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The Fundamental Grupoid

π_0.

Lemma 1.1. *There exists a functor* $\mathsf{Top} \to \mathsf{Set}$. *Moreover, if* $f, g \in \mathsf{Top}(X, Y)$ *are freely homotopic, then* $\pi_0(f) = \pi_0(g)$.

Proof. On objects $X \in \mathrm{ob}(\mathsf{Top})$, define $\pi_0(X)$ to be the set of equivalence classes of X under path connectedness. On morphisms $f : X \to Y$, define $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ by $\pi_0(f)\,[x] := [f(x)]$. This is well defined since if $[x] = [y]$, there exists a path u from x to y in X and it is easy to check that $f \circ u$ is a path from $f(x)$ to $f(y)$. Checking that π_0 is indeed a functor is left as an exercise. Suppose $H : f \simeq g$ and let $x \in X$. Then $H(x, t)$ is a path from $f(x)$ to $g(x)$ and thus $\pi_0(f)\,[x] = [f(x)] = [g(x)] = \pi_0(g)\,[x]$. □

Exercise 1.2. Check the functoriality of $\pi_0 : \mathsf{Top} \to \mathsf{Set}$.

Proposition 1.3. *If* $X, Y \in \mathrm{ob}(\mathsf{Top})$ *have the same homotopy type, then* $|\pi_0(X)| = |\pi_0(Y)|$, *i.e.* X *and* Y *have the same number of path components.*

Proof. Since X and Y are of the same homotopy type, they are isomorphic in hTop. By lemma 1.1, π_0 descends to hTop and since functors preserve isomorphisms, we have that $\pi_0(X) \cong \pi_0(Y)$. In Set, isomorphisms are bijections and thus the statement follows. □

Construction of the Fundamental Grupoid.

Lemma 1.4 (Gluing Lemma). *Let* $X, Y \in \mathrm{ob}(\mathsf{Top})$, $(X_\alpha)_{\alpha\in A}$ *a finite closed cover of* X *and* $(f_\alpha)_{\alpha\in A}$ *a finite family of maps* $f_\alpha \in \mathsf{Top}(X_\alpha, Y)$ *such that* $f_\alpha|_{X_\alpha\cap X_\beta} = f_\beta|_{X_\alpha\cap X_\beta}$ *for all* $\alpha, \beta \in A$. *Then there exists a unique* $f \in \mathsf{Top}(X, Y)$ *such that* $f|_{X_\alpha} = f_\alpha$ *for all* $\alpha \in A$.

Proof. Let $x \in X$. Since $(X_\alpha)_{\alpha\in A}$ is a cover of X, we find $\alpha \in A$ such that $x \in X_\alpha$. Define $f(x) := f_\alpha(x)$. This is well defined, since if $x \in X_\alpha \cap X_\beta$ for some $\beta \in A$, we have that $f(x) = f_\beta(x) = f_\alpha(x)$. Clearly $f|_{X_\alpha} = f_\alpha$ for all $\alpha \in A$ and f is unique. Let us show continuity. To this end, let $K \subseteq Y$ be closed. Then

$$\begin{aligned} f^{-1}(K) &= X \cap f^{-1}(K) \\ &= \bigcup_{\alpha \in A} X_\alpha \cap f^{-1}(K) \\ &= \bigcup_{\alpha \in A} \left(X_\alpha \cap f^{-1}(K)\right) \\ &= \bigcup_{\alpha \in A} \left(X_\alpha \cap f_\alpha^{-1}(K)\right). \end{aligned}$$

Since each f_α is continuous, $f_\alpha^{-1}(K)$ is closed in X_α for each $\alpha \in A$ and thus since X_α is closed, $f^{-1}(K)$ is closed as a finite union of closed sets. □

Theorem 1.5. *There is a functor* $\mathsf{Top} \to \mathsf{Grpd}$.

Proof. The proof is divided into several steps. Let us denote $\Pi : \mathsf{Top} \to \mathsf{Grpd}$ for the claimed functor.

Step 1: Definition of Π *on objects.* Let $X, Y \in \mathrm{ob}(\mathsf{Top})$, $f, g \in \mathsf{Top}(X, Y)$ and $A \subseteq X$. A map $F \in \mathsf{Top}(X \times I, Y)$ is called a ***homotopy from X to Y relative to A***, if

- $F(x, 0) = f(x)$, for all $x \in X$.
- $F(x, 1) = g(x)$, for all $x \in X$.
- $F(x, t) = f(x) = g(x)$, for all $x \in A$ and for all $t \in I$.

If there exists a homotopy between f and g relative to A we say that f and g are ***homotopic relative to A*** and write $f \simeq_A g$. If we want to emphasize the homotpoy relative to A, we write $F : f \simeq_A g$.

Lemma 1.6. *Let $X, Y \in \mathrm{ob}(\mathsf{Top})$ and $A \subseteq X$. Then being homotopic relative to A is an equivalence relation on* $\mathsf{Top}(X, Y)$.

Proof. Define a binary relation $R_A \subseteq \mathsf{Top}(X, Y) \times \mathsf{Top}(X, Y)$ by

$$f R_A g \quad :\Leftrightarrow \quad f \simeq_A g.$$

Let $f \in \mathsf{Top}(X, Y)$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x, t) := f(x).$$

Then clearly $F : f \simeq_A f$. Hence R_A is reflexive.
Let $g \in \mathsf{Top}(X, Y)$ and assume that $f R_A g$. Thus $G : f \simeq_A g$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x, t) := G(x, 1 - t).$$

Then it is easy to check that $F : g \simeq_A f$ and so R_A is symmetric.
Finally, let $h \in \mathsf{Top}(X, Y)$ and suppose that $f R_A g$ and $g R_A h$. Hence $F_1 : f \simeq_A g$ and

$F_2 : g \simeq_A h$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x,t) := \begin{cases} F_1(x,2t) & 0 \le t \le \frac{1}{2}, \\ F_2(x,2t-1) & \frac{1}{2} \le t \le 1. \end{cases}$$

Continuity of F follows by an application of the gluing lemma 1.4. Then it is easy to check that $F : f \simeq_A h$ and hence R_A is transitive. □

Let $X \in \mathrm{ob}(\mathsf{Top})$ and u a path in X from p to q. Define the ***path class*** $[u]$ ***of*** u by $[u] := [u]_{R_{\partial I}}$. Define now

- $\mathrm{ob}\big(\Pi(X)\big) := X$.
- $\Pi(X)(p,q) := \{[u] : u \text{ is a path from } p \text{ to } q\}$ for all $p, q \in X$.
- Let $p \in X$. Then define $\mathrm{id}_p \in \Pi(X)(p,p)$ by $\mathrm{id}_p := [c_p]$, where c_p is the constant path defined by $c_p(s) := p$ for all $s \in I$.
- And $\Pi(X)(q,r) \times \Pi(X)(p,q) \to \Pi(X)(p,r)$ by

  $$([v], [u]) \mapsto [u * v]$$

  Where $u * v \in \mathsf{Top}(p,r)$ is the ***concatenated path of u and v***, defined by

  $$(u * v)(s) := \begin{cases} u(2s) & 0 \le t \le \frac{1}{2}, \\ v(2s-1) & \frac{1}{2} \le t \le 1 \end{cases}.$$

  Continuity follows again from the gluing lemma 1.4 whereas well definedness follows from the next lemma.

Lemma 1.7. *Suppose that* $[u_1], [u_2] \in \Pi(X)(p,q)$ *and* $[v_1], [v_2] \in \Pi(X)(q,r)$ *such that* $[u_1] = [u_2]$ *and* $[v_1] = [v_2]$. *Then* $[u_1 * v_1] = [u_2 * v_2]$.

Proof. By assumption we have $G : u_1 \simeq_{\partial I} u_2$ and $H : v_1 \simeq_{\partial I} v_2$. Define $F \in \mathsf{Top}(I \times I, X)$ by

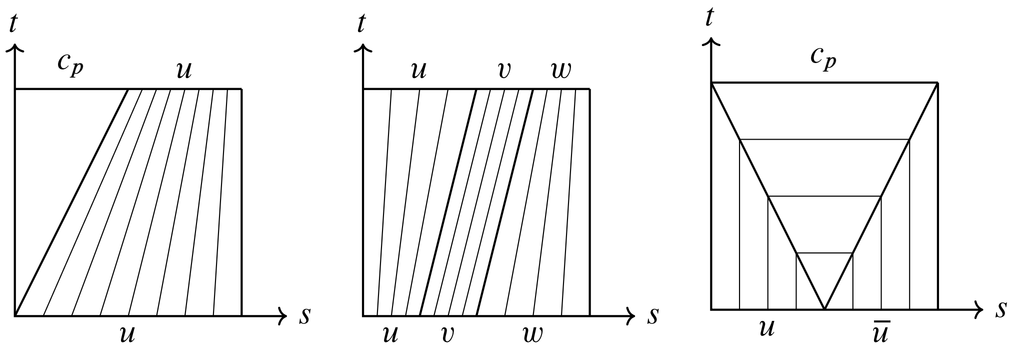
$$F(s,t) := \begin{cases} G(2s,t) & 0 \le s \le \frac{1}{2}, \\ H(2s-1,t) & \frac{1}{2} \le s \le 1. \end{cases}$$

Again, continuity follows from the gluing lemma 1.4 and it is easy to check that $F : u_1 * v_1 \simeq_{\partial I} u_2 * v_2$. □

Let us now check that $\Pi(X)$ is indeed a category. Let $[u] \in \Pi(X)(p,q)$. We want to show that $u \simeq_{\partial I} c_p * u$. To this end, we consider figure 1a and conclude that a suitable homotopy is given by $F \in \mathsf{Top}(I \times I, X)$ defined by

$$F(s,t) := \begin{cases} p & 0 \le 2s \le t, \\ u\left(\dfrac{2s-t}{2-t}\right) & t \le 2s \le 2. \end{cases}$$

Similarly, considering figure 1b leads to $F \in \mathsf{Top}(I \times I, X)$ defined by



(a) $u \simeq_{\partial I} c_p * u$. (b) $(u*v)*w \simeq_{\partial I} u*(v*w)$. (c) $u * \overline{u} \simeq_{\partial I} c_p$.

Figure 1. Visualization of the proof that $\Pi(X)$ is a grupoid object.

$$F(s,t) := \begin{cases} u\left(\dfrac{4s}{t+1}\right) & -1 \leq 4s-1 \leq t, \\ v(4s-t-1) & t \leq 4s-1 \leq t+1, \\ w\left(\dfrac{4s-t-2}{4-t-2}\right) & t+1 \leq 4s-1 \leq 3. \end{cases}$$

Lastly, we check that $\Pi(X)$ is a grupoid. To this end, for a path u from p to q, define its ***reverse path*** $\overline{u}$ by

$$\overline{u}(s) := u(1-s).$$

We claim that $u * \overline{u} \simeq_{\partial I} c_p$. From figure 1c we deduce that $F \in \mathsf{Top}(I \times I, X)$ is given by

$$F(s,t) := \begin{cases} u(2s) & 0 \leq 2s \leq 1-t, \\ u(1-t) & 1-t \leq 2s \leq t+1, \\ \overline{u}(2s-1) & t+1 \leq 2s \leq 2. \end{cases}$$

Step 2: Definition of Π *on morphisms.* Let $f \in \mathsf{Top}(X,Y)$. Then $\Pi(f)$ is a functor from $\Pi(X)$ to $\Pi(Y)$. Define $\Pi(f)$ as follows:
- Let $p \in \mathrm{ob}\big(\Pi(X)\big)$. Then define $\Pi(f)(p) := f(p) \in \mathrm{ob}\big(\Pi(Y)\big)$.
- Let $[u] \in \Pi(X)(p,q)$. Then define $\Pi(f)\,[u] := [f \circ u] \in$. We have to check that this definition is independent of the choice of the representative.

Lemma 1.8. *Let u and v be paths from p to q in X and suppose that $[u] = [v]$. Then for any $f \in \mathsf{Top}(X,Y)$ we also have that $[f \circ u] = [f \circ v]$.*

Proof. Suppose that $H : u \simeq_{\partial I} v$. Define $F \in \mathsf{Top}(I \times I, Y)$ by

$$F(s,t) := (f \circ F)(s,t).$$

Then $F : f \circ u \simeq_{\partial I} f \circ v$. □

Checking that Π satisfies the functorial properties is left as an exercise. $\square$

Exercise 1.9. Check that $\Pi : \mathsf{Top} \to \mathsf{Grpd}$ is indeed a functor.

Definition 1.10 (Free Homotopy). *Let $f, g \in \mathsf{Top}(X, Y)$. f and g are said to be **freely homotopic** if $f \simeq_{\varnothing} g$.*

Example 1.11 (Straight Line Homotopy). Let V be a real vector space. A subset $S \subseteq V$ is said to be ***convex***, if the line segment $\{(1-t)p + tq : 0 \leq t \leq 1\}$ is contained in S for all $p, q \in V$. Suppose now that V is finite dimensional and $S \subseteq V$ is convex. Moreover, let $f, g \in \mathsf{Top}(X, S)$ for some $X \in \mathrm{ob}(\mathsf{Top})$. Define $H : X \times I \to S$ by

$$H(x,t) := (1-t)f(x) + tg(x).$$

Then H is continuous and clearly $H : f \simeq g$. We call H the ***straight line homotopy between f and g***. Hence any two continuous maps defined on the same domain into a convex space are freely homotopic.

Remark 1.12. We will also write $f \simeq g$ for a free homotopy.

Definition 1.13 (Nullhomotopic). *A mapping $f \in \mathsf{Top}(X, Y)$ is said to be **nullhomotopic**, if f is freely homotopic to a constant map.*

Definition 1.14 (Contractible). *A topological space X is said to be **contractible**, if id_X is nullhomotopic.*

Definition 1.15 (Reparametrization). *Let u be a path in a topological space X. A **reparametrization** of u is a path $u \circ \varphi$, where $\varphi \in \mathsf{Top}(I, I)$ fixing 0 and 1.*

Lemma 1.16. *let u be a path in a topological space x and $u \circ \varphi$ a reparametrization of u. Then $u \simeq_{\partial I} u \circ \varphi$.*

Proof. Since I is convex, we find a straight line homotopy $H : \mathrm{id}_I \simeq \varphi$. Now $u \circ H$ is the homotopy we are looking for. $\square$

The Fundamental Group.

Lemma 1.17. *Let $\mathcal{G}$ be a locally small grupoid. Then for every $X \in \mathrm{ob}(\mathcal{G})$, $\mathcal{G}(X, X)$ can be equipped with a group structure.*

Proof. Since $\mathcal{G}$ is locally small, $\mathcal{G}(X, X)$ is a set for every $X \in \mathrm{ob}(\mathcal{G})$. Define a multiplication $\mathcal{G}(X, X) \times \mathcal{G}(X, X) \to \mathcal{G}(X, X)$ by $gh := h \circ g$. Clearly, this multiplication is associative. Moreover, the identity element is given by $\mathrm{id}_X \in \mathcal{G}(X, X)$ and since every $g \in \mathcal{G}(X, X)$ is an isomorphism, the multiplicative inverse is given by the inverse in $\mathcal{G}(X, X)$. $\square$

Proposition 1.18. *There is a functor $\mathsf{Top}_* \to \mathsf{Grp}$.*

Proof. Define $\pi_1 : \mathsf{Top}_* \to \mathsf{Grp}$ on objects $(X, p) \in \mathsf{Top}_*$ by

$$\pi_1(X, p) := \Pi(X)(p, p).$$

By theorem 1.5 together with lemma 1.17, $\pi_1(X, p)$ is actually a group, called the ***fundamental group of X with basepoint p***. On morphisms $f \in \mathsf{Top}_*\big((X, p), (Y, q)\big)$, define

$$\pi_1(f) := \Pi(f) : \Pi(X)(p, p) \to \Pi(Y)(q, q).$$

Let $[u], [v] \in \pi_1(X, p)$. Then

$$\begin{aligned}
\pi_1([u]\,[v]) &= \Pi(f)([u]\,[v]) \\
&= \Pi(f)\,[u * v] \\
&= [f \circ (u * v)] \\
&= [(f \circ u) * (f \circ v)] \\
&= \Pi(f)\,[u]\,\Pi(f)\,[v] \\
&= \pi_1(f)\,[u]\,\pi_1(f)\,[v]\,.
\end{aligned}$$

Thus $\pi_1(f)$ is a morphism in Grp. Functoriality of π_1 immediately follows from the functoriality of Π. $\square$

Definition 1.19 (Simply Connected). *A path connected topologial space X is said to be **simply connected**, if $\pi_1(X)$ is trivial.*

First Properties of the Fundamental Group.

Lemma 1.20. *Let $X \in \mathrm{ob}(\mathsf{Top})$, $p \in X$ and A be the path component of X containing p. Then $\pi_1(\iota)$, where $\iota : A \hookrightarrow X$ denotes the inclusion, is an isomorphism.*

Proof. Suppose $[u] \in \ker \pi_1(\iota)$. Then $[\iota \circ u] = [c_p]$ and Hence $F : \iota \circ u \simeq_{\partial I} c_p$. Since $I \times I$ is path connected and $p \in F(I \times I)$, it follows that $F(I \times I) \subseteq A$ and thus $F : u \simeq_{\partial I} c_p$ in A and hence $[u] = [c_p]$. To see that $\pi_1(\iota)$ is surjective, just observe that $u(I) \subseteq A$ for $[u] \in \pi_1(X, p)$ since $u(I)$ is path connected and $p \in u(I)$. $\square$

Lemma 1.21. *Let $X \in \mathrm{ob}(\mathsf{Top})$ be path connected and $p, q \in X$. Then*

$$\pi_1(X, p) \cong \pi_1(X, q).$$

Proof. Since X is path connected we find a path v from p to q in X. Define a mapping $\Phi_v : \pi_1(X, p) \to \pi_1(X, q)$

$$\Phi_v\,[u] := [\overline{v} * u * v]\,.$$

Clearly, Φ_v is invertible with inverse $\Phi_{\overline{v}}$. Moreover, for $[u], [w] \in \pi_1(X, p)$ we have that

$$\begin{aligned}
\Phi_v([u]\,[w]) &= \Phi_v\,[u * w] \\
&= [\overline{v} * u * w * v] \\
&= [\overline{v} * u * v * \overline{v} * w * v]
\end{aligned}$$

$$
\begin{aligned}
&= [\overline{v} * u * v]\, [\overline{v} * w * v] \\
&= \Phi_v\, [u]\, \Phi_v\, [w]\,.
\end{aligned}
$$

□

Lemma 1.22 (Square Lemma). *Let* $F \in \mathsf{Top}(I \times I, X)$*. Then*

$$F(0,\cdot) * F(\cdot,1) \simeq_{\partial I} F(\cdot,0) * F(1,\cdot).$$

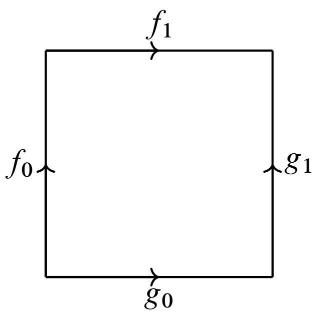
Proof. The idea is to consider first the case $F = \mathrm{id}_{I\times I}$. Hence define the paths f_0, f_1, g_0 and g_1 in $I \times I$ as indicated in figure 2a. Then there is a straight line homotopy $H : I \times I \to I \times I$ between them as indicated in figure 2b. Explicitly

$$H(s,t) := (1-t)(f_0 * f_1)(s) + t(g_0 * g_1)(s).$$

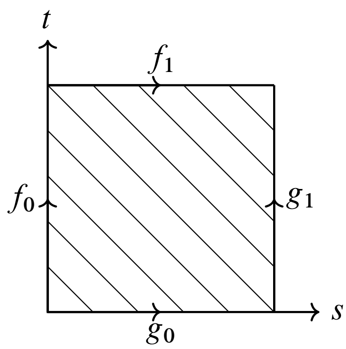
Then

$$(F \circ H)(s,t) = \begin{cases} F\left(2st, 2s(1-t)\right) & 0 \le s \le \frac{1}{2}, \\ F\left(t + (1-t)(2s-1), 1 + 2t(s-1)\right) & \frac{1}{2} \le s \le 1, \end{cases}$$

is the homotopy we are looking for. □



(a) The paths f_0, f_1, g_0 and g_1 in $I \times I$.



(b) $f_0 * f_1 \simeq_{\partial I} g_0 * g_1$.

Proposition 1.23. *Let* $f_0, f_1 \in \mathsf{Top}(X,Y)$ *such that* $F : f_0 \simeq f_1$*. Moreover, let* $p \in X$*. Then the diagram*

$$
\begin{tikzcd}
\pi_1(X,p) \arrow[r, "\pi_1(f_0)"] \arrow[dr, "\pi_1(f_1)"'] & \pi_1(Y, f_0(p)) \arrow[d, "\Phi_{F(p,\cdot)}"] \\
& \pi_1(Y, f_1(p))
\end{tikzcd}
$$

commutes, where $\Phi_\cdot$ *denotes the isomorphism in lemma 1.21.*

Proof. Let $[u] \in \pi_1(X,p)$. We have that

$$\pi_1(f_1)\,[u] = \left(\Phi_{F(p,\cdot)} \circ \pi_1(f_0)\right)[u] \Leftrightarrow [f_1 \circ u] = [\overline{F}(p,\cdot) * (f_0 \circ u) * F(p,\cdot)]$$

$$\Leftrightarrow [F(p,\cdot) * (f_1 \circ u)] = [(f_0 \circ u) * F(p,\cdot)]$$
$$\Leftrightarrow [F(u(0),\cdot) * F(u\cdot,1)] = [F(u\cdot,0) * F(u(1),\cdot)]\,,$$

where the last equality is true by the square lemma 1.22. □

Homotopy Invariance of π_1.

Lemma 1.24. *Being freely homotopic is a congruence on* Top.

Proof. (a) is immediate so we only have to check (b). Suppose $f_0 \in \mathsf{Top}(X,Y)$ and $g_0 \in \mathsf{Top}(Y,Z)$ such that $F : f_0 \simeq f_1$ and $G : g_0 \simeq g_1$. Consider $H_1 : X \times I \to Z$ defined by $H_1 := g_0 \circ F$. Then clearly $H_1 : g_0 \circ f_0 \simeq g_0 \circ f_1$. Moreover, we define $H_2 : X \times I \to Z$ by $H_2 := G(f_1\cdot,\cdot)$. Then $H_2 : g_0 \circ f_1 \simeq g_1 \circ f_1$. And we conclude by transitivity. □

Definition 1.25 (hTop). *The quotient category under the congruence of being freely homotopic is called the* ***homotopy category****, and is denoted by* hTop.

Definition 1.26 (Homotopy Type). *Two topological spaces X and Y are of the* ***same homotopy type****, if they are isomorphic in* hTop. *An explicit choice of such an isomorphism is called a* ***homotopy equivalence****.*

Exercise 1.27. Show that a topological space X has the same homotopy type as a one-point space if and only if X is contractible.

Theorem 1.28 (Homotopy Invariance of π_1). *Suppose X and Y are of the same homotopy type with homotopy equivalence $f : X \to Y$. Then for any $p \in X$ we have that $\pi_1(f) : \pi_1(X,p) \to (Y, f(p))$ is an isomorphism.*

Proof. By assumption there exists $g \in \mathsf{Top}(Y,X)$ such that $F : g \circ f \simeq \mathrm{id}_X$ and $G : f \circ g \simeq \mathrm{id}_Y$. By the functoriality of π_1 and proposition 1.23, the diagram

$$\begin{tikzcd}
 & \pi_1(Y, f(p)) \arrow[dr, "\pi_1(g)"] & \\
\pi_1(X,p) \arrow[ur, "\pi_1(f)"] \arrow[rr, "\pi_1(g\circ f)"] \arrow[dr, "\mathrm{id}_{\pi_1(X,p)}"'] & & \pi_1(X, g(f(p))) \arrow[dl, "\Phi_{F(p,\cdot)}"] \\
 & \pi_1(X,p) &
\end{tikzcd}$$

commutes. Since $\Phi_{F(p,\cdot)}$ is an isomorphism, $\pi_1(g \circ f)$ is an isomorphism, too. Hence $\pi_1(f)$ is injective. Using G instead of F and a similar argument yields that $\pi_1(f)$ is surjective. □

Lemma 1.29. *Let $G \in \mathrm{ob}(\mathsf{Grp})$, $S \in \mathsf{Set}$ and $\varphi : U(G) \to S$ a bijection. Then S can be given a group structure such that φ is an isomorphism.*

Proof. It is easy to show that $xy := \varphi\left(\varphi^{-1}(x)\varphi^{-1}(y)\right)$ defines a group structure on S with the requested property. □

Proposition 1.30. *Let $(X,p) \in \mathrm{ob}(\mathsf{Top}_*)$. Then $\pi_1(X,p) \cong \mathsf{hTop}_*\left((\mathbb{S}^1,1),(X,p)\right)$.*

Proof. Let $u \in \Omega(X,p)$. Then u passes to the quotient $\widetilde{u} : (\mathbb{S}^1,1) \to (X,p)$. Define now $\varphi[u] := [\widetilde{u}] \in \mathsf{hTop}_*\left((\mathbb{S}^1,1),(X,p)\right)$. This is well defined, since if $H : u \simeq_{\partial I} v$, it is easy to see that $\widetilde{H} : \widetilde{u} \simeq_{\{1\}} \widetilde{v}$. Moreover, if $f \in \mathsf{hTop}_*\left(\mathbb{S}^1,1),(X,p)\right)$, we define $\psi[f] := [f \circ \omega]$. Again, this is well defined since if $H : f \simeq_{\{1\}} g$, then $H \circ (\omega \times \mathrm{id}_I) : f \circ \omega \simeq_{\partial I} g \circ \omega$. It is easy to check that φ and ψ are inverses of each other and thus we have a bijection $\pi_1(X,p) \cong \mathsf{hTop}_*\left((\mathbb{S}^1,1),(X,p)\right)$ of sets. Hence an application of lemma 1.29 yields the result. □

$\pi_1(\mathbb{S}^1)$.

Definition 1.31 (Exponential Quotient Map and Fundamental Loop). *The mapping $\varepsilon : \mathbb{R} \to \mathbb{S}^1$ defined by*

$$\varepsilon(x) := e^{2\pi i x} \tag{1}$$

*is called the **exponential quotient map**. Moreover, the **fundamental loop ω** is defined to be the restriction $\omega := \varepsilon|_I$.*

Proposition 1.32 (Lifting Property of the Circle). *Let $n \in \mathbb{Z}$, $n \geq 0$, $X \subseteq \mathbb{R}^n$ compact and convex, $p \in X$, $f \in \mathsf{Top}_*\left((X,p),(\mathbb{S}^1,1)\right)$ and $m \in \mathbb{Z}$. Then there exists a unique map $\widetilde{f} \in \mathsf{Top}_*\left((X,p),(\mathbb{R},m)\right)$, called the **lifting of f**, such that*

$$\begin{tikzcd} & (\mathbb{R},m) \arrow[d, "\varepsilon"] \\ (X,p) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (\mathbb{S}^1,1) \end{tikzcd}$$

commutes.

Proof. We show first existence and then uniqueness.

Step 1: Existence. Since X is compact and f is continuous, f is uniformly continuous on X. Thus we find $\delta > 0$ such that $|f(x) - f(y)| < 2$, whenever $|x - y| < \delta$, i.e. $f(x)$ and $f(y)$ are not antipodal points. Moreover, since X is compact, X is bounded and hence we find $N \in \mathbb{N}$, such that $|x - y| < N\delta$ holds for all $x, y \in X$. Let $x \in X$. For $0 \leq k \leq N$, define $L_k : X \to X$ by

$$L_k(x) := \left(1 - \frac{k}{N}\right)p + \frac{k}{N}x.$$

Those are well defined functions since X is convex. Moreover, each L_k is continuous. Indeed, it is easy to check that L_k is Lipschitz. Also, for each $0 \leq k < N$, $f(L_k(x))$ and $f(L_{k+1}(x))$ are not antipodal for all $x \in X$. Indeed, it is easy to check that

$|L_k(x) - L_{k+1}(x)| < \delta$ holds for all $x \in X$. For $0 \leq k < N$ define $g_k : X \to \mathbb{S}^1 \setminus \{-1\}$ by

$$g_k(x) := \frac{f(L_{k+1}(x))}{f(L_k(x))}.$$

Clearly g_k is well defined and continuous as a composition of continuous functions. Let $\mathrm{Log} : \mathbb{S}^1 \setminus \{-1\} \to \mathbb{C}$ denote the principal branch of the logarithm. Define $\widetilde{f} : X \to \mathbb{R}$ by

$$\widetilde{f}(x) := m + \frac{1}{2\pi i} \sum_{k=0}^{N-1} \mathrm{Log}(g_k(x)).$$

Clearly, $\widetilde{f}$ is continuous and moreover we have that $\widetilde{f} = m$ since $g_k(p) = 1$ for all $0 \leq k < N$. Finally, for any $x \in X$ we have that

$$(\varepsilon \circ \widetilde{f})(x) = \varepsilon(m) \prod_{k=0}^{N-1} g_k(x) = \frac{f(L_N(x))}{f(L_0(x))} = \frac{f(x)}{f(p)} = f(x).$$

Step 2: Uniqueness. Suppose $\widetilde{g} \in \mathsf{Top}_* \big((X, p), (\mathbb{R}, m)\big)$ is another such function. Define $\varphi \in \mathsf{Top}_* \big((X, p), (\mathbb{R}, 0)\big)$ by

$$\varphi(x) := \widetilde{f}(x) - \widetilde{g}(x).$$

Then clearly $\varepsilon \circ \varphi = 1$ and thus $\varphi(X) \subseteq \mathbb{Z}$. Since X is convex, X is connected and so $\varphi = 0$.

