Yannis Bähni
yannis.baehni@uzh.ch

SOLUTIONS SHEET 8

Exercise 21.

a. The code can be found in the file `PECE2.m`.

b. First we may calculate the analytical solution of the initial value problem. For $\alpha \in \mathbb{R}_{>1}$ we have $dy_1(t) = \alpha t^{\alpha-1}$. Hence $\int\limits_0^t y_1'(\tau) d\tau = \alpha \int\limits_0^t \tau^{\alpha-1} d\tau$ and further by the substitution $u := y(\tau)$ we get $y_1(t) = \int\limits_0^{y_1(t)} du = t^\alpha$. In the same manner we arrive at $y_2 = t^{\alpha+1}$.

From figure 1 I can tell that if α increases so does the number of steps used by the predictor-corrector method. As a reference we have 6 successfull and 13 rejected steps for $\alpha = 1.01$, 9 successfull and 11 rejected steps for $\alpha = 1.1$, 14 successfull and 10 rejected steps for $\alpha = 1.5$ and 20 successfull and 22 rejected steps for $\alpha = 2$. The general accuracy is not the best the analytical solution and the discretization have a constant error around one.

Form figure 2 one can tell that if the tolerance ε gets smaller the global discretization error remains constant (this is contra intuitively, since the error should get smaller).The reason for this behaviour is the choice of the initial stepsize $h_0 = 0.5$. This stepsize is way too large for this problem and as I have read this exercise the first time it was immediately clear for me that the choice of h_0 of this size was bad. The starting stepsize algorithm applied to our problem can be seen in figure 3 and table 1. A large improvement of the accuracy can be observed and in comparison with the starting stepsize $h_0 = 0.5$ not much more steps were used to reach the demanded accuracy.

ε	h_0
10^{-3}	$1.0000 \cdot 10^{-4}$
10^{-5}	$7.9991 \cdot 10^{-5}$
10^{-7}	$1.7234 \cdot 10^{-5}$
10^{-9}	$3.7129 \cdot 10^{-6}$
10^{-11}	$7.9991 \cdot 10^{-7}$
10^{-13}	$1.7234 \cdot 10^{-7}$

TABLE 1. Starting stepsizes h_0 predicted by a starting stepsize algorithm for several tolerances ε.

Exercise 22. The source code can be found in the listings 1, 2 and 3. The final plot can be found in figure 4. We get

$$\bar{s} = 0.080420134626750 \tag{1}$$

A plot of the function $F(s) := u(1; s)$ can be found in figure 5.

Exercise 23.

Disclaimer: I did not manage to revise the whole proof of the Theorem below (only the first step) but anyway the conclusions are stated. In the future I will revise the proof since it is a nice example but not mandatory for this exercise.

