# SOLUTIONS SHEET 7

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Remark: We use here the results [Shi16, pp. 289–290], so our results may differ on a set of Lebesgue measure zero, since the Radon-Nikodým derivative (the density) is unique up to equality on a set of measure zero. Central is the following proposition.

**Proposition 0.1.** *Let $\varphi$ be defined on the set $\sum_{k=1}^{n}[a_k, b_k]$, continuously differentiable and either strictly increasing or strictly decreasing on each open interval $I_k := (a_k, b_k)$, and with $\varphi'(x) \neq 0$ for $x \in I_k$. Let $h_k(y)$ be the inverse of $\varphi$ on $I_k$. Then for $\eta := \varphi(\xi)$ we have*

$$\boxed{f_\eta(y) = \sum_{k=1}^{n} f_\xi(h_k(y))\,|h_k'(y)|\,\chi_{D_k}(y)} \tag{1}$$

*where $D_k$ denotes the domain of $h_k$.*

**Exercise 1.** Let $k \in \mathbb{N}$. Define $\varphi : \mathbb{R} \to \mathbb{R}$ by $\varphi(x) := x^k$. For $k$ odd, we have that $\varphi$ is strictly increasing on $(-\infty, 0)$ and $(0, \infty)$. Furthermore, $\varphi'(x) \neq 0$ on both intervals. Thus for $\eta := \xi^k$ and $y \in \mathbb{R}$ we have

$$\begin{aligned}
f_\eta(y) &= \frac{1}{k} f_\xi(y^{1/k})\,|y^{1/k-1}|\,\chi_{(-\infty,0)\cup(0,\infty)}(y)\\
&= \frac{1}{2k}\chi_{[-1,1]}(y^{1/k})\,|y^{1/k-1}|\,\chi_{(-\infty,0)\cup(0,\infty)}(y)\\
&= \frac{1}{2k}\chi_{[-1,1]}(y)\,|y^{1/k-1}|\,\chi_{(-\infty,0)\cup(0,\infty)}(y)\\
&= \frac{1}{2k}\,|y^{1/k-1}|\,\chi_{[-1,0)\cup(0,1]}(y)\\
&= \frac{1}{2k} y^{1/k-1}\chi_{[-1,0)\cup(0,1]}(y)
\end{aligned}$$

if we adapt the convention of taking always the positive root $y^{1/k-1}$ if it exists, which here is always the case, since if $k$ is odd we have $k = 2n+1$ for some $n \in \mathbb{N}_0$ and thus

$$\frac{1}{k} - 1 = \frac{1}{2n+1} - 1 = -\frac{2n}{2n+1} \tag{2}$$

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which has an even numerator. For $k$ even $\varphi$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$. Thus

$$f_\eta(y) = \begin{cases} \frac{1}{k}\left[f_\xi(-y^{1/k}) + f_\xi(y^{1/k})\right] y^{1/k} & y > 0, \\ 0 & y \leq 0. \end{cases}$$

Furthermore, for $y > 0$ we have

$$f_\eta(y) = \frac{1}{2k}\left[\chi_{(0,1]}(-y^{1/k}) + \chi_{(0,1]}(y^{1/k})\right] y^{1/k} = \frac{1}{k}\chi_{(0,1]}(y) y^{1/k-1}. \tag{3}$$

**Exercise 2.** If $\eta := |\xi|$, it is evident that $F_\eta(y) = 0$ for $y < 0$, while for $y \geq 0$

$$F_\eta(y) = \mathsf{P}(|\xi| \leq y) = \mathsf{P}(-y \leq \xi \leq y) = F_\xi(y) - F_\xi(-y) + \mathsf{P}(\xi = -y). \tag{4}$$

The function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) := |x|$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$. Furthermore the respective inverse functions $h_1 : (0, \infty) \to (-\infty, 0)$ and $h_2 : (0, \infty) \to (0, \infty)$ are given by

$$h_1(y) := -y \qquad \text{and} \qquad h_2(y) := y. \tag{5}$$

Thus we get

$$f_\eta(y) = \left[f_\xi(-y) + f_\xi(y)\right] \chi_{(0,\infty)}. \tag{6}$$

**Exercise 3.**

1. Define $\varphi : \big(-\pi/2, \pi/2\big) \to \mathbb{R}$ by $\varphi(x) := \tan x$. From

$$\varphi'(x) = \frac{1}{\cos^2 x} \tag{7}$$

we see that that $\varphi$ is strictly increasing on $\big(-\pi/2, \pi/2\big)$ and $\varphi'$ does not vanish on $\big(-\pi/2, \pi/2\big)$. The inverse function $h : \mathbb{R} \to \big(-\pi/2, \pi/2\big)$ is clearly given by $h(y) = \arctan y$. Thus for $\eta := \tan \xi$ we have

$$f_\eta(y) = f_\xi(\arctan y)\frac{1}{1+y^2} = \frac{1}{\pi}\frac{1}{1+y^2}\chi_{(-\pi/2,\pi/2)}(\arctan y) = \frac{1}{\pi}\frac{1}{1+y^2} \tag{8}$$

for all $y \in \mathbb{R}$.

