SOLUTIONS SHEET 4

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Remark: We will abreviate $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$.

Exercise 1.

(a) We use the quotient criterion [RS02, p. 100]. Consider the sequence

$$a_\nu := \frac{\nu!}{2^\nu (2\nu)!} \qquad \nu \in \mathbb{N}_{>0}. \tag{1}$$

Clearly $a_\nu \neq 0$ for all $\nu \in \mathbb{N}_{>0}$. We have

$$\begin{aligned} \lim_{\nu\to\infty} \frac{|a_\nu|}{|a_{\nu+1}|} &= \lim_{\nu\to\infty} \frac{\nu!}{2^\nu(2\nu)!} \frac{2^{\nu+1}(2(\nu+1))!}{(\nu+1)!} \\ &= \lim_{\nu\to\infty} \frac{2(2\nu+1)(2\nu+2)}{\nu+1} \\ &= 4 \lim_{\nu\to\infty} (2\nu+1) \\ &= \infty \end{aligned}$$

Thus by

$$\lim_{\nu\to\infty} \frac{|a_\nu|}{|a_{\nu+1}|} = \limsup_{\nu\to\infty} \frac{|a_\nu|}{|a_{\nu+1}|} = \liminf_{\nu\to\infty} \frac{|a_\nu|}{|a_{\nu+1}|} \tag{2}$$

and

$$\liminf_{\nu\to\infty} \frac{|a_\nu|}{|a_{\nu+1}|} \leq R \leq \limsup_{\nu\to\infty} \frac{|a_\nu|}{|a_{\nu+1}|} \tag{3}$$

we get

$$R = \infty. \tag{4}$$

(b) We use the Cauchy-Hadamard formula [RS02, p. 99]. Let $c \in \mathbb{C}^\times$. Consider the sequence

$$a_\nu := \begin{cases} \frac{1}{c^{\nu/2}} & \nu \equiv 0 \bmod 2 \\ 0 & \nu \equiv 1 \bmod 2 \end{cases}$$

Since $0 \leq 1/|c|^{\nu/2}$ we get

$$\limsup_{\nu\to\infty} |a_\nu|^{1/\nu} = \lim_{\nu\to\infty} \sup_{\mu\geq\nu} \{|a_\mu|^{1/\mu}\}$$

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$$
\begin{aligned}
&= \lim_{\nu\to\infty} \sup_{\substack{\mu\geq\nu \\ \mu\equiv 0 \bmod 2}} \{|a_\mu|^{1/\mu}\} \\
&= \limsup_{\nu\to\infty} \left|\frac{1}{c^\nu}\right|^{1/(2\nu)} \\
&= \limsup_{\nu\to\infty} \frac{1}{\sqrt{|c|}} \\
&= \lim_{\nu\to\infty} \frac{1}{\sqrt{|c|}} \\
&= \frac{1}{\sqrt{|c|}}
\end{aligned}
$$

and therefore

$$
R = \frac{1}{\limsup_{\nu\to\infty} |a_\nu|^{1/\nu}} = \sqrt{|c|}. \tag{5}
$$

Exercise 2. Proof by induction over $k \in \mathbb{N}$. If $k = 0$ we have

$$
\sum_{\nu=0}^{\infty} \binom{\nu}{0} z^\nu = \sum_{\nu=0}^{\infty} z^\nu = \frac{1}{1-z} \tag{6}
$$

by the well-known identity for the geometric series (see [RS02, p. 24]) which holds for all $z \in \mathbb{E}$. Now assume the stated identity is true for some $k \in \mathbb{N}$. Pascal's identity and the substitution $\mu := \nu - 1$ yields

$$
\begin{aligned}
\sum_{\nu=k+1}^{\infty} \binom{\nu}{k+1} z^{\nu-(k+1)} &= 1 + \sum_{\nu=k+2}^{\infty} \binom{\nu}{k+1} z^{(\nu-1)-k} \\
&= 1 + \sum_{\nu=k+2}^{\infty} \left[\binom{\nu-1}{k} + \binom{\nu-1}{k+1}\right] z^{(\nu-1)-k} \\
&= 1 + \sum_{\nu=k+2}^{\infty} \binom{\nu-1}{k} z^{(\nu-1)-k} + \sum_{\nu=k+2}^{\infty} \binom{\nu-1}{k+1} z^{(\nu-1)-k} \qquad (7) \\
&= \binom{k}{k} z^0 + \sum_{\mu=k+1}^{\infty} \binom{\mu}{k} z^{\mu-k} + \sum_{\mu=k+1}^{\infty} \binom{\mu}{k+1} z^{\mu-k} \\
&= \sum_{\mu=k}^{\infty} \binom{\mu}{k} z^{\mu-k} + z \sum_{\mu=k+1}^{\infty} \binom{\mu}{k+1} z^{\mu-k-1}
\end{aligned}
$$

Applying the induction hypothesis on (7) and rearranging yields

$$
\sum_{\nu=k+1}^{\infty} \binom{\nu}{k+1} z^{\nu-(k+1)} = \frac{1}{(1-z)(1-z)^{k+1}} = \frac{1}{(1-z)^{k+2}}. \tag{8}
$$

Therefore we conclude by the principle of induction.

