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SOLUTIONS SHEET 5

Exercise 1. We get

- $$\begin{aligned}P(x) &= y_0\frac{x-x_1}{x_0-x_1}\frac{x-x_2}{x_0-x_2} + y_1\frac{x-x_0}{x_1-x_0}\frac{x-x_2}{x_1-x_2} + y_2\frac{x-x_0}{x_2-x_0}\frac{x-x_1}{x_2-x_1}\\
&= 2\frac{x-2}{-2}\frac{x-3}{-3} + 3\frac{x}{2}\frac{x-3}{-1} + 8\frac{x}{3}\frac{x-2}{1}\\
&= \frac{1}{3}(2-x)(3-x) + \frac{3}{2}x(3-x) + \frac{8}{3}x(x-2)\\
&= \frac{1}{3}(x^2-5x+6) + \frac{3}{2}(3x-x^2) + \frac{8}{3}(x^2-2x)\\
&= \left(\frac{1}{3}-\frac{3}{2}+\frac{8}{3}\right)x^2 + \left(\frac{9}{2}-\frac{5}{3}-\frac{16}{3}\right)x + 2\\
&= \frac{3}{2}x^2 - \frac{5}{2}x + 2\end{aligned}$$
- One has the divided differences

$$f_{01} = \frac{f_1-f_0}{x_1-x_0} = \frac{1}{2} \quad f_{12} = \frac{f_2-f_1}{x_2-x_1} = 5 \quad f_{012} = \frac{f_{12}-f_{01}}{x_2-x_0} = \frac{3}{2} \tag{1}$$

Hence we get

$$P(x) = 2 + \frac{1}{2}x + \frac{3}{2}x(x-2) = \frac{3}{2}x^2 - \frac{5}{2}x + 2 \tag{2}$$

Exercise 2. **a.** The code can be found in the listings 1 to 3.

```
function [ y ] = mylagrange( x,xi,yi )
y = zeros(size(x));
for k = 1:length(x)
    for i = 1:length(xi)
        prod = 1;
        for j = 1:length(xi)
            if (j ~= i)
                prod = prod * (x(k) - xi(j))/(xi(i) - xi(j));
            end
        end
        y(k) = y(k) + yi(i) * prod;
    end
end
end
```

LISTING 1. `src/mylagrange.m`

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```
function [ y ] = mypwlinear( x,xi,yi )
y = zeros(size(x));
for k = 1:length(x)
   for i = 1:length(xi)-1
      if x(k) >= xi(i) && x(k) <= xi(i+1)
          rem = (yi(i+1) * xi(i) - yi(i) * xi(i+1))/(xi(i+1) - xi(i));
          y(k) = (yi(i+1) - yi(i))/(xi(i+1) - xi(i)) * x(k) - rem;
          break;
      end
   end
end
end
```

LISTING 2. `src/mypwlinear.m`

b. As one can see in figure 1 (or in figure 2) the Lagrange interpolant $L_8(x)$ has a peak between the first two nodes whereas the piece wise linear spline and the cubic spline are almost identical up to the difference between the fourth and fifth node. Nice to see is, that all the evaluated interpolants intersect in the provided values y_i for $i = 0, \ldots, n$. But this is clear, since this is the idea behind interpolation.

c. We get $\varepsilon(0.4) = -0.039582865144328$, $\varepsilon(0.65) = -0.001539115912818$ for the Lagrange interpolation, $\varepsilon(0.4) = 0.001150000000000$, $\varepsilon(0.65) = 0.001708333333333$ for the piece wise linear interpolation and for the cubic spline interpolation $\varepsilon(0.4) = 0.001009680243715$, $\varepsilon(0.65) = 0.000532507189652$.
As already mentioned in part **b.**, the piece wise linear spline and cubic spline coincide very well in $\sigma = 0.4$, whereas the Lagrange interpolant is very bad at this point since due to the C^∞ property of a polynomial interpolant, the slope can not increase drastically, hence we get a local minima around $\sigma = 0.4$. For $\sigma = 0.65$ however all the interpolated values for the three methods differ.

Exercise 3. (a) The plot can be found in figure 3.
(b) The plot can be found in figure 4. One observes that the Runge phenomenon occurs.