□

Corollary 1.33. *Let $u, v \in \Omega(\mathbb{S}^1, 1)$ such that $[u] = [v]$. If $\widetilde{u}, \widetilde{v} : (I, 0) \to (\mathbb{R}, 0)$ are the liftings of u and v, respectively, then $[\widetilde{u}] = [\widetilde{v}]$.*

Proof. Let $F : u \simeq_{\partial I} v$. By proposition 1.32, we find $\widetilde{F} \in \mathsf{Top}_* \big((I \times I, (0,0)), (\mathbb{R}, 0)\big)$, such that $\varepsilon \circ \widetilde{F} = F$. We claim that $\widetilde{F} : \widetilde{u} \simeq_{\partial I} \widetilde{v}$. For $s \in I$ define $\widetilde{u}_0(s) := \widetilde{F}(s, 0)$. Then $\widetilde{u}_0(0) = \widetilde{F}(0,0) = 0$ and since $\widetilde{u}_0$ is continuous we have that $\widetilde{u}_0 \in \mathsf{Top}_* \big((I, 0), (\mathbb{R}, 0)\big)$. Moreover

$$(\varepsilon \circ \widetilde{u}_0)(s) = \varepsilon\left(\widetilde{F}(s, 0)\right) = F(s, 0) = u(s)$$

for all $s \in I$ and thus $\widetilde{u}_0$ is a lifting of u. But by proposition 1.32, liftings are unique and thus $\widetilde{u}_0 = \widetilde{u}$. Next define $\widetilde{w}_0(t) := \widetilde{F}(0, t)$ for all $t \in I$. Then $\widetilde{w}_0(0) = \widetilde{F}(0,0) = 0$ and so $\widetilde{w}_0 \in \mathsf{Top}_* \big((I, 0), (\mathbb{R}, 0)\big)$. Moreover

$$(\varepsilon \circ \widetilde{w}_0)(t) = \varepsilon\left(\widetilde{F}(0, t)\right) = F(0, t) = u(0) = v(0) = 1.$$

for all $t \in I$. Thus

$$\begin{tikzcd} & (\mathbb{R}, 0) \arrow[d, "\varepsilon"] \\ (I, 0) \arrow[ur, "\widetilde{w}_0"] \arrow[r, "c_1"'] & (\mathbb{S}^1, 1) \end{tikzcd}$$

commutes. But also c_0 makes the above diagram commute. By uniqueness, $\widetilde{w}_0 = c_0$. Define $\widetilde{v}_0(s) := \widetilde{F}(s, 1)$ for all $s \in I$. Then $\widetilde{v}_0(0) = \widetilde{F}(0, 1) = \widetilde{w}_0(1) = 0$ and it is easy to check that $\widetilde{v}_0$ is a lift for v. Hence $\widetilde{v}_0 = \widetilde{v}$. Finally, define $\widetilde{w}_1(t) := \widetilde{F}(1, t)$ for all $t \in I$. Then $\widetilde{w}_1(0) = \widetilde{F}(1, 0) = \widetilde{u}(1)$ and thus $\widetilde{w}_1 \in \mathsf{Top}_* \big((I, 0), (\mathbb{R}, \widetilde{u}(1))\big)$. Moreover

$$(\varepsilon \circ \widetilde{w}_1)(t) = \varepsilon\left(\widetilde{F}(1, t)\right) = F(1, t) = v(1) = u(1) = 1$$

for all $t \in I$. By proposition 1.32, we have again that $\widetilde{w}_1 = c_{\widetilde{u}(1)}$. So $F : \widetilde{u} \simeq_{\partial I} \widetilde{v}$. □

Definition 1.34 (Degree). *Let $u \in \Omega(\mathbb{S}^1, 1)$. The **degree of u**, written $\deg u$, is defined by $\deg u := \widetilde{u}(1)$, where $\widetilde{u}$ is the unique lift of u such that $\widetilde{u}(0) = 0$.*

Theorem 1.35 (Fundamental Group of the Circle). *$\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.*

Proof. Define $\deg : \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}$ by $\deg\,[u] := \deg u$. This is well defined by corollary 1.33, since if $[u] = [v]$, then $[\widetilde{u}] = [\widetilde{v}]$ and in particular $\widetilde{u}(1) = \widetilde{v}(1)$.

Step 1: $\deg \in \mathsf{Grp}\big(\pi_1(\mathbb{S}^1, 1), (\mathbb{Z}, +)\big)$. Let $[u], [v] \in \pi_1(\mathbb{S}^1, 1)$. Moreover, let $\widetilde{u}$ and $\widetilde{v}$ denote the unique liftings of u and v, respectively, such that $\widetilde{u}(0) = 0$ and $\widetilde{v}(0) = 0$. Define $\widetilde{w} : I \to \mathbb{R}$ by

$$\widetilde{w}(s) := \begin{cases} \widetilde{u}(2s) & 0 \le s \le \frac{1}{2}, \\ \deg u + \widetilde{v}(2s - 1) & \frac{1}{2} \le s \le 1. \end{cases}$$

Then $\widetilde{w}$ is continuous by the gluing lemma and $\widetilde{w}(0) = 0$. Hence $\widetilde{w} \in \mathsf{Top}_* \big((I, 0), (\mathbb{R}, 0)\big)$. Also we have that $\varepsilon \circ \widetilde{w} = u * v$ and thus $\widetilde{w}$ is the lift of $u * v$. But $\widetilde{w}(1) = \deg u + \deg v$ and so

$$\deg([u]\,[v]) = \deg\,[u * v] = \deg(u * v) = \widetilde{w}(1) = \deg u + \deg v = \deg\,[u] + \deg\,[v]\,.$$

Step 2: $\deg$ *is injective.* Suppose $\deg\,[u] = 0$. Then $\widetilde{u}(1) = 0$ and thus $\widetilde{u} \in \Omega(\mathbb{R}, 0)$. Since $\mathbb{R}$ is contractible, we have that $[\widetilde{u}] = [c_0]$ and thus

$$[u] = [\varepsilon \circ \widetilde{u}] = \pi_1(\varepsilon)\,[\widetilde{u}] = \pi_1(\varepsilon)\,[c_0] = [\varepsilon \circ c_0] = [c_1]\,.$$

Thus $\ker(\deg)$ is trivial.
Step 3: $\deg$ *is surjective.* Let $m \in \mathbb{Z}$. Then $\deg\,[\varepsilon^m] = \deg \varepsilon^m = \widetilde{\varepsilon^m}(1) = m$.

□

The Seifert-Van Kampen Theorem

Coproducts and Pushouts in Grp.

Proposition 1.36 (Coproducts in Grp). Grp *has all small coproducts.*

Proof. Let $A \in \mathrm{ob}(\mathsf{Set})$ and A be the small category defined as the discrete category with $\mathrm{ob}(\mathsf{A}) := A$, i.e.

$$\bullet \qquad \bullet \qquad \bullet \qquad \cdots \qquad \bullet \qquad \bullet \qquad \bullet$$

Let $D : \mathsf{A} \to \mathsf{Grp}$ be a functor. Hence we get a family $(G_\alpha)_{\alpha\in A}$ in Grp, where $G_\alpha := D(\alpha)$ for all $\alpha \in A$. A ***word*** in $(G_\alpha)_{\alpha\in A}$ is a finite sequence in $\coprod_{\alpha\in A} G_\alpha$. A word in $(G_\alpha)_{\alpha\in A}$ will simply be written as $(g_1, \ldots, g_n)$, where $g_k \in G_\alpha$ for some $\alpha \in A$. The ***empty word*** is denoted by $()$. Let $\mathcal{W}$ denote the set of all words in $(G_\alpha)_{\alpha\in A}$. On $\mathcal{W}$ define a multiplication by ***concatenation***

$$(g_1, \ldots, g_n)(h_1, \ldots, h_m) := (g_1, \ldots, g_n, h_1, \ldots, h_m).$$

An ***elementary reduction*** is an operation of one of the following forms:

- $(g_1, \ldots, g_k, g_{k+1}, \ldots, g_n) \mapsto (g_1, \ldots, g_k g_{k+1}, \ldots, g_n$, where $g_k, g_{k+1} \in G_\alpha$ for some $\alpha \in A$.
- $(g_1, \ldots, g_{k-1}, 1_\alpha, g_{k+1}, \ldots, g_n) \mapsto (g_1, \ldots, g_{k-1}, g_{k+1}, \ldots, g_n)$.

Let $\sim$ denote the equivalence relation on $\mathcal{W}$ generated by elementary reductions.

Lemma 1.37. *$\mathcal{W}/\!\sim$ together with concatenation of representatives is an element of* Grp.

Proof. Define

$$[(g_1, \ldots, g_n)]\,[(h_1, \ldots, h_m)] := [(g_1, \ldots, g_n, h_1, \ldots, h_m)]\,.$$

It is left to the reader to show that this is well defined and that $\mathcal{W}/\!\sim$ is indeed a group. □

The group defined in lemma 1.37 will be denoted by $\ast_{\alpha\in A} G_\alpha$ and called the ***free product of $(G_\alpha)_{\alpha\in A}$***. Let us define a cocone on D. For this consider the inclusions $\iota_\alpha : G_\alpha \to \ast_{\alpha\in A} G_\alpha$ defined by

$$\iota_\alpha(g) := [(g)]$$

for all $\alpha \in A$. It is immediate from

$$\iota_\alpha(gh) = [(gh)] = [(g, h)] = [(g)]\,[(h)] = \iota_\alpha(g)\iota_\alpha(h)$$

for $g, h \in G_\alpha$, that ι_α is a morphism of groups. Since there are only the identity morphisms in A, $\left(\ast_{\alpha\in A} G_\alpha, (\iota_\alpha)_{\alpha\in A}\right)$ is a cocone on D. Let us show that this is in fact a universal cocone. To this end, suppose that $\left(C, (\varphi_\alpha)_{\alpha\in A}\right)$ is another cocone on D. Define a mapping $\bar{f} : \ast_{\alpha\in A} G_\alpha \to C$ by

$$\bar{f}\,[(g_1, \ldots, g_n)] := \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$$

where $g_k \in G_{\alpha_k}$. Then $\bar{f}$ is easily seen to be well defined since each φ_α is a morphism of groups. Moreover, if $g \in G_\alpha$, then

$$(\bar{f} \circ \iota_\alpha)(g) = \bar{f}\,[(g)] = \varphi_\alpha(g)$$

for all $\alpha \in A$. Suppose that $f : \mathop{\ast}_{\alpha \in A} G_\alpha \to C$ is another homomorphism of groups such that $f \circ \iota_\alpha = \varphi_\alpha$ for all $\alpha \in A$. Then for $[(g_1, \ldots, g_n)] \in \mathop{\ast}_{\alpha \in A} G_\alpha$ we have

$$\begin{aligned} f\,[(g_1, \ldots, g_n)] &= f([(g_1)] \cdots [(g_n)]) \\ &= f\,[(g_1)] \cdots f\,[(g_n)] \\ &= f\,(\iota_{\alpha_1}(g_1)) \cdots f\,(\iota_{\alpha_n}(g_n)) \\ &= \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n) \\ &= \bar{f}\,[(g_1, \ldots, g_n)]\,. \end{aligned}$$

$\square$

Exercise 1.38. Check that $\mathcal{W}/\sim$ is indeed a group with the declared group structure and that $\bar{f}$ is indeed well defined.

Proposition 1.39 (Pushouts in Grp). Grp *has all pushouts.*

Proof. Consider the diagram $D : \mathsf{A} \to \mathsf{Grp}$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \quad \xrightarrow{D} \quad \begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \\ H_2 & \end{tikzcd}$$

and define N to be the normal subgroup of $H_1 * H_2$ generated by elements of the form $[(\varphi_1(g^{-1}), \varphi_2(g))]$ for $g \in G$. Let $K := (H_1 * H_2)/N$. Then

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \arrow[d, "\pi\circ\iota_1"] \\ H_2 \arrow[r, "\pi\circ\iota_2"'] & K \end{tikzcd}$$

commutes. Indeed, if $g \in G$, we have that $(\pi \circ \iota_1 \circ \varphi_1)(g) = [(\varphi_1(g))]\,N$ and similarly $(\pi \circ \iota_2 \circ \varphi_2)(g) = [(\varphi_2(g))]\,N$. Then

$$[(\varphi_1(g))]^{-1}\,[(\varphi_2(g))] = [(\varphi_1(g)^{-1})]\,[(\varphi_2(g))] = [(\varphi_1(g^{-1}))]\,[(\varphi_2(g))] \in N.$$

Suppose that we have another cocone on D:

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \arrow[d, "\psi_1"] \\ H_2 \arrow[r, "\psi_2"'] & C \end{tikzcd}$$

By proposition 1.36, there exists a unique morphism of groups $f : H_1 * H_2 \to C$ and we thus get the following diagram:

$$\begin{tikzcd}
G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \arrow[d, "\iota_1"] \arrow[rrdd, bend left, "\psi_1"] & & \\
H_2 \arrow[r, "\iota_2"'] \arrow[rrrd, bend right, "\psi_2"'] & H_1 * H_2 \arrow[rd, "\pi"] \arrow[rrd, bend right, "f"'] & & \\
& & K \arrow[rd, dashed, "\bar{f}"] & \\
& & & C
\end{tikzcd}$$

To show that $N \subseteq \ker f$ is left as an exercise. Hence by the factorization theorem (see [Gri07, p. 23]), f factors uniquely through π, i.e. there exists a unique morphism of groups $\bar{f} : K \to C$ such that $\bar{f} \circ \pi = f$. $\square$

Exercise 1.40. In the previous proposition, verify that $N \subseteq \ker f$.

Definition 1.41 (Amalgamated Free Product). *The pushout of a diagram*

$$\begin{tikzcd}
G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \\
H_2 &
\end{tikzcd}$$

in Grp *is called the* ***amalgamated free product of H_1 and H_2 along $(G, \varphi_1, \varphi_2)$****, written* $H_1 *_{(G,\varphi_1,\varphi_2)} H_2$.

The Seifert-Van Kampen Theorem and its Consequences.

Theorem 1.42 (Seifert-Van Kampen). *Let $X \in \mathrm{ob}(\mathsf{Top})$, (U, V) an open cover for X, such that U, V and $U \cap V$ are path connected. Moreover, let $p \in U \cap V$. Then*

$$\pi_1(X, p) \cong \pi_1(U, p) *_{(\pi_1(U \cap V, p), \pi_1(\iota_U), \pi_1(\iota_V))} \pi_1(V, p), \tag{2}$$

where $\iota_U : U \cap V \hookrightarrow U$ and $\iota_V : U \cap V \hookrightarrow V$ denote inclusion.

Proof. Let $\jmath_U : U \hookrightarrow X$ and $\jmath_V : V \hookrightarrow X$ denote inclusions. We will show that $\big(\pi_1(X, p), \pi_1(\jmath_U), \pi_1(\jmath_V)\big)$ is a pushout of the diagram

$$\begin{tikzcd}
\pi_1(U \cap V, p) \arrow[r, "\pi_1(\iota_U)"] \arrow[d, "\pi_1(\iota_V)"'] & \pi_1(U, p) \\
\pi_1(V, p) &
\end{tikzcd} \tag{3}$$

in Grp and hence by proposition 1.39 and uniqueness, the statement follows. Clearly

$$\begin{tikzcd} \pi_1(U\cap V,p) \arrow[r, "\pi_1(\iota_U)"] \arrow[d, "\pi_1(\iota_V)"'] & \pi_1(U,p) \arrow[d, "\pi_1(\jmath_U)"] \\ \pi_1(V,p) \arrow[r, "\pi_1(\jmath_V)"'] & \pi_1(X,p) \end{tikzcd}$$

commutes. Suppose now that (G,φ_U,φ_V) is another cocone for the diagram (3). We want to show that there exists a unique homomorphism $\Phi : \pi_1(X,p) \to G$ such that $\Phi\circ\pi_1(\jmath_U) = \varphi_U$ and $\Phi\circ\pi_1(\jmath_V) = \varphi_V$. Let $[u] \in \pi_1(X,p)$. Choose a partition $0 = x_0 < \cdots < x_n = 1$ of I such that $u(x_k) \in U \cap V$ for all $k = 0,\dots,n$ and such that all $u|_{[x_{k-1},x_k]}$ take values either in U or in V for all $k = 1,\dots,n$. The existence of such a partition follows from an application of the Lebesgue number lemma on the open cover $(u^{-1}(U), u^{-1}(V))$ of I. Indeed, if $\delta > 0$ is the corresponding Lebesgue number of the cover, we find $n \in \omega$, $n > 0$, such that $1/n < \delta$. Thus $[(i-1)/n, i/n]$ is contained in either $u^{-1}(U)$ or $u^{-1}(V)$ for all $i = 1,\dots,n$. Now choose those i such that $u(i/n) \in U \cap V$. For $k = 1,\dots,n$, let $u_k : I \to X$ be defined by

$$u_k(s) := u\left((1-s)x_{k-1} + sx_k\right).$$

Moreover, for each $k = 1,\dots,n-1$ choose a path γ_k in $U \cap V$ from p to $u(x_k)$ and set $\gamma_0, \gamma_n := c_p$. Define now

$$\Phi\,[u] := \prod_{k=1}^{n} \varphi_\bullet\,[\gamma_{k-1} * u_k * \overline{\gamma}_k]\,, \tag{4}$$

where $\varphi_\bullet$ denotes either φ_U or φ_V depending on whether $\gamma_{k-1} * u_k * \overline{\gamma}_k$ is a loop in U or in V. If u is a loop in $U \cap V$, we can choose either φ_U or φ_V since (G,φ_U,φ_V) is a cocone of the diagram (3). Now there are some things to check.

Φ is a function. Suppose $H : u \simeq_{\partial I} v$.

$\Phi\,[u]$ does not depend on the choice of γ_k. Fix some $k = 1,\dots,n-1$ and suppose that γ'_k is another path from p to $u(x_k)$ in $U \cap V$. Then we have that

$$\varphi_\bullet\,[\gamma_{k-1} * u_k * \overline{\gamma}_k] = \varphi_\bullet\,[\gamma_{k-1} * u_k * \overline{\gamma}'_k * \gamma'_k * \overline{\gamma}_k] = \varphi_\bullet\,[\gamma_{k-1} * u_k * \overline{\gamma}'_k]\,\varphi_\bullet\,[\gamma'_k * \overline{\gamma}_k]$$

and

$$\begin{aligned} \varphi_\bullet\,[\gamma_k * u_{k+1} * \overline{\gamma}_{k+1}] &= \varphi_\bullet\,[\gamma_k * \overline{\gamma}'_k * \gamma'_k * u_{k+1} * \overline{\gamma}_{k+1}] \\ &= \varphi_\bullet\,[\gamma_k * \overline{\gamma}'_k]\,\varphi_\bullet\,[\gamma'_k * u_{k+1} * \overline{\gamma}_{k+1}] \\ &= \left(\varphi_\bullet\,[\gamma'_k * \overline{\gamma}_k]\right)^{-1} \varphi_\bullet\,[\gamma'_k * u_{k+1} * \overline{\gamma}_{k+1}]\,. \end{aligned}$$

Since $\gamma'_k * \overline{\gamma}_k$ is a loop in $U \cap V$, we have that

$$\varphi_\bullet\,[\gamma_{k-1} * u_k * \overline{\gamma}_k]\,\varphi_\bullet\,[\gamma_k * u_{k+1} * \overline{\gamma}_{k+1}] = \varphi_\bullet\,[\gamma_{k-1} * u_k * \overline{\gamma}'_k]\,\varphi_\bullet\,[\gamma'_k * u_{k+1} * \overline{\gamma}_{k+1}]\,.$$

$\Phi\,[u]$ *does not depend on the choice of a partition of I.* Suppose $\mathcal{P}_1$ and $\mathcal{P}_2$ are both partitions of I, their union $\mathcal{P}_1 \cup \mathcal{P}_2$ is a refinement of both $\mathcal{P}_1$ and $\mathcal{P}_2$. If we can show that adding a single point to a partition $\mathcal{P}$ of I does not affect the value $\Phi\,[u]$, then so it does not on $\mathcal{P}_1 \cup \mathcal{P}_2$ and hence is independent of the choice of a partition. Suppose we add $x_{k-1} < y < x_k$. Let us denote by u_y the reparametrized restriction of u from $u(x_{k-1})$ to $u(y)$ and by u'_k the reparametrized restriction of u from $u(y)$ to $u(x_k)$. Moreover, let γ_y be a path from p to $u(y)$ in $U \cap V$. We compute

$$\begin{aligned} \varphi_\bullet\,[\gamma_{k-1} * u_y * \overline{\gamma}_y]\,\varphi_\bullet\,[\gamma_y * u'_k * \overline{\gamma}_k] &= \varphi_\bullet\,[\gamma_{k-1} * u_y * \overline{\gamma}_y * \gamma_y * u'_k * \overline{\gamma}_k] \\ &= \varphi_\bullet\,[\gamma_{k-1} * u_y * u'_k * \overline{\gamma}_k] \\ &= \varphi_\bullet\,[\gamma_{k-1} * u_k * \overline{\gamma}_k]\,, \end{aligned}$$

since $u_y * u'_k$ is a reparametrization of u_k and $\gamma_{k-1} * u_y * \overline{\gamma}_y$, $\gamma_y * u'_k * \overline{\gamma}_k$ are both loops either in U or in V.

Φ *is a morphism of groups.* Let $[u]\,, [v] \in \pi_1(X, p)$. Let $0 = x_0 < \cdots < x_n = 1$ be a partition of I as above. By invariance under a change of partitions, we may assume that $0 = x_0 < \cdots < x_m = \frac{1}{2} < \cdots < x_n = 1$. Clearly $(u * v)(x_m) = p \in U \cap V$. Now both $0 = 2x_0 < \cdots < 2x_m = 1$ and $0 = 2x_m - 1 < \cdots < 2x_n - 1 = 1$ are partitions of I with $(u * v)_k = u_k$ for $k = 1, \ldots, m$ and $(u * v)_k = v_k$ for $k = m+1, \ldots, n$. By using invariance of the choice of a partition again and invariance of the choice of the γ_k yields

$$\begin{aligned} \Phi([u]\,[v]) &= \Phi\,[u * v] \\ &= \prod_{k=1}^{n} \varphi_\bullet\,[\gamma_{k-1} * (u * v)_k * \overline{\gamma}_k] \\ &= \prod_{k=1}^{m} \varphi_\bullet\,[\gamma_{k-1} * u_k * \overline{\gamma}_k] \prod_{k=m+1}^{n} \varphi_\bullet\,[\gamma_{k-1} * v_k * \overline{\gamma}_k] \\ &= \Phi\,[u]\,\Phi\,[v]\,. \end{aligned}$$

Checking commutativity. We have to show that $\Phi \circ \pi_1(\jmath_U) = \varphi_U$ and $\Phi \circ \pi_1(\jmath_V) = \varphi_V$ hold. Let us show the first identity, the second is similar. Let $[u] \in \pi_1(U, p)$. Then we can choose the trivial partition $0 = x_0 < x_1 = 1$ of I and thus get

$$(\Phi \circ \pi_1(\jmath_U))\,[u] = \Phi\,[u] = \varphi_U\,[\gamma_0 * u_1 * \overline{\gamma}_1] = \varphi_U\,[u]\,.$$

Showing uniqueness of Φ. Suppose $\Psi : \pi_1(X, p) \to G$ is another map with the smae properties as Φ. Let $[u] \in \pi_1(X, p)$. They keypoint is to observe that

$$[u] = \left[\prod_{k=1}^{n} (\gamma_{k-1} * u_k * \overline{\gamma}_k)\right]$$

holds. Thus

$$\begin{aligned}\Psi[u] &= \Psi\left[\prod_{k=1}^{n}(\gamma_{k-1} * u_k * \overline{\gamma}_k)\right] \\ &= \prod_{k=1} \Psi\left[\gamma_{k-1} * u_k * \overline{\gamma}_k\right] \\ &= \prod_{k=1}^{n} \varphi_{\bullet}\left[\gamma_{k-1} * u_k * \overline{\gamma}_k\right] \\ &= \Phi[u].\end{aligned}$$

□

Exercise 1.43. In the proof of the Seifert-Van Kampen theorem, show that $u_y * u'_k = u_k \circ \varphi$, where $\varphi \in \mathsf{Top}(I, I)$ is given by

$$\varphi(s) := \begin{cases} 2s(y - x_{k-1})/(x_k - x_{k-1}) & 0 \le s \le \frac{1}{2}, \\ 2(1-s)(y - x_{k-1})/(x_k - x_{k-1}) + 2s - 1 & \frac{1}{2} \le s \le 1. \end{cases}$$

CHAPTER 2

Homological Algebra

Diagram Lemmas

The Snake Lemma.

Proposition 2.1 (Snake Lemma). *Suppose we are given a commutative diagram in* AbGrp *with exact rows:*

$$\begin{tikzcd} & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "j"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i'"'] & B' \arrow[r, "j'"'] & C' & \end{tikzcd}$$

Then there exists $\delta \in \mathsf{AbGrp}(\ker h, \operatorname{coker} f)$ *such that the sequence*

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h \tag{5}$$

is exact.

Proof. Consider the augmented diagram in figure 10, where the morphisms k, l, p and q are induced by i, j, i' and j', respectively.
Step 1: Exactness at $\ker g$. Let $a \in \ker f$. Then $l(k(a)) = j(i(a)) = 0$ by exactness at B and thus $\operatorname{im} k \subseteq \ker l$. Conversly, let $b \in \ker l$. Then $j(b) = 0$ and by exactness at B, there exists $a \in A$ such that $i(a) = b$. Moreover $0 = g(b) = g(i(a)) = i'(f(a))$ since $b \in \ker g$ and thus $f(a) = 0$ by injectivity of i'. Hence $\ker j \subseteq \operatorname{im} k$.
Step 2: Exactness at $\operatorname{coker} g$. Let $a' + \operatorname{im} f \in \operatorname{coker} f$. Then

$$q(p(a' + \operatorname{im} f)) = j'(i'(a')) + \operatorname{im} h = \operatorname{im} h$$

by exactness at B' implies $\operatorname{im} p \subseteq \ker q$. Conversly, let $b' + \operatorname{im} g \in \ker q$. Then

$$0 = q(b' + \operatorname{im} g) = j'(b') + \operatorname{im} h$$

and thus $j'(b') \in \operatorname{im} h$. Hence there exists $c \in C$, such that $j'(b') = h(c)$. Since j is surjective, we find $b \in B$ such that $j(b) = c$. Therefore $j'(b') = h(j(b))$. By commutativity we get $j'(b') = j'(g(b))$ which is equivalent to $j'(b' - g(b)) = 0$. Thus $b' - g(b) \in \ker j'$ and exactness at B' yields the existence of $a' \in A'$ such that $i'(a') = b' - g(b)$. Now

$$p(a' + \operatorname{im} f) = i'(a') + \operatorname{im} g = b' - g(b) + \operatorname{im} g = b' + \operatorname{im} g$$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & \ker f \arrow[r, "k"] \arrow[d] & \ker g \arrow[r, "l"] \arrow[d] & \ker h \arrow[d] & \\
 & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "j"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "i'"'] \arrow[d] & B' \arrow[r, "j'"'] \arrow[d] & C' \arrow[d] & \\
 & \operatorname{coker} f \arrow[r, "p"'] \arrow[d] & \operatorname{coker} g \arrow[r, "q"'] \arrow[d] & \operatorname{coker} h \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Figure 3. Proof of the snake lemma.

and thus $\ker q \subseteq \operatorname{im} p$.
Step 3: Definition of δ. Consider the snakelike path indicated in figure 11a. Let $c \in \ker h$. Since j is surjective, we find $b \in B$ such that $j(b) = c$. Since $c \in \ker h$, we get that $0 = h(c) = h(j(b)) = j'(g(b))$ and thus $g(b) \in \ker j'$ which implies $g(b) \in \operatorname{im} i'$ by exactness at B'. Hence there exists $a' \in A'$ such that $i'(a') = g(b)$. Actually this a' is unique since i' is injective. Define $\delta : \ker h \to \operatorname{coker} f$ by

$$\delta(c) := a' + \operatorname{im} f.$$

Step 4: Checking that δ *is a morphism of groups.* Since j is only surjective, we have to show that δ is a function. So suppose we choose $b_0 \in B$ instead of $b \in B$ in figure 11b with $b_0 \neq b$. We want to show that $\delta(c) = a' + \operatorname{im} f = a'_0 + \operatorname{im} f$, or equivalently $a' - a'_0 \in \operatorname{im} f$. Since $c = j(b) = j(b_0)$, we have that $b - b_0 \in \ker j$. Hence by exactness at B there exists $a \in A$ such that $i(a) = b - b_0$. Applying g and invoking commutativity yields

$$g(b) - g(b_0) = g(i(a)) = i'(f(a))$$

Hence $i'(a') - i'(a'_0) = i'(f(a))$ and thus the injectivity of i' yields $a' - a'_0 = f(a)$. In the same manner one can show that δ is a morphism of groups.
Step 5: Exactness at $\ker h$. Let $b \in \ker g$. Then $\operatorname{im} l \subseteq \ker \delta$ immeddiately follows from figure 12a. Conversly, suppose $c \in \ker \delta$. From figure 12b we get that

$$g(b) = i'(a') = i'(f(a)) = g(i(a))$$

$$\begin{tikzcd}
\ker f \arrow[r, "k"] & \ker g \arrow[r, "l"] & \ker h \arrow[d] \\
& B \arrow[d, "g"] & C \arrow[l] \\
A' \arrow[d] & B' \arrow[l] & \\
\operatorname{coker} f \arrow[r, "p"'] & \operatorname{coker} g \arrow[r, "q"'] & \operatorname{coker} h
\end{tikzcd}$$

(a) Snakelike path.

$$\begin{tikzcd}
& & c \arrow[d, mapsto] \\
& b \arrow[d, mapsto] & c \arrow[l, mapsto] \\
a' \arrow[d, mapsto] & g(b) \arrow[l, mapsto] & \\
a' + \operatorname{im} f & &
\end{tikzcd}$$

(b) Definition of δ.

Figure 4

and thus $b - i(a) \in \ker g$. So $l(b - i(a)) = j(b) - j(i(a)) = j(b) = c$ by exactness at B and thus $\ker \delta \subseteq \operatorname{im} l$.

$$\begin{tikzcd}
& & j(b) \arrow[d, mapsto] \\
& b \arrow[d, mapsto] & j(b) \arrow[l, mapsto] \\
0 \arrow[d, mapsto] & g(b) = 0 \arrow[l, mapsto] & \\
\operatorname{im} f & &
\end{tikzcd}$$

(a) $\operatorname{im} l \subseteq \ker \delta$.

$$\begin{tikzcd}
& & c \arrow[d, mapsto] \\
a \arrow[d, mapsto] & b \arrow[d, mapsto] & c \arrow[l, mapsto] \\
a' = f(a) \arrow[d, mapsto] & g(b) \arrow[l, mapsto] & \\
a' + \operatorname{im} f & &
\end{tikzcd}$$

(b) $\ker \delta \subseteq \operatorname{im} l$.

