

ALEGBRA I - SUMMARY

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1. Groups

1.1. Subgroups.

DEFINITION 1.1. A subgroup of a group G is a subset $H \subseteq G$ such that

(1) $1 \in H$

(2) $x \in H$ implies $x^{-1} \in H$

(3) $x, y \in H$ implies $xy \in H$

PROPOSITION 1.1. $H \leq G$ if and only if $H \neq \emptyset$ and $x, y \in H$ implies $xy^{-1} \in H$.

PROPOSITION 1.2. For $H \neq \emptyset$ the following conditions are equivalent:

(1) $H \leq G$

(2) $HH \subseteq H$ and $H^{-1} \subseteq H$

$$(3) HH^{-1} \subseteq H$$

DEFINITION 1.2. Let G be a group and $X \subseteq G$. Define

$$\langle X \rangle := \bigcap_{X \subseteq H \leq G} H \quad (1)$$

PROPOSITION 1.3. Let X be a subset of a group G . Then

$$\langle X \rangle = \{x_1 \cdots x_n : \forall i \in I, x_i \in X \cup X^{-1}, n \in \mathbb{N}\} \quad (2)$$

1.1.1. Normal Subgroups.

1.2. Homomorphisms.

PROPOSITION 1.4. If $\varphi : A \rightarrow B$ is a group homomorphism, then $\varphi(1) = 1$, $\varphi(x^{-1}) = \varphi(x)^{-1}$ and $\varphi(x^n) = \varphi(x)^n$ for all $x \in A$ and $n \in \mathbb{Z}$.

PROPOSITION 1.5. A group homomorphism $\varphi : A \rightarrow B$ is injective if and only if $\ker(\varphi) = \{1\}$.

LEMMA 1.1. Let $\varphi : G \rightarrow H$ be an injective group homomorphism and let $a \in G$ be of finite order. Then $|\langle \varphi(a) \rangle| = |\langle a \rangle|$.

PROPOSITION 1.6. If $G = \langle X \rangle$ and $\varphi, \psi : G \rightarrow G'$ are group homomorphisms with $\varphi(x) = \psi(x)$ for every $x \in X$ then $\varphi = \psi$.

1.3. Cyclic Groups.

DEFINITION 1.3. A group or subgroup is cyclic when it is generated by a single element.

LEMMA 1.2. Every group of prime order is cyclic.

PROPOSITION 1.7. Every subgroup of $(\mathbb{Z}, +)$ is cyclic, generated by a unique nonnegative integer.

Proof. Let $H \leq \mathbb{Z}$. If $H = \{0\}$, then $H = \langle 0 \rangle$. So assume $H \neq \{0\}$. Thus H contains a nonzero integer. Since $H \leq \mathbb{Z}$ we have that H contains a positive integer. Let us denote the smallest positive integer in H by n . Every integer multiple of n belongs to H . Conversely, if $m \in H$ we have by integer division $m = nq + r$ for some $q, r \in \mathbb{Z}$ with $0 \leq r < n$. So $m - nq = r \in H$ contradicting the minimality of n being the smallest positive integer. Hence $m = nq$ and so $m \in \langle n \rangle$. \square

PROPOSITION 1.8. *Let G be a group and let $a \in G$. If $a^m \neq 1$ for all $m \neq 0$, then $\langle a \rangle \cong \mathbb{Z}$; in particular $\langle a \rangle$ is infinite. Otherwise, there is a smallest positive integer n such that $a^n = 1$, and then $a^m = 1$ if and only if n divides m , and $\langle a \rangle \cong \mathbb{Z}/n\mathbb{Z}$; in particular, $\langle a \rangle$ is finite of order n .*

COROLLARY 1.1. *Any two cyclic groups of order n are isomorphic.*

COROLLARY 1.2. *Every subgroup of a cyclic group is cyclic. Furthermore, either $H = \{1\}$ or $H = \langle x^n \rangle$, where n is the least positive integer with $x^n \in H$.*

Proof. Let $H \leq G := \langle x \rangle$, $H \neq \{1\}$. Then $H' := \{k \in \mathbb{Z} : x^k \in H\} \leq (\mathbb{Z}, +)$ since

- $0 \in H'$ by $1 = x^0 \in H \leq G$;
- If $k, k' \in H'$ then $k + k' \in H'$ by $x^{k+k'} = x^k x^{k'} \in H$;
- If $k \in H'$ then also $-k \in H'$ by $x^{-k} = (x^k)^{-1} \in H$.

Therefore $H' = \langle n \rangle$ for the least positive integer in H' . Now consider $\langle x^n \rangle$. We have that $\langle x^n \rangle \subseteq H$ by the previous observation and if $x^k \in H$ for some $k \in \mathbb{Z}$ we have $k \in H' = \langle n \rangle$ and so $k = mn$ for some $m \in \mathbb{Z}$ which yields $x^k = x^{mn} = (x^n)^m \in \langle x^n \rangle$. \square

PROPOSITION 1.9. *A cyclic group $G := \langle x \rangle$ of finite order n has a unique subgroup of order d , namely $\langle x^{n/d} \rangle = \{g \in G : g^d = 1\}$, for every divisor d of n .*

