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SOLUTIONS SHEET 2

Exercise 5.

a.

```
function [ t,y ] = euler( f,t0,tN,y0,h )
y(1) = y0;
t = t0:h:tN;
N = (tN - t0)/h;
for k = 1:N
    y(k+1) = y(k) + h * f(t(k),y(k));
end
end
```

Listing 1. `src/euler.m`

```
function [ t,y ] = heun( f,t0,tN,y0,h )
y(1) = y0;
t = t0:h:tN;
N = (tN - t0)/h;
for k = 1:N
    y(k+1) = y(k) + .5 * h * (f(t(k),y(k)) + ...
        f(t(k) + h,y(k) + h * f(t(k),y(k))));
end
end
```

Listing 2. `src/heun.m`

```
function [ t,y ] = modeul( f,t0,tN,y0,h )
y(1) = y0;
t = t0:h:tN;
N = (tN - t0)/h;
for k = 1:N
    y(k+1) = y(k) + h * f(t(k) + .5 * h, y(k) + .5 * h * f(t(k),y(k)));
end
end
```

Listing 3. `src/modeul.m`

```
function [ t,y ] = rk4( f,t0,tN,y0,h )
y(1) = y0;
t = t0:h:tN;
N = (tN - t0)/h;
for k = 1:N
    k1 = f(t(k),y(k));
    k2 = f(t(k) + .5 * h, y(k) + .5 * h * k1);
    k3 = f(t(k) + .5 * h, y(k) + .5 * h * k2);
    k4 = f(t(k) + h, y(k) + h * k3);
    y(k+1) = y(k) + 1/6 * h * (k1 + 2 * k2 + 2 * k3 + k4);
end
end
```

Listing 4. `src/rk4.m`

b. The exact solution of the *initial value problem*

$$\begin{cases} y'(t) = \alpha t^{\alpha-1} & \alpha \geqslant 1 \\ y(0) = 0 \end{cases} \tag{1}$$

is uniquely given by $y(t) = t^\alpha$ on $[0, 1]$.

c. We see in the console output of `src/ex_5_b.m`, that for $\alpha = 5$ the empirically determined convergence order agrees with the theoretical one. This is, for the Euler method one, Heun and modiefied Euler both two and the Runge Kutta method has convergence order four (this is due to the fact, that one-step method has the same convergence order as consistency order). From the Runge-Kutta method we see, that the convergence order for $\alpha = 1.1$ and $\alpha = 1.5$ are exactly those numbers. Hence the convergence order is limited by the exponent. The same phenomena occurs by heun and modified euler. Only in the euler method, the convergence rate is almost the same for each exponent. This is due to the fact, that $\alpha t^{\alpha-1} \notin C^1[0, 1]$ for $\alpha = 1.1, 1.5$ (only in $C^0[0, 1]$). For $f \in C^p(S, \mathbb{R}^d)$ and a method of order p we have for the *local discretization error*

$$|\tau(t, y, h, f)| \leqslant C\frac{h^p}{p!}\|f\|_p \tag{2}$$

Where $\|f\|_p := \sup\limits_{(t,y)\in S} \max\limits_{\nu_1+\nu_2\leqslant p} |\partial_1^{\nu_1}\partial_2^{\nu_2} f(t, y)|$. Hence in the cases of $\alpha = 1.1, 1.5$, the error cannot be bounded by such an expression. An interesting question would be, why $\tau(t, y, h, f) = Ch^\alpha$ in the method of Runge-Kutta. Maybe, I am able to clarify this another time.

Exercise 6. **a.** Consider

$$\Phi(x, y; h) = a_1 f(x, y) + a_2 f(x + p_1 h, y + p_2 h f(x, y)) \tag{3}$$

Taylor expansion yields

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```
clc;
clear;
format long;
t0 = 0;
tN = 1;
y0 = 0;
N = 10;
alphas = [1.1, 1.5, 5];
methods = {@euler, @heun, @modeul, @rk4};
hks = 2.^(-1 * (2:N));