Figure 5

Step 6: Exactness at coker f. Suppose that $a' + \operatorname{im} f \in \operatorname{im} \delta$. Then

$$p(a' + \operatorname{im} f) = i'(a') + \operatorname{im} g = g(b) + \operatorname{im} g = \operatorname{im} g$$

and thus $\operatorname{im} \delta \subseteq \ker p$. Conversly, suppose that $a' + \operatorname{im} f \in \ker p$. Hence $i'(a') \in \operatorname{im} g$ and we find $b \in B$ such that $g(b) = i'(a')$. Consider $j(b)$. By exactness at B' follows

$$h(j(b)) = j'(g(b)) = j'(i'(a')) = 0$$

So $j(b) \in \ker h$. Moreover, by construction $\delta(j(b)) = a' + \operatorname{im} f$ and thus $\ker p \subseteq \operatorname{im} \delta$.

□

The Five Lemma.

Proposition 2.2 (Five Lemma). *Suppose we are given a commutative diagram in* AbGrp *with exact rows and columns:*

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] \\
A \arrow[r, "\varphi_1"] \arrow[d, "f"] & B \arrow[r, "\varphi_2"] \arrow[d, "g"] & C \arrow[r, "\varphi_3"] \arrow[d, "h"] & D \arrow[r, "\varphi_4"] \arrow[d, "k"] & E \arrow[d, "l"] \\
A' \arrow[r, "\psi_1"'] \arrow[d] & B' \arrow[r, "\psi_2"'] \arrow[d] & C' \arrow[r, "\psi_3"'] & D' \arrow[r, "\psi_4"'] \arrow[d] & E' \\
0 & 0 & & 0 &
\end{tikzcd}$$

Then h is an isomorphism.

Proof. We show that h is bijective.

Step 1: h is injective. See figure 6. Let $c \in \ker h$. Hence $h(c) = 0$ and since φ_3 is a morphism of groups, we have that $(\varphi_3 \circ h)(c) = 0$. By commutativity, $(k \circ \varphi_3)(c) = 0$ and thus since k is injective, $\varphi_3(c) = 0$. Exactness at C implies that there exists some $b \in B$ such that $\varphi_2(b) = c$. Moreover, by commutativity $\psi_2(g(b)) = 0$ and thus we find $a' \in A'$ such that $\psi_1(a') = g(b)$. Sujectivity of f implies the existence of $a \in A$ such that $f(a) = a'$. Commutativity yields $g(b) = g(\varphi_1(a))$ and thus $b - \varphi_1(a) \in \ker g$. Since g is injective, $b = \varphi_1(a)$ and thus $c = \varphi_2(\varphi_1(a)) = 0$.

$$\begin{tikzcd}
a \arrow[r, mapsto] \arrow[d, mapsto] & b = \varphi_1(a) \arrow[r, mapsto] \arrow[d, mapsto] & c \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\
a' \arrow[r, mapsto] & g(b) \arrow[r, mapsto] & h(c) = 0 \arrow[r, mapsto] & 0
\end{tikzcd}$$

Figure 6. Proof of injectivity of h.

Step 2: h is surjective. See figure 7. Let $c' \in C'$. Since k is surjective, we find $d \in D$ such that $k(d) = \psi_3(c')$. Hence exactness at D' together with commutativity yields $(l \circ \varphi_4)(d) = 0$. Since l is injective, we get that $\varphi_4(d) = 0$. Thus by exactness at D we find $c \in C$ such that $\varphi_3(c) = d$. Hence by commutativity, $(\psi_3 \circ h)(c) = \psi_3(c')$ or equivalently, $c' - h(c) \in \ker \psi_3$. By exactness at C' we find $b' \in B'$ such that $\psi_2(b') = c' - h(c)$. Moreover, since g is surjective, we find $b \in B$ such that $g(b) = b'$. Finally, commutativity yields $(h \circ \varphi_2)(b) = c' - h(c)$ or equivalently $c' = h(c + \varphi_2(b))$.

□

$$\begin{tikzcd} c \arrow[r, mapsto] & d \arrow[r, mapsto] \arrow[d, mapsto] & 0 \arrow[d, mapsto] \\ c' \arrow[r, mapsto] & \psi_3(c') \arrow[r, mapsto] & 0 \end{tikzcd}$$

Figure 7. Proof of surjectivity of h.

The Barratt-Whitehead Lemma.

Proposition 2.3 (Barratt-Whitehead). *Suppose we are given a commutative diagram with exact rows in* AbGrp*:*

$$\begin{tikzcd} \dots \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "f_n"] & B_n \arrow[r, "j_n"] \arrow[d, "g_n"] & C_n \arrow[r, "k_n"] \arrow[d, "h_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\ \dots \arrow[r] & A'_n \arrow[r, "i'_n"'] & B'_n \arrow[r, "j'_n"'] & C'_n \arrow[r, "k'_n"'] & A'_{n-1} \arrow[r] & \dots \end{tikzcd}$$

where each h_n is an isomorphism. Then there exists a long exact sequence

$$\dots \longrightarrow A_n \xrightarrow{(i_n, f_n)} B_n \oplus A'_n \xrightarrow{g_n - i'_n} B'_n \xrightarrow{k_n h_n^{-1} j'_n} A_{n-1} \longrightarrow \dots .$$

Chain Complexes

Definition 2.4 (Chain Complex). *Let $\mathcal{A}$ be an abelian category. A **$\mathbb{Z}$-graded chain complex in $\mathcal{A}$** is a tuple $((C_n)_{n\in\mathbb{Z}}, (\partial_n)_{n\in\mathbb{Z}})$, consisting of a sequence $(C_n)_{n\in\mathbb{Z}}$ in* $\mathrm{ob}(\mathcal{A})$ *and a sequence $(\partial_n)_{n\in\mathbb{Z}}$ in* $\mathrm{mor}(\mathcal{A})$*, such that*

$$\partial_n \in \mathcal{A}(C_n, C_{n-1}) \qquad \text{and} \qquad \partial_n \circ \partial_{n+1} = 0$$

for all $n \in \mathbb{Z}$.
*Dually, a **$\mathbb{Z}$-graded cochain complex in $\mathcal{A}$** is a $\mathbb{Z}$-graded chain complex in $\mathcal{A}^{\mathrm{op}}$.*

Remark 2.5. Since we consider $\mathbb{Z}$-graded chain complexes only, we will just refer to them as chain complexes. Moreover, a chain complex $((C_n)_{n\in\mathbb{Z}}, (\partial_n)_{n\in\mathbb{Z}})$ will often be denoted by $(C_\bullet, \partial_\bullet)$ or even $(C_\bullet, \partial)$, for the sake of notational simplicity. To distinguish cochain complexes from chain complexes, we will use the notation $(C^\bullet, d^\bullet)$ or $(C^\bullet, d)$ for cochain complexes. However, without further restrictions, the two notions coincide. Indeed, if $(C^\bullet, d^\bullet)$ is a cochain complex, then $(C_\bullet, \partial_\bullet)$ defined by $C_n := C^{-n}$ and $\partial_n := d^{-n+1}$ for all $n \in \mathbb{Z}$ is a chain complex and vice versa. Thus chain and cochain complexes differ by a *sign change of the $\mathbb{Z}$-grading*.

Definition 2.6 (Chain Maps). *Let $\mathcal{A}$ be an abelian category and $(C_\bullet, \partial_\bullet)$, $(C'_\bullet, \partial'_\bullet)$ chain complexes in $\mathcal{A}$. A **chain map $f_\bullet : C_\bullet \to C'_\bullet$** is a sequence $(f_n)_{n\in\mathbb{Z}}$ in* $\mathrm{mor}(\mathcal{A})$ *such that*

$f_n \in \mathcal{A}(C_n, C'_n)$ and the diagram

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \end{tikzcd}$$

commutes for all $n \in \mathbb{Z}$.

Proposition 2.7. *Let $\mathcal{A}$ be an abelian category. Then there is a category with objects chain complexes in $\mathcal{A}$ and morphisms chain maps.*

Proof. Let $f_\bullet : C_\bullet \to C'_\bullet$ and $g_\bullet : C'_\bullet \to C''_\bullet$ be chain maps. Define a map $g_\bullet \circ f_\bullet$ by $g_n \circ f_n$ for each $n \in \mathbb{Z}$. This defines a chain map. Moreover, for each chain complex $C_\bullet$ define $\mathrm{id}_{C_\bullet}$ by id_{C_n} for all $n \in \mathbb{Z}$. It is easy to check, that then $\circ$ is associative and the identity laws hold. □

Definition 2.8 (Ch($\mathcal{A}$)). *Let $\mathcal{A}$ be an abelian category. The category in 2.7 is called the **category of chain complexes in $\mathcal{A}$** and we refer to it as* Ch($\mathcal{A}$).

Homology of Chain Complexes.

Definition 2.9 (Homology). *Let $\mathcal{A}$ be an abelian category and $(C^\bullet, \partial) \in \mathrm{ob}(\mathrm{Ch}(\mathcal{A}))$. For all $n \in \mathbb{Z}$ define the **n-th homology object of $(C^\bullet, \partial)$** to be*

$$H_n(C^\bullet, \partial) := \ker \partial_n / \operatorname{im} \partial_{n+1}. \tag{6}$$

Proposition 2.10. *For each $n \in \mathbb{Z}$ there exists a functor* $\mathsf{Comp} \to \mathsf{AbGrp}$.

Proof. Let $(C_\bullet, \partial_\bullet)$ be a chain complex. Let $x \in \operatorname{im} \partial_{n+1}$. Hence there exists $y \in C_{n+1}$ such that $x = \partial_{n+1} y$. But then $\partial_n x = (\partial_n \circ \partial_{n+1})(y) = 0$ and thus $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$. Define

$$H_n(C_\bullet, \partial_\bullet) := \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} \in \mathrm{ob}(\mathsf{AbGrp}).$$

Let $(C'_\bullet, \partial'_\bullet)$ be a chain complex and $f_\bullet : C_\bullet \to C'_\bullet$ a chain map. Then $f_n(\ker \partial_n) \subseteq \ker \partial'_n$. Indeed, if $y \in f_n(\ker \partial_n)$, there exists $x \in \ker \partial_n$, such that $y = f_n(x)$. Since $f_\bullet$ is a chain map, we thus have $\partial'_n y = (\partial'_n \circ f_n)(x) = (f_{n-1} \circ \partial_n)(x) = 0$. Moreover, we have that $\operatorname{im} \partial_{n+1} \subseteq \ker \pi'_n \circ f_n$, where $\pi'_n : \ker \partial'_n \to H_n(C'_\bullet, \partial'_\bullet)$ is the usual projection. Indeed, if $y \in \operatorname{im} \partial_{n+1}$, we find $x \in C_{n+1}$, such that $y = \partial_{n+1} x$. Since again $f_\bullet$ is a chain map, we have that $f_n y = (f_n \circ \partial_{n+1})(x) = (\partial'_{n+1} \circ f_{n+1})(x) \in \operatorname{im} \partial'_{n+1} = \ker \pi'_n$. Hence $\pi'_n \circ f_n$ factors uniquely through $\pi_n : \ker \partial_n \to H_n(C_\bullet, \partial_\bullet)$. Define $H_n(f_\bullet)$ to be this map. □

Remark 2.11. Let $(C_\bullet, \partial_\bullet)$ be a chain complex and $n \in \mathbb{Z}$. Then we will write $\langle x \rangle$ for an element in $H_n(C_\bullet, \partial_\bullet)$, the so-called *homology class*. Hence if $(C'_\bullet, \partial'_\bullet)$ is another chain complex and $f_\bullet : C_\bullet \to C'_\bullet$ a chain map, then $H_n(f)\langle c \rangle = \langle f_n c \rangle$.

Definition 2.12 (Cycles and Boundaries). *Let $(C_\bullet, \partial_\bullet)$ be a chain complex and $n \in \mathbb{Z}$. Then elements of $\ker \partial_n$ are called **n-cycles** and elements of $\operatorname{im} \partial_{n+1}$ are called **n-boundaries**.*

Definition 2.13 (Homology Functor). *Let $n \in \mathbb{Z}$ and $H_n : \mathsf{Comp} \to \mathsf{AbGrp}$ be the functor defined in proposition 2.10. We call H_n the **n-th homology functor**.*

Constructions.

Definition 2.14 (Subcomplex). *Let $(C_\bullet, \partial_\bullet)$ be a chain complex. A **subcomplex** of $(C_\bullet, \partial_\bullet)$ is a chain complex $(C'_\bullet, \partial'_\bullet)$, such that $C'_n \subseteq C_n$ for all $n \in \mathbb{Z}$ and that $\iota : C'_\bullet \to C_\bullet$ defined by $\iota_n : C'_n \hookrightarrow C_n$ for all $n \in \mathbb{Z}$, is a chain map.*

Definition 2.15 (Quotient Complex). *Let $(C'_\bullet, \partial'_\bullet)$ be a subcomplex of $(C_\bullet, \partial_\bullet)$. Then define the **quotient complex**, written $C_\bullet / C'_\bullet$, by setting*

$$(C_\bullet/C'_\bullet)_n := C_n/C'_n \qquad \text{and} \qquad \partial_n := C_n/C'_n \to C_{n-1}/C'_{n-1},$$

the induced function, for all $n \in \mathbb{Z}$.

Long Exact Sequence in Homology

Theorem 2.16 (Long Exact Sequence in Homology). *Let*

$$0 \longrightarrow C_\bullet \xrightarrow{f_\bullet} C'_\bullet \xrightarrow{g_\bullet} C''_\bullet \longrightarrow 0$$

be a short exact sequence in Comp. Then there exists a sequence $(\delta_n)_{n\in\mathbb{Z}}$, where for all $n \in \mathbb{Z}$, $\delta_n \in \mathsf{AbGrp}(H_n(C''_\bullet), H_{n-1}(C_\bullet))$ and such that

$$\cdots \longrightarrow H_n(C_\bullet) \xrightarrow{H_n(f)} H_n(C'_\bullet) \xrightarrow{H_n(g)} H_n(C''_\bullet) \xrightarrow{\delta_n} H_{n-1}(C_\bullet) \longrightarrow \cdots$$

is a long exact sequence in AbGrp.

Proof. Let $n \in \mathbb{Z}$ and consider the following diagram of induced morphisms:

$$\begin{tikzcd}
 & C_n/\operatorname{im}\partial_{n+1} \arrow[r, "f_n"] \arrow[d, "\partial_n"] & C'_n/\operatorname{im}\partial'_{n+1} \arrow[r, "g_n"] \arrow[d, "\partial'_n"] & C''_n/\operatorname{im}\partial''_{n+1} \arrow[r] \arrow[d, "\partial''_n"] & 0 \\
0 \arrow[r] & \ker\partial_{n-1} \arrow[r, "f_{n-1}"'] & \ker\partial'_{n-1} \arrow[r, "g_{n-1}"'] & \ker\partial''_{n-1} &
\end{tikzcd}$$

It is left to the reader to show that the induced maps are actually well defined, the diagram commutes and the rows are exact. Hence an application of the snake lemma 3.23 yields $\delta_n \in \mathsf{AbGrp}(\ker\partial''_n, \operatorname{coker}\partial_n)$ and an exact sequence

$$\ker\partial_n \xrightarrow{f_n} \ker\partial'_n \xrightarrow{g_n} \ker\partial''_n \xrightarrow{\delta_n} \operatorname{coker}\partial_n \xrightarrow{f_{n-1}} \operatorname{coker}\partial'_n \xrightarrow{g_{n-1}} \operatorname{coker}\partial''_n$$

It is easy to check that this exact sequence is the same as

$$H_n(C_\bullet) \xrightarrow{H_n(f)} H_n(C'_\bullet) \xrightarrow{H_n(g)} H_n(C''_\bullet) \xrightarrow{\delta_n} H_{n-1}(C_\bullet) \xrightarrow{H_{n-1}(f)} H_{n-1}(C'_\bullet) \xrightarrow{H_{n-1}(g)} H_{n-1}(C''_\bullet).$$

□

Exercise 2.17. In the proof of theorem 2.16 in the diagram, show that the induced maps are actually well defined, the diagram commutes and the two rows are exact.

Definition 2.18 (Connecting Homomorphism). *The sequence $(\delta_n)_{n\in\mathbb{Z}}$ of morphisms in* AbGrp *of theorem 2.16 is called the* ***connecting homomorphism of the short exact sequence*** $\mathbf{0 \to C_\bullet \to C'_\bullet \to C''_\bullet \to 0}$.

Proposition 2.19 (Naturality of the Connecting Homomorphism). *Suppose we are given a commutative diagram with exact rows in* Comp*:*

$$\begin{tikzcd}
0 \arrow[r] & A_\bullet \arrow[r, "f"] \arrow[d, "i"] & B_\bullet \arrow[r, "g"] \arrow[d, "j"] & C_\bullet \arrow[r] \arrow[d, "k"] & 0 \\
0 \arrow[r] & A'_\bullet \arrow[r, "f'"'] & B'_\bullet \arrow[r, "g'"'] & C'_\bullet \arrow[r] & 0.
\end{tikzcd}$$

Then there is a commutative diagram with exact rows in AbGrp*:*

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A_\bullet) \arrow[r, "H_n(f)"] \arrow[d, "H_n(i)"] & H_n(B_\bullet) \arrow[r, "H_n(g)"] \arrow[d, "H_n(j)"] & H_n(C_\bullet) \arrow[r, "\delta_n"] \arrow[d, "H_n(k)"] & H_{n-1}(A_\bullet) \arrow[r] \arrow[d, "H_{n-1}(i)"] & \cdots \\
\cdots \arrow[r] & H_n(A'_\bullet) \arrow[r, "H_n(f')"'] & H_n(B'_\bullet) \arrow[r, "H_n(g')"'] & H_n(C'_\bullet) \arrow[r, "\delta'_n"'] & H_{n-1}(A'_\bullet) \arrow[r] & \cdots ,
\end{tikzcd}$$

where δ and δ' are the corresponding connecting homomorphisms.

Proof. That the rows are exact is the content of proposition 3.24. Moreover, the first two squares commute because H_n is a functor. Hence left to check is only the commutativity of the third square. Let $\langle c\rangle \in H_n(C_\bullet)$. Using diagram 13 and figure 11b, we have $\delta_n\langle c\rangle = \langle a\rangle$ as in figure 8a. Hence

$$(H_{n-1}(i)\circ\delta_n)\langle c\rangle = H_{n-1}(i)\langle a\rangle = \langle i_{n-1}(a)\rangle.$$

By the commutativity of the initial diagram and the fact that j is a chain map, we have that

$$(f'_{n-1}\circ i_{n-1})\langle a\rangle = (j_{n-1}\circ f_{n-1})\langle a\rangle = j_{n-1}\partial_n(b) = \partial_n j_n(b).$$

Again, commutativity of the initial diagram implies $g'_n(j_n(b)) = k_n(g_n(b)) = k_n(c)$. Thus we get $\delta'_n\langle k_n(c)\rangle = \langle i_{n-1}(a)\rangle$ as indicated in figure 8b and so

$$(H_{n-1}(i)\circ\delta_n)\langle c\rangle = \langle i_{n-1}(a)\rangle = \delta'_n\langle k_n(c)\rangle = (\delta'_n\circ H_n(k))\langle c\rangle.$$

□

$$\begin{tikzcd}
& & \langle c\rangle \arrow[d, mapsto] \\
& \langle b\rangle \arrow[r, mapsto, "g_n"] \arrow[d, mapsto, "\partial_n"] & \langle c\rangle \\
a \arrow[r, mapsto, "f_{n-1}"'] \arrow[d, mapsto] & \partial_n b & \\
\langle a\rangle & &
\end{tikzcd}$$

(a) $\delta_n\langle c\rangle$.

$$\begin{tikzcd}
& & \langle k(c)\rangle \arrow[d, mapsto] \\
& \langle j_n(b)\rangle \arrow[r, mapsto, "g'_n"] \arrow[d, mapsto, "\partial_n"] & \langle k_n(c)\rangle \\
i_{n-1}(a) \arrow[r, mapsto, "f'_{n-1}"'] \arrow[d] & \partial_n j_n(b) & \\
\langle i_{n-1}(a)\rangle & &
\end{tikzcd}$$

(b) $\delta'\langle k_n(c)\rangle$.

Figure 8

Comparison Theorem

Resolutions.

Definition 2.20 (Projective Module). *A module $P \in \mathrm{ob}({}_R\mathsf{Mod})$ is said to be **projective**, if there is a diagram in ${}_R\mathsf{Mod}$ with exact rows:*

$$\begin{tikzcd}
& P \arrow[dl, dashed, "\exists"'] \arrow[d, "\forall"] & \\
M \arrow[r, "\forall"'] & N \arrow[r] & 0
\end{tikzcd}$$

Proposition 2.21. *Every free module is projective.*

Exercise 2.22. Prove proposition 2.21.

Definition 2.23 (Projective Resolution). *Let $A \in \mathrm{ob}({}_R\mathsf{Mod})$. A **projective resolution of A** is an exact sequence*

$$\ldots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

The Acyclic Models Theorem

Definition 2.24 (Models). *Let $\mathcal{C}$ be a category. A **family of models for $\mathcal{C}$** is a set A together with a family $(M_\alpha)_{\alpha\in A}$ of objects in $\mathcal{C}$.*

Definition 2.25 (F-Model). *Let $\mathcal{C}$ be a category with family of models $(M_\alpha)_{\alpha\in A}$ and $F : \mathcal{C} \to {}_R\mathsf{Mod}$ a functor. A **model for F** is a family $(x_\alpha)_{\alpha\in A}$ where $x_\alpha \in F(M_\alpha)$ for all $\alpha \in A$.*

Definition 2.26 (F-Model Basis). *Let $\mathcal{C}$ be a locally small category with family of models $(M_\alpha)_{\alpha\in A}$ and $F : \mathcal{C} \to {}_R\mathsf{Mod}$ a functor. F is called **free with basis in $(M_\alpha)_{\alpha\in A}$**, if for all $X \in \mathrm{ob}(\mathcal{C})$ there exists an F-model $(x_\alpha)_{\alpha\in A}$ such that*

$$\{F(f)(x_\alpha) : \alpha \in A, f \in \mathcal{C}(M_\alpha, X)\}$$

*is a basis for $F(X)$. The model $(x_\alpha)_{\alpha\in A}$ for F is then called a **model basis for F**.*

Example 2.27. Let $n \in \omega$. Then the one-element family (Δ^n) consisting of the standard n-simplex is a family of models for Top. Moreover, let $C_n : \mathsf{Top} \to \mathsf{AbGrp}$ be the functor which assigns to each topological space X the n-th singular chain group $C_n(X)$. Then C_n is free with basis (id_{Δ^n}). Indeed, we have that

$$F\left(\{C_n(\sigma)(\mathrm{id}_{\Delta^n}) : \sigma \in \mathsf{Top}(\Delta^n, X)\}\right) = F\left(\{\sigma : \sigma \in \mathsf{Top}(\Delta^n, X)\}\right) = C_n(X).$$

Proposition 2.28. *Let $\mathcal{C}$ be a category with a family of models $(M_\alpha)_{\alpha\in A}$. Moreover, suppose that*

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\xi"] & F' \arrow[r, Rightarrow, "\xi'"] \arrow[d, Rightarrow] & F'' \arrow[d, Rightarrow] \\ G \arrow[r, Rightarrow, "\eta"'] & G' \arrow[r, Rightarrow, "\eta'"'] & G'' \end{tikzcd}$$

in ${}_R\mathsf{Mod}^{\mathcal{C}}$, such that

(a) For all $X \in \mathrm{ob}(\mathcal{C})$, we have that $(\xi' \circ \xi)_X = 0$.
(b) For all $\alpha \in A$, we have that $\mathrm{im}\,\eta_M = \ker \eta'_M$.
(c) The diagram commutes for every $X \in \mathrm{ob}(\mathcal{C})$.
(d) F is free with basis in $(M_\alpha)_{\alpha\in A}$.

Then there exists $\zeta \in \mathrm{mor}({}_R\mathsf{Mod}^{\mathcal{C}})$ such that

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\xi"] \arrow[d, Rightarrow, "\zeta"'] & F' \arrow[r, Rightarrow, "\xi'"] \arrow[d, Rightarrow] & F'' \arrow[d, Rightarrow] \\ G \arrow[r, Rightarrow, "\eta"'] & G' \arrow[r, Rightarrow, "\eta'"'] & G'' \end{tikzcd}$$

commutes for every $X \in \mathrm{ob}(\mathcal{C})$.

Theorem 2.29 (Acyclic Models Theorem). *Let $\mathcal{C}$ be a locally small category with family of models $(M_\alpha)_{\alpha\in A}$ and $F, G : \mathcal{C} \to \mathrm{Ch}({}_R\mathsf{Mod})$ functors such that:*

- *$F(X)$ and $G(X)$ are non-negative chain complexes for all $X \in \mathrm{ob}(\mathcal{C})$.*
- *For all $n \in \omega$, the induced functor $F_n : \mathcal{C} \to {}_R\mathsf{Mod}$ is free with basis a subfamily of $(M_\alpha)_{\alpha\in A}$.*
- *For all $\alpha \in A$, $G(M_\alpha)$ is acyclic in positive degrees.*

If $\xi : H_0 \circ F \Rightarrow H_0 \circ G$ is a natural transformation, then there exists a natural chain map $F \Rightarrow G$ over ξ. Moreover, any two such natural chain maps are naturally chain homotopic.

CHAPTER 3

Singular Homology with Coefficients

The Eilenberg-Steenrod Axioms

Definition 3.1 (The Eilenberg-Steenrod Axioms). *A **homology theory** consist of two sequences $(\mathcal{H}_n)_{n\in\omega}$ and $(\delta_n)_{n\in\omega}$, where for each $n \in \omega$, $\mathcal{H}_n : \mathsf{Top}^2 \to \mathsf{AbGrp}$ is a functor and $\delta_n : \mathcal{H}_n \Rightarrow \mathcal{H}_{n-1} \circ R$ if $n > 0$ and $\delta_0 : \mathcal{H}_0 \Rightarrow 0$ is a natural transformation, with $R : \mathsf{Top}^2 \to \mathsf{Top}^2$ defined by $R(X, A) := (A, \varnothing)$, and subject to the following axioms:*

• ***The Exact Sequence Axiom.*** *Let $(X, A) \in \mathrm{ob}(\mathsf{Top}^2)$. Then there exists a long exact sequence*

$$\cdots \longrightarrow \mathcal{H}_n(A) \xrightarrow{\mathcal{H}_n(\iota_A)} \mathcal{H}_n(X) \xrightarrow{\mathcal{H}_n(\iota_X)} \mathcal{H}_n(X, A) \xrightarrow{\delta_n} \mathcal{H}_{n-1}(A) \longrightarrow \cdots$$

where $\iota_A : (A, \varnothing) \hookrightarrow (X, \varnothing)$ and $\iota_X : (X, \varnothing) \hookrightarrow (X, A)$ denote inclusions.
• ***The Dimension Axiom.*** *Let $*$ be the terminal object in Top. Then $\mathcal{H}_0(*) \cong \mathbb{Z}$ and $\mathcal{H}_n(*) = 0$ for all $n \in \omega$, $n > 0$.*

Theorem 3.2 (Baby Uniqueness Theorem). *Suppose $(\mathcal{H}, \delta)$ and $(\mathcal{K}, \varepsilon)$ satisfy the homotopy, exact sequence, excision and dimension axiom and suppose $\Phi_\bullet : (\mathcal{H}, \delta) \Rightarrow (\mathcal{K}, \varepsilon)$ is a natural transformation such that $\Phi_0(*) : \mathcal{H}_0(*) \to \mathcal{K}_0(*)$ is an isomorphism. Then*

$$\Phi_n(X, X') : \mathcal{H}_n(X, X') \to \mathcal{K}_n(X, X')$$

is an isomorphism for all pairs (X, X') consisting of a finite cell complex X and a subcomplex X'.

Proof.
Step 1: $X = \mathbb{S}^0$ and $X' = \varnothing$.
Step 2: $X = \mathbb{S}^k$ and $X' = \varnothing$.
Step 3: General case.

□

Singular Homology with Coefficients

Aim of this section is to construct for each $n \in \omega$ a functor $H_n : \mathsf{Top} \to \mathsf{AbGrp}$, called the *$n$-th singular homology functor*.

Simplices and Affinely Linear Mappings.

Definition 3.3 (Affinely Independent). *Let $n, k \in \omega$. A family $(v_0, \dots, v_k)$ in $\mathbb{R}^n$ is said to be **affinely independent**, iff the following condition is satisfied: Given $\lambda_0, \dots, \lambda_k \in \mathbb{R}$ such that*

$$\sum_{i=0}^{k} \lambda_i = 0 \qquad \text{and} \qquad \sum_{i=0}^{k} \lambda_i v_i = 0$$

implies $c_0 = \cdots = c_k = 0$.

Lemma 3.4. *Let $n, k \in \omega$. Then a family $(v_0, \dots, v_k)$ in $\mathbb{R}^n$ is affinely independent if and only if $(v_1 - v_0, \dots, v_k - v_0)$ is linearly independent in $\mathbb{R}^n$.*

Exercise 3.5. Prove lemma 3.4.

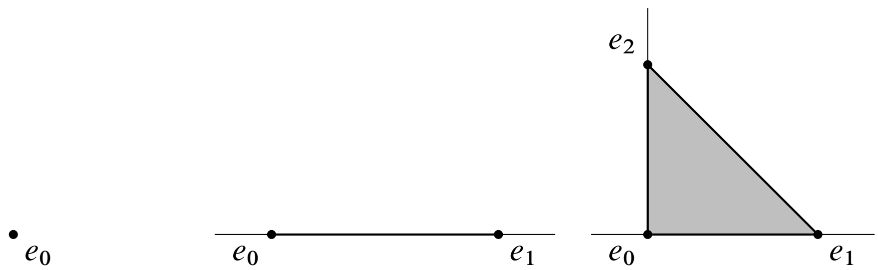
Definition 3.6 (Simplex). *Let $n, k \in \omega$ and $(v_0, \dots, v_k)$ affinely independent in $\mathbb{R}^n$. Define the **simplex spanned by** $(\boldsymbol{v_0, \dots, v_k})$, written $[v_0, \dots, v_n]$, to be the topological subspace*

$$[v_0, \dots, v_k] := \left\{ \sum_{i=0}^{k} \lambda_i v_i : \lambda_i \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=0}^{k} \lambda_i = 1 \right\} \subseteq \mathbb{R}^n.$$

*Moreover, each of the v_i's, $i = 0, \dots, k$, is called a **vertex** of the simplex $[v_0, \dots, v_k]$.*

Remark 3.7. Let $\sigma := [v_0, \dots, v_k]$ be a simplex spanned by $(v_0, \dots, v_k)$. Then we will also simply call σ a k-simplex in $\mathbb{R}^n$.

