LECTURE 7

Singular homology

In this lecture we finally get started on defining the *homology functors* H_n referred to in Lecture 1. Let us begin with some preliminaries on free abelian groups.

Definition 7.1. Let B be a subset of an abelian group F. We say F is **free abelian** with **basis** B if the subgroup generated by b is infinite cyclic for each $b \in B$ and $F = \bigoplus_{b \in B} \langle b \rangle$ as a direct sum.

Thus a free abelian group is a (possibly uncountable) direct sum of copies of $\mathbb{Z}$. A typical element $x \in F$ has a unique expression

$$x = \sum_{b \in B} m_b \, b, \qquad m_b \in \mathbb{Z}$$

where **almost all** (meaning all but a finite number) of the m_b are zero. The following trivial lemma will be crucial in all that follows.

Lemma 7.2. *Let F be a free abelian group with basis B. If A is an abelian group and $\phi \colon B \to A$ is a function then there exists a unique group homomorphism $\tilde{\phi} \colon F \to A$ such that*

$$\tilde{\phi}(b) = \phi(b), \qquad \forall \, b \in B,$$

that is, the following diagram commutes:

$$\begin{tikzcd} F \arrow[dr, dashed, "\tilde{\phi}"] & \\ B \arrow[u, hook] \arrow[r, "\phi"'] & A \end{tikzcd}$$

Moreover any abelian group A is isomorphic to a quotient group of the form F/R, where F is a free abelian group.

Proof. Define $\tilde{\phi}$ by

$$\tilde{\phi}\left(\sum_{b \in B} m_b \, b \right) := \sum_{b \in B} m_b \, \phi(b).$$

Then $\tilde{\phi}$ is well-defined since any element of F has a unique expression of this form, and it is obviously a homomorphism. Moreover $\tilde{\phi}$ is unique since any two homomorphisms that agree on a set of generators (in this case B) must coincide. The last statement is on Problem Sheet D. ∎

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We refer to the extension $\phi \mapsto \tilde{\phi}$ given in Lemma 7.2 as **extending by linearity**. By an abuse of notation, we will typically continue to write ϕ for the extension, rather than $\tilde{\phi}$.

LEMMA 7.3. *Given any set B, there exists a free abelian group having B as basis.*

Proof. If $B = \emptyset$, take $F = 0$. Otherwise, for each $b \in B$, let $\mathbb{Z}_b$ be a group whose elements are all symbols mb with $m \in \mathbb{Z}$ and addition defined by $mb + nb = (m+n)b$. Then $\mathbb{Z}_b$ is infinite cyclic with generator b. Now set

$$F := \bigoplus_{b \in B} \mathbb{Z}_b.$$

This is a free abelian group with basis given by the set $\{e_b \mid b \in B\}$, where e_b has a zero in each entry apart from the bth entry, where it is a 1. Identifying e_b with b, we see that F has basis B. ■

DEFINITION 7.4. The **rank** of a free abelian group F is the cardinality of any basis of B of F.

This is well-defined thanks to Problem D.1. Moreover two free abelian groups are isomorphic if and only if they have the same rank.

REMARK 7.5. One can extend the notion of rank to any abelian group: if G is an arbitrary abelian group then we say G has (possibly infinite) **rank** r if there exists a free abelian subgroup F of G such that F has rank r and G/F is torsion. Such subgroups F always exist (this is also part of Problem D.1). However it is not obvious that this definition is well-defined. Indeed, F is not unique, and it is by no means clear that the rank of F only depends on G. At the very end of the course we will develop one way of proving this.

Now let us define the notion of a simplex.