for j = 1:length(methods)
    disp(methods(j));
    figure;
    for i = 1:length(alphas)
        alpha = alphas(i);
        for k = 2:N
            [t,eta] = methods{j}(@(t,eta) alpha * t.^(alpha-1),...
                t0,tN,y0,hks(k-1));
            e(k-1) = max(abs(eta - t.^alpha));
        end
        loglog(hks, e, '-x');
        hold on;
        legendinfo{i} = ['$\alpha = ', num2str(alpha), '$'];
        exponent = (log(e(1)/e(end)))/(log(hks(1)/hks(end)));
        disp(['alpha = ', num2str(alpha), '; exponent: ', ...
            num2str(exponent)])
        xlim([hks(end), hks(1)]);
    end
    grid on;
    xlabel('$h_k$', 'interpreter', 'latex', 'fontsize', 16);
    ylabel('$e_{h_k}$', 'interpreter', 'latex', 'fontsize', 16)
    l = legend(legendinfo, 'location', 'southeast');
    set(l, 'fontsize', 14, 'interpreter', 'latex');
    saveas(gcf, ['ex_5_b_', num2str(j)], 'jpg');
end
```

LISTING 5. src/ex_5_b.m

$$\begin{aligned} f\left(x+p_1h, y+p_2hf\right) =& f+\left(p_1\partial_1 f+p_2 f\partial_2 f\right)h \\ &+\frac{1}{2}\left(p_1^2\partial_1^2 f+2p_1p_2 f\partial_1\partial_2 f+p_2^2 f^2\partial_2^2 f\right)h^2+O(h^3) \end{aligned}$$

Hence

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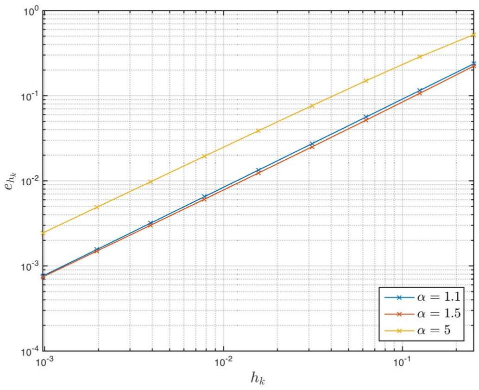
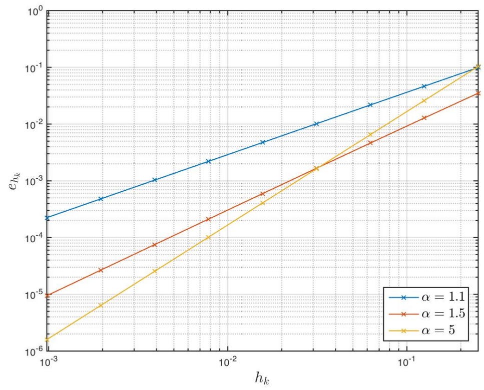
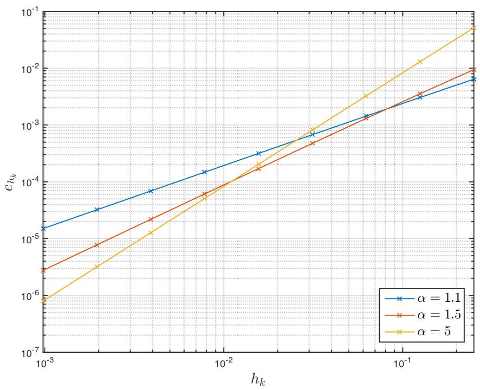


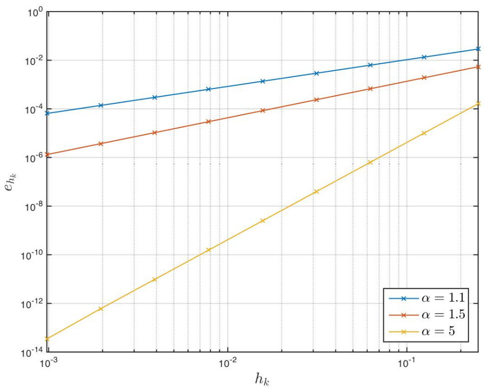
(A) Euler's Method.



(B) Heun Method.



(C) Modified Euler Method.



(D) Classical Runge-Kutta Method.

$$\begin{aligned}\Phi(x,y;h) =&(a_1+a_2)f + ha_2\left(p_1\partial_1 f + p_2 f\partial_2 f\right)\\ &+\frac{h^2}{2}a_2\left(p_1^2\partial_1^2 f + 2p_1p_2 f\partial_1\partial_2 f + p_2^2 f^2\partial_2^2 f\right) + O(h^3)\end{aligned}$$

Further the *exact relative increment* is given by

$$\begin{aligned}\Delta(x,y;h) =&f + \frac{h}{2}\left(\partial_1 f + f\partial_2 f\right)\\ &+\frac{h^2}{6}\left(\partial_1^2 f + 2f\partial_1\partial_2 f + \partial_1 f\partial_2 f + f\left(\partial_2 f\right)^2 + f^2\partial_2^2 f\right) + O(h^3)\end{aligned}$$

We see, that in $\Phi(x,y;h)$ the term $\frac{h^2}{6}\left(\partial_1 f\partial_2 f + f\left(\partial_2 f\right)^2\right)$ does not occur. Since in general this term is not equal to zero (consider for example $f(x,y) = xy^2$), we are not able to choose the parameters a_1, a_2, p_1 and p_2 so, that $\Phi(x,y;h)$ is of consistency order three.

b. Consider the trivial Cauchy problem

(4) $$\begin{cases} y'(x) = f(x)\\ y(x_0) = y_0\end{cases}$$

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```
        @euler

alpha = 1.1; exponent: 1.0352
alpha = 1.5; exponent: 1.0286
alpha = 5; exponent: 0.96746
        @heun

alpha = 1.1; exponent: 1.099
alpha = 1.5; exponent: 1.4821
alpha = 5; exponent: 1.9989
        @modeul

alpha = 1.1; exponent: 1.0922
alpha = 1.5; exponent: 1.4682
alpha = 5; exponent: 1.998
        @rk4

alpha = 1.1; exponent: 1.1
alpha = 1.5; exponent: 1.5
alpha = 5; exponent: 4.0174
```