Example 3.8 (Standard Simplex). Let $n \in \omega$. Then the family $(e_0, \dots, e_n)$ in $\mathbb{R}^n$, where $e_0 := 0$ and $(e_1, \dots, e_n)$ is the standard oriented basis of $\mathbb{R}^n$, is affinely independent by exercise 3.5. The n-simplex spanned by this family is called the ***standard n-simplex*** and is denoted by Δ^n.



(a) Δ^0. (b) Δ^1. (c) Δ^2.

Figure 9. Standard n-simplices.

Lemma 3.9. *Let $n, k \in \omega$ and $[v_0, \dots, v_k]$ a k-simplex in V. Then any $x \in [v_0, \dots, v_k]$ admits a unique representation $x = \sum_{i=0}^{k} \lambda_i v_i$.*

Exercise 3.10. Prove lemma 3.9.

Definition 3.11 (Affinely Linear Mapping). *Let $n, m in \omega$. A mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **affinely linear**, iff there exists an $\mathbb{R}$-linear vector space morphism $L : \mathbb{R}^n \to \mathbb{R}^m$ and $y \in \mathbb{R}^m$, such that*

$$A(x) = L(x) + y$$

holds for all $x \in \mathbb{R}^n$.

Exercise 3.12. Show that the composition of affinely linear mappings is again affinely linear.

Proposition 3.13 (Affine Map induced by Vertex Map). *Let $n, k, m \in \omega$ and $\sigma := [v_0, \dots, v_n]$ a k-simplex in $\mathbb{R}^n$. Given a function $f : \{v_0, \dots, v_k\} \to \mathbb{R}^m$, there exists a unique extension $\widetilde{f} : \sigma \to \mathbb{R}^m$, which is the restriction of an affinely linear map.*

Proof. We show first existence and then uniqueness.
Step 1: Existence. By exercise 3.5, $(v_1 - v_0, \dots, v_k - v_0)$ is linearly independent in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is finite dimensional, we may complete this linearly independent subset to a basis of $\mathbb{R}^n$. Hence there exists a unique vector space morphism $L : \mathbb{R}^n \to \mathbb{R}^m$, mapping

$$v_i - v_0 \mapsto f(v_i) - f(v_0),$$

for $i = 1, \dots, k$ and to the zero vector else. Now $A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$A := L - L(v_0) + f(v_0)$$

is the map we are looking for.
Step 2: Uniqueness. Given another such extension $\widetilde{g} : \sigma \to \mathbb{R}^m$ of f, say $\widetilde{g} = \widetilde{L} + y$, we have that $\widetilde{L}(v_i) = f(v_i) - y$ for all $i = 0, \dots, k$. Thus we compute

$$\widetilde{g}\left(\sum_{i=0}^{k} \lambda_i v_i\right) = \sum_{i=0}^{k} \lambda_i \widetilde{L}(v_i) + y = \sum_{i=0}^{k} \lambda_i f(v_i) - \sum_{i=0}^{k} \lambda_i y + y = \sum_{i=0}^{k} \lambda_i f(v_i).$$

□

Free Abelian Groups.

Proposition 3.14. *The forgetful functor U : $\mathsf{AbGrp} \to \mathsf{Set}$ admits a left adjoint.*

Proof. We have to construct a functor $F : \mathsf{Set} \to \mathsf{AbGrp}$. Let S be a set. Define

$$F(S) := \left\{f \in \mathbb{Z}^S : \text{supp } f \text{ is finite}\right\}.$$

Equipped with pointwise addition, $F(S)$ is an abelian group. There is a natural inclusion $\iota : S \hookrightarrow U\left(F(S)\right)$ sending $x \in S$ to the function taking the value one at x and zero else. Hence we may regard elements of $F(S)$ as formal linear combinations $\sum_{x\in S} m_x x$, where $m_x \in \mathbb{Z}$ for all $x \in S$. On morphisms $f : S \to T$ in Set, define $F(f) : F(S) \to F(T)$ simply by setting $F(f)\left(\sum_{x\in S} m_x x\right) := \sum_{x\in S} m_x f(x)$.
Let $G \in \text{ob}(\mathsf{AbGrp})$ be an abelian group and $\varphi \in \mathsf{AbGrp}\left(F(S), G\right)$ a morphism of groups.

Define $\overline{\varphi} \in \mathsf{Set}\big(S, U(G)\big)$ by $\overline{\varphi} := U(\varphi)$. Conversly, if we have $f \in \mathsf{Set}\big(S, U(G)\big)$, define $\overline{f} \in \mathsf{AbGrp}\big(F(S), G\big)$ by $\overline{f}\left(\sum_{x\in S} m_x x\right) := \sum_{x\in S} m_x f(x)$. This is well defined since all but finitely many m_x are zero and G is abelian. It is easy to check that $\overline{f}$ is indeed a morphism of groups. Let $\varphi \in \mathsf{AbGrp}\big(F(S), G\big)$. Then

$$\begin{aligned}\overline{\overline{\varphi}}\left(\sum_{x\in S} m_x x\right) &= \sum_{x\in S} m_x \overline{\varphi}(x)\\ &= \sum_{x\in S} m_x U(\varphi)(x)\\ &= \sum_{x\in S} m_x \varphi(x)\\ &= \varphi\left(\sum_{x\in S} m_x x\right).\end{aligned}$$

And for $f \in \mathsf{Set}\big(S, U(G)\big)$ we have that

$$\overline{\overline{f}}(x) = U(\overline{f})(x) = \overline{f}(x) = f(x).$$

Hence $\overline{\overline{\varphi}} = \varphi$ and $\overline{\overline{f}} = f$ and so we have a bijection

$$\mathsf{AbGrp}\big(F(S), G\big) \cong \mathsf{Set}\big(S, U(G)\big).$$

The mapping $f \mapsto \overline{f}$ will be referred to as ***extending by linearity***. To check naturality in S and G is left as an exercise. $\square$

Exercise 3.15. In proposition 3.14, check that $F : \mathsf{Set} \to \mathsf{AbGrp}$ is indeed a functor, called the ***free functor from* Set *to* AbGrp**, and the naturality of the bijection in both arguments.

Definition 3.16 (Free Abelian Group). *Let $F : \mathsf{Set} \to \mathsf{AbGrp}$ be the free functor. For any set S, we call $F(S)$ the* ***free group generated by S***.

Theorem 3.17. *There is a functor* $\mathsf{Top} \to \mathsf{Comp}$.

Proof. The proof is divided into several steps. Let us denote $C_\bullet : \mathsf{Top} \to \mathsf{Comp}$ for the claimed functor.

Step 1: Construction of a sequence of abelian groups. Let $v_0, \dots, v_k \in \mathbb{R}^n$ for some $n, k \in \omega$. We say that $(v_0, \dots, v_k)$ is ***affinely independent*** if $(v_1 - v_0, \dots, v_k - v_0)$ is linearly independent. We define the ***k-simplex spanned by $(v_0, \dots, v_k)$***, written $[v_0, \dots, v_k]$, to be

$$[v_0, \dots, v_k] := \left\{\textstyle\sum_{i=0}^k s_i v_i : s_i \geq 0 \text{ for all } i = 0, \dots, k \text{ and } \sum_{i=0}^k s_i = 1\right\}. \tag{7}$$

equipped with the subspace topology. Moreover, we define the ***standard n-simplex Δ^n*** to be the n-simplex spanned by $(e_0, \dots, e_n)$ where $e_0 := 0 \in \mathbb{R}^n$ and $(e_1, \dots, e_n)$ is the standard ordered basis of $\mathbb{R}^n$. Let $X \in \mathrm{ob}(\mathsf{Top})$. Define a ***singular n-simplex in X*** to be a morphism $\sigma \in \mathsf{Top}(\Delta^n, X)$. Let $n \in \mathbb{Z}$. Define

$$C_n(X) := \begin{cases} F\left(\mathsf{Top}(\Delta^n, X)\right) & n \geq 0, \\ 0 & n < 0. \end{cases} \tag{8}$$

We will call elements of $C_n(X)$ ***singular n-chains***.

Step 2: Construction of boundary operators. Let $X \in \mathrm{ob}(\mathsf{Top})$ and σ a singular n-simplex in X for $n \geq 1$. We define $\varphi_k^n : \Delta^{n-1} \to \Delta^n$, called the ***k-th face map***, to be the unique affine map determined by the vertex map

$$\begin{array}{ccc} & \varphi_k^n & \\ e_0 & \mapsto & e_0 \\ \vdots & & \vdots \\ e_{k-1} & \mapsto & e_{k-1} \\ e_k & \mapsto & e_{k+1} \\ \vdots & & \vdots \\ e_{n-1} & \mapsto & e_n. \end{array}$$

Explicitely, given $\sum_{i=0}^{n-1} s_i e_i \in \Delta^{n-1}$, we have that (see [Lee11, p. 152])

$$\varphi_k^n\left(\sum_{i=0}^{n-1} s_i e_i\right) = \sum_{i=0}^{n-1} s_i \varphi_k^n(e_i).$$

Define now

$$\partial\sigma := \sum_{k=0}^{n} (-1)^k \sigma \circ \varphi_k^n \in U\left(C_{n-1}(X)\right) \tag{9}$$

to be the ***boundary of σ***. Moreover, the ***singular boundary operator*** is defined to be $\overline{\partial}_n$ and $\partial_n := 0$ for $n \leq 0$.

Step 3: $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. It is enough to consider $n \geq 1$, since $\partial_n \circ \partial_{n+1} = 0$ holds trivially in the other cases. Let $X \in \mathrm{ob}(\mathsf{Top})$ and $\sigma \in \mathsf{Top}(\Delta^{n+1}, X)$. Then we have

$$\begin{aligned} (\partial_n \circ \partial_{n+1})(\sigma) &= \partial_n\left(\sum_{k=0}^{n+1}(-1)^k \sigma \circ \varphi_k^{n+1}\right) \\ &= \sum_{k=0}^{n+1}(-1)^k \partial_n\left(\sigma \circ \varphi_k^{n+1}\right) \\ &= \sum_{k=0}^{n+1}\sum_{j=0}^{n}(-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \end{aligned}$$

$$
\begin{aligned}
&= \sum_{0\le k\le j\le n} (-1)^{k+j}\sigma\circ\varphi_k^{n+1}\circ\varphi_j^n + \sum_{0\le j<k\le n+1} (-1)^{k+j}\sigma\circ\varphi_k^{n+1}\circ\varphi_j^n \\
&= \sum_{0\le j\le k\le n} (-1)^{k+j}\sigma\circ\varphi_j^{n+1}\circ\varphi_k^n + \sum_{0\le j<k\le n+1} (-1)^{k+j}\sigma\circ\varphi_k^{n+1}\circ\varphi_j^n \\
&= \sum_{0\le j<k\le n+1} \left((-1)^{k+j-1}\sigma\circ\varphi_j^{n+1}\circ\varphi_{k-1}^n + (-1)^{k+j}\sigma\circ\varphi_k^{n+1}\circ\varphi_j^n\right)
\end{aligned}
$$

Since $\varphi_j^{n+1}\circ\varphi_{k-1}^n = \varphi_k^{n+1}\circ\varphi_j^n$, it follows that

$$\partial_n\circ\partial_{n+1} = 0.$$

Indeed, consider the following chart of vertex maps:

$$
\begin{array}{ccccc}
 & \varphi_{k-1}^n & & \varphi_j^{n+1} & \\
e_0 & \mapsto & e_0 & \mapsto & e_0 \\
\vdots & & \vdots & & \vdots \\
e_{j-1} & \mapsto & e_{j-1} & \mapsto & e_{j-1} \\
e_j & \mapsto & e_j & \mapsto & e_{j+1} \\
\vdots & & \vdots & & \\
e_{k-1} & \mapsto & e_{k-1} & \mapsto & e_{k+1} \\
e_k & \mapsto & e_{k+1} & \mapsto & e_{k+2} \\
\vdots & & \vdots & & \\
e_{n-1} & \mapsto & e_n & \mapsto & e_{n+1}
\end{array}
\qquad
\begin{array}{ccccc}
 & \varphi_j^n & & \varphi_k^{n+1} & \\
e_0 & \mapsto & e_0 & \mapsto & e_0 \\
\vdots & & \vdots & & \vdots \\
e_{j-1} & \mapsto & e_{j-1} & \mapsto & e_{j-1} \\
e_j & \mapsto & e_{j+1} & \mapsto & e_{j+1} \\
\vdots & & \vdots & & \\
e_{k-1} & \mapsto & e_k & \mapsto & e_{k+1} \\
e_k & \mapsto & e_{k+1} & \mapsto & e_{k+2} \\
\vdots & & \vdots & & \\
e_{n-1} & \mapsto & e_n & \mapsto & e_{n+1}
\end{array}.
$$

Step 4: Construction of chain maps. Let $X, Y \in \mathrm{ob}(\mathsf{Top})$ and $f \in \mathsf{Top}(X,Y)$. For $n \ge 0$, define $f_n^\# : \mathsf{Top}(\Delta^n, X) \to U\left(C_n(Y)\right)$ by $f^\# := f\circ\sigma$. Extending this map by linearity yields a homomorphism $f_n^\# : C_n(X) \to C_n(Y)$. Moreover, set $f_n^\# := 0$ for $n < 0$. Let $n \ge 1$ and $\sigma \in \mathsf{Top}(\Delta^n, X)$. Then on one hand we have

$$(f_{n-1}^\#\circ\partial_n)(\sigma) = f_{n-1}^\#\left(\sum_{k=0}^n (-1)^k\sigma\circ\varphi_k^n\right) = \sum_{k=0}^n (-1)^k f\circ\sigma\circ\varphi_k^n$$

and on the other

$$(\partial_n\circ f_n^\#)(\sigma) = \partial_n(f\circ\sigma) = \sum_{k=0}^n (-1)^k f\circ\sigma\circ\varphi_k^n.$$

Checking, that $C_\bullet$ is indeed a functor is left as an exercise. □

Exercise 3.18. Show that $C_\bullet : \mathsf{Top} \to \mathsf{Comp}$ is a functor.

The Homology Functor.

Definition 3.19 (Singular Homology Functor). *Let $n \in \mathbb{Z}$. The composition*

$$H_n \circ C_\bullet : \mathsf{Top} \to \mathsf{AbGrp} \tag{10}$$

*of the singular chain complex functor $C_\bullet$ in theorem 3.17 and the n-th homology functor of proposition 2.10 is called the **n-th singular homology functor**, written H_n^{sing}.*

Remark 3.20. For notational purposes we will often refer to the functor H_n^{sing} simply as H_n.

Relative Homology.

Proposition 3.21. *There is a functor* $\mathsf{Top}^2 \to \mathsf{Comp}$.

Proof. Let $(X, A) \in \mathrm{ob}(\mathsf{Top}^2)$. Then we have an inclusion $\iota : A \hookrightarrow X$. Moreover, we have that $C_n(\iota)$ is injective for all $n \in \mathbb{Z}$. Indeed, this is obvious for $n < 0$ and for $n \geq 0$, suppose that $\sum_k m_k\sigma_k \in \ker C_n(\iota)$, where the σ_k are distinct. Then we have that $0 = C_n(\iota)\left(\sum_k m_k\sigma_k\right) = \sum_k m_k\iota \circ \sigma_k$, where the $\iota \circ \sigma_k$ are also distinct. Thus we conclude $\sum_k m_k\sigma_k = 0$. Hence we can see $C_\bullet(A)$ as a subcomplex of $C_\bullet(X)$ and so we can define

$$C_\bullet(X, A) := C_\bullet(X)/C_\bullet(A).$$

Moreover, on morphsisms $f \in \mathsf{Top}^2\left((X, A), (Y, B)\right)$ just let $C_\bullet(f)$ be the induced map. □

Definition 3.22 (Relative Homology Functor). *For $n \in \mathbb{Z}$, the functor*

$$H_n \circ C_\bullet : \mathsf{Top}^2 \to \mathsf{AbGrp} \tag{11}$$

*is called the **n-th relative singular homology functor.***

The Exact Sequence Axiom

Homological Algebra.

Diagram Lemmas.

Proposition 3.23 (Snake Lemma). *Suppose we are given a commutative diagram in* AbGrp *with exact rows:*

$$\begin{tikzcd}
 & A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "j"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "i'"'] & B' \arrow[r, "j'"'] & C' &
\end{tikzcd}$$

Then there exists $\delta \in \mathsf{AbGrp}(\ker h, \operatorname{coker} f)$ such that the sequence

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h \tag{12}$$

is exact.

Proof. Consider the augmented diagram in figure 10, where the morphisms k, l, p and q are induced by i, j, i' and j', respectively.

$$
\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
& \ker f \arrow[r, "k"] \arrow[d] & \ker g \arrow[r, "l"] \arrow[d] & \ker h \arrow[d] & \\
& A \arrow[r, "i"] \arrow[d, "f"] & B \arrow[r, "j"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "i'"'] \arrow[d] & B' \arrow[r, "j'"'] \arrow[d] & C' \arrow[d] & \\
& \operatorname{coker} f \arrow[r, "p"'] \arrow[d] & \operatorname{coker} g \arrow[r, "q"'] \arrow[d] & \operatorname{coker} h \arrow[d] & \\
& 0 & 0 & 0 &
\end{tikzcd}
$$

Figure 10. Proof of the snake lemma.

Step 1: Exactness at $\ker g$. Let $a \in \ker f$. Then $l(k(a)) = j(i(a)) = 0$ by exactness at B and thus $\operatorname{im} k \subseteq \ker l$. Conversly, let $b \in \ker l$. Then $j(b) = 0$ and by exactness at B, there exists $a \in A$ such that $i(a) = b$. Moreover $0 = g(b) = g(i(a)) = i'(f(a))$ since $b \in \ker g$ and thus $f(a) = 0$ by injectivity of i'. Hence $\ker j \subseteq \operatorname{im} k$.
Step 2: Exactness at $\operatorname{coker} g$. Let $a' + \operatorname{im} f \in \operatorname{coker} f$. Then

$$q(p(a' + \operatorname{im} f)) = j'(i'(a')) + \operatorname{im} h = \operatorname{im} h$$

by exactness at B' implies $\operatorname{im} p \subseteq \ker q$. Conversly, let $b' + \operatorname{im} g \in \ker q$. Then

$$0 = q(b' + \operatorname{im} g) = j'(b') + \operatorname{im} h$$

and thus $j'(b') \in \operatorname{im} h$. Hence there exists $c \in C$, such that $j'(b') = h(c)$. Since j is surjective, we find $b \in B$ such that $j(b) = c$. Therefore $j'(b') = h(j(b))$. By commutativity we get $j'(b') = j'(g(b))$ which is equivalent to $j'(b' - g(b)) = 0$. Thus $b' - g(b) \in \ker j'$ and exactness at B' yields the existence of $a' \in A'$ such that $i'(a') = b' - g(b)$. Now

$$p(a' + \operatorname{im} f) = i'(a') + \operatorname{im} g = b' - g(b) + \operatorname{im} g = b' + \operatorname{im} g$$

and thus $\ker q \subseteq \operatorname{im} p$.

Step 3: Definition of δ. Consider the snakelike path indicated in figure 11a. Let $c \in \ker h$. Since j is surjective, we find $b \in B$ such that $j(b) = c$. Since $c \in \ker h$, we get that $0 = h(c) = h(j(b)) = j'(g(b))$ and thus $g(b) \in \ker j'$ which implies $g(b) \in \operatorname{im} i'$ by exactness at B'. Hence there exists $a' \in A'$ such that $i'(a') = g(b)$. Actually this a' is unique since i' is injective. Define $\delta : \ker h \to \operatorname{coker} f$ by

$$\delta(c) := a' + \operatorname{im} f.$$

$$\begin{tikzcd}
\ker f \arrow[r, "k"] & \ker g \arrow[r, "l"] & \ker h \arrow[d] \\
 & B & C \arrow[l] \\
A' \arrow[d] & B' \arrow[l] & \\
\operatorname{coker} f \arrow[r, "p"'] & \operatorname{coker} g \arrow[r, "q"'] & \operatorname{coker} h
\end{tikzcd}$$

(a) Snakelike path.

$$\begin{tikzcd}
 & & c \arrow[d, mapsto] \\
 & b \arrow[d, mapsto] & c \arrow[l, mapsto] \\
a' \arrow[d, mapsto] & g(b) \arrow[l, mapsto] & \\
a' + \operatorname{im} f & &
\end{tikzcd}$$

(b) Definition of δ.

Figure 11

Step 4: Checking that δ *is a morphism of groups.* Since j is only surjective, we have to show that δ is a function. So suppose we choose $b_0 \in B$ instead of $b \in B$ in figure 11b with $b_0 \neq b$. We want to show that $\delta(c) = a' + \operatorname{im} f = a'_0 + \operatorname{im} f$, or equivalently $a' - a'_0 \in \operatorname{im} f$. Since $c = j(b) = j(b_0)$, we have that $b - b_0 \in \ker j$. Hence by exactness at B there exists $a \in A$ such that $i(a) = b - b_0$. Applying g and invoking commutativity yields

$$g(b) - g(b_0) = g(i(a)) = i'(f(a))$$

Hence $i'(a') - i'(a'_0) = i'(f(a))$ and thus the injectivity of i' yields $a' - a'_0 = f(a)$. In the same manner one can show that δ is a morphism of groups.
Step 5: Exactness at $\ker h$. Let $b \in \ker g$. Then $\operatorname{im} l \subseteq \ker \delta$ immeddiately follows from figure 12a. Conversely, suppose $c \in \ker \delta$. From figure 12b we get that

$$g(b) = i'(a') = i'(f(a)) = g(i(a))$$

and thus $b - i(a) \in \ker g$. So $l(b - i(a)) = j(b) - j(i(a)) = j(b) = c$ by exactness at B and thus $\ker \delta \subseteq \operatorname{im} l$.
Step 6: Exactness at $\operatorname{coker} f$. Suppose that $a' + \operatorname{im} f \in \operatorname{im} \delta$. Then

$$p(a' + \operatorname{im} f) = i'(a') + \operatorname{im} g = g(b) + \operatorname{im} g = \operatorname{im} g$$

$$\begin{tikzcd}
 & & j(b) \arrow[d, mapsto] \\
 & b \arrow[d, mapsto] & j(b) \arrow[l, mapsto] \\
0 \arrow[d, mapsto] & g(b) = 0 \arrow[l, mapsto] & \\
\operatorname{im} f & &
\end{tikzcd}$$

(a) $\operatorname{im} l \subseteq \ker \delta$.

$$\begin{tikzcd}
 & & c \arrow[d, mapsto] \\
a \arrow[d, mapsto] & b \arrow[d, mapsto] & c \arrow[l, mapsto] \\
a' = f(a) \arrow[d, mapsto] & g(b) \arrow[l, mapsto] & \\
a' + \operatorname{im} f & &
\end{tikzcd}$$

(b) $\ker \delta \subseteq \operatorname{im} l$.

Figure 12

and thus $\operatorname{im} \delta \subseteq \ker p$. Conversly, suppose that $a' + \operatorname{im} f \in \ker p$. Hence $i'(a') \in \operatorname{im} g$ and we find $b \in B$ such that $g(b) = i'(a')$. Consider $j(b)$. By exactness at B' follows

$$h(j(b)) = j'(g(b)) = j'(i'(a')) = 0$$

So $j(b) \in \ker h$. Moreover, by construction $\delta(j(b)) = a' + \operatorname{im} f$ and thus $\ker p \subseteq \operatorname{im} \delta$. □

Proposition 3.24 (Long Exact Sequence in Homology). *Let*

$$0 \longrightarrow C_\bullet \xrightarrow{f_\bullet} C'_\bullet \xrightarrow{g_\bullet} C''_\bullet \longrightarrow 0$$

be a short exact sequence in Comp. *Then there exists a sequence* $(\delta_n)_{n \in \mathbb{Z}}$, *where for all* $n \in \mathbb{Z}$, $\delta_n \in \mathsf{AbGrp}(H_n(C''_\bullet), H_{n-1}(C_\bullet))$ *and such that*

$$\cdots \longrightarrow H_n(C_\bullet) \xrightarrow{H_n(f)} H_n(C'_\bullet) \xrightarrow{H_n(g)} H_n(C''_\bullet) \xrightarrow{\delta_n} H_{n-1}(C_\bullet) \longrightarrow \cdots$$

is a long exact sequence in AbGrp.

Proof. Let $n \in \mathbb{Z}$ and consider the following diagram of induced morphisms:

$$\begin{tikzcd}
 & C_n / \operatorname{im} \partial_{n+1} \arrow[r, "f_n"] \arrow[d, "\partial_n"] & C'_n / \operatorname{im} \partial'_{n+1} \arrow[r, "g_n"] \arrow[d, "\partial'_n"] & C''_n / \operatorname{im} \partial''_{n+1} \arrow[r] \arrow[d, "\partial''_n"] & 0 \\
0 \arrow[r] & \ker \partial_{n-1} \arrow[r, "f_{n-1}"'] & \ker \partial'_{n-1} \arrow[r, "g_{n-1}"'] & \ker \partial''_{n-1} &
\end{tikzcd} \tag{13}$$

It is left to the reader to show that the induced maps are actually well defined, the diagram commutes and the rows are exact. Hence an application of the snake lemma 3.23 yields

$\delta_n \in \mathsf{AbGrp}(\ker \partial_n'', \operatorname{coker} \partial_n)$ and an exact sequence

$$\ker \partial_n \xrightarrow{f_n} \ker \partial_n' \xrightarrow{g_n} \ker \partial_n'' \xrightarrow{\delta_n} \operatorname{coker} \partial_n \xrightarrow{f_{n-1}} \operatorname{coker} \partial_n' \xrightarrow{g_{n-1}} \operatorname{coker} \partial_n''$$

It is easy to check that this exact sequence is the same as

$$H_n(C_\bullet) \xrightarrow{H_n(f)} H_n(C_\bullet') \xrightarrow{H_n(g)} H_n(C_\bullet'') \xrightarrow{\delta_n} H_{n-1}(C_\bullet) \xrightarrow{H_{n-1}(f)} H_{n-1}(C_\bullet') \xrightarrow{H_{n-1}(g)} H_{n-1}(C_\bullet'').$$

$\square$

Exercise 3.25. In the proof of theorem 3.24 in the diagram, show that the induced maps are actually well defined, the diagram commutes and the two rows are exact.

Definition 3.26 (Connecting Homomorphism). *The sequence $(\delta_n)_{n\in\mathbb{Z}}$ of morphisms in* AbGrp *of theorem 3.24 is called the* ***connecting homomorphism of the short exact sequence*** $\mathbf{0 \to C_\bullet \to C_\bullet' \to C_\bullet'' \to 0}$.

Corollary 3.27 (The Exact Sequence Axiom). *Consider the relative homology functors $(H_n)_{n\in\omega}$. Moreover, for each $(X, A) \in \mathrm{ob}(\mathsf{Top})^2$, let $(\delta_{n,(X,A)})_{n\in\omega}$ be the sequence of connecting homomorphisms of the short exact sequence*

$$0 \longrightarrow C_\bullet(A) \xrightarrow{C_\bullet(\iota_A)} C_\bullet(X) \xrightarrow{C_\bullet(\iota_X)} C_\bullet(X, A) \longrightarrow 0,$$

where $\iota_A : (A, \varnothing) \hookrightarrow (X, \varnothing)$ and $\iota_X : (X, \varnothing) \hookrightarrow (X, A)$ denote inclusions. Then δ_n is a natural transformation for $n > 0$ and there is a long exact sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{H_n(\iota_A)} H_n(X) \xrightarrow{H_n(\iota_X)} H_n(X, A) \xrightarrow{\delta_{n,(X,A)}} H_{n-1}(A) \longrightarrow \cdots$$

Proof. The only thing to show is that $\delta_n : H_n \Rightarrow H_{n-1} \circ R$ is a natural transformation for $n > 0$. Hence for any $f \in \mathsf{Top}^2\big((X, A), (Y, B)\big)$, we have to show that

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "\delta_{n,(X,A)}"] \arrow[d, "H_n(f)"'] & H_{n-1}(A) \arrow[d, "H_{n-1}(R(f))"] \\
H_n(Y, B) \arrow[r, "\delta_{n,(Y,B)}"'] & H_{n-1}(B)
\end{tikzcd}$$

commutes. To this end, consider the following commutative diagram with exact rows in Comp:

$$\begin{tikzcd}
0 \arrow[r] & C_\bullet(A) \arrow[r, "C_\bullet(\iota_A)"] \arrow[d, "C_\bullet(R(f))"] & C_\bullet(X) \arrow[r, "C_\bullet(\iota_X)"] \arrow[d, "C_\bullet(S(f))"] & C_\bullet(X, A) \arrow[r] \arrow[d, "C_\bullet(f)"] & 0 \\
0 \arrow[r] & C_\bullet(\iota_B) \arrow[r, "C_\bullet(\iota_B)"'] & C_\bullet(Y) \arrow[r, "C_\bullet(\iota_Y)"'] & C_\bullet(Y, B) \arrow[r] & 0,
\end{tikzcd}$$

where $S : \mathsf{Top}^2 \to \mathsf{Top}^2$ is th functor defined by $S(X, A) := (X, \varnothing)$. Applying proposition 2.19 to the above diagram yields the result. $\square$

The Dimension Axiom

In general, it is hard to compute $H_n(X)$ for an arbitrary topological space X and $n \in \omega$. However, as the next proposition shows, we can always compute $H_0(X)$ for a path connected space X. Combining this with lemma 3.65, we know how $H_0(X)$ looks for an arbitrary topological space X.

