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SOLUTIONS SHEET 5

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Exercise 1.

(a) We summarize the result in a lemma.

Lemma 0.1. *The power series*

$$\sum_{\nu=0}^{\infty}\frac{z^{2\nu}}{(2\nu)!} \qquad \text{and} \qquad \sum_{\nu=0}^{\infty}\frac{z^{2\nu+1}}{(2\nu+1)!} \tag{1}$$

have both radius of convergence $R=\infty$. Furthermore, for all $z\in\mathbb{C}$

$$\cosh z=\sum_{\nu=0}^{\infty}\frac{z^{2\nu}}{(2\nu)!} \qquad \text{and} \qquad \sinh z=\sum_{\nu=0}^{\infty}\frac{z^{2\nu+1}}{(2\nu+1)!} \tag{2}$$

holds.

Proof. Fix $z\in\mathbb{C}$. We have

$$\limsup_{\nu\to\infty}\left|\frac{z^{2\nu+2}}{(2\nu+2)!}\frac{(2\nu)!}{z^{2\nu}}\right|=|z|^2\limsup_{\nu\to\infty}\frac{1}{(2\nu+2)(2\nu+1)}=0<1 \tag{3}$$

and

$$\limsup_{\nu\to\infty}\left|\frac{z^{2\nu+3}}{(2\nu+3)!}\frac{(2\nu+1)!}{z^{2\nu+1}}\right|=|z|^2\limsup_{\nu\to\infty}\frac{1}{(2\nu+3)(2\nu+2)}=0<1. \tag{4}$$

Since z was arbitrary we conclude by the ratio test for series that both radii of convergence are ∞. Using the identities

$$\cosh z=\cos(iz) \qquad \text{and} \qquad \sinh z=-i\sin(iz) \qquad \forall z\in\mathbb{C} \tag{5}$$

and the definition of the trigonometric functions by series

$$\cos z:=\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(2\nu)!}z^{2\nu} \qquad \text{and} \qquad \sin z:=\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(2\nu+1)!}z^{2\nu+1} \tag{6}$$

we get

$$\cosh z=\cos(iz)=\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(2\nu)!}(iz)^{2\nu}=\sum_{\nu=0}^{\infty}\frac{(-1)^{2\nu}}{(2\nu)!}z^{2\nu}=\sum_{\nu=0}^{\infty}\frac{z^{2\nu}}{(2\nu)!} \tag{7}$$

and

$$\sinh z=-i\sin(iz)=-i\sum_{\nu=0}^{\infty}\frac{(-1)^{\nu}}{(2\nu+1)!}(iz)^{2\nu+1}=\sum_{\nu=0}^{\infty}\frac{(-1)^{2\nu}}{(2\nu+1)!}z^{2\nu+1}=\sum_{\nu=0}^{\infty}\frac{z^{2\nu+1}}{(2\nu+1)!} \tag{8}$$

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for all $z \in \mathbb{Z}$. □

Remark 0.1. *The power series given in lemma 0.1 can be rewritten into the standard form*

$$\sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu \tag{9}$$

by considering apropriate sequences $(a_\nu)_{\nu \in \mathbb{N}_0}$. Also it is clearly seen that $z_0 = 0$ is the point of expansion.

(b) Define $a_\nu := (-1)^{\nu-1}/\nu$ for $\nu \in \mathbb{N}$ and $a_0 := 0$. Since $(a_\nu)_{\nu \in \mathbb{N}}$ is convergent, the quotient criterion yields

$$R = \lim_{\nu \to \infty} \left| \frac{a_\nu}{a_{\nu+1}} \right| = \lim_{\nu \to \infty} \left| \frac{(-1)^{\nu-1}}{\nu} \frac{\nu+1}{(-1)^\nu} \right| = 1 + \lim_{\nu \to \infty} \frac{1}{\nu} = 1. \tag{10}$$

Thus the logarithmic series converges in $\mathbb{E}$ since the point of expansion z_0 is clearly 0. Since $R > 0$ we have that the limit function f is holomorphic in $\mathbb{E}$ by the theorem on the *interchangeability of differentiation and summation*. Furthermore, from the same theorem also follows that the derivative of the limit function coincides with the naive termwise differentiation of the power series within $\mathbb{E}$. Thus we get

$$f'(z) = \sum_{\nu=1}^{\infty} \nu a_\nu z^{\nu-1} = \sum_{\nu=1}^{\infty} (-z)^{\nu-1} = \sum_{\mu=0}^{\infty} (-z)^\mu = \frac{1}{1+z} \tag{11}$$

by the formula for the sum of a geometric series (if $z \in \mathbb{E}$ so is $-z \in \mathbb{E}$).

