SOLUTIONS SHEET 2

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Exercise 1. Recall, that $\mathcal{Z}$ is also called ***finite complement topology*** (see [Lee11, p. 45]).

(i) We have already showed that $(\mathbb{C}, \mathcal{Z})$ is not a Hausdorff space (see sheet 1 exercise 3), hence not compact. Therefore it is enough to show that any open cover of $(\mathbb{C}, \mathcal{Z})$ has a finite subcover.

Lemma 0.1. *$(\mathbb{C}, \mathcal{Z})$ is quasi-compact.*

Proof. Let $(U_\alpha)_{\alpha \in A}$ be an open cover of $(\mathbb{C}, \mathcal{Z})$, i.e.

$$\mathbb{C} = \bigcup_{\alpha \in A} U_\alpha \qquad \text{and} \qquad \forall \alpha \in A : U_\alpha \in \mathcal{Z}. \tag{1}$$

We can explicitely construct a finite subcover. Pick some $\alpha_0 \in A$ such that $U_{\alpha_0} \neq \varnothing$. Since $U_{\alpha_0} \in \mathcal{Z}$, $U_{\alpha_0}^c$ is finite, i.e. $U_{\alpha_0}^c = \{z_1, \ldots, z_n\} \subseteq \mathbb{C}$. Thus we can write

$$\mathbb{C} = U_{\alpha_0} \cup U_{\alpha_0}^c = U_{\alpha_0} \cup \{z_1, \ldots, z_n\}. \tag{2}$$

Since $\mathbb{C} = \bigcup_{\alpha \in A} U_\alpha$, we find $\alpha_i \in A$ for $i = 1, \ldots, n$ such that $z_i \in U_{\alpha_i}$. Hence $(U_{\alpha_\nu})_{\nu \in \{0, \ldots, n\}}$ is a finite subcover of $(U_\alpha)_{\alpha \in A}$. Since the construction was general, we conclude that $(\mathbb{C}, \mathcal{Z})$ is quasi-compact. ☐

(ii) The reasoning is similar to part i).

Lemma 0.2. *$(\{z_0\}^c, \{z_0\}^c \cap \mathcal{Z})$ is quasi-compact.*

Proof. Let $(U_\alpha)_{\alpha \in A}$ be an open cover of $(\{z_0\}^c, \{z_0\}^c \cap \mathcal{Z})$, i.e.

$$\{z_0\}^c = \bigcup_{\alpha \in A} U_\alpha \qquad \text{and} \qquad \forall \alpha \in A : U_\alpha \in \{z_0\}^c \cap \mathcal{Z}. \tag{3}$$

We can explicitely construct a finite subcover. Pick some $\alpha_0 \in A$ such that $U_{\alpha_0} \neq \varnothing$. Since $U_{\alpha_0} \in \{z_0\}^c \cap \mathcal{Z}$, there exists $V \in \mathcal{Z}$ such that $U_{\alpha_0} = \{z_0\}^c \cap V$. By considering the relative complement

$$U_{\alpha_0}^c = \{z_0\}^c \cap (\{z_0\}^c \cap V)^c = \{z_0\}^c \cap (\{z_0\} \cup V^c) = \{z_0\}^c \cap V^c \subseteq V^c \tag{4}$$

and using the fact that V^c is finite we conclude that $U_{\alpha_0}^c$ is finite, i.e. $U_{\alpha_0}^c = \{z_1, \ldots, z_n\} \subseteq \{z_0\}^c$. Thus we can write

$$\{z_0\}^c = U_{\alpha_0} \cup U_{\alpha_0}^c = U_{\alpha_0} \cup \{z_1, \ldots, z_n\}. \tag{5}$$

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Since $\{z_0\}^c = \bigcup_{\alpha\in A} U_\alpha$, we find $\alpha_i \in A$ for $i = 1, \dots, n$ such that $z_i \in U_{\alpha_i}$. Hence $(U_{\alpha_\nu})_{\nu\in\{0,\dots,n\}}$ is a finite subcover of $(U_\alpha)_{\alpha\in A}$. Since the construction was general, we conclude that $(\{z_0\}^c, \{z_0\}^c \cap \mathcal{Z})$ is quasi-compact. $\square$

