Yannis Bähni

Solution Book to Introduction to Smooth Manifolds by John M. Lee

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Contents

CHAPTER 1

Homotopy and the Fundamental Group

1. The Fundamental Group

Exercise 1.1. Let X be a topological space. For any points $p, q \in X$, path homotopy is an equivalence relation on the set of all paths in X from p to q.

Solution 1.1. Let f be a path in X from p to q. Define $H : I \times I \to X$ by $H(s,t) := f(s)$. Clearly H is continuous since f is. Indeed, take $U \subseteq X$ open, then $H^{-1}(U) = f^{-1}(U) \times I$ which is open in the box topology. Now clearly $H(s,0) = H(s,1) = f(s)$ for all $s \in I$ and $H(0,t) = f(0) = p$, $H(1,t) = f(1) = q$ for all $t \in I$. Hence $f \sim f$. If $f \sim g$ and F is a path homotopy from f to g, then $H : I \times I \to X$ defined by $H(s,t) := F(1-s, 1-t)$ is a path homotopy from g to f. Thus $g \sim f$. If $f \sim g$ and $g \sim h$ where F and G denote the path homotopies, respectively, then $H : I \times I \to X$ defined by

$$H(s,t) := \begin{cases} F(s,2t) & 0 \le t \le \frac{1}{2}, \\ G(s,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

is a path homotopy from f to h, hence $f \sim g$.

CHAPTER 2

Smooth Maps

1. Smooth Functions and Smooth Maps

1.1. Smooth Functions on Manifolds. We follow the terminology established in [Gri07, p. 515].

Exercise 1.1. Let M be a smooth manifold. $\mathscr{C}^\infty(M)$ is an associative and commutative $\mathbb{R}$-algebra with identity under the usual pointwise defined operations.

Solution 1.1. First we show that $\mathscr{C}^\infty(M)$ is a real vector space. Since $\mathscr{C}^\infty(M) \subseteq \mathbb{R}^M$ it is enough to show that $\mathscr{C}^\infty(M)$ is a linear subspace of the real vector space $\mathbb{R}^M$. Clearly, $\mathscr{C}^\infty(M) \neq \varnothing$, since $\chi_M \in \mathscr{C}^\infty(M)$. Indeed, for $p \in M$ we find a chart (U, φ) such that $p \in U$ and the composition $\chi_M \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is clearly the function $\chi_{\varphi(U)}$, which is smooth since it is constant. Now let $f, g \in \mathscr{C}^\infty(M)$, $\lambda \in \mathbb{R}$ and $p \in M$. By definition, there exist charts (U, φ), (V, ψ) such that $f \circ \varphi^{-1}$ and $g \circ \psi^{-1}$ are smooth. Now consider the chart $(U \cap V, \varphi)$. Then by

$$(\lambda f + g) \circ \varphi^{-1} = \lambda(f \circ \varphi^{-1}) + (g \circ \varphi^{-1}) = \lambda(f \circ \varphi^{-1}) + \big((g \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1})\big)$$

we have that $\lambda f + g \in \mathscr{C}^\infty(M)$. Hence $\mathscr{C}^\infty(M)$ is a real vector space.
Now define a product map $\cdot : \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ by pointwise multiplication. Indeed, similar to the previous reasoning, by

$$(f \cdot g) \circ \varphi^{-1} = (f \circ \varphi^{-1}) \cdot (g \circ \varphi^{-1}) = (f \circ \varphi^{-1}) \cdot \big((g \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1})\big)$$

we have that $f \cdot g$ is smooth. Let $f, g, h \in \mathscr{C}^\infty(M)$ and $\lambda \in \mathbb{R}$. Then for $p \in M$

$$\begin{aligned}
\big((\lambda f + g) \cdot h\big)(p) &= (\lambda f + g)(p) h(p) \\
&= \big(\lambda f(p) + g(p)\big) h(p) \\
&= \lambda f(p) h(p) + g(p) h(p) \\
&= \lambda (f \cdot h)(p) + (g \cdot h)(p) \\
&= \big(\lambda (f \cdot h)\big)(p) + (g \cdot h)(p) \\
&= \big(\lambda (f \cdot h) + (g \cdot h)\big)(p)
\end{aligned}$$

