Yannis Bähni
yannis.baehni@uzh.ch

SOLUTIONS SHEET 6

Exercise 17. The modified versions can be found in listing 1 to 4. For testing purposes I have solved the *pendulum equation with friction and external force* described by $\ddot{\varphi}(t) + \mu\dot{\varphi}(t) + \sin(\varphi(t)) = A\sin(\omega t)$,. Since this equation can be rewritten as a system of first order equations

$$\dot{\begin{bmatrix} \varphi(t) \\ \dot{\varphi}(t) \end{bmatrix}} = \begin{bmatrix} \dot{\varphi}(t) \\ A\sin(\omega t) - \mu\dot{\varphi}(t) - \sin(\varphi(t)) \end{bmatrix} \tag{1}$$

```
function [ t,y ] = euler( f,t0,tN,y0,N )
h = (tN - t0)/N;
t = t0:h:tN;
y = zeros(length(y0),N+1);
y(:,1) = y0;
for k = 1:N
    y(:,k+1) = y(:,k) + h * f(t(k),y(:,k));
end
end
```

Listing 1. `src/euler.m`

```
function [ t,y ] = heun( f,t0,tN,y0,N )
h = (tN - t0)/N;
t = t0:h:tN;
y = zeros(length(y0),N+1);
y(:,1) = y0;
for k = 1:N
    y(:,k+1) = y(:,k) + .5 * h * (f(t(k),y(:,k)) + ...
        f(t(k) + h,y(:,k) + h * f(t(k),y(:,k))));
end
end
```

Listing 2. `src/heun.m`

Exercise 18.
a. The weights can be found in the file `ex_18.mw`.
b. The code can be found in the file `SSCLMSM.m`. The only remarkable thing is that I have implemented a *starting step size estimator* according to [HNW93, p. 169]. I will give a short outline here of the most important steps. The estimator is based on the hypothesis local error $\approx Ch^{p+1}y^{(p+1)}(x_0)$. Then
a) Put $d_0 = \|y_0\|$ and $d_1 = \|f(x_0, y_0)\|$ Where for $y \in \mathbb{R}^n$

Yannis Bähni
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```
function [ t,y ] = modeul( f,t0,tN,y0,N )
h = (tN - t0)/N;
t = t0:h:tN;
y = zeros(length(y0),N+1);
y(:,1) = y0;
for k = 1:N
    y(:,k+1) = y(:,k) + h * f(t(:,k) + .5 * h, y(:,k) + ...
        .5 * h * f(t(k),y(:,k)));
end
end
```

LISTING 3. `src/modeul.m`