(c) The plot can be found in figure 5. The four derivatives are calculated using `MAPLE` (the third and the fourth derivative are quite ugly, see `ex_3_c.mw`). If we have a polynomial interpolant with nodes $x_0, \ldots, x_n$, the error is bounded by (in the equidistant case)

$$\begin{aligned}
|f(x) - P_{01\ldots n}(x)| &\leqslant \frac{1}{(n+1)!}\left|\prod_{k=0}^{n}(x - x_k)\right| \max_{x\in[a,b]} |f^{(n+1)}(x)| \\
&\leqslant \frac{1}{(n+1)!}\max_{x\in[a,b]} |f^{(n+1)}(x)| \frac{h^{n+1}n!}{4} \\
&= \frac{h^{n+1}}{4(n+1)}\max_{x\in[a,b]} |f^{(n+1)}(x)|
\end{aligned}$$

From figure 5 we can tell that the infinity norm of the derivatives on the interval $[-4, 4]$ monotoniously increases with the order. Thus the upper bound of the error estimation gets bigger and bigger. This must not mean that the Runge phenomenon occurs, but it is an indicator.
(d) The plot of the error can be found in figure 6. We observe convergence of the type Ca^x since the error is almost linear in a semi-logarithmic plot. This agrees quite well

```
function [ y ] = myspline( x,xi,yi )
%Cubic splinefunction from the the nodes and values (x_i,y_i), i = 0,...,n
%evaluated at the array x according to the chapter 2.5.2 in _[1], p. 117 -
%121.
%
%References:
%----------
%[1]    Roland W. Freund and Ronald H.W. Hoppe, Stoer/Bulirsch: Numerische
%       Mathematik I, 10., neu bearbeitete Auflage, Springer Verlag, 2007.
y = zeros(size(x));
n = length(xi);
h = diff(xi);
lambda = [0, h(2:end)./(h(1:end-1) + h(2:end))];
mu = [h(1:end-1)./(h(1:end-1) + h(2:end)), 0];
U = sparse(1:n-1,2:n,lambda,n,n);
L = sparse(2:n,1:n-1,mu,n,n);
D = sparse(1:n,1:n,2*ones(1,n),n,n);
d = sparse([0, 6./(h(1:end-1) + h(2:end)) .* ...
    ((yi(3:end) - yi(2:end-1))./h(2:end) - (yi(2:end-1) - yi(1:end-2)) ...
    ./h(1:end-1)), 0]');
M = (L + U + D)\d;
for k = 1:length(x)
    for j = 1:length(xi)-1
        if x(k) >= xi(j) && x(k) <= xi(j+1)
            beta_j = (yi(j+1)-yi(j))/h(j) - (2 * M(j) + M(j+1))/6 * h(j);
            gamma_j = M(j)/2;
            delta_j = (M(j+1) - M(j))/(6 * h(j));
            y(k) = yi(j) + beta_j * (x(k) - xi(j)) + ...
                gamma_j * (x(k) - xi(j))^2 + delta_j * (x(k) - xi(j))^3;
            break;
        end
    end
end
end
```

Listing 3. src/myspline.m

with the theoretical bound

$$\varepsilon_n \leqslant \frac{1}{2^n(n+1)!}\left(\frac{b-a}{2}\right)^{n+1} \max_{x\in[-4,4]} |f^{(n+1)}(x)| \tag{3}$$

considered as a function of $n \in \mathbb{N}$.

(e) The plot can be found in figure 7. The behaviour is of the form Cx^a since it is linear in a double logarithmic plot. This agrees quite well with the theoretical bound

$$\varepsilon_n \leqslant \frac{1}{8}H^2 \max_{x\in[-4,4]} |f^{(2)}(x)| \tag{4}$$

considered as a (discrete) function of $H \in \{(b-a)/n | n \in \mathbb{N}_{>0}\}$.