shows that $\cdot$ is bilinear in the first argument. A similar computation shows that $\cdot$ is bilinear. By

$$\begin{aligned}
\big((f \cdot g) \cdot\big)(p) &= (f \cdot g)(p) h(p) \\
&= f(p) g(p) h(p) \\
&= f(p)(g \cdot h)(p) \\
&= \big(f \cdot (g \cdot h)\big)(p)
\end{aligned}$$

we see that $\cdot$ is associative. Furthermore by

$$(f \cdot g)(p) = f(p)g(p) = g(p)f(p) = (g \cdot f)(p)$$

we see that $\cdot$ is commutative. Finally, the identity element is given by χ_M since

$$(\chi_M \cdot f)(p) = \chi_M(p)f(p) = 1f(p) = f(p).$$

CHAPTER 3

Tangent Vectors

1. Tangent Vectors

1.1. Tangent Vectors on Manifolds.

Exercise 1.1. Let M be a smooth manifold and $p \in M$. The set of all derivations at p, written T_pM, is a real vector space under the usual pointwise defined operations.

Solution 1.1. Clearly $T_pM \subseteq \mathrm{L}(\mathscr{C}^\infty(M); \mathbb{R})$ and thus it is enough to show that T_pM is a linear subspace of $\mathrm{L}(\mathscr{C}^\infty(M); \mathbb{R})$ (see [Lee13, p. 626]). We have $T_pM \neq \varnothing$, since $0 \in T_pM$ defined by $f \mapsto 0$. Let $u, v \in T_pM$, $\lambda \in \mathbb{R}$ and $f, g \in \mathscr{C}^\infty(M)$. Then by

$$\begin{aligned}(\lambda u + v)(fg) &= \lambda u(fg) + v(fg) \\ &= f(p)\big(\lambda u(g) + v(g)\big) + g(p)\big(\lambda u(f) + v(f)\big) \\ &= f(p)(\lambda u + v)(g) + g(p)(\lambda u + v)(f)\end{aligned}$$

we have that $\lambda u + v \in T_pM$.

Exercise 1.2. Suppose M is a smooth manifold. Let $p \in M$, $v \in T_pM$ and $f \in \mathscr{C}^\infty(M)$. If f is constant, then $v(f) = 0$.

Solution 1.2. First assume that $f = \chi_M$. Then

$$v(\chi_M) = v(\chi_M \cdot \chi_M) = f(p)v(\chi_M) + f(p)v(\chi_M) = v(\chi_M) + v(\chi_M) \tag{1}$$

implies that $v(f) = 0$. Hence if $f = \lambda\chi_M$ for $\lambda \in \mathbb{R}$, the $\mathbb{R}$-linearity of v implies that

$$v(f) = v(\lambda\chi_M) = \lambda v(\chi_M) = 0. \tag{2}$$

Exercise 1.3 (Properties of Differentials). Let M, N and P be smooth manifolds, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.

(a) $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is $\mathbb{R}$-linear.
(b) $\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p$.
(c) $\mathrm{d}(id_M)_p = id_{T_pM}$.
(d) If F is a diffeomorphism, then $\mathrm{d}F_p$ is an isomorphism and $(\mathrm{d}F_p)^{-1} = \mathrm{d}\left(F^{-1}\right)_{F(p)}$.

Solution 1.3. Let $u, v \in T_pM$, $\lambda \in \mathbb{R}$ and $f \in \mathscr{C}^\infty(N)$. Then

$$\begin{aligned}\mathrm{d}F_p(\lambda u + v)(f) &= (\lambda u + v)(f \circ F) \\ &= \lambda u(f \circ F) + v(f \circ F) \\ &= \lambda\, \mathrm{d}F_p(u)(f) + \mathrm{d}F_p(v)(f).\end{aligned}$$

This shows part (a). Let $v \in T_pM$ and $f \in \mathscr{C}^\infty(P)$. Then

$$\begin{aligned} \mathrm{d}(G \circ F)_p(v)(f) &= v\left(f \circ (G \circ F)\right) \\ &= v\left((f \circ G) \circ F\right) \\ &= \mathrm{d}F_p(f \circ G) \\ &= \mathrm{d}G_{F(p)}\left(\mathrm{d}F_p(v)\right)(f) \\ &= \left(\mathrm{d}G_{F(p)} \circ \mathrm{d}F_p\right)(v)(f). \end{aligned}$$