(c) Fix $z \in \mathbb{C}$ and let $a_\nu := (-1)^\nu/(2\nu+1)z^{2\nu+1}$ for $\nu \in \mathbb{N}_0$. By

$$\begin{aligned} \limsup_{\nu \to \infty} \left| \frac{a_{\nu+1}}{a_\nu} \right| &= \limsup_{\nu \to \infty} \left| \frac{(-1)^{\nu+1} z^{2\nu+3}}{2\nu+3} \frac{2\nu+1}{(-1)^\nu z^{2\nu+1}} \right| \\ &= |z|^2 \limsup_{\nu \to \infty} \frac{2\nu+1}{2\nu+3} \\ &= |z|^2 \end{aligned}$$

we deduce that $|z|^2 < 1$ must hold that the series is convergent. This is equivalent to $z \in \mathbb{E}$. Thus the arcustangens series converges in $\mathbb{E}$ since the point of expansion z_0 is clearly 0. Since $R > 0$ we have that the limit function g is holomorphic in $\mathbb{E}$ by the theorem on the *interchangeability of differentiation and summation*. Furthermore, from the same theorem also follows that the derivative of the limit function coincides with the naive termwise differentiation of the power series within $\mathbb{E}$. First of all we have to bring the power series in an apropriate form. We have

$$g(z) = \sum_{\nu=0}^{\infty} a_\nu = \sum_{\nu=0}^{\infty} b_\nu z^\nu \qquad \text{where} \qquad b_\nu := \begin{cases} 0 & \nu \equiv 0 \bmod 2, \\ 1/\nu & \nu \equiv 1 \bmod 4, \\ -1/\nu & \nu \equiv 3 \bmod 4. \end{cases}$$

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Hence

$$g'(z) = \sum_{\nu=1}^{\infty} \nu b_\nu z^{\nu-1} = \sum_{\nu=0}^{\infty} (-1)^\nu z^{2\nu} = \sum_{\nu=0}^{\infty} (-z^2)^\nu = \frac{1}{1+z^2} \tag{12}$$

by the formula for the sum of a geometric series (if $z \in \mathbb{E}$ so is $-z^2 \in \mathbb{E}$).

Exercise 2.

(a) Define $\gamma_0 * \cdots * \gamma_n : I \to U$ where

$$I := \left[a_0, b_0 + \textstyle\sum_{\nu=1}^{n}(b_\nu - a_\nu)\right] \tag{13}$$

by

$$\gamma_0 * \cdots * \gamma_n(t) := \begin{cases} \gamma_0(t) & t \in A_0, \\ \gamma_1(t + a_1 - b_0) & t \in A_1, \\ \gamma_\nu\left(t + a_\nu - b_0 - \sum_{\mu=1}^{\nu-1}(b_\mu - a_\mu)\right) & t \in A_\nu, \nu = 2, \ldots, n, \end{cases}$$

where

$$A_\nu := \begin{cases} [a_0, b_0] & \nu = 0, \\ [b_0, b_1 - a_1 + b_0] & \nu = 1, \\ \left[b_0 + \sum_{\mu=1}^{\nu-1}(b_\mu - a_\mu), b_0 + \sum_{\mu=1}^{\nu}(b_\mu - a_\mu)\right] & \nu = 2, \ldots, n. \end{cases}$$

Let $n \in \mathbb{N}_{>0}$. Recall, that for $z_0, \ldots, z_n$ the path $[z_0, \ldots, z_n] : [0, n] \to \mathbb{C}$ defined by

$$[z_0, \ldots, z_n](t) := z_\nu + (t - \nu)(z_{\nu+1} - z_\nu) \qquad t \in [\nu, \nu+1] \tag{14}$$

for $\nu = 0, \ldots, n-1$ is called a ***polygon***. Consider the paths $\gamma_\nu := [z_\nu, z_{\nu+1}]$, $\nu = 0, \ldots, n-1$. Then we have

$$I = \left[0, 1 + \textstyle\sum_{\nu=1}^{n-1} 1\right] = [0, n] \tag{15}$$

and

$$A_\nu = \begin{cases} [a_0, b_0] = [0, 1] & \nu = 0, \\ [b_0, b_1 - a_1 + b_0] = [1, 2] & \nu = 1, \\ \left[b_0 + \sum_{\mu=1}^{\nu-1}(b_\mu - a_\mu), b_0 + \sum_{\mu=1}^{\nu}(b_\mu - a_\mu)\right] = [\nu, \nu+1] & \nu = 2, \ldots, n-1. \end{cases}$$

