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SOLUTIONS SHEET 6

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Exercise 1. See separate sheet.

Exercise 2. See separate sheet.

Exercise 3. See separate sheet.

Exercise 4. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random variables. For $n, k \in \mathbb{N}$, $n \geq k$, set

$$S_n := \sum_{i=1}^{n} X_i \qquad \text{and} \qquad S_{n,k} := \sum_{i=k}^{n} X_i. \tag{1}$$

We claim, that

$$\{(S_n/n)_{n\in\mathbb{N}} \text{ converges}\} = \{(S_{n,k}/n)_{n\geq k} \text{ converges}\} \tag{2}$$

for all $k \in \mathbb{N}$. Assume that $\omega \in \Omega$ belongs to the set on the left. Then

$$\lim_{n\to\infty} \frac{S_{n,k}(\omega)}{n} = \lim_{n\to\infty} \frac{S_n(\omega) - S_{k-1}(\omega)}{n} = \lim_{n\to\infty} \frac{S_n(\omega)}{n} - \lim_{n\to\infty} \frac{S_{k-1}(\omega)}{n} = \lim_{n\to\infty} \frac{S_n(\omega)}{n} \tag{3}$$

implies that $(S_{n,k}(\omega)/n)_{n\geq k}$ also converges. Conversly

$$\lim_{n\to\infty} \frac{S_n(\omega)}{n} = \lim_{n\to\infty} \frac{S_n(\omega) - S_{k-1}(\omega) + S_{k-1}(\omega)}{n} = \lim_{n\to\infty} \frac{S_{n,k}(\omega)}{n} \tag{4}$$

implies that $(S_n(\omega)/n)_{n\in\mathbb{N}}$ converges whenever $(S_{n,k}(\omega)/n)_{n\geq k}$ converges. Now fix $k \in \mathbb{N}$. Then $\sigma(X_k, X_{k+1}, \ldots)$ is the smallest σ-algebra on Ω that makes each $X_k, X_{k+1}, \ldots$ measurable (see [Coh13, p. 309]). Hence each $S_{n,k}/n$, $n \geq k$, is measurable with respect to $\sigma(X_k, X_{k+1}, \ldots)$ as a scalar multiple of a finite sum of measurable functions. So by exercise 4 [Coh13, p. 49] we have that

$$\{(S_{n,k}/n)_{n\geq k} \text{ converges}\} \in \sigma(X_k, X_{k+1}, \ldots). \tag{5}$$

Thus $\{(S_n/n)_{n\in\mathbb{N}} \text{ converges}\} \in \bigcap_{k\in\mathbb{N}} \sigma(X_k, X_{k+1}, \ldots)$ and therefore $\mathsf{P}(\{(S_n/n)_{n\in\mathbb{N}} \text{ converges}\})$ is either 0 or 1 by Kolmogorov's zero-one law.

References

[Coh13] Donald L. Cohn. *Measure Theory*. Second edition. Springer, 2013.

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