This shows part (b). Part (c) is immediate by

$$\mathrm{d}(id_M)_p(v)(f) = v(f \circ id_M) = v(f)$$

for $f \in \mathscr{C}^\infty(M)$. Finally, part (b) and (c) yields

$$\mathrm{d}F_p \circ \mathrm{d}\left(F^{-1}\right)_{F(p)} = \mathrm{d}(F \circ F^{-1})_{F(p)} = \mathrm{d}(id_N)_{F(p)} = id_{T_{F(p)}N}$$

and

$$\mathrm{d}\left(F^{-1}\right)_{F(p)} \circ \mathrm{d}F_p = \mathrm{d}(F^{-1} \circ F)_p = \mathrm{d}(id_M)_p = id_{T_pM}$$

which shows that $\mathrm{d}F_p$ is bijective with inverse $(\mathrm{d}F_p)^{-1} = \mathrm{d}\left(F^{-1}\right)_{F(p)}$ by uniqueness. Since by part (a) $\mathrm{d}F_p$ is linear, we have that dF_p is an isomorphism (see [Lee13, p. 622]). This shows part (d).

CHAPTER 4

Vector Fields

1. Vector Fields on Manifolds

Exercise 1.1. Let M be a smooth manifold.

(a) If $X, Y \in \mathfrak{X}(M)$, $f, g \in \mathscr{C}^\infty(M)$, then $fX + gY \in \mathfrak{X}(M)$.

CHAPTER 5

Integral Curves and Flows

1. Integral Curves

Definition 1.1. *Let (M_1, d_1) and (M_2, d_2) be metric spaces. A mapping $f : M_1 \to M_2$ is said to be **Lipschitz continuous** if there exists $L \in \mathbb{R}_{>0}$ such that for all $x, y \in M_1$*

$$d_2\left(f(x), f(y)\right) \leq L d_1(x, y) \tag{3}$$

*holds. We say that f is **locally Lipschitz continuous** if for every point $x \in M_1$ there exists a neighbourhood on which f is Lipschitz continuous.*

Proposition 1.1. *Let (M_1, d_1) be a metric space and (M_2, d_2) a complete bounded metric space. For $f, g \in \mathscr{C}(M_1; M_2)$ define*

$$d_\infty(f, g) := \sup_{x \in M_1} d_2\left(f(x), g(x)\right). \tag{4}$$

Then $(\mathscr{C}(M_1; M_2), d_\infty)$ is a complete metric space.

Proof. Since M is bounded, there exists $C \in \mathbb{R}_{>0}$ such that $d_2(x, y) \leq R$ for all $x, y \in M_1$. Hence

$$d_\infty(f, g) = \sup_{x \in M_1} d_2\left(f(x), g(x)\right) \leq R < \infty$$

for all $f, g \in \mathscr{C}(M_1; M_2)$. The metric axioms are easily verified, so we only show the completeness property. Let $(f_\nu)_{\nu \in \mathbb{N}}$ be a Cauchy sequence in $\mathscr{C}(M_1; M_2)$. Fix $\varepsilon > 0$. Since $(f_\nu)_{\nu \in \mathbb{N}}$ is a Cauchy sequence, we find $N \in \mathbb{N}$, such that for all $\nu, \mu \geq N$

$$d_\infty\left(f_\nu, f_\mu\right) < \frac{\varepsilon}{2}$$

holds. So for all $y \in M_1$ we have

$$d_2\left(f_\nu(y), f_\mu(y)\right) \leq \sup_{x \in X} d_2\left(f_\nu(x), f_\mu(x)\right) = d_\infty(f_\nu, f_\mu) < \varepsilon.$$

whenever $\nu, \mu \geq N$. Thus $\left(f_\nu(y)\right)_{\nu \in \mathbb{N}}$ is a Cauchy sequence in M_2 for all $y \in M_1$. Since M_2 is complete