Hence $A_\nu = [\nu, \nu+1]$ for $\nu = 0, \ldots, n-1$. Furthermore

$$\gamma_0 * \cdots * \gamma_{n-1}(t) = \begin{cases} \gamma_0(t) = [z_0, z_1] & t \in A_0, \\ \gamma_1(t + a_1 - b_0) = z_1 + (t-1)(z_2 - z_1) & t \in A_1, \end{cases}$$

and

$$\gamma_0 * \cdots * \gamma_{n-1}(t) = \gamma_\nu\left(t + a_\nu - b_0 - \textstyle\sum_{\mu=1}^{\nu-1}(b_\mu - a_\mu)\right) = z_\nu + (t - \nu)(z_{\nu+1} - z_\nu) \tag{16}$$

for $t \in A_\nu, \nu = 2, \ldots, n-1$. Hence we conclude that

$$[z_0, \ldots, z_n] = [z_0, z_1] * \cdots * [z_{n-1}, z_n]. \tag{17}$$

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(b) An integration path in U is by definition a piecewise continuously differentiable mapping. Hence there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b]$ such that $\gamma|_{[t_\nu, t_{\nu+1}]}$ is continuously differentiable for $\nu = 0, \ldots, n-1$. Let $\gamma_\nu := \gamma|_{[t_\nu, t_{\nu+1}]}$ for $\nu = 0, \ldots, n-1$. Clearly

$$\gamma_\nu : [t_\nu, t_{\nu+1}] \to U \tag{18}$$

Using the terminology established in part (a) we get

$$I = \left[a, t_1 + \textstyle\sum_{\nu=1}^{n-1}(t_{\nu+1} - t_\nu)\right] = [a, t_n] = [a, b] \tag{19}$$

and

$$A_\nu = \begin{cases} [a_0, b_0] = [a, t_1] & \nu = 0, \\ [t_1, b_1 - a_1 + b_0] = [t_1, t_2] & \nu = 1, \\ \left[b_0 + \sum_{\mu=1}^{\nu-1}(b_\mu - a_\mu), b_0 + \sum_{\mu=1}^{\nu}(b_\mu - a_\mu)\right] = [t_\nu, t_{\nu+1}] & \nu = 2, \ldots, n-1. \end{cases}$$

Furthermore

$$\gamma_0 * \cdots * \gamma_{n-1}(t) = \begin{cases} \gamma_0(t) = \gamma(t) & t \in A_0, \\ \gamma_1(t + a_1 - b_0) = \gamma(t) & t \in A_1, \\ \gamma_\nu\left(t + a_\nu - b_0 - \sum_{\mu=1}^{\nu-1}(b_\mu - a_\mu)\right) = \gamma(t) & t \in A_\nu, \nu = 2, \ldots, n-1. \end{cases}$$

Hence we conclude

$$\gamma_0 * \cdots * \gamma_{n-1} = \gamma. \tag{20}$$

Exercise 3.

Lemma 0.2. *For $z_0 \in \mathbb{C}$ we have*

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\sum_{\nu=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^\nu \qquad \textit{for all } \zeta, z \textit{ with } |z - z_0| < |\zeta - z_0| \tag{21}$$

and

$$\frac{1}{\zeta - z} = -\frac{1}{z - z_0}\sum_{\nu=0}^{\infty}\left(\frac{\zeta - z_0}{z - z_0}\right)^\nu \qquad \textit{for all } \zeta, z \textit{ with } |\zeta - z_0| < |z - z_0|. \tag{22}$$

For $B_r(z_0)$ fixed, series (21) converges normally as a function series in the argument ζ on $\partial B_r(z_0)$ for any $z \in B_r(z_0)$ whereas series (22) converges normally as a function series in the argument ζ on $\partial B_r(z_0)$ for any $z \in \mathbb{C} \setminus \overline{B_r}(z_0)$.

Proof. In the first case we have

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\frac{\zeta - z_0}{\zeta - z} = \frac{1}{\zeta - z_0}\frac{1}{1 - (z - z_0)/(\zeta - z_0)} = \frac{1}{\zeta - z_0}\sum_{\nu=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^\nu \tag{23}$$

since by assumption $|(z - z_0)/(\zeta - z_0)| < 1$ and in the second

$$\frac{1}{\zeta - z} = \frac{1}{z - z_0}\frac{z - z_0}{\zeta - z} = -\frac{1}{\zeta - z_0}\frac{1}{1 - (\zeta - z_0)/(z - z_0)} = -\frac{1}{\zeta - z_0}\sum_{\nu=0}^{\infty}\left(\frac{\zeta - z_0}{z - z_0}\right)^\nu \tag{24}$$

