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SOLUTIONS SHEET 1

Exercise 1. First observe, that the ordinate values are almost zero (scaling factor is 10^{-14}). This is due to the fact, that $f(1) = 0$. Let $x, y \in \mathbb{R}$. Further let $\mathrm{fl}(x) := x(1+\varepsilon_x)$ and $\mathrm{fl}(y) := y(1+\varepsilon_y)$ with $|\varepsilon_x|, |\varepsilon_y| \leqslant \mathrm{eps}$ denote the *floating point representations* of x and y respectively (eps is the machine precision). We then have for the relative error of the difference $x - y$

$$\varepsilon_{x-y} = \frac{x}{x-y}\varepsilon_x - \frac{y}{x-y}\varepsilon_y \tag{1}$$

if $x - y \neq 0$. If $x \approx y$ we may have a large relative error. Since $f(1) = 0$ and the expression $f(x)$ is evaluated from left to right, we get that

$$f(x) = \underbrace{x^7 - 7x^6 + 21x^5 - 35x^4 + 35x^3 - 21x^2 + 7x}_{\approx 1} - 1 \tag{2}$$

Thus a large relative error results (of order 10^{-14}). Let us investigate this further. If we name the underbraced expression $f'(x)$, we get a relative error $\varepsilon_{f'(x)}(x)$. Further $\varepsilon_1 = 0$, since 1 is exactly representable in any binary system. Thus the oscillation of the graph is the result of a slightly varying relative error $\varepsilon_{f(x)}(x)$. This is the result of round-off errors in the subtractions and additions performed in $f'(x)$. Since, if we evaluate $(x-1)^7$ (which is the same as our $f(x)$), we get practically no error as one can see in the plot.

```
f = @(x) x.^7 - 7 * x.^6 + 21 * x.^5 - 35 * x.^4 + 35 * x.^3 ...
    - 21 * x.^2 + 7 * x - 1;
g = @(x) (x - 1).^7;
M = 2e-8;
steps = 2 * M/4e+2;
x = 1 - M:steps:1 + M;
plot(x, f(x), 'color', 'red');
hold on;
plot(x, g(x), 'color', 'green');

%Plot settings
set(gca, 'looseinset', get(gca, 'tightinset'));
leg = legend('$f(x)$', '$(x - 1)^7$');
set(leg, 'fontsize', 14, 'interpreter', 'latex');
ylim([-15e-15, 15e-15]);
xlabel('$x$', 'interpreter', 'latex', 'fontsize', 18);
ylabel('$f(x)$', 'interpreter', 'latex', 'fontsize', 18);
grid on;
saveas(gcf, 'ex_1.jpg');
```

Exercise 2. We see in the plot, that the convergence speed is slow. Remarkable is also, that there are peaks. This is due to the randomness of the algorithm, even when more points are used to approximate π, there is a certain possibility, that more points lay outside than with fewer points.