$$f(y) := \lim_{\nu \to \infty} f_\nu(y)$$

exists for all $y \in M_1$. Now we show that $f_\nu \to f$ with respect to d_∞. For all $\nu \geq N$ and $y \in M_1$ we have that

$$\begin{aligned} d_2\left(f_\nu(y), f(y)\right) &= \lim_{\mu\to\infty} d_2\left(f_\nu(y), f_\mu(y)\right) \\ &= \liminf_{\mu\to\infty} d_2\left(f_\nu(y), f_\mu(y)\right) \\ &\leq \liminf_{\mu\to\infty} d_\infty(f_\nu, f_\mu) \\ &\leq \frac{\varepsilon}{2} \\ &< \varepsilon. \end{aligned}$$

Hence

$$d_\infty(f_\nu, f) < \varepsilon$$

whenever $\nu \geq N$. So $f_\nu \to f$ with respect to d_∞. Left to show is that $f \in \mathscr{C}(M_1; M_2)$. Fix $x_0 \in M_1$. Since $f_\nu \to f$ with respect to d_∞, there exists $N \in \mathbb{N}$ such that

$$d_\infty(f_\nu, f) < \frac{\varepsilon}{3}$$

for all $\nu \geq N$. Fix $\nu_0 \geq N$. Since f_{ν_0} is continuous at x_0, there exists $\delta > 0$, such that

$$d_2\left(f_{\nu_0}(x_0), f_{\nu_0}(x)\right) < \frac{\varepsilon}{3}$$

whenever $d_1(x_0, x) < \delta$. Hence

$$\begin{aligned} d_2\left(f(x_0), f(x)\right) &= d_2\left(f(x_0), f_{\nu_0}(x)\right) + d_2\left(f_{\nu_0}(x_0), f_{\nu_0}(x)\right) + d_2\left(f_{\nu_0}(x), f(x)\right) \\ &< 2d_\infty(f, f_{\nu_0}) + \frac{\varepsilon}{3} \\ &< \varepsilon \end{aligned}$$

whenever $d_1(x_0, x) < \delta$. Thus $f \in \mathscr{C}(M_1; M_2)$. □

Lemma 1.1 (Integral Formulation of an ODE). *Let $n \in \mathbb{Z}$, $n > 0$, $U \subseteq \mathbb{R}^n$ and $f \in \mathscr{C}(U; \mathbb{R}^n)$. A mapping $y \in \mathscr{C}(J_0; U)$, for some interval J_0 containing t_0, is a solution of the initial value problem*

$$\begin{cases} y'(t) = f\left(y(t)\right) \\ y(t_0) = y_0 \end{cases} \tag{5}$$

if and only if

$$y(t) = y_0 + \int_{t_0}^{t} f\left(y(s)\right) \mathrm{d}s \tag{6}$$

holds for all $t \in J_0$.

Proof. Assume that $y \in \mathscr{C}^1(J_0; U)$ solves (5). Then

$$\int_{t_0}^{t} f\left(y(s)\right) \mathrm{d}s = \int_{t_0}^{t} y'(s)\, \mathrm{d}s = y(t) - y(t_0)$$

for all $t \in J_0$ by the corollary to the first fundamental theorem of calculus [Spi94, p. 284].

Conversly assume that

$$y(t) = y_0 + \int_{t_0}^{t} f\left(y(s)\right) \mathrm{d}s$$

for all $t \in J_0$. Since $f \circ y \in \mathscr{C}(J_0; \mathbb{R}^n)$, the first fundamental theorem of calculus [Spi94, p. 282] implies $y'(t) = f\left(y(s)\right)$ for all $t \in J_0$. Furthermore clearly $y(t_0) = t_0$ and $y \in \mathscr{C}^1(J_0; U)$. Hence y is a solution of (5). □

Lemma 1.2 (Contraction Lemma). *Let (M, d) be a nonempty complete metric space and T be a contraction. Then there exists a unique fixed point for T.*

