Classical Mechanics

Yannis Bähni*

Semester Paper under the Supervision of Prof. Dr. Ana Cannas Da Silva at ETH Zürich

*ETH Zürich, Rämistrasse 101, 8092 Zürich. *E-mail address*: baehniy@student.ethz.ch

Preface

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Contents

CHAPTER 1

Lagrangian Mechanics

Lagrangian Systems and the Principle of Least Action

Definition 1.1 (Lagrangian System). *A* ***Lagrangian system*** *is defined to be a tuple* (M, L) *consisting of an object* $M \in \mathsf{Diff}$ *and a morphism* $L \in \mathsf{Diff}(TM \times \mathbb{R}, \mathbb{R})$*, called a* ***Lagrangian function****.*

Definition 1.2 (Path Space). *Let* $M \in \mathsf{Diff}$*,* $p, q \in M$ *and* $t_p, t_q \in \mathbb{R}$ *with* $t_p \leq t_q$*. Define the* ***path space of*** $\boldsymbol{M}$ ***connecting*** $\boldsymbol{(p, t_p)}$ ***and*** $\boldsymbol{(q, t_q)}$ *to be the set*

$$\mathscr{P}(M)_{q,t_q}^{p,t_p} := \{\gamma \in \mathsf{Diff}([t_p, t_q], M) : \gamma(t_p) = p \text{ and } \gamma(t_q) = q\}. \tag{1}$$

Remark 1.3. For the sake of simplicity, we will just use the terminology *path space* for $\mathscr{P}(M)_{q,t_q}^{p,t_p}$ and simply write $\mathscr{P}(M)$. We implicitely assume the conditions of definition 1.2, however.

Proposition 1.4. *The path space* $\mathscr{P}(M)$ *is an infinite-dimensional real Fréchet manifold.*

Proof. □

Definition 1.5 (Variation). *Let* $\mathscr{P}(M)$ *be a path space and* $\gamma \in \mathscr{P}(M)$*. A* ***variation of*** $\boldsymbol{\gamma}$ *is defined to be a morphism* $\Gamma \in \mathsf{Diff}([t_p, t_q] \times [-\varepsilon_0, \varepsilon_0], M)$ *for some* $\varepsilon_0 > 0$ *and such that*

- $\Gamma(t, 0) = \gamma$ *for all* $t \in [t_p, t_q]$.
- $\Gamma(t_p, \varepsilon) = p$ *for all* $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.
- $\Gamma(t_q, \varepsilon) = q$ *for all* $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

Remark 1.6. If Γ is a variation of $\gamma \in \mathscr{P}(M)$, we write $\gamma_\varepsilon(-) := \Gamma(-, \varepsilon)$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

Definition 1.7 (Action Functional). *Let* (M, L) *be a Lagrangian system and* $\mathscr{P}(M)$ *be a path space. The morphism* $S : \mathscr{P}(M) \to \mathbb{R}$ *defined by*

$$S(\gamma) := \int_{t_p}^{t_q} L(\gamma(t), \gamma'(t), t)dt$$

is called the ***action functional****.*

Axiom 1 (Hamilton's Principle of Least Action). *Let* (M, L) *be a Lagrangian system and* $\mathcal{P}(M)$ *be a path space. A path* $\gamma \in \mathsf{Diff}([t_p, t_q], M)$ *describes a motion of* (M, L) *iff*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\gamma_\varepsilon) = 0 \tag{2}$$

for all variations γ_ε *of* γ.

CHAPTER 2

Hamiltonian Mechanics