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SOLUTIONS SHEET 2

Exercise 1.

a. Any remark here would be, that for any system of linear equations $Ax = b$, the right side b can also be a matrix.

```
function [ R ] = fweli_yannis( A )
%Input:
%-----
%A - Augmented matrix [A,b], where b can be of the same shape as A.
%
%Returns:
%-------
%R - Augmented system [R, b'], where R is upper triangular.
[m, ~] = size(A);
for j = 1:m-1
    for i = j+1:m
        A(i, :) = A(i, :) - A(i, j)./A(j,j) .* A(j,:);
    end
end
R = A;
end
```

Listing 1. `src/fweli_yannis.m`

```
function [ x ] = bksub_yannis( R )
%Input:
%-----
%R - Augmented matrix [R,b'], where b' can be of the same shape as A.
%
%Returns:
%-------
%x - Solution to the system Ax = b.
[m, n] = size(R);
x = zeros(m,n-m);
for i = m:-1:1
    x(i,:) = (R(i,m+1:end) - (R(i,i+1:m) * x(i+1:m,:)))/R(i,i);
end
end
```

Listing 2. `src/bksub_yannis.m`

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b. If we compute the inverse with the naive forward elimination, one obtains an error of

$$\left\|A^{-1} - \hat{A}^{-1}\right\| = 2.787702054069282e - 17$$

So the forward elimination algorithm yields a moderate inverse. But $A \in M_3(\mathbb{R})$, so for practical calculations, this is a very small matrix. Hence, not much can be said about the general behaviour of the error. Further we get

$$\|I - \hat{A}^{-1}A\| = 4.440892098500626e - 16 \tag{1}$$

This due to the computational errors in the built-in function `inv`.

```
format long;
A = [
  50, 1, 3;
  1, 6, 0;
  3, 0, 1
];
R = fweli_yannis([A, eye(3)]);
inverse = bksub_yannis(R);
norm(inv(A) - inverse), norm(eye(3) - inverse * A)
```

LISTING 3. `src/ex_1_b.m`

c. The error is between the gaussian elimination algorithm with pivoting and the inverse computed by `inv` is

$$\left\|A^{-1} - \hat{A}^{-1}\right\| = 4.510935197729729e - 16 \tag{2}$$

So again quite moderate. Again

$$\|I - \hat{A}^{-1}A\| = 2.426396072232912e - 15 \tag{3}$$

```
function [ R ] = fweliPivot_yannis( A )
[m,~] = size(A);
for j = 1:m
    [~, k] = max(A(j:end,j));
    A([j,k+j-1],:) = A([k+j-1,j],:);
    for i = j+1:m
        A(i, :) = A(i, :) - A(i, j)./A(j,j) .* A(j,:);
    end
end
R = A;
end
```

LISTING 4. `src/fweliPivot_yannis.m`

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```
A = [
 1, 1 + .5 * 1e-15, 3;
 2, 2, 20;
 3, 6, 4
];
R = fweliPivot_yannis([A, eye(3)]);
inverse = bksub_yannis(R);
norm(inv(A) - inverse), norm(eye(3) - inverse * A)
```

Listing 5. src/ex_1_c.m

Exercise 2. As one can see in the plot given below, the growth of the CPU time is approximately (a bit less) of order $O(n^3)$. Hence the theoretical value agrees with the experimental data. Since the theoretical order is considered for $n \to \infty$ one would have to use even more larger values for the size of the matrix.

