B_R(z_0)$. Furthermore, since $\hat{R} > R$ we have that $\sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu$ is convergent in $B_{\hat{R}}(z_0) \setminus \overline{B_R}(z_0)$, contradicting that $\sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu$ is divergent there by the definition of the radius of convergence. Contradiction.

**Exercise 4.** Central is Weierstrass' differentiation theorem for compact convergent series. For each $\nu \in \mathbb{N}_0$ let

$$f_\nu(z) := \sum_{\mu=0}^{\infty} c_{\nu\mu}(z - z_0)^\mu \tag{15}$$

be convergent in $B_r(z_0)$, $r > 0$, $z_0 \in \mathbb{C}$. Furthermore, assume that

$$f(z) := \sum_{\nu=0}^{\infty} f_\nu(z) = \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} c_{\nu\mu}(z - z_0)^\mu \tag{16}$$

is normally convergent in $B_r(z_0)$. Since $r > 0$, the theorem on interchangeability of differentiation and summation of power series implies that $f_\nu \in \mathcal{O}(B_r(z_0))$ for all $\nu \in \mathbb{N}_0$. Since $\sum_{\nu=0}^{\infty} f_\nu$ is normally convergent in $B_r(z_0)$, we have that $\sum_{\nu=0}^{\infty} f_\nu$ is locally uniformly convergent in $B_r(z_0)$ and thus compactly convergent in $B_r(z_0)$. Hence Weierstrass' theorem implies that the limit function $f$ is holomorphic in $B_r(z_0)$. Thus by the expansion theorem of Cauchy-Taylor, for any $z \in B_r(z_0)$ we find a disc centered at $z$ where $f$ is expandable in a Taylor seriers. This implies that $f$ is analytic in $B_r(z_0)$. Furthermore, the same theorem

Yannis Bähni
yannis.baehni@uzh.ch

implies that for any $k \in \mathbb{N}_0$ we have

$$f^{(k)}(z) = \sum_{\nu=0}^{\infty} f_\nu^{(k)}(z) = \sum_{\nu=0}^{\infty} \sum_{\mu=k}^{\infty} k! \binom{\mu}{k} c_{\nu\mu}(z - z_0)^{\mu-k} \tag{17}$$

for all $z \in B_r(z_0)$ by the theorem on interchangeability of differentiation and summation of power series. Since $f \in \mathcal{O}(B_r(z_0))$, the expansion theorem of Cauchy-Taylor implies that

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k = \sum_{k=0}^{\infty} \left( \sum_{\nu=0}^{\infty} c_{\nu k} \right) (z - z_0)^k \tag{18}$$

for all $z \in B_r(z_0)$.

## References

[IL03] Wolfgang Fischer and Ingo Lieb. *Funktionentheorie: Komplexe Analysis in einer Veränderlichen*. 8. Auflage. vieweg studium; Aufbaukurs Mathematik. Vieweg+Teubner Verlag, 2003.

[RS02] R. Remmert and G. Schumacher. *Funktionentheorie 1*. Springer-Lehrbuch. Springer Berlin Heidelberg, 2002. ISBN: 9783540418559.