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SOLUTIONS SHEET 3

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Exercise 1. Let $D \subseteq \mathbb{C}$ be non-empty and open in $\mathbb{C}$ and $f_1, f_2 : D \to \mathbb{C}$ be real differentiable. Fix some $z_0 \in D$. Since f_1 and f_2 are real differentiable in z_0 there exists $\varphi_1, \varphi_2, \psi_1, \psi_2 : D \to \mathbb{C}$ continuous at z_0 such that

$$f_1(z) = f_1(z_0) + (z - z_0)\varphi_1(z) + (\overline{z} - \overline{z_0})\psi_1(z) \tag{1}$$
$$f_2(z) = f_2(z_0) + (z - z_0)\varphi_2(z) + (\overline{z} - \overline{z_0})\psi_2(z) \tag{2}$$

for all $z \in D$.
(i) Let $a, b \in \mathbb{C}$. Multiplying (1) by a, (2) by b and adding both equations yields

$$af_1(z)+bf_2(z) = af_1(z_0)+bf_2(z_0)+(z-z_0)(a\varphi_1(z)+b\varphi_2(z))+(\overline{z}-\overline{z_0})(a\psi_1(z)+b\psi_2(z)) \tag{3}$$

for all $z \in D$. Clearly, $a\varphi_1 + b\varphi_2$ and $a\psi_1 + b\psi_2$ are continuous functions in z_0 and from (3) we deduce

$$\frac{\partial(af_1 + bf_2)}{\partial z}(z_0) = a\frac{\partial f_1}{\partial z}(z_0) + b\frac{\partial f_2}{\partial z}(z_0) \tag{4}$$

and

$$\frac{\partial(af_1 + bf_2)}{\partial \overline{z}}(z_0) = a\frac{\partial f_1}{\partial \overline{z}}(z_0) + b\frac{\partial f_2}{\partial \overline{z}}(z_0). \tag{5}$$

Since $z_0 \in D$ was arbitrary, we conclude

$$\frac{\partial(af_1 + bf_2)}{\partial z} = a\frac{\partial f_1}{\partial z} + b\frac{\partial f_2}{\partial z} \qquad \text{and} \qquad \frac{\partial(af_1 + bf_2)}{\partial \overline{z}} = a\frac{\partial f_1}{\partial \overline{z}} + b\frac{\partial f_2}{\partial \overline{z}}. \tag{6}$$

(ii) Multiplying (1) and (2) yields

$$\begin{aligned} f_1f_2 = f_1(z_0)f_2(z_0) &+ (z - z_0)\left[\varphi_1 f_2(z_0) + f_1(z_0)\varphi_2 + (z - z_0)\varphi_1\varphi_2 + (\overline{z} - \overline{z_0})\psi_1\varphi_2\right] \\ &+ (\overline{z} - \overline{z_0})\left[\psi_1 f_2(z_0) + f_1(z_0)\psi_2 + (z - z_0)\psi_2\varphi_1 + (\overline{z} - \overline{z_0})\psi_1\psi_2\right] \end{aligned}$$

where the argument z is omitted. Clearly, the two functions in the square brackets are continuous at z_0 and evaluating them at z_0 yields

$$\frac{\partial(f_1f_2)}{\partial z}(z_0) = \frac{\partial f_1}{\partial z}(z_0)f_2(z_0) + f_1(z_0)\frac{\partial f_2}{\partial z}(z_0) \tag{7}$$

and

$$\frac{\partial(f_1f_2)}{\partial \overline{z}}(z_0) = \frac{\partial f_1}{\partial \overline{z}}(z_0)f_2(z_0) + f_1(z_0)\frac{\partial f_2}{\partial \overline{z}}(z_0). \tag{8}$$

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Since $z_0 \in D$ was arbitrary, we conclude

$$\frac{\partial(f_1 f_2)}{\partial z} = \frac{\partial f_1}{\partial z} f_2 + f_1 \frac{\partial f_2}{\partial z} \qquad \text{and} \qquad \frac{\partial(f_1 f_2)}{\partial \overline{z}} = \frac{\partial f_1}{\partial \overline{z}} f_2 + f_1 \frac{\partial f_2}{\partial \overline{z}}. \tag{9}$$

(iii) Conjugating (1) yields

$$\overline{f_1}(z) = \overline{f_1}(z_0) + (\overline{z} - \overline{z_0})\overline{\varphi_1}(z) + (z - z_0)\overline{\psi_1}(z). \tag{10}$$

From (10) we deduce

$$\frac{\partial \overline{f_1}}{\partial \overline{z}}(z_0) = \overline{\varphi_1}(z_0) = \overline{\frac{\partial f_1}{\partial z}}(z_0) \tag{11}$$

since φ_1 and ψ_1 are also continuous at z_0. Taking conjugates in (11) and use that $z_0 \in D$ was arbitrary finally yields

$$\overline{\frac{\partial \overline{f_1}}{\partial \overline{z}}} = \frac{\partial f_1}{\partial z}. \tag{12}$$

(iv) This follows directly from

$$z = z_0 + (z - z_0) \qquad \text{and} \qquad \overline{z} = \overline{z_0} + (\overline{z} - \overline{z_0}). \tag{13}$$

(v) See separate sheet.