```
rng(3143647);
n = 1e+3:1e+2:5e+3;
for k = 1:length(n)
    A = rand(n(k));
    t = cputime;
    lu(A);
    time(k) = cputime - t;
end
f = @(n) time(1)/x(1)^3 * n.^3;
loglog(n, time, '-x', 'color', 'red');
hold on;
loglog(n, f(n), 'color', 'blue');
grid on;
xlabel('$n$', 'interpreter', 'latex', 'fontsize', 18);
ylabel('$t$', 'interpreter', 'latex', 'fontsize', 18);
leg = legend('$t(n)$', '$O(n^3)$', 'location', 'southeast');
set(leg, 'fontsize', 18, 'interpreter', 'latex');
saveas(gcf, 'ex_2.jpg');
```

Listing 6. src/ex_2.m

Exercise 3. Let $\|\cdot\|$ be any vector norm.

a. We have

$$\mathrm{cond}(I) = \|I\|\|I^{-1}\| = \|I\|^2 = \left(\max_{x\neq 0} \frac{\|Ix\|}{\|x\|}\right)^2 = \left(\max_{x\neq 0} \frac{\|x\|}{\|x\|}\right)^2 = 1$$

b. Using **a.**, **e.** and the fact, that $\mathrm{cond}(A^{-1}) = \mathrm{cond}(A)$, we have

$$1 = \mathrm{cond}(I) = \mathrm{cond}(AA^{-1}) \leqslant \mathrm{cond}(A)\mathrm{cond}(A^{-1}) = \mathrm{cond}(A)^2 \tag{4}$$

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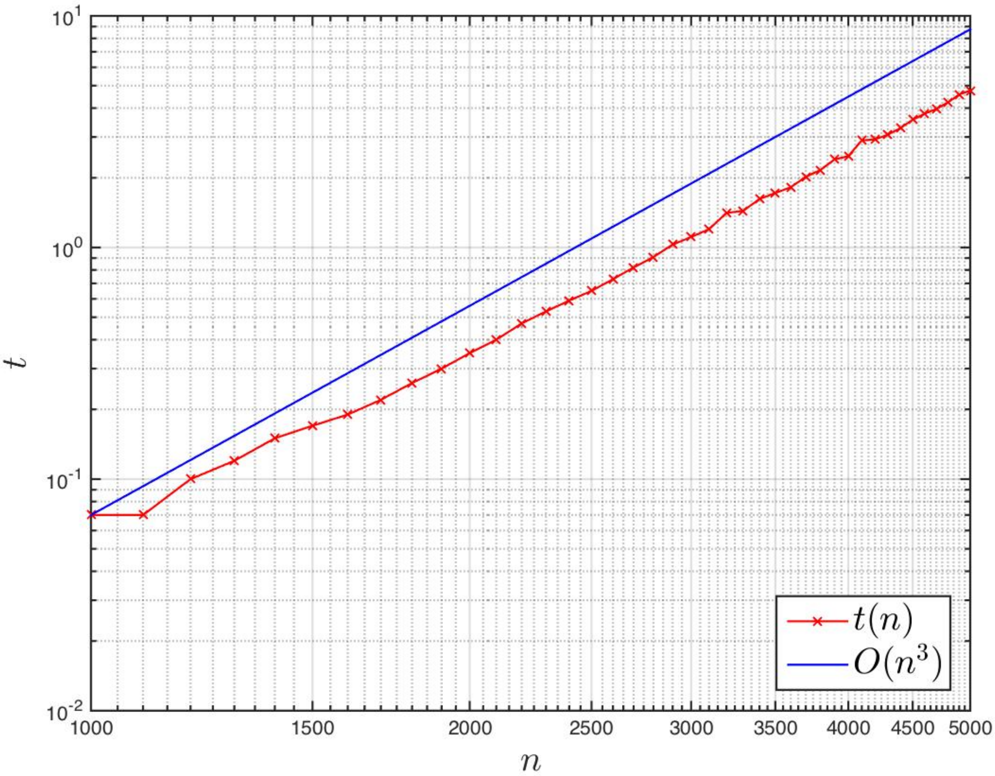

FIGURE 1. Plot of the CPU time and asymptotic computational complexity $O(n^3)$.

Taking the square root on both sides yields the requered inequality. This is possible since any norm is positive. Another quite nice proof can be given in terms of eigenvalues. Let $\lambda \in \mathbb{K}$ an eigenvalue of A and v a corresponding eigenvector. Then we get

$$\begin{aligned}\operatorname{cond}(A) &= \|A\|\|A^{-1}\| \\ &\geqslant \frac{\|Av\|}{\|v\|}\frac{\|A^{-1}v\|}{\|v\|} \\ &= \frac{\|\lambda v\|}{\|v\|}\frac{\|\lambda^{-1}v\|}{\|v\|} \\ &= |\lambda||\lambda^{-1}| \\ &= 1\end{aligned}$$

Remark 1.1. *In above proof the existence of eigenvalues is assumed. This is trivial, if we restrict our attention to linear operators on finite dimensional vector spaces.*

c. Let $\alpha \in \mathbb{R}$ and $\alpha \neq 0$. Then

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$$\begin{aligned}\operatorname{cond}(\alpha A) &= \|\alpha A\| \|(\alpha A)^{-1}\| = \max_{x\neq 0} \frac{\|\alpha A x\|}{\|x\|} \max_{x\neq 0} \frac{\|\alpha^{-1} A^{-1} x\|}{\|x\|} \\ &= |\alpha| \max_{x\neq 0} \frac{\|Ax\|}{\|x\|} \left|\alpha^{-1}\right| \max_{x\neq 0} \frac{\|A^{-1}x\|}{\|x\|} = \operatorname{cond}(A)\end{aligned}$$

