

## SOLUTIONS SHEET 2

YANNIS BÄHNI

**Exercise 1.** Recall, that  $\mathcal{Z}$  is also called *finite complement topology* (see [Lee11, p. 45]).

(i) We have already showed that  $(\mathbb{C}, \mathcal{Z})$  is not a Hausdorff space (see sheet 1 exercise 3), hence not compact. Therefore it is enough to show that any open cover of  $(\mathbb{C}, \mathcal{Z})$  has a finite subcover.

**Lemma 0.1.**  $(\mathbb{C}, \mathcal{Z})$  is quasi-compact.

*Proof.* Let  $(U_\alpha)_{\alpha \in A}$  be an open cover of  $(\mathbb{C}, \mathcal{Z})$ , i.e.

$$\mathbb{C} = \bigcup_{\alpha \in A} U_\alpha \quad \text{and} \quad \forall \alpha \in A : U_\alpha \in \mathcal{Z}. \quad (1)$$

We can explicitly construct a finite subcover. Pick some  $\alpha_0 \in A$  such that  $U_{\alpha_0} \neq \emptyset$ . Since  $U_{\alpha_0} \in \mathcal{Z}$ ,  $U_{\alpha_0}^c$  is finite, i.e.  $U_{\alpha_0}^c = \{z_1, \dots, z_n\} \subseteq \mathbb{C}$ . Thus we can write

$$\mathbb{C} = U_{\alpha_0} \cup U_{\alpha_0}^c = U_{\alpha_0} \cup \{z_1, \dots, z_n\}. \quad (2)$$

Since  $\mathbb{C} = \bigcup_{\alpha \in A} U_\alpha$ , we find  $\alpha_i \in A$  for  $i = 1, \dots, n$  such that  $z_i \in U_{\alpha_i}$ . Hence  $(U_{\alpha_\nu})_{\nu \in \{0, \dots, n\}}$  is a finite subcover of  $(U_\alpha)_{\alpha \in A}$ . Since the construction was general, we conclude that  $(\mathbb{C}, \mathcal{Z})$  is quasi-compact.  $\square$

(ii) The reasoning is similar to part i).

**Lemma 0.2.**  $(\{z_0\}^c, \{z_0\}^c \cap \mathcal{Z})$  is quasi-compact.

*Proof.* Let  $(U_\alpha)_{\alpha \in A}$  be an open cover of  $(\{z_0\}^c, \{z_0\}^c \cap \mathcal{Z})$ , i.e.

$$\{z_0\}^c = \bigcup_{\alpha \in A} U_\alpha \quad \text{and} \quad \forall \alpha \in A : U_\alpha \in \{z_0\}^c \cap \mathcal{Z}. \quad (3)$$

We can explicitly construct a finite subcover. Pick some  $\alpha_0 \in A$  such that  $U_{\alpha_0} \neq \emptyset$ . Since  $U_{\alpha_0} \in \{z_0\}^c \cap \mathcal{Z}$ , there exists  $V \in \mathcal{Z}$  such that  $U_{\alpha_0} = \{z_0\}^c \cap V$ . By considering the relative complement

$$U_{\alpha_0}^c = \{z_0\}^c \cap (\{z_0\}^c \cap V)^c = \{z_0\}^c \cap (\{z_0\} \cup V^c) = \{z_0\}^c \cap V^c \subseteq V^c \quad (4)$$

and using the fact that  $V^c$  is finite we conclude that  $U_{\alpha_0}^c$  is finite, i.e.  $U_{\alpha_0}^c = \{z_1, \dots, z_n\} \subseteq \{z_0\}^c$ . Thus we can write

$$\{z_0\}^c = U_{\alpha_0} \cup U_{\alpha_0}^c = U_{\alpha_0} \cup \{z_1, \dots, z_n\}. \quad (5)$$

Since  $\{z_0\}^c = \bigcup_{\alpha \in A} U_\alpha$ , we find  $\alpha_i \in A$  for  $i = 1, \dots, n$  such that  $z_i \in U_{\alpha_i}$ . Hence  $(U_{\alpha_\nu})_{\nu \in \{0, \dots, n\}}$  is a finite subcover of  $(U_\alpha)_{\alpha \in A}$ . Since the construction was general, we conclude that  $(\{z_0\}^c, \{z_0\}^c \cap \mathcal{Z})$  is quasi-compact.  $\square$

