Yannis Bähni

Solution Book to Introduction to Smooth Manifolds by John M. Lee

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Contents

CHAPTER 1

Foundations

1. Set Theory

1.1. Relations.

Exercise 1.1. Let X be a set and $\sim$ an equivalence relation on X. Then $X/\sim$ is a partition of X. Conversly, given any partition $\mathscr{C}$ of X, there exists a unique equivalence relation $\sim_{\mathscr{C}}$ on X such that $X/\sim_{\mathscr{C}} = \mathscr{C}$.

Solution 1.1. Let $\sim$ be an equivalence relation on X. If $X = \varnothing$, then $X/\sim = \varnothing$ which is a partition of the empty set $\varnothing$. So assume that $X \neq \varnothing$. Let $[x] \in X/\sim$. Then $[x] \neq \varnothing$ since $x \in [x]$ by reflexivity. Furthermore, let $[y] \in X/\sim$. Now we have to show that if $[x] \neq [y]$ then $[x] \cap [y] = \varnothing$. So assume that $z \in [x] \cap [y]$. Then $z \sim x$ and $z \sim y$ which implies $x \sim y$ by symmetry and transitivity from which easily follows that $[x] = [y]$. Also $X = \cup_{x \in X} [x]$ holds and thus $X/\sim$ is a partition of X.
Define a relation on X by

$$x \sim_{\mathscr{C}} y \quad :\Leftrightarrow \quad \exists A \in \mathscr{C} : x, y \in A.$$

Then it is easily seen that $\sim_{\mathscr{C}}$ is an equivalence relation on X where $[x] = A$ for some $A \in \mathscr{C}$ such that $x \in A$. Thus $X/\sim_{\mathscr{C}} = \mathscr{C}$.

CHAPTER 2

Connectedness and Compactness

1. Connectedness

Exercise 1.1. Let X be a nonempty connected topological space and $\sim$ an equivalence relation on X such that every equivalence class is open. Then there is exactly one equivalence class.

Solution 1.1. Let $x \in X$. If $[x] = X$, there is nothing to show. So assume that $[x] \neq X$. By exercise 1.1 we have that $X/\sim$ is a partition of X and thus $X = [x] \cup \left(\cup_{y \in [x]^c} [y]\right)$. Since $[x]$ and $\cup_{y \in [x]^c} [y]$ are nonempty, disjoint and open by assumption, they disconnect X, contradicting the connectedness of X.

2. Local Compactness

Exercise 2.1. In a Baire space, every meager subset has a dense complement.

Solution 2.1. Let F be a meager subset of the Baire space. Thus we can write $F = \cup_n F_n$, where each F_n is nowhere dense. Therefore $F^c = \cap_n F_n^c$ and $\cap_n \overline{F_n}^c \subseteq \cap_n F_n^c$. By the definition of a Baire space, $\cap_n \overline{F_n}^c$ is dense since each $\overline{F_n}$ is closed. Thus

$$X \supseteq \overline{F^c} \supseteq \overline{\cap_n \overline{F_n}^c} = X$$

and therefore $\overline{F^c}$ is dense.

CHAPTER 3

Homotopy and the Fundamental Group

1. The Fundamental Group

Exercise 1.1. Let X be a topological space. For any points $p, q \in X$, path homotopy is an equivalence relation on the set of all paths in X from p to q.

Solution 1.1. Let f be a path in X from p to q. Define $H : I \times I \to X$ by $H(s,t) := f(s)$. Clearly H is continuous since f is. Indeed, take $U \subseteq X$ open, then $H^{-1}(U) = f^{-1}(U) \times I$ which is open in the box topology. Now clearly $H(s,0) = H(s,1) = f(s)$ for all $s \in I$ and $H(0,t) = f(0) = p$, $H(1,t) = f(1) = q$ for all $t \in I$. Hence $f \sim f$. If $f \sim g$ and F is a path homotopy from f to g, then $H : I \times I \to X$ defined by $H(s,t) := F(1-s, 1-t)$ is a path homotopy from g to f. Thus $g \sim f$. If $f \sim g$ and $g \sim h$ where F and G denote the path homotopies, respectively, then $H : I \times I \to X$ defined by

$$H(s,t) := \begin{cases} F(s,2t) & 0 \le t \le \frac{1}{2}, \\ G(s,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

is a path homotopy from f to h, hence $f \sim g$.

Exercise 1.2. Let X be a path-connected topological space.

(a) Let $f, g : I \to X$ be two paths from p to q. Show that $f \sim g$ if and only if $f\overline{g} \sim c_p$.
(b) Show that X is simply connected if and only if any two paths in X with the same initial and terminal points are path-homotopic.
(c) Let $A \subseteq \mathbb{R}^n$ be convex. Then A is simply connected.

Solution 1.2. For (a), assume $f \sim g$. Hence $[f] = [g]$ and by the properties of path class products [Lee11, p. 189] we get

$$[f\overline{g}] = [f]\,[\overline{g}] = [g]\,[\overline{g}] = [c_p]$$

and thus $f\overline{g} \sim c_p$. Conversely, $f\overline{g} \sim c_p$ implies $[f\overline{g}] = [c_p]$ and thus

$$[g] = [c_p]\,[g] = ([f\overline{g}])\,[g] = ([f]\,[\overline{g}])\,[g] = [f]\,([\overline{g}]\,[g]) = [f]\,[c_q] = [f]\,.$$

For (b), assume that X is simply connected and let f and g be paths in X from p to q. Then $f\overline{g}$ is a loop based at p. Since $\pi_1(X,p) = \{[c_p]\}$, we get that $f\overline{g} \sim c_p$ and thus by part (a) that $f \sim g$. Conversely, let f be a loop based at p. Hence $f \sim c_p$ and so $\pi_1(X,p)$ is trivial. For (c), let f and g be paths in A from p to q. Then by example 7.4 [Lee11, pp. 185–186] we get that $f \sim g$. Hence by part (b) follows that A is simply connected.

Corollary 1.1. $\mathbb{R}^n$ *is simply connected.*

2. Categories and Functors

See [Lan71, pp. 57–58].

Exercise 2.1. Let G be a group and $N \trianglelefteq G$. Define $F : \mathsf{Grp} \to \mathsf{Set}$ by

$$F(H) := \{f \in \operatorname{Hom}(G, H) : N \subseteq \ker f\}\,. \tag{1}$$

(i) Show that F is a functor.
(ii) Show that $\langle {}^{G}\!/_{N}, \pi\rangle$ is a universal element of the functor F.

Solution 2.1. For (i), we have to define first the action of F on arrows of Grp. Consider $A \xrightarrow{\varphi} B$. Define $F(\varphi) : F(A) \to F(B)$ by

$$F(\varphi)(f) := \varphi \circ f.$$

Let $f \in F(A)$. Then $F(\mathrm{id}_A)(f) = \mathrm{id}_A \circ f = f$ and thus $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$. Furthermore, for $B \xrightarrow{\psi} C$ we have that

$$\begin{aligned} F(\psi \circ \varphi)(f) &= (\psi \circ \varphi) \circ f \\ &= \psi \circ (\varphi \circ f) \\ &= \psi \circ F(\varphi)(f) \\ &= F(\psi)\big(F(\varphi)(f)\big) \\ &= \big(F(\psi) \circ F(\varphi)\big)\,(f) \end{aligned}$$

and so $F(\psi\circ\varphi) = F(\psi)\circ F(\varphi)$. Hence F is a functor. For (ii), by proposition 4.7 [Gri07, p. 20] we get that $\pi \in F({}^{G}\!/_{N})$. Furthermore, consider $\langle A, \varphi\rangle$ for any A object and φ morphism in Grp such that $\varphi \in F(A)$. By the factorization theorem [Gri07, p. 23] there exists a unique homomorphism $\psi : G \to A$ such that $\varphi = \psi \circ \pi$. Thus

$$F(\psi)(\pi) = \psi \circ \pi = \varphi$$

and thus $\langle {}^{G}\!/_{N}, \pi\rangle$ is a universal element of the functor F.

Exercise 2.2. Let C be a category and $(X_\alpha)_{\alpha\in A}$ be a familiy of objects of C. If (S, ι_α) and (S', ι'_α) are two coproducts of $(X_\alpha)_{\alpha\in A}$, then there exists a unique isomorphism $f : S \to S'$ such that $f \circ \iota_\alpha = \iota'_\alpha$ for all $\alpha \in A$.

Solution 2.2. By the defining property of a coproduct there exist unique morphisms $f : S \to S'$ and $g : S' \to S$ as indicated in the commutative diagram below. Furthermore, above diagram yields

$$\begin{tikzcd} & S \arrow[dr, "f"] & \\ S' \arrow[ur, "g"] & X_\alpha \arrow[u, "\iota_\alpha"'] \arrow[l, "\iota'_\alpha"] \arrow[r, "\iota'_\alpha"'] \arrow[d, "\iota_\alpha"'] & S' \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

$$(g \circ f) \circ \iota_\alpha = \iota_\alpha \qquad \text{and} \qquad (f \circ g) \circ \iota'_\alpha = \iota'_\alpha.$$

for all $\alpha \in A$ and thus the commutative diagrams Also are commutative and

$$
\begin{tikzcd}
S \arrow[dr, "g\circ f"] & \\
X_\alpha \arrow[u, "\iota_\alpha"] \arrow[r, "\iota_\alpha"'] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
S' \arrow[dr, "f\circ g"] & \\
X_\alpha \arrow[u, "\iota'_\alpha"] \arrow[r, "\iota'_\alpha"'] & S'
\end{tikzcd}
$$

$$
\begin{tikzcd}
S \arrow[dr, "\mathrm{id}_S"] & \\
X_\alpha \arrow[u, "\iota_\alpha"] \arrow[r, "\iota_\alpha"'] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
S' \arrow[dr, "\mathrm{id}_{S'}"] & \\
X_\alpha \arrow[u, "\iota'_\alpha"] \arrow[r, "\iota'_\alpha"'] & S'
\end{tikzcd}
$$

so by the uniqueness property of the coproduct we get that

$$g \circ f = \mathrm{id}_S \qquad \text{and} \qquad f \circ g = \mathrm{id}_{S'} \,.$$

CHAPTER 4

The Seifert-Van Kampen Theorem

1. Fundamental Groups of Compact Surfaces

Exercise 1.1. Let G be a group. Recall, that for $g, h \in G$ the ***commutator of g and h***, written $[g, h]$, is defined to be

$$[g, h] := ghg^{-1}h^{-1}. \tag{2}$$

Furthermore, define

$$[G, G] := \langle \{[g, h] : g, h \in G\} \rangle. \tag{3}$$

(a) Show that $[G, G] \trianglelefteq G$. $[G, G]$ is called the ***commutator subgroup of G***.
(b) $[G, G]$ is trivial if and only if G is abelian.
(c) $G/[G, G]$ is abelian.

