LECTURE 3

Paths and the fundamental groupoid

In this lecture we define a rather pathetic functor, called π_0. We then define the *fundamental groupoid*. In the next lecture we will use the fundamental groupoid to define a much more interesting functor, the *fundamental group* π_1.

DEFINITION 3.1. A **path** u in a topological space X is a continuous map $u\colon I \to X$. If $u(0) = x$ and $u(1) = y$ we say u is a **path from** x **to** y. If $x = y$ then we say that u is a **loop**.

We will always use the letters u, v and w to denote paths (in contrast to f, g and h for arbitrary continuous maps). Moreover we will parametrise a path with the letter s, so u is the map $s \mapsto u(s)$, thus keeping the letter t for a homotopy parameter. This will hopefully help to keep the notation clear. Paths gives us a new notion of connectivity.

DEFINITION 3.2. A topological space X is **path connected** if for all $x, y \in X$ there exists a path from x to y.

Hopefully you are all easily able to prove the following result[1].

LEMMA 3.3. *Let X and Y be topological spaces. Then:*

1. *If X is path connected then X is connected (but the converse is not necessarily true).*
2. *If X and Y are path connected then so is $X \times Y$.*
3. *If $f\colon X \to Y$ is continuous and X is path connected then so is $f(X)$.*

Here we prove the following equally easy result:

PROPOSITION 3.4. *If X is a topological space then the binary relation $\sim$ on X defined by "$x \sim y$ if there exists a path from x to y" is an equivalence relation.*

Proof. A constant path based at x shows that $x \sim x$ for all $x \in X$. If u is a path from x to y then the path $\bar{u}(s) := u(1 - s)$ is a path from y to x, and hence $x \sim y$ implies $y \sim x$. Finally if u is a path from x to y and v is a path from y to z then

$$w(s) := \begin{cases} u(2s), & 0 \le s \le \frac{1}{2}, \\ v(2s - 1), & \frac{1}{2} \le s \le 1, \end{cases} \tag{3.1}$$

is a well-defined path from x to z (the gluing lemma shows that w is continuous.) Thus $x \sim y$ and $y \sim z$ implies $x \sim z$. ∎

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[1] I debated putting this on Problem Sheet B but decided it was too easy ...

Definition 3.5. The equivalence classes of X under the equivalence relation $\sim$ are called the **path components** of X.

We now construct the functor π_0.

Definition 3.6. Given a topological space X, let $\pi_0(X)$ denote the set of path components of X. If $f\colon X \to Y$ is a continuous map, define $\pi_0(f)\colon \pi_0(X) \to \pi_0(Y)$ to be the map that send a path component X' of X to the unique path component of Y containing $f(X')$ (this is well-defined due to Lemma 3.3.)

We then have:

Proposition 3.7. *$\pi_0\colon \mathsf{Top} \to \mathsf{Sets}$ is a functor. Moreover if $f \simeq g$ then $\pi_0(f) = \pi_0(g)$.*

Proof. The fact that π_0 is a functor is easy to check (i.e. that π_0 preserves identities and composition). Let us check that homotopic maps have the same image under π_0. Suppose $F\colon f \simeq g$. If X' is a path component of X then $X' \times I$ is path connected and hence so is $F(X' \times I)$ (here we are using Proposition 3.3 twice). Since

$$f(X') = F(X' \times \{0\}) \subseteq F(X' \times I)$$

and

$$g(X') = F(X' \times \{1\}) \subseteq F(X' \times I),$$

we see that the unique path component of Y containing $F(X' \times I)$ contains both $f(X')$ and $g(X')$. Thus $\pi_0(f) = \pi_0(g)$. ■

Corollary 3.8. *If X and Y have the same homotopy type then they have the same number of path components.*

Corollary 3.8 can be proved directly, but let us give an "abstract" proof using Problem A.2 and Problem A.3 from Problem Sheet A.

Proof. By the last part of Proposition 3.7 and Problem A.3, we may regard π_0 as a functor $\mathsf{hTop} \to \mathsf{Sets}$. If X and Y have the same homotopy type then there exists a continuous map $f\colon X \to Y$ such that $[f]$ is an isomorphism in hTop. Then by Problem A.2, $\pi_0([f])$ is an isomorphism in Sets. An isomorphism in Sets is a bijection; thus $\pi_0(X)$ and $\pi_0(Y)$ have the same cardinality. ■

Corollary 3.8 is about as interesting as it gets when it comes to the functor π_0. This is because π_0 has the misfortune of taking values in Sets, and there is not much one can with a set other than count it (i.e. the only obstruction to two sets being isomorphic is that they should have the same cardinality) Next lecture we will introduce another functor π_1 which takes values in Groups. As groups have many obstructions to being isomorphic, this functor will be considerably more interesting.