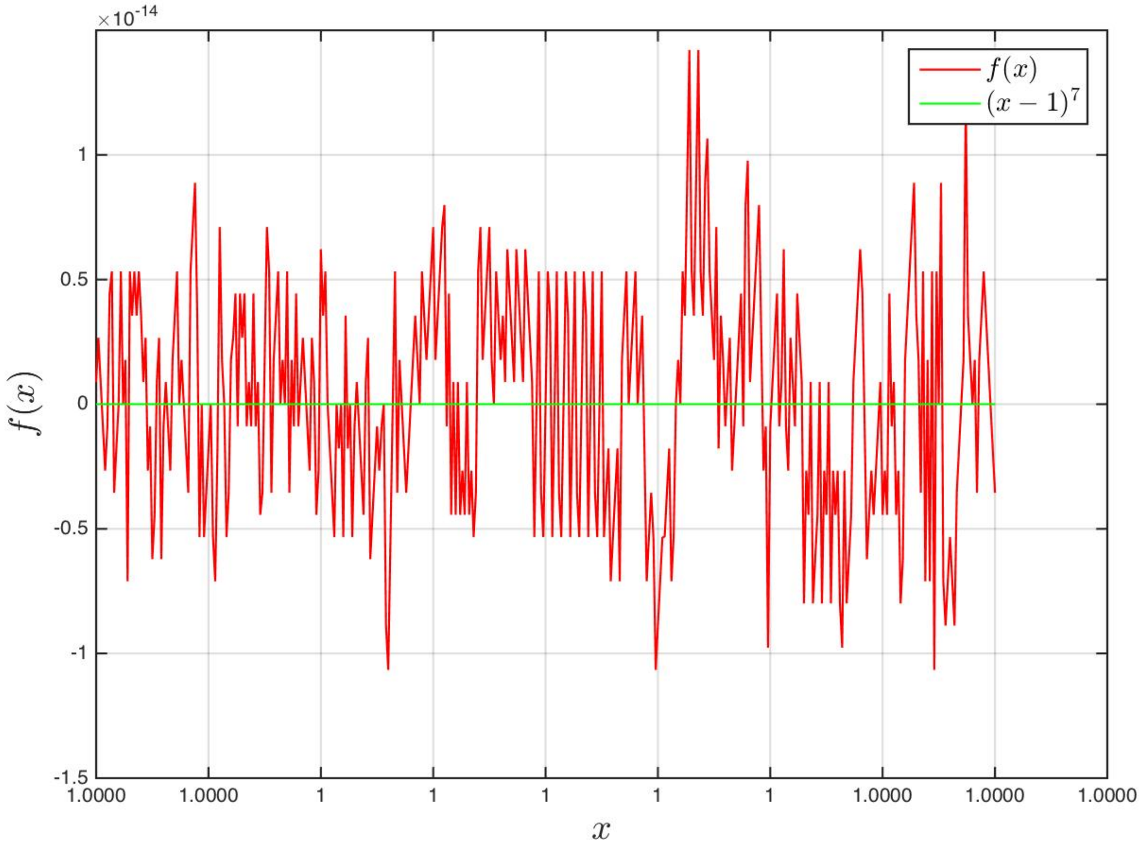

FIGURE 1. Plot of the polynomial function $f(x)$ with 401 equidistant grid points.

```
%Set the seed for getting always the same result
rng(9287641);
rel_err = @(approx, exact) abs(approx - exact)./abs(exact);
n = 1e+3:1e+3:1e+5;
values = arrayfun(@(n) pi_n(n), n);
semilogy(n, rel_err(values, pi), 'color', 'red');
grid on;

%Plot settings
set(gca, 'looseinset', get(gca, 'tightinset'));
ylabel('$\delta(\pi_n)$', 'interpreter', 'latex', 'fontsize', 18);
xlabel('$n$', 'interpreter', 'latex', 'fontsize', 18);
saveas(gcf, 'ex_2.jpg');
```

```
function [ out ] = pi_n( n )
coord = rand(2,n);
m = 0;
for i = 1:n
   if coord(1,i)^2 + coord(2,i)^2 <= 1
```

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```
        m = m + 1;
    else
        continue
    end
out = 4 * m/n;
end
```

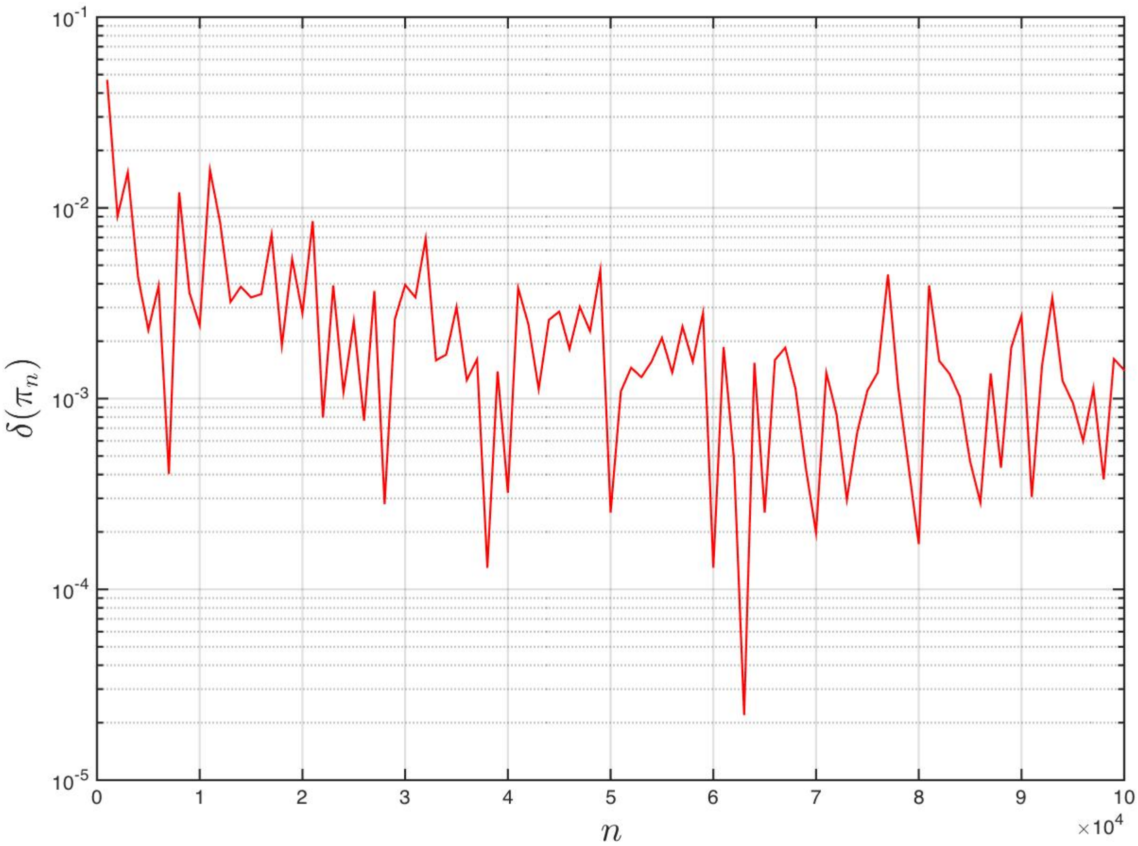

FIGURE 2. Relative error of the approximation of π in a semi-logarithmic plot for a more apealing visualization.

Exercise 3. As one can see in the plot for $n > 16$ the forward recursion diverges, formally $\lim_{n\to\infty} I_n = +\infty$, in contradiction to $\lim_{n\to\infty} I_n = 0$. The reason for this divergence is that the relative error of the subtraction $nI_{n-1} - 1$ for $n \in \mathbb{N}_{>0}$ accumulates. Since, if we inspect the sequence explicitely, we have

$$I_0 = \exp(1) - 1 \quad I_1 = \exp(1) - 2 \quad I_2 = 2\exp(1) - 5 \quad I_3 = 6\exp(1) - 16 \quad \dots \tag{3}$$

Hence the difference of the two terms goes to zero. This is necessary, since the limit of I_n is zero but thus results in the numerical evaluation of the subtraction of almost equal terms.

