SOLUTIONS SHEET 3

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Exercise 1. Let $D \subseteq \mathbb{C}$ be non-empty and open in $\mathbb{C}$ and $f_1, f_2 : D \to \mathbb{C}$ be real differentiable. Fix some $z_0 \in D$. Since f_1 and f_2 are real differentiable in z_0 there exists $\varphi_1, \varphi_2, \psi_1, \psi_2 : D \to \mathbb{C}$ continuous at z_0 such that

$$f_1(z) = f_1(z_0) + (z - z_0)\varphi_1(z) + (\overline{z} - \overline{z_0})\psi_1(z) \tag{1}$$

$$f_2(z) = f_2(z_0) + (z - z_0)\varphi_2(z) + (\overline{z} - \overline{z_0})\psi_2(z) \tag{2}$$

for all $z \in D$.

(i) Let $a, b \in \mathbb{C}$. Multiplying (1) by a, (2) by b and adding both equations yields

$$af_1(z)+bf_2(z) = af_1(z_0)+bf_2(z_0)+(z-z_0)(a\varphi_1(z)+b\varphi_2(z))+(\overline{z}-\overline{z_0})(a\psi_1(z)+b\psi_2(z)) \tag{3}$$

for all $z \in D$. Clearly, $a\varphi_1 + b\varphi_2$ and $a\psi_1 + b\psi_2$ are continuous functions in z_0 and from (3) we deduce

$$\frac{\partial(af_1 + bf_2)}{\partial z}(z_0) = a\frac{\partial f_1}{\partial z}(z_0) + b\frac{\partial f_2}{\partial z}(z_0) \tag{4}$$

and

$$\frac{\partial(af_1 + bf_2)}{\partial \overline{z}}(z_0) = a\frac{\partial f_1}{\partial \overline{z}}(z_0) + b\frac{\partial f_2}{\partial \overline{z}}(z_0). \tag{5}$$

Since $z_0 \in D$ was arbitrary, we conclude

$$\frac{\partial(af_1 + bf_2)}{\partial z} = a\frac{\partial f_1}{\partial z} + b\frac{\partial f_2}{\partial z} \qquad \text{and} \qquad \frac{\partial(af_1 + bf_2)}{\partial \overline{z}} = a\frac{\partial f_1}{\partial \overline{z}} + b\frac{\partial f_2}{\partial \overline{z}}. \tag{6}$$

(ii) Multiplying (1) and (2) yields

$$\begin{aligned} f_1 f_2 = f_1(z_0)f_2(z_0) &+ (z - z_0)\left[\varphi_1 f_2(z_0) + f_1(z_0)\varphi_2 + (z - z_0)\varphi_1\varphi_2 + (\overline{z} - \overline{z_0})\psi_1\varphi_2\right] \\ &+ (\overline{z} - \overline{z_0})\left[\psi_1 f_2(z_0) + f_1(z_0)\psi_2 + (z - z_0)\psi_2\varphi_1 + (\overline{z} - \overline{z_0})\psi_1\psi_2\right] \end{aligned}$$

where the argument z is omitted. Clearly, the two functions in the square brackets are continuous at z_0 and evaluating them at z_0 yields

$$\frac{\partial(f_1 f_2)}{\partial z}(z_0) = \frac{\partial f_1}{\partial z}(z_0)f_2(z_0) + f_1(z_0)\frac{\partial f_2}{\partial z}(z_0) \tag{7}$$

and

$$\frac{\partial(f_1 f_2)}{\partial \overline{z}}(z_0) = \frac{\partial f_1}{\partial \overline{z}}(z_0)f_2(z_0) + f_1(z_0)\frac{\partial f_2}{\partial \overline{z}}(z_0). \tag{8}$$

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Since $z_0 \in D$ was arbitrary, we conclude

$$\frac{\partial(f_1 f_2)}{\partial z} = \frac{\partial f_1}{\partial z} f_2 + f_1 \frac{\partial f_2}{\partial z} \qquad \text{and} \qquad \frac{\partial(f_1 f_2)}{\partial \overline{z}} = \frac{\partial f_1}{\partial \overline{z}} f_2 + f_1 \frac{\partial f_2}{\partial \overline{z}}. \tag{9}$$

(iii) Conjugating (1) yields

$$\overline{f_1}(z) = \overline{f_1}(z_0) + (\overline{z} - \overline{z_0})\overline{\varphi_1}(z) + (z - z_0)\overline{\psi_1}(z). \tag{10}$$

From (10) we deduce

$$\frac{\partial \overline{f_1}}{\partial \overline{z}}(z_0) = \overline{\varphi_1}(z_0) = \overline{\frac{\partial f_1}{\partial z}}(z_0) \tag{11}$$

since φ_1 and ψ_1 are also continuous at z_0. Taking conjugates in (11) and use that $z_0 \in D$ was arbitrary finally yields

$$\overline{\frac{\partial \overline{f_1}}{\partial \overline{z}}} = \frac{\partial f_1}{\partial z}. \tag{12}$$

(iv) This follows directly from

$$z = z_0 + (z - z_0) \qquad \text{and} \qquad \overline{z} = \overline{z_0} + (\overline{z} - \overline{z_0}). \tag{13}$$

(v)

Exercise 2. We show the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). Since the proofs are of a relatively simple nature, we focus on the formal part. The complex numbers $\mathbb{C}$ are a vector space over $\mathbb{R}$ (as a field extension). So the situation of the exercise can be sumarized by the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "T"] & \mathbb{C} \\
\mathbb{R}^2 \arrow[u, "\Phi_B"] \arrow[r, "M_B(T)"'] & \mathbb{R}^2 \arrow[u, "\Phi_B"']
\end{tikzcd}$$

where T is $\mathbb{R}$-linear, Φ_B denotes the basis-isomorphism which is in this case given by $\Phi_B(x, y) := x + ix$ and $M_B(T)$ is defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{14}$$

The first implication is evident by the definition of $\mathbb{C}$-linearity. Assume that (ii) holds. By

$$T(i) = (\Phi_B \circ M_B(T) \circ \Phi_B^{-1})(i) = b + id \tag{15}$$

and

$$iT(1) = i(\Phi_B \circ M_B(T) \circ \Phi_B^{-1})(1) = i(a + ic) = -c + ia \tag{16}$$

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we get the requirement $b + id = -c + ia$. Hence $b = -c$ and $a = d$. Assume that (iii) holds. Then we have for $z := x + iy \in \mathbb{C}$

$$T(z) = (\Phi_B \circ M_B(T) \circ \Phi_B^{-1})(x + iy) = (ax - cy) + i(cx + ay) = (a + ic)z. \tag{17}$$

Finally, assume that (iv) holds. Then T is clearly $\mathbb{C}$-linear since

$$T(\lambda z + w) = (a + ic)(\lambda z + w) = \lambda(a + ic)z + (a + ic)w = \lambda T(z) + T(w) \tag{18}$$

for $\lambda, z, w \in \mathbb{C}$.