The basic idea behind π_1 is that one can "multiply" paths if one ends where the other begins, via (3.1). Let us formalise this as a definition.

Definition 3.9. Let u and v be paths in X with $u(1) = v(0)$. Then we define

$$(u * v)(s) := \begin{cases} u(2s), & 0 \le s \le \frac{1}{2}, \\ v(2s-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

Remark 3.10. Note that the ordering here is the *opposite* to composition: $u * v$ means "first do u, then do v", meanwhile $g \circ f$ means "first do f, then do g".

Our aim is to construct a group whose elements are certain homotopy classes of paths in X with binary operation given by multiplying paths as above. However by Problem B.1 on Problem Sheet B, if X is path connected then since I is contractible, all paths $u\colon I \to X$ are homotopic, and thus if we tried to construct a group from homotopy classes of paths this group would have precisely one element (and so would be just as uninteresting as $\pi_0(X)$!) To recify this problem we use relative homotopy classes.

Definition 3.11. We define the **path class** of a path $u\colon I \to X$ to be the equivalence class $[u]$ of u, where the equivalence relation is being homotopic relative to $\partial I = \{0, 1\}$.

Remark 3.12. There is a potential for confusion here, in that we are using the same notation $[\cdot]$ to denote both the homotopy class and the relative homotopy class. However, it should always be clear from the context which is intended. In particular, for paths we will only ever talk about their path class, not their homotopy class, and thus the notation $[u]$ *always* means the path class.

The next result is similar to Proposition 2.6.

Proposition 3.13. *Suppose $u_0, u_1\colon I \to X$ and $v_0, v_1\colon I \to X$ are paths with*

$$u_0(1) = u_1(1) = v_0(0) = v_1(0).$$

Assume that

$$[u_0] = [u_1] \qquad \text{and} \qquad [v_0] = [v_1].$$

Then

$$[u_0 * v_0] = [u_1 * v_1].$$

Proof. If $U\colon u_0 \simeq u_1$ rel ∂I and $V\colon v_0 \simeq v_1$ rel ∂I then the map $W\colon I \times I \to X$ given by

$$W(s,t) := \begin{cases} U(2s, t), & 0 \le s \le \frac{1}{2}, \\ V(2s-1, t), & \frac{1}{2} \le s \le 1, \end{cases}$$

is a continuous map (the gluing lemma applies because functions agree on $\{\frac{1}{2}\} \times I$) which determines a homotopy from $u_0 * v_0$ to $u_1 * v_1$ rel ∂I. ∎

If u is a path from x to y, then running backwards along u gives a path from y to x. Let us fix some notation for this:

Definition 3.14. Given a path $u : I \to X$, we denote by $\bar{u} : I \to X$ the path u parametrised backwards:

$$\bar{u}(s) = u(1-s).$$

Next, let us given a name to the constant path:

Definition 3.15. Given a point $p \in X$, we denote by e_p the constant path $e_p(s) = p$. By a slight abuse of notation we denote by $[p]$ the path class $[e_p]$.

We now use this data to define a category. We will phrase this is as "definition" and then prove afterwards that it really is well-defined.

Definition 3.16. Let X be a topological space. We define the **fundamental groupoid** of X to be the category $\Pi(X)$ where:

- $\mathrm{obj}(\Pi(X)) = X$, that is, the objects of $\Pi(X)$ are the points in X themselves,
- $\mathrm{Hom}(x, y)$ is the set of path classes of paths from x to y:

$$\mathrm{Hom}(x,y) := \{[u] \mid u \text{ is a path from } x \text{ to } y\},$$

- and finally the composition

$$\mathrm{Hom}(x,y) \times \mathrm{Hom}(y,z) \to \mathrm{Hom}(x,z)$$

  is given by

$$\big([u],[v]\big) \mapsto [u * v]$$

  (note by assumption this concatenation makes sense as $u(1) = y = v(0)$).

Let us prove this really does form a category.

Proposition 3.17. *Let X be a topological space. Then $\Pi(X)$ is a well-defined category. The identity element of $\mathrm{Hom}(p,p)$ is $[p]$.*

Proof. From Definition 1.6, there are three things we need to verify:

1. the Hom sets are pairwise disjoint,
2. that composition is associative when defined,
3. that there exists an identity element in each Hom set.

