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# SOLUTIONS SHEET 4

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**Exercise 1.** To give the proof more structure, it is divided into three steps despite the structure of the exercise itself. I think this is more natural.

**Lemma 0.1.** $Q\mid_{\mathcal{A}_0}= P$.

*Proof.* Let $A \in \mathcal{A}_0$. Consider the sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_0$ defined by

$$B_n := \begin{cases} A & n = 1 \\ \varnothing & n > 1 \end{cases}.$$

Clearly $A \subseteq \bigcup_{n\in\mathbb{N}} B_n$ and thus

$$Q(A) \leq \sum_{n\in\mathbb{N}} P(B_n) = P(A). \tag{1}$$

Let $(B_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}_0$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} B_n$. Therefore $A = \bigcup_{n\in\mathbb{N}}(B_n \cap A)$ and thus by subadditivity of $P$

$$P(A) \leq \sum_{n\in\mathbb{N}} P(A_n \cap A) \leq \sum_{n\in\mathbb{N}} P(A_n). \tag{2}$$

Thus $P(A) \leq Q(A)$ since the sequences were arbitrary. □

**Lemma 0.2.** $Q : 2^\Omega \to [0,\infty]$ *is an outer measure.*

*Proof.* Clearly $Q(\varnothing) = 0$ by the observation tha $\varnothing \subseteq \bigcup_{n\in\mathbb{N}} B_n$ where $B_n := \varnothing$ for all $n \in \mathbb{N}$ and that $P(\varnothing) = 0$ since $P$ is a probability. Observe that if $(B_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}_0$ such that $B \subseteq \bigcup_{n\in\mathbb{N}} B_n$ we also have $A \subseteq \bigcup_{n\in\mathbb{N}} B_n$ since $A \subseteq B$. Hence the infimum in $Q(A)$ is taken on a large set than $Q(B)$, thus $Q(A) \leq Q(B)$. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $2^\Omega$. For any $A \in 2^\Omega$ and $\varepsilon > 0$ we find by definition of $Q(A)$ a sequence $(B_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_0$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} B_n$ and

$$Q(A) \leq \sum_{n\in\mathbb{N}} P(B_n) < Q(A) + \varepsilon. \tag{3}$$

Thus for any $n \in \mathbb{N}$ we find a sequence $(B_{n,k})_{k\in\mathbb{N}}$ in $\mathcal{A}_0$ such that

$$\sum_{k\in\mathbb{N}} P(B_{n,k}) \leq Q(A_n) + \frac{\varepsilon}{2^n} \qquad n \in \mathbb{N} \tag{4}$$

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and $A_n \subseteq \bigcup_{k\in\mathbb{N}} B_{n,k}$. Clearly $\bigcup_{n\in\mathbb{N}} A_n \subseteq \bigcup_{n\in\mathbb{N}}\bigcup_{k\in\mathbb{N}} B_{n,k}$ and so

$$Q\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} P(B_{n,k}) \leq \sum_{n\in\mathbb{N}} Q(A_n) + \varepsilon. \tag{5}$$

□

**Lemma 0.3.** *Each $B \in \mathcal{A}_0$ is $Q$-measurable.*

*Proof.* Let $A \subseteq \Omega$. Let $\varepsilon > 0$. In the same manner as in question 2 we find a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{A}_0$ such that

$$\sum_{n\in\mathbb{N}} P(A_n) - \varepsilon \leq Q(A). \tag{6}$$

Furthermore

$$Q(A) \geq \sum_{n\in\mathbb{N}} P(A_n \cap B) + \sum_{n\in\mathbb{N}} P(A_n \cap B^c) - \varepsilon \geq Q(A\cap B) + Q(A \cap B^c) - \varepsilon \tag{7}$$

by the additivity of $P$.

□

**Exercise 2.**

**Exercise 3.**

**Lemma 0.4.** *Let $F : \mathbb{R} \to \mathbb{R}$ be a distribution function. Then*

$$P\,(a,b) = F(b-) - F(a) \qquad \text{and} \qquad P\,[a,b) = F(b-) - F(a-). \tag{8}$$

*holds for all $-\infty \leq a < b \leq \infty$ in the first case and $-\infty < a < b \leq \infty$ in the second.*

*Proof.* Note that

$$(a,b) = \bigcup_{n\in\mathbb{N}_{>0}} \big(-\infty, b - 1/n\big] \setminus (-\infty, a]\,. \tag{9}$$

By (9) and the fact that $F$ induces a probability measure $P$ we have

$$\begin{aligned} P\,(a,b) &= P\left(\bigcup_{n\in\mathbb{N}_{>0}} \big(-\infty, b-1/n\big] \setminus (-\infty, a]\right) \\ &= P\left(\bigcup_{n\in\mathbb{N}_{>0}} \big(-\infty, b-1/n\big]\right) - P\,(-\infty, a] \\ &= \lim_{n\to\infty} P\,\big(-\infty, b-1/n\big] - P\,(-\infty,a] \\ &= \lim_{n\to\infty} F(b-1/n) - F(a) \\ &= F(b-) - F(a). \end{aligned}$$

The second statement follows similarly from

$$[a,b) = \bigcup_{n\in\mathbb{N}_{>0}} \big(-\infty, b-1/n\big] \setminus \bigcup_{n\in\mathbb{N}_{>0}} \big(-\infty, a-1/n\big] \tag{10}$$

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□

**Remark 0.1.** Note that for any $x \in \mathbb{R}$ the limit $F(x-)$ exists since $F$ is monotone increasing and bounded from above by 1. Indeed, towards a contradiction assume that there exists some $x_0 \in \mathbb{R}$ such that $F(x_0) > 1$. Hence $F(x_0) - 1 > 0$ and so there exsist $M \in \mathbb{R}$ such that $F(x) < F(x_0)$ for all $x > M$. Contradiction.

$F : \mathbb{R} \to \mathbb{R}$ can be rewritten as

$$F(x) = \frac{1}{4}\chi_{[0,1)}(x) + \frac{3}{4}\chi_{[1,2)}(x) + \chi_{[2,\infty)}(x). \tag{11}$$

Clearly $F$ is monotone increasing, continuous on the right and

$$\lim_{x\to-\infty} F(x) = 0 \qquad \text{and} \qquad \lim_{x\to\infty} F(x) = 1. \tag{12}$$

Thus there exists a unique probability measure $P$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that

$$F\,(a,b] = F(b) - F(a) \tag{13}$$

for all $-\infty \leq a < b < \infty$. From lemma 0.4 immediately follows

$$P(A) = 1 \qquad P(B) = 1 \qquad P(C) = 0 \qquad P(D) = \frac{3}{4} \qquad P(E) = 0.$$

**Exercise 4.**

1. An example would be the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) := \sum_{n=1}^{\infty}(1 - 1/n)\chi_{[n-1,n)}(x). \tag{14}$$

$F$ is discontinuous at any $n \in \mathbb{N}_{>0}$.