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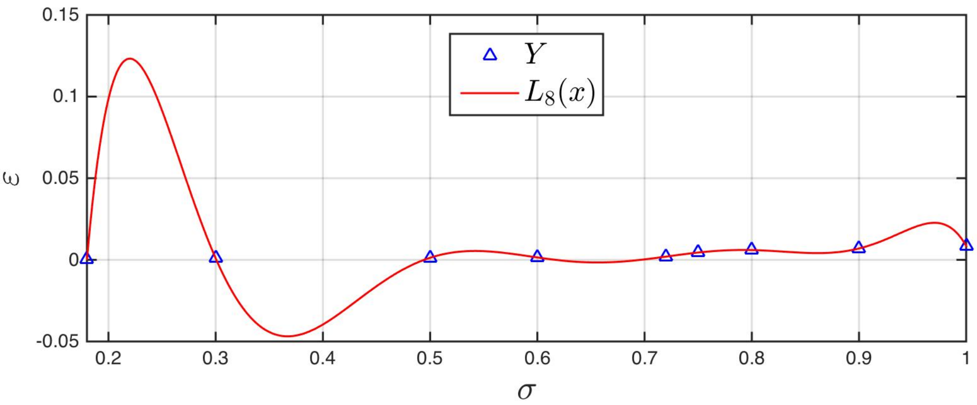

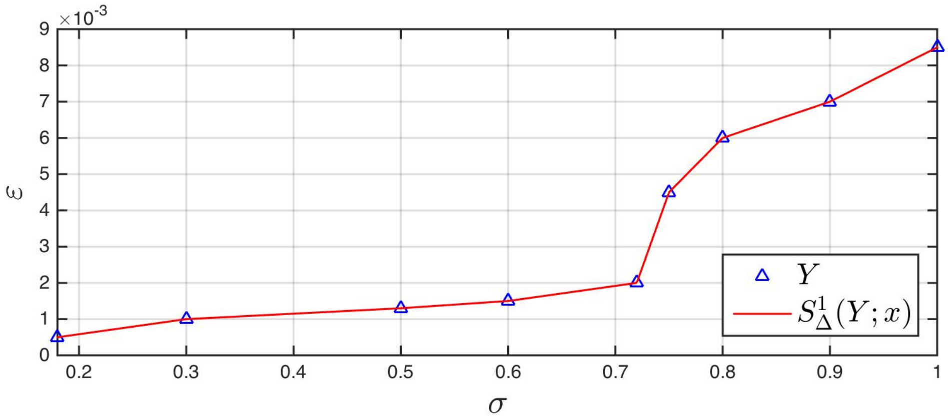

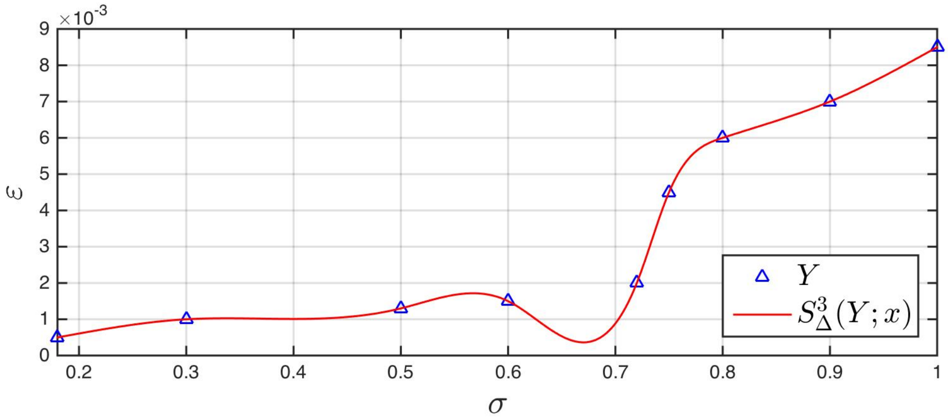

FIGURE 1. Plots of the interpolated test data using several methods.

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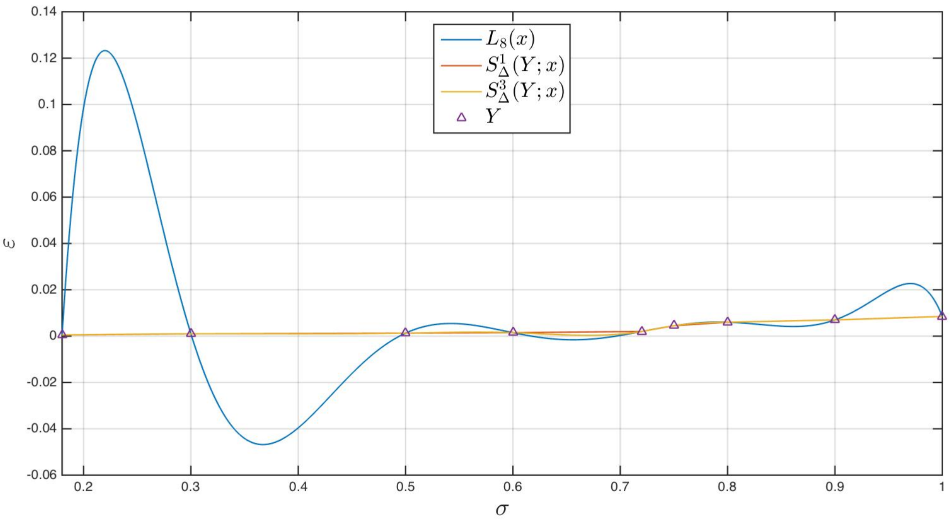

FIGURE 2. Plot of the interpolated test data using several methods.

