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SOLUTIONS SHEET 1

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Exercise 1. First we label the balls: the green ones with $\{1,\dots,17\}$, the blue ones with $\{18,\dots,22\}$ and the red ones with $\{23,\dots,33\}$.
1. Define the sample space as

$$\Omega := \{\omega : \omega = (a_1, a_2), a_j \neq a_k, j \neq k, a_i \in \{1,\dots,33\}\}.$$

Assume the outcomes are equally probable. Then

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{33 \cdot 32}$$

for any $A \in 2^\Omega$. Therefore

$$P(\{\text{none red}\}) = \frac{|\{\omega : \omega = (a_1, a_2), a_j \neq a_k, j \neq k, a_i \in \{1,\dots,22\}\}|}{33 \cdot 32} = \frac{22 \cdot 21}{33 \cdot 32} = \frac{7}{16}$$

2. Define the sample space as

$$\Omega := \{\omega : \omega = (a_1, a_2, a_3), a_i \in \{1,\dots,33\}\} \qquad |\Omega| = 33^3.$$

Now we have

$$P(\{\text{at most two green}\}) = 1 - P(\{\text{exactly three green}\})$$

and so by

$$P(\{\text{exactly three green}\}) = \left(\frac{17}{33}\right)^3$$

we get

$$P(\{\text{at most two green}\}) = 1 - \left(\frac{17}{33}\right)^3$$

Exercise 2. This can be modeled by sampling without replacement if we draw one card after the other. Therefore we consider the sample space

$$\Omega := \{\omega : \omega = (a_1, a_2, a_3, a_4, a_5), a_j \neq a_k, j \neq k, a_i \in \{1,\dots,52\}\}.$$

Thus

$$|\Omega| = 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48. \tag{1}$$

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1. We have

$$P(\{\text{same suit}\}) = \frac{|A_1 \cup A_2 \cup A_3 \cup A_4|}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{4 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{33}{16660}$$

where

$$\begin{aligned}
A_1 &:= \{\omega : \omega = (a_1, a_2, a_3, a_4, a_5), a_j \neq a_k, j \neq k, a_i \in \{1, \dots, 13\}\} \\
A_2 &:= \{\omega : \omega = (a_1, a_2, a_3, a_4, a_5), a_j \neq a_k, j \neq k, a_i \in \{14, \dots, 26\}\} \\
A_3 &:= \{\omega : \omega = (a_1, a_2, a_3, a_4, a_5), a_j \neq a_k, j \neq k, a_i \in \{27, \dots, 39\}\} \\
A_4 &:= \{\omega : \omega = (a_1, a_2, a_3, a_4, a_5), a_j \neq a_k, j \neq k, a_i \in \{40, \dots, 52\}\}
\end{aligned}$$

2. It is enough to consider the case $(a_1, a_2, a_3, a_4, a_5)$ where the first four have the same rank due to symmetry (formally we form a partition as in part a) of five subsets). For a_1 we have 52 possibilities, for a_2 we have 3 since there are only three cards of the same rank left, for a_3 we have 2, for a_4 we have exactly one and finally for a_5 we have 48 possibilities. Therefore we get

$$P(\{\text{four cards same rank}\}) = \frac{5 \cdot 52 \cdot 3 \cdot 2 \cdot 48}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{1}{4165}.$$

3. Again, let us consider the simplest case where the card with the lowest rank is a_1. Hence, for a_1 we have 40 possibilities. This is due to the fact, that if we start with 10, 11 or 12 we would not get an increasing sequence. For the four succeeding cards we have in total 4^4 possibilities. Now it does not matter if we pick first the highest card or the lowest, so we have to multiply the possibilities with $S\,|S_5| = 5!$ Therefore we get

$$P(\{\text{five cards sequential rank}\}) = \frac{5! \cdot 40 \cdot 4^4}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{128}{32487}.$$

4. Again, we consider a simple case where the first three cards are of the same rank and then the other two follow. Therefore

$$P(\{\text{full house}\}) = \frac{\binom{5}{3} \cdot 52 \cdot 3 \cdot 2 \cdot 48 \cdot 3}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{6}{4165}.$$

Exercise 3. We propose that

$$\sigma(\mathcal{C}) = \{\cup_{\iota \in I} C_\iota : I \subseteq \{1, \dots, n\}\} =: \mathcal{A}. \tag{2}$$

Firstly, $\Omega \in \mathcal{A}$ since $\Omega = \cup_{i=1}^n C_i$. Let $(A_i)_{i \in \mathbb{N}}$ be a family in $\mathcal{A}$. Then $A_i = \cup_{\iota \in I_j} C_\iota$ for $I_j \subseteq \{1, \dots, n\}$ and $j \in \mathbb{N}$. But then it is clearly seen that $\cup_{i \in \mathbb{N}} A_i \in \mathcal{A}$. Lastly, to show that $\mathcal{A}$ is also closed under complementation it is enough to show that $C_i^c \in \mathcal{A}$ for any $i = 1, \dots, n$ and that $\mathcal{A}$ is closed under finite intersections. Fix some $i \in \{1, \dots, n\}$. Then it is easy to see that $C_i^c = \cup_{j \neq i} C_j \in \mathcal{A}$. Furthermore, if $A, B \in \mathcal{A}$, we have $A = \cup_{\iota \in I_1} C_\iota$ and $B = \cup_{j \in J} C_j$. Hence

$$A \cap B = (\cup_{\iota \in I_1} C_\iota) \cap (\cup_{j \in J} C_j) = \cup_{\iota \in I \cap J} C_\iota \in \mathcal{A}.$$

Obviously, $\mathcal{C} \subseteq \mathcal{A}$ and thus $\sigma(\mathcal{C}) \subseteq \mathcal{A}$. The inclusion $\mathcal{A} \subseteq \sigma(\mathcal{C})$ is trivial.

Exercise 4.

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1. We will show

$$-P(A\Delta B) \leq P(A) - P(B) \leq P(A\Delta B). \tag{3}$$

The first inequality is equivalent to

$$P(B) - P(A) \leq P(A\Delta B)$$

which is easily verified by considering

$$P(A\Delta B) = P((A \cup B) \setminus (A \cap B)) = P(A \cup B) - P(A \cap B) \geq P(B) - P(A)$$

since $B \subseteq A \cup B$ and $A \cap B \subseteq A$. Similarly the second inequality in (3) follows from

$$P(A\Delta B) = P((A \cup B) \setminus (A \cap B)) = P(A \cup B) - P(A \cap B) \geq P(A) - P(B)$$

since $B \subseteq A \cup B$ and $A \cap B \subseteq B$.

2. Define a sequence

$$B_n := \cap_{i=1}^n A_i$$

for $n \in \mathbb{N}$. Clearly $(B_n)_{n\in\mathbb{N}}$ is decreasing and

$$\cap_{n\in\mathbb{N}} B_n = \cap_{n\in\mathbb{N}} A_n.$$

Hence

$$\begin{aligned} P(\cap_{n\in\mathbb{N}} A_n) &= P(\cap_{n\in\mathbb{N}} B_n) \\ &= \lim_{n\to\infty} P(B_n) \\ &= \lim_{n\to\infty} P(\cap_{i=1}^n A_i) \\ &= 1 - \lim_{n\to\infty} P(\cup_{i=1}^n A_i^c) \\ &\geq 1 - \lim_{n\to\infty} \sum_{i=1}^n P(A_i^c) \\ &= 1. \end{aligned}$$

by continuity from above of the probability measure P.