**Lemma 0.3.**  $(\{z_0\}^c, \{z_0\}^c \cap \mathcal{Z})$  is not Hausdorff.

*Proof.* Towards a contradiction assume that  $(\{z_0\}^c, \{z_0\}^c \cap \mathcal{Z})$  is Hausdorff. Thus for  $p, q \in \{z_0\}^c$  there exists open neighbourhoods  $U$  and  $V$  of  $p$  and  $q$  respectively such that  $U \cap V = \emptyset$ . From the latter it follows that  $U \subseteq V^c$ . Since  $V$  is open we find  $W_1 \in \mathcal{Z}$  such that  $V = \{z_0\}^c \cap W_1$ . Hence taking relative complements yields

$$V^c = \{z_0\}^c \cap (\{z_0\}^c \cap W_1)^c = \{z_0\}^c \cap W_1^c \subseteq W_1^c$$

So  $V^c$  is finite and therefore also  $U$ . Since  $U$  is open we have that there exists  $W_2 \in \mathcal{Z}$  such that  $U = \{z_0\}^c \cap W_2$ . Taking again relative complements yields

$$U^c = \{z_0\}^c \cap (\{z_0\}^c \cap W_2)^c = \{z_0\}^c \cap W_2^c \subseteq W_2^c$$

So  $U^c$  is also finite. Therefore the decomposition  $\{z_0\}^c = U \cup U^c$  implies that  $\{z_0\}^c$  is finite. Contradiction, since  $|\{z_0\}^c| \geq |\mathbb{R}| = \mathfrak{c}$ , which is clearly not finite.  $\square$

Therefore by lemma 0.2 we conclude that  $(\{z_0\}^c, \{z_0\}^c \cap \mathcal{Z})$  is quasi-compact, but from lemma 0.3 follows that  $(\{z_0\}^c, \{z_0\}^c \cap \mathcal{Z})$  is not compact.

(iii) By  $(\{z_0\}^c)^c = \{z_0\}$  which is finite immediately follows  $\{z_0\}^c \in \mathcal{Z}$ . But  $\{z_0\}^c = \mathbb{C} \setminus \{z_0\}$  is clearly not finite, thus  $\{z_0\} \notin \mathcal{Z}$ , hence  $\{z_0\}^c$  cannot be closed.

**Exercise 2.** Let  $G \subseteq \mathbb{C}$  be non-empty and open. We show the equivalences (i)  $\Leftrightarrow$  (ii) and (ii)  $\Leftrightarrow$  (iii). This is due to the fact that I am aware of the latter equivalence by considering [Lee11, p. 86] and the first one by [Lee11, p. 90] and the fact that every open connected subset of  $\mathbb{R}^n$  is path-connected. However, working out detailed and appropriate proofs is still a lot of work.

Assume that (i) holds. Proof by contradiction. Let  $G = G_1 \cup G_2$  for some open sets  $G_1, G_2 \subseteq \mathbb{C}$  with  $G_1 \cap G_2 = \emptyset$  and  $G_1, G_2 \neq G$ . Evidently  $G_1, G_2 \neq \emptyset$  and thus we find  $p \in G \cap G_1, q \in G \cap G_2$ . Let  $\gamma : [a, b] \rightarrow G$  be a path joining  $p$  and  $q$ , i.e.  $\gamma(a) = p$  and  $\gamma(b) = q$ . Since  $\gamma$  is continuous,  $G \cap G_1$  and  $G \cap G_2$  are relatively open in  $G$  we have that  $\gamma^{-1}(G \cap G_1)$  and  $\gamma^{-1}(G \cap G_2)$  are open in  $[a, b]$ . Furthermore, since  $a \in \gamma^{-1}(G \cap G_1)$  and  $b \in \gamma^{-1}(G \cap G_2)$  we have that both preimages are non-empty. By

$$\gamma^{-1}(G \cap G_1) \cup \gamma^{-1}(G \cap G_2) = \gamma^{-1}((G \cap G_1) \cup (G \cap G_2)) = \gamma^{-1}(G) = [a, b]$$

and

$$\gamma^{-1}(G \cap G_1) \cap \gamma^{-1}(G \cap G_2) = \gamma^{-1}((G \cap G_1) \cap (G \cap G_2)) = \gamma^{-1}(\emptyset) = \emptyset$$

we have that  $\gamma^{-1}(G \cap G_1)$  and  $\gamma^{-1}(G \cap G_2)$  disconnect  $[a, b]$  which is impossible since a real interval is always connected (see [Lee11, p. 89]).