Solution 1.1. For (a), set

$$X := \{[g, h] : g, h \in G\}.$$

Then by [KM13, p. 31] we have that

$$\langle X \rangle = \{x_1 \cdots x_n : n \in \mathbb{Z}, n \geq 1, x_1, \ldots, x_n \in X \cup X^{-1}\}.$$

Since for any $g \in G$ and $x \in \langle X \rangle$ we have

$$gxg^{-1} = gx_1 \cdots x_n g^{-1} = gx_1g^{-1}gx_2g^{-1} \cdots gx_{n-1}g^{-1}gx_ng^{-1}$$

it is enough to show that $g[h, k]g^{-1} \in \langle X \rangle$ for every $h, k \in G$. But

$$g[h, k]g^{-1} = [ghg^{-1}, gkg^{-1}]$$

and thus $[G, G] \trianglelefteq G$. For (b), assume that $[G, G] = \{1\}$. Since $X \subseteq [G, G]$, we have that $ghg^{-1}h^{-1} = 1$ for all $g, h \in G$ which is equivalent to $gh = hg$. Hence G is abelian. Conversely, assume that G is abelian. Then $[g, h] = 1$ for all $g, h \in G$, which implies $X = \{1\}$ and thus $[G, G]$ is trivial. For (c), let $x[G, G], y[G, G] \in G/[G, G]$. Then we see that

$$[x[G, G], y[G, G]] = [G, G].$$

Hence $[G/[G, G], G/[G, G]]$ is trivial and the claim follows from part (b).

CHAPTER 5

Covering Maps

1. Definitions and Basic Properties

Exercise 1.1. For $n \in \mathbb{Z}$, $n \geq 1$, define the ***n-th power map*** $p_n : \mathbb{S}^1 \to \mathbb{S}^1$ by $p_n(z) := z^n$. Show that p_n is a covering map.

Solution 1.1. The map p_n is surely continuous. We show surjectivity. Let $z^n \in p_n(\mathbb{S}^1)$. Then we have that $|z^n| = |z|^n = 1$ and thus $p_n(\mathbb{S}^1) \subseteq \mathbb{S}^1$. Conversly, every $z \in \mathbb{S}^1$ can be written as $e^{i\varphi}$ for some $\varphi \in \mathbb{R}$. Hence define $\widetilde{z} := e^{i\varphi/n} \in \mathbb{S}^1$. Then we have that $p_n(\widetilde{z}) = z$ and thus $\mathbb{S}^1 \subseteq p_n(\mathbb{S}^1)$. Now let $z_0 \in \mathbb{S}^1$. Define $U_{z_0} := \mathbb{S}^1 \setminus \{-z_0\}$. Then $p_n^{-1}(U_{z_0}) = \{z \in \mathbb{S}^1 : z^n \neq -z_0\}$, or equivalently $p_n^{-1}(U_{z_0}) = \{z \in \mathbb{S}^1 : z^n = -z_0\}^c$. Hence

Exercise 1.2. Let $f : (0, 2) \to \mathbb{S}^1$ be defined by $f(x) := e^{2\pi i x}$. Show that f is not a covering map.

Solution 1.2.

CHAPTER 6

Smooth Maps

1. Smooth Functions and Smooth Maps

1.1. Smooth Functions on Manifolds. We follow the terminology established in [Gri07, p. 515].

Exercise 1.1. Let M be a smooth manifold. $\mathscr{C}^\infty(M)$ is an associative and commutative $\mathbb{R}$-algebra with identity under the usual pointwise defined operations.

Solution 1.1. First we show that $\mathscr{C}^\infty(M)$ is a real vector space. Since $\mathscr{C}^\infty(M) \subseteq \mathbb{R}^M$ it is enough to show that $\mathscr{C}^\infty(M)$ is a linear subspace of the real vector space $\mathbb{R}^M$. Clearly, $\mathscr{C}^\infty(M) \neq \varnothing$, since $\chi_M \in \mathscr{C}^\infty(M)$. Indeed, for $p \in M$ we find a chart (U, φ) such that $p \in U$ and the composition $\chi_M \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is clearly the function $\chi_{\varphi(U)}$, which is smooth since it is constant. Now let $f, g \in \mathscr{C}^\infty(M)$, $\lambda \in \mathbb{R}$ and $p \in M$. By definition, there exist charts (U, φ), (V, ψ) such that $f \circ \varphi^{-1}$ and $g \circ \psi^{-1}$ are smooth. Now consider the chart $(U \cap V, \varphi)$. Then by

$$(\lambda f + g) \circ \varphi^{-1} = \lambda(f \circ \varphi^{-1}) + (g \circ \varphi^{-1}) = \lambda(f \circ \varphi^{-1}) + \big((g \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1})\big)$$

we have that $\lambda f + g \in \mathscr{C}^\infty(M)$. Hence $\mathscr{C}^\infty(M)$ is a real vector space.
Now define a product map $\cdot : \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ by pointwise multiplication. Indeed, similar to the previous reasoning, by

$$(f \cdot g) \circ \varphi^{-1} = (f \circ \varphi^{-1}) \cdot (g \circ \varphi^{-1}) = (f \circ \varphi^{-1}) \cdot \big((g \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1})\big)$$

we have that $f \cdot g$ is smooth. Let $f, g, h \in \mathscr{C}^\infty(M)$ and $\lambda \in \mathbb{R}$. Then for $p \in M$

$$\begin{aligned}
\big((\lambda f + g) \cdot h\big)(p) &= (\lambda f + g)(p) h(p) \\
&= \big(\lambda f(p) + g(p)\big) h(p) \\
&= \lambda f(p) h(p) + g(p) h(p) \\
&= \lambda (f \cdot h)(p) + (g \cdot h)(p) \\
&= \big(\lambda (f \cdot h)\big)(p) + (g \cdot h)(p) \\
&= \big(\lambda (f \cdot h) + (g \cdot h)\big)(p)
\end{aligned}$$

shows that $\cdot$ is bilinear in the first argument. A similar computation shows that $\cdot$ is bilinear. By

$$\begin{aligned}
\big((f \cdot g) \cdot\big)(p) &= (f \cdot g)(p) h(p) \\
&= f(p) g(p) h(p) \\
&= f(p)(g \cdot h)(p) \\
&= \big(f \cdot (g \cdot h)\big)(p)
\end{aligned}$$

we see that $\cdot$ is associative. Furthermore by

$$(f \cdot g)(p) = f(p)g(p) = g(p)f(p) = (g \cdot f)(p)$$

we see that $\cdot$ is commutative. Finally, the identity element is given by χ_M since

$$(\chi_M \cdot f)(p) = \chi_M(p)f(p) = 1f(p) = f(p).$$

CHAPTER 7

Tangent Vectors

1. Tangent Vectors

1.1. Tangent Vectors on Manifolds.

Exercise 1.1. Let M be a smooth manifold and $p \in M$. The set of all derivations at p, written T_pM, is a real vector space under the usual pointwise defined operations.

Solution 1.1. Clearly $T_pM \subseteq \mathrm{L}(\mathscr{C}^\infty(M); \mathbb{R})$ and thus it is enough to show that T_pM is a linear subspace of $\mathrm{L}(\mathscr{C}^\infty(M); \mathbb{R})$ (see [Lee13, p. 626]). We have $T_pM \neq \varnothing$, since $0 \in T_pM$ defined by $f \mapsto 0$. Let $u, v \in T_pM$, $\lambda \in \mathbb{R}$ and $f, g \in \mathscr{C}^\infty(M)$. Then by

$$\begin{aligned}(\lambda u + v)(fg) &= \lambda u(fg) + v(fg) \\ &= f(p)\left(\lambda u(g) + v(g)\right) + g(p)\left(\lambda u(f) + v(f)\right) \\ &= f(p)(\lambda u + v)(g) + g(p)(\lambda u + v)(f)\end{aligned}$$

we have that $\lambda u + v \in T_pM$.

Exercise 1.2. Suppose M is a smooth manifold. Let $p \in M$, $v \in T_pM$ and $f \in \mathscr{C}^\infty(M)$. If f is constant, then $v(f) = 0$.

Solution 1.2. First assume that $f = \chi_M$. Then

$$v(\chi_M) = v(\chi_M \cdot \chi_M) = f(p)v(\chi_M) + f(p)v(\chi_M) = v(\chi_M) + v(\chi_M) \tag{4}$$

implies that $v(f) = 0$. Hence if $f = \lambda\chi_M$ for $\lambda \in \mathbb{R}$, the $\mathbb{R}$-linearity of v implies that

$$v(f) = v(\lambda\chi_M) = \lambda v(\chi_M) = 0. \tag{5}$$

Exercise 1.3 (Properties of Differentials). Let M, N and P be smooth manifolds, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.

(a) $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is $\mathbb{R}$-linear.
(b) $\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p$.
(c) $\mathrm{d}(id_M)_p = id_{T_pM}$.
(d) If F is a diffeomorphism, then $\mathrm{d}F_p$ is an isomorphism and $(\mathrm{d}F_p)^{-1} = \mathrm{d}\left(F^{-1}\right)_{F(p)}$.

Solution 1.3. Let $u, v \in T_pM$, $\lambda \in \mathbb{R}$ and $f \in \mathscr{C}^\infty(N)$. Then

$$\begin{aligned}\mathrm{d}F_p(\lambda u + v)(f) &= (\lambda u + v)(f \circ F) \\ &= \lambda u(f \circ F) + v(f \circ F) \\ &= \lambda\, \mathrm{d}F_p(u)(f) + \mathrm{d}F_p(v)(f).\end{aligned}$$

This shows part (a). Let $v \in T_pM$ and $f \in \mathscr{C}^\infty(P)$. Then

$$\begin{aligned} \mathrm{d}(G \circ F)_p(v)(f) &= v\left(f \circ (G \circ F)\right) \\ &= v\left((f \circ G) \circ F\right) \\ &= \mathrm{d}F_p(f \circ G) \\ &= \mathrm{d}G_{F(p)}\left(\mathrm{d}F_p(v)\right)(f) \\ &= \left(\mathrm{d}G_{F(p)} \circ \mathrm{d}F_p\right)(v)(f). \end{aligned}$$

This shows part (b). Part (c) is immediate by

$$\mathrm{d}(id_M)_p(v)(f) = v(f \circ id_M) = v(f)$$

for $f \in \mathscr{C}^\infty(M)$. Finally, part (b) and (c) yields

$$\mathrm{d}F_p \circ \mathrm{d}\left(F^{-1}\right)_{F(p)} = \mathrm{d}(F \circ F^{-1})_{F(p)} = \mathrm{d}(id_N)_{F(p)} = id_{T_{F(p)}N}$$

and

$$\mathrm{d}\left(F^{-1}\right)_{F(p)} \circ \mathrm{d}F_p = \mathrm{d}(F^{-1} \circ F)_p = \mathrm{d}(id_M)_p = id_{T_pM}$$

which shows that $\mathrm{d}F_p$ is bijective with inverse $(\mathrm{d}F_p)^{-1} = \mathrm{d}\left(F^{-1}\right)_{F(p)}$ by uniqueness. Since by part (a) $\mathrm{d}F_p$ is linear, we have that dF_p is an isomorphism (see [Lee13, p. 622]). This shows part (d).

