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SOLUTIONS SHEET 1

Exercise 1. We use **Satz 1.4** from the lecture. For $b \in \mathbb{R}_{>0}$ consider

$$f : \begin{cases} [0,b] \times \mathbb{R} \longrightarrow \mathbb{R} \\ (x,y) \mapsto \exp(x^2)\cos(2y) \end{cases} \tag{1}$$

Observe, that

$$|f(x,y)| = \left|\exp(x^2)\cos(2y)\right| \leqslant \exp(x^2) \leqslant \exp(b^2) \tag{2}$$

on $[0,b] \times \mathbb{R}$ since $\exp(x^2)$ is monotone increasing by $\left(\exp(x^2)\right)' = 2x\exp(x^2) \geqslant 0$. Further f is continuous on $\mathbb{R} \times \mathbb{R}$ (and hence on the specified set) since for any $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}$ it holds, that

$$\begin{aligned} \lim_{(x,y)\to(x_0,y_0)} \exp(x^2)\cos(2y) &= \lim_{r\to 0^+} \exp\left(x_0^2 + 2r\cos(\varphi) + r^2\cos^2(\varphi)\right)\cos(2y_0 + 2r\sin(\varphi)) \\ &= \exp(x_0^2)\cos(2y_0) \end{aligned}$$

Since exp and cos are continuous. Left to show is, that f fullfills the *Lipschitz condition* (the continuity of f would also follow from that). Let $y_1, y_2 \in \mathbb{R}$. Then

$$\begin{aligned} \left|\exp(x^2)\cos(2y_1) - \exp(x^2)\cos(2y_2)\right| &= \exp(x^2)\left|\cos(2y_1) - \cos(2y_2)\right| \\ &= 2\exp(x^2)\left|\sin(\xi)(y_1 - y_2)\right| \\ &\leqslant 2\exp(x^2)|y_1 - y_2| \\ &\leqslant 2\exp(b^2)|y_1 - y_2| \end{aligned}$$

By the *mean value theorem* and some ξ between y_1 and y_2. Hence f is Lipschitz with $L = 2\exp(b^2)$. Thus, by **Satz 1.4** there exists for every $x_0 \in [0,b]$ (we have here $x_0 = 0$) and initial value $y_0 \in \mathbb{R}$ (here $y_0 = 2$) a *unique* $y \in C^1[0,b]$ which fullfills the initial value problem $y' = f(x,y)$ and $y(x_0) = y_0$. Especially this is true for any $b \in \mathbb{R}_{>0}$
We use **Satz 1.6**. Let $\hat{y}_0 \in \mathbb{R}$ denote the non-exact initial value and y_0 the exact. Further set $\hat{y}(x) := y(x_0, \hat{y}_0)$. Further let $y \in C^1[0,b]$ be the solution of the initial value problem $y' = f(x,y)$ and $y(0) = y_0$. Hence we get the estimation

$$|\hat{y}(x_1) - y(x_1)| \leqslant \exp(Lx_1)|\hat{y}_0 - y_0| = \exp\left(2\exp\left(b^2\right)x_1\right)|\hat{y}_0 - y_0| \overset{!}{=} 10^{-3} \tag{3}$$

Thus

$$|\hat{y}_0 - y_0| = 10^{-3}\exp\left(-2\exp\left(b^2\right)x_1\right) \tag{4}$$

Exercise 2. Use *separation of the variables.* Set $g(x) := x^2$ and $h(x) := x^3$. The solution is given by solving

$$\int_{x_0}^{x} \frac{y'(t)}{h(y(t))}dt = \int_{x_0}^{x} g(t)dt \tag{5}$$

Next, substitute $u := y(x)$. Hence $du = y'(x)dt$ and so

$$\int_c^{y(x)} \frac{1}{h(u)} du = \int_{x_0}^{x} g(t)dt \tag{6}$$

By the *fundamental theorem of calculus* we get

$$\int_0^x g(t)dt = \int_0^x t^2 dt = \left[\frac{t^3}{3}\right]_0^x = \frac{x^3}{3} \tag{7}$$

and for $c \neq 0$

$$\int_c^{y(x)} \frac{1}{h(u)} du = \int_c^{y(x)} \frac{1}{u^3} du = \left[-\frac{1}{2u^2}\right]_c^{y(x)} = \frac{1}{2}\left(\frac{1}{c^2} - \frac{1}{y^2(x)}\right) \tag{8}$$

We get

$$\boxed{y(x) = \pm\frac{1}{\sqrt{\frac{1}{c^2} - \frac{2x^3}{3}}} = \pm\sqrt{\frac{3c^2}{3 - 2x^3c^2}}} \tag{9}$$

For $c = 0$ we have the solution $y(x) \equiv 0$. The root in 9 is only real, if $\frac{1}{c^2} - \frac{2x^3}{3} > 0$, equivalently $x < \sqrt[3]{\frac{3}{2c^2}}$. Hence $I = \left[0, \sqrt[3]{\frac{3}{2c^2}}\right[$. For $c = 0$ we get $I = [0, +\infty[$.

