LECTURE 1

The Brouwer Fixed Point Theorem

Even the simplest problems in topology – for instance, whether two topological spaces X and Y are homeomorphic – are oftentimes very hard to answer. In order to show X and Y are homeomorphic, it suffices to find a single homeomorphism $f\colon X \to Y$. But in order to show that they are *not* homeomorphic, one needs to prove no such homeomorphism can exist. And how on earth are you meant to do that? Even if *you* can't find one, how do you know that tomorrow some really smart mathematician isn't going to magically come up with one? This is where *algebraic topology* comes in. The idea is to associate *algebraic invariants* of a topological space. Here "invariants" means that two homeomorphic spaces should have the same invariants. Thus to show two spaces are *not* homeomorphic, it suffices to show they have different invariants.

So, to summarise the entire course:

- Topology is hard.
- Algebra is easy.
- Algebraic topology converts topological problems into algebraic problems.
- Profit.

We illustrate this philosophy with an example. Let $B^n \subset \mathbb{R}^n$ denote the *closed* n-dimensional unit ball

$$B^n = \{x \in \mathbb{R}^n \mid |x| \le 1\}.$$

The boundary of B^n is the $(n-1)$-dimensional unit sphere S^{n-1}:

$$S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}.$$

The following famous theorem is due to Brouwer.

Theorem 1.1 (The Brouwer Fixed Point Theorem). *For all $n \ge 1$, every continuous map $f\colon B^n \to B^n$ has a fixed point.*

In the case $n = 1$, this theorem has a simple proof using connectivity:

Proof of Theorem 1.1 in the case $n = 1$. Suppose $f(-1) = a$ and $f(1) = b$. If $a = -1$ or $b = 1$ we are done, so assume that $a > -1$ and $b < 1$. Consider the graph of f:

$$\mathrm{Gr}(f) := \{(x, f(x)) \mid x \in [-1, 1]\}.$$

A fixed point of f is the same thing as a point of intersection between $\mathrm{Gr}(f)$ and the diagonal

$$\Delta := \{(x, x) \mid x \in [-1, 1]\}.$$

Will J. Merry, Algebraic Topology I, Autumn 2017, ETH Zürich. Last modified: October 22, 2017.

Since f is continuous, $\mathrm{Gr}(f)$ is connected[1]. Let

$$A := \{(x, f(x)) \mid f(x) > x\}, \qquad B := \{(x, f(x)) \mid f(x) < x\}.$$

Then $(-1, a) \in A$ and $(1, b) \in B$, so in particular A and B are both non-empty. If $\mathrm{Gr}(f) \cap \Delta = \emptyset$ then $\mathrm{Gr}(f) = A \cup B$. Since f is continuous, A and B are open[2] in $\mathrm{Gr}(f)$. This contradicts the fact that $\mathrm{Gr}(f)$ is connected. ∎

Interestingly, it is not known how to extend this simple argument to deal with the case $n > 1$. Nevertheless there are several different complicated arguments. For instance, there is an analytical argument that goes as follows: first approximate f by a sequence of *differentiable* functions g_k with the property that f has a fixed point if and only if all the g_k do for large k. Then prove directly that any differentiable function must have a fixed point.

The "cutest" proof uses methods from algebraic topology. Later on in the course we will construct a **homology functor** H_n for each $n \geq 0$, which associates to any topological space X an abelian group $H_n(X)$, and to any continuous map $f\colon X \to Y$ a homomorphism

$$H_n(f) : H_n(X) \to H_n(Y).$$

The induced maps $H_n(f)$ have the property that if $f\colon X \to Y$ and $g : Y \to Z$ then

$$H_n(g \circ f) = H_n(g) \circ H_n(f)\colon H_n(X) \to H_n(Z), \tag{1.1}$$

and

$$H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X)}\colon H_n(X) \to H_n(X). \tag{1.2}$$

Moreover the homology functor H_n vanishes on the ball B^{n+1} but not on the sphere S^n:

$$H_n(B^{n+1}) = 0, \qquad H_n(S^n) \neq 0, \qquad \forall\, n \geq 1. \tag{1.3}$$

The construction of H_n and the verification of (1.1), (1.2) and (1.3) will take some time. Nevertheless, armed with these only these properties, it is easy to prove Theorem 1.1 in all dimensions.

Definition 1.2. Suppose X is a subspace of a topological space Y. We say that X is a **retract** of Y if there exists a continuous map $r : Y \to X$ such that $r(x) = x$ for all $x \in X$. Equivalently, denoting by $\imath : X \hookrightarrow Y$ the inclusion, this means that the following diagram *commutes*:

$$\begin{tikzcd} & Y \arrow[dr, "r"] & \\ X \arrow[ur, hook, "\imath"] \arrow[rr, "\mathrm{id}"] & & X \end{tikzcd}$$

[1]It is the image of the continuous map $B^1 \to B^1 \times B^1$ given by $x \mapsto (x, f(x))$.

