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SOLUTIONS SHEET 4

Exercise 11. The *local discretisation error* of the stated two-step method for the IVP

$$\begin{cases} z'(t) = f(t, z(t)) \\ z(x) = y \end{cases} \tag{1}$$

where $f \in C^1[a,b]$, $x \in [a,b]$ and $y \in \mathbb{R}$ is given by

$$\begin{aligned} h\tau(x,y;h) &= z(x+2h) - a_0 z(x) - a_1 z(x+h) - h\left[b_0 f\left(x, z(x)\right) + b_1 f\left(x+h, z(x+h)\right)\right] \\ &= \underbrace{z(x+2h) - a_0 z(x) - a_1 z(x+h) - h\left[b_0 z'(x) + b_1 z'(x+h)\right]}_{=:\,\Gamma(h;x,y)} \end{aligned}$$

by stipulating $x := x_j = x + jh$. Now, expanding the function $\Gamma(h;x,y)$ in $h = 0$ yields

$$\begin{aligned} \Gamma(h;x,y) =& \sum_{k=0}^{N} \frac{1}{k!} \frac{d^k \Gamma(0;x,y)}{dh^k} h^k + O\left(h^{N+1}\right) \\ =& z(x) - a_0 z(x) - a_1 z(x) \\ &+ \left(2z'(x) - a_1 z'(x) - \left[b_0 z'(x) + b_1 z'(x)\right]\right) h \\ &+ \frac{1}{2}\left(4z''(x) - a_1 z''(x) - \left[b_1 z''(x) + b_1 z''(x)\right]\right) h^2 + O\left(h^3\right) \\ =& z(x)\left(1 - a_1 - a_0\right) \\ &+ z'(x)\left(2 - a_1 - b_1 - b_0\right) h \\ &+ z''(x)\left(2 - \frac{1}{2}a_1 - b_1\right) h^2 + O\left(h^3\right) \end{aligned}$$

We arrive at solving the linear system of three equations and three unknowns

$$\begin{array}{ccccc} a_0 & & & = & 1 - a_1 \\ & b_1 & + \; b_0 & = & 2 - a_1 \\ & b_1 & & = & 2 - \frac{1}{2}a_1 \end{array} \tag{2}$$

Hence we get

$$a_0 = 1 - a_1 \qquad b_1 = 2 - \frac{1}{2}a_1 \qquad b_0 = -\frac{1}{2}a_1 \tag{3}$$

in terms of $a_1 \in \mathbb{R}$. Back substitution yields the method of order at least two

$$\boxed{\eta_{j+2} = \left(1 - a_1\right)\eta_j + a_1 \eta_{j+1} + h\left[\left(2 - \frac{1}{2}a_1\right) f\left(x_{j+1}, \eta_{j+1}\right) - \frac{1}{2}a_1 f\left(x_j, \eta_j\right)\right]} \tag{4}$$

and further for the *stability condition* we have to consider

$$\Psi(\mu) = \mu^2 - a_1 \mu + (a_1 - 1) \tag{5}$$

Solving for the roots yields

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$$\mu_{1,2} = \frac{1}{2}\left[a_1 \pm \sqrt{a_1^2 - 4a_1 + 4}\right] = \frac{1}{2}\left[a_1 \pm |a_1 - 2|\right] \tag{6}$$

For the stability condition it must hold that $|\mu_{1,2}| \leqslant 1$ and if $\mu_1 = 1$ or $\mu_2 = 1$, $\mu_2 \neq 1$ or $\mu_1 \neq 1$ respectively. Let us study different cases.

- $\underline{a_1 > 2}$: Then $|\mu_1| = |a_1 - 1|$, hence $|\mu_1| > 1$ and the stability condition is not fullfilled. Thus μ_2 must not even be considered.
- $\underline{a_1 = 2}$: Then $\mu_{1,2} = 1$, hence $|\mu_{1,2}| = 1$ and multiplicity two. Thus the stability condition is also not fullfileld.
- $\underline{a_1 = 0}$: Then $\mu_{1,2} = \pm 1$, hence the stability condition is fullfilled, since $|\mu_{1,2}| = 1$ with multiplicity one.
- $\underline{a_1 < 0}$: Then $\mu_1 = 1$ and $\mu_2 = a_1 - 1$. But then $|\mu_2| = |a_1 - 1| = |1 - a_1| > 1$. Hence the stability condition is again not fullfilled.
- $\underline{0 < a_1 < 2}$: Then $\mu_1 = 1$ and $\mu_2 = a_1 - 1$. Hence $|\mu_2| = |a_1 - 1|$. Let us do a case study. $\underline{1 \leqslant a_1 < 2}$: Then we have $|\mu_2| < 1$. $\underline{0 < a_1 \leqslant 1}$: Then we have $|\mu_2| = 1 - a_1 < 1$.

In conclusion, the stability condition is fullfilled for $a_1 \in [0, 2[$.

