Yannis Bähni

Solution Book to Introduction to Smooth Manifolds by John M. Lee

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Contents

CHAPTER 1

Smooth Maps

1. Smooth Functions and Smooth Maps

1.1. Smooth Functions on Manifolds. We follow the terminology established in [Gri07, p. 515].

Exercise 1.1. Let M be a smooth manifold. $\mathscr{C}^\infty(M)$ is an associative and commutative $\mathbb{R}$-algebra with identity under the usual pointwise defined operations.

Solution 1.1. First we show that $\mathscr{C}^\infty(M)$ is a real vector space. Since $\mathscr{C}^\infty(M) \subseteq \mathbb{R}^M$ it is enough to show that $\mathscr{C}^\infty(M)$ is a linear subspace of the real vector space $\mathbb{R}^M$. Clearly, $\mathscr{C}^\infty(M) \neq \varnothing$, since $\chi_M \in \mathscr{C}^\infty(M)$. Indeed, for $p \in M$ we find a chart (U, φ) such that $p \in U$ and the composition $\chi_M \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is clearly the function $\chi_{\varphi(U)}$, which is smooth since it is constant. Now let $f, g \in \mathscr{C}^\infty(M)$, $\lambda \in \mathbb{R}$ and $p \in M$. By definition, there exist charts (U, φ), (V, ψ) such that $f \circ \varphi^{-1}$ and $g \circ \psi^{-1}$ are smooth. Now consider the chart $(U \cap V, \varphi)$. Then by

$$(\lambda f + g) \circ \varphi^{-1} = \lambda(f \circ \varphi^{-1}) + (g \circ \varphi^{-1}) = \lambda(f \circ \varphi^{-1}) + \big((g \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1})\big)$$

we have that $\lambda f + g \in \mathscr{C}^\infty(M)$. Hence $\mathscr{C}^\infty(M)$ is a real vector space.
Now define a product map $\cdot : \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ by pointwise multiplication. Indeed, similar to the previous reasoning, by

$$(f \cdot g) \circ \varphi^{-1} = (f \circ \varphi^{-1}) \cdot (g \circ \varphi^{-1}) = (f \circ \varphi^{-1}) \cdot \big((g \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1})\big)$$

we have that $f \cdot g$ is smooth. Let $f, g, h \in \mathscr{C}^\infty(M)$ and $\lambda \in \mathbb{R}$. Then for $p \in M$

$$\begin{aligned}
\big((\lambda f + g) \cdot h\big)(p) &= (\lambda f + g)(p) h(p) \\
&= \big(\lambda f(p) + g(p)\big) h(p) \\
&= \lambda f(p) h(p) + g(p) h(p) \\
&= \lambda (f \cdot h)(p) + (g \cdot h)(p) \\
&= \big(\lambda (f \cdot h)\big)(p) + (g \cdot h)(p) \\
&= \big(\lambda (f \cdot h) + (g \cdot h)\big)(p)
\end{aligned}$$

shows that $\cdot$ is bilinear in the first argument. A similar computation shows that $\cdot$ is bilinear. By

$$\begin{aligned}
\big((f \cdot g) \cdot\big)(p) &= (f \cdot g)(p) h(p) \\
&= f(p) g(p) h(p) \\
&= f(p) (g \cdot h)(p) \\
&= \big(f \cdot (g \cdot h)\big)(p)
\end{aligned}$$

we see that $\cdot$ is associative. Furthermore by

$$(f \cdot g)(p) = f(p)g(p) = g(p)f(p) = (g \cdot f)(p)$$

we see that $\cdot$ is commutative. Finally, the identity element is given by χ_M since

$$(\chi_M \cdot f)(p) = \chi_M(p)f(p) = 1f(p) = f(p).$$

CHAPTER 2

Tangent Vectors

1. Tangent Vectors

1.1. Tangent Vectors on Manifolds.

Exercise 1.1. Let M be a smooth manifold and $p \in M$. The set of all derivations at p, written T_pM, is a real vector space under the usual pointwise defined operations.

