SOLUTIONS SHEET 6

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Exercise 1.

(a) Let $r \in \mathbb{R}_{>0} \setminus \{1\}$. Partial fraction decomposition yields

$$\int_{\partial B_r(0)} \frac{2\zeta - 1}{\zeta(\zeta - 1)} \,\mathrm{d}\zeta = \int_{\partial B_r(0)} \frac{\mathrm{d}\zeta}{\zeta} + \int_{\partial B_r(0)} \frac{\mathrm{d}\zeta}{\zeta - 1} = \begin{cases} 4\pi i & 1 < r, \\ 2\pi i & 0 < r < 1. \end{cases} \tag{1}$$

(b) The solution of this exercise is based on the following proposition (see [RS02, p. 165]).

Proposition 0.1. *Let $D \subseteq \mathbb{C}$ be open and $f \in \mathscr{C}(D)$. For a function $F : D \to \mathbb{C}$ we have that F is holomorphic in D and $F' = f$ if and only if for all paths γ in D we have*

$$\int_\gamma f \,\mathrm{d}\zeta = F(\gamma(b)) - F(\gamma(a)). \tag{2}$$

(i) Clearly $f_1 \in \mathscr{C}(\mathbb{C})$ as a composition of continuous functions. Define $F_1 : \mathbb{C} \to \mathbb{C}$ by

$$F_1(z) := \frac{1}{1+i} \sin\left((1+i)z\right). \tag{3}$$

F_1 is clearly entire since

$$F_1'(z) = \frac{1}{1+i} \sin'\left((1+i)z\right) = \cos\left((1+i)z\right) = f_1(z). \tag{4}$$

exists for all $z \in \mathbb{C}$ since

$$\sin'(z) = \frac{1}{2i}\left((e^{iz})' - (e^{-iz})'\right) = \frac{1}{2i}\left(ie^{iz} + ie^{-iz}\right) = \cos(z). \tag{5}$$

Hence for any path γ starting at z_0 and ending at z_1 we have

$$\int_\gamma f_1 \,\mathrm{d}\zeta = F(2i) - F(1+i) = \frac{1}{1+i}\left(\sin(2i - 2) - \sin(2i)\right). \tag{6}$$

(ii) Again clearly $f_2 \in \mathscr{C}(\mathbb{C}^\times)$. Define $F_2 : \mathbb{C}^\times \to \mathbb{C}$ by

$$F_2(z) := \frac{i}{3} z^3 + z + 2i\frac{1}{z}. \tag{7}$$

Then we have $F_2' = f_2$ and so

$$\int_\gamma f_2 \,\mathrm{d}\zeta = F_2(2i) - F_2(1+i) = \frac{7}{3} + \frac{2}{3} i. \tag{8}$$

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(iii) Again $f_3 \in \mathscr{C}(\mathbb{C} \setminus \{-1\})$. Define $F_3 : \mathbb{C} \setminus \{-1\} \to \mathbb{C}$ by

$$F_3(z) := -\frac{1}{2}\frac{1}{(z+1)^2}. \tag{9}$$

Clearly $F_3' = f_3$ and so

$$\int_\gamma f_3 \,\mathrm{d}\zeta = F_3(2i) - F_3(1+i) = \frac{3}{25}. \tag{10}$$

(iv) Again $f_4 \in \mathscr{C}(\mathbb{C})$. Define $F_4 : \mathbb{C} \to \mathbb{C}$ by

$$F_4(z) := \frac{1}{2i}e^{iz^2}. \tag{11}$$

Clearly $F_4' = f_4$ and thus we get

$$\int_\gamma f_4 \,\mathrm{d}\zeta = F_4(2i) - F_4(1+i) = \frac{1}{2i}\left(e^{-4i} - e^{-2}\right). \tag{12}$$

Exercise 2.

