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SOLUTIONS SHEET 4

Exercise 1. Let $k \in \mathbb{N}_{>0}$ and $B_\sigma(\alpha) :=]\alpha - \sigma, \alpha + \sigma[$. Under the given conditions, for any $x^{(0)} \in B_\sigma(\alpha)$ it holds that $x^{(k)} \in B_\sigma(\alpha)$ for $k = 0, 1, 2, \ldots$. *Proof by induction.* Assume $k > 0$. Then we have

$$|x^{(k)} - \alpha| = |\Phi(x^{(k-1)}) - \Phi(\alpha)| \overset{\text{MVT}}{=} |\Phi'(\xi)||x^{(k-1)} - \alpha| \leqslant q|x^{(k-1)} - \alpha| \leqslant q\sigma < \sigma \tag{1}$$

for some $\xi \in B_\sigma(\alpha)$. Thus there exist some $\xi_1, \xi_2 \in B_\sigma(\alpha)$ such that

$$\begin{aligned} |x^{(k)} - \alpha| &\overset{\triangle}{\leqslant} |x^{(k+1)} - x^{(k)}| + |x^{(k+1)} - \alpha| \\ &= |\Phi(x^{(k)}) - \Phi(x^{(k-1)})| + |\Phi(x^{(k)}) - \Phi(\alpha)| \\ &= |\Phi'(\xi_1)||x^{(k)} - x^{(k-1)}| + |\Phi'(\xi_2)||x^{(k)} - \alpha| \\ &\leqslant q|x^{(k)} - x^{(k-1)}| + q|x^{(k)} - \alpha| \end{aligned}$$

Hence

$$|x^{(k)} - \alpha| \leqslant \frac{q}{1-q}|x^{(k)} - x^{(k-1)}| \tag{2}$$

$\square$

Exercise 2. Using Taylor's theorem we get for $k = 0, 1, \ldots$ (expansion around α)

$$x^{(k+1)} - \alpha = \Phi(x^{(k)}) - \Phi(\alpha) = \frac{1}{2}\Phi''(\alpha)(x^{(k)} - \alpha)^2 + o\left((x^{(k)} - \alpha)^2\right) \tag{3}$$

Dividing both sides by $(x^{(k)} - \alpha)^2$ and taking the limit $k \to \infty$ yields the stated property. The first part is prooven by considering

$$|x^{(k+1)} - \alpha| = |\Phi(x^{(k)}) - \alpha| = \frac{1}{2}|\Phi''(\alpha)||x^{(k)} - \alpha|^2 \leqslant C|x^{(k)} - \alpha|^2 \tag{4}$$

for some $C > 0$, since Φ is twice continuously differentiable in a neighbourhood of α, hence Φ is locally bounded. The requirement $|\Phi'(x)| \leqslant p$ is only necessary for $\lim_{k\to\infty} x^{(k)} = \alpha$. $\square$

Exercise 3.
a. Obviously, the exact solutions on $[-2, 2]$ are $\pm\sqrt{\frac{\pi}{2}}$.
b. The code can be found in listing 1.
c. I choose $[-1.3, -1.2]$ for $-\sqrt{\frac{\pi}{2}}$ and $[1.2, 1.3]$ for $\sqrt{\frac{\pi}{2}}$. The results are

$$\overline{x}_1 = 1.253308105468750 \qquad \left|\overline{x}_1 - \sqrt{\frac{\pi}{2}}\right| = 6.031846750076397e - 06 \tag{5}$$

and

$$\overline{x}_1 = -1.253308105468750 \qquad \left|\overline{x}_1 + \sqrt{\frac{\pi}{2}}\right| = 6.031846750076397e - 06 \tag{6}$$

Hence the bisection method yields accurate roots and obviously converges. <u>Remark:</u> I have used the termination criterion $(b - a)/2^k \leqslant \varepsilon$ for the k-th iteration.