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A more elegant proof can be deduced from the interchangeability of differentiation and summation of power series (see [RS02, p. 110]).

Lemma 0.1. *For $z \in \mathbb{E}$ and $k \in \mathbb{N}$ we have*

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\frac{1}{1-z} = \frac{k!}{(1-z)^{k+1}}. \tag{9}$$

Proof. Proof by induction over $k \in \mathbb{N}$. The statement obviously holds for $k = 0$. Assume the statement holds for $k \in \mathbb{N}$. Then we get

$$\begin{aligned}\frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}}\frac{1}{1-z} &= \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\mathrm{d}^k}{\mathrm{d}z^k}\frac{1}{1-z}\right]\\ &= \frac{\mathrm{d}}{\mathrm{d}z}\frac{k!}{(1-z)^{k+1}}\\ &= k!\frac{(k+1)(1-z)^k}{(1-z)^{2k+2}}\\ &= \frac{(k+1)!}{(1-z)^{k+2}}.\end{aligned}$$

□

By lemma 0.1 and the formula for the k-th derivative of a power series [RS02, p. 110] applied to the *geometric series* $\sum_{\nu=0}^{\infty} z^\nu$ we get

$$\frac{k!}{(1-z)^{k+1}} = \frac{\mathrm{d}^k}{\mathrm{d}z^k}\frac{1}{1-z} = \sum_{\nu=k}^{\infty} k!\binom{\nu}{k} z^{\nu-k} \tag{10}$$

for $k \in \mathbb{N}$. Dividing (10) by $k!$ yields

$$\sum_{\nu=k}^{\infty}\binom{\nu}{k} z^{\nu-k} = \frac{1}{(1-z)^{k+1}}. \tag{11}$$

Exercise 3.
(a) Consider the auxiliary function $\varphi : \mathbb{C} \to \mathbb{C}$ defined by

$$\varphi(z) := f(z)\exp(-z). \tag{12}$$

Clearly $\varphi \in \mathcal{O}(\mathbb{C})$ as a product of holomorphic functions. Furthermore by $f' = f$ we get

$$\varphi'(z) = f'(z)\exp(-z) - f(z)\exp(-z) = \varphi(z) - \varphi(z) = 0 \tag{13}$$

for any $z \in \mathbb{C}$. By [RS02, p. 55] φ is locally constant on $\mathbb{C}$ and since $\mathbb{C}$ is connected we have that φ is constant on $\mathbb{C}$ (see [RS02, p. 35]). Since $f(0) = 1$ we have

$$\varphi(0) = f(0)\exp(0) = 1. \tag{14}$$

Thus $\varphi \equiv 1$ which yields

$$\exp(z) = 1 \cdot \exp(z) = \varphi(z)\exp(z) = f(z)\exp(-z)\exp(z) = f(z) \tag{15}$$

for all $z \in \mathbb{C}$.

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(b) Differentiation of the addition formula with respect to w yields

$$g'(z+w) = g(z)g'(w). \tag{16}$$

for all $w, z \in \mathbb{C}$. Evaluating (16) at $w = 0$ gives

$$g'(z) = g(z)g'(0). \tag{17}$$

Define $\psi : \mathbb{C} \to \mathbb{C}$ by

$$\psi(z) := g(z)\exp(-g'(0)z). \tag{18}$$

Clearly $\psi \in \mathcal{O}(\mathbb{C})$ as a product of holomorphic functions. Similar to part (a) we get

$$\begin{aligned}\psi'(z) &= g'(z)\exp(-g'(0)z) - g(z)g'(0)\exp(-g'(0)z)\\ &= g(z)g'(0)\exp(-g'(0)z) - g(z)g'(0)\exp(-g'(0)z)\\ &= 0\end{aligned}$$

for all $z \in \mathbb{C}$. Again ψ is constant on $\mathbb{C}$ and with

$$g(0) = g(0+0) = g(0)g(0) \tag{19}$$

immediately follows $g(0) = 1$ by $g(0) \neq 0$. Thus $\psi(0) = 1$ and hence $\psi \equiv 1$ on $\mathbb{C}$ which implies

$$g(z) = \exp(g'(0)z) = \exp(bz) \tag{20}$$

for $b := g'(0)$.