CHAPTER 8

Vector Fields

1. Vector Fields on Manifolds

Exercise 1.1. Let M be a smooth manifold.

(a) If $X, Y \in \mathfrak{X}(M)$, $f, g \in \mathscr{C}^\infty(M)$, then $fX + gY \in \mathfrak{X}(M)$.

CHAPTER 9

Integral Curves and Flows

1. Integral Curves

Definition 1.1. *Let (M_1, d_1) and (M_2, d_2) be metric spaces. A mapping $f : M_1 \to M_2$ is said to be* ***Lipschitz continuous*** *if there exists $L \in \mathbb{R}_{>0}$ such that for all $x, y \in M_1$*

$$d_2\left(f(x), f(y)\right) \leq L d_1(x, y) \tag{6}$$

holds. We say that f is ***locally Lipschitz continuous*** *if for every point $x \in M_1$ there exists a neighbourhood on which f is Lipschitz continuous.*

Proposition 1.1. *Let (M_1, d_1) be a metric space and (M_2, d_2) a complete bounded metric space. For $f, g \in \mathscr{C}(M_1; M_2)$ define*

$$d_\infty(f, g) := \sup_{x \in M_1} d_2\left(f(x), g(x)\right). \tag{7}$$

Then $(\mathscr{C}(M_1; M_2), d_\infty)$ is a complete metric space.

Proof. Since M is bounded, there exists $C \in \mathbb{R}_{>0}$ such that $d_2(x, y) \leq R$ for all $x, y \in M_1$. Hence

$$d_\infty(f, g) = \sup_{x \in M_1} d_2\left(f(x), g(x)\right) \leq R < \infty$$

for all $f, g \in \mathscr{C}(M_1; M_2)$. The metric axioms are easily verified, so we only show the completeness property. Let $(f_\nu)_{\nu \in \mathbb{N}}$ be a Cauchy sequence in $\mathscr{C}(M_1; M_2)$. Fix $\varepsilon > 0$. Since $(f_\nu)_{\nu \in \mathbb{N}}$ is a Cauchy sequence, we find $N \in \mathbb{N}$, such that for all $\nu, \mu \geq N$

$$d_\infty\left(f_\nu, f_\mu\right) < \frac{\varepsilon}{2}$$

holds. So for all $y \in M_1$ we have

$$d_2\left(f_\nu(y), f_\mu(y)\right) \leq \sup_{x \in X} d_2\left(f_\nu(x), f_\mu(x)\right) = d_\infty(f_\nu, f_\mu) < \varepsilon.$$

whenever $\nu, \mu \geq N$. Thus $\left(f_\nu(y)\right)_{\nu \in \mathbb{N}}$ is a Cauchy sequence in M_2 for all $y \in M_1$. Since M_2 is complete

$$f(y) := \lim_{\nu \to \infty} f_\nu(y)$$

exists for all $y \in M_1$. Now we show that $f_\nu \to f$ with respect to d_∞. For all $\nu \geq N$ and $y \in M_1$ we have that

$$\begin{aligned} d_2\left(f_\nu(y), f(y)\right) &= \lim_{\mu\to\infty} d_2\left(f_\nu(y), f_\mu(y)\right) \\ &= \liminf_{\mu\to\infty} d_2\left(f_\nu(y), f_\mu(y)\right) \\ &\leq \liminf_{\mu\to\infty} d_\infty(f_\nu, f_\mu) \\ &\leq \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

Hence

$$d_\infty(f_\nu, f) < \varepsilon$$

whenever $\nu \geq N$. So $f_\nu \to f$ with respect to d_∞. Left to show is that $f \in \mathscr{C}(M_1; M_2)$. Fix $x_0 \in M_1$. Since $f_\nu \to f$ with respect to d_∞, there exists $N \in \mathbb{N}$ such that

$$d_\infty(f_\nu, f) < \frac{\varepsilon}{3}$$

for all $\nu \geq N$. Fix $\nu_0 \geq N$. Since f_{ν_0} is continuous at x_0, there exists $\delta > 0$, such that

$$d_2\left(f_{\nu_0}(x_0), f_{\nu_0}(x)\right) < \frac{\varepsilon}{3}$$

whenever $d_1(x_0, x) < \delta$. Hence

$$\begin{aligned} d_2\left(f(x_0), f(x)\right) &= d_2\left(f(x_0), f_{\nu_0}(x)\right) + d_2\left(f_{\nu_0}(x_0), f_{\nu_0}(x)\right) + d_2\left(f_{\nu_0}(x), f(x)\right) \\ &< 2d_\infty(f, f_{\nu_0}) + \frac{\varepsilon}{3} \\ &< \varepsilon \end{aligned}$$

whenever $d_1(x_0, x) < \delta$. Thus $f \in \mathscr{C}(M_1; M_2)$. □

Lemma 1.1 (Integral Formulation of an ODE). *Let $n \in \mathbb{Z}$, $n > 0$, $U \subseteq \mathbb{R}^n$ and $f \in \mathscr{C}(U; \mathbb{R}^n)$. A mapping $y \in \mathscr{C}(J_0; U)$, for some interval J_0 containing t_0, is a solution of the initial value problem*

$$\begin{cases} y'(t) = f\left(y(t)\right) \\ y(t_0) = y_0 \end{cases} \tag{8}$$

if and only if

$$y(t) = y_0 + \int_{t_0}^{t} f\left(y(s)\right) \mathrm{d}s \tag{9}$$

holds for all $t \in J_0$.

Proof. Assume that $y \in \mathscr{C}^1(J_0; U)$ solves (8). Then

$$\int_{t_0}^{t} f\left(y(s)\right) \mathrm{d}s = \int_{t_0}^{t} y'(s) \,\mathrm{d}s = y(t) - y(t_0)$$

for all $t \in J_0$ by the corollary to the first fundamental theorem of calculus [Spi94, p. 284].

Conversly assume that

$$y(t) = y_0 + \int_{t_0}^{t} f\left(y(s)\right) \mathrm{d}s$$

for all $t \in J_0$. Since $f \circ y \in \mathscr{C}(J_0; \mathbb{R}^n)$, the first fundamental theorem of calculus [Spi94, p. 282] implies $y'(t) = f\left(y(s)\right)$ for all $t \in J_0$. Furthermore clearly $y(t_0) = t_0$ and $y \in \mathscr{C}^1(J_0; U)$. Hence y is a solution of (8). $\square$

Lemma 1.2 (Contraction Lemma). *Let (M, d) be a nonempty complete metric space and T be a contraction. Then there exists a unique fixed point for T.*

Theorem 1.1 (Existence of ODE Solutions). *Let $n \in \mathbb{Z}$, $n > 0$, $U \subseteq \mathbb{R}^n$ open, $f \in \mathscr{C}(U; \mathbb{R}^n)$ locally Lipschitz continuous and $(t_0, x_0) \in \mathbb{R} \times U$. Then there exists an open interval $J_0 \subseteq \mathbb{R}$ and an open subset $U_0 \subseteq U$, such that $(t_0, x_0) \in J_0 \times U_0$ and for each $y_0 \in U_0$ a mapping $y \in \mathscr{C}^1(J_0; U)$ satisfying*

$$\begin{cases} y'(t) = f\left(y(t)\right) \\ y(t_0) = y_0 \end{cases}. \tag{10}$$

Proof. Since F is locally Lipschitz continuous on U, there exists a neighbourhood V of x_0, such that f is Lipschitz continuous on V. Since $(U, |\cdot|)$ has the same topology as the subspace $U \subseteq \mathbb{R}^n$ by [Lee11, p. 50], we find $W \subseteq \mathbb{R}^n$ open, such that $V = U \cap W$. But since U is open, so is V open in $\mathbb{R}^n$. Hence we may assume that f is Lipschitz continuous on U. Let $L > 0$ denote a Lipschitz constant of f. Now choose $r > 0$ so, such that $\overline{B}_r(x_0) \subseteq U$. Furthermore let

$$M := \sup_{x \in \overline{B}_r(x_0)} |f(x)| < \infty$$

since $\overline{B}_r(x_0)$ is compact and $\delta, \varepsilon > 0$ such that

$$\delta < \frac{r}{2} \qquad \text{and} \qquad \varepsilon < \min\left(\frac{r}{2M}, \frac{1}{L}\right).$$

Define

$$J_0 := (t_0 - \varepsilon, t_0 + \varepsilon) \subseteq \mathbb{R} \qquad \text{and} \qquad U_0 := B_\delta(x_0) \subseteq U.$$

For any $y_0 \in U_0$, let

$$A_{y_0} := \left\{ y \in \mathscr{C}(J_0; \overline{B}_r(x_0)) : y(t_0) = y_0 \right\}.$$

Clearly $A_{y_0} \neq \varnothing$ since $y = y_0$ is in A_{y_0}. $\overline{B}_r(x_0)$ is clearly bounded and complete since it is a closed subset of a complete metric space. Thus we can consider the metric space (A_{y_0}, d_∞), where d_∞ is defined as in proposition 1.1. From the proof of proposition 1.1 we also see that if $(y_\nu)_{\nu \in \mathbb{N}}$ is a Cauchy sequence in A_{y_0} and $y := \lim_{\nu \to \infty} y_\nu$, then $y(t_0) = \lim_{\nu \to \infty} y_\nu(t_0) = y_0$. Hence $y \in A_{y_0}$ and so (A_{y_0}, d_∞) is complete. For $y \in A_{y_0}$ define for $t \in J_0$

$$T(y)(t) := y_0 + \int_{t_0}^t f\left(y(s)\right) \mathrm{d}s.$$

Clearly T is continuous and $T(y)(t_0) = y_0$. Furthermore

$$\begin{aligned}
\big|T(y)(t) - x_0\big| &= \left|y_0 + \int_{t_0}^{t} f\big(y(s)\big)\,\mathrm{d}s - x_0\right| \\
&\leq |y_0 - x_0| + \int_{t_0}^{t} \big|f\big(y(s)\big)\big|\,\mathrm{d}s \\
&< \delta + M\,|t - t_0| \\
&< \delta + M\varepsilon \\
&< r.
\end{aligned}$$

for all $t \in J_0$. Hence $T : A_{y_0} \to A_{y_0}$. Furthermore for $y_1, y_2 \in A_{y_0}$ we have that

$$\begin{aligned}
d_\infty\big(T(y_1), T(y_2)\big) &= \sup_{t \in J_0}\left|\int_{t_0}^{t} f\big(y_1(s)\big)\,\mathrm{d}s - \int_{t_0}^{t} f\big(y_2(s)\big)\,\mathrm{d}s\right| \\
&\leq \sup_{t \in J_0}\int_{t_0}^{t} \big|f\big(y_1(s)\big) - f\big(y_2(s)\big)\big|\,\mathrm{d}s \\
&\leq L \sup_{t \in J_0}\int_{t_0}^{t} \big|y_1(s) - y_2(s)\big|\,\mathrm{d}s \\
&\leq L\varepsilon d_\infty(y_1, y_2).
\end{aligned}$$