```
function [ t,y ] = rk4( f,t0,tN,y0,N )
h = (tN - t0)/N;
t = t0:h:tN;
y = zeros(length(y0),N+1);
y(:,1) = y0;
for k = 1:N
    k1 = f(t(k),y(:,k));
    k2 = f(t(k) + .5 * h, y(:,k) + .5 * h * k1);
    k3 = f(t(k) + .5 * h, y(:,k) + .5 * h * k2);
    k4 = f(t(k) + h, y(:,k) + h * k3);
    y(:,k+1) = y(:,k) + 1/6 * h * (k1 + 2 * k2 + 2 * k3 + k4);
end
end
```

LISTING 4. `src/rk4.m`

$$\|y\| := \sqrt{\frac{1}{n}\sum_{k=0}^{n}\left(\frac{y_k}{\mathrm{sc}_k}\right)^2} \tag{2}$$

where for $k = 1, \ldots, n$

$$\mathrm{sc}_k := \mathrm{abstol}_k + |y_k| \cdot \mathrm{reltol}_k \tag{3}$$

For some *absolute* and *relative tolerance* $\mathrm{abstol}, \mathrm{reltol} \in \mathbb{R}^n$.

b) Let $h_0 := 0.01 \cdot (d_0/d_1)$. If either d_0 or d_1 is smaller than 10^{-5} we put $h_0 = 10^{-6}$.
c) Perform one explicit Euler step $y_1 := y_0 + h_0 f(x_0, y_0)$ and compute $f(x_0 + h_0, y_1)$.
d) Compute $d_2 := \|f(x_0 + h_0, y_1) - f(x_0, y_0)\|/h_0$.
e) Compute h_1 from the relation $h_1^{p+1} \max(d_1, d_2) = 0.01$. If $\max(d_1, d_2) \leqslant 10^{-15}$, put $h_1 := \max(10^{-6}, h_0 \cdot 10^{-3})$.
f) Finally, let $h := \min(100 \cdot h_0, h_1)$.

c. The constants can be found in the file `ex_18.mw`.

d. We have $f_{-1,p} := p_q(\mathbf{f}_p)(t_{p+1})$, where $\mathbf{f}_p := (f_{i,p})_{i=0}^{q}$ and p_q is the interpolation polynomial of degree q at the values $\mathbf{f}_p$ and nodes $(t_{p-q}, \ldots, t_p)$. Hence we get

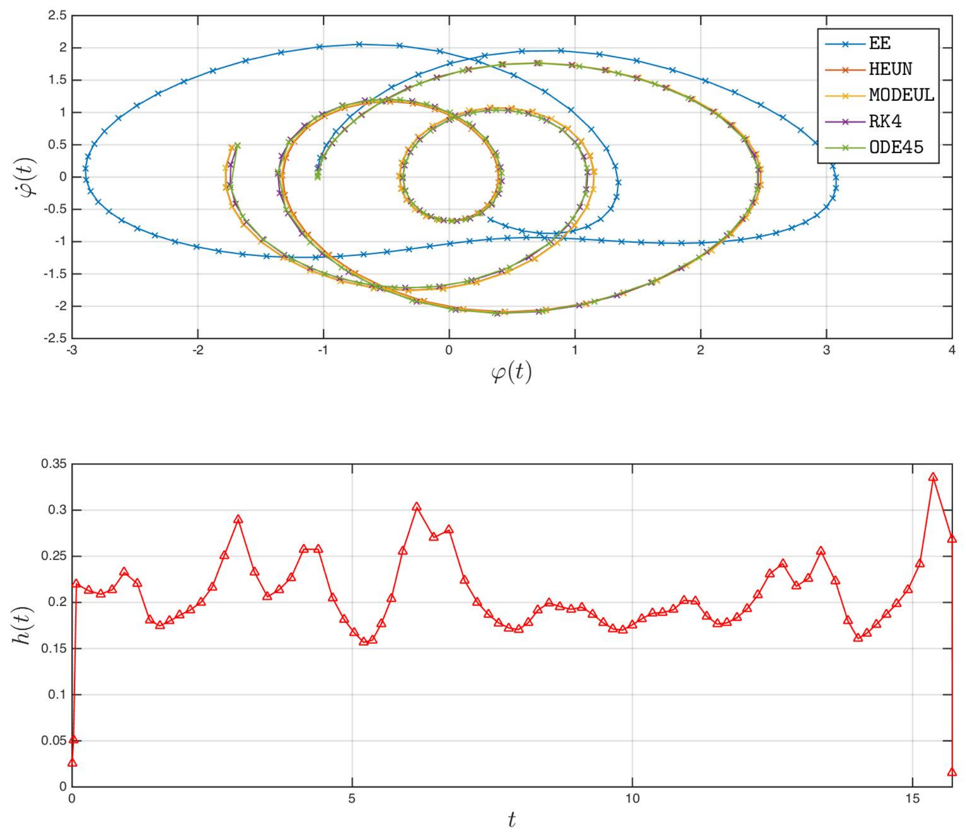

Figure 1. In the upper plot we see the solution of the *pendulum with friction and external force* described by the IVP $\ddot{\varphi}(t) + \mu\dot{\varphi}(t) + \sin(\varphi(t)) = A\sin(\omega t), \varphi(0) = (-\pi/3, 0)^t$, with $\mu = 0.1$, $A = 1$ and $\omega = 1.3$ on the interval $[0, 5\pi]$ using the four one step methods with stepsize $h \approx 0.157$ and the method of Fehlberg for abstol = reltol = 10^{-6}. In the lower plot the adaptive choice of the step size of the method of Fehlberg is shown.

