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SOLUTIONS SHEET 5

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Exercise 1.
(a) We summarize the result in a lemma.

Lemma 0.1. *The power series*

$$\sum_{\nu=0}^{\infty} \frac{z^{2\nu}}{(2\nu)!} \qquad \text{and} \qquad \sum_{\nu=0}^{\infty} \frac{z^{2\nu+1}}{(2\nu+1)!} \tag{1}$$

have both radius of convergence $R = \infty$. Furthermore, for all $z \in \mathbb{C}$

$$\cosh z = \sum_{\nu=0}^{\infty} \frac{z^{2\nu}}{(2\nu)!} \qquad \text{and} \qquad \sinh z = \sum_{\nu=0}^{\infty} \frac{z^{2\nu+1}}{(2\nu+1)!} \tag{2}$$

holds.

Proof. Fix $z \in \mathbb{C}$. We have

$$\limsup_{\nu\to\infty} \left| \frac{z^{2\nu+2}}{(2\nu+2)!} \frac{(2\nu)!}{z^{2\nu}} \right| = |z|^2 \limsup_{\nu\to\infty} \frac{1}{(2\nu+2)(2\nu+1)} = 0 < 1 \tag{3}$$

and

$$\limsup_{\nu\to\infty} \left| \frac{z^{2\nu+3}}{(2\nu+3)!} \frac{(2\nu+1)!}{z^{2\nu+1}} \right| = |z|^2 \limsup_{\nu\to\infty} \frac{1}{(2\nu+3)(2\nu+2)} = 0 < 1. \tag{4}$$

Since z was arbitrary we conclude by the ratio test for series that both radii of convergence are ∞. Using the identities

$$\cosh z = \cos(iz) \qquad \text{and} \qquad \sinh z = -i\sin(iz) \qquad \forall z \in \mathbb{C} \tag{5}$$

and the definition of the trigonometric functions by series

$$\cos z := \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu)!} z^{2\nu} \qquad \text{and} \qquad \sin z := \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)!} z^{2\nu+1} \tag{6}$$

we get

$$\cosh z = \cos(iz) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu)!}(iz)^{2\nu} = \sum_{\nu=0}^{\infty} \frac{(-1)^{2\nu}}{(2\nu)!} z^{2\nu} = \sum_{\nu=0}^{\infty} \frac{z^{2\nu}}{(2\nu)!} \tag{7}$$

and

$$\sinh z = -i\sin(iz) = -i\sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)!}(iz)^{2\nu+1} = \sum_{\nu=0}^{\infty} \frac{(-1)^{2\nu}}{(2\nu+1)!} z^{2\nu+1} = \sum_{\nu=0}^{\infty} \frac{z^{2\nu+1}}{(2\nu+1)!} \tag{8}$$

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for all $z \in \mathbb{Z}$. □

Remark 0.1. *The power series given in lemma 0.1 can be rewritten into the standard form*

$$\sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu \tag{9}$$

by considering apropriate sequences $(a_\nu)_{\nu\in\mathbb{N}}$. Also it is clearly seen that $z_0 = 0$ is the point of expansion.

(b) Define $a_\nu := (-1)^{\nu-1}/\nu$ for $\nu \in \mathbb{N}$ and $a_0 := 0$. Since $(a_\nu)_{\nu\in\mathbb{N}}$ is convergent, the quotient criterion yields

$$R = \lim_{\nu\to\infty}\left|\frac{a_\nu}{a_{\nu+1}}\right| = \lim_{\nu\to\infty}\left|\frac{(-1)^{\nu-1}}{\nu}\frac{\nu+1}{(-1)^\nu}\right| = 1 + \lim_{\nu\to\infty}\frac{1}{\nu} = 1. \tag{10}$$

Thus the logarithmic series converges in $\mathbb{E}$ since the point of expansion z_0 is clearly 0. Since $R > 0$ we have that the limit function f is holomorphic in $\mathbb{E}$ by the theorem on the *interchangeability of differentiation and summation*. Furthermore, from the same theorem also follows that the derivative of the limit function coincides with the naive termwise differentiation of the power series within $\mathbb{E}$. Thus we get

$$f'(z) = \sum_{\nu=1}^{\infty} \nu a_\nu z^{\nu-1} = \sum_{\nu=1}^{\infty}(-z)^{\nu-1} = \sum_{\mu=0}^{\infty}(-z)^\mu = \frac{1}{1+z} \tag{11}$$

by the formula for the sum of a geometric series (if $z \in \mathbb{E}$ so is $-z \in \mathbb{E}$).