Since $0 < L\varepsilon < 1$, T is a contraction. Hence by the contraction lemma 1.2 there exists a unique fixed point $y \in A_{y_0}$. This y is a solution to the initial value problem by lemma 1.1. □

CHAPTER 10

The Cotangent Bundle

1. Line Integrals

1.1. The Winding Number*.

Definition 1.1 (Winding Number). *Let $z_0 \in \mathbb{C}$ and $\gamma : [a, b] \to \mathbb{C} \setminus \{z_0\}$ be a piecewise continuously differentiable closed path. Then*

$$\mathrm{W}(\gamma, z_0) := \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - z_0} \tag{11}$$

is called the ***winding number*** *of γ around z_0.*

Proposition 1.1. *Let $z_0 := x_0 + iy_0 \in \mathbb{C}$ and $\gamma : [a, b] \to \mathbb{C} \setminus \{z_0\}$ be a piecewise continuously differentiable closed path. Then*

$$\mathrm{W}(\gamma, z_0) = \int_\gamma \omega \tag{12}$$

where $\omega \in \Omega^1\left(\mathbb{R}^2 \setminus \{z_0\}\right)$ is given by

$$\omega := \frac{1}{2\pi} \frac{(x - x_0)\,\mathrm{d}y - (y - y_0)\,\mathrm{d}x}{(x - x_0)^2 + (y - y_0)^2}. \tag{13}$$

Proof. This immediately follows from

$$\begin{aligned}
\int_\gamma \frac{\mathrm{d}z}{z - z_0} &= \int_\gamma \frac{\mathrm{d}x + i\,\mathrm{d}y}{(x + iy) - (x_0 + iy_0)} \\
&= \int_\gamma \frac{\mathrm{d}x + i\,\mathrm{d}y}{(x - x_0) + i(y - y_0)} \frac{(x - x_0) - i(y - y_0)}{(x - x_0) - i(y - y_0)} \\
&= \int_\gamma \frac{(x - x_0)\,\mathrm{d}x + \big((x - x_0)\,\mathrm{d}y - (y - y_0)\,\mathrm{d}x\big) + (y - y_0)\,\mathrm{d}y}{(x - x_0)^2 + (y - y_0)^2} \\
&= \int_\gamma \frac{(x - x_0)\,\mathrm{d}x + (y - y_0)\,\mathrm{d}y}{(x - x_0)^2 + (y - y_0)^2} + i \int_\gamma \frac{(x - x_0)\,\mathrm{d}y - (y - y_0)\,\mathrm{d}x}{(x - x_0)^2 + (y - y_0)^2} \\
&= i \int_\gamma \frac{(x - x_0)\,\mathrm{d}y - (y - y_0)\,\mathrm{d}x}{(x - x_0)^2 + (y - y_0)^2}
\end{aligned}$$

by the fundamental theorem for line integrals [Lee13, p. 291] since

$$\mathrm{d}\left(\frac{1}{2} \log\left((x - x_0)^2 + (y - y_0)^2\right)\right) = \frac{(x - x_0)\,\mathrm{d}x + (y - y_0)\,\mathrm{d}y}{(x - x_0)^2 + (y - y_0)^2}.$$

□

Remark 1.1. By proposition 1.1, the definition of the winding number in complex analysis given by definition 1.1 coincides with the one usually given in algebraic topology (see for example [Ful95, pp. 19–20]).

CHAPTER 11

Tensors

1. Multilinear Algebra

We follow the terminology established in [Lee13, p. 312].

Definition 1.1. *Let V be a finite-dimensional real vector space and $k, l \in \mathbb{Z}$ where $k, l \geq 0$. Then we define the* ***space of mixed tensors of type (k, l) on V*** *by*

$$T^{(k,l)}(V) := \underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l} \tag{14}$$

if $(k, l) \neq (0, 0)$ and

$$T^{(0,0)}(V) := \mathbb{R} \tag{15}$$

otherwise.

Proposition 1.1 (Tensor Characterization Lemma). *Let V be a finite-dimensional real vector space and $k, l \in \mathbb{Z}$ where $k \geq 1$, $l \geq 0$ and $(k, l) \neq (1, 0)$. Then*

$$\boxed{T^{(k,l)}(V) \cong \mathrm{L}\left((V^*)^{k-1}, V^l; V\right).} \tag{16}$$

Lemma 1.1.

Proof. Define

$$\Phi : V^k \times (V^*)^l \to \mathrm{L}\left((V^*)^{k-1}, V^l; V\right)$$

by letting

$$\Phi(v, \varphi)(\psi, w) := \varphi_1(w_1) \cdots \varphi_l(w_l) \psi_1(v_1) \cdots \psi_{k-1}(v_{k-1}) v_k.$$

It is easyily checked that $\Phi(v, \varphi) \in \mathrm{L}\left((V^*)^{k-1}, V^l; V\right)$ and that Φ is multilinear. By the characteristic property of the tensor product space [Lee13, p. 309] there exists a unique linear mapping

$$\widetilde{\Phi} : V^{\otimes k} \otimes (V^*)^{\otimes l} \to \mathrm{L}\left((V^*)^{k-1}, V^l; V\right)$$

such that

$$\Phi = \widetilde{\Phi} \circ \pi.$$

Now we claim that $\ker \widetilde{\Phi} = \{0\}$. Let $v \otimes \varphi \in \ker \widetilde{\Phi}$ and assume that $v, \varphi \neq 0$. Hence we find $w \in V^l$ such that $\varphi_i(w_i) \neq 0$ for all $i = 1, \ldots, l$. Furthermore since $v_1, \ldots, v_k \neq 0$, we find $\psi \in (V^*)^{k-1}$ such that $\psi_i(v_i) \neq 0$ for all $i = 1, \ldots, k-1$. For example, if (e_j) is a basis of V then $v_i = r_i^j e_i$ where at least one $r_i^j \neq 0$, say r_i^k. Then let $\psi_i := e_k^*$ where (e_j^*) denotes the corresponding basis of V^*. Then

$$\widetilde{\Phi}(v, \varphi)(\psi, w) = \varphi_1(w_1) \cdots \varphi_l(w_l) \psi_1(v_1) \cdots \psi_{k-1}(v_{k-1}) v_k \neq 0.$$

Contradiction. Thus the claim holds and we get that $\widetilde{\Phi}$ is injective. Since

$$\dim\left(V^{\otimes k} \otimes (V^*)^{\otimes l}\right) = (\dim V)^{k+l} = \dim\left(\mathrm{L}\left((V^*)^{k-1}, V^l; V\right)\right)$$

by [Lee13, p. 309] □

2. Pullbacks of Tensor Fields

Exercise 2.1 (Properties of Tensor Pullbacks). Suppose $F : M \to N$ is a smooth mapping and A, B are covariant tensor fields on N. Then

(a) $F^*(A \otimes B) = F^*A \otimes F^*B$.

Solution 2.1. Let $p \in M$. Then we have

$$\begin{aligned}
\left(F^*(A \otimes B)\right)_p (v_1, \ldots, v_{k+l}) &= (A \otimes B)_{F(p)} \left(\mathrm{d}F_p(v_1), \ldots, \mathrm{d}F_p(v_{k+l})\right) \\
&= \left(A_{F(p)} \otimes B_{F(p)}\right) \left(\mathrm{d}F_p(v_1), \ldots, \mathrm{d}F_p(v_{k+l})\right) \\
&= A_{F(p)} \left(\mathrm{d}F_p(v_1), \ldots, \mathrm{d}F_p(v_k)\right) B_{F(p)} \left(\mathrm{d}F_p(v_{k+1}), \ldots, \mathrm{d}F_p(v_{k+l})\right) \\
&= (F^*A)_p(v_1, \ldots, v_k)(F^*B)_p(v_{k+1}, \ldots, v_{k+l}) \\
&= \left((F^*A)_p \otimes (F^*B)_p\right) (v_1, \ldots, v_{k+l}) \\
&= (F^*A \otimes F^*B)_p(v_1, \ldots, v_{k+l})
\end{aligned}$$

for all $v_1, \ldots, v_{k+l} \in T_pM$.

CHAPTER 12

Orientations

1. Orientations of Vector Spaces

Exercise 1.1. Let V be a vector space of dimension $n \geq 1$. Define a relation $\sim$ on the set of all ordered bases of V by

$$(v_1, \dots, v_n) \sim (w_1, \dots, w_n) \quad :\Leftrightarrow \quad \det B > 0 \tag{17}$$

where B denotes the transition matrix defined by $w_j = B_j^i v_i$. Show that $\sim$ is an equivalence relation and that $|X/{\sim}| = 2$.

Solution 1.1. Clearly $(v_1, \dots, v_n) \sim (v_1, \dots, v_n)$ by $v_j = \delta_j^i v_i$. Assume $(v_1, \dots, v_n) \sim (w_1, \dots, w_n)$. Thus B defined by $w_j = B_j^i v_i$ has a positive determinant. But then by $\det(B^{-1}) = \big(\det(B)\big)^{-1}$ also $\det(B^{-1})$ is positive and $v_j = \big(B^{-1}\big)_j^i w_i$. Hence $(w_1, \dots, w_n) \sim (v_1, \dots, v_n)$. Lastly, assume that also $(w_1, \dots, w_n) \sim (u_1, \dots, u_n)$. Hence there exists a matrix A such that $u_j = A_j^i w_i$ where $\det(A) > 0$. Thus $u_j = A_j^i w_i = A_j^i (B_i^k v_k) = (A_j^i B_i^k) v_k$ and by $\det(AB) = \det(A)\det(B) > 0$ we get that $(v_1, \dots, v_n) \sim (u_1, \dots, u_n)$. Hence $\sim$ is an equivalence relation.
By [Gri07, p. 335] every vector space has a basis. Let us denote it by $(v_1, \dots, v_n)$. Therefore

$$(\widetilde{v}_1, \dots, \widetilde{v}_n) := (-v_1, \dots, v_n)$$

is also a basis for V simply by considering the transition matrix

$$\widetilde{B} := \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

defined by $v_j = \widetilde{B}_j^i \widetilde{v}_i$. Let $(w_1, \dots, w_n)$ be an ordered basis for V. Let the transition matrix B be defined by $w_j = B_j^i v_i$. If $\det(B) > 0$, we have that

$$(w_1, \dots, w_n) \sim (v_1, \dots, v_n).$$

Otherwise, if $\det(B) < 0$

$$w_j = B_j^i v_i = B_j^i \left(\widehat{B}_i^k \widehat{v}_k\right) = \left(B_j^i \widehat{B}_i^k\right) \widehat{v}_k$$

together with $\det(B\widehat{B}) = \det(B)\det(\widehat{B}) > 0$ yields

$$(w_1, \dots, w_n) \sim (\widetilde{v}_1, \dots, \widetilde{v}_n).$$

Since $\det(B) \neq 0$ by the nonsingularity of B, we have that there are exactly two equivalence classes

$$[(v_1, \dots, v_n)]_\sim \qquad \text{and} \qquad [(-v_1, \dots, v_n)]_\sim .$$

CHAPTER 13

Symplectic Forms

1. Symplectic Linear Algebra

Exercise 1.1. Let V be a finite dimensional real vector space and ω be a 2-covector on V. Then Ω is nondegenerate if and only if for each nonzero $v \in V$ there exists $w \in V$ such that $\omega(v, w) \neq 0$.