Proposition 3.28 (Zeroth Singular Homology Group). *Let $X \in \mathrm{ob}(\mathsf{Top})$ be non empty and path connected. Then $H_0(X) \cong \mathbb{Z}$ and any generator is of the form $\langle x \rangle$ for some $x \in X$.*

Proof. Since $\partial_0 : C_0(X) \to 0$, $\ker \partial_0 = C_0(X)$. Moreover, a map in $\mathsf{Top}(\Delta^0, X)$ can be identified with a point in X and hence an element of $C_0(X)$ can be written as $\sum_{x\in X} m_x x$. Define a mapping $\Phi : C_0(X) \to \mathbb{Z}$ by $\Phi\left(\sum_{x\in X} m_x x\right) := \sum_{x\in X} m_x$. This mapping is well defined since all but finitely many m_x are zero. It is also easy to check, that Φ is a morphism of groups and that Φ is surjective. We claim that $\ker \Phi = \mathrm{im}\, \partial_1$. Indeed, if $\sum_{x\in X} m_x x \in \ker \Phi$, then $\sum_{x\in X} m_x = 0$. Let $p \in X$. Since X is path connected, we find for each $x \in X$ a path σ_x from p to x. Consider the singular 1-chain $\sum_{x\in X} m_x \sigma_x$. Then we have

$$\partial_1\left(\sum_{x\in X} m_x \sigma_x\right) = \sum_{x\in X} m_x \left(\sigma_x(1) - \sigma_x(0)\right) = \sum_{x\in X} m_x(x - p) = \sum_{x\in X} m_x x.$$

Hence $\sum_{x\in X} m_x x \in \mathrm{im}\, \partial_1$. Conversely, it is enough to show the claim on basis elements $\sigma \in \mathsf{Top}(\Delta^1, X)$. We have

$$\Phi(\partial_1 \sigma) = \Phi\left(\sigma(1) - \sigma(0)\right) = 1 - 1 = 0.$$

Hence the first isomorphism theorem [Gri07, p. 23] implies that $H_0(X) \cong \mathbb{Z}$. Let $x \in X$. Then $\mathbb{Z}\langle x \rangle = H_0(X)$. Indeed, if $\sum_{y\in X} m_y y \in C_0(X)$, we have that $\sum_{y\in X} m_y \langle y \rangle = \sum_{y\in X} m_y \langle x \rangle$, since X is path connected we always find a path from x to y and hence $\langle x \rangle = \langle y \rangle$ for all $y \in X$. Suppose $\langle g \rangle$, where $g := \sum_{y\in X} m_y y \in C_0(X)$, is a generator of $H_0(X)$. Since isomorphisms map generators to generators,we have that $\Phi(g) = \pm 1$. Replacing g with $-g$, if necessary, we can assume that $\Phi(g) = 1$. Moreover, $g = x + (g - x)$ for any $x \in X$. Then $g - x \in \mathrm{im}\, \partial_1$. Indeed, we have that $\Phi(g - x) = 1 - 1 = 0$. □

Proposition 3.29. *Let $* \in \mathrm{ob}(\mathsf{Top})$ be a one point space. Then $H_n(*) = 0$ for all $n \in \omega$, $n > 0$.*

Proof. Since $*$ is a one-point space, we have that there is only one singular n-simplex in $*$, say σ_n. We compute

$$\partial_n \sigma_n = \sum_{k=0}^{n} (-1)^k \sigma_n \circ \varphi_k^n = \sum_{k=0}^{n} (-1)^k \sigma_{n-1} = \begin{cases} \sigma_{n-1} & n \text{ even}, \\ 0 & n \text{ odd}. \end{cases}$$

Hence

$$\ker \partial_n = \begin{cases} C_n(*) & n \text{ odd,} \\ 0 & n \text{ even.} \end{cases}$$

Moreover

$$\partial_{n+1}\sigma_{n+1} = \sum_{k=0}^{n+1}(-1)^k \sigma_{n+1} \circ \varphi_k^{n+1} = \sum_{k=0}^{n+1}(-1)^k \sigma_n = \begin{cases} 0 & n \text{ even,} \\ \sigma_n & n \text{ odd.} \end{cases}$$

So

$$\operatorname{im} \partial_{n+1} = \begin{cases} 0 & n \text{ even,} \\ C_n(*) & n \text{ odd,} \end{cases}$$

and thus $H_n(*) = 0$ for all $n > 0$. □

The Homotopy Axiom

The Acyclic Models Theorem.

Proposition 3.30. *Let $\mathcal{C}$ be a locally small category with family of models $(M_\alpha)_{\alpha\in A}$ and $K, L : \mathcal{C} \to \mathsf{AbGrp}$ two functors, where L is free with basis in $(M_\alpha)_{\alpha\in A}$ and model basis $(g_\alpha)_{\alpha\in A}$ for L. Moreover, let $(h_\alpha)_{\alpha\in A}$ be a family such that $h_\alpha \in K(M_\alpha)$ for all $\alpha \in A$. Then there exists a uniqe natural transformation $\Phi : L \Rightarrow K$ such that $\Phi_{M_\alpha}(g_\alpha) = h_\alpha$ for all $\alpha \in A$.*

Chain Homotopies and the Homotopy Axiom.

Definition 3.31 (Chain Homotopies). *Two chain maps $f_\bullet, g_\bullet : (C_\bullet, \partial_\bullet) \to (C'_\bullet, \partial'_\bullet)$ are said to be **chain homotopic**, written $f_\bullet \simeq g_\bullet$, if there exists a sequence $(F_n)_{n\in\mathbb{Z}}$, such that $F_n \in \mathsf{AbGrp}(C_n, C'_{n+1})$ and*

$$\partial'_{n+1} \circ F_n + F_{n-1} \circ \partial_n = f_n - g_n,$$

*for all $n \in \mathbb{Z}$. The sequence $(F_n)_{n\in\mathbb{Z}}$ is then called a **chain homotopy** and is written $F : f_\bullet \simeq g_\bullet$.*

Proposition 3.32. *Let $f_\bullet, g_\bullet \in \mathsf{Comp}(C_\bullet, C'_\bullet)$ with $f \simeq g$. Then $H_n(f_\bullet) = H_n(g_\bullet)$ for all $n \in \mathbb{Z}$.*

Proof. Assume $F : f_\bullet \simeq g_\bullet$ and let $\langle c\rangle \in H_n(C_\bullet) \to H_n(C'_\bullet)$. Then

$$\begin{aligned} H_n(f_\bullet)\langle c\rangle &= \langle f_n(c)\rangle \\ &= \langle g_n(c) + \partial'_{n+1}(F_n(c)) + F_{n-1}(\partial_n(c))\rangle \\ &= \langle g_n(c) + \partial'_{n+1}(F_n(c))\rangle \\ &= \langle g_n(c)\rangle \\ &= H_n(g_\bullet)\langle c\rangle \end{aligned}$$

since $c \in \ker \partial_n$. □

Proposition 3.33. *Let X be a topological space and define $\iota, \jmath : X \to X \times I$ by*

$$\iota(x) := (x, 0) \qquad \text{and} \qquad \jmath(x) := (x, 1).$$

Then $H_n(\iota) = H_n(\jmath)$ for all $n \in \omega$.

Theorem 3.34 (The Homotopy Axiom). *Let $f, g \in \mathsf{Top}(X, Y)$ be freely homotopic. Then $H_n(f) = H_n(g)$ for all $n \in \omega$.*

Proof. Using the notation and the result from proposition 3.32, we get that

$$H_n(f) = H_n(F \circ \iota) = H_n(F) \circ H_n(\iota) = H_n(F) \circ H_n(\jmath) = H_n(F \circ \jmath) = H_n(g).$$

□

The Excision Axiom

Barycentric Subdivision.

Definition 3.35 (Cone). *Let $m, n \in \omega$, $K \subseteq \mathbb{R}^m$ convex, $v_0, \ldots, v_n, w \in K$ and suppose $\alpha := A(v_0, \ldots, v_n)$ is an affine n-simplex. Then define the **cone on $\boldsymbol{\alpha}$ from $\mathbf{w}$**, written $\mathrm{Cone}_w(\alpha)$, by*

$$\mathrm{Cone}_w(\alpha) := A(w, v_0, \ldots, v_n).$$

Moreover, for an affine chain $c := \sum_i m_i \alpha_i$, define

$$\mathrm{Cone}_w(c) := \sum_i m_i \, \mathrm{Cone}_w(\alpha_i).$$

Lemma 3.36. *Let $m, n \in \omega$, $K \subseteq \mathbb{R}^m$ convex, $v_0, \ldots, v_n, w \in K$ and $\alpha := A(v_0, \ldots, v_n)$. Then*

$$\partial \, \mathrm{Cone}_w(\alpha) + \mathrm{Cone}_w(\partial \alpha) = \alpha.$$

Proof. We compute

$$\begin{aligned}\mathrm{Cone}_w(\partial\alpha) &= \sum_{k=0}^{n} (-1)^k \, \mathrm{Cone}_w\left(A(v_0, \ldots, \widehat{v}_k, \ldots, v_n)\right) \\ &= \sum_{k=0}^{n} (-1)^k A(w, v_0, \ldots, \widehat{v}_k, \ldots, v_n),\end{aligned}$$

since $\alpha \circ \varphi_k^n = A(v_0, \ldots, \widehat{v}_k, \ldots, v_n)$. Thus

$$\begin{aligned}\partial \, \mathrm{Cone}_w(\alpha) &= \sum_{k=0}^{n+1} (-1)^k \, \mathrm{Cone}_w(\alpha) \circ \varphi_k^{n+1} \\ &= \alpha + \sum_{k=1}^{n+1} (-1)^k A(w, v_0, \ldots, \widehat{v}_{k-1}, \ldots, v_n)\end{aligned}$$

$$
\begin{aligned}
&= \alpha - \sum_{k=0}^{n} (-1)^k A(w, v_0, \ldots, \widehat{v}_k, \ldots, v_n) \\
&= \alpha - \mathrm{Cone}_w(\partial \alpha).
\end{aligned}
$$

□

Definition 3.37 (Barycenter). *Let $\sigma := [v_0, \ldots, v_k] \subseteq \mathbb{R}^n$. Define the **barycenter of σ**, written b_σ, to be*

$$
b_\sigma := \frac{1}{1+k} \sum_{i=0}^{k} v_i .
$$

Definition 3.38 (Affine Barycentric Subdivision Operator). *Let $K \subseteq \mathbb{R}^m$ be convex. Define the **affine barycentric subdivision operator** $s : C_n^{\mathrm{aff}}(K) \to C_n^{\mathrm{aff}}(K)$ inductively by*

$$
s_n^{\mathrm{aff}}(\sigma) := \begin{cases} \sigma & n = 0, \\ \mathrm{Cone}_{\sigma(b_n)} \left(s_{n-1}^{\mathrm{aff}}(\partial \sigma) \right), & n > 0, \end{cases}
$$

where $b_n := b_{\Delta^n}$, on affine n-simplices $\sigma : \Delta^n \to K$ and extend by linearity.

Exercise 3.39. Let $m, n \in \omega$, $K \subseteq \mathbb{R}^m$ convex, $v_0, \ldots, v_n, w \in K$. Show that

$$
s_n^{\mathrm{aff}}(A(v_0, \ldots, v_n)) = \sum_{\sigma \in S_{n+1}} \mathrm{sgn}(\sigma) A(v_0^\sigma, \ldots, v_n^\sigma),
$$

where

$$
v_i^\sigma := \frac{1}{n-i+1} \sum_{k=i}^{n} v_{\sigma(k)}.
$$

Definition 3.40 (Barycentric Subdivision Operator). *Let $X \in \mathrm{ob}(\mathsf{Top})$. Define the **barycentric subdivision operator $s : C_n(X) \to C_n(X)$** by*

$$
s_n(\sigma) := \begin{cases} 0 & n < 0, \\ C_n(\sigma) \left(s_n^{\mathrm{aff}}(\mathrm{id}_{\Delta^n}) \right) & n \geq 0, \end{cases}
$$

on n-simplices σ.

Lemma 3.41. *Let $K \subseteq \mathbb{R}^m$ convex. Then for any affine n-simplex α we have that*

$$
s_n(\alpha) = s_n^{\mathrm{aff}}(\alpha).
$$

Proof. Let $\alpha := A(v_0, \ldots, v_n)$. Observe that exercise 3.39 yields

$$
\begin{aligned}
s(\alpha) &= C_n(\alpha)(s^{\mathrm{aff}}(\mathrm{id}_{\Delta^n})) \\
&= C_n(\alpha) \left(s^{\mathrm{aff}}(A(e_0, \ldots, e_n)) \right) \\
&= \sum_{\sigma \in S_{n+1}} \mathrm{sgn}(\sigma) \alpha \circ A(e_0^\sigma, \ldots, e_n^\sigma)
\end{aligned}
$$

$$= \sum_{\sigma \in S_{n+1}} \operatorname{sgn}(\sigma) A(v_0^\sigma, \dots, v_n^\sigma)$$
$$= s^{\mathrm{aff}}(\alpha).$$

□

Lemma 3.42. *Let $\sigma := [v_0, \dots, v_n] \subseteq \mathbb{R}^m$ be some n-simplex. Then for all $x, y \in \sigma$ we have that*

$$|x - y| \leq \sup_{k=0,\dots,n} |v_k - y| \,.$$

Moreover

$$|b_\sigma - v_k| \leq \frac{n}{n+1} \operatorname{diam} \sigma = \max_{i,j=0,\dots,n} |v_i - v_j| \,,$$

for all $k = 0, \dots, n$.

Proof. Let $x, y \in \sigma$ and write $x = \sum_{k=0}^n s_k v_k$. Since $\sum_{k=0}^n s_k = 1$, we have that

$$|x - y| = \left| \sum_{k=0}^n s_k v_k - y \right| \leq \sum_{k=0}^n s_k \, |v_k - y| \leq \sup_{k=0,\dots,n} |v_k - y| \,.$$

□

Definition 3.43 (Mesh). *Let $K \subseteq \mathbb{R}^m$ convex and $c \in C_n^{\mathrm{aff}}(K)$. Define the **mesh of c** to be the maximum of the diameters of the images of the affine simplices that appear in c.*

Proposition 3.44. *The barycentric subdivision operators $(s_n)_{n \in \mathbb{Z}}$ have the following properties:*

(a) For any $f \in \mathsf{Top}(X, Y)$, we have that $s_n \circ C_n(f) = C_n(f) \circ s_n$.
(b) $\partial_n \circ s_n = s_{n-1} \circ \partial_n$.
(c) Given an open cover $\mathcal{U}$ of X and some $c \in C_n(X)$, there exists $k \in \omega$, such that $s_n^k(c) \in C_n^{\mathcal{U}}(X)$.

Proof. For proving (a), let $\sigma : \Delta^n \to X$ be an n-simplex. Then by the functoriality of C_n we get that

$$\begin{aligned}
(s_n \circ C_n(f))\,(\sigma) &= s_n(f \circ \sigma) \\
&= C_n(f \circ \sigma)\,(s_n^{\mathrm{aff}}(\mathrm{id}_{\Delta^n})) \\
&= (C_n(f) \circ C_n(\sigma))\,(s_n^{\mathrm{aff}}(\mathrm{id}_{\Delta^n}) \\
&= (C_n(f) \circ s_n)(\sigma).
\end{aligned}$$

For proving (b), we do an induction on n. For $n = 0$, this is trivially true. Hence assume (b) holds for some $n \in \omega$. Using lemma 3.36, lemma 3.41 and part (a) we compute

$$\begin{aligned}
(\partial \circ s)(\sigma) &= \partial C_{n+1}(\sigma) \left(s^{\mathrm{aff}}(\mathrm{id}_{\Delta^{n+1}}) \right) \\
&= C_n(\sigma) \partial \operatorname{Cone}_{b_{n+1}} \left(s^{\mathrm{aff}}(\partial \,\mathrm{id}_{\Delta^{n+1}}) \right)
\end{aligned}$$

$$
\begin{aligned}
&= C_n(\sigma)\left(s^{\text{aff}}(\partial \operatorname{id}_{\Delta^{n+1}}) - \operatorname{Cone}_{b_{n+1}}(\partial s^{\text{aff}} \partial \operatorname{id}_{\Delta^{n+1}})\right) \\
&= C_n(\sigma)\left(s(\partial \operatorname{id}_{\Delta^{n+1}}) - \operatorname{Cone}_{b_{n+1}}(\partial s \partial \operatorname{id}_{\Delta^{n+1}})\right) \\
&= C_n(\sigma)\left(s(\partial \operatorname{id}_{\Delta^{n+1}}) - \operatorname{Cone}_{b_{n+1}}(s \partial^2 \operatorname{id}_{\Delta^{n+1}})\right) \\
&= C_n(\sigma) s(\partial \operatorname{id}_{\Delta^{n+1}}) \\
&= s C_n(\sigma)(\partial \operatorname{id}_{\Delta^{n+1}}) \\
&= s \partial C_{n+1}(\sigma)(\operatorname{id}_{\Delta^{n+1}}) \\
&= (s \circ \partial)(\sigma),
\end{aligned}
$$

for any $n+1$-simplex σ.
For proving (c), let $\sigma : \Delta^n \to X$ be any singular n-simplex. Then $\sigma^{-1}(\mathcal{U})$ is an open cover for Δ^n and hence there exists a Lebesgue number $\delta > 0$. $\square$

Theorem 3.45. *For each $n \in \mathbb{Z}$, $H_n(s_n) = \operatorname{id}_{H_n}$.*

Theorem 3.46. *Let $\mathcal{U}$ be an open cover for a topologial space X. Then the inclusion $\iota_\bullet : C_\bullet^{\mathcal{U}}(X) \hookrightarrow C_\bullet(X)$ induces an isomorphism $H_n^{\mathcal{U}}(X) \to H_n(X)$.*

Proof.
Step 1: Construction of a chain homotopy between s_n and id. We define inductively $F_n : C_n(X) \to C_{n+1}(X)$ by

$$
F_n \sigma := \begin{cases} 0 & n = 0, \\ C_{n+1}(\sigma) \operatorname{Cone}_{b_n}(\operatorname{id}_{\Delta^n} - s_n \operatorname{id}_{\Delta^n} - F_{n-1} \partial_n \operatorname{id}_{\Delta^n}) & n > 0. \end{cases}
$$

We have to show now that

$$
\partial_{n+1} \circ F_n + F_{n-1} \circ \partial_n = \operatorname{id} - s_n
$$

holds. We proceed by induction over $n \in \omega$. The case $n = 0$ is clear. Let us suppose that above identity holds for some $n - 1 \in \omega$. Then proposition 3.44 and the induction hypothesis yields

$$
\begin{aligned}
\partial F_n \sigma =& \partial C_{n+1}(\sigma) \operatorname{Cone}_{b_n}(\operatorname{id}_{\Delta^n} - s_n \operatorname{id}_{\Delta^n} - F_{n-1} \partial \operatorname{id}_{\Delta^n}) \\
=& C_n(\sigma) \partial \operatorname{Cone}_{b_n}(\operatorname{id}_{\Delta^n} - s_n \operatorname{id}_{\Delta^n} - F_{n-1} \partial \operatorname{id}_{\Delta^n}) \\
=& \sigma - s_n \sigma - C_n(\sigma) F_{n-1} \partial \operatorname{id}_{\Delta^n} \\
& - C_n(\sigma) \operatorname{Cone}_{b_n}\left(\partial \operatorname{id}_{\Delta^n} - \partial s_n \operatorname{id}_{\Delta^n} - \partial F_{n-1} \partial_n \operatorname{id}_{\Delta^n}\right) \\
=& \sigma - s_n \sigma - C_n(\sigma) F_{n-1} \partial \operatorname{id}_{\Delta^n} \\
& - C_n(\sigma) \operatorname{Cone}_{b_n}\left(\partial \operatorname{id}_{\Delta^n} - s_{n-1} \partial \operatorname{id}_{\Delta^n} - \partial F_{n-1} \partial_n \operatorname{id}_{\Delta^n} - F_{n-2} \partial \partial \operatorname{id}_{\Delta^n}\right) \\
=& \sigma - s_n \sigma - C_n(\sigma) F_{n-1} \partial \operatorname{id}_{\Delta^n} \\
=& \sigma - s_n \sigma - F_{n-1} \partial \sigma,
\end{aligned}
$$

since for any continuous map $f : X \to X$ it is easy to show that

$$C_n(f) \circ F_{n-1} = F_{n-1} \circ C_{n-1}(f).$$

Step 2: Injectivity. Let $\langle c \rangle \in \ker H_n^{\mathcal{U}}(X)$. Hence there exists some $b \in C_{n+1}(X)$ such that $\partial b = c$. By part (c) of proposition 3.44 we find $k \in \omega$ such that $s^k(b) \in C_{n+1}^{\mathcal{U}}(X)$. Moreover, $\partial s^k b = s^k \partial b = s^k c$. Now using the chain homotopy of the first step and induction, one can show that $s^k c$ and c differ by a boundary and hence that $\langle s^k c \rangle = \langle c \rangle$. Thus $\langle c \rangle = \langle \partial s^k b \rangle = 0$.
Step 3: Surjectivity. Let $\langle c \rangle \in H_n(X)$. Then by part (c) of proposition 3.44 we find $k \in \omega$ such that $s^k c \in C_n^{\mathcal{U}}(X)$. Moreover, $\partial s^k c = s^k \partial c = 0$. Hence by the previous reasoning we have that $\langle s^k c \rangle = \langle c \rangle$.

□

The Excision Axiom.

Proposition 3.47. *Let $(X, A) \in \mathsf{ob}(\mathsf{Top}^2)$ and $\mathcal{U}$ an open cover for X. Then the inclusion $C_\bullet^{\mathcal{U}}(X, Y) \hookrightarrow C_\bullet(X, A)$, where $C_\bullet^{\mathcal{U}}(X, A) := C_\bullet^{\mathcal{U}}(X)/C_\bullet^{\mathcal{U}\cap A}(A)$, induces an ismorphism $H_n^{\mathcal{U}}(X, A) \cong H_n(X, A)$.*

Theorem 3.48 (The Excision Axiom). *Let U and V be an open cover for a topological space X. Then the inclusion $(U, U \cap V) \hookrightarrow (X, V)$ induces an isomorphism in homology $H_n(U, U \cap V) \cong H_n(X, V)$*

Proof. Let $\mathcal{U} := \{U, V\}$. Then we have that $C_\bullet^{\mathcal{U}}(X) = C_\bullet(U) + C_\bullet(V)$. Consider the short exact sequence

$$0 \longrightarrow C_\bullet(U) + C_\bullet(V) \xrightarrow{\iota_\bullet} C_\bullet(X) \longrightarrow C_\bullet(X)/\left(C_\bullet(U) + C_\bullet(V)\right) \longrightarrow 0$$

in Comp. Using proposition 3.24 we get a long exact sequence in homology and theorem 3.46 implies that $H_n(\iota_\bullet)$ is an isomorphism. Since this is every third map in the long exact sequence, it is easy to show that

$$H_n\left(C_\bullet(X)/(C_\bullet(U) + C_\bullet(V))\right) = 0.$$

Moreover, let us consider the short exact sequence

$$0 \longrightarrow \frac{C_\bullet(U) + C_\bullet(V)}{C_\bullet(V)} \xrightarrow{\iota'_\bullet} \frac{C_\bullet(X)}{C_\bullet(V)} \longrightarrow \frac{C_\bullet(X)}{C_\bullet(U) + C_\bullet(V)} \longrightarrow 0$$

in Comp. Again, there is an associated long exact sequence in homology by proposition 3.24, where by the above, every third term is vanishing. Thus $H_n(\iota'_\bullet)$ is an isomorphism. Now the second isomorphism theorem together with the fact that $C_\bullet(U \cap V) = C_\bullet(U) \cap C_\bullet(V)$ implies

$$f_\bullet : \frac{C_\bullet(U)}{C_\bullet(U \cap V)} \cong \frac{C_\bullet(U) + C_\bullet(V)}{C_\bullet(V)}.$$

Thus for any $n \in \mathbb{Z}$, we have that

$$H_n(\iota'_\bullet \circ f_\bullet) = H_n(\iota'_\bullet) \circ H_n(f_\bullet) : H_n\left(\frac{C_\bullet(U)}{C_\bullet(U \cap V)}\right) \cong H_n\left(\frac{C_\bullet(X)}{C_\bullet(V)}\right).$$

□

Theorem 3.49 (The Excision Axiom, Second Formulation). *Let $X'' \subseteq X' \subseteq X$ be subspaces such that $\bar{X}'' \subseteq \mathring{X}'$. Then the inclusion $(X \setminus X'', X' \setminus X'') \hookrightarrow (X, X')$ induces an isomorphism*

$$H_n(X \setminus X'', X' \setminus X'') \cong H_n(X, X'),$$

for all $n \in \omega$.

Reduced Homology.

The Mayer-Vietoris Theorem.

Proposition 3.50 (Commutative Braid). *Consider the diagram*

$$\begin{tikzcd}
\cdots \arrow[dr, color=green] \arrow[rr, bend left, color=red] & & E_{n+1} \arrow[dr, color=red] \arrow[rr, bend left] & & C_n \arrow[dr, "i_n"] \arrow[rr, bend left, color=blue] & & F_n \arrow[dr, color=blue] \arrow[rr, bend left, color=green] & & B_{n-1} \arrow[dr, color=green] \arrow[rr, bend left, color=red] & & \cdots \\
& D_{n+1} \arrow[ur, "j_{n+1}"] \arrow[dr, color=green] & & A_n \arrow[ur, color=blue] \arrow[dr, color=red] & & D_n \arrow[ur, color=green] \arrow[dr, "j_n"] & & A_{n-1} \arrow[ur, color=red] \arrow[dr, color=blue] & & D_{n-1} \arrow[ur, "j_{n-1}"] \arrow[dr, color=green] & \\
\cdots \arrow[ur, "i_{n+1}"] \arrow[rr, bend right, color=blue] & & F_{n+1} \arrow[ur, color=blue] \arrow[rr, bend right, color=green] & & B_n \arrow[ur, color=green] \arrow[rr, bend right, color=red] & & E_n \arrow[ur, color=red] \arrow[rr, bend right] & & C_{n-1} \arrow[ur, "i_{n-1}"] \arrow[rr, bend right, color=blue] & & \cdots
\end{tikzcd}$$

in AbGrp*, where the blue, green and red strands are long exact sequences. Moreover, assume that either*

$$\operatorname{im} i_n \subseteq \ker j_n \qquad \text{or} \qquad \ker j_n \subseteq \operatorname{im} i_n$$

holds for all $n \in \mathbb{Z}$. Then the black strand is also a long exact sequence.

Proof.

Step 1: Exactness at C_n.

• $\operatorname{im}(E_{n+1} \to C_n) \subseteq \ker i_n$. Suppose $e_{n+1} \mapsto c_n$. Then we have by commutativity that $e_{n+1} \mapsto a_n \mapsto c_n$. By exactness we have that $a_n \mapsto 0 \mapsto 0$ and thus again by commutativity, $a_n \mapsto c_n \mapsto 0$.

• $\ker i_n \subseteq \operatorname{im}(E_{n+1} \to C_n)$. Suppose $c_n \mapsto 0$. Then $c_n \mapsto 0 \mapsto 0$, hence by commutativity and exactness we find $a_n \mapsto c_n$. Again, commutativity yields $a_n \mapsto b_n \mapsto 0$. Hence by exactness at B_n we find $f_{n+1} \mapsto b_n$ and by commutativity, $f_{n+1} \mapsto a'_n \mapsto b_n$. Thus $a_n - a'_n \in \ker(A_n \to B_n)$ and thus $e_{n+1} \mapsto a_n - a'_n$. By exactness at A_n, $e_{n+1} \mapsto a_n - a'_n \mapsto c_n$ and so by commutativity, $e_{n+1} \mapsto c_n$.

Step 2: Exactness at D_n, assuming $\operatorname{im} i_n \subseteq \ker j_n$. Left to show is that $\ker j_n \subseteq \operatorname{im} i_n$. Suppose $d_n \mapsto 0$. Then $d_n \mapsto 0 \mapsto 0$. Hence by commutativity, $d_n \mapsto f_n \mapsto 0$. Thus exactness at F_n implies $c_n \mapsto f_n$. By commutativity, $c_n \mapsto d'_n \mapsto f_n$. By assumption, $d'_n \mapsto 0$. Now $d_n - d'_n \mapsto 0$ and thus by exactness at D_n, we have that $b_n \mapsto d_n - d'_n$. By commutativity, $b_n \mapsto 0$ and hence by exactness at B_n we get that $a_n \mapsto b_n$. By commutativity, $a_n \mapsto c'_n \mapsto d_n - d'_n$ and so $c_n + c'_n$ does the job.
Step 3: Exactness at D_n, assuming $\ker j_n \subseteq \operatorname{im} i_n$. Left to show is that $\operatorname{im} i_n \subseteq \ker j_n$. Suppose $c_n \mapsto d_n$. Then $d_n \mapsto f_n$ and by commutativity we have that $c_n \mapsto f_n$. Hence $d_n \mapsto f_n 0$ by exactness at F_n. Hence commutativity yields $d_n \mapsto e_n \mapsto 0$. By exactness we have that $b_n \mapsto e_n$. Commutativity implies $b_n \mapsto d'_n \mapsto e_n$. Hence $d_n - d'_n \mapsto 0$ and thus by assumption $c'_n \mapsto d_n - d'_n$. Now $d_n - d'_n \mapsto f_n$ and hence by commutativity $c'_n \mapsto f_n$. So $c_n - c' \mapsto 0$ and by exactness at C_n we have that $a_n \mapsto c_n - c'_n$. By commutativity $a_n \mapsto b'_n \mapsto d'_n$. Now $b'_n \mapsto d'_n \mapsto e_n$. By commutativity $b'_n \mapsto e_n$, but by exactness at B_n, we have that $b'_n \mapsto 0$. so $e_n = 0$ and thus $d_n \mapsto 0$.
Step 4: Exactness at E_n.
• $\operatorname{im} j_n \subseteq \ker(E_n \to C_{n-1})$. Suppose $d_n \mapsto e_n$. Consider $d_n \mapsto f_n \mapsto a_{n-1}$. Then by commutativity, $d_n \mapsto e_n \mapsto a_{n-1}$. Hence by exactness $e_n \mapsto a_{n-1} \mapsto 0$ and thus by commutativity, $e_n \mapsto 0$.
• $\ker(E_n \to C_{n-1}) \subseteq \operatorname{im} j_n$. Suppose $e_n \mapsto 0$. Hence by commutativity $e_n \mapsto a_{n-1} \mapsto 0$. Hence by exactness at A_{n-1} we get that $f_n \mapsto a_{n-1}$. Again, exactness at A_{n-1} and commutativity implies that $f_n \mapsto 0$. Hence by exactness at F_n we have that $d_n \mapsto f_n$. Commutatvity yields $d_n \mapsto e'_n \mapsto a_{n-1}$ and hence $e_n - e'_n \mapsto 0$. Thus $b_n \mapsto e_n - e'_n$. Commutativity now implies that $b_n \mapsto d'_n \mapsto e_n - e'_n$ and hence $d_n + d'_n$ does the job.

