Yannis Bähni
yannis.baehni@uzh.ch

SOLUTIONS SHEET 3

YANNIS BÄHNI

Exercise 1. Let $D \subseteq \mathbb{C}$ be non-empty and open in $\mathbb{C}$ and $f_1, f_2 : D \to \mathbb{C}$ be real differentiable. Fix some $z_0 \in D$. Since f_1 and f_2 are real differentiable in z_0 there exists $\varphi_1, \varphi_2, \psi_1, \psi_2 : D \to \mathbb{C}$ continuous at z_0 such that

$$f_1(z) = f_1(z_0) + (z - z_0)\varphi_1(z) + (\overline{z} - \overline{z_0})\psi_1(z) \tag{1}$$
$$f_2(z) = f_2(z_0) + (z - z_0)\varphi_2(z) + (\overline{z} - \overline{z_0})\psi_2(z) \tag{2}$$

for all $z \in D$.
(i) Let $a, b \in \mathbb{C}$. Multiplying (1) by a, (2) by b and adding both equations yields

$$af_1(z)+bf_2(z) = af_1(z_0)+bf_2(z_0)+(z-z_0)(a\varphi_1(z)+b\varphi_2(z))+(\overline{z}-\overline{z_0})(a\psi_1(z)+b\psi_2(z)) \tag{3}$$

for all $z \in D$. Clearly, $a\varphi_1 + b\varphi_2$ and $a\psi_1 + b\psi_2$ are continuous functions in z_0 and from (3) we deduce

$$\frac{\partial(af_1 + bf_2)}{\partial z}(z_0) = a\frac{\partial f_1}{\partial z}(z_0) + b\frac{\partial f_2}{\partial z}(z_0) \tag{4}$$

and

$$\frac{\partial(af_1 + bf_2)}{\partial \overline{z}}(z_0) = a\frac{\partial f_1}{\partial \overline{z}}(z_0) + b\frac{\partial f_2}{\partial \overline{z}}(z_0). \tag{5}$$

Since $z_0 \in D$ was arbitrary, we conclude

$$\frac{\partial(af_1 + bf_2)}{\partial z} = a\frac{\partial f_1}{\partial z} + b\frac{\partial f_2}{\partial z} \qquad \text{and} \qquad \frac{\partial(af_1 + bf_2)}{\partial \overline{z}} = a\frac{\partial f_1}{\partial \overline{z}} + b\frac{\partial f_2}{\partial \overline{z}}. \tag{6}$$

(ii) Multiplying (1) and (2) yields

$$\begin{aligned} f_1 f_2 = f_1(z_0)f_2(z_0) &+ (z - z_0)\left[\varphi_1 f_2(z_0) + f_1(z_0)\varphi_2 + (z - z_0)\varphi_1\varphi_2 + (\overline{z} - \overline{z_0})\psi_1\varphi_2\right] \\ &+ (\overline{z} - \overline{z_0})\left[\psi_1 f_2(z_0) + f_1(z_0)\psi_2 + (z - z_0)\psi_2\varphi_1 + (\overline{z} - \overline{z_0})\psi_1\psi_2\right] \end{aligned}$$

where the argument z is omitted. Clearly, the two functions in the square brackets are continuous at z_0 and evaluating them at z_0 yields

$$\frac{\partial(f_1 f_2)}{\partial z}(z_0) = \frac{\partial f_1}{\partial z}(z_0)f_2(z_0) + f_1(z_0)\frac{\partial f_2}{\partial z}(z_0) \tag{7}$$

and

$$\frac{\partial(f_1 f_2)}{\partial \overline{z}}(z_0) = \frac{\partial f_1}{\partial \overline{z}}(z_0)f_2(z_0) + f_1(z_0)\frac{\partial f_2}{\partial \overline{z}}(z_0). \tag{8}$$

(Yannis Bähni) University of Zurich, Rämistrasse 71, 8006 Zurich
E-mail address: yannis.baehni@uzh.ch.

Yannis Bähni
yannis.baehni@uzh.ch

Since $z_0 \in D$ was arbitrary, we conclude

$$\frac{\partial(f_1 f_2)}{\partial z} = \frac{\partial f_1}{\partial z} f_2 + f_1 \frac{\partial f_2}{\partial z} \qquad \text{and} \qquad \frac{\partial(f_1 f_2)}{\partial \overline{z}} = \frac{\partial f_1}{\partial \overline{z}} f_2 + f_1 \frac{\partial f_2}{\partial \overline{z}}. \tag{9}$$

(iii) Conjugating (1) yields

$$\overline{f_1}(z) = \overline{f_1}(z_0) + (\overline{z} - \overline{z_0})\overline{\varphi_1}(z) + (z - z_0)\overline{\psi_1}(z). \tag{10}$$

From (10) we deduce

$$\frac{\partial \overline{f_1}}{\partial \overline{z}}(z_0) = \overline{\varphi_1}(z_0) = \overline{\frac{\partial f_1}{\partial z}}(z_0) \tag{11}$$

since φ_1 and ψ_1 are also continuous at z_0. Taking conjugates in (11) and use that $z_0 \in D$ was arbitrary finally yields

$$\overline{\frac{\partial \overline{f_1}}{\partial \overline{z}}} = \frac{\partial f_1}{\partial z}. \tag{12}$$

(iv) This follows directly from

$$z = z_0 + (z - z_0) \qquad \text{and} \qquad \overline{z} = \overline{z_0} + (\overline{z} - \overline{z_0}). \tag{13}$$