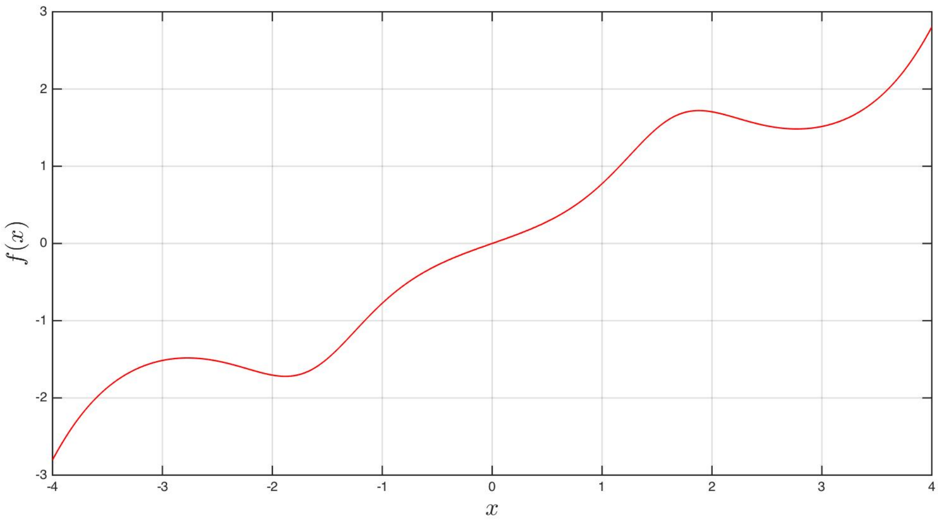

FIGURE 3. Plot of the graph of the function $f(x)$ on the interval $[-4, 4]$.

Exercise 4. As one can see in figure 8, the exact solution and the interpolation values agree almost identically. In figure 9 however, the Runge phenomenon occurs for the Lagrange polynomial (its degree is 20, so one would expect that).

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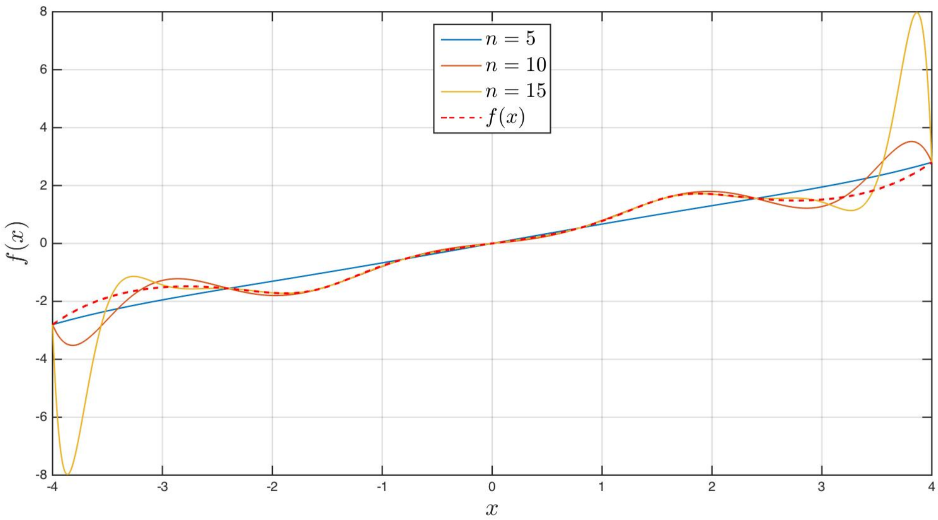

FIGURE 4. Plot of the interpolated graph of the function $f(x)$ on the interval $[-4, 4]$.