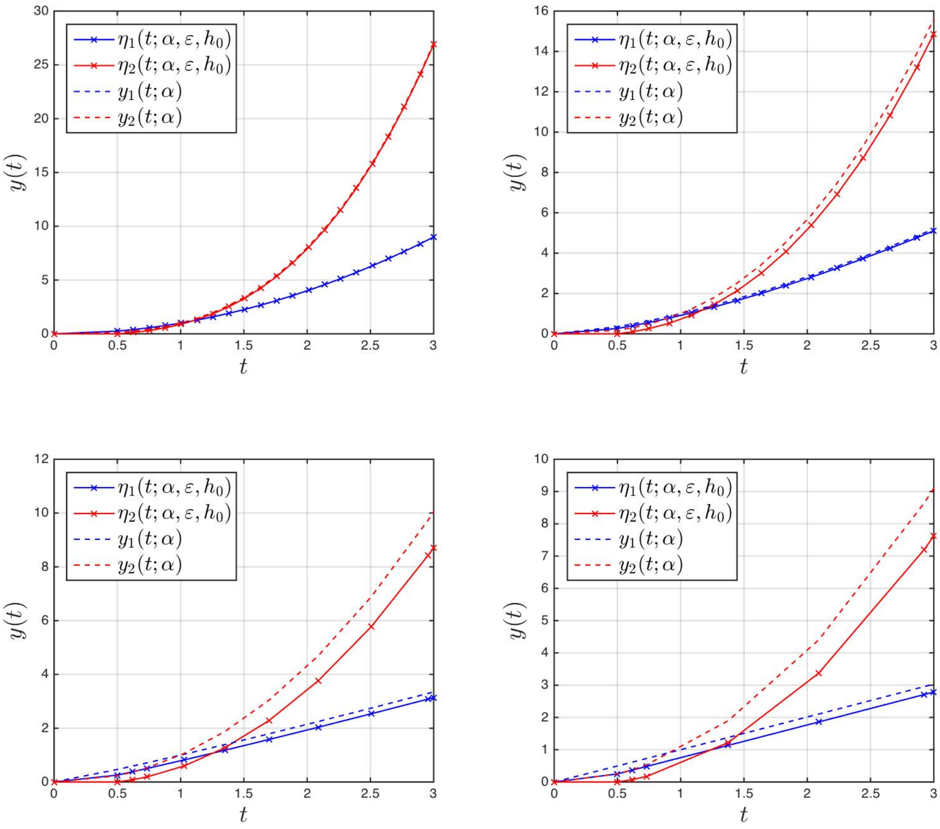

Figure 1. Plots of the discretization of the initial value problem for fixed tolerance $\varepsilon = 10^{-3}$ and variable exponent α. In the upper left we have $\alpha = 2$, in the upper right $\alpha = 1.5$, in the bottom left $\alpha = 1.1$ and finally in the bottom right plot $\alpha = 1.01$ was choosen. The stepsize $h_0 = 0.5$ was used to initialize the predictor-corrector method `PECE2` of order 2.

I grab on the idea of the Article *The Convergence Of Shooting Methods For Singular Boundary Value Problems by Othmar Koch and Ewa B. Weinmüller**, last accessed April 28, 2016, I give the

Theorem 1.1. *For the solution of the nonlinear operator equation $f(x) = 0$, $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, D convex, we consider the* perturbed Newton iteration

$$\left(Df(x^{(k)}) + E(x^{(k)})\right)\left(x^{(k+1)} - x^{(k)}\right) = -f(x^{(k)}) + e(x^{(k)}), \qquad k = 0, 1, \ldots \tag{2}$$

Assume that there exists $x^* \in \mathbb{R}^n$ such that $f(x^*) = 0$ and that the following hypothesis hold in a suitably chosen closed ball $\overline{B_r(x^*)}$:

*http://www.ams.org/journals/mcom/2003-72-241/S0025-5718-01-01407-7/S0025-5718-01-01407-7.pdf

