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SOLUTIONS SHEET 7

Exercise 1.

a. Let $f \in C^5(\mathbb{R})$, $h > 0$, $\psi(h;x) := f(x+h)$ and $\hat{\psi}(h;x) := f(x-h)$. Then for $x \in \mathbb{R}$ we have by *Taylor's Theorem*

$$f(x+h) = \psi(h;x) = \sum_{k=0}^{4} \frac{1}{k!}\frac{d^k\psi}{dh^k}(0;x)h^k + \frac{1}{5!}\frac{d^5\psi}{dh^5}(\hat{\xi}_1;x)h^5 = \sum_{k=0}^{4} \frac{f^{(k)}(x)}{k!}h^k + \frac{f^{(5)}(\xi_1)}{5!}h^5 \tag{1}$$

for $\hat{\xi}_1 \in]0,h[$ and $\xi_1 \in]x, x+h[$ and

$$f(x-h) = \hat{\psi}(h;x) = \sum_{k=0}^{4} \frac{1}{k!}\frac{d^k\hat{\psi}}{dh^k}(0;x)h^k + \frac{1}{5!}\frac{d^5\hat{\psi}}{dh^5}(\hat{\xi}_2;x)h^5 = \sum_{k=0}^{4} \frac{(-1)^k f^{(k)}(x)}{k!}h^k - \frac{f^{(5)}(\xi_2)}{5!}h^5 \tag{2}$$

for $\hat{\xi}_2 \in]0,h[$ and $\xi_2 \in]x-h, x[$. Let us now consider the symmetric finite difference $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$. Substituting equations (1) and (2) in the symmetric finite difference yields

$$\begin{aligned}
\Phi_1(x,h) :&= \frac{f(x+h)-f(x-h)}{2h} \\
&= \frac{1}{2h}\sum_{k=0}^{4}\left[\frac{f^{(k)}(x)}{k!}h^k + \frac{f^{(5)}(\xi_1)}{5!}h^5 - \frac{(-1)^k f^{(k)}(x)}{k!}h^k + \frac{f^{(5)}(\xi_2)}{5!}h^5\right] \\
&= \frac{1}{2h}\left[2hf'(x) + \frac{1}{3}f^{(3)}(x)h^3 + \frac{1}{120}\left(f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right)h^5\right] \\
&= f'(x) + \frac{1}{6}f^{(3)}(x)h^2 + \frac{1}{240}\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right]h^4
\end{aligned}$$

Further we get by stipulating $h = 2\hat{h}$, $\hat{h} > 0$

$$\begin{aligned}
\Phi_2(x,h) :&= \frac{f(x+2\hat{h})-f(x-2\hat{h})}{4\hat{h}} \\
&= \frac{1}{4\hat{h}}\sum_{k=0}^{4}\left[2^k\frac{f^{(k)}(x)}{k!}\hat{h}^k - 2^k\frac{(-1)^k f^{(k)}(x)}{k!}\hat{h}^k + \frac{2^5}{5!}\left(f^{(5)}(\xi_3) + f^{(5)}(\xi_4)\right)h^5\right] \\
&= \frac{1}{2\hat{h}}\sum_{k=0}^{4}\left[2^{k-1}\frac{f^{(k)}(x)}{k!}\hat{h}^k - 2^{k-1}\frac{(-1)^k f^{(k)}(x)}{k!}\hat{h}^k + \frac{2^4}{5!}\left(f^{(5)}(\xi_3) + f^{(5)}(\xi_4)\right)\hat{h}^5\right] \\
&= \frac{1}{2\hat{h}}\left[2f'(x)\hat{h} + \frac{4}{3}f^{(3)}(x)\hat{h}^3 + \frac{2^4}{5!}\left(f^{(5)}(\xi_3) + f^{(5)}(\xi_4)\right)\hat{h}^5\right] \\
&= f'(x) + \frac{2}{3}f^{(3)}(x)\hat{h}^2 + \frac{2}{3}\left(f^{(5)}(\xi_3) + f^{(5)}(\xi_4)\right)\hat{h}^4
\end{aligned}$$

Now let $\hat{h} = h$. We see that in $\Phi_2(x,h) - 4\Phi_1(x,h)$ the term $O(h^2)$ vanishes. To get a discretization for $f'(x)$ the scaling factor $-\frac{1}{3}$ is needed. Hence

$$
\begin{aligned}
-\frac{1}{3}\left[\Phi_2(x,h) - 4\Phi_1(x,h)\right] &= -\frac{1}{3}\left[\frac{f(x+2h)-f(x-2h)}{4h} - \frac{8}{h}\left(\frac{f(x+h)-f(x-h)}{4h}\right)\right] \\
&= \frac{f(x-2h) - 8f(x-h) + 8f(x+h) - f(x+2h)}{12h}
\end{aligned}
$$

In general the $O(h^4)$ term does not vanish as can be seen by considering the difference

$$
\frac{2}{3}\left(f^{(5)}(\xi_3) + f^{(5)}(\xi_4)\right)h^4 - \frac{1}{240}\left[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)\right]h^4 = \frac{159}{240}\left(f^{(5)}(\xi_3) + f^{(5)}(\xi_4)\right)h^4 \tag{3}
$$

b. The source code can be found in listing 1.

```
function [ y ] = SFD4( fi,h )
n = length(fi);
y = zeros(1,n-4);
for k = 1:(n-4)
    y(k) = (fi(k) - 8 * fi(k + 1) + 8 * fi(k + 3) - fi(k + 4))/(12 * h);
end
end
```

Listing 1. `src/SFD4.m`

c. The plot can be found in figur 1. Obviously the convergence rate is of the type Ch^4 as predicted by **a.** and does agree quite well.