(c) Fix $z \in \mathbb{C}$ and let $a_\nu := (-1)^\nu/(2\nu+1)z^{2\nu+1}$ for $\nu \in \mathbb{N}_0$. By

$$\begin{aligned}\limsup_{\nu\to\infty}\left|\frac{a_{\nu+1}}{a_\nu}\right| &= \limsup_{\nu\to\infty}\left|\frac{(-1)^{\nu+1}z^{2\nu+3}}{2\nu+3}\frac{2\nu+1}{(-1)^\nu z^{2\nu+1}}\right| \\ &= |z|^2 \limsup_{\nu\to\infty}\frac{2\nu+1}{2\nu+3} \\ &= |z|^2\end{aligned}$$

we deduce that $|z|^2 < 1$ must hold that the series is convergent. This is equivalent to $z \in \mathbb{E}$. Thus the arcustangens series converges in $\mathbb{E}$ since the point of expansion z_0 is clearly 0. Since $R > 0$ we have that the limit function g is holomorphic in $\mathbb{E}$ by the theorem on the *interchangeability of differentiation and summation*. Furthermore, from the same theorem also follows that the derivative of the limit function coincides with the naive termwise differentiation of the power series within $\mathbb{E}$. First of all we have to bring the power series in an apropriate form. We have

$$g(z) = \sum_{\nu=0}^{\infty} a_\nu = \sum_{\nu=0}^{\infty} b_\nu z^\nu \qquad \text{where} \qquad b_\nu := \begin{cases} 0 & \nu \equiv 0 \bmod 2, \\ 1/\nu & \nu \equiv 1 \bmod 4, \\ -1/\nu & \nu \equiv 3 \bmod 4. \end{cases}$$

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Hence

$$g'(z) = \sum_{\nu=1}^{\infty} \nu b_\nu z^{\nu-1} = \sum_{\nu=0}^{\infty} (-1)^\nu z^{2\nu} = \sum_{\nu=0}^{\infty} (-z^2)^\nu = \frac{1}{1+z^2} \tag{12}$$

by the formula for the sum of a geometric series (if $z \in \mathbb{E}$ so is $-z^2 \in \mathbb{E}$).

Exercise 2.

(a) Define $\gamma_0 * \cdots * \gamma_n : I \to U$ where

$$I := \left[a_0, b_0 + \textstyle\sum_{\nu=1}^{n} (b_\nu - a_\nu)\right] \tag{13}$$

by

$$\gamma_0 * \cdots * \gamma_n(t) := \begin{cases} \gamma_0(t) & t \in A_0, \\ \gamma_1(t + a_1 - b_0) & t \in A_1, \\ \gamma_\nu\left(t + a_\nu - b_0 - \sum_{\mu=1}^{\nu-1}(b_\mu - a_\mu)\right) & t \in A_\nu, \nu = 2, \dots, n, \end{cases}$$

where

$$A_\nu := \begin{cases} [a_0, b_0] & \nu = 0, \\ [b_0, b_1 - a_1 + b_0] & \nu = 1, \\ \left[b_0 + \sum_{\mu=1}^{\nu-1}(b_\mu - a_\mu), b_0 + \sum_{\mu=1}^{\nu}(b_\mu - a_\mu)\right] & \nu = 2, \dots, n. \end{cases}$$