Solution 1.1. We have that

$$\ker \widehat{\omega} = \{v \in V : \forall w \in V\, (\omega(v, w) = 0)\}.$$

Hence if ω is nondegenerate we have that $\widehat{\omega}$ is an isomorphism and thus $\ker \widehat{\omega} = \{0\}$. Conversly, we have that $\ker \widehat{\omega} = \{0\}$ and since $\dim V = \dim V^*$, we have that $\widehat{\omega}$ is an isomorphism.

Exercise 1.2. Let (V, ω) be a symplectic vector space and $S, T \subseteq V$ be linear subspaces.

(a) $\dim S + \dim S^\omega = \dim V$.
(b) $\left(S^\omega\right)^\omega = S$.
(c) $S \subseteq T \Leftrightarrow T^\omega \subseteq S^\omega$.
(d) $\omega|_S$ nondegenerate $\Leftrightarrow S \cap S^\omega = \{0\} \Leftrightarrow V = S \oplus S^\omega$.
(e) If $S \subseteq S^\omega$, then $\dim S \leq \frac{1}{2} \dim V$.
(f) If S is of codimension 1, then S is coisotropic.
(g) S lagrangian $\Leftrightarrow S$ isotropic and coisotropic $\Leftrightarrow S = S^\omega$.

Solution 1.2. For proving (a), consider the mapping $\Phi : V \to S^*$ defined by $\Phi(v) := \omega(v, \cdot)|_S$. Clearly, $\ker \Phi = S^\omega$. Let $\varphi \in S^*$. By exercise B.13 [Lee13, p. 623], there exists an extension $\widehat{\varphi} \in V^*$ of φ. Since $\widehat{\omega}$ is an isomorphism, there exists $v \in V$ such that $\widehat{\varphi} = \omega(v, \cdot)$. This implies $\widehat{\varphi}|_S = \omega(v, \cdot)|_S$. Hence we get that Φ is surjective and thus $\Phi(V) = S^*$. Hence the rank-nullity law [Lee13, p. 627] implies that

$$\dim V = \dim S^* + \dim S^\omega = \dim S + \dim S^\omega.$$

For proving (b), let $v \in S$. Then for any $u \in S^\omega$ we have that $\omega(v, u) = -\omega(u, v) = 0$ and thus $S \subseteq \left(S^\omega\right)^\omega$. Hence S is a linear subspace of $\left(S^\omega\right)^\omega$. Furthermore part (a) yields

$$\dim S = \dim V - \dim S^\omega = \dim \left(S^\omega\right)^\omega$$

Thus exercise B.4. (b) [Lee13, p. 620] implies that $\left(S^\omega\right)^\omega = S$.
For (c), suppose that $S \subseteq T$ and let $v \in T^\omega$. Then for any $u \in S$ we have that $\omega(v, u) = 0$ and thus $T^\omega \subseteq S^\omega$. Conversly, suppose that $T^\omega \subseteq S^\omega$. By part (b) we can also show that $\left(S^\omega\right)^\omega \subseteq \left(T^\omega\right)^\omega$. But this holds as one can easily see. Thus $S \subseteq T$ and the statement follows.
For (d), we show the two equivalences separately. We have that

$$\ker \widehat{\omega|_S} = \{v \in S : \forall w \in S\, (\omega(v, w) = 0)\} = S \cap S^\omega.$$

So $\omega|_S$ is nondegenerate if and only if $S \cap S^\omega = \{0\}$. For the second equivalence, assume that $S \cap S^\omega = \{0\}$. Then by [Fis14, p. 100] and part (a) we have that

$$\dim(S + S^\omega) = \dim S + \dim S^\omega - \dim(S \cap S^\omega) = \dim S + \dim S^\omega = \dim V.$$

Thus exercise B.4. (b) [Lee13, p. 620] implies that $S + S^\omega = V$. Since $S \cap S^\omega = \{0\}$ holds, we have $V = S \oplus S^\omega$ by [Fis14, p. 101]. The other implication follows simply by definition of the direct sum.

(e) directly follows from (a) and [Lee13, p. 620] since

$$2 \dim S \leq \dim S + \dim S^\omega = \dim V.$$

For (f) let S have codimension 1. Hence by part (a) we get that $\dim S^\omega = 1$. Thus any element in S^ω can be written as λv, where $\lambda \in \mathbb{R}$ and $v \in S^\omega \setminus \{0\}$. Hence $\omega(\lambda v, \mu v) = \lambda\mu\omega(v, v) = 0$ and thus $S^\omega \subseteq \left(S^\omega\right)^\omega$ which is by part (b) equivalent to $S^\omega \subseteq S$. For proving (g), we first observe that the second equivalence is trivial. Now assume that S is lagrangian. From part (a) immediately follows that $\dim S = \dim S^\omega$. Since $S \subseteq S^\omega$ we get that $S = S^\omega$. Conversely, assume that $S = S^\omega$. Using again part (a) we get that $2 \dim S = \dim V$.

CHAPTER 14

Symplectic Form on the Cotangent Bundle

1. Symplectic Volume

Exercise 1.1.

(a) If $\omega \in \Lambda^2(V^*)$, then $\omega = \sum_{i=1}^n e_i^* \wedge f_i^*$.
(b)
(c) Deduce that any symplectic manifold (M, ω) is canonically oriented. Does the Möbius band admit a symplectic structure?
(d)

Solution 1.1. For (a), we adapt the notation introduced in [Lee13, pp. 351–354] and use the result about a basis of $\Lambda^k(V^*)$. Letting

$$(\varepsilon^1, \dots, \varepsilon^{k+2n}) := (u_1^*, \dots, u_k^*, e_1^*, \dots, e_n^*, f_1^*, \dots, f_n^*)$$

where $(u_1, \dots, u_k, e_1, \dots, e_n, f_1, \dots, f_n)$ is the basis of V obtained in [Sil08, p. 3]. Then we get

$$\begin{aligned}
\omega &= \sum_{\{I:0\le i_1<i_2\le k+2n\}} \omega_I \varepsilon^I \\
&= \sum_{\{I:1\le i_1\le k, i_1<i_2\le k+2n\}} \omega(u_{i_1}, \varepsilon^{i_2})\varepsilon^I + \sum_{\{I:k<i_1<i_2\le k+n\}} \omega(e_{i_1}, e_{i_2})\varepsilon^I \\
&\quad + \sum_{\{I:k<i_1\le k+n<i_2\le k+2n\}} \omega(e_{i_1}, f_{i_2})\varepsilon^I + \sum_{\{I:k+n<i_1<i_2\le k+2n\}} \omega(f_{i_1}, f_{i_2})\varepsilon^I \\
&= \sum_{\{I:k<i_1\le k+n<i_2\le k+2n\}} \delta^{i_1}_{i_2-n}\varepsilon^I \\
&= \sum_{\{k<i_1\le k+n\}} \varepsilon^{i_1(i_1+n)} \\
&= \sum_{i=1}^n e_i^* \wedge f_i^*
\end{aligned}$$

by [Lee13, p. 356].
For (c), part (a) implies that $(\omega_p)^n \neq 0$ for all $p \in M$. Thus $\omega^n \neq 0$. Clearly, ω^n is a top form. Thus by [Lee13, p. 381], ω^n induces a unique orientation on M. Since the Möbius band is not orientable by [Lee13, p. 393], we have that the Möbius band does not admit a symplectic structure.

Exercise 1.2. Let (M, ω) be a $2n$-dimensional compact symplectic manifold.

(a) Show that $[\omega^n] \in H^{2n}_{\mathrm{dR}}(M)$ is nonzero.
(b) Conclude that $[\omega] \neq 0$.
(c) $\mathbb{S}^{2n}$ does not admit a symplectic structure for $n > 1$.

Solution 1.2. For (a), assume that $[\omega^n] = 0$. Thus there exists an exact form $\alpha \in \Omega^{2n}(M)$, such that $\omega^n + \alpha = 0$. Hence there exists $\beta \in \Omega^{2n-1}(M)$, such that $\omega^n + \mathrm{d}\beta = 0$. By exercise 1.1 (c) we have that ω^n determines a unique orientation of M for which ω^n is positively oriented. Hence linearity, positivity and Stoke's theorem [Lee13, pp. 407,411] yield

$$0 < \int_M \omega = -\int_M \mathrm{d}\beta = \int_{\partial M} \beta = 0.$$

since $\partial M = \varnothing$. Contradiction.

For (b), we use that one can define a product for cohomology classes (see [Lee13, p. 464]). Then one has that $[\omega^n] = [\omega]^n$.

For (c), by [Lee13, p. 450] we have that $H^2_{\mathrm{dR}}(\mathbb{S}^{2n}) \cong 0$. Hence if $\mathbb{S}^{2n}$ admits a symplectic structure ω, then by part (b) we would have $[w] \neq 0$, which contradicts the fact that $H^2_{\mathrm{dR}}(\mathbb{S}^{2n}) \cong 0$.

CHAPTER 15

Lagrangian Submanifolds

1. Tautological Form and Symplectomorphisms

Exercise 1.1. Let M and N be smooth manifolds, $F : M \to N$ a diffeomorphism and $A \in \Gamma\left(T^{(0,k)}TN\right)$, $k \in \mathbb{Z}$, $k \geq 1$. Then

$$F^*A(X_1, \ldots, X_k) = A(F_*X_1, \ldots, F_*X_k) \circ F \tag{18}$$

holds for all $X_1, \ldots, X_k \in \mathfrak{X}(M)$.

Solution 1.1. Let $p \in M$. Then

$$\begin{aligned} F^*A(X_1, \ldots, X_k)(p) &= (F^*A)_p(X_1|_p, \ldots, X_k|_p) \\ &= A_{F(p)}\left(\mathrm{d}F_p(X_1|_p), \ldots, \mathrm{d}F_p(X_k|_p)\right) \\ &= A_{F(p)}\left((F_*X_1)_{F(p)}, \ldots, (F_*X_k)_{F(p)}\right) \\ &= A\left(F_*X_1, \ldots, F_*X_k\right)\left(F(p)\right). \end{aligned}$$

Exercise 1.2.

(a) Let (M, ω) be a symplectic manifold and $\alpha \in \Omega^1(M)$ such that $\omega = -\,\mathrm{d}\alpha$. Furthermore, let $g : M \to M$ be a diffeomorphism such that $g^*\alpha = \alpha$. Then there exists a unique vector field $X \in \mathfrak{X}(M)$, such that $X \lrcorner\, \omega = -\alpha$ and

$$\rho_t \circ g = g \circ \rho_t \tag{19}$$

holds, where $\rho : D \to M$ is the local flow generated by X.