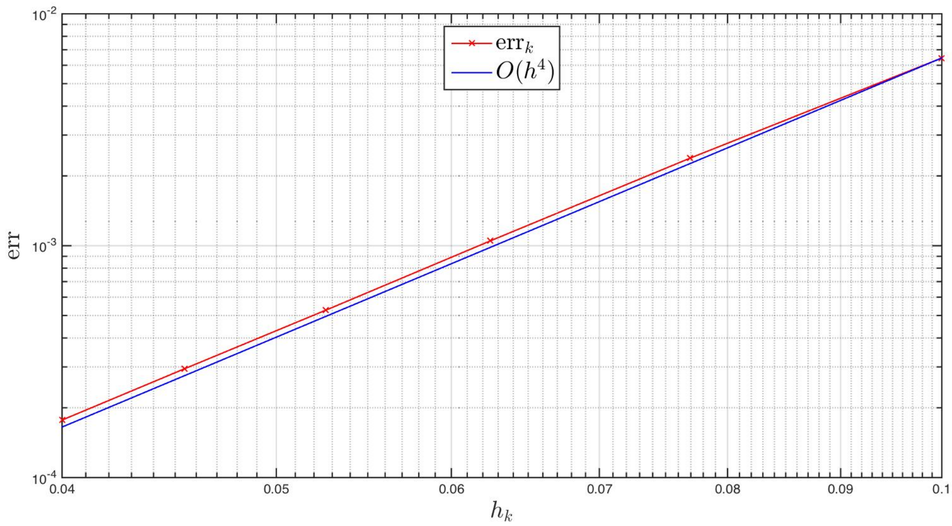

Figure 1. Plot of the error of the discretized derivative of the function $f(x) = \frac{\cos(5x)}{x+1}$ on the interval $[0,2]$ for decreasing stepsize h_k.

Exercise 2. **a.** Let $f \in C^2[a,b]$. For $x \in [a,b]$ consider the tangent line at the point $x = \frac{a+b}{2}$ which is given by truncating the Taylor expansion after the second term

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(4)
$$T(x) = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right)$$

Definite integration yields

$$\begin{aligned}
\int_a^b T(x)dx &= f\left(\frac{a+b}{2}\right)(b-a) + \frac{1}{2}f'\left(\frac{a+b}{2}\right)(b^2-a^2) - \frac{1}{2}f'\left(\frac{a+b}{2}\right)(a+b)(b-a) \\
&= f\left(\frac{a+b}{2}\right)(b-a)
\end{aligned}$$

Since $f \in C^2[a,b]$ there exists some $C > 0$ such that $|f''(x)| \leqslant C$ for all $x \in [a,b]$ since $[a,b]$ is compact by Heine-Borel. Thus for some $\xi_x \in]a,b[$

$$\begin{aligned}
\left|\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a)\right| &= \left|\int_a^b [f(x) - T(x)]\,dx\right| \\
&= \left|\frac{1}{2}\int_a^b f''(\xi_x)\left(x - \frac{a+b}{2}\right)^2 dx\right| \\
&\leqslant \frac{1}{2}\int_a^b |f''(\xi_x)|\left|x - \frac{a+b}{2}\right|^2 dx \\
&\leqslant \frac{C}{2}\left[\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2} - x\right)^2 + \int_{\frac{a+b}{2}}^b \left(x - \frac{a+b}{2}\right)^2\right] \\
&= \frac{C}{2}\left[\frac{1}{24}(b-a)^3 + \frac{1}{24}(b-a)^3\right] \\
&= \frac{C}{24}(b-a)^3
\end{aligned}$$

□

b. By **a.** for $h := \frac{b-a}{n}$, $n \in \mathbb{N}_{>0}$ we get

$$\begin{aligned}
\left|\int_a^b f(x)dx - h\sum_{k=1}^n f\left(\frac{x_{k-1}+x_k}{2}\right)\right| &\leqslant \sum_{k=1}^n \left|\int_{x_{k-1}}^{x_k} f(x)dx - hf\left(\frac{x_{k-1}+x_k}{2}\right)\right| \\
&\leqslant \frac{1}{24}\sum_{k=1}^n C_k h^3 \\
&\leqslant \frac{C}{24}nh^3 \\
&= \frac{C(b-a)}{24}h^2
\end{aligned}$$

for $C := \max\limits_{k=1,\dots,n}\{C_k\} > 0$. □

Exercise 3. **a.** The source code can be found in listing 2.

