Yannis Bähni
yannis.baehni@uzh.ch

SOLUTIONS SHEET 3

YANNIS BÄHNI

Exercise 1. 1. We can write $\xi = \xi_1 + \cdots + \xi_n$ and $\eta = \eta_1 + \cdots + \eta_m$. Let $0 \leq k \leq m+n$. Using exercise 6 [Shi16, p. 217] yields

$$\begin{aligned}
P\{\xi + \eta = k\} &= \sum_{\substack{x \in \{0,1\}^{m+n} \\ \sum_\iota x_\iota = k}} P\{\xi_1 = x_1, \ldots, \xi_n = x_n, \eta_1 = x_{n+1}, \ldots, \eta_m = x_{m+n}\} \\
&= \sum_{\substack{x \in \{0,1\}^{m+n} \\ \sum_\iota x_\iota = k}} P\{\xi_1 = x_1\} \cdots P\{\xi_n = x_n\} P\{\eta_1 = x_{n+1}\} \cdots P\{\eta_m = x_{m+n}\} \\
&= \binom{m+n}{k} p^k (1-p)^{m+n-k}.
\end{aligned}$$

Hence $\xi + \eta \sim \operatorname{Bin}(m+n, p)$.

2. Assuming ξ_1 and ξ_2 are two independent Binomial random variables (a random variable ξ is a Binomial random variable if it takes values $k = 0, 1, \ldots, n$ with probabilities $p_k := \binom{n}{k} p^k (1-p)^{n-k}$ for some $0 \leq p \leq 1$). Let $0 \leq k \leq m+n$. Note that

$$\{\xi_1 + \xi_2 = k\} = \bigsqcup_{i+j=k} \{\xi_1 = i, \xi_2 = j\}. \tag{1}$$

Equation (1), the mutually independence of ξ_1 and ξ_2 and the stated binomial identity yields

$$\begin{aligned}
P\{\xi_1 + \xi_2 = k\} &= \sum_{i+j=k} P\{\xi_1 = i, \xi_2 = j\} \\
&= \sum_{i+j=k} P\{\xi_1 = i\} P\{\xi_2 = j\} \\
&= p^k (1-p)^{m+n-k} \sum_{i+j=k} \binom{n}{i} \binom{m}{j} \\
&= \binom{m+n}{k} p^k (1-p)^{m+n-k}.
\end{aligned}$$

Hence $\xi_1 + \xi_2 \sim \operatorname{Bin}(m+n, p)$.

(Yannis Bähni) University of Zurich, Rämistrasse 71, 8006 Zurich
E-mail address: yannis.baehni@uzh.ch.

Yannis Bähni
yannis.baehni@uzh.ch

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4. Assuming ξ_1 and ξ_2 are two independent Poisson random variables (a random variable ξ is a Poisson random variable if it takes values $k = 0, 1, 2, \ldots$ with probabilities $p_k := e^{-\lambda}\lambda^k/k!$ for some $\lambda > 0$) with parameters, respectively, $\lambda_1 > 0$ and $\lambda_2 > 0$. Let $k \in \mathbb{N}_0$. Note that

$$\{\xi_1 + \xi_2 = k\} = \bigsqcup_{i+j=k} \{\xi_1 = i, \xi_2 = j\} = \bigsqcup_{i=0}^{k} \{\xi_1 = i, \xi_2 = k - i\}. \tag{2}$$

Equation (2) and the mutually independence of ξ_1 and ξ_2 yields

$$\begin{aligned} P\{\xi_1 + \xi_2 = k\} &= \sum_{i=0}^{k} P\{\xi_1 = i, \xi_2 = k - i\} \\ &= \sum_{i=0}^{k} P\{\xi_1 = i\} P\{\xi_2 = k - i\} \\ &= e^{-(\lambda_1+\lambda_2)} \sum_{i=0}^{k} \frac{\lambda_1^i}{i!} \frac{\lambda_2^{k-i}}{(k-i)!} \\ &= \frac{1}{k!} e^{-(\lambda_1+\lambda_2)} \sum_{i=0}^{k} \binom{k}{i} \lambda_1^i \lambda_2^{k-i} \\ &= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1 + \lambda_2)^k}{k!}. \end{aligned}$$

Hence $\xi_1 + \xi_2 \sim \mathrm{Poi}(\lambda_1 + \lambda_2)$.

References

[Shi16] A.N. Shiryaev. *Probability-1*. Graduate Texts in Mathematics. Springer New York, 2016. ISBN: 9780387722061.