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SOLUTIONS SHEET 7

Exercise 19. Remark: Here I use the *spectral norm* for linear operators defined by $\|A\|_2 := \max_{\|x\|_2=1} \|Ax\|_2$.

a. The map $f(x,y) := By$ defined on $S := \{(x,y) | x \in [a,b], y \in \mathbb{R}^2\}$ with values $\mathbb{R}^2$ is obviously continuous in each coordinate and hence continuous on S. If we write $y := (y_1, y_2) \in \mathbb{R}^2$ we have

$$D_y f(x,y) = \begin{pmatrix} \frac{\partial}{\partial y_1}((x-1)y_1 + y_2) & \frac{\partial}{\partial y_2}((x-1)y_1 + y_2) \\ \frac{\partial}{\partial y_1}(y_1 + (x-1)y_2) & \frac{\partial}{\partial y_2}(y_1 + (x-1)y_2) \end{pmatrix} = \begin{pmatrix} x-1 & 1 \\ 1 & x-1 \end{pmatrix} = B \tag{1}$$

Again $D_y f(x,y)$ is continuous since each partial derivative is.

b. We get

$$\|D_y f(x,y)\|_2 = \|B\|_2 = \sqrt{\mu_{\max}} = \sqrt{\max_{x \in [0,1/2]} \{(x-2)^2, x^2\}} = \sqrt{(x-2)^2} = |x-2| =: k(x)$$

where μ_k are the eigenvalues of $B^t B$.

c. We have

$$\begin{aligned} P(u,v) &:= D_u r(u,v) + D_v r(u,v) \\ &= \begin{pmatrix} 5 + \cos(u_1+u_2) & \cos(u_1+u_2) \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ \cos(v_1 - v_2) & 7 - \cos(v_1 - v_2) \end{pmatrix} \\ &= \begin{pmatrix} 9 + \cos(u_1+u_2) & \cos(u_1+u_2) \\ \cos(v_1 - v_2) & 9 - \cos(v_1 - v_2) \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 9 & 0 \\ 0 & 9 \end{pmatrix}}_{=: P_0} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \underbrace{\frac{1}{9} \begin{pmatrix} \cos(u_1+u_2) & \cos(u_1+u_2) \\ \cos(v_1 - v_2) & \cos(v_1 - v_2) \end{pmatrix}}_{=: M(u,v)} \right) \end{aligned}$$

Further $\|M(u,v)\|_2 \leqslant 2/9 =: \mu$ and $\|P_0^{-1} D_v r(u,v)\|_2 \leqslant \sqrt{67/162 + (1/54)\sqrt{165}} =: m$.

d. Now $\int_0^{1/2} k(t)dt = \int_0^{1/2} |t-2| dt = \int_0^{1/2} (2-t) = 7/8$. Hence reformulation yields $\lambda \overset{!}{\geqslant} m(\exp(7/8) - 1)$. I get $\lambda \overset{!}{\geqslant} 1.1291\ldots$. However it is not possible to choose λ so that $\lambda + \mu < 1$. The upper boundaries are hence too generous.