Theorem 1.1 (Existence of ODE Solutions). *Let $n \in \mathbb{Z}$, $n > 0$, $U \subseteq \mathbb{R}^n$ open, $f \in \mathscr{C}(U; \mathbb{R}^n)$ locally Lipschitz continuous and $(t_0, x_0) \in \mathbb{R} \times U$. Then there exists an open interval $J_0 \subseteq \mathbb{R}$ and an open subset $U_0 \subseteq U$, such that $(t_0, x_0) \in J_0 \times U_0$ and for each $y_0 \in U_0$ a mapping $y \in \mathscr{C}^1(J_0; U)$ satisfying*

$$\begin{cases} y'(t) = f\left(y(t)\right) \\ y(t_0) = y_0 \end{cases} . \tag{7}$$

Proof. Since F is locally Lipschitz continuous on U, there exists a neighbourhood V of x_0, such that f is Lipschitz continuous on V. Since $(U, |\cdot|)$ has the same topology as the subspace $U \subseteq \mathbb{R}^n$ by [Lee11, p. 50], we find $W \subseteq \mathbb{R}^n$ open, such that $V = U \cap W$. But since U is open, so is V open in $\mathbb{R}^n$. Hence we may assume that f is Lipschitz continuous on U. Let $L > 0$ denote a Lipschitz constant of f. Now choose $r > 0$ so, such that $\overline{B}_r(x_0) \subseteq U$. Furthermore let

$$M := \sup_{x \in \overline{B}_r(x_0)} |f(x)| < \infty$$

since $\overline{B}_r(x_0)$ is compact and $\delta, \varepsilon > 0$ such that

$$\delta < \frac{r}{2} \qquad \text{and} \qquad \varepsilon < \min\left(\frac{r}{2M}, \frac{1}{L}\right).$$

Define

$$J_0 := (t_0 - \varepsilon, t_0 + \varepsilon) \subseteq \mathbb{R} \qquad \text{and} \qquad U_0 := B_\delta(x_0) \subseteq U.$$

For any $y_0 \in U_0$, let

$$A_{y_0} := \left\{ y \in \mathscr{C}(J_0; \overline{B}_r(x_0)) : y(t_0) = y_0 \right\}.$$

Clearly $A_{y_0} \neq \varnothing$ since $y = y_0$ is in A_{y_0}. $\overline{B}_r(x_0)$ is clearly bounded and complete since it is a closed subset of a complete metric space. Thus we can consider the metric space (A_{y_0}, d_∞), where d_∞ is defined as in proposition 1.1. From the proof of proposition 1.1 we also see that if $(y_\nu)_{\nu \in \mathbb{N}}$ is a Cauchy sequence in A_{y_0} and $y := \lim_{\nu \to \infty} y_\nu$, then $y(t_0) = \lim_{\nu \to \infty} y_\nu(t_0) = y_0$. Hence $y \in A_{y_0}$ and so (A_{y_0}, d_∞) is complete. For $y \in A_{y_0}$ define for $t \in J_0$

$$T(y)(t) := y_0 + \int_{t_0}^{t} f\left(y(s)\right) \mathrm{d}s.$$

Clearly T is continuous and $T(y)(t_0) = y_0$. Furthermore

$$\begin{aligned}
\big|T(y)(t) - x_0\big| &= \left|y_0 + \int_{t_0}^{t} f\big(y(s)\big)\,\mathrm{d}s - x_0\right| \\
&\leq |y_0 - x_0| + \int_{t_0}^{t} \big|f\big(y(s)\big)\big|\,\mathrm{d}s \\
&< \delta + M\,|t - t_0| \\
&< \delta + M\varepsilon \\
&< r.
\end{aligned}$$

for all $t \in J_0$. Hence $T : A_{y_0} \to A_{y_0}$. Furthermore for $y_1, y_2 \in A_{y_0}$ we have that

$$\begin{aligned}
d_\infty\big(T(y_1), T(y_2)\big) &= \sup_{t\in J_0}\left|\int_{t_0}^{t} f\big(y_1(s)\big)\,\mathrm{d}s - \int_{t_0}^{t} f\big(y_2(s)\big)\,\mathrm{d}s\right| \\
&\leq \sup_{t\in J_0}\int_{t_0}^{t} \big|f\big(y_1(s)\big) - f\big(y_2(s)\big)\big|\,\mathrm{d}s \\
&\leq L\sup_{t\in J_0}\int_{t_0}^{t} \big|y_1(s) - y_2(s)\big|\,\mathrm{d}s \\
&\leq L\varepsilon d_\infty(y_1, y_2).
\end{aligned}$$