Proof. We prove the equality $\langle x^{n/d} \rangle = \{g \in G : g^d = 1\}$ only. An element of $\langle x^{n/d} \rangle$ has the form $x^{kn/d}$ for some integer k . Therefore

$$(x^{kn/d})^d = (x^n)^k = 1^k = 1$$

Conversely if $g \in G$ with $g^d = 1$ we have that $g = x^k$ for some integer k since G is cyclic. Hence $1 = g^d = x^{kd}$ and so $n \mid kd$. So

$$g = x^k = (x^{kd/n})^{n/d} = (x^{n/d})^{kd/n} \in \langle x^{n/d} \rangle$$

\square

LEMMA 1.3. *For all $n \in \mathbb{N}$*

$$\mathbb{Z}_n^\times = \{\bar{k} : \gcd(n, k) = 1\} \quad \text{and} \quad |\mathbb{Z}_n^\times| = \varphi(n)$$

1.4. Symmetric Groups.

1.4.1. Cycles.

LEMMA 1.4. *Let*

1.5. Automorphisms.

DEFINITION 1.4. Let G be a group. For $a \in G$ define

$$\iota_a : \begin{cases} G \rightarrow G \\ x \mapsto axa^{-1} \end{cases} \quad (3)$$

Further

$$\text{Inn}(G) := \{\iota_a : a \in G\} \quad (4)$$

LEMMA 1.5. We have $\iota_a \in \text{Aut}(G)$ for any $a \in G$. Furthermore $\text{Inn}(G) \trianglelefteq \text{Aut}(G)$.

Proof. $\iota_a \in \text{Aut}(G)$ and $\text{Inn}(G) \leq \text{Aut}(G)$ is obvious. Let $\iota_a \in \text{Inn}(G)$ and $\varphi \in \text{Aut}(G)$. Then for $x \in G$

$$\varphi \iota_a \varphi^{-1}(x) = \varphi(a \varphi^{-1}(x) a^{-1}) = \varphi(a) x \varphi(a)^{-1}$$

so $\varphi \iota_a \varphi^{-1} = \iota_{\varphi(a)} \in \text{Inn}(G)$. □

LEMMA 1.6. The mapping

$$\iota : \begin{cases} G \rightarrow \text{Aut}(G) \\ a \mapsto \iota_a \end{cases} \quad (5)$$

is a homomorphism. Furthermore $\ker(\iota) = Z(G)$.

Proof. That $\iota \in \text{Hom}(G, \text{Aut}(G))$ is obvious. For $a \in \ker(\iota)$ we must have that $\iota_a = \text{id}$, so $axa^{-1} = x$ for any $x \in G$. Hence $a \in Z(G)$. The converse is trivial. □

Thus by the first isomorphism theorem we have

$$G/Z(G) \cong \text{Inn}(G)$$

PROPOSITION 1.10. Let G be a group and assume $G/Z(G)$ is cyclic. Then G is abelian.

Proof. Since $G/Z(G)$ is cyclic we have $G/Z(G) = \langle gZ(G) \rangle$ for some $g \in G$. Furthermore, each element of $G/Z(G)$ has the form $(gZ(G))^k = g^k Z(G)$ since $Z(G) \trianglelefteq G$ and $k \in \mathbb{Z}$. Let $x, y \in G$. Since $G/Z(G)$ provides a partition of G we have $x = g^m z$ and $y = g^n z'$ for some $m, n \in \mathbb{Z}$, $z, z' \in Z(G)$. Therefore

$$xy = g^m z g^n z' = g^m g^n z z' = g^{m+n} z z' = g^n g^m z z' = g^n g^m z' z = g^n z' g^m z = yx$$

since center elements commute with every group element. □

1.6. Direct Products.

PROPOSITION 1.11. A group G is isomorphic to the direct product $G_1 \times G_2$ of two groups G_1, G_2 if and only if it contains normal subgroups $A \cong G_1$ and $B \cong G_2$ such that $A \cap B = \{1\}$ and $AB = G$.

LEMMA 1.7. Let G be a group. For every $a \in G$ the mappings

$$\vartheta_a : \begin{cases} G \rightarrow G \\ x \mapsto ax \end{cases} \quad \vartheta'_a : \begin{cases} G \rightarrow G \\ x \mapsto xa \end{cases} \quad (6)$$

are bijections.

THEOREM 1.1.

LEMMA 1.8. Let G be a group. If $x^2 = 1$ for every $x \in G$ then G is abelian.

PROPOSITION 1.12. In a finite group, the inverse of an element is a positive power of that element.

PROPOSITION 1.13. If $G = \langle X \rangle$ and the elements of X are pairwise interchangeable then G is abelian. Hence every cyclic group is abelian.

DEFINITION 1.5. Let G be a group. The order of an element $x \in G$ is defined by $|\langle x \rangle|$.

DEFINITION 1.6. Relative to $H \leq G$ the left coset of an element $x \in G$ is the subset xH of G ; the right coset of an element $x \in G$ is the subset Hx of G .

PROPOSITION 1.14. The left cosets of $H \leq G$ constitute a partition of G and so do the right cosets.

PROPOSITION 1.15. The number of left cosets of a subgroup is equal to the number of right cosets.

DEFINITION 1.7. The index $[G : H]$ of $H \leq G$ is the cardinal number of its left or right cosets.

PROPOSITION 1.16. (Lagrange's Theorem) If $H \leq G$, then $|G| = [G : H]|H|$. Hence if $|G| < \infty$, the order and the index of a subgroup divide the order of G .

THEOREM 1.2. *In a finite group G we have $g^{|G|} = 1$ for any $g \in G$.*

DEFINITION 1.8. *Let $N \trianglelefteq G$. The group of all cosets of N is the quotient group G/N of G by N . The homomorphism $x \mapsto xN = Nx$ is the canonical projection of G onto G/N .*

PROPOSITION 1.17. *Let $N \trianglelefteq G$. Every subgroup of G/N is the quotient H/N of a unique subgroup H of G that contains N .*