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since again by assumption $|(\zeta - z_0)/(z - z_0)| < 1$.
Fix some $B_r(z_0)$. Let $z \in B_r(z_0)$. For any $\nu \in \mathbb{N}_0$ and $q := |z - z_0|/r$ we have

$$\max_{\zeta \in \partial B_r(z_0)} \left| \left(\frac{z - z_0}{\zeta - z_0} \right)^\nu \right| = \frac{|z - z_0|^\nu}{\min_{\zeta \in \partial B_r(z_0)} |\zeta - z_0|^\nu} = \left(\frac{|z - z_0|}{r} \right)^\nu = q^\nu \tag{25}$$

Hence if we define $f_\nu : \partial B_r(z_0) \to \mathbb{C}$ by

$$f_\nu(\zeta) := \left(\frac{z - z_0}{\zeta - z_0} \right)^\nu \qquad \nu \in \mathbb{N}_0 \tag{26}$$

and let $\zeta \in \partial B_r(z_0)$ fixed, we have

$$\sum_{\nu=0}^{\infty} |f_\nu|_{\partial B_r(z_0)} = \sum_{\nu=0}^{\infty} q^\nu < \infty \tag{27}$$

since $\partial B_r(z_0)$ is compact and $q < 1$. Thus the series (21) is normally convergent. If $z \in \mathbb{C} \setminus \overline{B_r}(z_0)$, we have

$$\max_{\zeta \in \partial B_r(z_0)} \left| \left(\frac{\zeta - z_0}{z - z_0} \right)^\nu \right| = \frac{\max_{\zeta \in \partial B_r(z_0)} |\zeta - z_0|^\nu}{|z - z_0|^\nu} = \left(\frac{r}{|z - z_0|} \right)^\nu = p^\nu \tag{28}$$

for $p := r/|z - z_0|$. Observe again that $p < 1$ since $|z - z_0| > r$ by assumption. Hence we conclude similarly as in the previous case that the series (22) is normally convergent. $\square$

Proposition 0.1. *For $z_0 \in \mathbb{C}$ we have*

$$\frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{\mathrm{d}\zeta}{\zeta - z} = \begin{cases} 1 & z \in B_r(z_0), \\ 0 & z \in \mathbb{C} \setminus \overline{B_r}(z_0). \end{cases} \tag{29}$$

Proof. We use the fact, that if γ is a path, $\sum f_\nu$, $f_\nu \in \mathscr{C}(|\gamma|)$, a series of functions, which is uniformly convergent in $|\gamma|$, then $\sum \int_\gamma f_\nu = \int_\gamma \sum f_\nu$ (see [RS02, p. 163]). By lemma 0.2 we know that the two series are normally convergent in $\partial B_r(z_0)$ and since every normally convergent series is locally uniformly convergent, the compactness of $\partial B_r(z_0)$ implies that the two series are uniformly convergent in $\partial B_r(z_0)$ (see [RS02, p. 92] and [RS02, p. 85]). Consider first the case where $z \in B_r(z_0)$. Then lemma 0.2 yields

$$\begin{aligned}
\int_{\partial B_r(z_0)} \frac{\mathrm{d}\zeta}{\zeta - z} &= \int_{\partial B_r(z_0)} \frac{1}{\zeta - z_0} \sum_{\nu=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^\nu \mathrm{d}\zeta \\
&= \sum_{\nu=0}^{\infty} \int_{\partial B_r(z_0)} \frac{1}{\zeta - z_0} \left(\frac{z - z_0}{\zeta - z_0} \right)^\nu \mathrm{d}\zeta \\
&= \sum_{\nu=0}^{\infty} (z - z_0)^\nu \int_{\partial B_r(z_0)} \frac{\mathrm{d}\zeta}{(\zeta - z_0)^{\nu+1}} \\
&= 2\pi i
\end{aligned}$$

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since for $n \in \mathbb{Z}$

$$\int_{\partial B_r(z_0)} (\zeta - z_0)^n \, \mathrm{d}\zeta = \begin{cases} 0 & n \neq -1, \\ 2\pi & n = -1. \end{cases} \tag{30}$$

In the case $z \in \mathbb{C} \setminus \overline{B_r}(z_0)$ we get

$$\begin{aligned} \int_{\partial B_r(z_0)} \frac{\mathrm{d}\zeta}{\zeta - z} &= -\int_{\partial B_r(z_0)} \frac{1}{z - z_0} \sum_{\nu=0}^{\infty} \left(\frac{\zeta - z_0}{z - z_0} \right)^{\nu} \mathrm{d}\zeta \\ &= -\sum_{\nu=0}^{\infty} \frac{1}{(z - z_0)^{\nu+1}} \int_{\partial B_r(z_0)} (\zeta - z_0)^{\nu} \, \mathrm{d}\zeta \\ &= 0. \end{aligned}$$

□

References

[RS02] R. Remmert and G. Schumacher. *Funktionentheorie 1*. Springer-Lehrbuch. Springer Berlin Heidelberg, 2002. ISBN: 9783540418559.