Now assume that (ii) holds. Let  $z_0 \in G$ . Since joinability by paths in  $G$  is an equivalence relation (let us denote it simply by  $\sim$ ) define

$$G_1 := [z_0]_{\sim}. \tag{6}$$

**Lemma 0.4.**  $G_1$  is open.

*Proof.* Let  $z_1 \in G_1$ . Since  $G$  is open we find  $\varepsilon > 0$  such that  $B_\varepsilon(z_1) \subseteq G$ .  $B_\varepsilon(z_1)$  is evidently path connected (consider just straight lines joining different points). Since  $z_1 \in G_1$ , we have that there is a path joining  $z_0$  and  $z_1$ . By concatenating paths, there is a path from  $z_0$  to every point in  $B_\varepsilon(z_1)$  and thus  $B_\varepsilon(z_1) \subseteq G_1$ .  $\square$

**Lemma 0.5.** The relative complement  $G_1^c$  in  $G$  is open.

*Proof.* Let  $z_1 \in G_1^c$ . Again we find  $\varepsilon > 0$  such that  $B_\varepsilon(z_1) \subseteq G$  by the openness of  $G$ . Towards a contradiction assume that  $B_\varepsilon(z_1) \cap G_1 \neq \emptyset$ . Hence we find  $z_2 \in B_\varepsilon(z_1) \cap G_1$ . This means, that there is a path joining  $z_0$  and  $z_2$ . But since  $B_\varepsilon(z_1)$  is path connected there would be a path joining  $z_1$  and  $z_0$  which yields a contradiction. Hence  $B_\varepsilon(z_1) \subseteq G_1^c$ .  $\square$

Since evidently  $G = G_1 \cup G_1^c$  and by lemma 0.4 and 0.5  $G_1, G_1^c$  are open and clearly disjoint, (ii) implies that either  $G_1 = G$  or  $G_1^c = G$ . The latter is impossible since  $z_0 \notin G_1^c$ . Hence we conclude that  $G = G_1$ . Thus  $G$  is a single equivalence class under joinability by paths, hence path-connected.

Next we show that (ii)  $\Rightarrow$  (iii). To be completely rigorous, we state the following lemma which can be found as an exercise in [Lee11, p. 50].

**Lemma 0.6.** Let  $(X, \mathcal{T})$  be a topological space and  $S \subseteq X$ . Then  $B \subseteq S$  is closed in  $S$  if and only if  $B = S \cap A$  for some closed set  $A$  in  $X$ .

*Proof.* Assume  $B \subseteq S$  is closed in  $S$ . Hence the relative complement  $B^c$  is open in  $S$ . Therefore we have  $B^c = S \cap U$  for some open set  $U$  in  $X$ . Thus

$$B = (S \cap U)^c = S \cap (S \cap U)^c = S \cap (S^c \cup U^c) = (S \cap S^c) \cup (S \cap U^c) = S \cap U^c.$$

But since  $U$  is open in  $X$  we have that  $U^c$  is closed in  $X$ . Conversely assume that  $B = S \cap A$  for some closed set  $A$  in  $X$ . Taking relative complements yields

$$B^c = (S \cap A)^c = S \cap (S \cap A)^c = S \cap A^c$$

and since  $A$  is closed in  $X$  we have that  $A^c$  is open in  $X$  which means  $B^c$  is open in  $S$ .  $\square$

Now assume that (ii) holds. Let  $U \subseteq G$  be a non-empty, open and relatively closed (with respect to  $G$ ) subset. Since  $U$  is relatively closed, by lemma 0.6 there exists a closed set  $A \subseteq \mathbb{C}$  such that  $U = G \cap A$ . Observe, that by

$$U^c = G \cap (G \cap A)^c = G \cap (G^c \cup A^c) = G \cap A^c$$

$U^c$  is open in  $\mathbb{C}$  since  $G$  and  $A^c$  are open in  $\mathbb{C}$ . Clearly,  $G = U \cup U^c$  and  $U \cap U^c = \emptyset$ . Therefore (ii) implies that either  $U = G$  or  $U^c = G$  where the latter is impossible since by assumption  $U \neq \emptyset$ . Hence we conclude that  $U = G$ .

Finally we show (iii)  $\Rightarrow$  (ii). This is equivalent to showing that not (ii) implies not (iii). So we have  $G = G_1 \cup G_2$  for some open disjoint sets  $G_1, G_2 \subseteq \mathbb{C}$  where  $G_1, G_2 \neq G$ . Now clearly  $\emptyset \subsetneq G_1 \subsetneq G$ ,  $G_1$  is open and  $G_1$  is relatively closed in  $G$  since

$$G \cap G_2^c = (G_1 \cup G_2) \cap G_2^c = G_1 \cap G_2^c = G_1$$

since  $G_2^c$  is closed and  $G_1 \cap G_2 = \emptyset$ .