2. Define $\varphi : \big[-\pi/2, \pi/2\big] \to [0, 1]$ by $\varphi(x) := \sin^2 x$. Then by

$$\varphi'(x) = 2\sin x \cos x = \sin(2x) \tag{9}$$

it is easily seen that $\varphi$ is strictly decreasing on $\big(-\pi/2, 0\big)$ and strictly increasing on $\big(0, \pi/2\big)$. Furthermore the inverse functions $h_1 : (0, 1) \to \big(-\pi/2, 0\big)$ and $h_2 : (0, 1) \to \big(0, \pi/2\big)$ are given by

$$h_1(y) := \arcsin(-\sqrt{y}) \qquad \text{and} \qquad h_2(y) := \arcsin(\sqrt{y}). \tag{10}$$

Thus we get

$$f_\eta(y) = \begin{cases} \frac{1}{2\sqrt{y}\sqrt{1-y}}\left[f_\xi(\arcsin(-\sqrt{y})) + f_\xi(\arcsin(\sqrt{y}))\right] & x \in (0, 1) \\ 0 & y \in (0, 1)^c. \end{cases}$$

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Furthermore for $y \in (0,1)$ this yields

$$f_\eta(y) = \frac{1}{2\sqrt{y}\sqrt{1-y}}\left[f_\xi(\arcsin(-\sqrt{y})) + f_\xi(\arcsin(\sqrt{y}))\right] = \frac{1}{\pi\sqrt{y}\sqrt{1-y}} \tag{11}$$

which is for sure continuous on $(0,1)$.

**Exercise 4.** Let $\xi$ be Cauchy distributed with parameters $(\alpha, 1)$, $\alpha \in \mathbb{R}$, i.e.

$$f_\xi(x) = \frac{1}{\pi}\frac{1}{1+(x-\alpha)^2}. \tag{12}$$

Furthermore we can assume that $\xi \neq 0$. Consider the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(x) := \begin{cases} 1/x & x \in \mathbb{R}^\times, \\ 0 & x = 0. \end{cases}$$

Then we have

$$\begin{aligned} f_\eta(y) &= \frac{|a|}{y^2}\left[f_\xi(a/y)\chi_{(-\infty,0)} + f_\xi(a/y)\chi_{(0,\infty)}\right] \\ &= \frac{|a|}{\pi y^2}\frac{1}{1+(a/y-\alpha)^2}\chi_{\mathbb{R}^\times} \\ &= \frac{|a|}{\pi y^2}\frac{1}{1+a^2/y^2-2a\alpha/y+\alpha^2}\chi_{\mathbb{R}^\times} \\ &= \frac{|a|}{\pi}\frac{1}{(1+\alpha^2)y^2+a^2-2a\alpha y}\chi_{\mathbb{R}^\times} \\ &= \frac{|a|}{\pi}\frac{1}{(1+\alpha^2)y^2-2a\alpha y+a^2\alpha^2/(1+\alpha^2)+a^2-a^2\alpha^2/(1+\alpha^2)}\chi_{\mathbb{R}^\times} \\ &= \frac{|a|}{\pi}\frac{1}{(1+\alpha^2)(y-a\alpha/(1+\alpha^2))^2+a^2/(1+\alpha^2)}\chi_{\mathbb{R}^\times} \\ &= \frac{1}{\pi}\frac{|a|/(1+\alpha^2)}{(y-a\alpha/(1+\alpha^2))^2+|a|^2/(1+\alpha^2)^2}\chi_{\mathbb{R}^\times} \end{aligned}$$

and thus $\eta$ is Cauchy distributed with parameters

$$\left(\frac{a\alpha}{1+\alpha^2}, \frac{|a|}{1+\alpha^2}\right). \tag{13}$$

**Exercise 5.** Consider the function $\varphi : (0,\infty) \to \mathbb{R}$ defined by $\varphi(x) := 1/(x+1)$. By

$$\varphi'(x) = -\frac{1}{(x+1)^2} \tag{14}$$

we see that $\varphi$ is strictly decreasing on $(0,\infty)$ and $\varphi'$ does not vanish on $(0,\infty)$. Furthermore by $\lim_{x\to\infty}\varphi(x) = 0$ and $\lim_{x\searrow 0}\varphi(x) = 1$ (this is immediate by extending $\varphi$) we have that $\varphi((0,\infty)) = (0,1)$. Furthermore the inverse function $h : (0,1) \to (0,\infty)$ of $\varphi$ is seen to be

$$h(y) = \frac{1-y}{y}. \tag{15}$$

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Thus we get for $\eta := 1/(\xi + 1)$, $\xi > 0$,

$$f_\eta(y) = \begin{cases} \frac{1}{y^2} f_\xi((1-y)/y) & y \in (0,1)\,, \\ 0 & y \in (-\infty, 0] \cup [1, \infty)\,. \end{cases}$$

## References

[Shi16] A.N. Shiryaev. *Probability-1*. Third Edition. Graduate Texts in Mathematics. Springer New York, 2016. ISBN: 9780387722061.