□

Proposition 3.51 (Long Exact Sequence of Triples). *Let X be a topological space and $X'' \subseteq X' \subseteq X$ subspaces. Then there is a long exact sequence*

$$\dots \longrightarrow H_n(X', X'') \longrightarrow H_n(X, X'') \longrightarrow H_n(X, X') \longrightarrow H_{n-1}(X', X'') \longrightarrow \dots$$

Proof. This immediately follows from proposition 3.50 by considering the braid

$$\begin{tikzcd}[column sep=tiny]
\dots \arrow[dr] \arrow[rr, bend left] & & H_{n+1}(X,X') \arrow[dr] \arrow[rr, bend left] & & H_n(X',X'') \arrow[dr] \arrow[rr, bend left] & & H_{n-1}(X'') \arrow[dr] & \\
& H_{n+1}(X,X'') \arrow[ur] \arrow[dr] & & H_n(X') \arrow[ur] \arrow[dr] & & H_n(X,X'') \arrow[ur] \arrow[dr] & & H_{n-1}(X' \\
\dots \arrow[ur] \arrow[rr, bend right] & & H_n(X'') \arrow[ur] \arrow[rr, bend right] & & H_n(X) \arrow[ur] \arrow[rr, bend right] & & H_n(X,X') \arrow[ur] &
\end{tikzcd}$$

where the red strand is the long exact sequence corresponding to the pair (X, X'), the green corresponds to (X, X'') and the blue to (X, X''). Commutativity follows from the fact that H_n is a functor and that the connecting homomorphisms are natural. Hence an application of the commutative braid proposition yields the result. $\square$

Theorem 3.52 (Mayer-Vietoris). *Let U and V be an open cover for a topological space X. Define four inclusions*

$$\iota_U : U \cap V \hookrightarrow U, \iota_V : U \cap V \hookrightarrow V, \jmath_U : U \hookrightarrow X, \text{ and } \jmath_V : V \hookrightarrow X.$$

Then there is a long exact sequence

$$\dots \longrightarrow H_n(U \cap V) \xrightarrow{(H_n(\iota_U), H_n(\iota_V))} H_n(U) \oplus H_n(V) \xrightarrow{H_n(\jmath_U) - H_n(\jmath_V)} H_n(X) \xrightarrow{D} \dots$$

where $D : H_n(X) \to H_{n-1}(U \cap V)$.

Definition 3.53 (Retract). *Let $(X, S) \in \mathrm{ob}(\mathsf{Top}^2)$. We say that **$S$ is a retract of X**, iff the inclusion $\iota : S \hookrightarrow X$ admits a retraction in* Top.

Definition 3.54 (Deformation Retract). *Let $(X, S) \in \mathrm{ob}(\mathsf{Top}^2)$. We say that **$S$ is a deformation retract of X**, iff S is a retract of X such that $\iota \circ r \simeq \mathrm{id}_X$ holds, where r denotes the retraction of the inclusion $\iota : S \hookrightarrow X$.*

Definition 3.55 (Strong Deformation Retract). *Let $(X, S) \in \mathrm{ob}(\mathsf{Top}^2)$. We say that **$S$ is a strong deformation retract of X**, iff S is a deformation retract of X such that $\iota \circ r \simeq_S \mathrm{id}_X$.*

Example 3.56. Let $n \in \omega$, $n \geq 1$. Define $N := e_{n+1}$ and $S := -e_{n+1}$, where e_{n+1} denotes the $n+1$-th standard basis vector of $\mathbb{R}^{n+1}$. Moreover, set

$$U_+ := \mathbb{S}^n \setminus S \qquad \text{and} \qquad U_- := \mathbb{S}^n \setminus N.$$

Then U_+ and U_- are open subsets of $\mathbb{S}^n$, the extended upper and lower hemisphere, respectively. We claim that $\mathbb{S}^{n-1}$ *is a strong deformation retract of* $U_+ \cap U_-$, where we include $\mathbb{S}^{n-1}$ into $U_+ \cap U_-$ via $\iota : \mathbb{S}^{n-1} \hookrightarrow U_+ \cap U_-$ defined by

$$\iota(x) := (x, 0).$$

Define $r : U_+ \cap U_- \to \mathbb{S}^{n-1}$ by setting

$$r(x) := \frac{1}{\sqrt{1 - x_{n+1}^2}}(x_1, \dots, x_n).$$

Then it is easy to see that $r \circ \iota = \mathrm{id}_{\mathbb{S}^{n-1}}$. Consider $H : (U_+ \cap U_-) \times I \to U_+ \cap U_-$ defined by

$$H(x, t) := \frac{1}{\sqrt{1 + (t^2 - 1)x_{n+1}^2}}(x_1, \dots, x_n, t x_{n+1}).$$

Then it is also not hard to check that $H : \iota \circ r \simeq_{\mathbb{S}^{n-1}} \mathrm{id}_{U_+ \cap U_-}$.

Proposition 3.57 (Homology of Spheres). *For $n \in \omega$, we have*

$$\widetilde{H}_k(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & k = n, \\ 0 & k \neq n. \end{cases}$$

Proof. We induct over $n \in \omega$. If $n = 0$, □

The Additivity Axiom

Proposition 3.58. *In* Grp*, all small products exist.*

Proof. It is easy to show that if $(G_\alpha)_{\alpha \in A}$ is a family of objects in Grp, then the cartesian product $\prod_{\alpha \in A} G_\alpha$ with componentwise product $(gh)_\alpha := g_\alpha h_\alpha$ together with the natural projections $\pi_\alpha : \prod_{\alpha \in A} G_\alpha \to G_\alpha$, is a universal cone. □

Definition 3.59 (Direct Product). *Small products in* Grp *are called **direct products**, written $\Pi_{\alpha \in A} G_\alpha$.*

Proposition 3.60. *In* AbGrp*, all small coproduct exist.*

Proof. It is easy to show that if $(G_\alpha)_{\alpha \in A}$ is a family of objects in AbGrp, then the subgroup of $\Pi_{\alpha \in A} G_\alpha$ defined by the elements which entries are almost all zero together with the natural inclusions ι_α is a universal cocone. □

Definition 3.61 (Direct Sum). *The small coproducts in* AbGrp *are called **direct sums**, written $\bigoplus_{\alpha \in A} G_\alpha$.*

Definition 3.62 (Direct Sum Chain Complex). *Let $(C_\bullet^\alpha, \partial_\bullet^\alpha)_{\alpha \in A}$ be a family of chain complexes. Then the chain complex $\left(\bigoplus_{\alpha \in A} C_\bullet^\alpha, \bigoplus_{\alpha \in A} \partial_\bullet^\alpha\right)$ defined by*

$$\left(\bigoplus_{\alpha \in A} C_\bullet^\alpha\right)_n := \bigoplus_{\alpha \in A} C_n^\alpha \qquad \text{and} \qquad \left(\bigoplus_{\alpha \in A} \partial_\bullet^\alpha\right)_n := \bigoplus_{\alpha \in A} \partial_n^\alpha,$$

*for all $n \in \mathbb{Z}$, is called the **direct sum chain complex** of the family $(C_\bullet^\alpha, \partial_\bullet^\alpha)_{\alpha \in A}$.*

Lemma 3.63. *Let $(C_\bullet^\alpha, \partial_\bullet^\alpha)_{\alpha \in A}$ be a family in* Comp*. Then*

$$H_n\left(\bigoplus_{\alpha \in A} C_\bullet^\alpha\right) \cong \bigoplus_{\alpha \in A} H_n(C_\bullet^\alpha),$$

for all $n \in \mathbb{Z}$.

Exercise 3.64. Prove lemma 3.63.

Lemma 3.65. *Let X be a topological space and let $\{X_\alpha\}_{\alpha \in A}$ denote the set of path components of X. Then*

$$H_n(X) \cong \bigoplus_{\alpha \in A} H_n(X_\alpha)$$

for all $n \in \omega$.

Proof. Let $\iota_\alpha : X_\alpha \hookrightarrow X$ denote inclusion for all $\alpha \in A$. Consider

$$\sum_{\alpha\in A} C_n(\iota_\alpha) : \bigoplus_{\alpha\in A} C_n(X_\alpha) \to C_n(X)$$

and let $\varphi : C_n(X_\alpha) \to \bigoplus_{\alpha\in A} C_n(X_\alpha)$ the map extended by linearity defined as follows on elements $\sigma \in \mathsf{Top}(\Delta^n, X)$: since Δ^n is path connected, we have that $\sigma(\Delta^n) \subseteq X_\alpha$ for some unique $\alpha \in A$. Just set $x_\alpha := \sigma_k : \Delta^n \to X_\alpha$ if $\sigma(\Delta^n) \subseteq X_\alpha$ and $x_\alpha := 0$ else. Thus it is easy to show that $\bigoplus_{\alpha\in A} C_n(X_\alpha) \cong C_n(X)$. Then we have $\bigoplus_{\alpha\in A} C_\bullet(X_\alpha) \cong C_\bullet(X)$ as chain complexes, and since functors preserve isomorphisms, the result follows from lemma 3.63. □

The Brouwer Fixed Point Theorem

Definition 3.66 (Retract). *Let $X \in \mathrm{ob}(\mathsf{Top})$ and $S \subseteq X$ a subspace. We say that **S is a retract of X**, if the inclusion $\iota : S \hookrightarrow X$ admits a retraction in* Top.

Lemma 3.67. *Let $n \in \mathbb{Z}$, $n \geq 1$. Then $\mathbb{S}^n$ is not a retract of $\mathbb{B}^{n+1}$.*

Proof.

□

Proposition 3.68. *Let $n \in \omega$, $X \in \mathrm{ob}(\mathsf{Top})$ and $f \in \mathsf{Top}(\mathbb{S}^n, X)$. Then the followin conditions are equivalent:*

(*a*) *f is nullhomotopic.*
(*b*) *f admits a continuous extension to $\mathbb{B}^{n+1}$.*
(*c*) *Let $p \in \mathbb{S}^n$. Then $f \simeq_p c_{f(p)}$.*

Proof. We show (a)⇒(b)⇒(c)⇒(a). Assume that (a) holds. Hence we have that $H : f \simeq c_p$ for some $p \in X$. Define $g : \mathbb{B}^{n+1} \to X$ by

$$g(x) := \begin{cases} p & 0 \leq |x| \leq \frac{1}{2}, \\ H(x/|x|, 2-2|x|) & \frac{1}{2} \leq |x| \leq 1. \end{cases}$$

Then $g \in \mathsf{Top}(\mathbb{B}^{n+1}, X)$ by the gluing lemma and $g|_{\mathbb{S}^n} = f$. Assume that (b) holds. So let $g \in \mathsf{Top}(\mathbb{B}^{n+1}, X)$ be an extension of f. Define $H : \mathbb{S}^n \times I \to X$ by

$$H(x,t) := g((1-t)x + tp, t).$$

Then it is easy to check that $H : f \simeq_p c_{f(p)}$. Finally, (c)⇒(a) is immediate. □

Theorem 3.69 (Brouwer Fixed Point Theorem). *Let $n \in \mathbb{Z}$, $n \geq 1$. Then every mapping $f \in \mathsf{Top}(\mathbb{B}^n, \mathbb{B}^n)$ has a fixed point.*

Proof.

□

The Hurewicz Theorem

Abelianizations.

Proposition 3.70. *The forgetful functor U : $\mathsf{AbGrp} \to \mathsf{Grp}$ admits a left adjoint.*

Proof. Let $G \in \mathrm{ob}(\mathsf{Grp})$. For $g, h \in G$, define the ***commutator of g and h***, written $[g, h]$, by $[g, h] := ghg^{-1}h^{-1}$. Moreover, set

$$X_G := \{[g, h] : g, h \in G\}$$

and define the ***commutator subgroup of G***, written $[G, G]$, by $[G, G] := \langle X_G \rangle$.

Lemma 3.71. *For all $G \in \mathrm{ob}(\mathsf{Grp})$, $[G, G] \trianglelefteq G$.*

Proof. We follow [Lee11, p. 265]. Clearly, $[G, G] \le G$. By [KM13, p. 31] we have that

$$\langle X \rangle = \{x_1 \cdots x_n : n \in \omega \setminus \{0\}, x_1, \dots, x_n \in X_G \cup X_G^{-1}\}.$$

It is easy to check that $X_G = X_G^{-1}$ and thus

$$\langle X \rangle = \{x_1 \cdots x_n : n \in \omega \setminus \{0\}, x_1, \dots, x_n \in X_G\}.$$

Let $k \in G$ and $x_1 \cdots x_n \in [G, G]$. Since

$$kx_1 \cdots x_n k^{-1} = kx_1k^{-1}kx_2k^{-1}k \cdots kx_nk^{-1}$$

it is enough to show that $k[g, h]k^{-1} \in [G, G]$ for all $g, h \in G$. But this immediately follows from

$$k[g, h]k^{-1} = kghg^{-1}h^{-1}k^{-1} = [kgk^{-1}, khk^{-1}].$$

Thus $[G, G] \trianglelefteq G$. $\square$

Lemma 3.72. *$G \in \mathrm{ob}(\mathsf{AbGrp})$ if and only if $[G, G] = \{1\}$.*

Proof. Let $G \in \mathrm{ob}(\mathsf{AbGrp})$. Then $[g, h] = 1$ for all $g, h \in G$, which implies $X_G = \{1\}$ and thus $\langle X_G \rangle = \{1\}$. Conversly, since $X_G \subseteq [G, G] = \{1\}$, we have that $[g, h] = 1$ for all $g, h \in G$ which is equivalent to $gh = hg$ for all $g, h \in G$. $\square$

Corollary 3.73. *The quotient group $G/[G, G]$ is abelian.*

Proof. By lemma 3.72 it is enough to show that $[G/[G, G], G/[G, G]]$ is trivial. We actually show that $X_{G/[G,G]} = \{1\}$. This immediately follows from

$$[g[G, G], h[G, G]] = ghg^{-1}h^{-1}[G, G] = [G, G]$$

for $g[G, G], h[G, G] \in G/[G, G]$. $\square$

Hence define $\mathrm{Ab} : \mathsf{Grp} \to \mathsf{AbGrp}$ on objects by

$$\mathrm{Ab}(G) := G/[G, G].$$

The abelian group $\mathrm{Ab}(G)$ is called the ***abelianization of G***. On morphisms $\varphi : G \to H$ in Grp define $\mathrm{Ab}(\varphi) : \mathrm{Ab}(G) \to \mathrm{Ab}(H)$ by setting $\mathrm{Ab}(\varphi)(g[G, G]) := \varphi(g)[H, H]$. It is easy to check that this is a well defined morphism of abelian groups.

Let $H \in \mathrm{ob}(\mathsf{AbGrp})$ and $\psi \in \mathsf{AbGrp}(\mathrm{Ab}(G), H)$. Define $\overline{\psi} \in \mathsf{Grp}(G, U(H))$ by setting $\overline{\psi}(g) := \psi(g\,[G,G])$. If $\varphi \in \mathsf{Grp}(G, U(H))$, define $\overline{\varphi} \in \mathsf{AbGrp}(\mathrm{Ab}(G), H)$ by $\overline{\varphi}(g\,[G,G]) := \varphi(g)$. It is easy to check that this mapping is actually well defined and that $\overline{\overline{\psi}} = \psi$ and $\overline{\overline{\varphi}} = \varphi$ holds. □

Exercise 3.74. In proposition 3.70, check that Ab : $\mathsf{Grp} \to \mathsf{AbGrp}$ is indeed a functor and the naturality of the bijection in both arguments.

The Hurewicz Morphism. Since elements of $H_1(X)$ are homology classes of loops, one might suspect that there is a connection between the fundamental group $\pi_1(X, p)$ of a path connected space X at p and the first singular homology group $H_1(X)$. However, since $H_1(X)$ is always abelian and $\pi_1(X, p)$ is not necessarily abelian, they cannot be equal. In this section we use a little trick which makes matters simpler: if c is any singular n-chain, not necessarily an n-cycle, we can also take its equivalence class modulo n-boundaries. We shall denote this class also with $\langle c\rangle$. Clearly, if c is an n-cycle, then $\langle c\rangle$ is the usual homology class.

Theorem 3.75 (Hurewicz Theorem). *Let $X \in \mathrm{ob}(\mathsf{Top})$ be path connected and $p \in X$. Then $\mathrm{Ab}(\pi_1(X, p)) \cong H_1(X)$.*

Proof. We show the result in a sequence of lemmata.

Lemma 3.76. *The mapping $h : \pi_1(X, p) \to H_1(X)$ defined by $h([u]) := \langle u\rangle$ is well defined.*

Proof. First of all, since $u \in \Omega(X, p)$, we have that $u \in C_1(X)$. Moreover, $\partial u = u(1) - u(0) = p - p = 0$. Thus u has a homology class $\langle u\rangle$. Let us check that h is well defined. Suppose that $[u] = [v]$. Hence $F : u \simeq_{\partial I} v$. Consider the fundamental loop $\omega \in \Omega(\mathbb{S}^1, 1)$. By [Lee11, p. 70], ω is a quotient map. Since $u, v \in \Omega(X, p)$, there exist $\widetilde{u}, \widetilde{v} \in \mathsf{Top}(\mathbb{S}^1, X)$, such that $\widetilde{u} \circ \omega = u$ and $\widetilde{v} \circ \omega = v$ (see [Lee11, p. 72]). Since I is a locally compact Hausdorff space [Lee11, p. 107] implies that $\omega \times \mathrm{id}_I$ is a quotient map. Thus F passes to the quotient and yields a map $\widetilde{F} \in \mathsf{Top}(\mathbb{S}^1 \times I, X)$. Now it is easy to check that $\widetilde{F} : \widetilde{u} \simeq_{\{1\}} \widetilde{v}$. Thus an application of the homotopy axiom yields

$$\langle u\rangle = \langle \widetilde{u} \circ \omega\rangle = H_1(\widetilde{u})\langle \omega\rangle = H_1(\widetilde{v})\langle \omega\rangle = \langle \widetilde{v} \circ \omega\rangle = \langle v\rangle.$$

□

Lemma 3.77. *Let u be a path in X from p to q. Then $\langle \overline{u}\rangle = -\langle u\rangle$.*

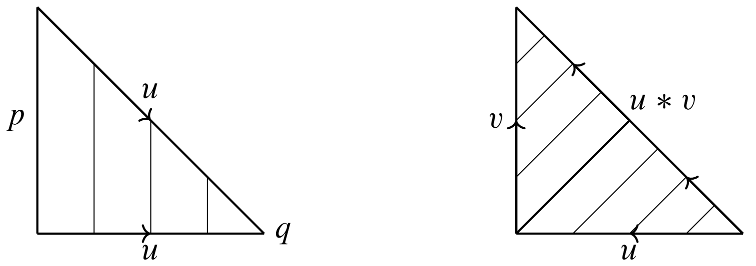
Proof. From figure 13a, we deduce that an appropriate definition of a singular 2-simplex σ would be

$$\sigma(x, y) := u(x).$$

Indeed

$$\partial\sigma = \overline{u} - c_p + u$$

and since c_p is the boundary of $\sigma_p \in \mathsf{Top}(\Delta^2, X)$ defined by $\sigma_p(x, y) := p$, we have that $\overline{u} + u$ is a boundary. □



(a) $\langle \overline{u} \rangle = -\langle u \rangle$. (b) $\langle u * v \rangle = \langle u \rangle + \langle v \rangle$.

Lemma 3.78. *Let u and v be paths in X from p to q and from q to r, respectively. Then $\langle u * v \rangle = \langle u \rangle + \langle v \rangle$.*

Proof. Consider figure 13b. The thin lines correspond to where $y - x$ is constant. Hence define $\sigma : \Delta^2 \to X$ by

$$\sigma(x, y) := \begin{cases} u(y - x + 1) & 0 \leq y \leq x \leq 1, \\ v(y - x) & 0 \leq x \leq y \leq 1. \end{cases}$$

An application of the gluing lemma shows that σ is actually a singular 2-simplex. Moreover

$$\partial \sigma = u * v - v + \overline{u}.$$

Hence lemma 3.77 yield

$$0 = \langle u * v - v + \overline{u} \rangle = \langle u * v \rangle - \langle v \rangle - \langle u \rangle.$$

□

Corollary 3.79. *h is a morphism of groups.*

Corollary 3.80. *Let u, v, w be composable paths in X. Then $\langle (u * v) * w \rangle = \langle u * (v * w) \rangle$.*

Lemma 3.81. *h is surjective.*

Proof. Let $x \in X$. If $x = p$, define $\gamma_p := c_p$. If $x \neq p$, by the path connectedness of X we can choose a path γ_x from p to x. Hence we get a map $\gamma : X \to \mathsf{Top}(\Delta^1, X)$. Extending by linearity yields a mapping $\gamma : C_0(X) \to C_1(X)$. Let $c := \sum_{k=1}^n m_k \sigma_k$ be a 1-cycle in X. Consider

$$[u] := [\gamma_{\sigma_1(0)} * \sigma_1 * \overline{\gamma_{\sigma_1(1)}}]^{m_1} \cdots [\gamma_{\sigma_n(0)} * \sigma_n * \overline{\gamma_{\sigma_n(1)}}]^{m_n} \in \pi_1(X, p).$$

Now lemma 3.77 and 3.78, corollary 3.79 and 3.80 yields

$$\begin{aligned} h([u]) &= \sum_{k=1}^n m_k \langle \gamma_{\sigma_k(0)} * \sigma_k * \overline{\gamma_{\sigma_k(1)}} \rangle \\ &= \sum_{k=1}^n m_k \left(\langle \gamma_{\sigma_k(0)} \rangle + \langle \sigma_k \rangle + \langle \overline{\gamma_{\sigma_k(1)}} \rangle \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} m_k \left(\langle \gamma_{\sigma_k(0)} \rangle + \langle \sigma_k \rangle - \langle \gamma_{\sigma_k(1)} \rangle\right) \\
&= \langle c \rangle - \sum_{k=1}^{n} m_k \langle \gamma_{\sigma_k(1)-\sigma_k(0)} \rangle \\
&= \langle c \rangle - \sum_{k=1}^{n} m_k \langle \gamma_{\partial \sigma_k} \rangle \\
&= \langle c \rangle - \langle \gamma_{\partial c} \rangle \\
&= \langle c \rangle .
\end{aligned}
$$

□

Lastly, we want to show that $\ker h = [\pi_1(X,p), \pi_1(X,p)]$. Since then the first isomorphism theorem implies $\mathrm{Ab}(\pi_1(X,p)) \cong H_1(X)$. Since $H_1(X)$ is abelian, clearly $[\pi_1(X,p), \pi_1(X,p)] \subseteq \ker h$ and thus h factors uniquely $\widetilde{h} : \mathrm{Ab}(\pi_1(X,p)) \to H_1(X)$. The next lemma will be useful.

Lemma 3.82. *Let $\sigma : \Delta^2 \to X$ be a singular 2-simplex. Define $\sigma^{(k)} := \sigma \circ \varphi_k^2$ for $k = 0, 1, 2$. Then $[\sigma^{(0)} * \overline{\sigma^{(1)}} * \sigma^{(2)}] = [c_{\sigma(e_1)}]$.*

Proof. Let $u := \sigma^{(0)} * \overline{\sigma^{(1)}} * \sigma^{(2)}$. Since $\mathbb{B}^2 \approx \Delta^2$, we can consider $\sigma : \mathbb{B}^2 \to X$. One can check that the circle representative $\widetilde{u}$ of u is the reparametrized restriction $\sigma|_{\mathbb{S}^1}$. Since reparametrizations are invariant under homotopies, we have that u is a nullhomotopic loop. □

Let $\sigma \in \mathsf{Top}(\Delta^1, X)$. Define $g(\sigma) := [\gamma_{\sigma(0)} * \sigma * \overline{\gamma_{\sigma(1)}}]_{\mathrm{Ab}}$, where $[u]_{\mathrm{Ab}}$ denotes the equivalence class of $[u]$ in $\mathrm{Ab}(\pi_1(X,p))$. Since $\mathrm{Ab}(\pi_1(X,p))$ is abelian, extension by linearity yields a map $g : C_1(X) \to \mathrm{Ab}(\pi_1(X,p))$.

Lemma 3.83. *g vanishes on* $\operatorname{im} \partial_2$.

Proof. Let $\sigma \in \mathsf{Top}(\Delta^2, X)$. Then lemma 3.82 yields

$$
\begin{aligned}
g(\partial\sigma) &= g\left(\sigma^{(0)}\right) g\left(\sigma^{(1)}\right)^{-1} g\left(\sigma^{(2)}\right) \\
&= \left[\gamma_{\sigma(e_1)} * \sigma^{(0)} * \overline{\gamma_{\sigma(e_2)}} * \gamma_{\sigma(e_2)} * \overline{\sigma^{(1)}} * \overline{\gamma_{\sigma(e_0)}} * \gamma_{\sigma(e_0)} * \sigma^{(2)} * \overline{\gamma_{\sigma(e_1)}}\right]_{\mathrm{Ab}} \\
&= \left[\gamma_{\sigma(e_1)} * \sigma^{(0)} * \overline{\sigma^{(1)}} * \sigma^{(2)} * \overline{\gamma_{\sigma(e_1)}}\right]_{\mathrm{Ab}} \\
&= \left[\gamma_{\sigma(e_1)} * c_{\sigma(e_1)} * \overline{\gamma_{\sigma(e_1)}}\right]_{\mathrm{Ab}} \\
&= \left[c_p\right]_{\mathrm{Ab}} .
\end{aligned}
$$

□

By lemma 3.83, g passes to the quotient and yields a map $\widetilde{g} : H_1(X) \to \mathrm{Ab}(\pi_1(X, p))$. Moreover

$$(\widetilde{g} \circ \widetilde{h})\,[u]_{\mathrm{Ab}} = \widetilde{g}\left(h\,[u]\right) = \widetilde{g}\langle u\rangle = g(u) = [c_p * u * \overline{c_p}]_{\mathrm{Ab}} = [u]_{\mathrm{Ab}}$$

and thus $\widetilde{h}$ admits a retraction in AbGrp which implies that $\widetilde{h}$ is injective. Hence $\ker \widetilde{h}$ is trivial and thus if we write $\pi : \pi_1(X, p) \to \mathrm{Ab}(\pi_1(X, p))$ for the canoncial projection

$$\ker h = \ker(\widetilde{h} \circ \pi) = (\widetilde{h} \circ \pi)^{-1}(0) = \pi^{-1}\left(\widetilde{h}^{-1}(0)\right) = \pi^{-1}(0) = [\pi_1(X, p), \pi_1(X, p)]\,.$$

□

Definition 3.84 (Hurewicz Homomorphism). *Let $X \in \mathrm{ob}(\mathsf{Top})$ and $p \in X$. The homomorphism $h : \pi_1(X, p) \to H_1(X)$ defined in theorem 3.75 is called the **Hurewicz homomorphism**.*

Proposition 3.85. *Let $U : \mathsf{AbGrp} \to \mathsf{Grp}$ denote the forgetful functor. Then the Hurewicz homomorphism is a natural transformation $\pi_1 \Rightarrow U \circ H_1$.*

Proof.

□

The Jordan-Brouwer Separation Theorem

Lemma 3.86. *Let $X \in \mathrm{ob}(\mathsf{Top})$, $S \in \mathrm{ob}(\mathsf{Set})$ and $f : U(X) \to S$ a bijection. Then S can be equipped with a topology such that f becomes a homeomorphism.*

Proof. Let $\mathcal{T}$ be the topology on X. Then it is easy to see, that

$$\varphi(\mathcal{T}) := \{\varphi(U) : U \in \mathcal{T}\}$$

is the right topology on S. □

Lemma 3.87. *Given a sequential diagram*

$$X_0 \overset{\iota_0}{\hookrightarrow} X_1 \overset{\iota_1}{\hookrightarrow} X_2 \overset{\iota_2}{\hookrightarrow} \cdots$$

in Top, we have that $\varinjlim_n X_n \cong \bigcup_{n\in\omega} X_n$.

Proof. Using the construction in proposition A.18, we have that a sequential colimit in Set is given by $\coprod_{n\in\omega} X_n/\sim$, where it is easy to check that in this case $x \in X_n \sim y \in X_m$ if and only if $x = y$. Moreover, we have that $\coprod_{n\in\omega} X_n/\sim \;\cong \bigcup_{n\in\omega} X_n$ in Set, as one can easily show by considering the map $[x_n] \mapsto x_n$. Using lemma 3.86, we get that $\coprod_{n\in\omega} X_n/\sim \;\cong \bigcup_{n\in\omega} X_n$ in Top. Moreover, it is easy to check that a set $U \subseteq \bigcup_{n\in\omega} X_n$ is open if and only if $U \cap X_n$ is open in X_n for all $n \in \omega$. □

Definition 3.88 (T_1). *A topological space X is said to be a **T_1-space**, if $\{x\}$ is closed in X for every $x \in X$.*

Definition 3.89 (Weakly Hausdorff). *A topological space X is said to be a* ***weakly Hausdorff space****, if for any map $f \in \mathsf{Top}(K, X)$ for a compact Hausdorff space K, $f(K)$ is closed in X.*

Exercise 3.90. Show that any Hausdorff space is a weakly Hausdorff space and that any weakly Hausdorff space is a T_1-space, but both contraries are not true.

Exercise 3.91. Let X be a weakly Hausdorff space. Assume that $f \in \mathsf{Top}(K, X)$ for a compact Hausdorff space K. Show that $f(K)$ is a compact subspace of X.