Lemma 0.3. *$(\{z_0\}^c, \{z_0\}^c \cap \mathcal{Z})$ is not Hausdorff.*

Proof. Towards a contradiction assume that $(\{z_0\}^c, \{z_0\}^c \cap \mathcal{Z})$ is Hausdorff. Thus for $p, q \in \{z_0\}^c$ there exists open neighbourhoods U and V of p and q respectively such that $U \cap V = \varnothing$. From the latter it follows that $U \subseteq V^c$. Since V is open we find $W_1 \in \mathcal{Z}$ such that $V = \{z_0\}^c \cap W_1$. Hence taking relative complements yields

$$V^c = \{z_0\}^c \cap (\{z_0\}^c \cap W_1)^c = \{z_0\}^c \cap W_1^c \subseteq W_1^c$$

So V^c is finite and therefore also U. Since U is open we have that there exists $W_2 \in \mathcal{Z}$ such that $U = \{z_0\}^c \cap W_2$. Taking again relative complements yields

$$U^c = \{z_0\}^c \cap (\{z_0\}^c \cap W_2)^c = \{z_0\}^c \cap W_2^c \subseteq W_2^c$$

So U^c is also finite. Therefore the decomposition $\{z_0\}^c = U \cup U^c$ implies that $\{z_0\}^c$ is finite. Contradiction, since $|\{z_0\}^c| \geq |\mathbb{R}| = \mathfrak{c}$, which is clearly not finite. $\square$

Therefore by lemma 0.2 we conclude that $(\{z_0\}^c, \{z_0\}^c \cap \mathcal{Z})$ is quasi-compact, but from lemma 0.3 follows that $(\{z_0\}^c, \{z_0\}^c \cap \mathcal{Z})$ is not compact.

(iii) By $(\{z_0\}^c)^c = \{z_0\}$ which is finite immediately follows $\{z_0\}^c \in \mathcal{Z}$. But $\{z_0\}^c = \mathbb{C} \setminus \{z_0\}$ is clearly not finite, thus $\{z_0\} \notin \mathcal{Z}$, hence $\{z_0\}^c$ cannot be closed.

Exercise 2. Let $G \subseteq \mathbb{C}$ be non-empty and open. We show the equivalences (i) $\Leftrightarrow$ (ii) and (ii) $\Leftrightarrow$ (iii). This is due to the fact that I am aware of the latter equivalence by considering [Lee11, p. 86] and the first one by [Lee11, p. 90] and the fact that every open connected subset of $\mathbb{R}^n$ is path-connected. However, working out detailed and appropriate proofs is still alot of work.
Assume that (i) holds. Proof by contradiction. Let $G = G_1 \cup G_2$ for some open sets $G_1, G_2 \subseteq \mathbb{C}$ with $G_1 \cap G_2 = \varnothing$ and $G_1, G_2 \neq G$. Evidently $G_1, G_2 \neq \varnothing$ and thus we find $p \in G \cap G_1$, $q \in G \cap G_2$. Let $\gamma : [a, b] \to G$ be a path joining p and q, i.e. $\gamma(a) = p$ and $\gamma(b) = q$. Since γ is continuous, $G \cap G_1$ and $G \cap G_2$ are relatively open in G we have that $\gamma^{-1}(G \cap G_1)$ and $\gamma^{-1}(G \cap G_2)$ are open in $[a, b]$. Furthermore, since $a \in \gamma^{-1}(G \cap G_1)$ and $b \in \gamma^{-1}(G \cap G_2)$ we have that both preimages are non-empty. By

$$\gamma^{-1}(G \cap G_1) \cup \gamma^{-1}(G \cap G_2) = \gamma^{-1}((G \cap G_1) \cup (G \cap G_2)) = \gamma^{-1}(G) = [a, b]$$

and

$$\gamma^{-1}(G \cap G_1) \cap \gamma^{-1}(G \cap G_2) = \gamma^{-1}((G \cap G_1) \cap (G \cap G_2)) = \gamma^{-1}(\varnothing) = \varnothing$$

we have that $\gamma^{-1}(G \cap G_1)$ and $\gamma^{-1}(G \cap G_2)$ disconnect $[a, b]$ which is impossible since a real interval is always connected (see [Lee11, p. 89]).