Solution 1.1. Clearly $T_pM \subseteq \mathrm{L}(\mathscr{C}^\infty(M);\mathbb{R})$ and thus it is enough to show that T_pM is a linear subspace of $\mathrm{L}(\mathscr{C}^\infty(M);\mathbb{R})$ (see [Lee13, p. 626]). We have $T_pM \neq \varnothing$, since $0 \in T_pM$ defined by $f \mapsto 0$. Let $u, v \in T_pM$, $\lambda \in \mathbb{R}$ and $f, g \in \mathscr{C}^\infty(M)$. Then by

$$\begin{aligned}(\lambda u + v)(fg) &= \lambda u(fg) + v(fg) \\ &= f(p)\big(\lambda u(g) + v(g)\big) + g(p)\big(\lambda u(f) + v(f)\big) \\ &= f(p)(\lambda u + v)(g) + g(p)(\lambda u + v)(f)\end{aligned}$$

we have that $\lambda u + v \in T_pM$.

Exercise 1.2. Suppose M is a smooth manifold. Let $p \in M$, $v \in T_pM$ and $f \in \mathscr{C}^\infty(M)$. If f is constant, then $v(f) = 0$.

Solution 1.2. First assume that $f = \chi_M$. Then

$$v(\chi_M) = v(\chi_M \cdot \chi_M) = f(p)v(\chi_M) + f(p)v(\chi_M) = v(\chi_M) + v(\chi_M) \tag{1}$$

implies that $v(f) = 0$. Hence if $f = \lambda\chi_M$ for $\lambda \in \mathbb{R}$, the $\mathbb{R}$-linearity of v implies that

$$v(f) = v(\lambda\chi_M) = \lambda v(\chi_M) = 0. \tag{2}$$

Exercise 1.3 (Properties of Differentials). Let M, N and P be smooth manifolds, let $F : M \to N$ and $G : N \to P$ be smooth maps, and let $p \in M$.

(a) $\mathrm{d}F_p : T_pM \to T_{F(p)}N$ is $\mathbb{R}$-linear.
(b) $\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p$.
(c) $\mathrm{d}(id_M)_p = id_{T_pM}$.
(d) If F is a diffeomorphism, then $\mathrm{d}F_p$ is an isomorphism and $(\mathrm{d}F_p)^{-1} = \mathrm{d}\left(F^{-1}\right)_{F(p)}$.

Solution 1.3. Let $u, v \in T_pM$, $\lambda \in \mathbb{R}$ and $f \in \mathscr{C}^\infty(N)$. Then

$$\begin{aligned}\mathrm{d}F_p(\lambda u + v)(f) &= (\lambda u + v)(f \circ F) \\ &= \lambda u(f \circ F) + v(f \circ F) \\ &= \lambda\, \mathrm{d}F_p(u)(f) + \mathrm{d}F_p(v)(f).\end{aligned}$$

This shows part (a). Let $v \in T_pM$ and $f \in \mathscr{C}^\infty(P)$. Then

$$\begin{aligned} \mathrm{d}(G \circ F)_p(v)(f) &= v\left(f \circ (G \circ F)\right) \\ &= v\left((f \circ G) \circ F\right) \\ &= \mathrm{d}F_p(f \circ G) \\ &= \mathrm{d}G_{F(p)}\left(\mathrm{d}F_p(v)\right)(f) \\ &= \left(\mathrm{d}G_{F(p)} \circ \mathrm{d}F_p\right)(v)(f). \end{aligned}$$

This shows part (b). Part (c) is immediate by

$$\mathrm{d}(id_M)_p(v)(f) = v(f \circ id_M) = v(f)$$

for $f \in \mathscr{C}^\infty(M)$. Finally, part (b) and (c) yields

$$\mathrm{d}F_p \circ \mathrm{d}\left(F^{-1}\right)_{F(p)} = \mathrm{d}(F \circ F^{-1})_{F(p)} = \mathrm{d}(id_N)_{F(p)} = id_{T_{F(p)}N}$$

and

$$\mathrm{d}\left(F^{-1}\right)_{F(p)} \circ \mathrm{d}F_p = \mathrm{d}(F^{-1} \circ F)_p = \mathrm{d}(id_M)_p = id_{T_pM}$$

which shows that $\mathrm{d}F_p$ is bijective with inverse $(\mathrm{d}F_p)^{-1} = \mathrm{d}\left(F^{-1}\right)_{F(p)}$ by uniqueness. Since by part (a) $\mathrm{d}F_p$ is linear, we have that dF_p is an isomorphism (see [Lee13, p. 622]). This shows part (d).