Since $(\alpha A)^{-1} = \alpha^{-1}A^{-1}$ by $(\alpha A)\,\alpha^{-1}A^{-1} = \alpha\alpha^{-1}AA^{-1} = I$.

d. Let $\lambda_{\max}, \lambda_{\min} \in \mathbb{K}$ be the largest and smallest eigenvalues respectively (formally $\lambda_{\max} := \max_i |\lambda_i|$ and $\lambda_{\min} := \min_i |\lambda_i|$) and $v_{\max}, v_{\min}$ the corresponding eigenvectors, then

$$\begin{aligned}\operatorname{cond}(A) &= \|A\|\|A^{-1}\| = \max_{x\neq 0} \frac{\|Ax\|}{\|x\|} \max_{x\neq 0} \frac{\|A^{-1}x\|}{\|x\|} \\ &\geqslant \frac{\|Av_{\max}\|}{\|v_{\max}\|} \frac{\|A^{-1}v_{\min}\|}{\|v_{\min}\|} \\ &= |\lambda_{\max}| \frac{\|v_{\max}\|}{\|v_{\max}\|} \frac{1}{|\lambda_{\min}|} \frac{\|v_{\min}\|}{\|v_{\min}\|} \\ &= \left|\frac{\lambda_{\max}}{\lambda_{\min}}\right|\end{aligned}$$

Since $A\lambda_{\min} = \lambda_{\min} v_{\min}$ is equivalent to $A^{-1}v_{\min} = \lambda_{\min}^{-1} v_{\min}$.

e. We have

$$\operatorname{cond}(AB) = \|AB\|\|B^{-1}A^{-1}\| \leqslant \|A\|\|B\|\|A^{-1}\|\|B^{-1}\| = \operatorname{cond}(A)\operatorname{cond}(B) \tag{5}$$

Since $\|AB\| \leqslant \|A\|\|B\|$.

Exercise 4. **a.** See **c.**.

b. See **c.**.

c. We get

$$\delta(\hat{x}) = \frac{\|\hat{x} - x\|}{\|x\|} = 1.096122480449067e + 43 \tag{6}$$

and

$$\delta(\hat{y}) = \frac{\|\hat{y} - y\|}{\|y\|} = 2.920311908056222e - 13 \tag{7}$$

Where $\hat{x}$ and $\hat{y}$ are the computed solutions of $Ax = b$ and $By = c$ respectively. This is due to the fact, that the Vandermonde matrix is very ill-conditioned, since its condition in this particular example is

$$\operatorname{cond}(A) = 7.661394127343940e + 20 \tag{8}$$

Whereas

$$\operatorname{cond}(B) = 1.378706441139780e + 04 \tag{9}$$

d. One observes, that the condition numbers of the Vandermonde and the random matrix differ in a factor of 10^{14}. That is quite large. Further the condition numbers of the Vandermonde matrix increase drastically up to the size 15×15, then it is more or

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```
format long;
rng(7978045);
n = 1e+2;
A = vander(rand(1,n));
B = rand(n);
b = sum(A,2);
c = sum(B,2);
x = A\b;
y = B\c;
v = ones(n,1);
error_x = norm(x - v)/norm(v)
error_y = norm(y - v)/norm(v)
cond(A, 2)
cond(B, 2)
```