[2]Consider the map $g : \mathrm{Gr}(f) \to \mathbb{R}$ given by $g(x, f(x)) = x - f(x)$. Then $A = g^{-1}\big((-\infty, 0)\big)$ and $B = g^{-1}\big((0, \infty)\big)$.

Lemma 1.3. *For all $n \geq 1$, S^n is not a retract of B^{n+1}.*

Proof. Suppose for contradiction that there exists a retraction $r\colon B^{n+1} \to S^n$, so that the following diagram commutes:

$$\begin{tikzcd} & B^{n+1} \arrow[dr, "r"] & \\ S^n \arrow[ur, hook, "\imath"] \arrow[rr, "\mathrm{id}"] & & S^n \end{tikzcd}$$

Equation (1.1) means that we can "apply the homology functor H_n" to this commutative diagram to obtain another one:

$$\begin{tikzcd} & H_n(B^{n+1}) \arrow[dr, "H_n(r)"] & \\ H_n(S^n) \arrow[ur, "H_n(\imath)"] \arrow[rr, "H_n(\mathrm{id})"] & & H_n(S^n) \end{tikzcd}$$

Note this diagram is a commutative diagram of group homomorphisms between abelian groups, rather than a commutative diagram of continuous maps between topological spaces. Since $H_n(B^{n+1}) = 0$ by (1.3) the map $H_n(r)\colon H_n(B^{n+1}) \to H_n(S^n)$ is the zero map. But since $H_n(\mathrm{id}) = \mathrm{id}$ by (1.2) and $H_n(S^n) \neq 0$, this is a contradiction. ■

Remark 1.4. In fact, Lemma 1.3 is also true for $n = 0$. The 0-dimensional sphere is just $\{-1, 1\}$, which is disconnected. Since $[-1, 1]$ is connected and the image of a connected subset under a continuous map is connected, it follows there does not exist *any* continuous surjective map $r\colon B^1 \to S^0$ (and thus in particular there does not exist a retraction.)

We now show how Theorem 1.1 follows from Lemma 1.3.

Proof of Theorem 1.1. Take $n \geq 0$. Suppose $f\colon B^{n+1} \to B^{n+1}$ has no fixed points. Then for every point $x \in B^{n+1}$, there is a unique line that starts at $f(x)$, goes through x, and then hits a point on the boundary S^n of B^{n+1}. Let us denote by $r\colon B^{n+1} \to S^n$ the map that sends x to the point on S^n that this line hits. See Figure 1.1. Since f is continuous, the map r is also continuous[3]. If $x \in S^n$ then clearly $r(x) = x$. Thus r is a retraction. This contradicts[4] Lemma 1.3. ■

Let us now formalise the notion of a "homology functor", by introducing elements of a field of mathematics called **category theory**. In this course, we will only ever use category theory as a convenient "language" to phrase theorems from algebraic topology in—we will never actually use any genuine theorems in category theory.

[3]This is an easy exercise.

[4]If $n = 0$, apply Remark 1.4 instead of Lemma 1.3.

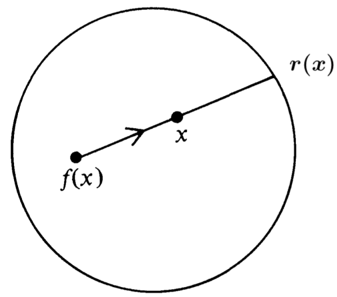

Figure 1.1: The retract r.

Remark 1.5. A word of warning: category theory is often (lovingly) referred to as abstract nonsense. But fear not: nothing we will do will ever be *that* abstract!

Definition 1.6. A **category** C consists of three ingredients. The first is a *class* $\mathrm{obj}(\mathsf{C})$ of **objects**. Secondly, for each ordered pair of objects (A, B) there is a *set* $\mathrm{Hom}(A, B)$ of **morphisms** from A to B. Sometimes instead of $f \in \mathrm{Hom}(A, B)$ we write $f : A \to B$ or $A \xrightarrow{f} B$. Finally, there is a rule, called **composition**, which associates to every ordered triple (A, B, C) of objects a map

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C),$$

written

$$(f, g) \mapsto g \circ f,$$

which satisfies the following three axioms:

1. The Hom sets are pairwise disjoint; that is, each $f \in \mathrm{Hom}(A, B)$ has a unique **domain** A and a unique **target** B.
2. Composition is associative whenever defined, i.e. given

   $$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

   one has

   $$(h \circ g) \circ f = h \circ (g \circ f).$$

3. For each $A \in \mathrm{obj}(\mathsf{C})$ there is a unique morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$ called the *identity* which has the property that $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_B \circ f = f$ for every $f : A \to B$.