CHAPTER 3

Vector Fields

1. Vector Fields on Manifolds

Exercise 1.1. Let M be a smooth manifold.

(a) If $X, Y \in \mathfrak{X}(M)$, $f, g \in \mathscr{C}^\infty(M)$, then $fX + gY \in \mathfrak{X}(M)$.

CHAPTER 4

The Cotangent Bundle

1. Line Integrals

1.1. The Winding Number*.

Definition 1.1 (Winding Number). *Let $z_0 \in \mathbb{C}$ and $\gamma : [a,b] \to \mathbb{C} \setminus \{z_0\}$ be a piecewise continuously differentiable closed path. Then*

$$\mathrm{W}(\gamma, z_0) := \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - z_0} \tag{3}$$

*is called the **winding number** of γ around z_0.*

Proposition 1.1. *Let $z_0 := x_0 + iy_0 \in \mathbb{C}$ and $\gamma : [a,b] \to \mathbb{C} \setminus \{z_0\}$ be a piecewise continuously differentiable closed path. Then*

$$\mathrm{W}(\gamma, z_0) = \int_\gamma \omega \tag{4}$$

where $\omega \in \Omega^1\left(\mathbb{R}^2 \setminus \{z_0\}\right)$ is given by

$$\omega := \frac{1}{2\pi} \frac{(x - x_0)\,\mathrm{d}y - (y - y_0)\,\mathrm{d}x}{(x - x_0)^2 + (y - y_0)^2}. \tag{5}$$

Proof. This immediately follows from

$$\begin{aligned}
\int_\gamma \frac{\mathrm{d}z}{z - z_0} &= \int_\gamma \frac{\mathrm{d}x + i\,\mathrm{d}y}{(x + iy) - (x_0 + iy_0)} \\
&= \int_\gamma \frac{\mathrm{d}x + i\,\mathrm{d}y}{(x - x_0) + i(y - y_0)} \frac{(x - x_0) - i(y - y_0)}{(x - x_0) - i(y - y_0)} \\
&= \int_\gamma \frac{(x - x_0)\,\mathrm{d}x + \big((x - x_0)\,\mathrm{d}y - (y - y_0)\,\mathrm{d}x\big) + (y - y_0)\,\mathrm{d}y}{(x - x_0)^2 + (y - y_0)^2} \\
&= \int_\gamma \frac{(x - x_0)\,\mathrm{d}x + (y - y_0)\,\mathrm{d}y}{(x - x_0)^2 + (y - y_0)^2} + i \int_\gamma \frac{(x - x_0)\,\mathrm{d}y - (y - y_0)\,\mathrm{d}x}{(x - x_0)^2 + (y - y_0)^2} \\
&= i \int_\gamma \frac{(x - x_0)\,\mathrm{d}y - (y - y_0)\,\mathrm{d}x}{(x - x_0)^2 + (y - y_0)^2}
\end{aligned}$$

by the fundamental theorem for line integrals [Lee13, p. 291] since

$$\mathrm{d}\left(\frac{1}{2}\log\left((x - x_0)^2 + (y - y_0)^2\right)\right) = \frac{(x - x_0)\,\mathrm{d}x + (y - y_0)\,\mathrm{d}y}{(x - x_0)^2 + (y - y_0)^2}.$$

$\square$

Remark 1.1. By proposition 1.1, the definition of the winding number in complex analysis given by definition 1.1 coincides with the one usually given in algebraic topology (see for example [Ful95, pp. 19–20]).

CHAPTER 5

Tensors

1. Pullbacks of Tensor Fields

Exercise 1.1 (Properties of Tensor Pullbacks). Suppose $F : M \to N$ is a smooth mapping and A, B are covariant tensor fields on N. Then

(a) $F^*(A \otimes B) = F^*A \otimes F^*B$.

