Solutions to Problem Sheet A

This Problem Sheet is based on Lecture 1 and Lecture 2. A (†) means I will use the problem in lectures; a ($\star$) means I think the problem is challenging.

Problem A.1. Let X be a topological space. Assume X can be written as an arbitrary union

$$X = \bigcup_i X_i,$$

where each X_i is an open subspace of X. Assume we given a topological space Y and continuous functions

$$f_i \colon X_i \to Y,$$

with the property that

$$f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}, \qquad \forall\, i, j \text{ such that } X_i \cap X_j \neq \emptyset.$$

Then there exists a unique continuous function $f \colon X \to Y$ such that

$$f|_{X_i} = f_i, \qquad \forall\, i \in \mathbb{N}.$$

Solution. First we prove the existence. For any $x \in X$ there exists an i such that $x \in X_i$. Set $f(x) = f_i(x)$. Clearly, f is well-defined, since for $j \neq i$ with $x \in X_j$ we have by assumption that $f_i(x) = f_j(x)$. Since x is arbitrary it suffices to prove continuity of f at x. Note that X_i is open and $f(x) = f|_{X_i}(x) = f_i(x)$. Since f_i is continuous at x and $x \in X_i = \operatorname{int}(X_i)$ it follows that also f is continuous at x. Now suppose that g is another such map with the same properties. Then for every $x \in X$ we have $f(x) = f_i(x) = g(x)$. Hence $f = g$ which proves uniqueness.

Problem A.2 (†). Let C and D be categories and $T \colon \mathsf{C} \to \mathsf{D}$ a functor. Suppose f is an isomorphism in C. Prove that $T(f)$ is an isomorphism in D.

Solution. Let A and B be objects of the category C such that f is a morphism between them. By assumption $f \colon A \to B$ is an isomorphism in C, hence there exists a morphism $g \colon B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Since T is a functor we see that $\mathrm{id}_{T(A)} = T(\mathrm{id}_A) = T(g \circ f) = T(g) \circ T(f)$ and similarly $\mathrm{id}_{T(B)} = T(f) \circ T(g)$. This proves that $T(f)$ is an isomorphism in D.

Problem A.3 (†). Let C and D be categories. Suppose $\sim$ is a congruence on C and $T \colon \mathsf{C} \to \mathsf{D}$ is a functor. Assume that whenever $f \sim g$ one has $T(f) = T(g)$. Prove that T induces a functor $T' \colon \mathsf{C}' \to \mathsf{D}$, where C' denotes the quotient category.

Solution. On objects of the category C the functor T' is equal to T. On morphisms we define $T'([f]) \colon = T(f)$. This is well-defined, indeed for $[f] = [g]$ we have that $T(f) = T(g)$ by assumption. We need to show that T' satisfies the properties of a functor. Clearly $T'([g] \circ [f]) = T'([g \circ f]) = T(g \circ f) = T(g) \circ T(f) = T'([g]) \circ T'([f])$ since T is a functor, and for any object $A \in \mathsf{C}$, $T'([\mathrm{id}_A]) = T(\mathrm{id}_A) = \mathrm{id}_{T(A)} = \mathrm{id}_{T'(A)}$.

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Problem A.4 (†). Show that a topological space X has the same homotopy type as a point if and only if X is contractible.

Solution.

- "$\Rightarrow$" There exists $f\colon X \to \{*\}$ and $g\colon \{*\} \to X$ continuous such that $g \circ f \simeq \mathrm{id}_X$. But $g \circ f\colon X \to \{*\} \to X$ is necessarily a constant map. Hence id_X is homotopic to a constant map, which proves that X is contractible.
- "$\Leftarrow$" Let $c\colon X \to X$ be the constant map sending every point $x \in X$ to a fixed point $q \in X$ and assume $\mathrm{id}_X \simeq c$. Define $f\colon X \to \{*\}$ the constant map and $g\colon \{*\} \to X$ the constant map sending $*$ to q. Clearly $g \circ f = c \simeq \mathrm{id}_X$ and $f \circ g = \mathrm{id}_{\{*\}}$. This shows that X has the homotopy type of a point.

Problem A.5. Let X a topological space. Define an equivalence relation on $X \times I$ by $(x,t) \sim (x',t')$ if $t = t' = 1$. Let CX denote the quotient space $(X \times I)/\sim$. We call CX the **cone** on X. Prove that CX is always contractible, and deduce that any topological space can be embedded inside a contractible one.

Solution. Let $c\colon CX \to CX$ denote the constant map sending every point to the equivalence class $[x,1]$. (Note that $[x,1] = [x',1]$ for any two points x and x' in X.) We define the homotopy $H\colon CX \times [0,1] \to CX$ by $H([x,t],s) := [x, s + (1-s)t]$. One can see that H is a homotopy between id_{CX} and c, which proves that CX is contractible. Moreover, every topological space X can be embedded into the contractible space CX via the map $i\colon X \to CX$ given by $x \mapsto [x,0]$.