SOLUTIONS SHEET 8

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Exercise 1. We follow [IL03, pp. 99–101]. Let $U := \mathbb{C} \setminus \{\pm e^{\pm i\pi/4}\}$ and define $F : U \to \mathbb{C}$ by

$$F(z) := \frac{1}{1+z^4}. \tag{1}$$

Clearly $F \in \mathcal{O}(U)$ as a well-defined rational function, U is open in $\mathbb{C}$ and $\mathbb{R} \subseteq U$. Furthermore $F|_{\mathbb{R}} = f$. Hence F is a holomorphic continuation of f. Since havin an analytic continuation is equivalent to be real-analytic (see [IL03, p. 100]), we have that f is real-analytic.
Let $x_0 \in \mathbb{R}$. The Taylor series expansion of f is completely determined by the one of F. So the only thing which restricts the radius of convergence of the Taylor series expansions are the singularities of F. I will again formalize why this is the case. Let

$$F(z) = \sum_{\nu=0}^{\infty} a_\nu (z - x_0)^\nu \tag{2}$$

be the Taylor expansion of F around x_0. By Cauchy-Taylor the radius of convergence of the expansion (2) is at least $|x_0 - e^{i\pi/2}|$ if $x_0 \geq 0$ and $|x_0 + e^{i\pi/4}|$ if $x_0 \leq 0$. Let $r := |x_0 - e^{i\pi/4}|$ and assume $x_0 \geq 0$. (the case $x_0 \leq 0$ is similar) and $R > r$. Hence the series in (2) converges in $B_R(x_0)$. Hence it defines a function $G : B_R(x_0) \to \mathbb{C}$ by

$$G(z) := \sum_{\nu=0}^{\infty} a_\nu (z - x_0)^\nu \tag{3}$$

with $G|_{B_r}(x_0) = F$. Since G is expandable in a power series, we have $G \in \mathcal{O}(B_R(x_0))$ by [RS02, p. 187]. Since any holomorphic functionis continuous, we have $G \in \mathscr{C}(B_R(x_0))$. Let $(z_\nu)_{\nu\in\mathbb{N}}$ be a sequence in $B_r(x_0)$ such that $\lim_{\nu\to\infty} z_\nu = e^{i\pi/4}$. Clearly

$$\lim_{\nu\to\infty} F(z_\nu) = \infty \tag{4}$$

and since $G|_{B_r(x_0)} = F$ we have

$$\lim_{\nu\to\infty} G(z_\nu) = \infty. \tag{5}$$

But since $R > r$, G is continuous at $e^{i\pi/4}$ and so we must have

$$G(e^{i\pi/4}) = \lim_{\nu\to\infty} G(z_\nu) = \infty. \tag{6}$$

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Thus the series G diverges at $e^{i\pi/4}$, contradicting that $e^{i\pi/4} \in B_R(x_0)$. Now for general $x_0 \in \mathbb{R}$, the radius of convergence R of the Taylor series expansion of f in x_0 is the radius of convergence of the restriction of the Taylor series expansion of F in x_0 on $\mathbb{R}$, hence

$$R = \begin{cases} |x_0 - e^{i\pi/4}| & x_0 \geq 0, \\ |x_0 + e^{i\pi/4}| & x_0 \leq 0. \end{cases}$$

References

[IL03] Wolfgang Fischer and Ingo Lieb. *Funktionentheorie: Komplexe Analysis in einer Veränderlichen*. 8. Auflage. vieweg studium; Aufbaukurs Mathematik. Vieweg+Teubner Verlag, 2003.

[RS02] R. Remmert and G. Schumacher. *Funktionentheorie 1*. Springer-Lehrbuch. Springer Berlin Heidelberg, 2002. ISBN: 9783540418559.