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SOLUTIONS SHEET 5

Exercise 14. I will use the

> **Theorem 1.1.** *A linear multistep method is convergent for $f \in C^1[a,b]$ if and only if it fulfills the stability condition for Ψ and is consistent, this means $\Psi(1) = 0$ and $\Psi'(1) - \chi(1) = 0$.*

a. We have $\Psi(\mu) = \mu^4 - 1 = (\mu^2+1)(\mu^2-1) = (\mu+i)(\mu-i)(\mu+1)(\mu-1)$. So we have the roots $\lambda_{1,2} = \pm i$ and $\lambda_{3,4} = \pm 1$. Thus, for any root it holds that $|\lambda| = 1$. Since the roots are distinct (multiplicity one), the stability condition is fulfilled. Further we have $\Psi(1) = 0$ and $\chi(\mu) = \frac{8}{3}\mu^3 - \frac{4}{3}\mu^2 + \frac{8}{3}\mu$. Thus $\Psi'(1) - \chi(1) = 4 - \frac{8}{3} + \frac{4}{3} - \frac{8}{3} = 0$. Hence the method *is* convergent.

b. We have $\Psi(\mu) = \mu^2 - \frac{2}{3}\mu - \frac{1}{3} = (\mu-1)\left(\mu + \frac{1}{3}\right)$. Thus the stability condition is fulfilled and further $\Psi(1) = 0$. For $\chi(\mu) = \frac{7}{12}\mu + \frac{3}{4}$, we have $\Psi'(1) - \chi(1) = 2 - \frac{2}{3} - \frac{7}{12} - \frac{3}{4} = 0$. Hence the method *is* convergent.

Exercise 15. **a.** The code can be found in listing 1.

```
function [ t,y ] = LMSM( f,t0,tN,y0,h,OSM,a,b )
%Implementation of a linear multistep method.
%Implementation of a linear (predictor) multistep method of the form
%   y_(j+r) + a_(r-1)y_(j+r-1) + ... + a_0y_j =
%                           h[b_rf(x_(j+rh),y_(j+r)) + ... + b_0f(x,y_j)]
%Input:
%-----
%f,t0,tN,y0,h -- Standard input for solving ODEs.
%a -- The vector [a_(r-1,...,a_0)] in above abstract method.
%b -- The vector [b_(r-1),...,b_0] in above abstract method.
t = t0:h:tN;
N = (tN - t0)/h;
y = zeros(1,N+1);
r = length(a);

[~,eta] = OSM(f,t0,t0 + (r - 1) * h,y0,h);
y(1:r) = eta;

for k = r:N
    F = arrayfun(f, t(k:-1:k-r+1), y(k:-1:k-r+1));
    y(k+1) = -a * y(k:-1:k-r+1)' + h * b * F';
end
end
```

Listing 1. `src/LMSM.m`

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b. As one can see in figure 1, the computed soution agrees quite well with the theoretical solution. Remark: This multistep-method is an *explicit Adams method* and can be found among others in [HNW93, p. 358]

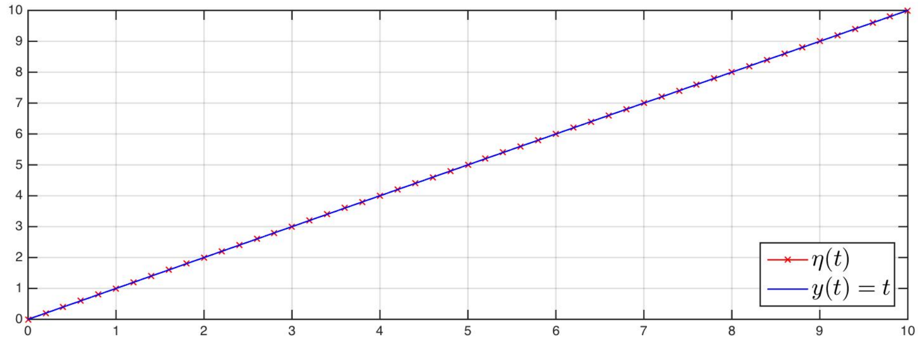

FIGURE 1. Plot of the discretized solution of the IVP $y'(t) = 1$, $y(0) = 0$ using a linear multi-step method and the classical Runge-Kutta method of order four for the starting values.

Exercise 16. By the *Dahlquist barrier* we can tell, that if we have an r-step method, the order p is bounded either by $r+1$ if r is odd or by $r+2$ if r is even. Hence since we must construct a method of order 5, we must construct a four step method. Hence we make the ansatz

$$\Psi(\mu) = (\mu - 1)(\mu - \alpha)(\mu - \beta)(\mu - \gamma) \tag{1}$$

As one can see in `ex_16.mw`, for $\alpha = \beta = \gamma = 0$ we get a convergent four step method of order five given by

$$\boxed{\eta_{j+4} - \eta_{j+3} = h\left(\frac{251}{720}f_{j+4} + \frac{323}{360}f_{j+3} - \frac{11}{30}f_{j+2} + \frac{53}{360}f_{j+1} - \frac{19}{720}f_j\right)} \tag{2}$$

For $\alpha = \beta = \frac{1}{3}$ and $\gamma = 0$ we get

$$\boxed{\eta_{j+4} - \frac{5}{3}\eta_{j+3} + \frac{7}{9}\eta_{j+2} - \frac{1}{9}\eta_{j+1} = h\left(\frac{149}{405}f_{j+4} + \frac{229}{405}f_{j+3} - \frac{97}{135}f_{j+2} + \frac{109}{405}f_{j+1} - \frac{16}{405}f_j\right)} \tag{3}$$

Those method are not of order six in general, since the so choosen parameters α, β, γ do not cancel out the coefficient of $(\mu - 1)^5$ as one can see on the last two lines of `ex_16.mw`.

REFERENCES

[HNW93] E. Hairer, S. P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I (2nd Revised. Ed.): Nonstiff Problems.* New York, NY, USA: Springer-Verlag New York, Inc., 1993. ISBN: 0-387-56670-8.