Listing 7. src/ex_4_c.m

less constant. In contrast, the condirion number of the random matrix only slightly increases with higher dimension.

```
rng(2312431);
for k = 2:50
    semilogy(k, cond(vander(rand(1,k)), 2), 'x', 'color', 'blue');
    hold on;
    semilogy(k, cond(rand(k), 2), 'x', 'color', 'red');
    hold on;
end
grid on;
xlabel('$n \times n$', 'interpreter', 'latex', 'fontsize', 18);
ylabel('$\mathrm{cond}(A)$', 'interpreter', 'latex', 'fontsize', 18);
saveas(gcf, 'ex_4_d.jpg');
```

Listing 8. src/ex_4_d.m

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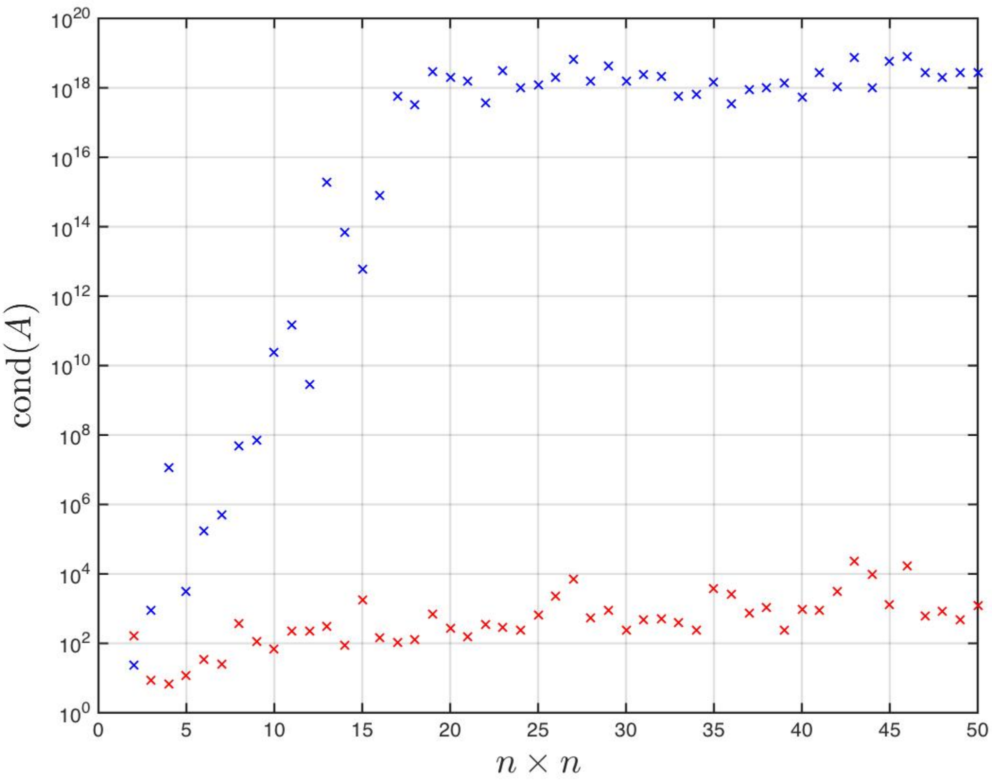

FIGURE 2. Plot of the condition numbers of the different matrices, in blue the Vandermonde matric and in red the random matrix.