Exercise 4.

(a)

Proposition 0.1. *For all $w, z \in \mathbb{C}$ holds:*

$$\cos(w+z) = \cos w \cos z - \sin w \sin z \tag{21}$$

$$\sin(w+z) = \sin w \cos z + \cos w \sin z. \tag{22}$$

Proof. First we prove (21). We have

$$\begin{aligned}\cos w \cos z &= \frac{1}{4}\left(e^{iw} + e^{-iw}\right)\left(e^{iz} + e^{-iz}\right)\\ &= \frac{1}{4}\left(e^{i(w+z)} + e^{i(w-z)} + e^{i(z-w)} + e^{-i(w+z)}\right)\end{aligned}$$

and

$$\begin{aligned}\sin w \sin z &= -\frac{1}{4}\left(e^{iw} - e^{-iw}\right)\left(e^{iz} - e^{-iz}\right)\\ &= -\frac{1}{4}\left(e^{i(w+z)} - e^{i(w-z)} - e^{i(z-w)} + e^{-i(w+z)}\right)\end{aligned}$$

Thus

$$\cos w \cos z - \sin w \sin z = \frac{1}{2}(e^{i(w+z)} + e^{-i(w+z)}) = \cos(w+z). \tag{23}$$

Now we prove (22). We have

$$\sin w \cos z = \frac{1}{4i}\left(e^{iw} - e^{-iw}\right)\left(e^{iz} + e^{-iz}\right)$$

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$$= \frac{1}{4i}\left(e^{i(w+z)} + e^{i(w-z)} - e^{i(z-w)} - e^{-i(w+z)}\right)$$

and

$$\begin{aligned}\cos w \sin z &= \frac{1}{4i}\left(e^{iw} + e^{-iw}\right)\left(e^{iz} - e^{-iz}\right)\\ &= \frac{1}{4i}\left(e^{i(w+z)} - e^{i(w-z)} + e^{i(z-w)} - e^{-i(w+z)}\right)\end{aligned}$$

Thus

$$\sin w \cos z + \cos w \sin z = \frac{1}{2i}\left(e^{i(w+z)} - e^{-i(w+z)}\right) = \sin(w+z). \tag{24}$$

□

(b)

(c) Let $z \in \mathbb{C} \setminus 2\pi\mathbb{Z}$. Then the index shift $\mu := \nu + n$ and application of the formula for the partial sum of a geometric series yields

$$\begin{aligned}\sum_{\nu=0}^{n} \cos(\nu z) &= \frac{1}{2}\sum_{\nu=0}^{n}\left(e^{i\nu z} + e^{-i\nu z}\right)\\ &= \frac{1}{2} + \frac{1}{2}\sum_{\nu=-n}^{n} e^{i\nu z}\\ &= \frac{1}{2} + \frac{1}{2}\sum_{\mu=0}^{2n} e^{i(\mu-n)z}\\ &= \frac{1}{2} + \frac{1}{2}e^{-inz}\sum_{\mu=0}^{2n} e^{i\mu z}\\ &= \frac{1}{2} + \frac{1}{2}e^{-inz}\frac{1-e^{iz(2n+1)}}{1-e^{iz}}\\ &= \frac{1}{2} + \frac{1}{2}e^{-inz}\frac{e^{iz(2n+1)}-1}{e^{iz}-1}\\ &= \frac{1}{2} + \frac{1}{2}e^{-inz}\frac{e^{-iz(n+1/2)}}{e^{-iz(n+1/2)}}\frac{e^{2iz(n+1/2)}-1}{e^{iz}-1}\\ &= \frac{1}{2} + \frac{1}{2}\frac{1}{e^{-iz/2}}\frac{e^{iz(n+1/2)}-e^{-iz(n+1/2)}}{e^{iz}-1}\\ &= \frac{1}{2} + \frac{1}{2}\frac{e^{iz(n+1/2)}-e^{-iz(n+1/2)}}{e^{iz/2}-e^{-iz/2}}\\ &= \frac{1}{2} + \frac{1}{2}\frac{\sin(nz+z/2)}{\sin(z/2)}\end{aligned}$$

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References

[RS02] R. Remmert and G. Schumacher. *Funktionentheorie 1*. Springer-Lehrbuch. Springer Berlin Heidelberg, 2002. ISBN: 9783540418559.