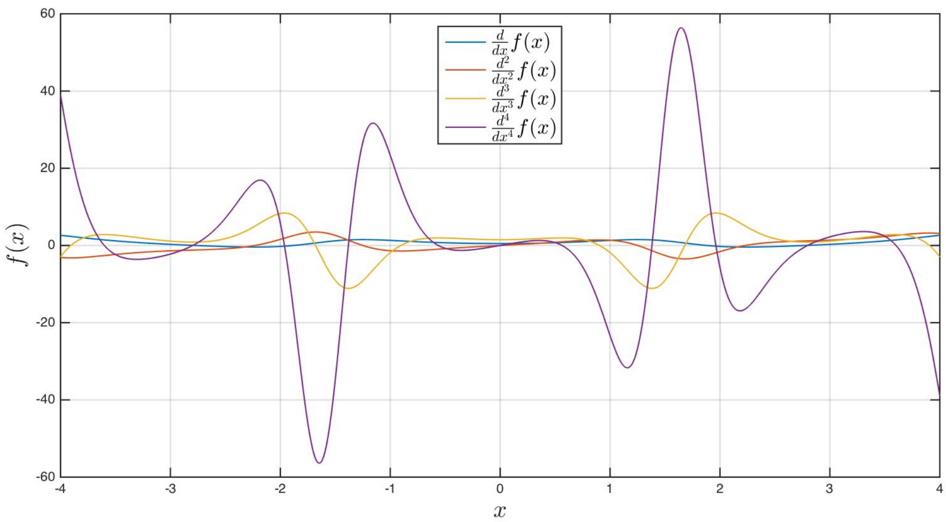

FIGURE 5. Plot of the first four derivatives of the function $f(x)$.

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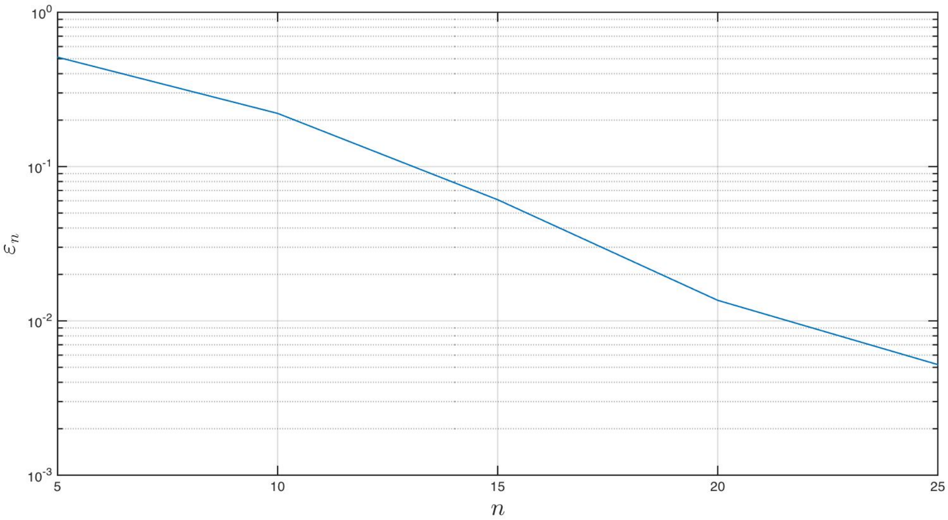

FIGURE 6. Plot of the error of the Lagrange interpolant for a various number of nodes.