Proposition 0.2 (Zentrierungslemma for Rectangles). *Let $D \subseteq \mathbb{C}$ be open and $f : D \to \mathbb{C}$ holomorphic in D. Furthermore let $R \subseteq D$ be a rectangle in D such that $\overline{R} \subseteq D$. Let $z \in R$. If $B_r(z) \subseteq R$, we have*

$$\int_{\partial R} f \,\mathrm{d}\zeta = \int_{\partial B_r(z)} f \,\mathrm{d}\zeta. \tag{13}$$

Proof. We make use of the labeling on the separate sheet. Clearly

$$\partial R = [z_0, z_1, z_2, z_3, z_0] \qquad \text{and} \qquad B_r(z) = \alpha + \beta. \tag{14}$$

Define

$$\begin{aligned}
\gamma_1 &:= [w_3, z_1] + [z_1, z_2] + [z_2, w_0] + [w_0, w_1] - \alpha + [w_2, w_3],\\
\gamma_2 &:= [z_0, w_3] - [w_2, w_3] - \beta - [w_0, w_1] + [w_0, z_3] + [z_3, z_0].
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{\gamma_1+\gamma_2} f \,\mathrm{d}\zeta &= \int_{[z_0,w_3]+[w_3,z_1]+[z_1,z_2]+[z_2,w_0]+[w_0,z_3]+[z_3,z_0]} f \,\mathrm{d}\zeta - \int_{\alpha+\beta} f \,\mathrm{d}\zeta\\
&= \int_{\partial R} f \,\mathrm{d}\zeta - \int_{\partial B_r(z)} f \,\mathrm{d}\zeta.
\end{aligned}$$

Since $\overline{R} \subseteq D$, there exists a rectangle R' with $\overline{R} \subseteq R' \subseteq D$. Clearly a rectangle is a star-shaped domain with any center since it is convex. Hence the Cauchy integral theorem for star-shaped domains implies that

$$\int_{\gamma_1} f \,\mathrm{d}\zeta = 0 \qquad \text{and} \qquad \int_{\gamma_2} f \,\mathrm{d}\zeta = 0 \tag{15}$$

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since γ_1 and γ_2 are closed. Thus

$$\int_{\partial R} f \,\mathrm{d}\zeta - \int_{\partial B_r(z)} f \,\mathrm{d}\zeta = \int_{\gamma_1+\gamma_2} f \,\mathrm{d}\zeta = \int_{\gamma_1} f \,\mathrm{d}\zeta + \int_{\gamma_2} f \,\mathrm{d}\zeta = 0. \tag{16}$$

This implies

$$\int_{\partial R} f \,\mathrm{d}\zeta = \int_{\partial B_r(z)} f \,\mathrm{d}\zeta. \tag{17}$$

□

Theorem 0.1 (Cauchy Integral Formula for Rectangles). *Let $f : D \to \mathbb{C}$ be holomorphic in D and let R be a rectangle in R such that $\overline{R} \subseteq D$. Then we have for any $z \in R$:*

$$f(z) = \frac{1}{2\pi i} \int_{\partial R} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta. \tag{18}$$

Proof. Let $z \in R$. Define $g : D \to \mathbb{C}$ by

$$g(\zeta) := \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \zeta \in D \setminus \{z\}, \\ f'(z) & \zeta = z. \end{cases} \tag{19}$$

Then g is holomorphic in $D\setminus\{z\}$ and continuous at z. Now we find $\overline{B_r}(z) \subseteq R$. Since $\overline{B_r}(z)$ is compact we have that $|g| : \overline{B_r}(z) \to \mathbb{R}$ is bounded, say $|g|_{\overline{B_r}(z)} \leq M$. Fix $0 < \varepsilon < r$. Then the standard estimate and proposition 0.2 yields

$$\left|\int_{\partial R} g \,\mathrm{d}\zeta\right| = \left|\int_{\partial B_\varepsilon(z)} g \,\mathrm{d}\zeta\right| \leq |g|_{\partial B_\varepsilon(z)} 2\pi\varepsilon \leq |g|_{B_r(z)} 2\pi\varepsilon \leq 2\pi M\varepsilon. \tag{20}$$

Hence

$$\int_{\partial R} g \,\mathrm{d}\zeta = 0. \tag{21}$$

Using again proposition 0.2 we find

$$\int_{\partial R} \frac{\mathrm{d}\zeta}{\zeta - z} = 2\pi i. \tag{22}$$

Putting (21) and (22) together yields

$$0 = \int_{\partial R} g \,\mathrm{d}\zeta = \int_{\partial R} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta - f(z) \int_{\partial R} \frac{\mathrm{d}\zeta}{\zeta - z} = \int_{\partial R} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta - 2\pi i f(z). \tag{23}$$