Yannis Bähni
yannis.baehni@uzh.ch

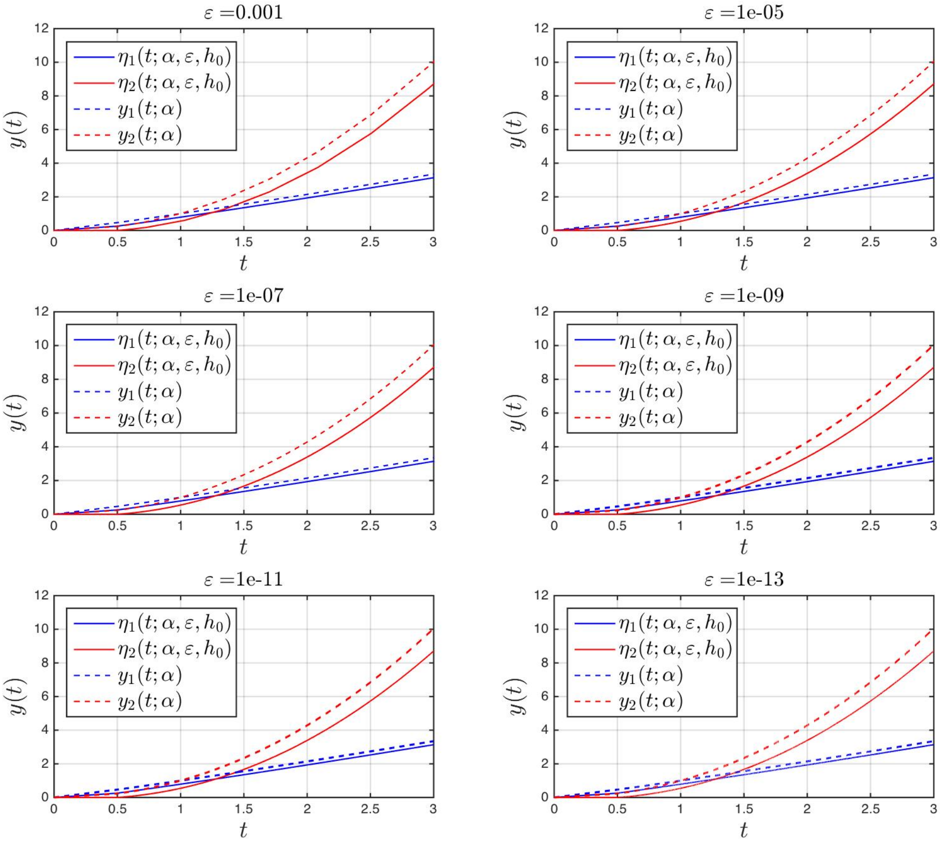

FIGURE 2. Plots of the discretization of the initial value problem for fixed exponent $\alpha = 1.1$ and variable tolerance ε. Again $h_0 = 0.5$ was used.

(i.) $\exists \gamma > 0 \forall x, y \in \overline{B_r(x^*)}, \|Df(x) - Df(y)\| \leqslant \gamma \|x - y\|$
(ii.) $\|E(x)\|, \|e(x)\| \leqslant \varepsilon$, *for* ε *small and* $x \in \overline{B_r(x^*)}$,
(iii.) $\forall x \in \overline{B_r(x^*)}, \|f(x)\| \leqslant \delta_0$,
(iv.) $f(x) \in C^2(\overline{B_r(x^*)})$ *and* $\|D^2 f(x)\| \leqslant K$ *for all* $x \in \overline{B_r(x^*)}$,
(v.) $Df(x^*)$ *is nonsingular and* $\|Df(x^*)^{-1}\| \leqslant \beta$,
(vi.) $\frac{\beta\varepsilon}{1-\beta\gamma r} < 1$

This implies that
- $\|x^{(k+1)} - x^*\| \leqslant C\left(\|x^{(k)} - x^*\|^2 + \varepsilon\right)$,
- *For* $\|x^{(k)} - x^*\| \gg \varepsilon$ *convergence is quadratic,*
- *near* x^* *convergence is only linear,*
- *The sequence* $\left(x^{(k)}\right)_{k\in\mathbb{N}}$ *approaches a ball* $\mathfrak{B}$ *with center* x^* *and radius* $r^* = O(\varepsilon)$,
- $x^{(k)}$ *is confined to* $\mathfrak{B}$ *for sufficiently large* k, *but the sequence does not converge to a single point in* $\mathfrak{B}$ *in general.*

I will try to give a more detailed proof as in the paper. First we need the following