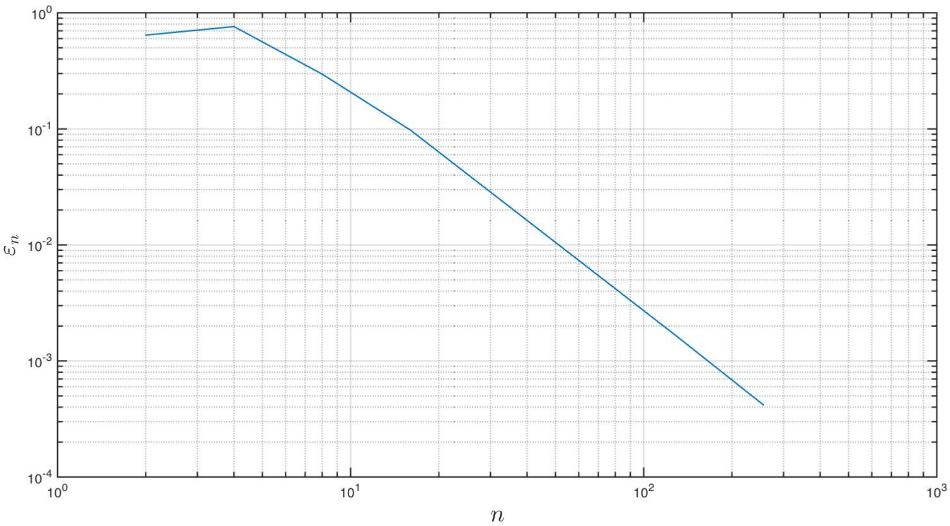

FIGURE 7. Plot of the error of the piece wise linear spline interpolant for a various number of nodes.

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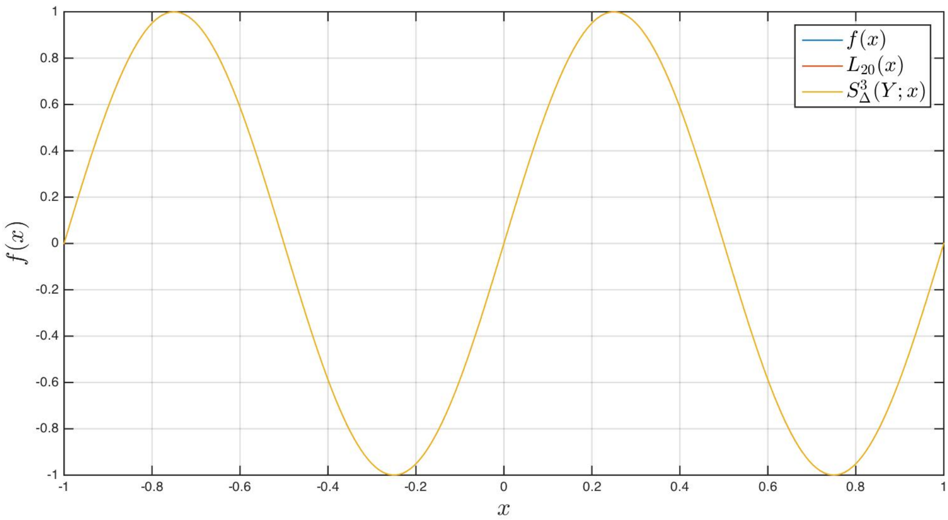

FIGURE 8. Plot of the nonperturbed interpolation.

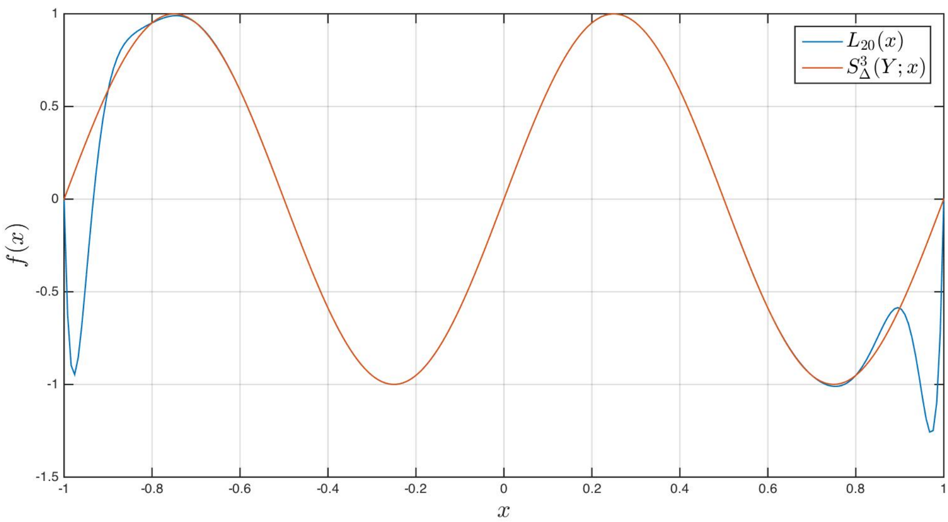

FIGURE 9. Plot of the perturbed interpolation.