**Exercise 3.** The proof is given as a sequence of lemmata.

**Lemma 0.7.** *The mapping  $h$  is well-defined, i.e.  $h(\mathbb{H}) \subseteq \mathbb{E}$ .*

*Proof.* Let  $z \in \mathbb{H}$ . Then  $\operatorname{Im}(z) > 0$  and thus

$$\begin{aligned} \left| \frac{z-i}{z+i} \right|^2 &= \frac{(z-i)(\bar{z}+i)}{(z+i)(\bar{z}-i)} \\ &= \frac{|z|^2 + i(z-\bar{z}) + 1}{|z|^2 + i(\bar{z}-z) + 1} \\ &= \frac{|z|^2 - 2\operatorname{Im}(z) + 1}{|z|^2 + 2\operatorname{Im}(z) + 1} \\ &\leq \frac{|z|^2 - 2\operatorname{Im}(z) + 1}{|z|^2 + 1} \\ &= 1 - \frac{2\operatorname{Im}(z)}{|z|^2 + 1} \\ &< 1. \end{aligned}$$

□

**Lemma 0.8.** *The mapping  $h$  is invertible with inverse*

$$h^{-1} : \mathbb{E} \rightarrow \mathbb{H}, w \mapsto i \frac{1+w}{1-w}. \quad (7)$$

*Proof.* Let  $z \in \mathbb{H}$ . Then we have

$$1 + h(z) = \frac{2z}{z+i} \quad \text{and} \quad 1 - h(z) = \frac{2i}{z+i}. \quad (8)$$

Therefore

$$\frac{1+h(z)}{1-h(z)} = \frac{z}{i} \quad \Leftrightarrow \quad z = i \frac{1+h(z)}{1-h(z)}. \quad (9)$$

This quotient is well-defined since  $h(z) \neq 1$  for  $z \in \mathbb{H}$ . Thus consider the mapping

$$g : \mathbb{E} \rightarrow \mathbb{C}, w \mapsto i \frac{1+w}{1-w}.$$

Now let  $w \in \mathbb{E}$ . Then

$$\begin{aligned} \operatorname{Im} \left( i \frac{1+w}{1-w} \right) &= \frac{1}{2i} \left[ i \frac{1+w}{1-w} + i \frac{1+\bar{w}}{1-\bar{w}} \right] \\ &= \frac{1-|w|^2}{1-(w+\bar{w})+|w|^2} \\ &= \frac{1-|w|^2}{1-2\operatorname{Re}(w)+|w|^2} \end{aligned}$$

$$\begin{aligned} &\geq \frac{1 - |w|^2}{(1 + |w|)^2} \\ &= \frac{1 - |w|}{1 + |w|} \\ &> 0 \end{aligned}$$

since  $|\operatorname{Re}(w)| \leq |w| < 1$ . Hence  $g(\mathbb{E}) \subseteq \mathbb{H}$ . Furthermore, for  $z \in \mathbb{H}$  and  $w \in \mathbb{E}$  we have

$$g(h(z)) = i \frac{1 + (z - i)/(z + i)}{1 - (z - i)/(z + i)} = z \quad \text{and} \quad h(g(w)) = \frac{(1 + w)/(1 - w) - 1}{(1 + w)/(1 - w) + 1} = w.$$

Thus  $g \circ h = \operatorname{id}_{\mathbb{H}}$  and  $h \circ g = \operatorname{id}_{\mathbb{E}}$  which implies that  $g$  and  $h$  are bijective and inverse to each other, thus  $g = h^{-1}$ .  $\square$

**Lemma 0.9.** *It holds that  $h \in \mathcal{O}(\mathbb{H})$  and  $h^{-1} \in \mathcal{O}(\mathbb{E})$ .*

*Proof.*  $h$  as well as  $h^{-1}$  are well-defined rational functions, hence holomorphic in their respective domains.  $\square$

By lemma 0.7, 0.8 and 0.9 we conclude that the Cayley-map is biholomorphic.

**Exercise 4.** In both cases we use that a function  $f : D \rightarrow \mathbb{C}$ , where  $D \subseteq \mathbb{C}$  is non-empty and open, is complex-differentiable in  $c \in D$  if and only if  $f$  is real-differentiable in  $c$  and the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x}(c) = \frac{\partial v}{\partial y}(c) \quad \text{and} \quad \frac{\partial u}{\partial y}(c) = -\frac{\partial v}{\partial x}(c)$$

holds, where  $f = u + iv$  (see [RS02, p. 47]). Furthermore, there is a useful sufficiency criterion (see [RS02, p. 48]).

