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ALEGBRA I - SUMMARY

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Contents

1. Groups

Lemma 1.1. *Let G be a group. For every $a \in G$ the mappings*

$$\vartheta_a : \begin{cases} G \to G \\ x \mapsto ax \end{cases} \qquad \vartheta'_a : \begin{cases} G \to G \\ x \mapsto xa \end{cases} \tag{1}$$

are bijections.

Theorem 1.1.

Lemma 1.2. *Let G be a group. If $x^2 = 1$ for every $x \in G$ then G is abelian.*

Definition 1.1. *A* subgroup *of a group G is a subset $H \subseteq G$ such that*

(1) $1 \in H$

(2) $x \in H$ *implies* $x^{-1} \in H$

(3) $x, y \in H$ *implies* $xy \in H$

Proposition 1.1. $H \leq G$ *if and only if* $H \neq \varnothing$ *and* $x, y \in H$ *implies* $xy^{-1} \in H$.

Proposition 1.2. *For* $H \neq \varnothing$ *the following conditions are equivalent:*

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(1) $H \leq G$

(2) $HH \subseteq H$ *and* $H^{-1} \subseteq H$

(3) $HH^{-1} \subseteq H$

PROPOSITION 1.3. *In a finite group, the inverse of an element is a positive power of that element.*

DEFINITION 1.2. *Let* G *be a group and* $X \subseteq G$*. Define*

$$\langle X \rangle := \bigcap_{X \subseteq H \leq G} H \tag{2}$$

PROPOSITION 1.4. *Let* X *be a subset of a group* G*. Then*

$$\langle X \rangle = \{x_1 \cdots x_n : \forall i \in I \; x_i \in X \cup X^{-1}, n \in \mathbb{N}\} \tag{3}$$

DEFINITION 1.3. *A group or subgroup is* cyclic *when it is generated by a single element.*

PROPOSITION 1.5. *Every subgroup of* $\mathbb{Z}$ *is cyclic, generated by a unique nonnegative integer.*

PROPOSITION 1.6. *If* $G = \langle X \rangle$ *and the elements of* X *are pairwise interchangeable then* G *is abelian. Hence every cyclic group is abelian.*

DEFINITION 1.4. *Let* G *be a group. The* order of an element $x \in G$ *is defined by* $|\langle x \rangle|$*.*

DEFINITION 1.5. *Relative to* $H \leq G$ *the* left coset *of an element* $x \in G$ *is the subset* xH *of* G*; the* right coset *of an element* $x \in G$ *is the subset* Hx *of* G*.*

PROPOSITION 1.7. *The left cosets of* $H \leq G$ *constitute a partition of* G *and so do the right cosets.*

PROPOSITION 1.8. *The number of left cosets of a subgroup is equal to the number of right cosets.*

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Definition 1.6. *The* index $[G : H]$ *of* $H \leq G$ *is the cardinal number of its left or right cosets.*

Proposition 1.9. (Lagrange's Theorem) *If* $H \leq G$*, then* $|G| = [G : H]|H|$*. Hence if* $|G| < \infty$*, the order and the index of a subgroup divide the order of* G*.*

Definition 1.7. *Let* $N \trianglelefteq G$*. The group of all cosets of* N *is the* quotient group G/N *of* G *by* N*. The homomorphism* $x \mapsto xN = Nx$ *is the* canonical projection *of* G *onto* G/N*.*

Proposition 1.10. *Let* $N \trianglelefteq G$*. Every subgroup of* G/N *is the quotient* H/N *of a unique subgroup* H *of* G *that contains* N*.*

2. Homomorphisms

Proposition 2.1. *If* $\varphi : A \to B$ *is a group homomorphism, then* $\varphi(1) = 1$, $\varphi(x^{-1}) = \varphi(x)^{-1}$ *and* $\varphi(x^n) = \varphi(x)^n$ *for all* $x \in A$ *and* $n \in \mathbb{Z}$*.*

Proposition 2.2. *If* $G = \langle X \rangle$ *and* $\varphi, \psi : G \to G'$ *are group homomorphisms with* $\varphi(x) = \psi(x)$ *for every* $x \in X$ *then* $\varphi = \psi$*.*

Proposition 2.3. *A group homomorphism* $\varphi : A \to B$ *is injective if and only if* $\ker(\varphi) = \{1\}$*.*