Now assume that (ii) holds. Let $z_0 \in G$. Since joinability by paths in G is an equivalence relation (let us denote it simply by $\sim$) define

$$G_1 := [z_0]_\sim . \tag{6}$$

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Lemma 0.4. *G_1 is open.*

Proof. Let $z_1 \in G_1$. Since G is open we find $\varepsilon > 0$ such that $B_\varepsilon(z_1) \subseteq G$. $B_\varepsilon(z_1)$ is evidently path connected (consider just straight lines joining different points). Since $z_1 \in G_1$, we have that there is a path joining z_0 and z_1. By concatenating paths, there is a path from z_0 to every point in $B_\varepsilon(z_1)$ and thus $B_\varepsilon(z_1) \subseteq G_1$. □

Lemma 0.5. *The relative complement G_1^c in G is open.*

Proof. Let $z_1 \in G_1^c$. Again we find $\varepsilon > 0$ such that $B_\varepsilon(z_1) \subseteq G$ by the openness of G. Towards a contradiction assume that $B_\varepsilon(z_1) \cap G_1 \neq \varnothing$. Hence we find $z_2 \in B_\varepsilon(z_1) \cap G_1$. This means, that there is a path joining z_0 and z_2. But since $B_\varepsilon(z_1)$ is path connected there would be a path joining z_1 and z_0 which yields a contradiction. Hence $B_\varepsilon(z_1) \subseteq G_1^c$. □

Since evidently $G = G_1 \cup G_1^c$ and by lemma 0.4 and 0.5 G_1, G_1^c are open and clearly disjoint, (ii) implies that either $G_1 = G$ or $G_1^c = G$. The latter is impossible since $z_0 \notin G_1^c$. Hence we conclude that $G = G_1$. Thus G is a single equivalence class under joinability by paths, hence path-connected.
Next we show that (ii) $\Rightarrow$ (iii). To be completely rigorous, we state the following lemma which can be found as an exercise in [Lee11, p. 50].

Lemma 0.6. *Let $(X, \mathcal{T})$ be a topological space and $S \subseteq X$. Then $B \subseteq S$ is closed in S if and only if $B = S \cap A$ for some closed set A in X.*

Proof. Assume $B \subseteq S$ is closed in S. Hence the relative complement B^c is open in S. Therefore we have $B^c = S \cap U$ for some open set U in X. Thus

$$B = (S \cap U)^c = S \cap (S \cap U)^c = S \cap (S^c \cup U^c) = (S \cap S^c) \cup (S \cap U^c) = S \cap U^c.$$

But since U is open in X we have that U^c is closed in X. Conversly assume that $B = S \cap A$ for some closed set A in X. Taking relative complements yields

$$B^c = (S \cap A)^c = S \cap (S \cap A)^c = S \cap A^c$$

and since A is closed in X we have that A^c is open in X which means B^c is open in S. □

Now assume that (ii) holds. Let $U \subseteq G$ be a non-empty, open and relatively closed (with respect to G) subset. Since U is relatively closed, by lemma 0.6 there exists a closed set $A \subseteq \mathbb{C}$ such that $U = G \cap A$. Observe, that by

$$U^c = G \cap (G \cap A)^c = G \cap (G^c \cup A^c) = G \cap A^c$$

U^c is open in $\mathbb{C}$ since G and A^c are open in $\mathbb{C}$. Clearly, $G = U \cup U^c$ and $U \cap U^c = \varnothing$. Therefore (ii) implies that either $U = G$ or $U^c = G$ where the latter is impossible since by assumption $U \neq \varnothing$. Hence we conclude that $U = G$.
Finally we show (iii) $\Rightarrow$ (ii). This is equivalent to showing that not (ii) implies not (iii). So we have $G = G_1 \cup G_2$ for some open disjoint sets $G_1, G_2 \subseteq \mathbb{C}$ where $G_1, G_2 \neq G$. Now clearly $\varnothing \subsetneq G_1 \subsetneq G$, G_1 is open and G_1 is relatively closed in G since

$$G \cap G_2^c = (G_1 \cup G_2) \cap G_2^c = G_1 \cap G_2^c = G_1$$

since G_2^c is closed and $G_1 \cap G_2 = \varnothing$.

