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SOLUTIONS SHEET 8

Exercise 1. The source code can be found in listing 1.

```
format long;
f = @(x) exp(-x.^2./2);
a = 0;
b = 2;
h1 = 1;
h2 = .5;
exact = 1.196288013322608;
%Simpson
I1 = composite_simpson(f,a,b,h1);
I2 = composite_simpson(f,a,b,h2);
IR = I2 + (I2 - I1)/(2^4 - 1);
disp(abs(I1 - exact));disp(abs(I2 - exact));disp(abs(IR - exact));
%Gauss
I1 = composite_gaussian_quadrature(f,a,b,h1);
I2 = composite_gaussian_quadrature(f,a,b,h2);
IR = I2 + (I2 - I1)/(2^4 - 1);
disp(abs(I1 - exact));disp(abs(I2 - exact));disp(abs(IR - exact));
```

Listing 1. `src/ex_1.m`

Let $I^* := 1.196288013322608$. As one can see in table 1 I_R is more accurate than I_1 and I_2 in both cases.

Exercise 2. **a.** By the Lagrange interpolation formula for the two nodes $x_0 := 0$ and $x_1 := b$ we get on the interval $[0, b]$ the approximation

$$\frac{f(x)}{\sqrt{x}} \approx \frac{1}{\sqrt{x}}\left(f(0)\frac{x - x_1}{x_0 - x_1} + f(b)\frac{x - x_0}{x_1 - x_0}\right) \tag{1}$$

for the function $f(x)$. Definite integration from 0 to b yields

	$\lvert I_1 - I^*\rvert$	$\lvert I_2 - I^*\rvert$	$\lvert I_R - I^*\rvert$
Simpson	$1.223329169979248 \cdot 10^{-4}$	$6.279754676263849 \cdot 10^{-6}$	$1.457122811743261 \cdot 10^{-6}$
Gauss-Legendre	$8.284213132592200 \cdot 10^{-5}$	$4.204645835725884 \cdot 10^{-6}$	$1.037853196939054 \cdot 10^{-6}$

Table 1. Table of errors of discretized integration.

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$$
\begin{aligned}
\int_0^b \frac{f(x)}{\sqrt{x}}dx &\approx \int_0^b \frac{1}{\sqrt{x}}\left(f(0)\frac{x-x_1}{x_0-x_1}+f(b)\frac{x-x_0}{x_1-x_0}\right)dx \\
&= \frac{f(0)}{x_0-x_1}\left(\int_0^b \sqrt{x}dx - x_1\int_0^b \frac{dx}{\sqrt{x}}\right)+\frac{f(b)}{x_1-x_0}\left(\int_0^b \sqrt{x}dx - x_0\int_0^b \frac{dx}{\sqrt{x}}\right) \\
&= \frac{f(b)}{b}\int_0^b \sqrt{x}dx - \frac{f(0)}{b}\left(\int_0^b \sqrt{x}dx - b\int_0^b \frac{dx}{\sqrt{x}}\right) \\
&= \frac{2\sqrt{b}}{3}f(b)-\left(\frac{2\sqrt{b}}{3}-2\sqrt{b}\right)f(0) \\
&= \frac{4\sqrt{b}}{3}f(0)+\frac{2\sqrt{b}}{3}f(b)
\end{aligned}
$$

For $\alpha x+\beta \in \mathbb{R}[x]$ we have

$$
\begin{aligned}
\int_0^b \frac{\alpha x+\beta}{\sqrt{x}}dx &= \alpha\int_0^b \sqrt{x}dx+\beta\int_0^b \frac{dx}{\sqrt{x}}dx \\
&= \frac{2\alpha b^{3/2}}{3}+2\beta\sqrt{b} \\
&= \frac{4\beta\sqrt{b}}{3}+\frac{2\sqrt{b}}{3}(\alpha b+\beta) \\
&= \frac{4\sqrt{b}}{3}f(0)+\frac{2\sqrt{b}}{3}f(b)
\end{aligned}
$$

Hence the integration formula is exact for polynomials of degree one.