Solution 1.2. For (a), we observe that $\widehat{\omega} : TM \to T^*M$ is a smooth bundle isomorphism (see [Lee13, p. 341]). Thus we define $X : M \to TM$ by

$$X := -\widehat{\omega}^{-1}(\alpha).$$

As a composition of smooth maps, X is smooth and clearly, it is a section of the projection $\pi : TM \to M$ by definition. Hence $X \in \mathfrak{X}(M)$. Furthermore $X \lrcorner\, \omega = \widehat{\omega}(X) = -\alpha$.
Let ρ denote the flow of X and define

$$\theta_t := g \circ \rho_t \circ g^{-1}, \qquad t \in \mathbb{R}.$$

Then we have that

$$\theta_0 = g \circ \rho_0 \circ g^{-1} = g \circ \mathrm{id}_M \circ g^{-1} = \mathrm{id}_M$$

and for $t \in \mathbb{R}$, $p \in M$

$$\begin{aligned}
\left(\theta^{(p)}\right)'(t) &= \left(g \circ \rho^{(g^{-1}(p))}\right)'(t) \\
&= \mathrm{d}g_{\rho^{(g^{-1}(p))}(t)} \left(\rho^{(g^{-1}(p))}\right)'(t) \\
&= \mathrm{d}g_{\rho^{(g^{-1}(p))}(t)} X_{\rho^{(g^{-1}(p))}(t)} \\
&= (g_* X)_{g(\rho^{(g^{-1}(p))})(t)} \\
&= (g_* X)_{\theta^{(p)}(t)} .
\end{aligned}$$

Since $g * \alpha = \alpha$ and $\omega = -\mathrm{d}\alpha$, we have that g is a symplectomorphism. Indeed, we have that

$$g^*\omega = g^*(-\mathrm{d}\alpha) = -\mathrm{d}(g^*\alpha) = -\mathrm{d}\alpha = \omega.$$

by [Lee13, p. 366]. Let $Y \in \mathfrak{X}(M)$. Then by exercise 1.1 we have that

$$\begin{aligned}
\omega(g_* X, Y) \circ g &= (g^*\omega)(X, g_*^{-1} Y) \\
&= \omega\left(X, g_*^{-1} Y\right) \\
&= (X \lrcorner\, \omega)\left(g_*^{-1} Y\right) \\
&= -\alpha\left(g_*^{-1} Y\right) \circ g^{-1} \\
&= -(g^*\alpha)\left(g_*^{-1} Y\right) \\
&= -\alpha(Y) \circ g \\
&= (X \lrcorner\, \omega)(Y) \circ g \\
&= \omega(X, Y) \circ g.
\end{aligned}$$

Thus $\omega(g_* X, Y) = \omega(X, Y)$ for all $Y \in \mathfrak{X}(M)$. Since $\widehat{\omega} : \mathfrak{X}(M) \to \mathfrak{X}^*(M)$ is an isomorphism, we get that $g_* X = X$. We deduce that the local flow θ is also generated by X and thus by uniqueness [Lee13, p. 212] we deduce that $\theta = \rho$ which implies $\theta_t = \rho_t$ and thus by definition of θ, $\rho_t \circ g = g \circ \rho_t$.
For (b), let $X := X^i \frac{\partial}{\partial x^i} + Y^i \frac{\partial}{\partial \xi^i}$. We calculate

$$\begin{aligned}
X \lrcorner\, \omega &= \sum_{i=1}^{n} \left(X \lrcorner\, (\mathrm{d}x^i \wedge \mathrm{d}\xi^i)\right) \\
&= \sum_{i=1}^{n} \left((X \lrcorner\, \mathrm{d}x^i) \wedge \mathrm{d}y^i) - \mathrm{d}x^i \wedge (X \lrcorner\, \mathrm{d}\xi^i)\right) \\
&= \sum_{i=1}^{n} \left(X^i \,\mathrm{d}\xi^i - Y^i \,\mathrm{d}x^i\right).
\end{aligned}$$

Since $X \lrcorner\, \omega = -\alpha$, we get that

$$X = \xi^i \frac{\partial}{\partial \xi^i}.$$

Define an isotopy $\rho : \mathbb{R} \times T^*M \to T^*M$ by $\rho(t,p) := (x, e^t \xi)$, where $p = (x, \xi)$. Then we have that $\rho_0 = \mathrm{id}_M$ and

CHAPTER 16

Dolbeault Theory

1. Tensor Characterization Lemma

Definition 1.1. *Let $k, l \in \mathbb{Z}$, $k, l \geq 0$ and M a smooth manifold. Then the* ***bundle of mixed tensors of type (k, l)*** *is defined by*

$$T^{(k,l)}TM := \coprod_{p \in M} T^{(k,l)}(T_pM). \tag{20}$$

Proposition 1.1. *The bundle of mixed tensors of type (k, l) has a unique natural structure as a smooth vector bundle of rank n^{k+l} over M.*

Proof. For each $p \in M$ let $E_p := T^{(k,l)}(T_pM)$. By [Lee13, p. 57] and [Lee13, p. 313] $\dim E_p = n^{k+l}$. Furthermore, let $E := T^{(k,l)}TM$ and $\pi : E \to M$ be defined by $\pi(p, A) := p$. Let $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ be an atlas for M. For each $\alpha \in A$ define

$$\Phi_\alpha : \begin{cases} \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{n^{k+l}} \\ (p, A) \mapsto \big(p, (A^{i_1 \ldots i_k}_{j_1 \ldots j_l})\big) \end{cases}$$

Clearly, the inverse is given by

$$\Phi_\alpha^{-1} : \begin{cases} U_\alpha \times \mathbb{R}^{n^{k+l}} \to \pi^{-1}(U_\alpha) \\ \big(p, (A^{i_1 \ldots i_k}_{j_1 \ldots j_l})\big) \mapsto (p, A) \end{cases} .$$

Hence each Φ_α is bijective. Now we have to check, that $\Phi_\alpha|_{E_p}$ is an isomorphism. So let $\lambda \in \mathbb{R}$ and $B \in E_p$. Then

$$\begin{aligned} \Phi_\alpha|_{E_p}(p, \lambda A + B) &= \big(p, (\lambda A + B)^{i_1 \ldots i_k}_{j_1 \ldots j_l})\big) \\ &= \big(p, \lambda(A^{i_1 \ldots i_k}_{j_1 \ldots j_l}) + (B^{i_1 \ldots i_k}_{j_1 \ldots j_l})\big) \\ &= \lambda \Phi_\alpha|_{E_p}(p, A) + \Phi_\alpha|_{E_p}(p, B). \end{aligned}$$

Now let $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$. We consider the mapping

$$\Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^{n^{k+l}} \to (U_\alpha \cap U_\beta) \times \mathbb{R}^{n^{k+l}}.$$

Define $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n^{k+l}, \mathbb{R})$ by

$$\tau_{\alpha\beta} := (\delta^i_j).$$

Then we have that

$$\left(\Phi_\alpha \circ \Phi_\beta^{-1}\right)\big(p, (A^{i_1 \ldots i_k}_{j_1 \ldots j_l})\big) = \big(p, (A^{i_1 \ldots i_k}_{j_1 \ldots j_l})\big) = \big(p, \tau_{\alpha\beta}(p)(A^{i_1 \ldots i_k}_{j_1 \ldots j_l})\big).$$

Since $\tau_{\alpha\beta}$ is constant, it is smooth (see [Lee13, p. 36]). Hence we can apply the vector bundle chart lemma [Lee13, p. 253] and the result follows. □

What follows is a reformulation of the smoothness criteria for tensor fields ([Lee13, pp. 317–318]) for tensor fields of type $(1, k)$.

Proposition 1.2 (Smoothness Criteria for Tensor Fields). *Let M be smooth manifold and let $A : M \to T^{(1,k)}TM$ be a rough section. Then the following are equivalent:*

(*a*) $A \in \Gamma(T^{(1,k)}TM)$.
(*b*) *In every smooth coordinate chart, the component functions of A are smooth.*
(*c*) *For all $X_1, \dots, X_k \in \mathfrak{X}(M)$, the rough section $A(X_1, \dots, X_k) : M \to TM$ defined by*

$$A(X_1, \dots, X_k)(p) := A_p(X_1|_p, \dots, X_k|_p) \tag{21}$$

is a smooth vector field.
(*d*) *If $X_1, \dots, X_k$ are smooth vector fields on some open subset $U \subseteq M$, then also $A(X_1, \dots, X_k)$ is a smooth vector field on U.*

Proof. We prove (a)⇔(b) and (b)⇒(c)⇒(d)⇒(b).
To prove (a)⇔(b), let $(U, (x^i))$ be a smooth chart. Actually, we can prove this for general tensor fields of type (k, l). Proposition 1.1 together with the proof of the vector bundle chart lemma [Lee13, pp. 253–254] implies, that the corresponding chart on $T^{(k,l)}TM$ is given by $\left(\pi^{-1}(U), \widetilde{\varphi}\right)$, where $\widetilde{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^{n^{k+l}}$ is defined by

$$\widetilde{\varphi} := (\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}}) \circ \Phi$$

where $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^{n^{k+l}}$ is given as in the proof of proposition 1.1. Now we consider the coordinate representation $\widehat{A}$ in the given charts (see [Lee13, p. 35]). Since A is a rough section, we have that

$$A^{-1}\left(\pi^{-1}(U)\right) = (\pi \circ A)^{-1}(U) = \mathrm{id}_M^{-1}(U) = U.$$

Hence $\varphi\left(U \cap A^{-1}(\pi^{-1}(U))\right) = \varphi(U)$, which is open, and $\widehat{A} : \varphi(U) \to \widetilde{\varphi}\left(\pi^{-1}(U)\right)$ is given by

$$\begin{aligned}
\widehat{A}(x) &= (\widetilde{\varphi} \circ A \circ \varphi^{-1})(x) \\
&= (\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}}) \left(\Phi(\varphi^{-1}(x), A_{\varphi^{-1}(x)})\right) \\
&= (\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}}) \left(\varphi^{-1}(x), \left(A^{i_1 \dots i_k}_{j_1 \dots j_l}(\varphi^{-1}(x))\right)\right) \\
&= \left(x, \left(\widehat{A}^{i_1 \dots i_k}_{j_1 \dots j_l}(x)\right)\right).
\end{aligned}$$

By [Lee13, p. 35] A is smooth if and only if in any chart $\widehat{A}$ is smooth. This is furthermore equivalent to that each $\widehat{A}^{i_1 \dots i_k}_{j_1 \dots j_l}$ is smooth and thus equivalent to that $A^{i_1 \dots i_k}_{j_1 \dots j_l}$ is smooth (see [Lee13, p. 33]).
To prove (b)⇒(c), let $(U, (x^i))$ be a smooth chart. Then write $X_1, \dots, X_k \in \mathfrak{X}(M)$ as

$$X_\nu = X_\nu^{\mu_\nu} \frac{\partial}{\partial x^{\mu_\nu}}.$$

for $\nu = 1, \ldots, k$. For $p \in U$ lemma **??** implies

$$\begin{aligned} A(X_1, \ldots, X_n)(p) &= A_p(X_1|_p, \ldots, X_k|_p) \\ &= A_p\left(X_1^{\mu_1}(p) \frac{\partial}{\partial x^{\mu_1}}\bigg|_p, \ldots, X_1^{\mu_k}(p) \frac{\partial}{\partial x^{\mu_k}}\bigg|_p \right) \\ &= X_1^{\mu_1}(p) \cdots X_k^{\mu_k}(p) A_p \left(\frac{\partial}{\partial x^{\mu_1}}\bigg|_p, \ldots, \frac{\partial}{\partial x^{\mu_k}}\bigg|_p \right) \\ &= X_1^{\mu_1}(p) \cdots X_k^{\mu_k}(p) A^i_{\mu_1 \ldots \mu_k}(p) \frac{\partial}{\partial x^i}\bigg|_p . \end{aligned}$$

By the smoothness criterion for vector fields [Lee13, p. 175] we have that each component function $X_\nu^{\mu_n}$ is smooth. Thus if A is smooth, we have by that each $A^i_{j_1 \ldots j_k}$ is smooth and since $\mathscr{C}^\infty(M)$ is an $\mathbb{R}$-algebra (see [Lee13, p. 33]), we have that

$$X_1^{\mu_1} \cdots X_k^{\mu_k} A^i_{\mu_1 \ldots \mu_k}$$

is smooth for $i = 1, \ldots, n$. Thus again by the smoothness criterion together with the localness of smoothness [Lee13, p. 35] we get that $A(X_1, \ldots, X_k) \in \mathfrak{X}(M)$.

