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SOLUTIONS SHEET 2

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Exercise 1. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Recall, that for $A \in \mathcal{A}$ with $P(A) > 0$ the *conditional probability of B with respect to A* is defined by

$$P(B|A) := \frac{P(B \cap A)}{P(A)}. \tag{1}$$

1. Let $A_1, A_2 \in \mathcal{A}$ with $0 < P(A_2) < 1$. Observe, that by $0 < P(A_2) < 1$ the conditional probability $P(B|A_2^c)$ is well-defined since $P(A_2^c) = 1 - P(A_2) > 0$. Thus

$$\begin{aligned} P(A_1) &= P((A_1 \cap A_2) \cup (A_1 \cap A_2^c)) \\ &= P(A_1 \cap A_2) + P(A_1 \cap A_2^c) \\ &= \frac{P(A_1 \cap A_2)}{P(A_2)} P(A_2) + \frac{P(A_1 \cap A_2^c)}{P(A_2^c)} P(A_2^c) \\ &= P(A_1|A_2)P(A_2) + P(A_1|A_2^c)P(A_2^c). \end{aligned}$$

2. We have

$$P(A_3|A_1 \cap A_2) = \frac{P(A_3 \cap A_1 \cap A_2)}{P(A_1 \cap A_2)} = \frac{P(A_3)P(A_1)P(A_2)}{P(A_1)P(A_2)} = P(A_3).$$

3. First we prove two auxiliary results.

Lemma 0.1. *Let $A_1, \dots, A_n \in \mathcal{A}$ be independent. Then $A_1^c, \dots, A_n^c$ are independent.*

Proof. It is enough to consider the case $A_1, \dots, A_{i-1}, A_i^c, A_{i+1}, \dots, A_n$ for some $i \in \{1, \dots, n\}$. Let $I \subseteq \{1, \dots, n\}$. If $i \notin I$, there is nothing to prove. So assume $i \in I$. Then we have

$$\begin{aligned} P\left(A_i^c \cap \bigcap_{\iota \in I \setminus \{i\}} A_\iota\right) &= P\left(\bigcap_{\iota \in I \setminus \{i\}} A_\iota\right) - P\left(\bigcap_{\iota \in I} A_\iota\right) \\ &= \prod_{\iota \in I \setminus \{i\}} P(A_\iota) - \prod_{\iota \in I} P(A_\iota) \\ &= (1 - P(A_i)) \prod_{\iota \in I \setminus \{i\}} P(A_\iota) \\ &= P(A_i^c) \prod_{\iota \in I \setminus \{i\}} P(A_\iota). \end{aligned}$$

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□

Lemma 0.2. *Let $f : [0,1) \to \mathbb{R}$ be defined by $f(x) := \log(1-x) + x$. Then $f \leq 0$.*

Proof. f is clearly differentiable on $[0,1)$ with

$$f'(x) = 1 - \frac{1}{1-x} = \frac{x}{x-1} \leq 0. \tag{2}$$

Hence f is monotonically decreasing on $[0,1)$. By $f(0) = 0$ we conclude $f \leq 0$. □

If $P(A_i) = 1$ for some $i \in \{1, \ldots, n\}$ we have

$$P\left(\left(\bigcup_{j=1}^n A_j\right)^c\right) \leq P(A_i^c) = 0 \leq \exp\left(-\sum_{j=1}^n P(A_j)\right)$$

since $A_i \subseteq \bigcup_{j=1}^n A_j$. Therefore it is enough to consider $P(A_i) < 1$ for $i = 1, \ldots, n$. Using lemma 0.1 and 0.2 we get

$$\begin{aligned}
P\left(\left(\bigcup_{i=1}^n A_i\right)^c\right) &= P\left(\bigcap_{i=1}^n A_i^c\right) \\
&= \prod_{i=1}^n P(A_i^c) \\
&= \exp\left(\log\left(\prod_{i=1}^n P(A_i^c)\right)\right) \\
&= \exp\left(\sum_{i=1}^n \log(P(A_i^c))\right) \\
&= \exp\left(\sum_{i=1}^n \log(1 - P(A_i))\right) \\
&\leq \exp\left(-\sum_{i=1}^n P(A_i)\right)
\end{aligned}$$