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Exercise 3. The proof is given as a sequence of lemmata.

Lemma 0.7. *The mapping h is well-defined, i.e. $h(\mathbb{H}) \subseteq \mathbb{E}$.*

Proof. Let $z \in \mathbb{H}$. Then $\operatorname{Im}(z) > 0$ and thus

$$\begin{aligned}\left|\frac{z-i}{z+i}\right|^2 &= \frac{(z-i)(\overline{z}+i)}{(z+i)(\overline{z}-i)} \\ &= \frac{|z|^2 + i(z-\overline{z}) + 1}{|z|^2 + i(\overline{z}-z) + 1} \\ &= \frac{|z|^2 - 2\operatorname{Im}(z) + 1}{|z|^2 + 2\operatorname{Im}(z) + 1} \\ &\leq \frac{|z|^2 - 2\operatorname{Im}(z) + 1}{|z|^2 + 1} \\ &= 1 - \frac{2\operatorname{Im}(z)}{|z|^2 + 1} \\ &< 1.\end{aligned}$$

□

Lemma 0.8. *The mapping h is invertible with inverse*

$$h^{-1} : \mathbb{E} \to \mathbb{H}, w \mapsto i\frac{1+w}{1-w}. \tag{7}$$

Proof. Let $z \in \mathbb{H}$. Then we have

$$1 + h(z) = \frac{2z}{z+i} \qquad \text{and} \qquad 1 - h(z) = \frac{2i}{z+i}. \tag{8}$$

Therefore

$$\frac{1+h(z)}{1-h(z)} = \frac{z}{i} \qquad \Leftrightarrow \qquad z = i\frac{1+h(z)}{1-h(z)}. \tag{9}$$

This quotient is well-defined since $h(z) \neq 1$ for $z \in \mathbb{H}$. Thus consider the mapping

$$g : \mathbb{E} \to \mathbb{C}, w \mapsto i\frac{1+w}{1-w}.$$

Now let $w \in \mathbb{E}$. Then

$$\begin{aligned}\operatorname{Im}\left(i\frac{1+w}{1-w}\right) &= \frac{1}{2i}\left[i\frac{1+w}{1-w} + i\frac{1+\overline{w}}{1-\overline{w}}\right] \\ &= \frac{1-|w|^2}{1-(w+\overline{w})+|w|^2} \\ &= \frac{1-|w|^2}{1-2\operatorname{Re}(w)+|w|^2}\end{aligned}$$

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$$\geq \frac{1-|w|^2}{(1+|w|)^2}$$
$$= \frac{1-|w|}{1+|w|}$$
$$> 0$$

since $|\mathrm{Re}(w)| \leq |w| < 1$. Hence $g(\mathbb{E}) \subseteq \mathbb{H}$. Furthermore, for $z \in \mathbb{H}$ and $w \in \mathbb{E}$ we have

$$g(h(z)) = i\frac{1+(z-i)/(z+i)}{1-(z-i)/(z+i)} = z \qquad \text{and} \qquad h(g(w)) = \frac{(1+w)/(1-w)-1}{(1+w)/(1-w)+1} = w.$$

Thus $g \circ h = \mathrm{id}_{\mathbb{H}}$ and $h \circ g = \mathrm{id}_{\mathbb{E}}$ which implies that g and h are bijective and inverse to each other, thus $g = h^{-1}$. ☐

Lemma 0.9. *It holds that $h \in \mathcal{O}(\mathbb{H})$ and $h^{-1} \in \mathcal{O}(\mathbb{E})$.*

Proof. h aswell as h^{-1} are well-defined rational functions, hence holomorphic in their respective domains. ☐

By lemma 0.7, 0.8 and 0.9 we conclude that the Cayley-map is biholomorphic.