Here (1) is obvious. Let us first prove (3). We claim that $[p]$ (i.e. the path class of e_p) is the identity element in $\mathrm{Hom}(p,p)$. For this we must prove that for any path u with $u(0) = p$ we have $e_p * u \simeq u$ rel ∂I, and similarly for any path v with $v(1) = p$ we have $v * e_p \simeq v$ rel ∂I. We will prove the first statement only, as the second is similar. Consider Figure 3.1. The shaded triangle is the set $\{(s,t) \mid 2s \leq 1 - t\}$. For fixed t, consider the horizontal line L_t that runs from the start of the shaded region to the right-hand edge (the point $(1,t)$). The function

$$l_t(s) := \frac{s - \frac{1}{2}(1-t)}{1 - \frac{1}{2}(1-t)}$$

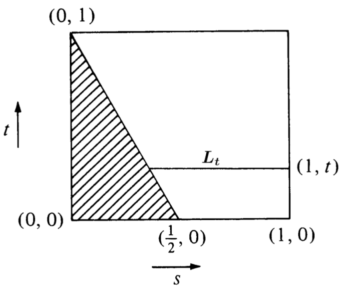

Figure 3.1: Proving $e_p * u \simeq u$ rel ∂I.

maps L_t onto $[0,1]$. Now consider the map $U\colon I \times I \to X$ given by

$$U(s,t) := \begin{cases} p, & 2s \leq 1-t, \\ u(l_t(s)), & 2s \geq 1-t. \end{cases}$$

The gluing lemma shows that U is continuous, and by construction $U\colon e_p * u \simeq u$ rel ∂I.

Now let us prove associativity. Suppose u, v and w are three paths with $u(1) = v(0)$ and $v(1) = w(0)$. This is a similar but slightly trickier argument, and we will not write out the formulae precisely. Consider Figure 3.2. Draw two slanted lines, one

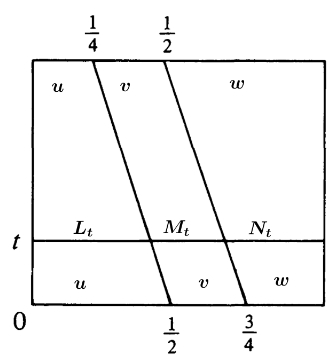

Figure 3.2: Proving $(u * v) * w \simeq u * (v * w)$ rel ∂I.

that starts at $(1/4, 1)$ and runs to $(1/2, 0)$, and one that starts at $(1/2, 1)$ and runs to $(3/4, 0)$. Now let L_t, M_t and N_t denote the three horizontal lines as marked that come from intersecting the horizontal line with t fixed. Then let l_t, m_t and n_t denote reparametrisations that map L_t, M_t and N_t onto $[0,1]$ respectively. The desired homotopy is U is obtained by setting $U(t,s) = u(l_t(s))$ on the left-hand region, setting $U(s,t) = v(m_t(s))$ on the middle region and finally setting $U(t,s) = w(n_t(s))$ on the right-hand region. The gluing lemma shows that U is continuous, and by construction we have $U\colon (u*v)*w \simeq u*(v*w)$ rel ∂I. This completes the proof. ■

In fact, the category $\Pi(X)$ has an additional special property:

Proposition 3.18. *Every morphism in $\Pi(X)$ is an isomorphism. More precisely, for any path u from x to y, one has*

$$[u] * [\bar{u}] = [x], \qquad [\bar{u}] * [u] = [y].$$

Proof. We must show $u * \bar{u} \simeq e_x$ rel ∂I and $\bar{u} * u \simeq e_y$ rel ∂I. Again, I will prove only the first statement. Moreover this time round, I will give the formulae but not the picture[2]. To this end consider the function $U \colon I \times I \to X$ given by

$$U(s,t) := \begin{cases} u\big(2s(1-t)\big), & 0 \le s \le \frac{1}{2}, \\ u\big(2(1-s)(1-t)\big), & \frac{1}{2} \le s \le 1. \end{cases}$$

The gluing lemma shows that U is continuous, and one checks that $U \colon u * \bar{u} \simeq e_x$ rel ∂I. This completes the proof. ∎

Categories with this property have a special name.

Definition 3.19. Let C be a category. We say that C is a **groupoid category** if:

- C is a **small** category[3], which by definition means that $\mathrm{obj}(\mathsf{C})$ is a *set* and not just a class, cf. Remark 1.7.
- Every morphism $f : A \to B$ in C is an isomorphism.

Thus Proposition 3.18 can alternatively be rephrased as: the fundamental groupoid is a groupoid cateogory.

[2] And thus you should draw the picture out!

[3] Nothing we ever do in this course will ever need to worry about the distinction between a set and a class, so you are free to ignore this part of the definition if you want ...