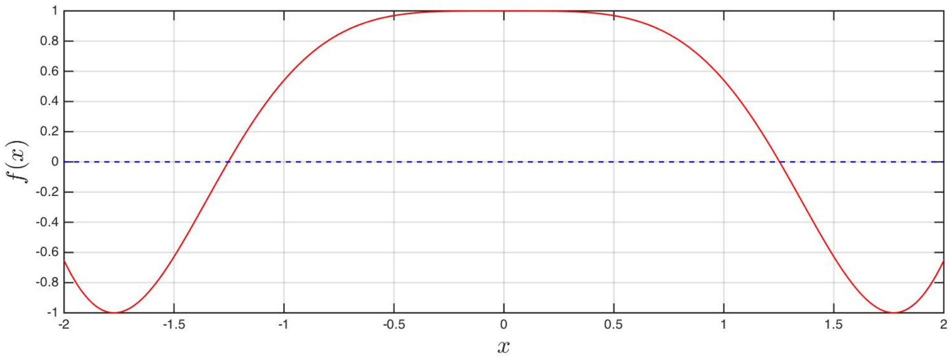

FIGURE 1. Plot of the graph of the function $f(x) = \cos(x^2)$ on the interval $[-2, 2]$.

```
function [ x,it ] = bisect_baehni( a,b,tol,f )
it = 0;
while (b - a)/2^it > tol
    m = (a + b)/2;
    M = f(m);
    if f(a) * M < 0
        b = m;
    else
        a = m;
    end
    it = it + 1;
end
x = m;
end
```

LISTING 1. `src/bisect_baehni.m`

Exercise 4. **a.** The code can be found in listing 2. Remark: I have implemented the termination criterion $|f(x^{(k)})/f'(x^{(k)})| \leqslant \varepsilon$.

b. The computed solution is given by $\overline{x} = 1.570796326794900$. The absolute error is given by $|\overline{x} - \pi/2| = 3.330669073875470e - 15$.

c. This wont work. The bisection method only works for an interval, where the endpoints have different sign. This is not the case here as one can see in figure 2.

Exercise 5. **a.** As one can tell from figure 3, the roots are located around $\pm\frac{1}{2}$. Those are the only roots since $\lim\limits_{x\to\pm\infty} f(x) = -\infty$.

b. We get $\overline{x} = 0.51493326$.

c. Φ has to be a contraction in some neighbourhood of the fixed point $\alpha \approx 0.5$. Thus considering $|\Phi'(x)| < 1$ yields