Proposition 3.92. *Given a sequence*

$$X_0 \xrightarrow{i_0} X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} \dots$$

of closed embeddings of weakly Hausdorff spaces, then

$$\varinjlim_n C_\bullet(X_n) = C_\bullet\left(\varinjlim_n X_n\right).$$

Proof. Let $X := \bigcup_{n\in\omega} X_n$. The main part is the following lemma:

Lemma 3.93. *Let $f \in \mathsf{Top}(K, X)$ for a compact Hausdorff space K. Then $f(K)$ is contained in one of the X_n.*

Proof. Towards a contradiction, assume that $f(K)$ is not conatined in any X_n. Hence we find a sequence $(x_n)_{n\in\omega}$ in K, such that $f(x_n) \notin X_n$ for all $n \in \omega$. Define

$$S_m := \{f(x_k) : k \geq m\},$$

for $m \in \omega$. Then $S_{m+1} \subseteq S_m$, $\bigcap_{m\in\omega} S_m = \varnothing$ and $S_m \cap X_n$ is finite for all $n \in \omega$. By exercise 3.90, we get that $S_m \cap X_n$ is closed in X_n for all $n \in \omega$. Hence by the definition of the colimit topology, S_m is closed in X for all $m \in \omega$. Thus $Y_m := X \setminus S_m$ is open in X and easily seen to be an open cover for X. By construction, no finite subcover of it can cover $f(K)$, and hence we have a contradiction to the fact that $f(K)$ is compact by exercise 3.91. □

By the previous lemma, any singular n-simplex $\sigma : \Delta^n \to X$ is contained in some X_n and so the result follows from the definition of the colimit in Comp. □

Proposition 3.94. *Let $X \subseteq \mathbb{S}^n$ homeomorphic to I^m, $0 \leq m \leq n$. Then $\widetilde{H}_k(\mathbb{S}^n \setminus X) = 0$ for $k \in \omega$.*

Proof. If $m = 0$, then X is a single point in $\mathbb{S}^n$. Hence $\mathbb{S}^n \setminus X \cong \mathbb{R}^n$ and thus $\widetilde{H}_k(\mathbb{S}^n \setminus X) = 0$, since $\mathbb{R}^n$ is contractible. Now let $0 < m \leq n$ and suppose the claim holds for $m - 1$. Let $f : X \to I^m$ be a homeomorphism and define

$$I^+ := \{x \in I^m : x_1 \geq 1/2\} \qquad \text{and} \qquad I^- := \{x \in I^m : x_1 \leq 1/2\}.$$

Moreover, define $X^\pm := f^{-1}(I^\pm)$ and $Y := X^+ \cap X^-$. Then $Y \approx I^{m-1}$ and $\mathbb{S}^n \setminus X^+$, $\mathbb{S}^n \setminus X^-$ is an open cover for $\mathbb{S}^n \setminus Y$. Since $\mathbb{S}^n \setminus X^+ \cap \mathbb{S}^n \setminus X^- = \mathbb{S}^n \setminus X$, by Mayer-Vietoris we get a long exact sequence in reduced homology:

$$\dots \widetilde{H}_{k+1}(\mathbb{S}^n \setminus Y) \to \widetilde{H}_k(\mathbb{S}^n \setminus X) \to \widetilde{H}_k(\mathbb{S}^n \setminus X^+) \oplus \widetilde{H}_k(\mathbb{S}^n \setminus X^-) \to \widetilde{H}_k(\mathbb{S}^n \setminus Y) \dots$$

By hypothesis, the end terms vanish and thus we get an isomorphism

$$\widetilde{H}_k(\mathbb{S}^n \setminus X) \xrightarrow{(H_k(\iota^+), H_k(\iota^-))} \widetilde{H}_k(\mathbb{S}^n \setminus X^+) \oplus \widetilde{H}_k(\mathbb{S}^n \setminus X^-).$$

Now take some nonzero element $\langle c \rangle \in \widetilde{H}_k(\mathbb{S}^n \setminus X)$ (if there exists no nonzero element, we are done). Since we have an isomorphism, either $H_k(\iota^+)\langle c \rangle$ or $H_k(\iota^-)\langle c \rangle$ must be nonzero. Without loss of generality, assume $H_k(\iota^+)\langle c \rangle \neq 0$. In the same manner we can split X^+ in two parts and thus getting a decreasing sequence $(X_n)_{n\in\omega}$ of closed subsets of $\mathbb{S}^n$ such that $\langle c \rangle$ is taken to $\langle c_n \rangle \neq 0$ by the homomorphism on homology induced by inclusion. Now each $\mathbb{S}^n \setminus X_n$ is open, and thus we get

$$\widetilde{H}_k(\mathbb{S}^n \setminus \textstyle\bigcap_{n\in\omega} X_n) = \underrightarrow{\operatorname{colim}}_j\, \widetilde{H}_k(\mathbb{S}^n \setminus X_j).$$

Since $\widetilde{H}_k(\mathbb{S}^n \setminus X_j) \to \widetilde{H}_k(\mathbb{S}^n \setminus X_{j+1})$ sends $\langle c_j \rangle$ to $\langle c_{j+1} \rangle$, we end up with a nonzero element $\langle c_\infty \rangle$ in $\widetilde{H}_k(\mathbb{S}^n \setminus \bigcap_{n\in\omega} X_n)$. But $\bigcap_{n\in\omega} X_n$ is homeomorphic to I^{m-1}. Hence by hypothesis, $\widetilde{H}_k(\mathbb{S}^n \setminus \bigcap_{n\in\omega} X_n)$, contradicting the existence of a nonzero element $\langle c_\infty \rangle$. ntradiction. □

Corollary 3.95. *Let $X \subseteq \mathbb{S}^n$ be homeomorphic to $\mathbb{S}^m$ for some $0 \leq m \leq n-1$. Then*

$$\widetilde{H}_k(\mathbb{S}^n \setminus X) = \begin{cases} \mathbb{Z} & k = n-m-1, \\ 0 & k \neq n-m-1. \end{cases}$$

Proof. Let $m = 0$. Then X consists of two distinct points and thus $\mathbb{S}^n \setminus X \approx \mathbb{S}^{n-1}$. Hence

$$\widetilde{H}_k(\mathbb{S}^n \setminus X) = \widetilde{H}_k(\mathbb{S}^{n-1}) = \begin{cases} \mathbb{Z} & k = n-1, \\ 0 & k \neq n-1. \end{cases}$$

Now suppose the claim holds for $m-1$. Then if $X \approx \mathbb{S}^m$, write $X = X^+ \cup X^-$, where $X^\pm$ is homeomorphic to the upper an lower hemisphere of $\mathbb{S}^m$. Applying Mayer-Vietoris to the cover $\mathbb{S}^n \setminus X^+$ and $\mathbb{S}^n \setminus X^-$ of $\mathbb{S}^n \setminus Y$, where $Y := X^+ \cap X^-$, yields a long exact sequence

$$\begin{tikzcd} \dots \widetilde{H}_{k+1}(\mathbb{S}^n \setminus X^+) \oplus \widetilde{H}_{k+1}(\mathbb{S}^n \setminus X^-) \arrow[r] & \widetilde{H}_{k+1}(\mathbb{S}^n \setminus Y) \arrow[dl, rounded corners] \\ \widetilde{H}_k(\mathbb{S}^n \setminus X) \arrow[r] & \widetilde{H}_k(\mathbb{S}^n \setminus X^+) \oplus \widetilde{H}_k(\mathbb{S}^n \setminus X^-) \dots \end{tikzcd}$$

The end terms of above sequence both vanish by proposition 3.94 and thus we get an isomorphism

$$\widetilde{H}_k(\mathbb{S}^n \setminus X) \cong \widetilde{H}_{k-1}(\mathbb{S}^n \setminus Y).$$

Since $Y \approx \mathbb{S}^{m-1}$, we are done. □

Theorem 3.96 (The Jordan-Brouwer Separation Theorem). *Let $f \in \mathsf{Top}(\mathbb{S}^{n-1}, \mathbb{S}^n)$ is an embedding. Then $\mathbb{S}^n \setminus f(\mathbb{S}^{n-1})$ has two components, where $f(\mathbb{S}^{n-1})$ is the boundary of each component.*

Proof. Define $X := f(\mathbb{S}^{n-1})$. Hence corollary 3.95 yields

$$\widetilde{H}_k(\mathbb{S}^n \setminus X) = \begin{cases} \mathbb{Z} & k = 0, \\ 0 & k > 0. \end{cases}$$

Thus the dimension axiom together with additivity on path components implies that $\mathbb{S}^n \setminus X$ has two path-components. Since X is closed by the closed map lemma, $\mathbb{S}^n \setminus X$ is open, and since $\mathbb{S}^n$ is a topological manifold, $\mathbb{S}^n$ is locally path-connected and thus by [Lee11, p. 93], $\mathbb{S}^n \setminus X$ is locally path-connected and hence path-components and components coincide.

Let X_1 and X_2 denote the two components of $\mathbb{S}^n \setminus X$. Since $X \cup X_1$ is closed, the boundary of ∂X_1 is contained in X. Conversly, suppose $p \in X$ and let U be a neighbourhood of p in $\mathbb{S}^n$. Now there exists some $V \subseteq X \cap U$, such that $p \in V$ and $X \setminus V \approx \mathbb{B}^{n-1}$ (X is a copy of $\mathbb{S}^{n-1}$). Since $\mathbb{B}^{n-1} \approx [0,1]^{n-1}$, we get by proposition 3.94, that $\widetilde{H}_0(\mathbb{S}^n \setminus (X \setminus V)) = 0$ and hence the dimension axiom yields that $\mathbb{S}^n \setminus (X \setminus V)$ has one path component. Now for $x_1 \in X_1$ and $x_2 \in X_2$ we find a path with image in $\mathbb{S}^n \setminus (X \setminus V)$, since X_1 and X_2 are distinct path-components, we must have that the path intersects V. Hence V contains points belonging to $\overline{X}_1$ and $\overline{X}_2$, thus $p \in \partial X_1$. □

CHAPTER 4

Cellular Homology

Cell Complexes

Adjunction Spaces.

Definition 4.1 (Adjunction Space). *Let X and Y be topologial spaces and let $A \subseteq X$ be a closed subspace. Moreover, let $f \in \mathsf{Top}(A, Y)$. Define the* ***adjunction space of X and Y along f****, written $X \cup_f Y$, to be*

$$X \cup_f Y := \left(X \coprod Y\right)/\sim,$$

where $\sim$ is the smallest equivalence relation on $X \coprod Y$ generated by $a \sim f(a)$, for $a \in A$.

Lemma 4.2. *Let X and Y be topological spaces, $A \subseteq X$ a closed subspace and $f \in \mathsf{Top}(A, Y)$. Then:*

(*a*) *$X \cup_f Y$ with obvious inclusions is the pushout of the diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, hook, "\iota"'] & Y \\ X & \end{tikzcd}$$

in Top.

(*b*) *The inclusion $q \circ \iota_Y : Y \to X \cup_f Y$ is a closed embedding.*

(*c*) *$q \circ \iota_X|_{X \setminus A}$ is an open embedding.*

(*d*) *$X \cup_f Y$ is the disjoint union of $(q \circ \iota_X)(X \setminus A)$ and $(q \circ \iota_Y)(Y)$.*

Proof. To prove (a), simply use that $X \coprod Y$ is a coproduct in Top. Indeed, if we have another cocone for the diagram, we have also a cocone for the coproduct diagram of X and Y. Hence there exists a unique continuous map from $X \coprod Y$ to the other vertex, and it is easy to check that this map passes to the quotient.

To prove (b), observe that $q \circ \iota_Y$ with restricted codomain has an obvious inverse defined by $[y] \mapsto y$. This is well defined since if $y \sim y'$, we must have $y = y'$ by definition of the equivalence relation generated by $a \sim f(a)$. Let $B \subseteq Y$ closed. Then $q^{-1}\left(q(\iota_Y(B))\right) = f^{-1}(B) \coprod B$, and thus since $f^{-1}(B)$ is closed in A and A is closed in X, $f^{-1}(B)$ is closed in X. Hence $f^{-1}(B) \coprod B$ is closed in $X \coprod Y$ by definition of the disjoint union space topology. From this also follows that $q(\iota_Y(Y))$ is closed in $X \cup_f Y$. Note that since A is closed in X, $X \setminus A$ is open in X. Similar to part (b), we see that an inverse is given by $[x] \mapsto x$. Let $U \subseteq X \setminus A$ be open. Then $q^{-1}\left(q(\iota_X(U))\right) = U$, which is open in $X \setminus A$ and hence in X.

□

Definition 4.3 (Cells). *Let $n \in \omega$, $n \geq 1$. Then $\mathbb{E}^n := \mathbb{B}^n \setminus \mathbb{S}^{n-1}$ is called the **standard n-cell**. If X is a topological space and $E \subseteq X$ is homeomorphic to $\mathbb{E}^n$, then E is called an **n-cell in X**. Moreover, if $f \in \mathsf{Top}(\mathbb{S}^{n-1}, Y)$, the adjunction space $\mathbb{B}^n \cup_f Y$ is said to be obtained from Y by **attaching an n-cell**.*

Examples 4.4.

(a) For all $n \geq 1$, $\mathbb{S}^n$ is obtained by attaching an n-cell to a point.
(b) $\mathbb{RP}^n = E_0 \cup E_1 \cup \ldots E_{n-1} \cup E_n$.
(c) $\mathbb{CP}^n = E_0 \cup E_2 \cup E_4 \cup \cdots \cup E_{2n}$.
(d) For any $m, n \in \omega$, the space $\mathbb{S}^m \times \mathbb{S}^n$ is obtained from $\mathbb{S}^m \vee \mathbb{S}^n$ by attaching a $m + n$-cell.

Proposition 4.5. *Let Y be a Hausdorff space, $n \in \omega$, $n > 0$, and $f \in \mathsf{Top}(\mathbb{S}^{n-1}, Y)$. Then if $\iota : Y \hookrightarrow \mathbb{B}^n \cup_f Y$ denotes inclusion, there is a long exact sequence*

$$\ldots H_k(\mathbb{S}^{n-1}) \xrightarrow{H_k(f)} H_k(Y) \xrightarrow{H_k(\iota)} H_k(\mathbb{B}^n \cup_f Y) \longrightarrow H_{k-1}(\mathbb{S}^{n-1}) \ldots$$

Proof. By lemma 4.2, we know that $\mathbb{B}^n \cup_f Y = q(\mathbb{E}^n) \dot\cup q(Y)$. Hence we can write $\mathbb{B}^n \cup_f Y = U \cup V$, where

$$U := q(\mathbb{B}^n_{1/2}) \qquad \text{and} \qquad V := (\mathbb{B}^n \cup_f Y) \setminus q(0),$$

where $0 \in \mathbb{B}^n$. We claim that $q(Y)$ is a deformation retract of V. Indeed, consider $H : V \times I \to V$ defined by

$$H(v, t) := \begin{cases} v & v \in q(Y), \\ q\left((1-t)x + tx/|x|\right) & v = q(x) \in q(\mathbb{E}^n \setminus \{0\}). \end{cases}$$

By lemma 4.2 (b), we can identify $q(Y)$ with Y. Since U is contractible, Mayer-Vietoris yields a sequence

$$\ldots H_k(U \cap V) \longrightarrow H_k(V) \longrightarrow H_k\mathbb{B}^n \cup_f Y \longrightarrow H_{k-1}(U \cap V) \ldots$$

Since $U \cap V$ has the same homotopy type as $\mathbb{S}^{n-1}$ and $H_k(Y) \cong H_k(V)$, we are almost done. Left to check is that $H_k(\iota_V)$ and $H_k(f)$ can be identified. To this end consider the commutative diagram

$$\begin{tikzcd}
U \setminus \{0\} \arrow[r] \arrow[d, hook] & U \cap V \arrow[d, "\iota_V"] \\
\mathbb{B}^n \setminus \{0\} \arrow[r] & V \\
\mathbb{S}^{n-1} \arrow[r, "f"'] \arrow[u, hook] & Y. \arrow[u, hook]
\end{tikzcd}$$

We see that all the vertical maps and the top horizontal map induce isomorphisms on homology, thus we have

$$\begin{tikzcd}
H_k(U \cap V) \arrow[r, "H_k(\iota_V)"] \arrow[d] & H_k(V) \\
H_k(\mathbb{S}^{n-1}) \arrow[r, "H_k(f)"'] & H_k(Y) \arrow[u]
\end{tikzcd}$$

in homology, where the vertical maps are isomorphisms. □

The Relative Homeomorphism Theorem.

Definition 4.6 (Strong Deformation Retract). *Let X be a topological space and $A \subseteq X$ a subspace. We say that **A is a strong deformation retract of X**, if A is a deformation retract of X and $\iota \circ r \simeq_A \mathrm{id}_X$.*

Theorem 4.7. *Let X be a topological space and $A \subseteq X$ a closed subspace, such that there exists a neighbourhood U of A in X such that A is a strong deformation retract of U. If $q : X \to X/A$ denotes the quotient map and $*$ corresponds the equivalence relation of an element in A, we have that*

$$H_n(q) : H_n(X, A) \to H_n(X/A, *)$$

is an isomorphism.

Proof. There is an obvious commutative diagram comming from the long exact sequences of pairs in homology:

$$\begin{tikzcd}
\dots H_n(A) \arrow[r] \arrow[d] & H_n(X) \arrow[r] \arrow[d, "\mathrm{id}"] & H_n(X, A) \arrow[r] \arrow[d] & H_{n-1}(A) \arrow[r] \arrow[d] & H_{n-1}(X) \dots \arrow[d, "\mathrm{id}"] \\
\dots H_n(U) \arrow[r] & H_n(X) \arrow[r] & H_n(X, U) \arrow[r] & H_{n-1}(U) \arrow[r] & H_{n-1}(X) \dots
\end{tikzcd}$$

Since all vertical maps except the middle one are isomorphisms, the five lemma yields that the middle one is also an isomorphism. Now $\{*\}$ is a strong deformation retract of U/A in X/A. Indeed, if $H : U \times I \to U$ is the map corresponding to the strong deformation retract of A in U, we see that $H : U \times I \to U/A$ passes to the quotient to yield a map

$\widetilde{H} : U/A \times I \to U/A$ with the desired properties. Hence applying the above argument to the induced case, yields

$$H_n(X/A, *) \cong H_n(X/A, U/A).$$

Consider

$$\begin{tikzcd}
H_n(X,A) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "H_n(q)"'] & H_n(X,U) \arrow[r, leftrightarrow, "\cong"] & H_n(X \setminus A, U \setminus A) \arrow[d, "H_n(q)"] \\
H_n(X/A, *) \arrow[r, leftrightarrow, "\cong"'] & H_n(X/A, U/A) \arrow[r, leftrightarrow, "\cong"'] & H_n\big((X/A) \setminus \{*\}, (U/A) \setminus \{*\}\big).
\end{tikzcd}$$

Now the right-hand side $H_n(q)$ is an isomorphism since q is a homeomorphism restricted to $X \setminus A$. □

Definition 4.8 (Relative Homeomorphism). *Let $f \in \mathsf{Top}^2\big((X,A),(Y,B)\big)$. We say that f is a **relative homeomorphism**, if $f : X \setminus A \to Y \setminus B$ is a homeomorphism.*

Theorem 4.9 (The Relative Homeomorphism Theorem). *Let $f \in \mathsf{Top}^2\big((X,A),(Y,B)\big)$ be a relative homeomorphism between a compact space X and a compact Hausdorff space Y and where A and B are closed subspaces. Moreover, assume that there exist neighbourhoods U and V, such that A and B are strong deformation retracts of U and V, respectively. Then*

$$H_n(f) : H_n(X,A) \to H_n(Y,B)$$

is an isomorphism for all $n \in \omega$.

Proof. The assumptions imply the existence of a well-defined continuous bijective map $\widetilde{f}$, such that the following diagram commutes:

$$\begin{tikzcd}
(X,A) \arrow[r, "f"] \arrow[d] & (Y,B) \arrow[d] \\
(X/A, *) \arrow[r, "\widetilde{f}"'] & (Y/B, *).
\end{tikzcd}$$

Since every quotient of a compact space is compact (see [Lee11, p. 96]), we get that X/A is compact. Moreover, by [Lee11, p. 102], we have that Y/B is Hausdorff. Hence the closed map lemma implies that $\widetilde{f}$ is a homeomorphism. Applying homology yields

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "H_n(f)"] \arrow[d] & H_n(Y,B) \arrow[d] \\
H_n(X/A, *) \arrow[r, "H_n(\widetilde{f})"'] & H_n(Y/B, *),
\end{tikzcd}$$

where the two vertical maps are isomorphisms by theorem 4.7. □

The Degree

Definition 4.10 (Degree). *Let $n \geq 1$ and $f : \mathbb{S}^n \to \mathbb{S}^n$ continuous. Then since $H_n(\mathbb{S}^n)$ is infinite cyclic, there is a unique integer* $\deg f$ *such that $H_n(f)$ is the multiplication by* $\deg f$. *This integer is called the* ***degree of f***.

Proposition 4.11. *Let $n \geq 1$ and $f, g \in \mathsf{Top}(\mathbb{S}^n, \mathbb{S}^n)$. Then:*

(*a*) $\deg(g \circ f) = \deg g \deg f$.
(*b*) $\deg \mathrm{id}_{\mathbb{S}^n} = 1$.
(*c*) *If f is constant, then* $\deg f = 0$.
(*d*) *If $f \simeq g$, then* $\deg f = \deg g$.
(*e*) *If f is a homotopy equivalence, then* $\deg f = \pm 1$.

Proposition 4.12. *Let $n \geq 1$ and $A \in \mathrm{O}(n+1)$. Set $f := A|_{\mathbb{S}^n}$. Then* $\deg f = \det A$.

Proof. The group $\mathrm{O}(n+1)$ has two connected components distinguished by det. By homotopy invariance it is enough to check the result for one A in each component. Since the identity matrix induces degree 1, we have to check it only for the other component. We take A the reflection in some hyperplane in $\mathbb{R}^{n+1}$. Divide $\mathbb{S}^n$ in two hemispheres which are preserved by A. Then f induces a reflection f' in the equatorial $\mathbb{S}^{n-1}$. Using Mayer-Vietorsi we obtain the commutative diagram

$$\begin{tikzcd} H_n(\mathbb{S}^n) \arrow[r, leftrightarrow, "\cong"] \arrow[d, "H_n(f)"'] & H_{n-1}(\mathbb{S}^{n-1}) \arrow[d, "H_{n-1}(f')"] \\ H_n(\mathbb{S}^n) \arrow[r, leftrightarrow, "\cong"'] & H_{n-1}(\mathbb{S}^{n-1}). \end{tikzcd}$$

Hence $\deg f = \deg f'$. Thus it suffices to prove the result for $n = 1$. ☐

Corollary 4.13. *Let $n \geq 1$. Then the antipodal map $\alpha : \mathbb{S}^n \to \mathbb{S}^n$, $\alpha(x) := -x$, has degree* $(-1)^{n+1}$.

Theorem 4.14 (The Hairy Ball Theorem). *There exists a nowhere vanishing vector field on $\mathbb{S}^n$ if and only if n is odd.*

Proof. Let V be a nowhere vanishing vector field on $\mathbb{S}^n$. Define $H : \mathbb{S}^n \times I \to \mathbb{S}^n$ by

$$H(x,t) := (\cos \pi t)x + (\sin \pi t)V(x)/\left|V(x)\right|.$$

It is easy to check that $H : \mathrm{id}_{\mathbb{S}^n} \simeq \alpha$, but then we have that

$$(-1)^{n+1} = \deg \alpha = \deg \mathrm{id}_{\mathbb{S}^n} = 1,$$

which is only possible if n is odd. Conversely, if $n = 2m-1$, define $V : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ by

$$V(x_1, y_1, \ldots, x_m, y_m) := (-y_1, x_1, \ldots, -y_m, x_m).$$

☐

Theorem 4.15. *An odd map has odd degree.*

Cellular Homology

Definition 4.16 (Cell-Like Filtration). *Let X be a topological space. A* ***cell-like filtration $\mathcal{F}$ of X*** *is an increasing sequence $\mathcal{F}$, $\varnothing =: F^{-1} \subseteq F^0 \subseteq F^1 \subseteq \dots$ of weakly Hausdorff closed subspaces such that $X = \bigcup_{n\in\omega} F^n$ and*

$$H_k(F^n, F^{n-1}) = 0,$$

for all $k \neq n$.

Proposition 4.17. *Let X be a topological space with cell-like filtration $\mathcal{F}$. Then:*

(*a*) *$H_k(F^n) = 0$ for $k > n$.*
(*b*) *The inclusion $F^n \hookrightarrow X$ induces isomorphisms $H_k(F^n) \cong H_k(X)$ for $k < n$.*

Proof. For proving (a), we consider the long exact sequence associated to the pair (F^n, F^{n-1}):

$$\dots H_{k+1}(F^n, F^{n-1}) \longrightarrow H_k(F^{n-1}) \longrightarrow H_k(F^n) \longrightarrow H_k(F^n, F^{n-1}) \dots$$

If $k \neq n, n-1$, both end terms vanish by definition of a cell-like filtration. Hence we get that

$$H_k(F^{n-1}) \cong H_k(F^n)$$

for all $k \neq n, n-1$. In particular, if $k > n$ we get a chain of isomorphisms

$$H_k(F^n) \cong H_k(F^{n-1}) \cong \dots \cong H_k(F^0) = H_k(F^0, \varnothing) = H_k(F^0, F^{-1}) = 0.$$

Also we have that $H_k(F^{k+m}) \cong H_k(F^{k+1})$ for all $m \geq 1$. Hence

$$H_k(F^{k+1}) \cong \varinjlim_{n\geq k+1} H_k(F^n) = H_k(X).$$

$\square$

Theorem 4.18. *Let X be a topological space with cell-like filtration $\mathcal{F}$. Then*

$$H_n(X) \cong H_n(X, \mathcal{F})$$

for all $n \geq 0$.

Proof. Consider

$$\begin{tikzcd}
 & H_{n+1}(F^{n+1}, F^n) \arrow[d, "\delta_{n+1}"'] \arrow[dr, "\partial^{\mathcal{F}}_{n+1}"] & & 0 \arrow[d] \\
0 \arrow[r] & H_n(F^n) \arrow[r, "\eta_n"] \arrow[d] & H_n(F^n, F^{n-1}) \arrow[r, "\delta_n"] \arrow[dr, "\partial^{\mathcal{F}}_n"'] & H_{n-1}(F^{n-1}) \arrow[d, "\eta_{n-1}"] \\
 & H_n(F^{n+1}) \arrow[d] & & H_{n-1}(F^{n-1}, F^{n-2}) \\
 & 0. & &
\end{tikzcd}$$

Then we have

$$\begin{aligned} H_n(X) &\cong H_n(F^{n+1}) \\ &\cong H_n(F^n)/\operatorname{im}\delta_{n+1} \\ &\cong \operatorname{im}\eta_n/\operatorname{im}\eta_n\circ\delta_{n+1} \\ &\cong \ker\delta_n/\operatorname{im}\partial^{\mathcal{F}}_{n+1} \\ &\cong \ker\eta_{n-1}\circ\delta_n/\operatorname{im}\partial^{\mathcal{F}}_{n+1} \\ &\cong \ker\partial^{\mathcal{F}}_n/\operatorname{im}\partial^{\mathcal{F}}_{n+1} \\ &= H_n(X,\mathcal{F}). \end{aligned}$$

☐

Definition 4.19. *Let $Y, Z \in \mathrm{ob}(\mathsf{Top})$ and $n \geq 1$. We say that **Z is obtained from Y by attaching n-cells**, if there exists a pushout*

$$\begin{tikzcd} \coprod_{\alpha\in A}\mathbb{S}^{n-1}_\alpha \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d] \\ \coprod_{\alpha\in A}\mathbb{B}^n_\alpha \arrow[r, "g"'] & Z. \end{tikzcd}$$

*Moreover, for all $\alpha \in A$ set $f_\alpha := f|_{\mathbb{S}^{n-1}_\alpha}$ and $g_\alpha := g|_{\mathbb{B}^n_\alpha}$. We call g_α the **characteristic map** of the n-cell $g(\mathbb{E}^n_\alpha)$ and f_α its **attaching map**.*

Definition 4.20. *Let X be a cell-complex and $n \geq 1$. Moreover, suppose e_λ is an n-cell and e_ν is an $n-1$-cell. Define $h_{\lambda,\nu} : \mathbb{S}^{n-1}_\lambda \to \mathbb{S}^{n-1}_\nu$, where $\mathbb{S}^{n-1}_\nu := g_\nu(\mathbb{B}^{n-1}_\nu)/f_\nu(\partial\mathbb{B}^{n-1}_\nu)$, to be the composition*

$$\mathbb{S}^{n-1}_\lambda \xrightarrow{f_\lambda} X^{n-1} \longrightarrow X^{n-1}/X^{n-1} \xrightarrow{q_\nu} \mathbb{S}^{n-1}_\nu.$$

Also, define

$$[e_\lambda : e_\nu] := \deg(h_{\lambda,\nu}).$$

Theorem 4.21 (Cellular Boundary Formula). *Let X be a cell-complex. Then the cellular boundary operator $\partial^{\mathrm{cell}} : C^{\mathrm{cell}}_n(X) \to C^{\mathrm{cell}}_{n-1}(X)$ is given by*

$$\partial^{\mathrm{cell}} e_\lambda = \sum_\nu [e_\lambda : e_\nu]\, e_\nu,$$

on generators.

Proof.

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CHAPTER 5

Homology with Coefficients

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Tensor Products in AbGrp.

Definition 5.1 (Tensor Product in AbGrp). *Let $G, H \in \mathrm{ob}(\mathsf{AbGrp})$. We say that a tuple $(G \otimes H, \otimes)$ consisting of an abelian group $G \otimes H$ and a bilinear map $\otimes : G \times H \to G \otimes H$ is a* ***tensor product in*** **AbGrp**, *if $(G \otimes H, \otimes)$ satisfies the following universal property: For any bilinear mapping $\varphi : G \times H \to K$ for some $K \in \mathrm{ob}(\mathsf{AbGrp})$, there exists a unique map $\overline{\varphi} \in \mathsf{AbGrp}(G \otimes K, K)$, such that $\overline{\varphi} \circ \otimes = \varphi$.*

Proposition 5.2. *There exists a tensor product in* AbGrp.

Proof. Set $G \otimes H := F(G \times H)/N$, where $N \trianglelefteq F(G \times H)$ is the subgroup containing

$$(g + g', h) - (g, h) - (g', h) \qquad \text{and} \qquad (g, h + h') - (g, h) - (g, h'),$$

for all $g, g' \in G$ and $h, h' \in H$. Moreover, define $\otimes : G \times H \to G \otimes H$ by setting

$$\otimes(g, h) := g \otimes h := (g, h) + N.$$

Then it is easy to check that $\otimes$ is a bilinear map. By extending $\varphi : G \times H \to K$ by linearity we get a unique homomorphism $\overline{\varphi} : F(G \times H) \to K$. It is easy to check, that $\overline{\varphi}$ passes to the quotient since $N \subseteq \ker \overline{\varphi}$. $\square$

Let $G \in \mathrm{ob}(\mathsf{AbGrp})$. Then there exists a functor $G \otimes - : \mathsf{AbGrp} \to \mathsf{AbGrp}$ defined by $G \otimes H$ on objects and $\mathrm{id}_G \otimes \varphi$ on morphisms.