Remark 1.7. Note that we said that $\mathrm{obj}(\mathsf{C})$ was a *class* and $\mathrm{Hom}(A, B)$ was a *set*. There is (an important, but technical) difference between a class and a set. If you've ever taken a class on logic/set theory, you'll know that not every "collection" of objects is formally a set. For instance, the collection of all sets is itself not a set! A class is a more general concept (the collection of all sets is a class). Nevertheless, as far as this course is concerned, the distinction is irrelevant, and you are free to ignore this remark!

Here are four examples of categories:

Example 1.8. The category Sets of **sets**. The objects of Sets are all the sets, and $\mathrm{Hom}(A, B)$ is just the set of all functions from A to B, and composition is just the usual composition of functions.

Example 1.9. The category Top of **topological spaces**. The objects of Top are all the topological spaces, and $\mathrm{Hom}(X, Y)$ is just the set $C(X, Y)$ of all *continuous* functions from X to Y, and composition is just the usual composition of functions.

Example 1.10. The category Groups of **groups**. The objects of Groups are just groups, and $\mathrm{Hom}(G, H)$ is just the set $\mathrm{Hom}(G, H)$ of all *homomorphisms* from G to H, and composition is just the usual composition of homomorphisms.

Example 1.11. The category Ab of **abelian groups**. The objects of Ab are just abelian groups, and $\mathrm{Hom}(G, H)$ is again just the set $\mathrm{Hom}(G, H)$ of all *homomorphisms* from G to H, and composition is just the usual composition of homomorphisms.

Remark 1.12. The fact that we require the morphism sets to be pairwise disjoint has several pedantic consequences. For example, suppose $A \subsetneq B$ are two sets. Then the inclusion $\imath \colon A \hookrightarrow B$ and the identity map $\mathrm{id}_A \colon A \to A$ are different morphisms, since they have different targets. One should be aware that we only allow the composition $g \circ f$ when the range of f is *exactly* the same as the domain of g. Suppose X, Y, Y' and Z are topological spaces with $Y \subsetneq Y'$. From the point of view of analysis, say, if $f \colon X \to Y$ and $g \colon Y' \to Z$ are continuous functions then the composition $g \circ f \colon X \to Z$ is clearly a well-defined continuous function. But from the point of view of category theory, the composition $g \circ f$ *does not exist!* Rather, one must first take the inclusion $\imath \colon Y \hookrightarrow Y'$ and then consider the composition $g \circ \imath \circ f$, which is a well-defined element of the morphism space $C(X, Z)$.

A **functor** is a map from one category to another:

Definition 1.13. Suppose C and D are two categories. A **functor** $T \colon \mathsf{C} \to \mathsf{D}$ associates to each $A \in \mathrm{obj}(\mathsf{C})$ an object $T(A) \in \mathrm{obj}(\mathsf{D})$, and to each morphism $A \xrightarrow{f} B$ in C a morphism $T(A) \xrightarrow{T(f)} T(B)$ in D which satisfies the following two axioms:

1. If $A \xrightarrow{f} B \xrightarrow{g} C$ in C then $T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C)$ in D and
$$T(g \circ f) = T(g) \circ T(f).$$
2. $T(\mathrm{id}_A) = \mathrm{id}_{T(A)}$ for every $A \in \mathrm{obj}(\mathsf{C})$.

The easiest example of a functor is a **forgetful functor**:

Example 1.14. The forgetful functor $\mathsf{Top} \to \mathsf{Sets}$ simply "forgets" the topological structure. Thus it assigns to each topological space its underlying set, and to each continuous function it assigns the same function, considered now simply as a map between two *sets* (i.e. it "forgets" the function is continuous).

We can now make sense of the homology functor mentioned earlier.

THEOREM 1.15. *For each $n \geq 0$ there exists a functor $H_n : \mathsf{Top} \to \mathsf{Ab}$ called a* **homology functor** *with the property that for all $n \geq 0$,*

$$H_n(B^{n+1}) = 0, \qquad H_n(S^n) \neq 0.$$

I say "a" homology functor since H_n is not (quite) unique (we will construct several different ones eventually). In fact, before constructing homology functors we will first construct an "easier" functor called the **fundamental group**. This will (almost[5]) be a functor

$$\pi_1 \colon \mathsf{Top} \to \mathsf{Groups},$$

and its construction will take us up the end of Lecture 4.

[5]Strictly speaking π_1 will be a functor from the category of *pointed topological spaces*, more on this later.