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SOLUTIONS SHEET 5

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Exercise 1. Let $\beta \in \mathbb{R}_{>0}$ and $\alpha \in \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \frac{c}{1 + (x-\alpha)^2/\beta^2} \tag{1}$$

for some $c \in \mathbb{R}$. Clearly $f \in \mathscr{C}(\mathbb{R})$ and thus Borel-measurable.. From a standard fact of real analysis follows that the function $\mathsf{P} : \mathscr{B}(\mathbb{R}) \to \overline{\mathbb{R}}$ defined by

$$\mathsf{P}(A) := \int_A f \,\mathrm{d}\lambda \tag{2}$$

is a measure. We now determine $c \in \mathbb{R}$ such that P is a probability measure. The substitution $s = (x-\alpha)/\beta$ yields

$$\begin{aligned}
\mathsf{P}(\mathbb{R}) &= \int_{-\infty}^{\infty} f \,\mathrm{d}\lambda \\
&= c\int_{-\infty}^{\infty} \frac{1}{1 + (x-\alpha)^2/\beta^2} \,\mathrm{d}\lambda(x) \\
&= c\beta \int_{-\infty}^{\infty} \frac{1}{1+s^2} \,\mathrm{d}\lambda(s) \\
&= c\beta \arctan|_{-\infty}^{\infty} \\
&= c\beta\pi
\end{aligned}$$

and by $\mathsf{P}(\mathbb{R}) = 1$ we conclude $c = 1/(\beta\pi)$. The distribution function F of P is now given by

$$\begin{aligned}
F(t) &= \mathsf{P}((-\infty, t]) \\
&= \int_{-\infty}^{t} f \,\mathrm{d}\lambda \\
&= \frac{1}{\beta\pi}\int_{-\infty}^{t} \frac{1}{1 + (x-\alpha)^2/\beta^2} \,\mathrm{d}\lambda(x) \\
&= \frac{1}{\pi}\int_{-\infty}^{(t-\alpha)/\beta} \frac{1}{1+s^2} \,\mathrm{d}\lambda(s) \\
&= \frac{1}{\pi}\arctan((t-\alpha)/\beta) + \frac{1}{2}
\end{aligned}$$

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for any $t \in \mathbb{R}$.

Exercise 2.

1.

Exercise 3. Let $\varepsilon > 0$. Since $\mathsf{E}(X) = np$ and $\mathrm{Var}(X) = np(1-p)$ Chebychev's inequality implies

$$\begin{aligned}
\lim_{n\to\infty} \mathsf{P}(|X - np| \leq n\varepsilon) &= \lim_{n\to\infty} \mathsf{P}(|X - \mathsf{E}(X)| \leq n\varepsilon) \\
&= 1 - \lim_{n\to\infty} \mathsf{P}(|X - \mathsf{E}(X)| > n\varepsilon) \\
&\geq 1 - \lim_{n\to\infty} \frac{\mathrm{Var}(X)}{n^2\varepsilon^2} \\
&= 1 - \lim_{n\to\infty} \frac{np(1-p)}{n^2\varepsilon^2} \\
&= 1 - \lim_{n\to\infty} \frac{p(1-p)}{n\varepsilon^2} \\
&= 1
\end{aligned}$$

Since $\mathsf{P}(|X - np| \leq n\varepsilon) \leq 1$ for all $n \in \mathbb{N}$ we conclude that

$$\lim_{n\to\infty} \mathsf{P}(|X - np| \leq n\varepsilon) = 1. \tag{3}$$