The Universal Coefficient Theorem

Theorem 5.3 (The Universal Coefficient Theorem). *Let $(C_\bullet, \partial_\bullet)$ be a free chain complex and $A \in \mathrm{ob}(\mathsf{AbGrp})$. For every $n \in \omega$ there is a split short exact sequence*

$$0 \longrightarrow H_n(C_\bullet) \otimes A \longrightarrow H_n(C_\bullet \otimes A) \longrightarrow \mathrm{Tor}(H_{n-1}(C_\bullet), A) \longrightarrow 0.$$

CHAPTER 6

Cohomology

The archetypical example of a cohomology theory arieses in differential topology: The *de Rham cohomology*. It is the homology of the non-negative cochain complex

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d^0} \Omega^1(M) \xrightarrow{d^1} \dots$$

where M is a smooth manifold, $\Omega^k(M) := \Gamma(\Lambda^k T^\vee M)$ denotes the vector space of *smooth differential k-forms* and $d : \Omega^k(M) \to \Omega^{k+1}(M)$ denotes the *exterior differentiation operators*. These extend the notion of a differential of a function and hence provide a more intuitive approach to cohomology than the mere algebraic one. For more on this topic, see for example [Lee13].

The Cohomology Ring

The Cup Product.

Proposition 6.1. *Let $X \in \mathrm{ob}(\mathsf{Top})$ and $R \in \mathrm{ob}(\mathsf{Ring})$. Then there exists a contravariant functor*

$$C(-; R) : \mathsf{Top} \to \mathsf{GRing}.$$

Proof.
Step 1: Definition on objects. Let $X \in \mathsf{Top}$. For $\alpha \in C^n(X; R)$ and $\beta \in C^m(X; R)$ define

$$(\alpha \cup \beta)(\sigma) := \alpha(\sigma \circ A(e_0, \dots, e_n))\beta(\sigma \circ A(e_n, \dots, e_{n+m})),$$

for all singular $n + m$-simplices σ in X. Hence extending by linearity yields a map

$$\cup : C^n(X; R) \times C^m(X; R) \to C^{n+m}(X; R).$$

Moreover, if

$$C(X; R) := \bigoplus_{n\in\omega} C^n(X; R),$$

we define $\cup : C(X; R) \times C(X; R) \to C(X; R)$ by

$$\sum_i \alpha_i \cup \sum_j \beta_j := \sum_{i,j} \alpha_i \cup \beta_j .$$

This is called the ***cup product on $C(X; R)$***. It is easily verified that $(C(X; R), \cup) \in \mathsf{GRing}$.

Step 2: Definition on morphisms. Let $n \in \omega$ and $f \in \mathsf{Top}(X, Y)$. For $\alpha \in C^n(Y; R)$ define

$$C(f; R)(\alpha) := C^n(f)(\alpha) \in C^n(X; R),$$

and extend by linearity.

$\square$

CHAPTER 7

Homotopy Theory

APPENDIX A

Basic Category Theory

Categories. We use the first order theory of Neumann-Bernays-Gödel (BNG) as described in [Men15, p. 231].

Definition A.1 (Category). *A **category** $\mathcal{C}$ consists of*

- *A class* $\mathrm{ob}(\mathcal{C})$*, called the **objects of** $\mathcal{C}$.*
- *A class* $\mathrm{mor}(\mathcal{C})$*, called the **morphisms of** $\mathcal{C}$.*
- *Two functions* $\mathrm{dom} : \mathrm{mor}(\mathcal{C}) \to \mathrm{ob}(\mathcal{C})$ *and* $\mathrm{cod} : \mathrm{mor}(\mathcal{C}) \to \mathrm{ob}(\mathcal{C})$*, which assign to each morphism f in $\mathcal{C}$ its **domain** and **codomain**, respectively.*
- *For each $X \in \mathrm{ob}(\mathcal{C})$ a function* $\mathrm{ob}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C})$ *which assigns a morphism* id_X *such that* $\mathrm{dom}\,\mathrm{id}_X = \mathrm{cod}\,\mathrm{id}_X = X$.
- *A function*

$$\circ : \{(g, f) \in \mathrm{mor}(\mathcal{C}) \times \mathrm{mor}(\mathcal{C}) : \mathrm{dom}\, g = \mathrm{cod}\, f\} \to \mathrm{mor}(\mathcal{C}) \tag{14}$$

  *mapping (g, f) to $g \circ f$, called **composition**, such that* $\mathrm{dom}(g \circ f) = \mathrm{dom}\, f$ *and* $\mathrm{cod}(g \circ f) = \mathrm{cod}\, g$.

Subject to the following axioms:

- ***(Associativity Axiom)** For all $f, g, h \in \mathrm{mor}(\mathcal{C})$ with* $\mathrm{dom}\, h = \mathrm{cod}\, g$ *and* $\mathrm{dom}\, g = \mathrm{cod}\, f$*, we have that*

$$(h \circ g) \circ f = h \circ (g \circ f). \tag{15}$$

- ***(Unit Axiom)** For all $f \in \mathrm{mor}(\mathcal{C})$ with* $\mathrm{dom}\, f = X$ *and* $\mathrm{cod}\, f = Y$ *we have that*

$$f = f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f. \tag{16}$$

Remark A.2. Let $\mathcal{C}$ be a category. For $X, Y \in \mathrm{ob}(\mathcal{C})$ we will abreviate

$$\mathcal{C}(X, Y) := \{f \in \mathrm{mor}(\mathcal{C}) : \mathrm{dom}\, f = X \text{ and } \mathrm{cod}\, f = Y\}.$$

Moreover, $f \in \mathcal{C}(X, Y)$ is depicted as

$$f : X \to Y. \tag{17}$$

Example A.3. Let $*$ be a single, not nearer specified object. Consider as morphisms the class of all cardinal numbers and as composition cardinal addition. By [Hal12, pp. 112–113], cardinal addition is associative and $\varnothing$ serves for the identity id_*.

Definition A.4 (Locally Small, Hom-Set). *A category $\mathcal{C}$ is said to be* ***locally small*** *if for all $X, Y \in \mathcal{C}$, $\mathcal{C}(X, Y)$ is a set. If $\mathcal{C}$ is locally small, $\mathcal{C}(X, Y)$ is called a* ***hom-set*** *for all $X, Y \in \mathcal{C}$.*

Definition A.5 (Isomorphism). *Let $\mathcal{C}$ be a category. An* ***isomorphism in $\mathcal{C}$*** *is a morphism $f : X \to Y$ in $\mathcal{C}$, such that there exists a morphism $g : Y \to X$ in $\mathcal{C}$ with*

$$g \circ f = \mathrm{id}_X \qquad \text{and} \qquad f \circ g = \mathrm{id}_Y .$$

In algebraic topology, there is a very useful construction on categories.

Definition A.6 (Congruence). *Let $\mathcal{C}$ be a category. A* ***congruence on $\mathcal{C}$*** *is an equivalence relation $\sim$ on $\mathrm{mor}(\mathcal{C})$ such that*

(a) *If $f \in \mathcal{C}(X, Y)$ and $f \sim g$, then $g \in \mathcal{C}(X, Y)$.*
(b) *If $f_0 : X \to Y$ and $g_0 : Y \to Z$ such that $f_0 \sim f_1$ and $g_0 \sim g_1$, then $g_0 \circ f_0 \sim g_1 \circ f_1$.*

Exercise A.7. Let $\mathcal{C}$ be a category. Show that for any congruence on $\mathcal{C}$, there exists a category $\mathcal{C}'$, called ***quotient category***, with $\mathrm{ob}(\mathcal{C}') = \mathrm{ob}(\mathcal{C})$, for any objects $X, Y \in \mathcal{C}'$

$$\mathcal{C}'(X, Y) = \{[f] : f \in \mathcal{C}(X, Y)\},$$

and pointwise composition.

Functors.

Definition A.8 (Functor). *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A* ***functor $F : \mathcal{C} \to \mathcal{D}$*** *is a pair of functions (F_1, F_2), $F_1 : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, called the* ***object function*** *and $F_2 : \mathrm{mor}(\mathcal{C}) \to \mathrm{mor}(\mathcal{D})$, called the* ***morphism function****, such that for every morphism $f : X \to Y$ we have that $F_2(f) : F_1(X) \to F_1(Y)$ and (F_1, F_2) is subject to the following* ***compatibility conditions****:*

- *For all $X \in \mathrm{ob}(\mathcal{C})$, $F_2(\mathrm{id}_X) = \mathrm{id}_{F_1(X)}$.*
- *For all $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ we have that $F_2(g \circ f) = F_2(g) \circ F_2(f)$.*

Remark A.9. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. It is convenient to denote the components F_1 and F_2 also with F.

Subcategories.

Definition A.10 (Subcategory). *Let $\mathcal{C}$ be a category. A* ***subcategory $\mathcal{S}$ of $\mathcal{C}$*** *consists of*

- *A subclass $\mathrm{ob}(\mathcal{S}) \subseteq \mathrm{ob}(\mathcal{C})$.*
- *A subclass $\mathrm{mor}(\mathcal{S}) \subseteq \mathrm{mor}(\mathcal{C})$.*

Subject to the following conditions:

- *For all $X \in \mathcal{S}$, $\mathrm{id}_S \in \mathrm{mor}(\mathcal{S})$.*

Example A.11 (Top^2). Define the objects of Top^2 to be the class of tuple (X, A), where $X \in \mathrm{ob}(\mathsf{Top})$ and A is a subspace of X. Moreover, given objects (X, A) and (Y, B) in

Top^2, a morphism between (X, A) and (Y, B) is a tuple (f, g), where $f \in \mathsf{Top}(X, Y)$ and $g \in \mathsf{Top}(A, B)$, such that

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, "g"'] & X \arrow[d, "f"] \\ B \arrow[r, hook] & Y \end{tikzcd}$$

commutes.

Example A.12 (Top_*). Define the objects of Top_* to be the class of all tuple (X, p), where X is a topological space and $p \in X$. Moreover, given objects (X, p) and (Y, q) in Top_*, define $\mathsf{Top}_*\big((X, p), (Y, q)\big) := \{f \in \mathsf{Top}(X, Y) : f(p) = q\}$. It is easy to check that Top_* is a category, called the ***category of pointed topological spaces***.

Limits.

Definition A.13 (Diagram). *Let $\mathscr{C}$ be a category and A a small category. A functor $\mathsf{A} \to \mathscr{C}$ is called a **diagram in $\mathscr{C}$ of shape A**.*

Definition A.14 (Cone and Limit). *Let $\mathscr{C}$ be a category and $D : \mathsf{A} \to \mathscr{C}$ a diagram in $\mathscr{C}$ of shape A. A **cone on D** is a tuple $\big(C, (f_\alpha)_{\alpha \in \mathsf{A}}\big)$, where $C \in \mathscr{C}$ is an object, called the **vertex** of the cone, and a family of arrows in $\mathscr{C}$*

$$\big(C \xrightarrow{f_\alpha} D(\alpha)\big)_{\alpha \in \mathsf{A}}. \tag{18}$$

such that for all morphisms $f \in \mathsf{A}$, $f : \alpha \to \beta$, the triangle

$$\begin{tikzcd} & D(\alpha) \arrow[dd, "D(f)"] \\ C \arrow[ur, "f_\alpha"] \arrow[dr, "f_\beta"'] & \\ & D(\beta) \end{tikzcd}$$

*commutes. A **(small) limit of D** is a cone $\big(L, (\pi_\alpha)_{\alpha \in \mathsf{A}}\big)$ with the property that for any other cone $\big(C, (f_\alpha)_{\alpha \in \mathsf{A}}\big)$ there exists a unique morphism $\overline{f} : C \to L$ such that $\pi_\alpha \circ \overline{f} = f_\alpha$ holds for every $\alpha \in \mathsf{A}$.*

Remark A.15. In the setting of definition A.14, if $\big(L, (\pi_\alpha)_{\alpha \in \mathsf{A}}\big)$ is a limit of D, we sometimes reffering to L only as the limit of D and we write

$$L = \lim_{\leftarrow \mathsf{A}} D. \tag{19}$$

Filtered Colimits.

Definition A.16 (Filtered Category). *A category $\mathcal{J}$ is* ***filtered****, if $\mathcal{J}$ is not empty and*
(a) To any two objects j and j' in $\mathcal{J}$ there exists $k \in \mathrm{ob}(\mathcal{J})$ and morphisms $j \to k$ and $j' \to k$.
(b) To any two parallel arrows $u, v : i \to j$ in $\mathcal{J}$, there exists $k \in \mathrm{ob}(\mathcal{J})$ and a morphism $w : j \to k$ in $\mathcal{J}$, such that $w \circ u = w \circ v$.

Definition A.17 (Filtered Colimit). *Let $\mathcal{C}$ be a category. A* ***filtered diagram in $\mathcal{C}$ of shape $\mathcal{J}$*** *is a diagram in $\mathcal{C}$ of shape $\mathcal{J}$, where $\mathcal{J}$ is a small filtered category. A* ***filtered colimit of D*** *is a colimit of a filtered diagram D in $\mathcal{C}$, written $\varinjlim D$.*

Proposition A.18. *In Set, all filtered limits exist.*

Proof. Let $D : \mathcal{J} \to \mathsf{Set}$ be a filtered diagram. Define

$$X := \coprod_{j \in \mathrm{ob}(\mathcal{J})} D(j),$$

and define a relation $\sim$ on X by

$$x \in D(j) \sim y \in D(j') :\Leftrightarrow \exists f : j \to k, g : j' \to k \text{ such that } D(f)(x) = D(g)(y).$$

Then it is easy to check that $\sim$ is an equivalence relation and that $\varinjlim D \cong X/\sim$. $\square$

Proposition A.19. *In Top, all filtered colimits exist.*

Proof. Let $D : \mathcal{J} \to \mathsf{Top}$ be a filtered diagram. If $U : \mathsf{Top} \to \mathsf{Set}$ denotes the forgetful functor, $U \circ D$ is a filtered diagram in Set. Hence using proposition A.18, we know that $\varinjlim(U \circ D)$ exists. Hence we get a limiting cocone $\big(\varinjlim(U \circ D), (q_j)_{j \in \mathrm{ob}(\mathcal{J})}\big)$ in Set. Define a topology $\varinjlim(U \circ D)$ by letting $U \subseteq \varinjlim(U \circ D)$ to be open if and only if $q_j^{-1}(U)$ open in $U(D(j))$ for all $j \in \mathrm{ob}(\mathcal{J})$. Then it is easy to check that we have a limiting cocone in Top. $\square$

Definition A.20 (Preorder). *Let P be a set. A* ***preorder on P*** *is a reflexive and transitive binary relation $\preccurlyeq$ on P. A set equipped with a preorder is called a* ***preordered set****.*

Definition A.21 (Directed Preorder). *A preorder $\preccurlyeq$ on a set P is said to be* ***directed*** *if for every two elements $p, q \in P$ there exists $r \in P$, such that $p \preccurlyeq r$ and $q \preccurlyeq r$ holds. A set equipped with a directed preorder is called a* ***directed set****.*

Lemma A.22. *Let $(P, \preccurlyeq)$ be a directed set. Define*
- $\mathrm{ob}(\mathcal{J}(P, \preccurlyeq)) := P$.
- *For $p, q \in P$, if $p \not\preccurlyeq q$ let $\mathcal{J}(P, \preccurlyeq)(p, q) := \varnothing$ and if $p \preccurlyeq q$ let $\mathcal{J}(P, \preccurlyeq)(p, q)$ be the unique arrow $p \to q$.*
- *For $p, q, r \in P$, let $q \to r \circ p \to q := p \to r$.*

Then $\mathcal{J}(P, \preccurlyeq)$ is a filtered category.

Exercise A.23. Prove lemma A.22.

Definition A.24 (Sequential Colimit). *A* ***sequential colimit of D*** *is a colimit of a diagram D of shape $(\omega, \leq)$. We will write $\varinjlim_n D(n)$ for a sequential colimit of D.*

1. Preadditive Catgeories

Let $G, H \in \mathrm{ob}(\mathsf{AbGrp})$ and $\varphi, \psi \in \mathsf{AbGrp}(G, H)$. Define $\varphi + \psi$ pointwise. Since H is abelian, it follows that $\varphi + \psi \in \mathsf{AbGrp}(G, H)$. Moreover, it is easy to check, that with this operation defined above, $\mathsf{AbGrp}(G, H)$ is an abelian group and

$$\circ : \mathsf{AbGrp}(H, K) \times \mathsf{AbGrp}(G, H) \to \mathsf{AbGrp}(G, K)$$

is bilinear for each $K \in \mathrm{ob}(\mathsf{AbGrp})$. This motivates the following definition.

Definition A.25 (Preadditive Category [Lan78]). *A* ***preadditive category*** *is a locally small category $\mathcal{C}$ in which all hom-sets $\mathcal{C}(X, Y)$ can be equipped with the structure of an abelian group and composition is bilinear, i.e. for all mophisms $f, f' : X \to Y$ and $g, g' : Y \to Z$ in $\mathcal{C}$ we have that*

$$(g + g') \circ (f + f') = g \circ f + g \circ f' + g' \circ f + g' \circ f'. \tag{20}$$

Remark A.26. In a preadditive category $\mathcal{C}$, we have that $\mathcal{C}(X, Y) \neq \varnothing$ for all $X, Y \in \mathrm{ob}(\mathcal{C})$.

Lemma A.27. *Let $\mathcal{C}$ be a preadditive category. Then compositions with the zero elements are again zero elements of the correpsonding abelian groups.*

Proof. This simply follows from $0 \circ f = (0 + 0) \circ f = 0 \circ f + 0 \circ f$. The other case is similar. □

2. Additive Categories

As in Grp, the trivial group in AbGrp is both an initial and a terminal object. Objects with this property have a special name.

Definition A.28 (Null Object [Lan78]). *Let $\mathcal{C}$ be a category. A* ***null object in $\mathcal{C}$*** *is a an object of $\mathcal{C}$ which is both initial and terminal.*

Definition A.29 (Zero Arrow [Lan78]). *Let $\mathcal{C}$ be a category with a null object 0. For $X, Y \in \mathrm{ob}(\mathcal{C})$, the unique composition $X \to 0 \to Y$ is called the* ***zero arrow from X to Y****, denoted by $0 : X \to Y$.*

Lemma A.30. *Let $\mathcal{C}$ be a preadditive category with null object and $X, Y \in \mathrm{ob}(\mathcal{C})$. Then the zero arrow $0 : X \to Y$ is the zero element of the group $\mathcal{C}(X, Y)$.*

Proof. The zero arrow $0 : X \to Y$ is the unique composition

$$X \longrightarrow 0 \longrightarrow Y.$$

However, since 0 is a null object, we have that the two morphisms are the two zero objects in the corresponding abelian group structures. Hence lemma A.27 yields the result. □

Let $A, B \in \mathrm{ob}(\mathsf{AbGrp})$. Then we have seen that $A \coprod B \cong A \prod B$. This can be generalized to preadditive categories.

Proposition A.31. *Let $\mathcal{C}$ be a preadditive category admitting all finite coproducts. Then any n-ary coproduct is also an n-ary product for all $n \in \omega$. In particular, $\mathcal{C}$ admits all finite products and a null object.*

Proof.

Step 1: Zero-ary case [Lan78, p. 194]. Since $\mathcal{C}$ has the empty coproduct, $\mathcal{C}$ has an initial object $\varnothing$. Since $\varnothing$ is initial, there exists a unique map $\varnothing \to \varnothing$, namely $\mathrm{id}_\varnothing$. But $\mathcal{C}(\varnothing, \varnothing)$ is a group and thus $\mathrm{id}_\varnothing = 0$. Hence for any morphism $f : X \to \varnothing$, lemma A.27 yields $f = \mathrm{id}_\varnothing \circ f = 0 \circ f = 0$.

Step 2: Binary case. By the zero-ary case we know that $\mathcal{C}$ admits a null object 0. Let $X, Y \in \mathrm{ob}(\mathcal{C})$. We want to show that $X \coprod Y$ is also a product of X and Y. By the universal property of the coproduct we have a commutative diagram

$$\begin{tikzcd}
 & X & \\
X \arrow[ur, "\mathrm{id}_X"] \arrow[r, "\iota_X"] \arrow[dr, "0"'] & X \coprod Y \arrow[u, "(\mathrm{id}_X, 0)"'] \arrow[d, "(0, \mathrm{id}_Y)"] & Y \arrow[ul, "0"'] \arrow[l, "\iota_Y"'] \arrow[dl, "\mathrm{id}_Y"] \\
 & Y. &
\end{tikzcd}$$

Suppose (Z, p_X, p_Y) is another product cone. Define $f : Z \to X \coprod Y$ by

$$f := \iota_X \circ p_X + \iota_Y \circ p_Y.$$

Using lemma A.30 and A.27, we compute

$$\begin{aligned}
(\mathrm{id}_X, 0) \circ f &= (\mathrm{id}_X, 0) \circ \iota_X \circ p_X + (\mathrm{id}_X, 0) \circ \iota_Y \circ p_Y \\
&= \mathrm{id}_X \circ p_X + 0 \circ p_Y \\
&= p_X + 0 \\
&= p_X,
\end{aligned}$$

and simiarly $(0, \mathrm{id}_Y) \circ f = p_Y$. Now we have to check uniqueness. This is the hardest part of the proof and involves the *Yoneda embedding* $\mathcal{Y} : \mathcal{C} \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$. We want to show that

$$\iota_X \circ (\mathrm{id}_X, 0) + \iota_Y \circ (0, \mathrm{id}_Y) = \mathrm{id}_{X \coprod Y}.$$

Applying the Yoneda embedding to the category $\mathcal{C}^{\mathrm{op}}$, we get that it is enough to show that

$$f \circ \iota_X \circ (\mathrm{id}_X, 0) + \iota_Y \circ (0, \mathrm{id}_Y) = f \circ \mathrm{id}_{X \coprod Y}$$

holds for all morphisms $f \in \mathcal{C}(X \coprod Y, C)$. Let $(\alpha, \beta) : X \coprod Y \to C$ be any morphism (where $(\alpha, \beta) \circ \iota_X) = \alpha$ and $(\alpha, \beta) \circ \iota_Y = \beta$). Using the universal property of the corpoduct it is easy to show that

$$(\alpha, \beta) \circ \iota_X \circ (\mathrm{id}_X, 0) = (\alpha, 0) \quad \text{and} \quad (\alpha, \beta) \circ \iota_Y \circ (0, \mathrm{id}_Y) = (0, \beta),$$

and moreover one can show that for any other morphism $(\alpha', \beta') : X \coprod Y \to C$ we have

$$(\alpha, \beta) + (\alpha', \beta') = (\alpha + \alpha', \beta + \beta').$$

Thus

$$(\alpha, \beta) \circ \mathrm{id}_{X \coprod Y} = (\alpha, \beta) = (\alpha, 0) + (0, \beta) = (\alpha, \beta) \circ \left(\iota_X \circ (\mathrm{id}_X, 0) + \iota_Y \circ (0, \mathrm{id}_Y)\right).$$

Now if $f' : Z \to X \coprod Y$ is another morphism making the diagram commute, we have that

$$\begin{aligned} f - f' &= \mathrm{id}_{X \coprod Y} \circ (f - f') \\ &= \left(\iota_X \circ (\mathrm{id}_X, 0) + \iota_Y \circ (0, \mathrm{id}_Y)\right) \circ (f - f') \\ &= \iota_X \circ (p_X - p_X) + \iota_Y \circ (p_Y - p_Y) \\ &= \iota_X \circ 0 + \iota_Y \circ 0 \\ &= 0, \end{aligned}$$

by lemma A.27.
Step 3: n-ary case. Induction over $n \in \omega$.

□

Definition A.32 (Additive Category). *An **additive category** is a preadditive category which admits all finite coproducts.*

Remark A.33. Let $\mathcal{C}$ be an additive category. Then by proposition A.31, $\mathcal{C}$ admits all finite products which coincide with the coproducts.

3. Abelian Categories

Definition A.34 (Kernel and Cokernel [Lan78]). *Let $\mathcal{C}$ be a category with a null object 0. A **kernel of a morphism $f : X \to Y$** is defined to be an equalizer of*

$$X \underset{0}{\overset{f}{\rightrightarrows}} Y.$$

*Dually, a **cokernel of a morphism $f : X \to Y$** is a coequalizer of the above diagram.*

Lemma A.35. *In* Grp*, every monic is a kernel and every epic is a cokernel.*

Proof. Let $m : G \to H$ be a monic in Grp. Consider the fork

$$G \xrightarrow{m} H \overset{\pi}{\underset{0}{\rightrightarrows}} \operatorname{coker} m.$$

Then one can check that this is in fact a universal fork. Similarly, one can check that

$$\ker e \overset{\iota}{\underset{0}{\rightrightarrows}} G \xrightarrow{e} H$$

is a universal cofork for any epic $e : G \to H$ in Grp. $\square$

Definition A.36 (Abelian Category [Lan78]). *An* ***abelian category*** *is an additive category satisfying the following additional conditions:*

(*a*) *Every morphism admits a kernel.*
(*b*) *Every morphism admits a cokernel.*
(*c*) *Every monic is a kernel.*
(*d*) *Every epic is a cokernel.*

Examples A.37. AbGrp, Vect_K, ${}_R\mathsf{Mod}$ and Mod_R.

4. Exact Sequences

We follow [Fre64, p. 44].

Lemma A.38. *Given a diagram*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in AbGrp*, we have that above sequence is exact at B if and only if $g \circ f = 0$ and*

$$\ker g \overset{\iota}{\hookrightarrow} B \xrightarrow{\pi} \operatorname{coker} f \quad = 0.$$

Proof. Trivial. $\square$

In lemma A.38, the second condition involves statements about the kernel and the cokernel in the categorical sense. Indeed, in Grp we have that

$$\ker g \overset{\iota}{\hookrightarrow} B \overset{g}{\underset{0}{\rightrightarrows}} C \qquad \text{and} \qquad A \overset{f}{\underset{0}{\rightrightarrows}} B \xrightarrow{\pi} \operatorname{coker} f$$

are an equalizer and a coequalizer, respectively. Hence

$$\ker g = \ker g \overset{\iota}{\hookrightarrow} B \qquad \text{and} \qquad \operatorname{coker} f = B \xrightarrow{\pi} \operatorname{coker} f.$$

Definition A.39 (Exactness). *Let $\mathcal{C}$ be an abelian category. A sequence*

$$X \longrightarrow Y \longrightarrow Z$$

of objects in $\mathcal{C}$ is said to be ***exact at Y****, if*

$$X \longrightarrow Y \longrightarrow Z \ = 0 \qquad \textit{and} \qquad K \longrightarrow Y \longrightarrow C \ = 0,$$

where

$$K \to Y = \ker(Y \to Z) \qquad \textit{and} \qquad Y \to C = \operatorname{coker}(X \to Y).$$

Definition A.40 (Left and Right Exactness). *Let $\mathcal{C}$ be an abelian category. A* ***left exact sequence*** *is an exact sequence of the form*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z.$$

Similarly, a ***right exact sequence*** *is an exact sequence*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

Definition A.41 (Exact Functor). *Let $\mathcal{C}$, $\mathcal{D}$ be abelian categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a* ***left exact functor****, if it preserves left exact sequences. Similarly, F is called a* ***right exact functor****, if it preserves right exact sequences. Finally, F is called an* ***exact functor****, if it preserves exact sequences.*

Remark A.42. A functor between abelian categories is left exact if and only if it preserves kernels. Similarly, it is right exact if and only if it preserves cokernels. A functor between abelian categories is exact, if and only if it is both left and right exact.

APPENDIX B

Basic Group Theory

APPENDIX C

Basic Point-Set Topology

The Category of Topological Spaces

Topologies. We follow [Lee11, p. 20].

Definition C.1 (Topology). *Let X be a set. A* ***topology on X*** *is a collection $\mathcal{T} \subseteq 2^X$ of subsets of X, satisfying the following properties:*
(i) $X, \varnothing \in \mathcal{T}$.
(ii) If $U_1, \dots, U_n \in \mathcal{T}$, then $U_1 \cap \dots \cap U_n \in \mathcal{T}$.
(iii) If $(U_\alpha)_{\alpha \in A}$ is a family in $\mathcal{T}$ for an arbitrary set A, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.

Definition C.2 (Topological Space). *A* ***topological space*** *is a tuple $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a topology on X.*

Remark C.3. In practice, the topology $\mathcal{T}$ on X in the topologial space $(X, \mathcal{T})$ is often omitted for shortness.

Definition C.4 (Point). *Let $(X, \mathcal{T})$ be a topological space. The elements of X are called* ***points****.*

Definition C.5 (Open and Closed Subsets). *Let $(X, \mathcal{T})$ be a topologial space. A subset $U \in \mathcal{T}$ is called an* ***open subset of X****. A subset $A \subseteq X$ is called a* ***closed subset of X****, if its complement $X \setminus A$ is an open subset of X.*

Definition C.6 (Neighbourhood). *Let X be a topological space and $A \subseteq X$ a subset. A* ***neighbourhood of A in X*** *is an open subset U such that $A \subseteq U$. In particular, for some point p in X, a neighbourhood of $\{p\}$ in X is simply called a* ***neighbourhood of p****.*

Continuity.

Definition C.7 (Continuity). *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topologial spaces. A function $f : X \to Y$ is called* ***continuous****, if for all $U \in \mathcal{T}_Y$, $f^{-1}(U) \in \mathcal{T}_X$.*

The Category Top.

Proposition C.8. *There exists a locally small category with objects topologial spaces and continuous maps as morphisms.*

Exercise C.9. Prove proposition C.8.

Definition C.10 (Top). *The category defined in the proof of proposition C.8 is called the* ***category of topologial spaces****, denoted by* Top.

<++>

The Lebesgue Number Lemma.

Definition C.11 (Lebesgue Number). *Let (M, d) be a metric space with an open cover $(U_\alpha)_{\alpha \in A}$. A number $\delta > 0$ is called a* ***Lebesgue number*** *for the cover, if every subset of M whose diameter is less than δ is contained in U_α for some $\alpha \in A$.*

Lemma C.12 (Lebesgue Number Lemma). *Every open cover of a compact metric space admits a Lebesgue number.*

The Closed Map Lemma.

Lemma C.13 (Closed Map Lemma). *Let $X, Y \in \mathsf{ob}(\mathsf{Top})$ such that X is compact and Y is Hausdorff, and $f \in \mathsf{Top}(X, Y)$. Then:*

(*a*) *f is a closed map.*
(*b*) *If f is injective, it is a topological embedding.*
(*c*) *If f is surjective, it is a quotient map.*
(*d*) *If f is bijective, it is a homeomorphism.*

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