(vi) See separate sheet.

(vii) Let $t_0 \in I$. The function $\varphi : I \to U \subseteq \mathbb{C}$ is differentiable if and only if there exists a function $\varphi_1 : I \to U$ which is continuous at t_0 and such that

$$\varphi(t) = \varphi(t_0) + (t - t_0)\varphi_1(t) \tag{14}$$

for all $t \in I$ (this was proven in Analysis I). Furthermore there exists $f_1, f_2 : U \to \mathbb{C}$ continuous at $\varphi(t_0)$ such that

$$f(z) = f(\varphi(t_0)) + (z - \varphi(t_0))f_1(z) + (\overline{z} - \overline{\varphi(t_0)})f_2(z) \tag{15}$$

for all $z \in U$. Combining (14) and (15) yields

$$\begin{aligned} f(\varphi(t)) &= f(\varphi(t_0)) + (\varphi(t) - \varphi(t_0))f_1(z) + (\overline{\varphi(t)} - \overline{\varphi(t_0)})f_2(z) \\ &= f(\varphi(t_0)) + (t - t_0)\left[\varphi_1(t)f_1(\varphi(t)) + \overline{\varphi_1(t)}f_2(\varphi(t))\right] \end{aligned}$$

for all $t \in I$. Again, $\varphi_1(t)f(\varphi(t)) + \overline{\varphi_1(t)}f_2(\varphi(t))$ is clearly continuous at t_0 since composited functions are and thus we conclude

$$\frac{\mathrm{d}(f \circ \varphi)}{\mathrm{d}t}(t_0) = \frac{\mathrm{d}\varphi}{\mathrm{d}t}(t_0)\frac{\partial f}{\partial z}(\varphi(t_0)) + \frac{\mathrm{d}\overline{\varphi}}{\mathrm{d}t}(t_0)\frac{\partial f}{\partial \overline{z}}(\varphi(t_0)). \tag{16}$$

Since $t_0 \in I$ was arbitrary we conclude

$$\frac{\mathrm{d}(f \circ \varphi)}{\mathrm{d}t} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\partial f}{\partial z} + \frac{\mathrm{d}\overline{\varphi}}{\mathrm{d}t}\frac{\partial f}{\partial \overline{z}}. \tag{17}$$

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Exercise 2. We show the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). Since the proofs are of a relatively simple nature, we focus on the formal part. The complex numbers $\mathbb{C}$ are a vector space over $\mathbb{R}$ (as a field extension). So the situation of the exercise can be sumarized by the following commutative diagram:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "T"] & \mathbb{C} \\ \mathbb{R}^2 \arrow[u, "\Phi_B"] \arrow[r, "M_B(T)"'] & \mathbb{R}^2 \arrow[u, "\Phi_B"'] \end{tikzcd}$$

where T is $\mathbb{R}$-linear, Φ_B denotes the basis-isomorphism which is in this case given by $\Phi_B(x,y) := x + iy$ and $M_B(T)$ is defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{18}$$

The first implication is evident by the definition of $\mathbb{C}$-linearity. Assume that (ii) holds. By

$$T(i) = (\Phi_B \circ M_B(T) \circ \Phi_B^{-1})(i) = b + id \tag{19}$$

and

$$iT(1) = i(\Phi_B \circ M_B(T) \circ \Phi_B^{-1})(1) = i(a + ic) = -c + ia \tag{20}$$

we get the requirement $b + id = -c + ia$. Hence $b = -c$ and $a = d$. Assume that (iii) holds. Then we have for $z := x + iy \in \mathbb{C}$

$$T(z) = (\Phi_B \circ M_B(T) \circ \Phi_B^{-1})(x + iy) = (ax - cy) + i(cx + ay) = (a + ic)z. \tag{21}$$

Finally, assume that (iv) holds. Then T is clearly $\mathbb{C}$-linear since

$$T(\lambda z + w) = (a + ic)(\lambda z + w) = \lambda(a + ic)z + (a + ic)w = \lambda T(z) + T(w) \tag{22}$$

for $\lambda, z, w \in \mathbb{C}$.

Exercise 3. See separate sheet.

Exercise 4. We show this in two steps: first $\liminf_{\nu\to\infty} |a_\nu| \,/\, |a_{\nu+1}| \leq R$ and second $R \leq \limsup_{\nu\to\infty} |a_\nu| \,/\, |a_{\nu+1}|$. Define

$$S := \liminf_{\nu\to\infty} \frac{|a_\nu|}{|a_{\nu+1}|} \qquad \text{and} \qquad T := \limsup_{\nu\to\infty} \frac{|a_\nu|}{|a_{\nu+1}|}. \tag{23}$$

If $S = 0$ there is nothing to prove. First we assume that $0 < S < \infty$. Since $a_\nu \neq$ for almost all $\nu \in \mathbb{N}$ we find ν_0' such that $a_\nu \neq 0$ for all $\nu \geq \nu_0'$. For any $\varepsilon > 0$ we find $v_0 \in \mathbb{N}$, $\nu_0 \geq \nu_0'$ such that

$$\left| \inf_{\mu\geq\nu} \left\{ \frac{|a_\mu|}{|a_{\mu+1}|} \right\} - S \right| < \varepsilon \qquad \forall \nu \geq \nu_0 \tag{24}$$

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Setting $\nu = \nu_0$ in (24) yields

$$S - \varepsilon < \frac{|a_\nu|}{|a_{\nu+1}|} \qquad \forall \nu \geq \nu_0. \tag{25}$$

Now let $0 < \varepsilon < S$.