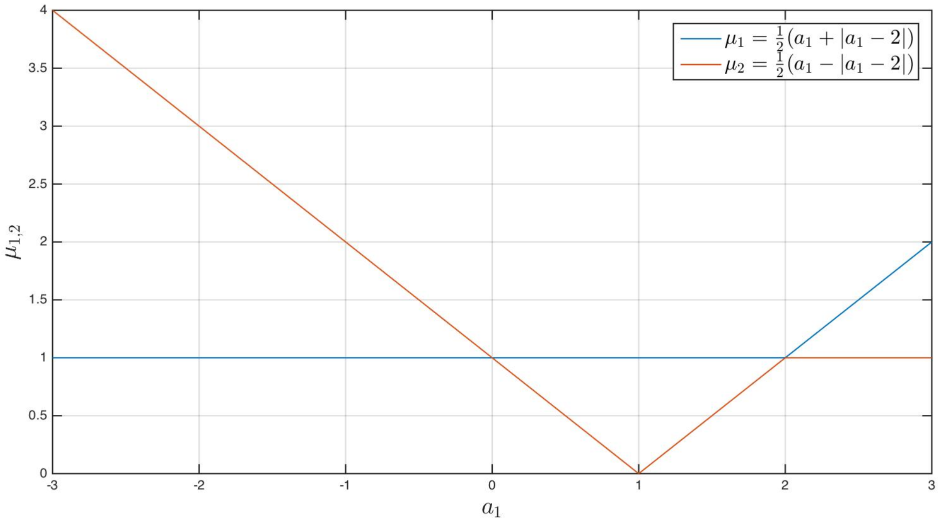

FIGURE 1. Plot of the roots $\mu_{1,2}$ of the function $\Psi(\mu)$.

Exercise 12. **a.** Let us write the equation as

$$u_{j+4} + \langle \tilde{a}, u_j^3 \rangle = u_{j+4} - \frac{7}{2}u_{j+3} + \frac{9}{2}u_{j+2} - \frac{5}{2}u_{j+1} + \frac{1}{2}u_j = 0 \tag{7}$$

Further write $p_2(j) \equiv \delta \in \mathbb{C}$ and $p_1(j) = \gamma_2 j^2 + \gamma_1 j + \gamma_0 \in \mathbb{C}[j]$. If $\alpha \in \mathbb{C}^4$, we have the linear system of equations

$$\begin{array}{ccccccccc} & & & & \gamma_0 & + & \delta & = \alpha_0 \\ \gamma_2 & + & \gamma_1 & + & \gamma_0 & + & \frac{\delta}{2} & = \alpha_1 \\ 4\gamma_2 & + & 2\gamma_1 & + & \gamma_0 & + & \frac{\delta}{4} & = \alpha_2 \\ 9\gamma_3 & + & 3\gamma_1 & + & \gamma_0 & + & \frac{\delta}{8} & = \alpha_3 \end{array} \tag{8}$$

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By considering the determinant is non-zero, hence the system is uniquely solvable in terms of a general starting vector α. Using `MAPLE` to solve above system yields

$$\begin{bmatrix}\gamma_2\\ \gamma_1\\ \gamma_0\\ \delta\end{bmatrix} = \begin{bmatrix} 2\,\alpha_1 - 5/2\,\alpha_2 - 1/2\,\alpha_0 + \alpha_3 \\ 29/2\alpha_2 - 13\,\alpha_1 + 7/2\,\alpha_0 - 5\,\alpha_3 \\ -7\,\alpha_0 + 8\,\alpha_3 + 24\,\alpha_1 - 24\,\alpha_2 \\ -8\,\alpha_3 - 24\,\alpha_1 + 24\,\alpha_2 + 8\,\alpha_0 \end{bmatrix} \tag{9}$$

Again using `MAPLE` we can proove that the sequence u_j fullfills the difference equation (see `ex_12.mw`).

b. We have to consider

$$\lim_{n\to\infty}\frac{u_n}{n} = \lim_{n\to\infty}\frac{1}{n}\left(\gamma_2 n^2 + \gamma_1 n + \gamma_0 + \delta\left(\frac{1}{2}^n\right)\right) = \lim_{n\to\infty}\gamma_2 n + \gamma_1 + \frac{\gamma_0}{n} + \frac{\delta}{n}\left(\frac{1}{2}\right)^n \overset{!}{=} 0 \tag{10}$$

From the condition 10 we can immediately conclude that $\gamma_2 = 0$ and $\gamma_1 = 0$, since $\lim\limits_{n\to\infty}\frac{\delta}{n}\left(\frac{1}{2}\right)^n = 0$ for any choice of $\delta \in \mathbb{C}$. Hence for the starting values we get a two dimensional subvector space of $\mathbb{C}^4$ from the system of linear equations

$$\begin{aligned} -\frac{1}{2}\alpha_0 + 2\alpha_1 - \frac{5}{2}\alpha_2 + \alpha_3 &= 0 \\ \frac{7}{2}\alpha_0 - 13\alpha_1 + \frac{29}{2}\alpha_2 - 13\alpha_1 - 5\alpha_3 &= 0 \end{aligned}$$