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```
function [ I ] = composite_simpson( f,a,b,h )
x = a:h:b;
M = 1/2 * (x(1:(end - 1)) + x(2:end));
I = h/6 * (f(x(1)) + f(x(end)) + 4 * sum(f(M)) + 2 * sum(f(x(2:(end-1)))));
end
```

LISTING 2. `src/composite_simpson.m`

b. Since $\frac{d}{dx}2^x = \log(2)2^x$ we have $\int_0^2 2^x dx = \frac{1}{\log(2)}\left[2^x\right]_0^2 = \frac{3}{\log(2)}$. The plot can be found in figure 2. As one can see the theoretical order $O(h^4)$ agrees precisely with experimentational convergence rate.

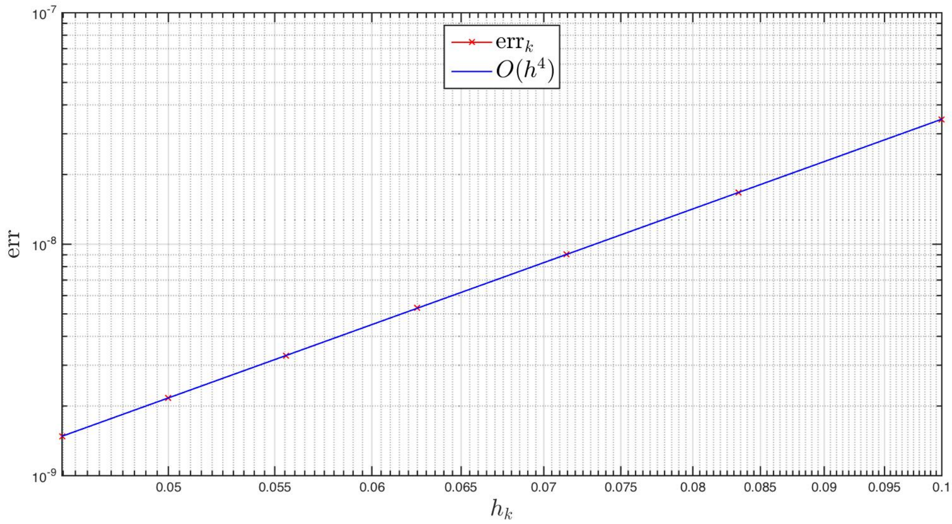

FIGURE 2. Plot of the error of th discretized integral of the function $f(x) = 2^x$ on $[0, 2]$ using the composited Simpson rule.

Exercise 4. (a) Let us have a look at the *affine transformation* $\psi(\tau) : [-1, 1] \to [a, b]$ defined by $\psi(\tau) := \frac{1}{2}((b - a)\tau + (a + b))$ (this is simple Lagrange interpolation of the two nodes $(-1, a)$ and $(1, b)$). Then

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{\psi^{-1}(a)}^{\psi^{-1}(b)} f(\psi(\tau))d\tau = \frac{b-a}{2}\int_{-1}^{1} f(\psi(\tau))d\tau \tag{5}$$

and thus

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\left[f\left(\frac{1}{2}\left(\frac{1}{\sqrt{3}}(a-b) + (a+b)\right)\right) + f\left(\frac{1}{2}\left(\frac{1}{\sqrt{3}}(b-a) + (a+b)\right)\right)\right] \tag{6}$$

(b) The code can be found in listing 3.

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```
function [ I ] = composite_gaussian_quadrature( f,a,b,n )
h = (b - a)/n;
xi = a:h:b;
I = 0;
for k = 1:n
I = I + (xi(k+1) - xi(k))/2 * (f(1/2 * (1/sqrt(3) * (xi(k) - xi(k+1)) ...
    + (xi(k) + xi(k+1)))) + f(1/2 * (1/sqrt(3) * (xi(k+1) - xi(k)) ...
    + (xi(k) + xi(k+1)))));
end
end
```

Listing 3. `src/composite_gaussian_quadrature.m`

(c) We have $\int\limits_0^5 \frac{1}{1+(x-\pi)^2}dx = \int\limits_{-\pi}^{5-\pi} \frac{1}{1+u^2}du = \arctan(5-\pi) + \arctan(\pi)$. The number of subintervals needed to reach an accuracy lower than 10^{-4} is nine as can be told from the source code in listing 4.

```
f = @(x) 1./(1 + (x - pi).^2);
a = 0;
b = 5;
tol = 1e-4;
steps = 1;
I = composite_gaussian_quadrature(f,a,b,steps);
error = abs(I - (atan(5 - pi) + atan(pi)));
steps = steps + 1;
while error >= tol
    I = composite_gaussian_quadrature(f,a,b,steps);
    error = abs(I - (atan(5 - pi) + atan(pi)));
    steps = steps + 1;
end
disp(['Number of steps used to reach the accuracy ',num2str(tol), ...
    ': ', num2str(steps)])
```

Listing 4. `src/ex_4_c.m`