

SOLUTIONS SHEET 3

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Exercise 1. Let $D \subseteq \mathbb{C}$ be non-empty and open in \mathbb{C} and $f_1, f_2 : D \rightarrow \mathbb{C}$ be real differentiable. Fix some $z_0 \in D$. Since f_1 and f_2 are real differentiable in z_0 there exists $\varphi_1, \varphi_2, \psi_1, \psi_2 : D \rightarrow \mathbb{C}$ continuous at z_0 such that

$$f_1(z) = f_1(z_0) + (z - z_0)\varphi_1(z) + (\bar{z} - \bar{z}_0)\psi_1(z) \quad (1)$$

$$f_2(z) = f_2(z_0) + (z - z_0)\varphi_2(z) + (\bar{z} - \bar{z}_0)\psi_2(z) \quad (2)$$

for all $z \in D$.

(i) Let $a, b \in \mathbb{C}$. Multiplying (1) by a , (2) by b and adding both equations yields

$$af_1(z) + bf_2(z) = af_1(z_0) + bf_2(z_0) + (z - z_0)(a\varphi_1(z) + b\varphi_2(z)) + (\bar{z} - \bar{z}_0)(a\psi_1(z) + b\psi_2(z)) \quad (3)$$

for all $z \in D$. Clearly, $a\varphi_1 + b\varphi_2$ and $a\psi_1 + b\psi_2$ are continuous functions in z_0 and from (3) we deduce

$$\frac{\partial(af_1 + bf_2)}{\partial z}(z_0) = a \frac{\partial f_1}{\partial z}(z_0) + b \frac{\partial f_2}{\partial z}(z_0) \quad (4)$$

and

$$\frac{\partial(af_1 + bf_2)}{\partial \bar{z}}(z_0) = a \frac{\partial f_1}{\partial \bar{z}}(z_0) + b \frac{\partial f_2}{\partial \bar{z}}(z_0). \quad (5)$$

Since $z_0 \in D$ was arbitrary, we conclude

$$\frac{\partial(af_1 + bf_2)}{\partial z} = a \frac{\partial f_1}{\partial z} + b \frac{\partial f_2}{\partial z} \quad \text{and} \quad \frac{\partial(af_1 + bf_2)}{\partial \bar{z}} = a \frac{\partial f_1}{\partial \bar{z}} + b \frac{\partial f_2}{\partial \bar{z}}. \quad (6)$$

(ii) Multiplying (1) and (2) yields

$$\begin{aligned} f_1 f_2 &= f_1(z_0)f_2(z_0) + (z - z_0)[\varphi_1 f_2(z_0) + f_1(z_0)\varphi_2 + (z - z_0)\varphi_1\varphi_2 + (\bar{z} - \bar{z}_0)\psi_1\varphi_2] \\ &\quad + (\bar{z} - \bar{z}_0)[\psi_1 f_2(z_0) + f_1(z_0)\psi_2 + (z - z_0)\psi_2\varphi_1 + (\bar{z} - \bar{z}_0)\psi_1\psi_2] \end{aligned}$$

where the argument z is omitted. Clearly, the two functions in the square brackets are continuous at z_0 and evaluating them at z_0 yields

$$\frac{\partial(f_1 f_2)}{\partial z}(z_0) = \frac{\partial f_1}{\partial z}(z_0)f_2(z_0) + f_1(z_0)\frac{\partial f_2}{\partial z}(z_0) \quad (7)$$

and

$$\frac{\partial(f_1 f_2)}{\partial \bar{z}}(z_0) = \frac{\partial f_1}{\partial \bar{z}}(z_0)f_2(z_0) + f_1(z_0)\frac{\partial f_2}{\partial \bar{z}}(z_0). \quad (8)$$

Since $z_0 \in D$ was arbitrary, we conclude

$$\frac{\partial(f_1 f_2)}{\partial z} = \frac{\partial f_1}{\partial z} f_2 + f_1 \frac{\partial f_2}{\partial z} \quad \text{and} \quad \frac{\partial(f_1 f_2)}{\partial \bar{z}} = \frac{\partial f_1}{\partial \bar{z}} f_2 + f_1 \frac{\partial f_2}{\partial \bar{z}}. \quad (9)$$

(iii) Conjugating (1) yields

$$\overline{f_1}(z) = \overline{f_1}(z_0) + (\bar{z} - \bar{z}_0)\overline{\varphi_1}(z) + (z - z_0)\overline{\psi_1}(z). \quad (10)$$

From (10) we deduce

$$\frac{\partial \overline{f_1}}{\partial \bar{z}}(z_0) = \overline{\varphi_1}(z_0) = \frac{\partial \overline{f_1}}{\partial z}(z_0) \quad (11)$$

since φ_1 and ψ_1 are also continuous at z_0 . Taking conjugates in (11) and use that $z_0 \in D$ was arbitrary finally yields

$$\overline{\frac{\partial \overline{f_1}}{\partial \bar{z}}} = \frac{\partial f_1}{\partial z}. \quad (12)$$

(iv) This follows directly from

$$z = z_0 + (z - z_0) \quad \text{and} \quad \bar{z} = \bar{z}_0 + (\bar{z} - \bar{z}_0). \quad (13)$$

(v) See separate sheet.