```
N = 18;
forward = arrayfun(@(n) forward_rec(n), 1:N);
```

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```
backward = arrayfun(@(n) backward_rec(n, 50), 1:N);
plot(1:N, forward);
hold on;
plot(1:N, backward);
grid on;
leg = legend('Forward', 'Backward');
set(leg, 'fontsize', 14);
xlabel('$n$', 'interpreter', 'latex', 'fontsize', 18);
ylabel('$I_n$', 'interpreter', 'latex', 'fontsize', 18);
saveas(gcf, 'ex_3.jpg');
```

```
function [ I_n ] = forward_rec( n )
    if n == 0
        I_n = exp(1) - 1;
    else
        I_n = n * forward_rec(n - 1) - 1;
    end
end
```

```
function [ I_n ] = backward_rec( n, N )
    if n == N
        I_n = 0;
    else
        I_n = (backward_rec(n + 1, N) + 1)/(n+1);
    end
end
```

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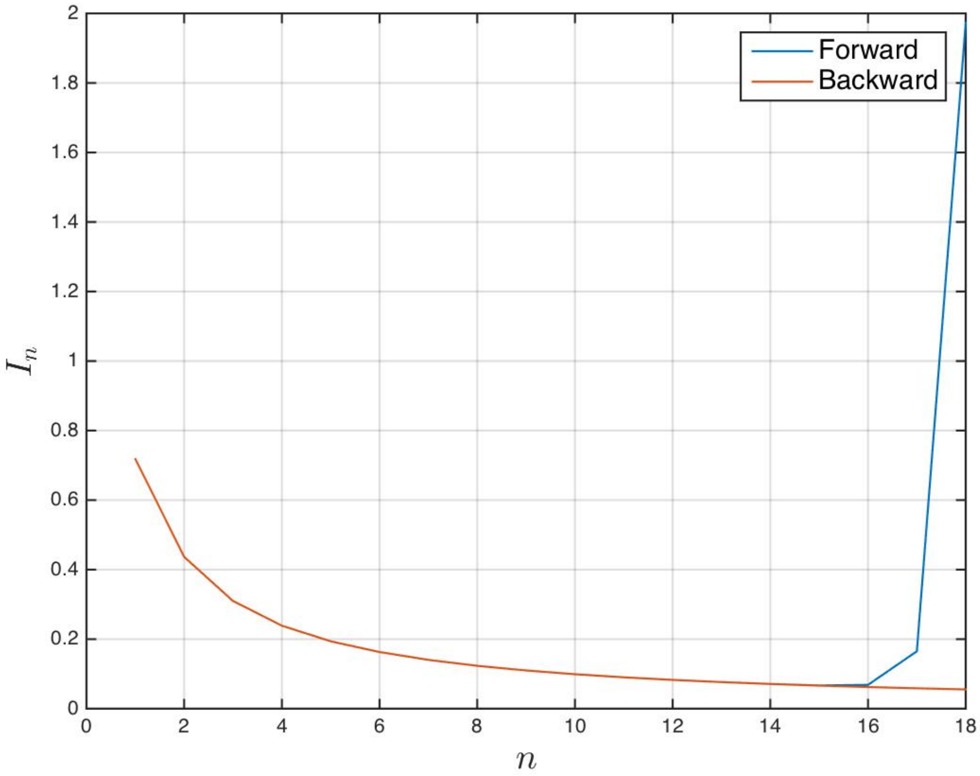

FIGURE 3. Plot of forward and backward recurrence relation of the integrals I_n up to $n = 50$.