Exercise 4. In both cases we use that a function $f : D \to \mathbb{C}$, where $D \subseteq \mathbb{C}$ is non-empty and open, is complex-differentiable in $c \in D$ if and only if f is real-differentiable in c and the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}(c) = \frac{\partial v}{\partial y}(c) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(c) = -\frac{\partial v}{\partial x}(c)$$

holds, where $f = u + iv$ (see [RS02, p. 47]). Furthermore, there is a usefull sufficiency criterion (see [RS02, p. 48]).

Lemma 0.10 (Sufficient Criterion for complex Differentiability). *If u and v have continuous partial derivatives in D, then $f := u + iv$ is real differentiable in D. If additionally the Cauchy Riemann equations hold for some subset $\hat{D} \subseteq D$, then f is complex differentiable in $\hat{D}$.*

Proof. The first statement follows immediately from the well know fact of multivariable calculus, that the existence and continuity of all partial derivatives is sufficient for real differentiability (see [Zor04, p. 457]), by considering $f : D \to \mathbb{R}^2$ defined by

$$f(x,y) = u(x,y) + iv(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

whereas the second statement follows from the above discussion. ☐

First we consider the function f. We can decompose $f = u + iv$ where $u, v : \mathbb{C} \to \mathbb{R}$ are defined by

$$u(x+iy) := x^3y^2 \qquad \text{and} \qquad v(x+iy) := x^2y^3. \tag{10}$$

Calculating the partial derivatives in $c := x + iy = (x,y)$ yields

$$\frac{\partial u}{\partial x}(x,y) = 3x^2y^2 \qquad \frac{\partial u}{\partial y}(x,y) = 2x^3y \qquad \frac{\partial v}{\partial x}(x,y) = 2xy^3 \qquad \frac{\partial v}{\partial y}(x,y) = 3x^2y^2.$$

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We immediately see that all partial derivatives are continuous on $\mathbb{R}^2$ and thus u and v are (continuously) real differentiable in $\mathbb{R}^2$ and so is f. Clearly

$$\frac{\partial u}{\partial x}(x,y) = 3x^2y^2 = \frac{\partial v}{\partial y}(x,y) \tag{11}$$

holds for any $x, y \in \mathbb{R}$ and thus for any $c \in \mathbb{C}$. However, for the second equation we get the requirement

$$\frac{\partial u}{\partial y}(x,y) = 2x^3y = -2xy^3 = -\frac{\partial v}{\partial x}(x,y). \tag{12}$$

or equivalently $x^3y = -xy^3$. If $x = 0$, then the equality holds for any $y \in \mathbb{R}$ and if $y = 0$ then the equality holds for any $x \in \mathbb{R}$. If $x, y \neq 0$ we get the equality $x^2 = -y^2$ which cannot be true since $y^2 > 0$ would imply $x^2 < 0$. Hence the set in which f is complex differentiable is exactly $\{z \in \mathbb{C} : \mathrm{Re}(z) = 0 \vee \mathrm{Im}(z) = 0\}$.
Now consider the function g. Similar to the previous part we can write $g = u + iv$ where $u, v : \mathbb{C} \to \mathbb{R}$ are defined by

$$u(x+iy) := e^x \cos y \qquad \text{and} \qquad v(x+iy) := e^x \sin y. \tag{13}$$

Calculating the partial derivatives yields

$$\begin{aligned}
\tfrac{\partial u}{\partial x}(x,y) &= e^x \cos y = u(x+iy) \\
\tfrac{\partial u}{\partial y}(x,y) &= -e^x \sin y = -v(x+iy) \\
\tfrac{\partial v}{\partial x}(x,y) &= e^x \sin y = v(x+iy) \\
\tfrac{\partial v}{\partial y}(x,y) &= e^x \cos y = u(x+iy)
\end{aligned}$$

which immediately implies that u and v are real differentiable on $\mathbb{R}^2$ (since again all partial derivatives are continuous on $\mathbb{R}^2$), hence f is real differentiable, and that the Cauchy-Riemann equations holds on $\mathbb{C}$. Thus g is holomorphic in $\mathbb{C}$ or *entire*.

References

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