Exercise 3. First, we have to determine the solution of the homogenous problem $y'(x) = 3x^2y(x)$. Define $h(x) := x$ and $g(x) = 3x^2$. Then the homogenous problem is equivalent to the equation $y'(x) = g(x)h(y(x))$. Further we get $\frac{y'(x)}{h(y(x))} = g(x)$. Next we consider for $x_0 := 0$

$$\int_{x_0}^{x} \frac{y'(t)}{h(y(t))} dt = \int_{x_0}^{x} g(t)dt \tag{10}$$

Next, substitute $u := y(x)$. Hence $du = y'(x)dt$ and so

$$\int_c^{y(x)} \frac{1}{h(u)} du = \int_{x_0}^{x} g(t)dt \tag{11}$$

By the *fundamental theorem of calculus* we get

$$\int_{x_0}^{x} g(t)dt = 3\int_{x_0}^{x} t^2 dt = 3\left[\frac{t^3}{3}\right]_{x_0}^{x} = x^3 - x_0^3 \tag{12}$$

and

$$\int_c^{y(x)} \frac{1}{h(u)} du = \int_c^{y(x)} \frac{1}{u} du = [\log(u)]_c^{y(x)} = \log\left(\frac{y(x)}{c}\right) \tag{13}$$

Taking the exponential on 11 we get

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$$\boxed{y(x) = c\exp(x^3)} \tag{14}$$

For solving the inhomogene problem, we consider the real-valued function $c(x)$. We then have by the *Leibniz rule*

$$y'(x) = c'(x)\exp(x^3) + 3x^2c(x)\exp(x^3) = c'(x)\exp(x^3) + 3x^2y(x) \tag{15}$$

By comparing we get that $c'(x) = 2x^5\exp(-x^3)$. Definit integration yields

$$\int_{x_0}^{x} c'(t)dt = 2\int_{x_0}^{x} t^5\exp(-t^3)dt \tag{16}$$

Now we use *substitution* and *partial integration* to compute above integral. First, set $u = t^3$. Then we have, that $du = 3t^2dt = 3t^{2/3}dt$. Hence

$$\begin{aligned}
2\int_{x_0}^{x} t^5\exp(-t^3)dt &= \frac{2}{3}\int_{x_0^3}^{x^3} u\exp(-u)du \\
&= \frac{2}{3}\left((-u\exp(-u))|_{x_0^3}^{x^3} + \int_{x_0^3}^{x^3}\exp(-u)du\right) \\
&= \frac{2}{3}\left((-u\exp(-u))|_{x_0^3}^{x^3} + [-\exp(u)]_{x_0^3}^{x^3}\right) \\
&= \frac{2}{3}\left(-x^3\exp(-x^3) + 1 - \exp(x^3)\right) \\
&= \frac{2}{3} - \frac{2}{3}\exp(-x^3)\left(x^3 + 1\right)
\end{aligned}$$

Thus the solution is given by

$$\boxed{y(x) = \left(c + \frac{2}{3}\right)\exp(x^3) - \frac{2x^3}{3} - \frac{2}{3}} \tag{17}$$

Exercise 4. We have that $y^{(1)}(x) = c(1 + x^2)$ and $y^{(2)}(x) = c + cx^2 + c\frac{x^4}{2}$. I state, that

$$y^{(k)}(x) = c + \int_0^x f(t, y^{(k-1)}(t))dt = c\sum_{j=0}^{k}\frac{x^{2j}}{j!} \tag{18}$$

Proof by induction on $k \in \mathbb{N}_{>0}$. For the base case take $k = 1$ or $k = 2$. Let $k > 2$. Then

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$$\begin{aligned}
y^{(k)}(x) &= c + \int\limits_0^x f(t, y^{(k-1)}(t))dt = c + \int\limits_0^x f\left(t, c\sum_{j=0}^{k-1}\frac{x^{2j}}{j!}\right)dt \\
&= c + 2c\int\limits_0^x t\sum_{j=0}^{k-1}\frac{t^{2j}}{j!}dt = c\left(1 + 2\sum_{j=0}^{k-1}\int\limits_0^x\frac{t^{2j+1}}{j!}dt\right) \\
&= c\left(1 + 2\sum_{j=0}^{k-1}\frac{1}{2(j+1)}\frac{t^{2(j+1)}}{j!}dt\right) = c\left(1 + \sum_{j=0}^{k-1}\frac{t^{2(j+1)}}{(j+1)!}dt\right) \\
&= c\left(1 + \sum_{j=1}^{k}\frac{t^2}{j!}dt\right) = c\sum_{j=0}^{k}\frac{x^{2j}}{j!}
\end{aligned}$$

Taking the limit $k \to \infty$ yields

$$\lim_{k\to\infty} y^{(k)}(x) = c\lim_{k\to\infty}\sum_{j=0}^{k}\frac{x^{2j}}{j!} = c\exp(x^2) \tag{19}$$

Since $\sum\limits_{j=0}^{\infty}\frac{x^j}{j!} = \exp(x)$ converges uniformly and absolutely for all $x \in \mathbb{R}$.