To prove (c)$\Rightarrow$(d), we use that smoothness is a local property (see [Lee13, p. 35]). Let $p \in U$. Then by [Cat17, p. 14] we find a smooth bump function ψ supported in U and identically equal to 1 on some neighbourhood V of p. Set

$$\widetilde{X}_\nu|_p := \begin{cases} \psi(p) X_\nu|_p & p \in \operatorname{supp} \psi \\ 0 & p \in M \setminus \operatorname{supp} \psi \end{cases} .$$

Then the gluing lemma for smooth maps [Lee13, p. 35] implies $\widetilde{X}_1, \ldots, \widetilde{X}_k \in \mathfrak{X}(M)$. Hence by (c) we get that $A(\widetilde{X}_1, \ldots, \widetilde{X}_k) \in \mathfrak{X}(M)$ and so the restriction $A(\widetilde{X}_1, \ldots, \widetilde{X}_k)|_V$ is smooth. But $A(\widetilde{X}_1, \ldots, \widetilde{X}_k)|_V = A(X_1, \ldots, X_k)$ and so we are done.

Lasty to prove (d)$\Rightarrow$(b), each vector field locally defined by

$$X_{j_\nu} = \delta^{\mu_\nu}_{j_\nu} \frac{\partial}{\partial x^{\mu_\nu}} .$$

is smooth. Thus by

$$A(X_1, \ldots, X_n)(p) = \delta^{\mu_1}_{j_1} \cdots \delta^{\mu_k}_{j_k} A^i_{\mu_1 \ldots \mu_k}(p) \frac{\partial}{\partial x^i}\bigg|_p = A^i_{j_1 \ldots j_k}(p) \frac{\partial}{\partial x^i}\bigg|_p$$

we get that $A^i_{j_1 \ldots j_k}$ is smooth and hence by (b) also A. ☐

Theorem 1.1 (Tensor Characterization Lemma). *A mapping*

$$\underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k} \to \mathscr{C}^\infty(M) \qquad \textit{or} \qquad \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k} \to \mathfrak{X}(M)$$

is induced by an element of $\Gamma(T^{(0,k)}TM)$ or $\Gamma(T^{(1,k)}TM)$, respectively, if and only if they are multilinear over $\mathscr{C}^\infty(M)$.

Proof. We are proving only the second statement. Any element in $\Gamma(T^{(1,k)}TM)$ induces a mapping $\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by part (c) of the smoothness criteria for tensor fields 1.2. Thus we have to show that

$\mathscr{A}$ is multilinear over $\mathscr{C}^\infty(M)$. Let $f \in \mathscr{C}^\infty(M)$ and $X_\nu, \widetilde{X}_\nu \in \mathfrak{X}(M)$, $\nu = 1, \dots, k$. Then for any $p \in M$ we have that

$$\begin{aligned}\mathscr{A}(X_1, \dots, fX_\nu + \widetilde{X}_\nu, \dots, X_k)_p =& A_p(X_1|_p, \dots, (fX_\nu + \widetilde{X}_\nu)_p, \dots, X_k|_p) \\ =& A_p(X_1|_p, \dots, f(p)X_\nu|_p + \widetilde{X}_\nu|_p, \dots, X_k|_p) \\ =& f(p)A_p(X_1|_p, \dots, X_\nu|_p, \dots, X_k|_p) \\ &+ A_p(X_1|_p, \dots, \widetilde{X}_\nu|_p, \dots, X_k|_p) \\ =& f(p)\mathscr{A}(X_1, \dots, X_\nu, \dots, X_k)_p \\ &+ \mathscr{A}(X_1, \dots, \widetilde{X}_\nu, \dots, X_k)_p \\ =& \big(f\mathscr{A}(X_1, \dots, X_\nu, \dots, X_k)\big)_p \\ &+ \mathscr{A}(X_1, \dots, \widetilde{X}_\nu, \dots, X_k)_p.\end{aligned}$$

Conversly, suppose that $\mathscr{A} : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is multilinear over $\mathscr{C}^\infty(M)$. Let $p \in M$. First we show that $\mathscr{A}$ acts locally, i.e. if $X_\nu = \widetilde{X}_\nu$ in some neighbourhood U of p implies that also

$$\mathscr{A}(X_1, \dots, X_\nu, \dots, X_k) = \mathscr{A}(X_1, \dots, \widetilde{X}_\nu, \dots, X_k)$$

on U. By the multilinearity of $\mathscr{A}$ it is enough to show that if X_ν vanishes on U then so does $\mathscr{A}$. There exists a smooth bump function ψ for $\{p\}$ supported in U (see [Lee13, p. 44]). Hence $\psi X_\nu = 0$ on M and $\psi(p) = 1$. Thus

$$0 = \mathscr{A}(X_1, \dots, \psi X_\nu, \dots, X_k)_p = \psi(p)\mathscr{A}(X_1, \dots, X_\nu, \dots, X_k)_p.$$

and since $\psi(p) = 1$ we have that

$$\mathscr{A}(X_1, \dots, X_\nu, \dots, X_k)_p = 0$$

for any $p \in U$.
Next we show that $\mathscr{A}$ actually acts pointwise, i.e. if $X_\nu|_p$ vanishes so does $\mathscr{A}$. Let $(U, (x^i))$ be a chart containing p and $X_\nu = X^i_\nu \frac{\partial}{\partial x^i}$ on U. The same construction as used showing the implication (c)$\Rightarrow$(d) in the proof of proposition 1.2 yields the existence of $f^1, \dots, f^n \in \mathscr{C}^\infty(M)$ and $\widetilde{X}_1, \dots, \widetilde{X}_n \in \mathfrak{X}(M)$ such that $f^i = X^i_\nu$ and $\widetilde{X}_i = \frac{\partial}{\partial x^i}$ on a neighbourhood $V \subseteq U$ of p. Thus by the previous localization, we get that

$$\mathscr{A}(X_1, \dots, X_\nu, \dots, X_k) = \mathscr{A}(X_1, \dots, f^i\widetilde{X}_i, \dots, X_k) = f^i\mathscr{A}(X_1, \dots, \widetilde{X}_i, \dots, X_k)$$

in U. Since $0 = X^i_\nu(p) = f^i(p)$, $\mathscr{A}$ vanishes at p. Hence $\mathscr{A}$ depends only on the value of X_ν at p. Thus define a rough section $A : M \to T^{(1,k)}TM$ by

$$A_p(v_1, \dots, v_k) := \mathscr{A}(V_1, \dots, V_k)(p)$$

where $V_1, \dots, V_k \in \mathfrak{X}(M)$ are any extensions of $v_1, \dots, v_k \in T_pM$ (see [Lee13, p. 177]). By the above, the choice of the extensions does not matter and the resulting rough section is smooth by proposition 1.2 part (c), hence $A \in \Gamma(T^{(1,k)}TM)$. $\square$

2. Integrability

Definition 2.1. *Let (M, J) be an almost complex manifold. For $X, Y \in \mathfrak{X}(M)$ define the **Nijenhuis tensor N** as*

$$N(X,Y) := [JX, JY] - J\,[X, JY] - J\,[JX, Y] - [X, Y] \tag{22}$$

where $[X, Y]$ denotes the usual Lie-bracket of vector fields.

Exercise 2.1.

(a) If $[\cdot, Y] \circ J = J \circ [\cdot, Y]$ for all $Y \in \mathfrak{X}(M)$, then $N(X,Y) = 0$ for all $X \in \mathfrak{X}(M)$.

(b) N is actually a tensor.

Solution 2.1. Part (a) simply follows from

$$N(X,Y) = J\,[X, JY] - J\,[X, JY] + [X, Y] - [X, Y] = 0.$$

To prove (b), we observe that by the tensor characterization lemma 1.1 it is enough to show that N is multilinear over $\mathscr{C}^\infty(M)$. Let $f \in \mathscr{C}^\infty(M)$ and $X, Y, Z \in \mathfrak{X}(M)$. Then

$$\begin{aligned} N(fX+Y, Z) =& [J(fX+Y), JZ] - J\,[fX+Y, JZ] - J\,[J(fX+Y), Z] \\ & - [fX+Y, Z] \\ =& [fJX + JY, JZ] - J\,[fX+Y, JZ] - J\,[fJX+JY, Z] \\ & - [fX+Y, Z] \\ =& [fJX, JZ] + [JY, JZ] - J\,[fX, JZ] - J\,[Y, JZ] - J\,[fJX, Z] \\ & - J\,[JY, Z] - [fX, Z] - [Y, Z] \\ =& f\,[JX, JZ] - (JZf)JX + [JY, JZ] - fJ\,[X, JZ] + (JZf)JX \\ & - [Y, JZ] - fJ\,[JX, Z] + (Zf)JJX - J\,[JY, Z] - f\,[X, Z] \\ & + (Zf)X - [Y, Z] \\ =& fN(X, Z) + N(Y, Z). \end{aligned}$$

by [Lee13, pp. 187–188]. Linearity in the second argument is shown similarly. Hence $N : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is bilinear over $\mathscr{C}^\infty(M)$.

CHAPTER 17

Complex Manifolds

1. Complex Projective Space

This is problem 2-11 [Lee11].

Exercise 1.1. Let $f : X \to Y$ be a continuous map between topological spaces, and let $\mathcal{B}$ be a basis for X. Then $f(\mathcal{B}) := \{f(B) : B \in \mathcal{B}\}$ is a basis for Y if and only if f is open and surjective. Deduce that if X is second countable and f open and surjective, then Y is second countable.