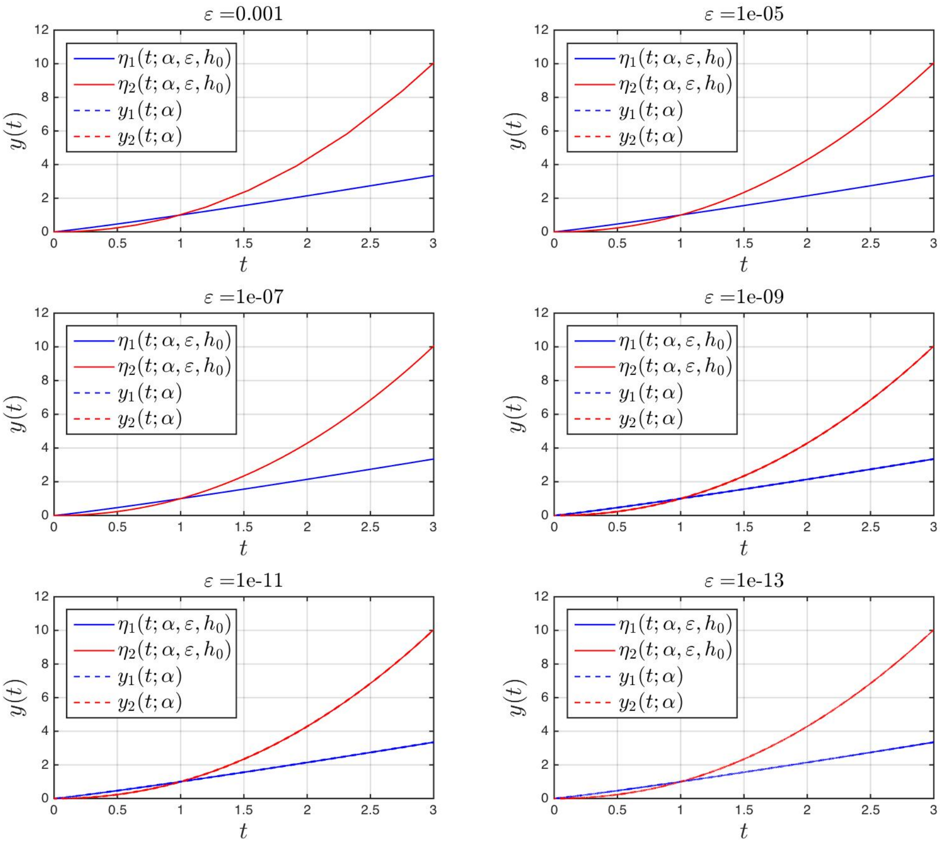

FIGURE 3. Plots of the discretization of the initial value problem for fixed exponent $\alpha = 1.1$ and variable tolerance ε. Additionally a starting stepsize algorithm was used.

Lemma 1.1. (Banach perturbation lemma) *Let $A \in M_n(\mathbb{C})$ with $\|A\| \leqslant q < 1$. Then $(I - A) \in \mathrm{GL}_n(\mathbb{C})$ and $(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$* (Neumann series) *and $\|(I-A)^{-1}\| \leqslant 1/(1-q)$.*

Proof 1.1. *Assume $x \in \mathbb{C}^n$ so that $(I - A)x = 0$. Hence $x = Ax$ and thus $\|x\| = \|Ax\| \leqslant \|A\|\|x\| < \|x\|$. The only posiibility is $\|x\| = 0$ and hence this impkies $x = 0$. Thus A considered as a linear map is injective. Since $\mathbb{C}^n$ is of finite dimension A is also surjective and hence bijective. Further assume*

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k \tag{3}$$

Multiplication of above equation by $(I - A)$ yields $I = \lim_{k\to\infty}(I - A^{k+1}) = I$ and thus is obviously true by considering $\lim_{k\to\infty} A^k = 0$. □

Now we return to the proof of the main theorem.

Yannis Bähni
yannis.baehni@uzh.ch

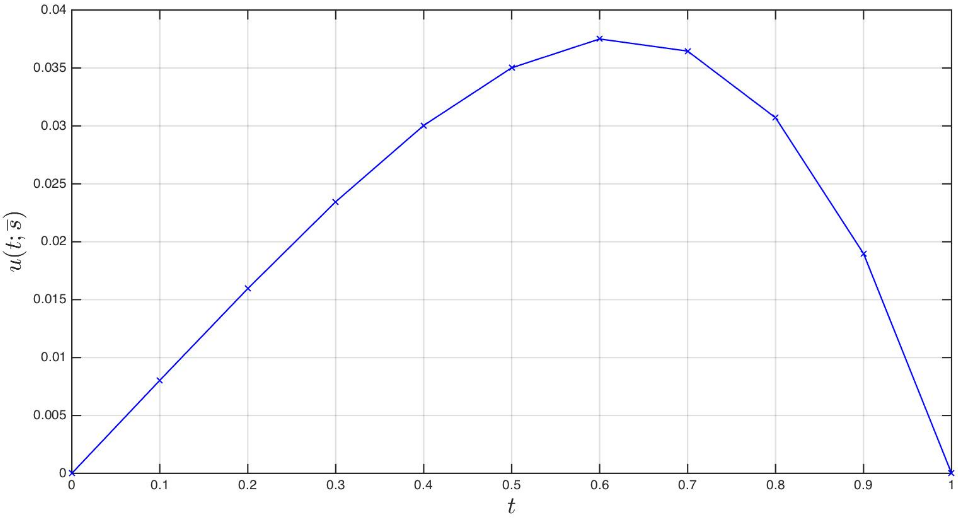

FIGURE 4. Plot of the function $u(x)$ as a solution of the ODE $-u''(x) = \sin(x^2)$ on $[0,1]$ with boundary conditions $u(0) = u(1) = 0$ and $h = 10^{-1}$.

