

YANNIS BÄHNI

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SOLUTION  
BOOK TO  
INTRODUCTION  
TO SMOOTH  
MANIFOLDS BY  
JOHN M. LEE

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## Contents

<b>Chapter 1. Tangent Vectors</b> . . . . .	<b>2</b>
1 Tangent Vectors . . . . .	2
1.1 Tangent Vectors on Manifolds . . . . .	2
<b>Chapter 2. Vector Fields</b> . . . . .	<b>4</b>
1 Vector Fields on Manifolds . . . . .	4
<b>Chapter 3. Tensors</b> . . . . .	<b>5</b>
1 Pullbacks of Tensor Fields . . . . .	5
Appendix. Bibliography . . . . .	<b>6</b>

## Contents

## CHAPTER 1

### Tangent Vectors

#### 1. Tangent Vectors

##### 1.1. Tangent Vectors on Manifolds.

**Exercise 1.1.** Let  $M$  be a smooth manifold and  $p \in M$ . The set of all derivations at  $p$ , written  $T_p M$ , is a real vector space under the usual pointwise defined operations.

**Solution 1.1.** Clearly  $T_p M \subseteq L(\mathcal{C}^\infty(M); \mathbb{R})$  and thus it is enough to show that  $T_p M$  is a linear subspace of  $L(\mathcal{C}^\infty(M); \mathbb{R})$  (see [Lee13, p. 626]). We have  $T_p M \neq \emptyset$ , since  $0 \in T_p M$  defined by  $f \mapsto 0$ . Let  $u, v \in T_p M$ ,  $\lambda \in \mathbb{R}$  and  $f, g \in \mathcal{C}^\infty(M)$ . Then by

$$\begin{aligned} (\lambda u + v)(fg) &= \lambda u(fg) + v(fg) \\ &= f(p) (\lambda u(g) + v(g)) + g(p) (\lambda u(f) + v(f)) \\ &= f(p) (\lambda u + v)(g) + g(p) (\lambda u + v)(f) \end{aligned}$$

we have that  $\lambda u + v \in T_p M$ .

**Exercise 1.2.** Suppose  $M$  is a smooth manifold. Let  $p \in M$ ,  $v \in T_p M$  and  $f \in \mathcal{C}^\infty(M)$ . If  $f$  is constant, then  $v(f) = 0$ .

**Solution 1.2.** First assume that  $f = \chi_M$ . Then

$$v(\chi_M) = v(\chi_M \cdot \chi_M) = f(p)v(\chi_M) + f(p)v(\chi_M) = v(\chi_M) + v(\chi_M) \quad (1)$$

implies that  $v(f) = 0$ . Hence if  $f = \lambda \chi_M$  for  $\lambda \in \mathbb{R}$ , the  $\mathbb{R}$ -linearity of  $v$  implies that

$$v(f) = v(\lambda \chi_M) = \lambda v(\chi_M) = 0. \quad (2)$$

**Exercise 1.3 (Properties of Differentials).** Let  $M$ ,  $N$  and  $P$  be smooth manifolds, let  $F : M \rightarrow N$  and  $G : N \rightarrow P$  be smooth maps, and let  $p \in M$ .

- (a)  $dF_p : T_p M \rightarrow T_{F(p)} N$  is  $\mathbb{R}$ -linear.
- (b)  $d(G \circ F)_p = dG_{F(p)} \circ dF_p$ .
- (c)  $d(id_M)_p = id_{T_p M}$ .

**Solution 1.3.** Let  $u, v \in T_p M$ ,  $\lambda \in \mathbb{R}$  and  $f \in \mathcal{C}^\infty(N)$ . Then

$$\begin{aligned} dF_p(\lambda u + v)(f) &= (\lambda u + v)(f \circ F) \\ &= \lambda u(f \circ F) + v(f \circ F) \\ &= \lambda dF_p(u)(f) + dF_p(v)(f). \end{aligned}$$

This shows part (a). Let  $v \in T_p M$  and  $f \in \mathcal{C}^\infty(P)$ . Then

$$\begin{aligned} d(G \circ F)_p(v)(f) &= v(f \circ (G \circ F)) \\ &= v((f \circ G) \circ F) \\ &= dF_p(f \circ G) \\ &= dG_{F(p)}(dF_p(v))(f) \\ &= (dG_{F(p)} \circ dF_p)(v)(f). \end{aligned}$$

This shows part (b). Part (c) is immediate by

$$d(id_M)_p(v)(f) = v(f \circ id_M) = v(f)$$

for  $f \in \mathcal{C}^\infty(M)$ . Finally, part (b) and (c) yields

$$dF_p \circ d(F^{-1})_{F(p)} = d(F \circ F^{-1})_{F(p)} = d(id_N)_{F(p)} = id_{T_{F(p)}N}$$

and

$$d(F^{-1})_{F(p)} \circ dF_p = d(F^{-1} \circ F)_p = d(id_M)_p = id_{T_p M}$$

which shows that  $dF_p$  is bijective with inverse  $(dF_p)^{-1} = d(F^{-1})_{F(p)}$  by uniqueness. Since by part (a)  $dF_p$  is linear, we have that  $dF_p$  is an isomorphism (see [Lee13, p. 622]). This shows part (d).

## CHAPTER 2

### Vector Fields

#### 1. Vector Fields on Manifolds

**Exercise 1.1.** Let  $M$  be a smooth manifold.

- (a) If  $X, Y \in \mathfrak{X}(M)$ ,  $f, g \in \mathcal{C}^\infty(M)$ , then  $fX + gY \in \mathfrak{X}(M)$ .

## CHAPTER 3

### Tensors

#### 1. Pullbacks of Tensor Fields

**Exercise 1.1 (Properties of Tensor Pullbacks).** Suppose  $F : M \rightarrow N$  is a smooth mapping and  $A, B$  are covariant tensor fields on  $N$ . Then

(a)  $F^*(A \otimes B) = F^*A \otimes F^*B$ .

**Solution 1.1.** Let  $p \in M$ . Then we have

$$\begin{aligned} (F^*(A \otimes B))_p(v_1, \dots, v_{k+l}) &= (A \otimes B)_{F(p)}(dF_p(v_1), \dots, dF_p(v_{k+l})) \\ &= (A_{F(p)} \otimes B_{F(p)})(dF_p(v_1), \dots, dF_p(v_{k+l})) \\ &= A_{F(p)}(dF_p(v_1), \dots, dF_p(v_k)) B_{F(p)}(dF_p(v_{k+1}), \dots, dF_p(v_{k+l})) \\ &= (F^*A)_p(v_1, \dots, v_k) (F^*B)_p(v_{k+1}, \dots, v_{k+l}) \\ &= ((F^*A)_p \otimes (F^*B)_p)(v_1, \dots, v_{k+l}) \\ &= (F^*A \otimes F^*B)_p(v_1, \dots, v_{k+l}) \end{aligned}$$

for all  $v_1, \dots, v_{k+l} \in T_pM$ .



## Bibliography

- [Lee13] John M. Lee. *Introduction to Smooth Manifolds*. Second Edition. Graduate Texts in Mathematics. Springer, 2013.