DEFINITION 7.6. An ordered tuple $(z_0, z_1, \ldots, z_n)$ of points in $\mathbb{R}^m$ is said to be **affinely independent** if the set $\{z_1 - z_0, z_2 - z_0, \ldots, z_n - z_0\}$ is linearly independent (thus necessarily $n \le m$). Given an affinely independent tuple $(z_0, z_1, \ldots, z_n)$ of vectors in $\mathbb{R}^m$, we denote by $[z_0, z_1, \ldots, z_n]$ the n**-simplex spanned by** $(z_0, z_1, \ldots, z_n)$, namely the set

$$[z_0, z_1, \ldots, z_n] := \left\{ x \in \mathbb{R}^m \mid x = \sum_{i=0}^{n} s_i\, z_i, \text{ where } 0 \le s_i \le 1, \ \sum_{i=0}^{n} s_i = 1 \right\}.$$

We call the points z_i the **vertices** of the n-simplex $[z_0, z_1, \ldots, z_n]$. The expression $x = \sum_{i=0}^{n} s_i\, z_i$ of any point $x \in [z_0, z_1, \ldots, z_n]$ is unique[1]. We call the $(n+1)$-tuple $(s_0, s_1, \ldots, s_m)$ the **barycentric coordinates** of x. The **barycentre** of the n-simplex $[z_0, z_1, \ldots, z_n]$ is the unique point where all the s_i are equal, namely

$$\frac{1}{n+1}(z_0 + z_1 + \cdots + z_n). \tag{7.1}$$

[1]Exercise: Why?

DEFINITION 7.7. Let $[z_0, z_1, \dots, z_n]$ be an n-simplex. The **face opposite to** z_i is the $(n-1)$-simplex[2] $[z_0, \dots \hat{z_i}, \dots, z_n]$. Here the circumflex $\hat{}$ means[3] "delete". Equivalently

$$[z_0, \dots \hat{z_i}, \dots, z_n] := \{x \in [z_0, z_1, \dots, z_n] \mid s_i = 0.\}.$$

An n-simplex thus has $n+1$ faces. The **boundary** of an n-simplex is the union of its faces.

DEFINITION 7.8. The **standard n-simplex** in $\mathbb{R}^{n+1}$ is the n-simplex $[e_0, e_1, \dots, e_n]$, where e_i is the vector coordinates are all zero, apart from the $i+1$st position, which is 1. We denote the standard n-simplex by Δ^n.

So much for a simplex in $\mathbb{R}^{n+1}$. What about in an arbitrary topological space X?

DEFINITION 7.9. Let X be a topological space. A **singular n-simplex in X** is a continuous map $\sigma \colon \Delta^n \to X$.

Since Δ^0 is a point, a 0-simplex in X is simply a point in X. Since Δ^1 is a closed interval, a 1-simplex is[4] the same thing as a path in X. The adjective "singular" is added to emphasis that the image $\sigma(\Delta^n)$ does not need to "look" anything like Δ^n, i.e. we do *not* require σ to be a homeomorphism. In particular, there is nothing stopping σ being a constant map.

DEFINITION 7.10. Let X be a topological space and $n \geq 0$. Let $C_n(X)$ denote the free abelian group with basis the singular n-simplices in X (cf. Lemma 7.3.) We call an element of $C_n(X)$ a **singular n-chain**. It is convenient for notational reasons to also define $C_{-1}(X) = 0$.

Note that (as a group), $C_n(X)$ is typically huge: if X is an uncountable set then $C_n(X)$ is itself uncountable for all $n \geq 0$. We will shortly replace $C_n(X)$ with a (usually smaller) abelian group $H_n(X)$. First, let us explain how to obtain a singular $(n-1)$-simplex from a singular n-simplex.

DEFINITION 7.11. Let $\sigma \colon \Delta^n \to X$ be a singular n-simplex in X. If we restrict σ to one of the faces of Δ^n, we get a continuous map from an $n-1$-simplex into X.