□

Exercise 3.
(i) Consider the function $f : U \to \mathbb{C}$ defined by

$$f(z) := \frac{(z-2)(z^7+1)}{z^2(z^4+1)} \tag{24}$$

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where U is $\mathbb{C}$ without the roots of the denominator. The roots are given by $0, e^{\pm i\pi/4}$ and $e^{\pm i3\pi/4}$. Hence $\overline{B_1}(2) \subseteq U$. Since f is holomorphic in U as a well-defined rational function, the Cauchy integral formula yields

$$\int_{\partial B_1(2)} \frac{z^7+1}{z^2(z^4+1)}\,\mathrm{d}z = \int_{\partial B_1(2)} \frac{f(z)}{z-2}\,\mathrm{d}z = 2\pi i f(2) = 0. \tag{25}$$

(ii)

(iii)

(iv) Partial fraction decomposition yields

$$\int_{\partial B_3(0)} \frac{\cos(\pi z)}{z^2-1}\,\mathrm{d}z = \frac{1}{2}\left[\int_{\partial B_3(0)} \frac{\cos(\pi z)}{z-1}\,\mathrm{d}z - \int_{\partial B_3(0)} \frac{\cos(\pi z)}{z+1}\,\mathrm{d}z\right] \tag{26}$$

Now $f(z) := \cos(\pi z)$ is entire, and since $\pm 1 \in B_3(0)$ we get

$$\int_{\partial B_3(0)} \frac{\cos(\pi z)}{z^2-1}\,\mathrm{d}z = \pi i\,[f(1) - f(-1)] = 0. \tag{27}$$

Exercise 4.
(a) Partial fraction decomposition yields for $z \in \mathbb{C} \setminus \overline{\mathbb{E}}$ fixed

$$f(z) = -\frac{1}{2\pi i z}\left[\int_{\partial\mathbb{E}} \frac{\mathrm{d}\zeta}{\zeta} - \int_{\partial\mathbb{E}} \frac{\mathrm{d}\zeta}{\zeta - z}\right] = -\frac{1}{2\pi i z}2\pi i = -\frac{1}{z}. \tag{28}$$

(b) This can directly be copied from my solution to exercise 2 on sheet 4 with slight improvements.

Lemma 0.1. *For $z \in \mathbb{C} \setminus \{1\}$ and $k \in \mathbb{N}_0$ we have*

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\frac{1}{1-z} = \frac{k!}{(1-z)^{k+1}}. \tag{29}$$

Proof. Proof by induction on $k \in \mathbb{N}_0$. The statement obviously holds for $k = 0$. Assume the statement holds for some $k \in \mathbb{N}_0$. Then we get

$$\begin{aligned}
\frac{\mathrm{d}^{k+1}}{\mathrm{d}z^{k+1}}\frac{1}{1-z} &= \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\mathrm{d}^k}{\mathrm{d}z^k}\frac{1}{1-z}\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\frac{k!}{(1-z)^{k+1}} \\
&= k!\frac{(k+1)(1-z)^k}{(1-z)^{2k+2}} \\
&= \frac{(k+1)!}{(1-z)^{k+2}}.
\end{aligned}$$

□

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The geometric series $\sum_{\nu=0}^{\infty} z^\nu$ converges for all $z \in \mathbb{E}$. Hence by the theorem on the interchangeability of differentiation and summation [RS02, p. 110] we have that the limit function is differentiable within the radius of convergence (here $R = 1$) and the k-th derivative is given by

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}\frac{1}{1-z} = \frac{\mathrm{d}^k}{\mathrm{d}z^k}\sum_{\nu=0}^{\infty} z^\nu = \sum_{\nu\geq k} k!\binom{\nu}{k} z^{\nu-k} \qquad k \in \mathbb{N}_0, z \in \mathbb{E}. \tag{30}$$

References

[RS02] R. Remmert and G. Schumacher. *Funktionentheorie 1*. Springer-Lehrbuch. Springer Berlin Heidelberg, 2002. ISBN: 9783540418559.