$$
\begin{aligned}
f_{-1,p} &= \sum_{k=0}^{q} f_{k,p} L_{k,p,q} \\
&= \sum_{k=0}^{q} f\left(t_{p-k}, \eta_{p-k}^{(1)}\right) L_{k,p,q} \\
&= \sum_{k=0}^{q} f\left(t_{p-k}, \eta_{p-k}^{(1)}\right) \prod_{\substack{i=0 \\ i \neq k}}^{q} \frac{t - t_{p-i}}{t_{p-k} - t_{p-i}}
\end{aligned}
$$

Hence in the case $\underline{q = 1}$ we get

Yannis Bähni
yannis.baehni@uzh.ch

$$f_{-1,p} = f\left(t_{p-1}, \eta_{p-1}^{(1)}\right) \frac{t - t_p}{t_{p-1} - t_p} + f\left(t_p, \eta_p^{(1)}\right) \frac{t - t_{p-1}}{t_p - t_{p-1}} \tag{4}$$

In large scale problems we may use the Newton algorithm to calculate the interpolation polynomial.

e. As in few talks with you and Prof. Sauter, this exercise seems to be quite hard. The estimation $O(h^{q+2})$ makes sense but it is difficult to proove it in an explicit case like $q = 1$. Despite my long and hard thinking I was not able to solve it.

Exercise 19. **a.** The `MAPLE` implementation can be found in the file `ex_19.mw`. A visualization of the solution can be found in figure 2. An optimal visualization is in the *phase space*, hence plotting $(y_1(t), y_2(t))$.

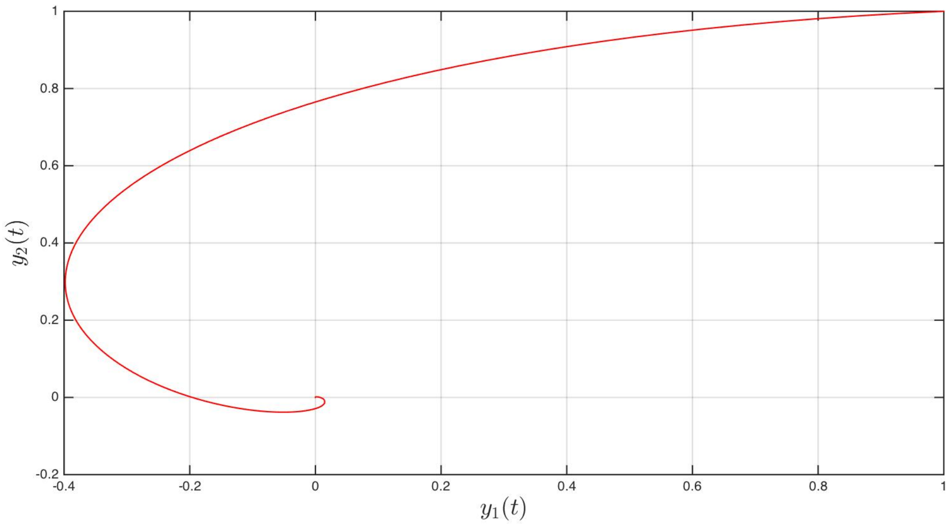

FIGURE 2. Plot of the IVP for $0 \leqslant t \leqslant 1$. For 10^6 evaluation points, since the solution tends rapidly to zero.

b. I consider only the interval $[0, 1]$, since the major characteristics of the analytical solution of the IVP can be well seen there. As one can see in figure 4 the predictor-corrector with tolerance $\varepsilon = 10^{-3}$ yields a quite nice discretization of the analytical