LISTING 6. Console output of `src/ex_5_b.m`

The solution to 4 is given by $y(x) = y_0 + \int\limits_{x_0}^{x} f(t)dt$. Hence for a general Cauchy problem we may write

$$y(x+h) = y(x) + \int\limits_{x}^{x+h} f(t, y(t))dt = y(x) + h\int\limits_{0}^{1} f(x+h\tau, y(x+h\tau))d\tau$$

by the substitution $t = x + h\tau$. Taking the *Trapezoidal rule* yields

$$\begin{aligned}
y(x+h) &= y(x) + \int\limits_{x}^{x+h} f(t, y(t))dt \\
&= y(x) + h\int\limits_{0}^{1} f(x+h\tau, y(x+h\tau))d\tau \\
&\approx y(x) + \frac{h}{2}\left(f(x, y(x)) + f(x+h, y(x+h))\right) \\
&\approx y(x) + \frac{h}{2}\left(f(x, y(x)) + f(x+h, y(x) + hf(x, y(x)))\right)
\end{aligned}$$

By stipulating $y(x+h) \approx y(x) + hf(x, y(x))$ (Euler's method) since we do not know the exact solution $y(x+h)$. This is the method of *Heun*. In the same manner, using the *Midpoint rule* yields

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$$
\begin{aligned}
y(x+h) &= y(x) + \int_x^{x+h} f(t, y(t))dt \\
&= y(x) + h\int_0^1 f(x+h\tau, y(x+h\tau))d\tau \\
&\approx y(x) + hf\left(x+\frac{h}{2}, y\left(x+\frac{h}{2}\right)\right) \\
&\approx y(x) + hf\left(x+\frac{h}{2}, y(x)+\frac{h}{2}f(x,y)\right)
\end{aligned}
$$

which is the *Modified Euler method.* The last approximation is again due to the Euler method and the fact, that $y\left(x+\frac{h}{2}\right)$ is unknown.

Exercise 7. Substituting k_1, k_2 and k_3 in the increment function $\Phi(x,y;h)$ using `Maple` yields

$$
\begin{aligned}
\Phi(x,y;h) = 1/6\, f\left(x, y\left(x\right)\right) \\
&+ 2/3\, f\left(x + h/2, y\left(x\right) + 1/2\, hf\left(x, y\left(x\right)\right)\right) \\
&+ 1/6\, f[x+h, y\left(x\right) + h\left(-f\left(x, y\left(x\right)\right) + 2\, f\left(x+h/2, y\left(x\right) + 1/2\, hf\left(x, y\left(x\right)\right)\right)\right)]
\end{aligned}
$$

Expanding $\Phi(x,y;h)$ in a Taylor series around $h=0$ yields

$$
\begin{aligned}
\Phi(x,y;h) = f\left(x, y\left(x\right)\right) &+ \left(1/2 D_1\left(f\right)\left(x, y\left(x\right)\right) + 1/2 D_2\left(f\right)\left(x, y\left(x\right)\right) f\left(x, y\left(x\right)\right)\right) h \\
&+ [1/6\left(D_{1,1}\right)\left(f\right)\left(x, y\left(x\right)\right) \\
&+ 1/3\left(D_{1,2}\right)\left(f\right)\left(x, y\left(x\right)\right) f\left(x, y\left(x\right)\right) \\
&+ 1/6\left(D_{2,2}\right)\left(f\right)\left(x, y\left(x\right)\right)\left(f\left(x, y\left(x\right)\right)\right)^2 \\
&+ 1/6\left(D_2\left(f\right)\left(x, y\left(x\right)\right)\right)^2 f\left(x, y\left(x\right)\right) \\
&+ 1/6 D_2\left(f\right)\left(x, y\left(x\right)\right) D_1\left(f\right)\left(x, y\left(x\right)\right)] h^2 + O\left(h^3\right)
\end{aligned}
$$

Comparing with the exact relative increment

$$
\begin{aligned}
\Delta(x,y;h) =& f + \frac{h}{2}\left(\partial_1 f + f\partial_2 f\right) \\
&+ \frac{h^2}{6}\left(\partial_1^2 f + 2f\partial_1\partial_2 f + \partial_1 f\partial_2 f + f\left(\partial_2 f\right)^2 + f^2\partial_2^2 f\right) + O(h^3)
\end{aligned}
$$

yields

$$
\Delta(x,y;h) - \Phi(x,y;h) = O\left(h^3\right) \tag{5}
$$

And thus, the given one-step method is of consistency order three.