**Lemma 0.10 (Sufficient Criterion for complex Differentiability).** *If  $u$  and  $v$  have continuous partial derivatives in  $D$ , then  $f := u + iv$  is real differentiable in  $D$ . If additionally the Cauchy Riemann equations hold for some subset  $\hat{D} \subseteq D$ , then  $f$  is complex differentiable in  $\hat{D}$ .*

*Proof.* The first statement follows immediately from the well known fact of multivariable calculus, that the existence and continuity of all partial derivatives is sufficient for differentiability (see [Zor04, p. 457]), by considering  $f : D \rightarrow \mathbb{R}^2$  defined by

$$f(x, y) = u(x, y) + iv(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

whereas the second statement follows from the above discussion.  $\square$

First we consider the function  $f$ . We can decompose  $f = u + iv$  where  $u, v : \mathbb{C} \rightarrow \mathbb{R}$  are defined by

$$u(x + iy) := x^3 y^2 \quad \text{and} \quad v(x + iy) := x^2 y^3. \quad (10)$$

Calculating the partial derivatives in  $c := x + iy = (x, y)$  yields

$$\frac{\partial u}{\partial x}(x, y) = 3x^2 y^2 \quad \frac{\partial u}{\partial y}(x, y) = 2x^3 y \quad \frac{\partial v}{\partial x}(x, y) = 2xy^3 \quad \frac{\partial v}{\partial y}(x, y) = 3x^2 y^2.$$

We immediately see that all partial derivatives are continuous on  $\mathbb{R}^2$  and thus  $u$  and  $v$  are (continuously) real differentiable in  $\mathbb{R}^2$  and so is  $f$ . Clearly

$$\frac{\partial u}{\partial x}(x, y) = 3x^2y^2 = \frac{\partial v}{\partial y}(x, y) \quad (11)$$

holds for any  $x, y \in \mathbb{R}$  and thus for any  $c \in \mathbb{C}$ . However, for the second equation we get the requirement

$$\frac{\partial u}{\partial y}(x, y) = 2x^3y = -2xy^3 = -\frac{\partial v}{\partial x}(x, y). \quad (12)$$

or equivalently  $x^3y = -xy^3$ . If  $x = 0$ , then the equality holds for any  $y \in \mathbb{R}$  and if  $y = 0$  then the equality holds for any  $x \in \mathbb{R}$ . If  $x, y \neq 0$  we get the equality  $x^2 = -y^2$  which cannot be true since  $y^2 > 0$  would imply  $x^2 < 0$ . Hence the set in which  $f$  is complex differentiable is exactly  $\{z \in \mathbb{C} : \operatorname{Re}(z) = 0 \vee \operatorname{Im}(z) = 0\}$ .

Now consider the function  $g$ . Similar to the previous part we can write  $g = u + iv$  where  $u, v : \mathbb{C} \rightarrow \mathbb{R}$  are defined by

$$u(x + iy) := e^x \cos y \quad \text{and} \quad v(x + iy) := e^x \sin y. \quad (13)$$

Calculating the partial derivatives yields

$$\begin{aligned} \frac{\partial u}{\partial x}(x, y) &= e^x \cos y = u(x + iy) \\ \frac{\partial u}{\partial y}(x, y) &= -e^x \sin y = -v(x + iy) \\ \frac{\partial v}{\partial x}(x, y) &= e^x \sin y = v(x + iy) \\ \frac{\partial v}{\partial y}(x, y) &= e^x \cos y = u(x + iy) \end{aligned}$$

which immediately implies that  $u$  and  $v$  are real differentiable on  $\mathbb{R}^2$  (since again all partial derivatives are continuous on  $\mathbb{R}^2$ ), hence  $f$  is real differentiable, and that the Cauchy-Riemann equations holds on  $\mathbb{C}$ . Thus  $g$  is holomorphic in  $\mathbb{C}$  or *entire*.

#### REFERENCES

- [Lee11] John M. Lee. *Introduction to Topological Manifolds*. Second Edition. Springer Science+Business Media, 2011.
- [RS02] R. Remmert and G. Schumacher. *Funktionentheorie 1*. Springer-Lehrbuch. Springer Berlin Heidelberg, 2002. ISBN: 9783540418559.
- [Zor04] Vladimir A. Zorich. *Mathematical Analysis I*. Springer-Verlag Berlin Heidelberg, 2004.