solution. I have used $h_0 = 2.2485 \cdot 10^{-6}$. This is a result of a *starting step size algorithm* found in [HNW93, p. 169]. The predictor-corrector method uses 23677 steps for the integration. If we use this number for the number of steps in the one step methods we get figure 3. Despite the small stepsize (on the unit interval with 23677 steps $h = (t_N - t_0)/N$ we have $h \approx 10^{-5}$) the Euler method blows up to 10^{145}, the discretization by the Heun and modified Euler method are bad, since they use only two steps to get to zero and do not match the analytical solution at all on the way to it. The Runge Kutta method is also bad. Hence no of the one step method is accurate for this stepsize. This is due to the fact, that the given IVP is to some degree stiff and hence better solvable with an implicit method provided by the corrector method of the predictor-corrector method. Remark: The results would be more clear if a predictor-corrector method of an higher order would have been used. The case $q = 1$ is the smallest non-trivial one and thus, since the Runge-Kutta method is of order four, $q = 2$ would have been a better (but obviously more complicating) choice.

Yannis Bähni
yannis.baehni@uzh.ch

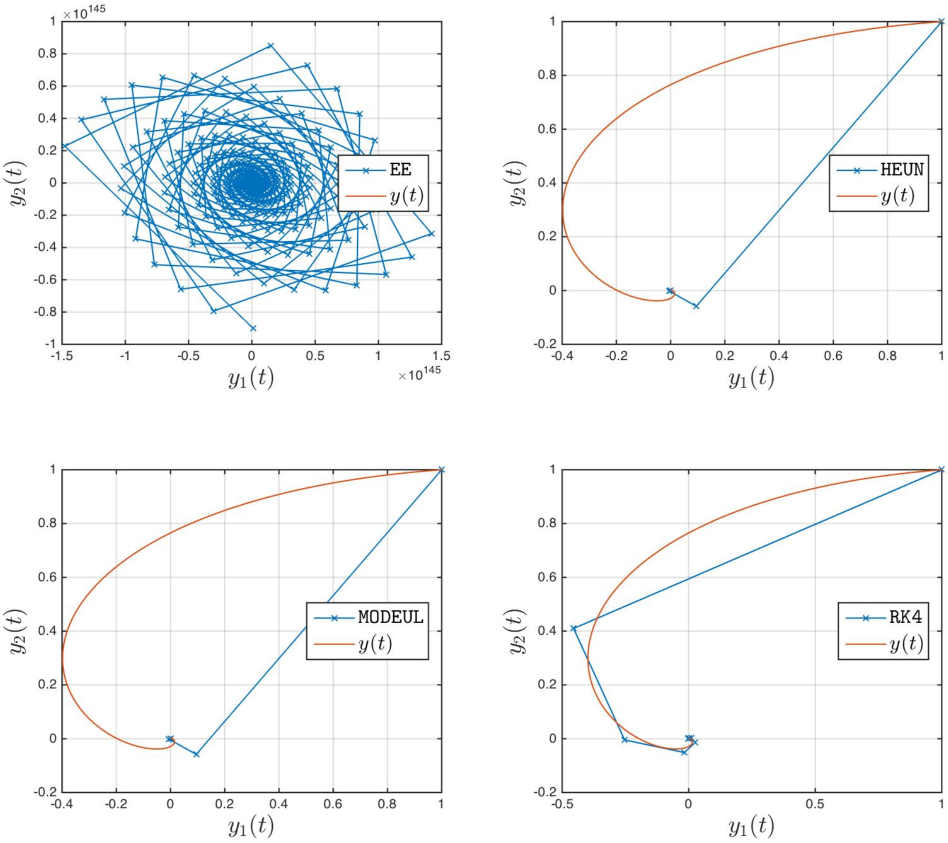

FIGURE 3. Plot of the IVP for $0 \leqslant t \leqslant 10$ using several one step methods.

REFERENCES

[HNW93] E. Hairer, S. P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I (2nd Revised. Ed.): Nonstiff Problems.* New York, NY, USA: Springer-Verlag New York, Inc., 1993. ISBN: 0-387-56670-8.

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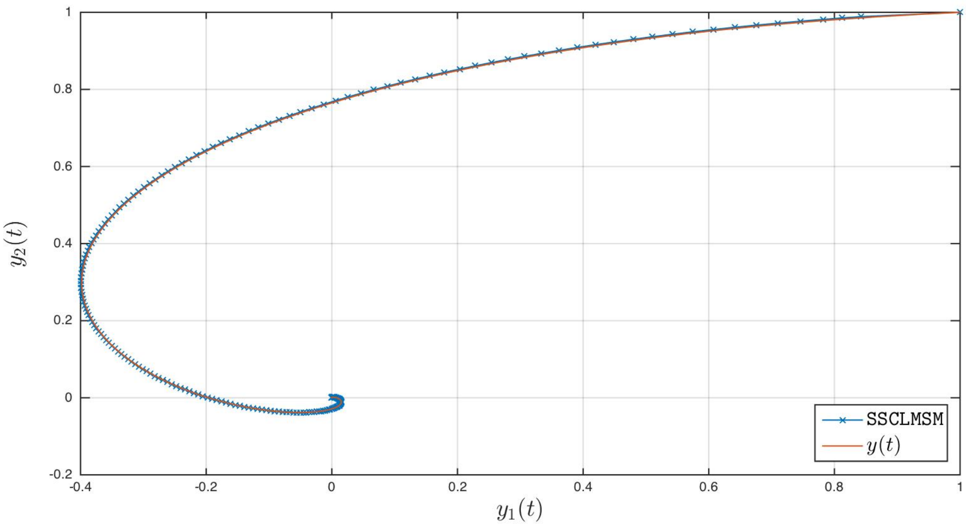

FIGURE 4. Plot of the IVP for $0 \leqslant t \leqslant 1$ using a predictor-corrector method for $\varepsilon = 10^{-6}$.