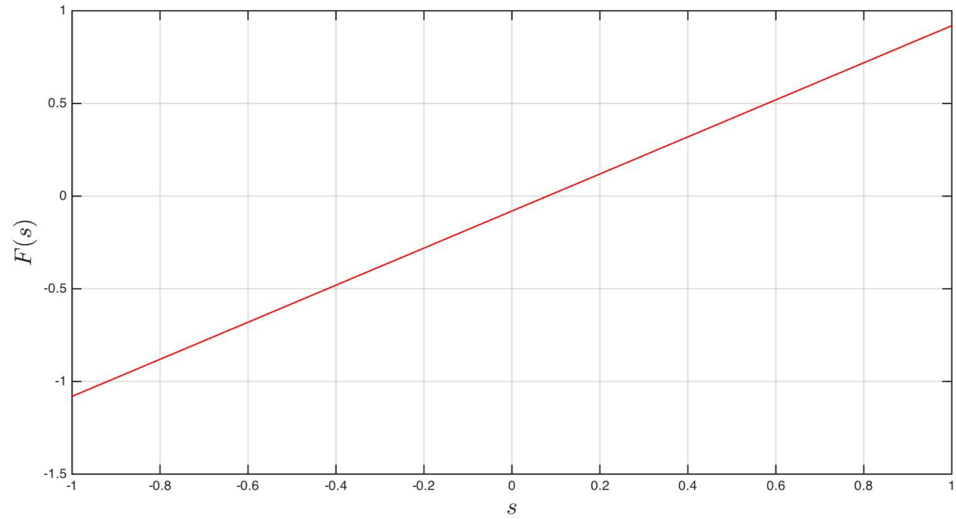

FIGURE 5. Plot of the function $F(s) := u(1;s)$.

Proof 1.2. *The proof is divided in three steps. In the first step we want to rewrite the iteration as an iteration with only one error term.* ***Step 1.*** *We may write* $Df(x) = Df(x_0)\left(I - Df(x_0)^{-1}\left(Df(x_0) - Df(x)\right)\right)$*. Now we chose* $r < 1/(\beta\gamma)$*. Then*

$$\begin{aligned}\|Df(x_0)^{-1}\left(Df(x_0) - Df(x)\right)\| &\leqslant \|Df(x_0)^{-1}\|\|Df(x_0) - Df(x)\| \\ &\leqslant \beta\gamma\|x_0 - x\| \\ &\leqslant \beta\gamma r < 1\end{aligned}$$

```
format long;
N = 10;

%s0
u0 = [0;0];
a = 0;
b = 1;
[ ~,u ] = RK4( @fun,a,b,u0,N );
s0 = u(1,end);

%s0
u0 = [0;1];
a = 0;
b = 1;
[ ~,u ] = RK4( @fun,a,b,u0,N );
s1 = u(1,end);

ub = bisect( @F, s0, s1, 1e-6);
disp('sbar:')
disp(ub);

%Final plot
u0 = [0;ub];
[ x,u ] = RK4( @fun,a,b,u0,N );
figure(1);
plot(x,u(1,:),'-x','color','blue');
grid on;
xlabel('$t$','interpreter','latex','fontsize',18);
ylabel('$u(t;\overline{s})$','interpreter','latex','fontsize',18);
fig = figure(1);
fig.PaperUnits = 'inches';
fig.PaperPosition = [0, 0, 12, 6];
saveas(fig, 'ex_22.jpg');
```

LISTING 1. ex_22.m

```
function [ val ] = F( s )
u0 = [0;s];
[ ~,u ] = RK4( @fun,0,1,u0,10 );
val = u(1,end);
end
```

LISTING 2. F.m

holds. Using the Banach perturbation lemma *thus yields that* $I - Df(x_0)^{-1}(Df(x_0) - Df(x)) \in \mathrm{GL}_n(\mathbb{R})$. *Hence*

```
function [ x ] = bisect( f,a,b,epsilon )
while (b - a) > epsilon
    x = (a + b)/2;
    if f(a) * f(x) < 0
        b = x;
    else
        a = x;
    end
end
end
```