Solution 1.1. Let $p \in M$. Then we have

$$\begin{aligned}
\big(F^*(A \otimes B)\big)_p (v_1, \ldots, v_{k+l}) &= (A \otimes B)_{F(p)} \big(\mathrm{d}F_p(v_1), \ldots, \mathrm{d}F_p(v_{k+l})\big) \\
&= \big(A_{F(p)} \otimes B_{F(p)}\big) \big(\mathrm{d}F_p(v_1), \ldots, \mathrm{d}F_p(v_{k+l})\big) \\
&= A_{F(p)} \big(\mathrm{d}F_p(v_1), \ldots, \mathrm{d}F_p(v_k)\big)\, B_{F(p)} \big(\mathrm{d}F_p(v_{k+1}), \ldots, \mathrm{d}F_p(v_{k+l})\big) \\
&= (F^*A)_p(v_1, \ldots, v_k)(F^*B)_p(v_{k+1}, \ldots, v_{k+l}) \\
&= \big((F^*A)_p \otimes (F^*B)_p\big) (v_1, \ldots, v_{k+l}) \\
&= (F^*A \otimes F^*B)_p(v_1, \ldots, v_{k+l})
\end{aligned}$$

for all $v_1, \ldots, v_{k+l} \in T_pM$.

CHAPTER 6

Orientations

1. Orientations of Vector Spaces

Exercise 1.1. Let V be a vector space of dimension $n \geq 1$. Define a relation $\sim$ on the set of all ordered bases of V by

$$(v_1, \ldots, v_n) \sim (w_1, \ldots, w_n) \quad :\Leftrightarrow \quad \det B > 0 \tag{6}$$

where B denotes the transition matrix defined by $w_j = B^i_j v_i$. Show that $\sim$ is an equivalence relation and that $|X/{\sim}| = 2$.

Solution 1.1. Clearly $(v_1, \ldots, v_n) \sim (v_1, \ldots, v_n)$ by $v_j = \delta^i_j v_i$. Assume $(v_1, \ldots, v_n) \sim (w_1, \ldots, w_n)$. Thus B defined by $w_j = B^i_j v_i$ has a positive determinant. But then by $\det(B^{-1}) = \big(\det(B)\big)^{-1}$ also $\det(B^{-1})$ is positive and $v_j = \left(B^{-1}\right)^i_j w_i$. Hence $(w_1, \ldots, w_n) \sim (v_1, \ldots, v_n)$. Lastly, assume that also $(w_1, \ldots, w_n) \sim (u_1, \ldots, u_n)$. Hence there exists a matrix A such that $u_j = A^i_j w_i$ where $\det(A) > 0$. Thus $u_j = A^i_j w_i = A^i_j (B^k_i v_k) = (A^i_j B^k_i) v_k$ and by $\det(AB) = \det(A)\det(B) > 0$ we get that $(v_1, \ldots, v_n) \sim (u_1, \ldots, u_n)$. Hence $\sim$ is an equivalence relation.
By [Gri07, p. 335] every vector space has a basis. Let us denote it by $(v_1, \ldots, v_n)$. Therefore

$$(\widetilde{v}_1, \ldots, \widetilde{v}_n) := (-v_1, \ldots, v_n)$$

is also a basis for V simply by considering the transition matrix

$$\widetilde{B} := \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

defined by $v_j = \widetilde{B}^i_j \widetilde{v}_i$. Let $(w_1, \ldots, w_n)$ be an ordered basis for V. Let the transition matrix B be defined by $w_j = B^i_j v_i$. If $\det(B) > 0$, we have that

$$(w_1, \ldots, w_n) \sim (v_1, \ldots, v_n).$$

Otherwise, if $\det(B) < 0$

$$w_j = B^i_j v_i = B^i_j \left(\widehat{B}^k_i \widehat{v}_k\right) = \left(B^i_j \widehat{B}^k_i\right) \widehat{v}_k$$

together with $\det(B\widehat{B}) = \det(B)\det(\widehat{B}) > 0$ yields

$$(w_1, \ldots, w_n) \sim (\widetilde{v}_1, \ldots, \widetilde{v}_n).$$

Since $\det(B) \neq 0$ by the nonsingularity of B, we have that there are exactly two equivalence classes

$$[(v_1, \ldots, v_n)]_\sim \qquad \text{and} \qquad [(-v_1, \ldots, v_n)]_\sim \, .$$

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