b. Define $x_0 := \frac{1}{7}\left(3-2\sqrt{\frac{6}{5}}\right)$ and $x_1 := \frac{1}{7}\left(3+2\sqrt{\frac{6}{5}}\right)$. By **a.** we get

$$
\begin{aligned}
\int_0^1 \frac{f(x)}{\sqrt{x}}dx &\approx \int_0^1 \frac{1}{\sqrt{x}}\left(f(x_0)\frac{x-x_1}{x_0-x_1}+f(x_1)\frac{x-x_0}{x_1-x_0}\right)dx \\
&= \frac{f(x_0)}{x_0-x_1}\left(\int_0^1 \sqrt{x}dx - x_1\int_0^1 \frac{dx}{\sqrt{x}}\right)+\frac{f(x_1)}{x_1-x_0}\left(\int_0^1 \sqrt{x}dx - x_0\int_0^1 \frac{dx}{\sqrt{x}}\right) \\
&= \frac{f(x_0)}{x_0-x_1}\left(\frac{2}{3}-2x_1\right)+\frac{f(x_1)}{x_1-x_0}\left(\frac{2}{3}-2x_0\right) \\
&= \left(1+\frac{1}{3}\sqrt{\frac{5}{6}}\right)f(x_0)+\left(1-\frac{1}{3}\sqrt{\frac{5}{6}}\right)f(x_1)
\end{aligned}
$$

Since only verification is requested and not explicitely verification by hand I use `MAPLE` for that task. Consider the ansatz $f(x) := \alpha x^3+\beta x^2+\gamma x+\delta \in \mathbb{R}_3[x]$. We get

$$
\int_0^1 \frac{f(x)}{\sqrt{x}}dx = \frac{2}{7}\alpha+\frac{2}{5}\beta+\frac{2}{3}\gamma+2\delta = \left(1+\frac{1}{3}\sqrt{\frac{5}{6}}\right)f(x_0)+\left(1-\frac{1}{3}\sqrt{\frac{5}{6}}\right)f(x_1) \tag{2}
$$

as can be seen in the file `ex_2_b.mw`.

Exercise 3. Define $I_C^* := 1.80904847580054$ and $I_S^* := 0.62053660344676$. For the error decay assume $\varepsilon = Ch^b$.

a. The code can be found in listing 2.

```
function [ I ] = composite_midpoint( f,a,b,h )
xi = a:h:b;
M = 1/2 * (xi(1:(end-1)) + xi(2:end));
I = h * sum(f(M));
end
```

Listing 2. `src/composite_midpoint.m`

b. We have

h	$\lvert I_C - I_C^*\rvert$	$\lvert I_S - I_S^*\rvert$
0.2	0.0096277	0.0013432
0.1	0.0061012	0.00057142
0.05	0.0042499	0.00021392
0.025	0.0030092	0.000077517

For I_C we get the rates

$$0.658116078264471 \qquad 0.521645128395790 \qquad 0.498037439756502 \tag{3}$$

and for I_S

$$1.233022241755271 \qquad 1.417502784497086 \qquad 1.464478005457666 \tag{4}$$

Thus the convergence rate is remarkably lower than the theoretical one of $O(h^2)$. This is due to the singularity at $x = 0$.

c. The integrals are given by $2\int_0^1 \cos(t^2)dt$ and $2\int_0^1 \sin(t^2)dt$ respectively. We have

h	$\lvert I_C - I_C^*\rvert$	$\lvert I_S - I_S^*\rvert$
0.2	0.0056101	0.0036584
0.1	0.0014025	0.00090401
0.05	0.00035062	0.00022535
0.025	0.000087653	0.000056295

For I_C we get the rates

$$2.000024781741962 \qquad 2.000036571814530 \qquad 2.000011012369889 \tag{5}$$

and for I_S

$$2.016811827544148 \qquad 2.004209444877414 \qquad 2.001052724560917 \tag{6}$$

Hence the experimental computed rate is approximately $O(h^2)$ which agrees quite well with the theoretical convergence rate.

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d. We have

$$|I_C - I_C^*| = 0.00043208 \qquad |I_S - I_S^*| = 0.00020559 \tag{7}$$

Hence the discretizations obtained by the weighted Gauss rule are quite accurate.

Conclusions: We see that it is better to transform an integral with singularity to an integral without singularity (if possible). This is due to the behaviour of the function at that point: it can be that the function goes up to infinity (or down) and hence is badly approximated by a discretization since it is a limit (thus infinite) process. Thus strategy **c.** is here clearly more efficient than strategy **b.**. Further we see that the integral I_C behaves much worser than I_S. However this is clear by considering the graphs of the functions: I_C goes up to infinity at $x = 0$ whereas I_S goes to zero. In terms of errors and calculation complexity (number of function evaluations and operations) strategy **d.** is the most efficient. This is due to the high order and non-compositedness. With strategy **b.** one never reaches the exactness of **d.** for I_C and for I_S one needs more than 20 subintervals. In strategy **c.** also around 20 subintervals are needed to reach the accuracy of **d.**.