Since $0 < L\varepsilon < 1$, T is a contraction. Hence by the contraction lemma 1.2 there exists a unique fixed point $y \in A_{y_0}$. This y is a solution to the initial value problem by lemma 1.1. $\square$

CHAPTER 6

The Cotangent Bundle

1. Line Integrals

1.1. The Winding Number*.

Definition 1.1 (Winding Number). *Let $z_0 \in \mathbb{C}$ and $\gamma : [a,b] \to \mathbb{C} \setminus \{z_0\}$ be a piecewise continuously differentiable closed path. Then*

$$\mathrm{W}(\gamma, z_0) := \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - z_0} \tag{8}$$

is called the ***winding number*** *of γ around z_0.*

Proposition 1.1. *Let $z_0 := x_0 + iy_0 \in \mathbb{C}$ and $\gamma : [a,b] \to \mathbb{C} \setminus \{z_0\}$ be a piecewise continuously differentiable closed path. Then*

$$\mathrm{W}(\gamma, z_0) = \int_\gamma \omega \tag{9}$$

where $\omega \in \Omega^1\left(\mathbb{R}^2 \setminus \{z_0\}\right)$ is given by

$$\omega := \frac{1}{2\pi} \frac{(x - x_0)\,\mathrm{d}y - (y - y_0)\,\mathrm{d}x}{(x - x_0)^2 + (y - y_0)^2}. \tag{10}$$

Proof. This immediately follows from

$$\begin{aligned}
\int_\gamma \frac{\mathrm{d}z}{z - z_0} &= \int_\gamma \frac{\mathrm{d}x + i\,\mathrm{d}y}{(x + iy) - (x_0 + iy_0)} \\
&= \int_\gamma \frac{\mathrm{d}x + i\,\mathrm{d}y}{(x - x_0) + i(y - y_0)} \frac{(x - x_0) - i(y - y_0)}{(x - x_0) - i(y - y_0)} \\
&= \int_\gamma \frac{(x - x_0)\,\mathrm{d}x + \big((x - x_0)\,\mathrm{d}y - (y - y_0)\,\mathrm{d}x\big) + (y - y_0)\,\mathrm{d}y}{(x - x_0)^2 + (y - y_0)^2} \\
&= \int_\gamma \frac{(x - x_0)\,\mathrm{d}x + (y - y_0)\,\mathrm{d}y}{(x - x_0)^2 + (y - y_0)^2} + i \int_\gamma \frac{(x - x_0)\,\mathrm{d}y - (y - y_0)\,\mathrm{d}x}{(x - x_0)^2 + (y - y_0)^2} \\
&= i \int_\gamma \frac{(x - x_0)\,\mathrm{d}y - (y - y_0)\,\mathrm{d}x}{(x - x_0)^2 + (y - y_0)^2}
\end{aligned}$$

by the fundamental theorem for line integrals [Lee13, p. 291] since

$$\mathrm{d}\left(\frac{1}{2}\log\left((x - x_0)^2 + (y - y_0)^2\right)\right) = \frac{(x - x_0)\,\mathrm{d}x + (y - y_0)\,\mathrm{d}y}{(x - x_0)^2 + (y - y_0)^2}.$$

$\square$

Remark 1.1. By proposition 1.1, the definition of the winding number in complex analysis given by definition 1.1 coincides with the one usually given in algebraic topology (see for example [Ful95, pp. 19–20]).

CHAPTER 7

Tensors

1. Pullbacks of Tensor Fields

Exercise 1.1 (Properties of Tensor Pullbacks). Suppose $F : M \to N$ is a smooth mapping and A, B are covariant tensor fields on N. Then

(a) $F^*(A \otimes B) = F^*A \otimes F^*B$.

