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SOLUTIONS SHEET 5

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Exercise 1.
(a) We summarize the result in a lemma.

Lemma 0.1. *The power series*

$$\sum_{\nu=0}^{\infty} \frac{z^{2\nu}}{(2\nu)!} \qquad \textit{and} \qquad \sum_{\nu=0}^{\infty} \frac{z^{2\nu+1}}{(2\nu+1)!} \tag{1}$$

have both radius of convergence $R = \infty$. Furthermore, for all $z \in \mathbb{C}$

$$\cosh z = \sum_{\nu=0}^{\infty} \frac{z^{2\nu}}{(2\nu)!} \qquad \textit{and} \qquad \sinh z = \sum_{\nu=0}^{\infty} \frac{z^{2\nu+1}}{(2\nu+1)!} \tag{2}$$

holds.

Proof. Fix $z \in \mathbb{C}$. We have

$$\limsup_{\nu\to\infty} \left| \frac{z^{2\nu+2}}{(2\nu+2)!} \frac{(2\nu)!}{z^{2\nu}} \right| = |z|^2 \limsup_{\nu\to\infty} \frac{1}{(2\nu+2)(2\nu+1)} = 0 < 1 \tag{3}$$

and

$$\limsup_{\nu\to\infty} \left| \frac{z^{2\nu+3}}{(2\nu+3)!} \frac{(2\nu+1)!}{z^{2\nu+1}} \right| = |z|^2 \limsup_{\nu\to\infty} \frac{1}{(2\nu+3)(2\nu+2)} = 0 < 1. \tag{4}$$

Since z was arbitrary we conclude by the ratio test for series that both radii of convergence are ∞. Using the identities

$$\cosh z = \cos(iz) \qquad \text{and} \qquad \sinh z = -i\sin(iz) \qquad \forall z \in \mathbb{C} \tag{5}$$

and the definition of the trigonometric functions by series

$$\cos z := \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu)!} z^{2\nu} \qquad \text{and} \qquad \sin z := \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)!} z^{2\nu+1} \tag{6}$$

we get

$$\cosh z = \cos(iz) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu)!} (iz)^{2\nu} = \sum_{\nu=0}^{\infty} \frac{(-1)^{2\nu}}{(2\nu)!} z^{2\nu} = \sum_{\nu=0}^{\infty} \frac{z^{2\nu}}{(2\nu)!} \tag{7}$$

and

$$\sinh z = -i\sin(iz) = -i \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(2\nu+1)!} (iz)^{2\nu+1} = \sum_{\nu=0}^{\infty} \frac{(-1)^{2\nu}}{(2\nu+1)!} z^{2\nu+1} = \sum_{\nu=0}^{\infty} \frac{z^{2\nu+1}}{(2\nu+1)!} \tag{8}$$

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for all $z \in \mathbb{Z}$. $\square$

Remark 0.1. *The power series given in lemma 0.1 can be rewritten into the standard form*

$$\sum_{\nu=0}^{\infty} a_\nu (z - z_0)^\nu \tag{9}$$

by considering apropriate sequences $(a_\nu)_{\nu \in \mathbb{N}}$. Also it is clearly seen that $z_0 = 0$ is the point of expansion.

(b) Define $a_\nu := (-1)^{\nu-1}/\nu$ for $\nu \in \mathbb{N}$ and $a_0 := 0$. Since $(a_\nu)_{\nu \in \mathbb{N}}$ is convergent, the quotient criterion yields

$$R = \lim_{\nu \to \infty} \left| \frac{a_\nu}{a_{\nu+1}} \right| = \lim_{\nu \to \infty} \left| \frac{(-1)^{\nu-1}}{\nu} \frac{\nu+1}{(-1)^\nu} \right| = 1 + \lim_{\nu \to \infty} \frac{1}{\nu} = 1. \tag{10}$$

Thus the logarithmic series converges in $\mathbb{E}$ since the point of expansion z_0 is clearly 0. Since $R > 0$ we have that the limit function f is holomorphic in $\mathbb{E}$ by the theorem on the *interchangeability of differentiation and summation*. Furthermore, from the same theorem also follows that the derivative of the limit function coincides with the naive termwise differentiation of the power series within $\mathbb{E}$. Thus we get

$$f'(z) = \sum_{\nu=1}^{\infty} \nu a_\nu z^{\nu-1} = \sum_{\nu=1}^{\infty} (-z)^{\nu-1} = \sum_{\mu=0}^{\infty} (-z)^\mu = \frac{1}{1+z} \tag{11}$$

by the formula for the sum of a geometric series (if $z \in \mathbb{E}$ so is $-z \in \mathbb{E}$).

(c) Fix $z \in \mathbb{C}$ and let $a_\nu := (-1)^\nu/(2\nu+1)z^{2\nu+1}$ for $\nu \in \mathbb{N}_0$. By

$$\begin{aligned} \limsup_{\nu \to \infty} \left| \frac{a_{\nu+1}}{a_\nu} \right| &= \limsup_{\nu \to \infty} \left| \frac{(-1)^{\nu+1} z^{2\nu+3}}{2\nu+3} \frac{2\nu+1}{(-1)^\nu z^{2\nu+1}} \right| \\ &= |z|^2 \limsup_{\nu \to \infty} \frac{2\nu+1}{2\nu+3} \\ &= |z|^2 \end{aligned}$$

we deduce that $|z|^2 < 1$ must hold that the series is convergent. This is equivalent to $z \in \mathbb{E}$. Thus the arcustangens series converges in $\mathbb{E}$ since the point of expansion z_0 is clearly 0. Since $R > 0$ we have that the limit function g is holomorphic in $\mathbb{E}$ by the theorem on the *interchangeability of differentiation and summation*. Furthermore, from the same theorem also follows that the derivative of the limit function coincides with the naive termwise differentiation of the power series within $\mathbb{E}$. First of all we have to bring the power series in an apropriate form. We have

$$g(z) = \sum_{\nu=0}^{\infty} a_\nu = \sum_{\nu=0}^{\infty} b_\nu z^\nu \qquad \text{where} \qquad b_\nu := \begin{cases} 0 & \nu \equiv 0 \bmod 2, \\ 1/\nu & \nu \equiv 1 \bmod 4, \\ -1/\nu & \nu \equiv 3 \bmod 4. \end{cases}$$

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Hence

$$g'(z) = \sum_{\nu=1}^{\infty} \nu b_\nu z^{\nu-1} = \sum_{\nu=0}^{\infty} (-1)^\nu z^{2\nu} = \sum_{\nu=0}^{\infty} (-z^2)^\nu = \frac{1}{1+z^2} \tag{12}$$

by the formula for the sum of a geometric series (if $z \in \mathbb{E}$ so is $-z^2 \in \mathbb{E}$).