Actually this definition is cheating a little bit; whilst any face of Δ^n is an $(n-1)$-simplex, it is not the *standard* $(n-1)$-simplex, since the domain is wrong. Thus strictly speaking, the restriction of a n-simplex σ in X to a face is not actually a singular $(n-1)$-simplex in X, since it is not a continuous map from Δ^{n-1} into X. There are two ways round this tedious pedantry:

1. Ignore it. After all, it's clear what we mean.
2. Fix it by making the notation more complicated.

[2] This is clearly an $(n-1)$-simplex as a subset of a linearly independent set is also linearly independent.
[3] This is a convention we will use throughout the course.
[4] Not quite! We will come back to this in Lecture 9.

We shall go[5] for option (2). To this end, let us define the i**th face map**

$$\varepsilon_i\colon \Delta^{n-1} \to \Delta^n, \qquad i = 0, 1, \ldots, n$$

that maps the standard $(n-1)$-simplex Δ^{n-1} homeomorphically onto the ith face of Δ^n. Explicitly,

$$\varepsilon_0(s_0, s_1, \ldots s_{n-1}) = (0, s_0, s_1, \ldots s_{n-1}),$$

for $i = 0$, and for $1 \le i \le n-1$,

$$\varepsilon_i(s_0, s_1, \ldots s_{n-1}) = (s_0, s_1, \ldots, s_{i-1}, 0, s_i, \ldots, s_{n-1}),$$

and finally

$$\varepsilon_n(s_0, s_1, \ldots, s_{n-1}) = (s_0, s_1, \ldots, s_{n-1}, 0).$$

Where necessary we will write $\varepsilon_i^n\colon \Delta^{n-1} \to \Delta^n$ (this is needed for instance in (7.2) below).

We can now "improve" Definition 7.11:

Definition 7.12. Let $\sigma\colon \Delta^n \to X$ be a singular n-simplex in X and let $0 \le i \le n$. The composition $\sigma \circ \varepsilon_i\colon \Delta^{n-1} \to X$ is then a singular $(n-1)$-simplex in X, which we call the **restriction of σ to the ith face**.

We can now define the boundary of a singular n-simplex.

Definition 7.13. Let $\sigma\colon \Delta^n \to X$ be a singular n-simplex in X. The **boundary** of σ is the alternating sum of the restriction of σ to the faces:

$$\partial\sigma := \sum_{i=0}^{n} (-1)^i \sigma \circ \varepsilon_i.$$

Thus the boundary of σ is *not* a singular $(n-1)$-simplex, but rather a formal *sum* of singular $(n-1)$-simplices, and hence (by definition) a singular $(n-1)$-chain: $\partial\sigma \in C_{n-1}(X)$. We define the boundary of a singular 0-simplex to be zero.

Remark 7.14. If we omit the face maps (which we will occasionally do, cf. in Proposition 8.5 next lecture), the formula is slightly more intuitive (albeit formally incorrect):

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma|_{[e_0, \ldots, \hat{e}_i, \ldots, e_n]}.$$

Applying Lemma 7.2 we obtain a well defined map on the free abelian group $C_n(X)$.

[5]Being pedantic is an important quality for a mathematician to have (or at least, to pretend to have when teaching others ...)

Definition 7.15. The **singular boundary operator**

$$\partial\colon C_n(X) \to C_{n-1}(X)$$

is the unique homomorphism extending the operator from Definition 7.13. Occasionally for clarity we will write $\partial_n\colon C_n(X) \to C_{n-1}(X)$.

Thus for each $n \geq 0$ we have constructed a sequence of free abelian groups and homomorphisms. We illustrate this pictorially as

$$\cdots \longrightarrow C_n(X) \xrightarrow{\partial} C_{n-1}(X) \longrightarrow \cdots \longrightarrow C_1(X) \xrightarrow{\partial} C_0(X) \longrightarrow 0.$$

Anticipating the category Comp of *chain complexes* that we will introduce in Lecture 10, we will bundle all the groups $C_n(X)$ together and write $(C_\bullet(X), \partial)$ to denote all the groups and maps at once.

Proposition 7.16. *$\partial^2 = 0$, that is, for any $n \geq 0$ the composition*

$$C_{n+1}(X) \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} C_{n-1}(X)$$

is always zero.