$$|\Phi'(x)| = |1 + Af'(x)| = |1 - 2A(\sin(4x) + x)| \overset{!}{<} 1 \tag{7}$$

Some algebraic manipulations further yield

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```
function [ x, it] = hybrid_baehni( a,b,nmax_b,nmax_n,tol,f,df )
%Bisection
it = 0;
while (b - a)/2^it > tol && it <= nmax_b
    m = (a + b)/2;
    M = f(m);
    if f(a) * M < 0
        b = m;
    else
        a = m;
    end
    it = it + 1;
end
%Newton-Raphson
x0 = m;
it = 0;
while abs(f(x0)/df(x0)) > tol && it <= nmax_n
    xit = x0 - f(x0)/df(x0);
    x0 = xit;
    it = it + 1;
end
x = x0;
end
```

LISTING 2. src/hybrid_baehni.m

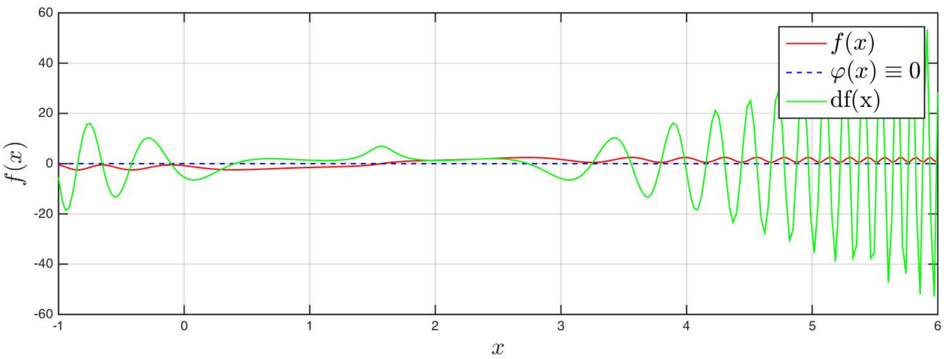

FIGURE 2. Plot of the graph of the function $f(x)$ on the interval $[-1, 6]$.

$$0 < A < \frac{1}{\sin(4x) + x} < 0.71 \tag{8}$$

for $x \in]0.4, 0.6[$.

d. In the third case the method does not converge.

Exercise 6. **a.** The code can be found in listing 3.

b. The Newton method converges to a local minima of the function $f(x)$ which is not a root. This may happen for a bad choice of $x^{(0)}$, if it is not in a small enough

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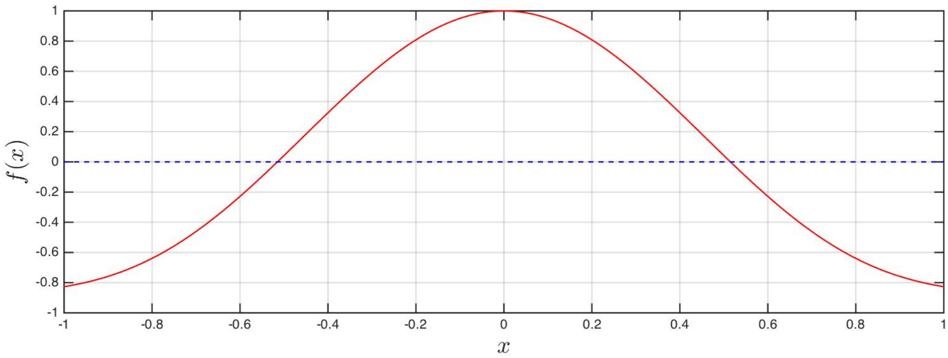

FIGURE 3. Plot of the graph of the function $f(x)$ on the interval $[-1, 1]$.

```
function [ x,it ] = secant( f,x0,x1,tol,nmax )
it = 1;x(1) = x0;x(2) = x1;
while abs(x1 - x0) > tol && it <= nmax
    quotient = f(x1) * (x1 - x0)/(f(x1) - f(x0));
    xit = x1 - quotient;
    x0 = x1;
    x1 = xit;
    x(end + 1) = xit;
    it = it + 1;
end
end
```

LISTING 3. `src/secant.m`

neighbourhood of the root. We obtain an oscillating sequence of iterations $x^{(k)}$ of ones and zeros. The code for the Newton method can be found in listing 4.

```
function [ x,it ] = newton( f,df,x0,tol,nmax )
it = 0;x(1) = x0;
while abs(f(x0)/df(x0)) > tol && it <= nmax
    xit = x0 - f(x0)/df(x0);
    x0 = xit;
    x(end + 1) = xit;
    it = it + 1;
end
end
```

LISTING 4. `src/newton.m`

c. Since the exact solution is in this case not known (not algebraically determinable), we have to stick to an approximation. This will be the last value attained in the iteration process. Let us denote this approximation by x^*. If we have a method of order p,

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and define $e_k := |x^{(k)} - x^*|$, we have $e_{k+1} \approx Ce_k^p$ (by the definition of the convergence order) and thus $\log(e_{k+1}) \approx \log(C) + p\log(e_k)$. Hence

$$\boxed{p \approx \frac{\log(e_{k+1}) - \log(e_k)}{\log(e_k) - \log(e_{k-1})}} \tag{9}$$

For the Newton method we get

$$\begin{gathered} 1.378194362924571 \\ 1.595203314298199 \\ 1.840229819606686 \\ 1.976574837258011 \\ 1.999372430379578 \end{gathered}$$

which agrees quite well with the theoretical order of $p = 2$. The secant method however is more difficult. We get

$$\begin{gathered} -1.368954928628122 \\ 0.353496085466967 \\ -3.842629675388165 \\ -1.549839511226760 \\ 4.534191360553605 \\ -0.442672744409319 \\ -2.038156440662885 \\ 0.493361260772762 \\ 3.131863108143211 \\ 1.320854970058889 \\ 1.756801932829158 \end{gathered}$$

The theoretical value would be $\frac{1}{2}\left(1 + \sqrt{5}\right)$.