Hence using the *Gauss algorithm* yields

$$\begin{bmatrix}1 & -4 & 5 & -2\\ 7 & -26 & 29 & -10\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & -4 & 5 & -2\\ 0 & 2 & -6 & 4\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & -4 & 5 & -2\\ 0 & 1 & -3 & 2\end{bmatrix}$$

Now, let $\alpha_3 := \lambda$ and $\alpha_2 := \mu$ where $\lambda, \mu \in \mathbb{C}$. Backwards substitution yields

$$\alpha \in \left\langle \begin{bmatrix}-6\\-2\\0\\1\end{bmatrix}, \begin{bmatrix}7\\3\\1\\0\end{bmatrix} \right\rangle \subseteq \mathbb{C}^4 \tag{11}$$

Exercise 13. First I state as mentioned in the hint the

> **Theorem 1.1.** (Banach fixed-point theorem) *Let (M, d) be a complete metric space and $\varphi : M \to M$ with the property, that there exists some $0 < C < 1$, such that for all $x, x' \in M$ holds*
>
> $$d(\varphi(x), \varphi(x')) \leqslant C d(x, x') \tag{12}$$
>
> *Then there exists a unique $x^* \in M$ with $\varphi(x^*) = x^*$.*

Set $\Phi\left(\eta_{i+1}^{(k)}; t_{i+1}, \hat{\eta}_i, h\right) := \hat{\eta}_i + hf\left(t_{i+1}, \eta_{i+1}^{(k)}\right)$. Then we have for $y_1, y_2 \in \mathbb{R}^n$

$$\|\Phi(y_1; t_{i+1}, \hat{\eta}_i, h) - \Phi(y_2; t_{i+1}, \hat{\eta}_i, h)\| = h\,\|f(t_{i+1}, y_1) - f(t_{i+1}, y_2)\| \leqslant hL\|y_1 - y_2\|$$

Since $h < \frac{1}{L}$, we can now apply theorem 1.1 on Φ. Thus there exists some $x^* \in \mathbb{R}^n$, such that $x^* = \Phi(x^*)$. Left to show is, that $\lim\limits_{k\to\infty}\eta_{i+1}^{(k)} = x^*$. We have

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$$
\begin{aligned}
\left\|\eta_{i+1}^{(k+1)} - x^*\right\| &= \left\|\Phi\left(\eta_{i+1}^{(k)}; t_{i+1}, \hat{\eta}_i, h\right) - \Phi\left(x^*; t_{i+1}, \hat{\eta}_i, h\right)\right\| \\
&= h\left\|f\left(t_{i+1}, \eta_{i+1}^{(k)}\right) - f\left(t_{i+1}, x^*\right)\right\| \\
&\leqslant hL\left\|\eta_{i+1}^{(k)} - x^*\right\| \\
&= \left\|\Phi\left(\eta_{i+1}^{(k-1)}; t_{i+1}, \hat{\eta}_i, h\right) - \Phi\left(x^*; t_{i+1}, \hat{\eta}_i, h\right)\right\| \\
&\leqslant (hL)^2\left\|\eta_{i+1}^{(k-1)} - x^*\right\| \\
&\leqslant \dots \\
&\leqslant (hL)^{k+1}\left\|\hat{\eta}_i - x^*\right\|
\end{aligned}
$$

Hence we have $0 \leqslant \left\|\eta_{i+1}^{(k+1)} - x^*\right\| \leqslant (hL)^{k+1}\left\|\hat{\eta}_i - x^*\right\| \to 0$ as $k \to \infty$ and thus $\lim_{k\to\infty} \eta_{i+1}^{(k)} = x^*$. Divergence for $h = \frac{1}{L}$ can be shown for the IVP

$$
\begin{cases} y'(t) = y \\ y(0) = y_0 \end{cases} \tag{13}
$$

for $y_0 \in \mathbb{R}$. Consider the first step provided by the implicit Euler method

$$
\eta_1 = \eta_0 + hf\left(t_1, \eta_1\right) = y_0 + hf\left(h, \eta_1\right) \tag{14}
$$

Now, the function $f(t,y) = y$ is Lipschitz on $\mathbb{R} \times \mathbb{R}^n$, since for any $y_1, y_2 \in \mathbb{R}^n$

$$
\|f(t,y_1) - f(t,y_2)\| = \|y_1 - y_2\| \tag{15}
$$

If we set $h = 1$, the fixpoint iteration yields

$$
\begin{aligned}
\eta_1^{(0)} &= y_0 \\
\eta_1^{(1)} &= y_0 + hf\left(h, y_0\right) = 2y_0 \\
\eta_1^{(2)} &= y_0 + hf\left(h, 2y_0\right) = 3y_0 \\
&\vdots \\
\eta_1^{(k)} &= y_0 + hf\left(h, ky_0\right) = (k+1)y_0
\end{aligned}
$$

Thus $\lim_{k\to\infty} \eta_1^{(k)} = \lim_{k\to\infty} (k+1)y_0 = \pm\infty$ for $y_0 \neq 0$. Hence the fixpoint iteration diverges.