Proof. Since $C_{n+1}(X)$ is generated by all the $(n+1)$-simplices, by Lemma 7.2 it suffices to show that if $\sigma\colon \Delta^{n+1} \to X$ is a singular $(n+1)$-simplex then $\partial^2\sigma = 0$. As you can probably guess, the point it that since the boundary operator was defined via an alternating sum, when you apply it twice things cancel. Indeed, if $k < j$ then one has the following *face relation*:

$$\varepsilon_j^{n+1} \circ \varepsilon_k^n = \varepsilon_k^{n+1} \circ \varepsilon_{j-1}^n\colon \Delta^{n-1} \to \Delta^{n+1}. \tag{7.2}$$

To prove (7.2), it suffices to observe that both sides give the same answer when fed a vertex e_i for $i = 0, 1, \ldots, n-1$. Now we compute:

$$\begin{aligned}
\partial^2\sigma &= \partial\left(\sum_{j=0}^{n+1}(-1)^j \sigma \circ \varepsilon_j^{n+1}\right) \\
&= \sum_{k=0}^{n}\sum_{j=0}^{n+1}(-1)^{j+k}\sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n \\
&= \underbrace{\sum_{j\leq k}(-1)^{j+k}\sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n}_{(*)} + \underbrace{\sum_{k<j}(-1)^{j+k}\sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n}_{(\dagger)}.
\end{aligned}$$

We claim that the two terms $(*)$ and $(\dagger)$ cancel. Indeed, to see this first apply (7.2) to $(**)$ and change variables by setting $l = k$ and $m = j - 1$ to obtain:

$$\sum_{k<j}(-1)^{j+k}\sigma \circ \varepsilon_j^{n+1} \circ \varepsilon_k^n = \sum_{k<j}(-1)^{j+k}\sigma \circ \varepsilon_k^{n+1} \circ \varepsilon_{j-1}^n = \sum_{l\leq m}(-1)^{l+m+1}\sigma \circ \varepsilon_l^{n+1} \circ \varepsilon_m^n.$$

The last expression is the same as $(*)$, only every term appears with the opposite sign. This completes the proof. ∎

DEFINITION 7.17. A **singular n-cycle** in X is a singular n-chain that lies in the kernel of ∂. We denote by $Z_n(X)$ the set of all singular n-cycles. A **singular n-boundary** in X is a singular n-chain that lies in the image of ∂. We denote by $B_n(X)$ the set of all singular n-boundaries[6]. Both $Z_n(X)$ and $B_n(X)$ are subgroups of $C_n(X)$. Moreover since $\partial^2 = 0$, we have

$$B_n(X) \subseteq Z_n(X) \subseteq C_n(X).$$

We can therefore form the quotient group. This will be the eponymous *singular homology*.

DEFINITION 7.18. We define the **n-singular homology** group of X, written $H_n(X)$, to be the quotient group

$$H_n(X) = Z_n(X) \big/ B_n(X).$$

Thus $H_n(X)$ is an abelian (not free abelian!) group for each n. Given a singular n-cycle c, we denote[7] by $\langle c \rangle$ the coset $c + B_n(X) \in H_n(X)$ and call $\langle c \rangle$ the **homology class** determined by c.

We will conclude this lecture by showing that H_n is a functor. This means that we need to associate to each continuous map $f \colon X \to Y$ a homomorphism $H_n(f) \colon H_n(X) \to H_n(Y)$.

DEFINITION 7.19. If $f \colon X \to Y$ is a continuous map and $\sigma \colon \Delta^n \to X$ is a singular n-simplex in X then $f \circ \sigma \colon \Delta^n \to Y$ is a singular n-simplex in Y. We therefore obtain an induced map $f_n^{\#} \colon C_n(X) \to C_n(Y)$ by extending this by linearity (Lemma 7.2:

$$f_n^{\#}\left(\sum m_\sigma \, \sigma\right) := \sum m_\sigma f \circ \sigma.$$

Just as with the entire complex $(C_\bullet(X), \partial)$, we can bundle all the maps $f_n^{\#}$ together and write $f_\bullet^{\#}$. You will not be surprised to learn that $f \mapsto f_\bullet^{\#}$ is a functor. We will study this in Lecture 10 (it's a functor $\mathsf{Top} \to \mathsf{Comp}$.) For now though, let us prove that $f_n^{\#}$ descends to the quotient to define a map on $H_n(X) \to H_n(Y)$. This is the content of the following proposition.