Let $n \in \mathbb{N}_{>0}$. Recall, that for $z_0, \dots, z_n$ the path $[z_0, \dots, z_n] : [0, n] \to \mathbb{C}$ defined by

$$[z_0, \dots, z_n](t) := z_\nu + (t - \nu)(z_{\nu+1} - z_\nu) \qquad t \in [\nu, \nu+1] \tag{14}$$

for $\nu = 0, \dots, n-1$ is called a ***polygon***. Consider the paths $\gamma_\nu := [z_\nu, z_{\nu+1}]$, $\nu = 0, \dots, n-1$. Then we have

$$I = \left[0, 1 + \textstyle\sum_{\nu=1}^{n-1} 1\right] = [0, n] \tag{15}$$

and

$$A_\nu = \begin{cases} [a_0, b_0] = [0, 1] & \nu = 0, \\ [b_0, b_1 - a_1 + b_0] = [1, 2] & \nu = 1, \\ \left[b_0 + \sum_{\mu=1}^{\nu-1}(b_\mu - a_\mu), b_0 + \sum_{\mu=1}^{\nu}(b_\mu - a_\mu)\right] = [\nu, \nu+1] & \nu = 2, \dots, n-1. \end{cases}$$

Hence $A_\nu = [\nu, \nu+1]$ for $\nu = 0, \dots, n-1$. Furthermore

$$\gamma_0 * \cdots * \gamma_{n-1}(t) = \begin{cases} \gamma_0(t) = [z_0, z_1] & t \in A_0, \\ \gamma_1(t + a_1 - b_0) = z_1 + (t-1)(z_2 - z_1) & t \in A_1, \end{cases}$$

and

$$\gamma_0 * \cdots * \gamma_{n-1}(t) = \gamma_\nu\left(t + a_\nu - b_0 - \textstyle\sum_{\mu=1}^{\nu-1}(b_\mu - a_\mu)\right) = z_\nu + (t - \nu)(z_{\nu+1} - z_\nu) \tag{16}$$

for $t \in A_\nu, \nu = 2, \dots, n-1$. Hence we conclude that

$$[z_0, \dots, z_n] = [z_0, z_1] * \cdots * [z_{n-1}, z_n]. \tag{17}$$

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(b) An integration path in U is by definition a piecewise continuously differentiable mapping. Hence there exists a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a, b]$ such that $\gamma|_{[t_\nu, t_{\nu+1}]}$ is continuously differentiable for $\nu = 0, \ldots, n-1$. Let $\gamma_\nu := \gamma|_{[t_\nu, t_{\nu+1}]}$ for $\nu = 0, \ldots, n-1$. Clearly

$$\gamma_\nu : [t_\nu, t_{\nu+1}] \to U \tag{18}$$

Using the terminology established in part (a) we get

$$I = \left[a, t_1 + \textstyle\sum_{\nu=1}^{n-1}(t_{\nu+1} - t_\nu)\right] = [a, t_n] = [a, b] \tag{19}$$

and

$$A_\nu = \begin{cases} [a_0, b_0] = [a, t_1] & \nu = 0, \\ [t_1, b_1 - a_1 + b_0] = [t_1, t_2] & \nu = 1, \\ \left[b_0 + \sum_{\mu=1}^{\nu-1}(b_\mu - a_\mu), b_0 + \sum_{\mu=1}^{\nu}(b_\mu - a_\mu)\right] = [t_\nu, t_{\nu+1}] & \nu = 2, \ldots, n-1. \end{cases}$$

Furthermore

$$\gamma_0 * \cdots * \gamma_{n-1}(t) = \begin{cases} \gamma_0(t) = \gamma(t) & t \in A_0, \\ \gamma_1(t + a_1 - b_0) = \gamma(t) & t \in A_1, \\ \gamma_\nu\left(t + a_\nu - b_0 - \sum_{\mu=1}^{\nu-1}(b_\mu - a_\mu)\right) = \gamma(t) & t \in A_\nu, \nu = 2, \ldots, n-1. \end{cases}$$

Hence we conclude

$$\gamma_0 * \cdots * \gamma_{n-1} = \gamma. \tag{20}$$

Exercise 3.