Lemma 0.1. *For all $\nu \geq \nu_0$ we have $|a_\nu|(S-\varepsilon)^\nu \leq |a_{\nu_0}|(S-\varepsilon)^{\nu_0}$.*

Proof. Proof by induction on $\nu \geq \nu_0$. The case $\nu = \nu_0$ is clear. By (25) we have that

$$|a_{\nu+1}|(S-\varepsilon)^{\nu+1} = |a_{\nu+1}|(S-\varepsilon)^\nu(S-\varepsilon) \leq |a_\nu|(S-\varepsilon)^\nu \leq |a_{\nu_0}|(S-\varepsilon)^{\nu_0}. \tag{26}$$

□

Hence

$$|a_\nu|(S-\varepsilon)^\nu \leq \max\left\{|a_0|, \dots, |a_{\nu_0-1}|(S-\varepsilon)^{\nu_0-1}, |a_{\nu_0}|(S-\varepsilon)^{\nu_0}\right\} \qquad \forall \nu \in \mathbb{N}. \tag{27}$$

Taking the limit $\varepsilon \searrow 0$ on inequality (27) yields

$$|a_\nu| S^\nu \leq \max\left\{|a_0|, \dots, |a_{\nu_0-1}| S^{\nu_0-1}, |a_{\nu_0}| S^{\nu_0}\right\} \qquad \forall \nu \in \mathbb{N} \tag{28}$$

and thus

$$S \leq R. \tag{29}$$

Now we consider the case $S = \infty$. By definition, for every $M > 0$ we find $\nu_0 \in \mathbb{N}$, $\nu_0 \geq \nu_0'$ such that

$$\inf_{\mu \geq \nu}\left\{\frac{|a_\mu|}{|a_{\mu+1}|}\right\} > M \qquad \forall \nu \geq \nu_0. \tag{30}$$

Again, this is equivalent to

$$M < \frac{|a_\nu|}{|a_{\nu+1}|} \qquad \forall \nu \geq \nu_0. \tag{31}$$

Similarly to the statement of lemma 0.1 one proves that

$$|a_\nu| M^\nu \leq |a_{\nu_0}| M^{\nu_0} \qquad \forall \nu \geq \nu_0. \tag{32}$$

Hence

$$|a_\nu| M^\nu \leq \max\left\{a_0, \dots, a_{\nu_0-1} M^{\nu_0-1}, |a_{\nu_0}| M^{\nu_0}\right\} \qquad \forall M > 0. \tag{33}$$

Thus the sequence $(|a_\nu| M^\nu)_{\nu\in\mathbb{N}}$ is bounded for any $M > 0$ which implies $R = \infty$.
If $T = \infty$ there is nothing to prove. So assume $0 < T < \infty$. Fix $\varepsilon > 0$. By the definition of the limit superior we find an index $\nu_0 \in \mathbb{N}$, $\nu_0 \geq \nu_0'$, such that

$$\left|\sup_{\mu \geq \nu}\left\{\frac{|a_\mu|}{|a_{\mu+1}|}\right\} - T\right| < \varepsilon \qquad \forall \nu \geq \nu_0. \tag{34}$$

Estimate (34) is equivalent to

$$\frac{|a_\nu|}{|a_{\nu+1}|} < T + \varepsilon \qquad \forall \nu \geq \nu_0. \tag{35}$$

We have an analogous version of lemma 0.1.

Lemma 0.2. *For all $\nu \geq \nu_0$ we have $|a_\nu|(T+\varepsilon)^\nu \geq |a_{\nu_0}|(T+\varepsilon)^{\nu_0}$.*

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Proof. Proof by induction on $\nu \geq \nu_0$. If $\nu = \nu_0$ there is nothing to prove. Furthermore, estimate (35) implies

$$|a_{\nu+1}|\,(T+\varepsilon)^{\nu+1} \geq |a_\nu|\,(T+\varepsilon)^{\nu} \geq |a_{\nu_0}|\,(T+\varepsilon)^{\nu_0}. \tag{36}$$

□

An immediate consequence of lemma 0.2 is that if $T \neq 0$ we have that $(|a_\nu|\,T^\nu)_{\nu\in\mathbb{N}} \not\to 0$ (take the limit $\varepsilon \searrow 0$). So $T + z_0 \in \mathbb{C} \setminus B_R(z_0)$, which immediately implies $T \geq R$. Finally we consider the case $T = 0$.