(vi) See separate sheet.

(vii) Let $t_0 \in I$. The function $\varphi : I \rightarrow U \subseteq \mathbb{C}$ is differentiable if and only if there exists a function $\varphi_1 : I \rightarrow U$ which is continuous at t_0 and such that

$$\varphi(t) = \varphi(t_0) + (t - t_0)\varphi_1(t) \quad (14)$$

for all $t \in I$ (this was proven in Analysis I). Furthermore there exists $f_1, f_2 : U \rightarrow \mathbb{C}$ continuous at $\varphi(t_0)$ such that

$$f(z) = f(\varphi(t_0)) + (z - \varphi(t_0))f_1(z) + (\bar{z} - \overline{\varphi(t_0)})f_2(z) \quad (15)$$

for all $z \in U$. Combining (14) and (15) yields

$$\begin{aligned} f(\varphi(t)) &= f(\varphi(t_0)) + (\varphi(t) - \varphi(t_0))f_1(z) + (\overline{\varphi(t)} - \overline{\varphi(t_0)})f_2(z) \\ &= f(\varphi(t_0)) + (t - t_0) [\varphi_1(t)f_1(\varphi(t)) + \overline{\varphi_1(t)}f_2(\varphi(t))] \end{aligned}$$

for all $t \in I$. Again, $\varphi_1(t)f_1(\varphi(t)) + \overline{\varphi_1(t)}f_2(\varphi(t))$ is clearly continuous at t_0 since composed functions are and thus we conclude

$$\frac{d(f \circ \varphi)}{dt}(t_0) = \frac{d\varphi}{dt}(t_0) \frac{\partial f}{\partial z}(\varphi(t_0)) + \frac{d\overline{\varphi}}{dt}(t_0) \frac{\partial f}{\partial \bar{z}}(\varphi(t_0)). \quad (16)$$

Since $t_0 \in I$ was arbitrary we conclude

$$\frac{d(f \circ \varphi)}{dt} = \frac{d\varphi}{dt} \frac{\partial f}{\partial z} + \frac{d\overline{\varphi}}{dt} \frac{\partial f}{\partial \bar{z}}. \quad (17)$$

Exercise 2. We show the implications (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (i). Since the proofs are of a relatively simple nature, we focus on the formal part. The complex numbers \mathbb{C} are a vector space over \mathbb{R} (as a field extension). So the situation of the exercise can be summarized by the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{T} & \mathbb{C} \\ \uparrow \Phi_B & & \uparrow \Phi_B \\ \mathbb{R}^2 & \xrightarrow{M_B(T)} & \mathbb{R}^2 \end{array}$$

where T is \mathbb{R} -linear, Φ_B denotes the basis-isomorphism which is in this case given by $\Phi_B(x, y) := x + iy$ and $M_B(T)$ is defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \quad (18)$$

The first implication is evident by the definition of \mathbb{C} -linearity. Assume that (ii) holds. By

$$T(i) = (\Phi_B \circ M_B(T) \circ \Phi_B^{-1})(i) = b + id \quad (19)$$

and

$$iT(1) = i(\Phi_B \circ M_B(T) \circ \Phi_B^{-1})(1) = i(a + ic) = -c + ia \quad (20)$$

we get the requirement $b + id = -c + ia$. Hence $b = -c$ and $a = d$. Assume that (iii) holds. Then we have for $z := x + iy \in \mathbb{C}$

$$T(z) = (\Phi_B \circ M_B(T) \circ \Phi_B^{-1})(x + iy) = (ax - cy) + i(cx + ay) = (a + ic)z. \quad (21)$$

Finally, assume that (iv) holds. Then T is clearly \mathbb{C} -linear since

$$T(\lambda z + w) = (a + ic)(\lambda z + w) = \lambda(a + ic)z + (a + ic)w = \lambda T(z) + T(w) \quad (22)$$

for $\lambda, z, w \in \mathbb{C}$.