Solution 1.1. Let $p \in M$. Then we have

$$\begin{aligned}
\big(F^*(A \otimes B)\big)_p (v_1, \ldots, v_{k+l}) &= (A \otimes B)_{F(p)} \big(\mathrm{d}F_p(v_1), \ldots, \mathrm{d}F_p(v_{k+l})\big) \\
&= \big(A_{F(p)} \otimes B_{F(p)}\big) \big(\mathrm{d}F_p(v_1), \ldots, \mathrm{d}F_p(v_{k+l})\big) \\
&= A_{F(p)} \big(\mathrm{d}F_p(v_1), \ldots, \mathrm{d}F_p(v_k)\big)\, B_{F(p)} \big(\mathrm{d}F_p(v_{k+1}), \ldots, \mathrm{d}F_p(v_{k+l})\big) \\
&= (F^*A)_p(v_1, \ldots, v_k)(F^*B)_p(v_{k+1}, \ldots, v_{k+l}) \\
&= \big((F^*A)_p \otimes (F^*B)_p\big) (v_1, \ldots, v_{k+l}) \\
&= (F^*A \otimes F^*B)_p(v_1, \ldots, v_{k+l})
\end{aligned}$$

for all $v_1, \ldots, v_{k+l} \in T_pM$.

CHAPTER 8

Orientations

1. Orientations of Vector Spaces

Exercise 1.1. Let V be a vector space of dimension $n \geq 1$. Define a relation $\sim$ on the set of all ordered bases of V by

$$(v_1, \dots, v_n) \sim (w_1, \dots, w_n) \quad :\Leftrightarrow \quad \det B > 0 \tag{11}$$

where B denotes the transition matrix defined by $w_j = B_j^i v_i$. Show that $\sim$ is an equivalence relation and that $|X/{\sim}| = 2$.

Solution 1.1. Clearly $(v_1, \dots, v_n) \sim (v_1, \dots, v_n)$ by $v_j = \delta_j^i v_i$. Assume $(v_1, \dots, v_n) \sim (w_1, \dots, w_n)$. Thus B defined by $w_j = B_j^i v_i$ has a positive determinant. But then by $\det(B^{-1}) = \big(\det(B)\big)^{-1}$ also $\det(B^{-1})$ is positive and $v_j = \big(B^{-1}\big)_j^i w_i$. Hence $(w_1, \dots, w_n) \sim (v_1, \dots, v_n)$. Lastly, assume that also $(w_1, \dots, w_n) \sim (u_1, \dots, u_n)$. Hence there exists a matrix A such that $u_j = A_j^i w_i$ where $\det(A) > 0$. Thus $u_j = A_j^i w_i = A_j^i (B_i^k v_k) = (A_j^i B_i^k) v_k$ and by $\det(AB) = \det(A)\det(B) > 0$ we get that $(v_1, \dots, v_n) \sim (u_1, \dots, u_n)$. Hence $\sim$ is an equivalence relation.

By [Gri07, p. 335] every vector space has a basis. Let us denote it by $(v_1, \dots, v_n)$. Therefore

$$(\widetilde{v}_1, \dots, \widetilde{v}_n) := (-v_1, \dots, v_n)$$

is also a basis for V simply by considering the transition matrix

$$\widetilde{B} := \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

defined by $v_j = \widetilde{B}_j^i \widetilde{v}_i$. Let $(w_1, \dots, w_n)$ be an ordered basis for V. Let the transition matrix B be defined by $w_j = B_j^i v_i$. If $\det(B) > 0$, we have that

$$(w_1, \dots, w_n) \sim (v_1, \dots, v_n).$$

Otherwise, if $\det(B) < 0$

$$w_j = B_j^i v_i = B_j^i \left(\widehat{B}_i^k \widehat{v}_k\right) = \left(B_j^i \widehat{B}_i^k\right) \widehat{v}_k$$

together with $\det(B\widehat{B}) = \det(B)\det(\widehat{B}) > 0$ yields

$$(w_1, \dots, w_n) \sim (\widetilde{v}_1, \dots, \widetilde{v}_n).$$

Since $\det(B) \neq 0$ by the nonsingularity of B, we have that there are exactly two equivalence classes

$$[(v_1, \dots, v_n)]_\sim \qquad \text{and} \qquad [(-v_1, \dots, v_n)]_\sim .$$

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