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SOLUTIONS SHEET 7

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Remark: We use here the results [Shi16, pp. 289–290], so our results may differ on a set of Lebesgue measure zero, since the Radon-Nikodým derivative (the density) is unique up to equality on a set of measure zero. Central is the following proposition.

Proposition 0.1. *Let φ be defined on the set $\sum_{k=1}^{n} [a_k, b_k]$, continuously differentiable and either strictly increasing or strictly decreasing on each open interval $I_k := (a_k, b_k)$, and with $\varphi'(x) \neq 0$ for $x \in I_k$. Let $h_k(y)$ be the inverse of φ on I_k. Then for $\eta := \varphi(\xi)$ we have*

$$\boxed{f_\eta(y) = \sum_{k=1}^{n} f_\xi(h_k(y))\,|h_k'(y)|\,\chi_{D_k}(y)} \tag{1}$$

where D_k denotes the domain of h_k.

Exercise 1. Let $k \in \mathbb{N}$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(x) := x^k$. For k odd, we have that φ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$. Furthermore, $\varphi'(x) \neq 0$ on both intervals. Thus for $\eta := \xi^k$ and $y \in \mathbb{R}$ we have

$$\begin{aligned} f_\eta(y) &= \frac{1}{k} f_\xi(y^{1/k})\,|y^{1/k-1}|\,\chi_{(-\infty,0)\cup(0,\infty)}(y) \\ &= \frac{1}{2k}\chi_{[-1,1]}(y^{1/k})\,|y^{1/k-1}|\,\chi_{(-\infty,0)\cup(0,\infty)}(y) \\ &= \frac{1}{2k}\chi_{[-1,1]}(y)\,|y^{1/k-1}|\,\chi_{(-\infty,0)\cup(0,\infty)}(y) \\ &= \frac{1}{2k}\,|y^{1/k-1}|\,\chi_{[-1,0)\cup(0,1]}(y) \\ &= \frac{1}{2k} y^{1/k-1}\chi_{[-1,0)\cup(0,1]}(y) \end{aligned}$$

if we adapt the convention of taking always the positive root $y^{1/k-1}$ if it exists, which here is always the case, since if k is odd we have $k = 2n+1$ for some $n \in \mathbb{N}_0$ and thus

$$\frac{1}{k} - 1 = \frac{1}{2n+1} - 1 = -\frac{2n}{2n+1} \tag{2}$$

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which has an even numerator. For k even φ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$. Thus

$$f_\eta(y) = \begin{cases} \frac{1}{k}\left[f_\xi(-y^{1/k}) + f_\xi(y^{1/k})\right] y^{1/k} & y > 0, \\ 0 & y \leq 0. \end{cases}$$

Furthermore, for $y > 0$ we have

$$f_\eta(y) = \frac{1}{2k}\left[\chi_{(0,1]}(-y^{1/k}) + \chi_{(0,1]}(y^{1/k})\right] y^{1/k} = \frac{1}{k}\chi_{(0,1]}(y) y^{1/k-1}. \tag{3}$$

Exercise 2. If $\eta := |\xi|$, it is evident that $F_\eta(y) = 0$ for $y < 0$, while for $y \geq 0$

$$F_\eta(y) = \mathsf{P}(|\xi| \leq y) = \mathsf{P}(-y \leq \xi \leq y) = F_\xi(y) - F_\xi(-y) + \mathsf{P}(\xi = -y). \tag{4}$$

The function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) := |x|$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$. Furthermore the respective inverse functions $h_1 : (0, \infty) \to (-\infty, 0)$ and $h_2 : (0, \infty) \to (0, \infty)$ are given by

$$h_1(y) := -y \qquad \text{and} \qquad h_2(y) := y. \tag{5}$$

Thus we get

$$f_\eta(y) = \left[f_\xi(-y) + f_\xi(y)\right] \chi_{(0,\infty)}. \tag{6}$$

Exercise 3.

Exercise 4.

Exercise 5. Consider the function $\varphi : (0, \infty) \to \mathbb{R}$ defined by $\varphi(x) := 1/(x + 1)$. By

$$\varphi'(x) = -\frac{1}{(x+1)^2} \tag{7}$$

we see that φ is strictly decreasing on $(0, \infty)$ and φ' does not vanish on $(0, \infty)$. Furthermore by $\lim_{x\to\infty} \varphi(x) = 0$ and $\lim_{x\searrow 0} \varphi(x) = 1$ (this is immediate by extending φ) we have that $\varphi((0, \infty)) = (0, 1)$. Furthermore the inverse function $h : (0, 1) \to (0, \infty)$ of φ is seen to be

$$h(y) = \frac{1-y}{y}. \tag{8}$$

Thus we get for $\eta := 1/(\xi + 1)$, $\xi > 0$,

$$f_\eta(y) = \begin{cases} \frac{1}{y^2} f_\xi((1-y)/y) & y \in (0, 1)\,, \\ 0 & y \in (-\infty, 0] \cup [1, \infty)\,. \end{cases}$$

References

[Shi16] A.N. Shiryaev. *Probability-1*. Third Edition. Graduate Texts in Mathematics. Springer New York, 2016. ISBN: 9780387722061.