PROPOSITION 7.20. *If $f \colon X \to Y$ is continuous, then the following diagrams commutes for every n:*

$$\begin{tikzcd} C_n(X) \arrow[r, "\partial"] \arrow[d, "f_n^{\#}"'] & C_{n-1}(X) \arrow[d, "f_{n-1}^{\#}"] \\ C_n(Y) \arrow[r, "\partial"'] & C_{n-1}(Y) \end{tikzcd}$$

[6]Boundary begins with a "b", hence the notation $B_n(X)$. Similarly cycle begins with a "c", hence the notation ... wait a second ... Damnit, we already used $C_n(X)$ for the chain groups! Next best option: *Zykel* begins with a "z" ...

[7]We use angle brackets $\langle \cdot \rangle$ rather than square brackets $[\cdot]$ to distinguish between homotopy and homology classes.

Proof. It suffice to evaluate both $f^\#_{n-1} \circ \partial$ and $\partial \circ f^\#_n$ on a singular n-simplex σ in X. Now

$$f^\#_{n-1} \circ \partial\sigma = f^\#_{n-1}\left(\sum_{i=0}^{n}(-1)^i \sigma \circ \varepsilon_i\right) = \sum_{i=0}^{n}(-1)^i f \circ \sigma \circ \varepsilon_i.$$

Similarly

$$\partial \circ f^\#_n \sigma = \partial(f \circ \sigma) = \sum_{i=0}^{n}(-1)^i f \circ \sigma \circ \varepsilon_i.$$

■

Corollary 7.21. *If $f\colon X \to Y$ is continuous then both $f^\#_n(Z_n(X)) \subseteq Z_n(Y)$ and $f^\#_n(B_n(X)) \subseteq B_n(Y)$. Thus $f^\#_n$ induces a map $H_n(f)\colon H_n(X) \to H_n(Y)$.*

Proof. If $\partial c = 0$ then $\partial(f^\#_n c) = f^\#_{n-1}(\partial c) = 0$, so that $f^\#_n c \in Z_n(Y)$. Similarly if $b = \partial c$ then $f^\#_n b = f^\#_n(\partial c) = \partial(f^\#_{n+1} c)$, so that $f^\#_n b \in B_n(Y)$. ■

Thus $f^\#_n$ induces a map $H_n(f)\colon H_n(X) \to H_n(Y)$, given by

$$H_n(f)\langle c\rangle := \langle f^\#_n c\rangle.$$

Corollary 7.22. *For each $n \geq 0$, $H_n\colon \mathsf{Top} \to \mathsf{Ab}$ is a functor.*

Proof. We need only check that $H_n(g \circ f) = H_n(g) \circ H_n(f)$ and that $H_n(\mathrm{id}_X) = \mathrm{id}_{H_n(X)}$. Both of these are immediate from the definitions. ■

Corollary 7.23. *If X and Y are homeomorphic then $H_n(X) \cong H_n(Y)$ for all $n \geq 0$.*

Proof. Immediate from Problem A.2. ■

Thinking back to Lecture 1, we have now constructed the singular homology functors from Theorem 1.15. In order for our proof of the Brouwer Fixed Point Theorem 1.1 to be complete, we need to verify that $H_n(B^{n+1}) = 0$ and $H_n(S^n) \neq 0$. We will prove that $H_n(B^{n+1}) = 0$ next lecture; the fact that $H_n(S^n) \neq 0$ will take much longer (Lecture 15).