Listing 3. `bisect.m`

$$Df(x)^{-1} = \left(I - Df(x_0)^{-1}\left(Df(x_0) - Df(x)\right)\right)^{-1} Df(x_0)^{-1} \tag{4}$$

and

$$\begin{aligned} \|Df(x_0)^{-1}\| &= \|\left(I - Df(x_0)^{-1}\left(Df(x_0) - Df(x)\right)\right)^{-1} Df(x_0)^{-1}\| \\ &\leqslant \|\left(I - Df(x_0)^{-1}\left(Df(x_0) - Df(x)\right)\right)^{-1}\|\|Df(x_0)^{-1}\| \\ &\leqslant \frac{\beta}{1-\beta\gamma r} \end{aligned}$$

We conclude that

$$\|Df(x)^{-1}E(x)\| \leqslant \|Df(x)^{-1}\|\|E(x)\| \leqslant \frac{\beta\varepsilon}{1-\beta\gamma r} < 1 \tag{5}$$

For $z, w \in \mathbb{C}$ consider the complex-valued rational function $\psi(z;w) := (z+w)^{-1}$. Series expansion around $z = 0$ thus yields

$$\frac{1}{z+w} = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{d^k\psi}{dz^k}(0;w)z^k = \frac{1}{w}\sum_{k=0}^{\infty}\left(\frac{z}{w}\right)^k \tag{6}$$

The series converges if and only if $\left|\frac{z}{w}\right| < 1$. Using 5 and truncating after the first term yields

$$\left(Df(x) + E(x)\right)^{-1} = Df(x)^{-1} + E_1(x) \tag{7}$$

where $\|E_1(x)\| \leqslant C\varepsilon$. We thus may write the iteration as

$$\begin{aligned} x^{(k+1)} &= x^k - \left(Df(x^{(k)})^{-1} + E_1(x^{(k)})\right)\left(f(x^{(k)}) - e(x^{(k)})\right) \\ &= x^{(k)} - Df(x^{(k)})^{-1}f(x^{(k)}) + Df(x^{(k)})^{-1}e(x^{(k)}) - E_1(x^{(k)})f(x^{(k)}) + E_1(x^{(k)})e(x^k) \\ &= x^{(k)} - Df(x^{(k)})^{-1}f(x^{(k)}) + Df(x^{(k)})^{-1}e_1(x^{(k)}) \end{aligned}$$

for $k = 0, 1, \ldots$ and $\|e_1(x)\| \leqslant C\varepsilon$.

Step 2. *To be completed.*

Step 3. *To be completed.*

Yannis Bähni
yannis.baehni@uzh.ch

Conclusions. If the abstract approximations of $f(x)$ and $Df(x)$ are $\tilde{f}(x)$ and $D\tilde{f}(x)$ where the relations $\tilde{f}(x) = f(x) + e(x)$ and $D\tilde{f} = Df(x) + E(x)$ with some error functions $E(x), e(x)$ satisfying $\|E(x)\|, \|e(x)\| \leqslant \varepsilon$ for some small $\varepsilon > 0$ holds we can conclude from the above theorem, that for $\|x^{(k)} - x^*\| \gg \varepsilon$ the convergence of the Newton iterates defined by

$$x^{(k+1)} = x^{(k)} - \left(Df(x^{(k)}) + E(x^{(k)})\right)^{-1} \left(f(x^{(k)}) - e(x^{(k)})\right) \qquad k = 0, 1, \ldots \tag{8}$$

for some starting value x_0 (formaly in a suitable chosen closed neighbourhood of the root x^*) is quadratic and for $x^{(k)}$ near the fixpoint x^* convergence is however only linear. Thus if we use approximations (e.g. finite difference schemes) we get a significand loss in convergence speed which also depends on the magnitude of the error functions $\|E(x)\|$ and $\|e(x)\|$. In general the sequence $(x^{(k)})_{k\in\mathbb{N}}$ does not converge to a unique solution x^* which solves $f(x^*) = 0$. In comparison with the *Newton-Kantorovich Theorem* this is a difference since the Newton-Kantorovich Theorem guarantees a unique solution in the so called *Kantorovich-neighbourhood* Thus we may get several solutions s_k^* to the initial condition $w'(b; s_k^*) = \beta$. However the best idea is to plot the function $F(s)$ dependent on $s \in \mathbb{R}$ to get an idea where the roots s^* of $F(s)$ may be located. This may be a problem in higher dimensions, but for that we have improved shooting methods (parallel, fitting or multiple).