Solution 1.1. Assume first that $f(\mathcal{B})$ is a basis for Y. Let $U \subseteq X$ be open. Since $\mathcal{B}$ is a basis for X, we have that $U = \cup_{\iota \in I} B_\iota$. Thus by exercise A4. (h) [Lee11, p. 388], we have that $f(U) = \cup_{\iota\in I} f(B_\iota)$, which is open since $f(B_\iota)$ is open for each $\iota \in I$. Assume that f is not surjective. Hence we find $y \in Y \setminus f(X)$. Let U be a neighbourhood of y. By exercise 240 [Lee11, p. 33], there exists $f(B) \in f(\mathcal{B})$ such that $y \in f(B) \subseteq U$. But by exercise A.4 (g) [Lee11, p. 388], this implies that $y \in f(X)$. contradiction.
Conversly, suppose that f is open and surjective. Thus $f(B)$ is open for any $B \in B$. Let $U \subseteq Y$ be open. Since f is continuous, $f^{-1}(U)$ is open in X and thus $U = \cup_{\iota \in U} B_\iota$. Therefore $f\left(f^{-1}(U)\right) = \cup_{\iota\in I} f(B_\iota)$ and by the surjectivity of f we get $f\left(f^{-1}(U)\right) = U$ by exercise A.7 [Lee11, p. 388].
If X is second countable, there exists a countable basis $\mathcal{B}$ for X. Since f is open and surjective, $f(\mathcal{B})$ is a countable basis for Y.

Definition 1.1. *Let $n \in \mathbb{Z}$, $n \geq 0$. On $\mathbb{C}^{n+1} \setminus \{0\}$ define an equivalence relation $\sim$ by*

$$z \sim w \quad :\Leftrightarrow \quad \exists \lambda \in \mathbb{C}^\times (z = \lambda w). \tag{23}$$

The quotient space of $\mathbb{C}^{n+1} \setminus \{0\}$ under $\sim$ is called the ***complex projective space*** *and is denoted by $\mathbb{CP}^n$.*

Exercise 1.2.

(a) $\mathbb{CP}^n$ is an n-dimensional complex manifold.

Solution 1.2. To prove (a), we show first that $\mathbb{CP}^n$ is Hausdorff and second countable. To this end, we observe that $\mathbb{CP}^n$ is precisely the orbit space of the action of $\mathbb{C}^\times$ on $\mathbb{C}^{n+1} \setminus \{0\}$ by scalar multiplication, which we will denote by θ. Assume that $(\lambda_n, z_n) \to (\lambda, z)$ in $\mathbb{C}^\times \times (\mathbb{C}^{n+1} \setminus \{0\})$. By [Eng89, p. 260], this is equivalent to $\lambda_n \to \lambda$ and $z_n \to z$. Therefore, we have that

$$\theta(\lambda_n, z_n) = \lambda_n z_n \to \lambda z = \theta(\lambda, z).$$

Thus θ is a continuous action and by [Lee13, p. 541] we have that the canonical projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is an open map. Hence by exercise 1.1 $\mathbb{CP}^n$ is second countable.
Let $z_n \to z$ in $\mathbb{C}^{n+1} \setminus \{0\}$ and $(\lambda_n)_{n\in\mathbb{N}}$ a sequence in $\mathbb{C}^\times$ such that $\theta(\lambda_n, z_n)$ converges. Since $z_n \to z$ in $\mathbb{C}^{n+1} \setminus \{0\}$, we find a component z^i of z which

is nonzero. Now take a subsequence $z^i_{n_k}$ of z^i_n, such that $z^i_{n_k} \neq 0$. Then $1/z^i_{n_k} \to 1/z^i$ and since $\theta(\lambda_n, z_n)$ converges, also $\lambda_{n_k} z^i_{n_k}$ converges. But then $\lambda_{n_k} = \lambda_{n_k} z^i_{n_k}(1/z^i_{n_k})$ converges and thus by the characterizations of proper actions [Lee13, p. 543], θ is a proper action. Thus by [Lee13, p. 543], the orbit space $\mathbb{CP}^n$ is Hausdorff.

For $\nu = 1, \dots, n+1$ define

$$U_\nu := \{\pi(z) : z_\nu \neq 0\}$$

and $\varphi_\nu : U_\nu \to \mathbb{C}^n$

$$\varphi_\nu([z_1, \dots, z_{n+1}]) := \frac{1}{z_\nu}(z_1, \dots, z_{\nu-1}, z_{\nu+1}, \dots, z_{n+1}).$$

Since π is an open map, each U_ν is open and φ_ν is easily checked to be well defined. The inverse $\varphi_\nu^{-1} : \mathbb{C}^n \to U_\nu$ is given by

$$\varphi_\nu^{-1}(z_1, \dots, z_n) = [z_1, \dots, z_{\nu-1}, 1, z_\nu, \dots, z_n]\,.$$

CHAPTER 18

Kähler Forms

1. The Fubini-Study Structure

Lemma 1.1. *For every $z \in \mathbb{C}^n$ there exists $A \in \mathrm{U}(n)$ such that $Az \in \mathbb{C} \times \{0\}^{n-1}$.*

Proof. If $z = 0$, for example $Iz = 0 \in \mathbb{C} \times \{0\}^{n-1}$. So let us assume that $|z| = 1$. Let (e_ν) denote the standard basis of $\mathbb{C}^n$. Since $z \neq 0$, the set $\{z\}$ is linearly independent and thus by exercise B.4 [Lee13, p. 620] contained in a basis for $\mathbb{C}^n$, which can be made into an orthonormal basis using the Gram-Schmidt algorithm, say $(\widetilde{e}_\nu)$, where $\widetilde{e}_1 = z$. Define a linear mapping $\widetilde{A} : \mathbb{C}^n \to \mathbb{C}^n$ by matrix multiplication with $\widetilde{A} := (\widetilde{e}_1, \ldots, \widetilde{e}_n)$. Clearly $\widetilde{A}e_\nu = \widetilde{e_\nu}$ and $\widetilde{A} \in \mathrm{U}(n)$. Thus $\widetilde{A}^{-1} \in \mathrm{U}(n)$ and $\widetilde{A}^{-1}z = e_1 \in \mathbb{C} \times \{0\}^{n-1}$. Now set $A := \widetilde{A}^{-1}$. If $|z| \neq 1$, we have that

$$Az = A\left(|z|\,\frac{z}{|z|}\right) = |z|\,A\left(\frac{z}{|z|}\right) = |z|\,e_1 \in \mathbb{C} \times \{0\}^{n-1}\,. \tag{24}$$

□

Exercise 1.1. (a) The form

$$\frac{i}{2}\partial\overline{\partial}\log\left(|z|^2 + 1\right) \tag{25}$$

on $\mathbb{C}^n$ is a Kähler form.

Solution 1.1. For (a), define a smooth function $\rho : \mathbb{C}^n \to \mathbb{R}$ by

$$\rho(z) := \log\left(|z|^2 + 1\right).$$

Then we have

$$\frac{\partial^2 \rho}{\partial z_\mu \partial \overline{z}_\nu} = \frac{\partial}{\partial z_\mu}\frac{z_\nu}{|z|^2 + 1} = \frac{\delta_{\nu\mu}}{|z|^2 + 1} - \frac{z_\nu \overline{z}_\mu}{(|z|^2 + 1)^2}. \tag{26}$$

Let $A \in \mathrm{U}(n)$. Then

$$A^*\omega_{\mathrm{FS}} = \frac{i}{2}A^*\partial\overline{\partial}\rho = \frac{i}{2}\partial\overline{\partial}A^*\rho = \frac{i}{2}A\partial\overline{\partial}(\rho \circ A) = \frac{i}{2}\partial\overline{\partial}\rho = \omega_{\mathrm{FS}}$$

since

$$|Az|^2 = \langle Az, Az\rangle = z^t A^t \overline{A}\overline{z} = z^t\overline{z} = \langle z, z\rangle = |z|^2$$

for any $z \in \mathbb{C}^n$. Let $p := (a, 0, \ldots, 0) \in \mathbb{C} \times \{0\}^{n-1}$. From 26 we deduce that

$$\frac{\partial^2 \rho}{\partial z_\mu \partial \overline{z}_\nu}(p) = \begin{cases} \frac{1}{(|a|^2+1)^2} & \nu = \mu = 1 \\ \frac{1}{|a|^2+1} & \nu = \mu > 1 \\ 0 & \nu \neq \mu \end{cases}$$

Therefore the matrix $\left(\frac{\partial^2\rho}{\partial z_\mu \partial \overline{z}_\nu}(p)\right)$ does have the eigenvalues $\frac{1}{(|a|^2+1)^2}$ and $\frac{1}{|a|^2+1}$ which are both positive. Thus $\left(\frac{\partial^2\rho}{\partial z_\mu \partial \overline{z}_\nu}(p)\right)$ is positive definite.
For (b), we have that $\varphi \circ \varphi = \mathrm{id}_U$ and φ is holomorphic. Furthermore

$$\begin{aligned}
\varphi^* \log\left(|z|^2 + 1\right) &= \log\left(|\varphi(z)|^2 + 1\right) \\
&= \log\left(\frac{1}{|z_1|^2}\left(1 + \sum_{\nu=2}^{n} |z_\nu|^2\right) + 1\right) \\
&= \log\left(\frac{1}{|z_1|^2}\left(|z|^2 + 1\right)\right) \\
&= \log\left(|z|^2 + 1\right) + \log\left(\frac{1}{|z_1|^2}\right).
\end{aligned}$$

For (c), we have that

$$\begin{aligned}
\partial\overline{\partial}\varphi^* \log\left(|z|^2 + 1\right) &= \partial\overline{\partial} \log\left(|z|^2 + 1\right) + \partial\overline{\partial} \log\left(\frac{1}{|z_1|^2}\right) \\
&= \partial\overline{\partial} \log\left(|z|^2 + 1\right) - \partial\overline{\partial} \log z_1 - \partial\overline{\partial} \log \overline{z}_1 \\
&= \partial\overline{\partial} \log\left(|z|^2 + 1\right)
\end{aligned}$$

by part (b) and so

$$\varphi^*\omega_{\mathrm{FS}} = \frac{i}{2}\varphi^*\partial\overline{\partial}\rho = \frac{i}{2}\partial\overline{\partial}\varphi^*\rho = \frac{i}{2}\partial\overline{\partial}\rho = \omega_{\mathrm{FS}}.$$

For (f), we have that

$$\begin{aligned}
\int_{\mathbb{CP}^1} \omega_{\mathrm{FS}} &= \int_{\mathbb{R}^2} \frac{\mathrm{d}x \wedge \mathrm{d}y}{(x^2 + y^2 + 1)^2} \\
&= \int_{\mathbb{R}^2} \frac{\mathrm{d}x\,\mathrm{d}y}{(x^2 + y^2 + 1)^2} \\
&= \int_0^\infty \int_0^{2\pi} \frac{r}{(r^2+1)^2}\,\mathrm{d}\theta\,\mathrm{d}r \\
&= 2\pi \int_0^\infty \frac{r}{(r^2+1)^2}\,\mathrm{d}r \\
&= \pi \int_1^\infty \frac{\mathrm{d}s}{s^2} \\
&= \pi.
\end{aligned}$$

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