

## SOLUTIONS SHEET 4

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**Exercise 1.** To give the proof more structure, it is divided into three steps despite the structure of the exercise itself. I think this is more natural.

**Lemma 0.1.**  $Q|_{\mathcal{A}_0} = P$ .

*Proof.* Let  $A \in \mathcal{A}_0$ . Consider the sequence  $(B_n)_{n \in \mathbb{N}}$  in  $\mathcal{A}_0$  defined by

$$B_n := \begin{cases} A & n = 1 \\ \emptyset & n > 1 \end{cases}.$$

Clearly  $A \subseteq \bigcup_{n \in \mathbb{N}} B_n$  and thus

$$Q(A) \leq \sum_{n \in \mathbb{N}} P(B_n) = P(A). \quad (1)$$

Let  $(B_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{A}_0$  such that  $A \subseteq \bigcup_{n \in \mathbb{N}} B_n$ . Therefore  $A = \bigcup_{n \in \mathbb{N}} (B_n \cap A)$  and thus by subadditivity of  $P$

$$P(A) \leq \sum_{n \in \mathbb{N}} P(A_n \cap A) \leq \sum_{n \in \mathbb{N}} P(A_n). \quad (2)$$

Thus  $P(A) \leq Q(A)$  since the sequences were arbitrary.  $\square$

**Lemma 0.2.**  $Q : 2^\Omega \rightarrow [0, \infty]$  is an outer measure.

*Proof.* Clearly  $Q(\emptyset) = 0$  by the observation that  $\emptyset \subseteq \bigcup_{n \in \mathbb{N}} B_n$  where  $B_n := \emptyset$  for all  $n \in \mathbb{N}$  and that  $P(\emptyset) = 0$  since  $P$  is a probability. Observe that if  $(B_n)_{n \in \mathbb{N}}$  is a sequence in  $\mathcal{A}_0$  such that  $B \subseteq \bigcup_{n \in \mathbb{N}} B_n$  we also have  $A \subseteq \bigcup_{n \in \mathbb{N}} B_n$  since  $A \subseteq B$ . Hence the infimum in  $Q(A)$  is taken on a large set than  $Q(B)$ , thus  $Q(A) \leq Q(B)$ . Let  $(A_n)_{n \in \mathbb{N}}$  be a sequence in  $2^\Omega$ . For any  $A \in 2^\Omega$  and  $\varepsilon > 0$  we find by definition of  $Q(A)$  a sequence  $(B_n)_{n \in \mathbb{N}}$  in  $\mathcal{A}_0$  such that  $A \subseteq \bigcup_{n \in \mathbb{N}} B_n$  and

$$Q(A) \leq \sum_{n \in \mathbb{N}} P(B_n) < Q(A) + \varepsilon. \quad (3)$$

Thus for any  $n \in \mathbb{N}$  we find a sequence  $(B_{n,k})_{k \in \mathbb{N}}$  in  $\mathcal{A}_0$  such that

$$\sum_{k \in \mathbb{N}} P(B_{n,k}) \leq Q(A_n) + \frac{\varepsilon}{2^n} \quad n \in \mathbb{N} \quad (4)$$

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and  $A_n \subseteq \bigcup_{k \in \mathbb{N}} B_{n,k}$ . Clearly  $\bigcup_{n \in \mathbb{N}} A_n \subseteq \bigcup_{n \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} B_{n,k}$  and so

$$Q\left(\bigcup_{n \in \mathbb{N}} A_n\right) \leq \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} P(B_{n,k}) \leq \sum_{n \in \mathbb{N}} Q(A_n) + \varepsilon. \quad (5)$$

□

**Lemma 0.3.** *Each  $B \in \mathcal{A}_0$  is  $Q$ -measurable.*

*Proof.*

□

**Exercise 2.**

**Exercise 3.**

**Lemma 0.4.** *Let  $F : \mathbb{R} \rightarrow \mathbb{R}$  be a distribution function. Then*

$$P(a, b) = F(b-) - F(a) \quad \text{and} \quad P[a, b) = F(b) - F(a). \quad (6)$$

*holds for all  $-\infty \leq a < b \leq \infty$ .*

*Proof.* Note that

$$(a, b) = \bigcup_{n \in \mathbb{N}_{>0}} (-\infty, b - 1/n] \setminus (-\infty, a]. \quad (7)$$

By (7) and the fact that  $F$  induces a probability measure  $P$  we have

$$\begin{aligned} P(a, b) &= P\left(\bigcup_{n \in \mathbb{N}_{>0}} (-\infty, b - 1/n] \setminus (-\infty, a]\right) \\ &= P\left(\bigcup_{n \in \mathbb{N}_{>0}} (-\infty, b - 1/n]\right) - P(-\infty, a] \\ &= \lim_{n \rightarrow \infty} P(-\infty, b - 1/n] - P(-\infty, a] \\ &= \lim_{n \rightarrow \infty} F(b - 1/n) - F(a) \\ &= F(b-) - F(a). \end{aligned}$$

The second statement follows similarly from

$$[a, b) = \bigcup_{n \in \mathbb{N}_{>0}} (-\infty, b - 1/n] \setminus \bigcup_{n \in \mathbb{N}_{>0}} (-\infty, a - 1/n] \quad (8)$$

□

**Remark 0.1.** Note that for any  $x \in \mathbb{R}$  the limit  $F(x-)$  exists since  $F$  is monotone increasing and bounded from above by 1. Indeed, towards a contradiction assume that there exists some  $x_0 \in \mathbb{R}$  such that  $F(x_0) > 1$ . Hence  $F(x_0) - 1 > 0$  and so there exist  $M \in \mathbb{R}$  such that  $F(x) < F(x_0)$  for all  $x > M$ . Contradiction.

$F : \mathbb{R} \rightarrow \mathbb{R}$  can be rewritten as

$$F(x) = \frac{1}{4}\chi_{[0,1)}(x) + \frac{3}{4}\chi_{[1,2)}(x) + \chi_{[2,\infty)}(x). \quad (9)$$

Clearly  $F$  is monotone increasing, continuous on the right and

$$\lim_{x \rightarrow -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} F(x) = 1. \quad (10)$$

Thus there exists a unique probability measure  $P$  on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  such that

$$F(a, b] = P(a, b] = F(b) - F(a) \quad (11)$$

for all  $-\infty \leq a < b < \infty$ . From lemma 0.4 immediately follows

$$P(A) = 1 \quad P(B) = 1 \quad P(C) = 0 \quad P(D) = \frac{3}{4} \quad P(E) = 0.$$

**Exercise 4.**

1. An example would be the function  $F : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$F(x) := \sum_{n=1}^{\infty} (1 - 1/n)\chi_{[n-1,n)}(x). \quad (12)$$

$F$  is discontinuous at any  $n \in \mathbb{N}_{>0}$ .