# Mathematical Aspects of Classical Mechanics

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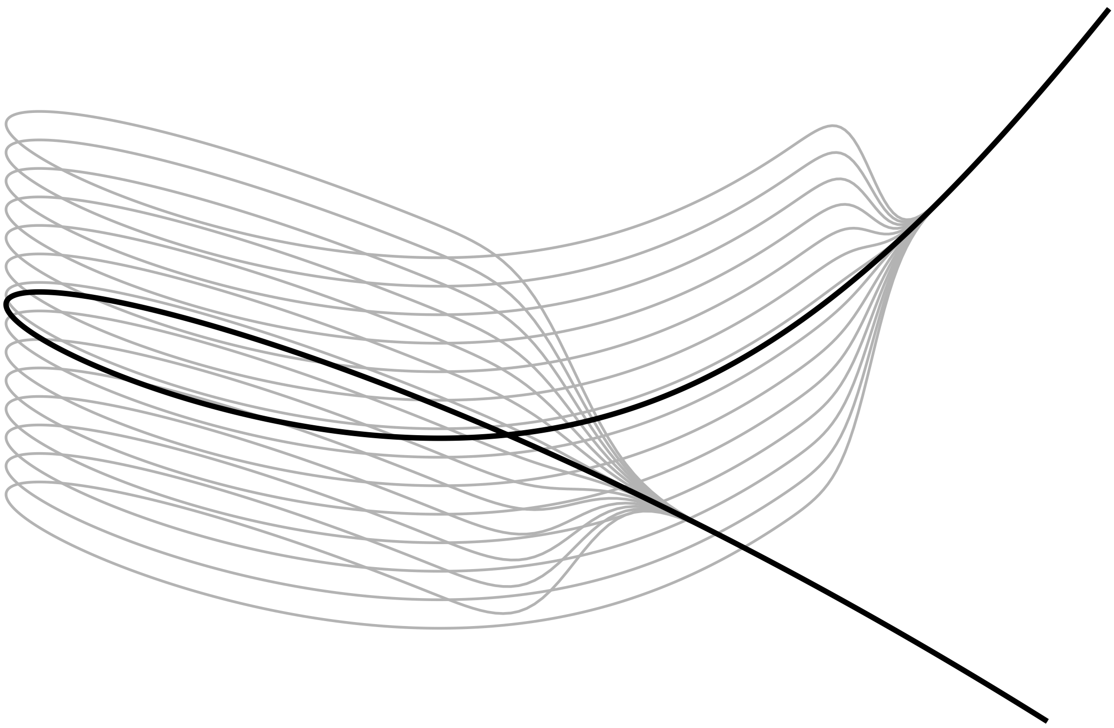

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# Preface

These notes are the product of a semester project done at the *ETH Zürich* in the autumn semester of 2018 under the supervision of *Dr. Ana Cannas da Silva*. I will roughly follow the first chapter of the book *Quantum Mechanics for Mathematicians* by *Leon A. Takhtajan* [Tak08], which serves as an introduction to classical mechanics. Since this introduction is very brief, understandable by considering its purpose, I additionally rely on the classic *Mathematical Methods of Classical Mechanics* by *Vladimir I. Arnold* [Arn89]. As the title already suggests, this is not a treatment of the physical part of classical mechanics, but rather a mathematical one. Hence the aim of these notes is to give a thoughtful introduction to the mathematical methods used in the realm of classical mechanics and their strong connection to differential topology and differential geometry, especially *symplectic geometry*. Therefore it is only natural to consider also the book *Lectures on Symplectic Geometry* by *Ana Cannas da Silva* [Sil08].

I would like to thank first of all my supervisor Dr. Cannas da Silva for granting me this opportunity of writing these notes, and also for introducting me to symplectic geometry back in the autumn semester 2017. Moreover, I would like to thank *Prof. Dr. Will J. Merry*, whose brilliant lectures on *Algebraic Topology* as well as *Differential Geometry* helped me alot in understanding this and related subjects. Also, he was a great help in answering questions and clarifying concepts. A big help was also the marvelous trilogy of books from *John M. Lee* ([Lee11], [Lee13] and [Lee97]), which clear, thoughtful and highly formal exposition of the subject give an in-depth understanding of the matter. I won't deny the obvious: My style of writing and even the typeset of this document is highly inspired, sometimes even copied, from the style used by Jack Lee. The simple reason is, that I appreciate his work very much and try to achieve the same fineness. A prominent indicator of this fact is also the numerous citations of his books in these notes. Lastly, I would like to thank both the mathematics institute at the *University of Zürich* as well as the mathematics institute here at ETH Zürich, for teaching me mathematics. Without whom, maybe I would never have experienced the passion for doing mathematics. In this sense, happy reading (shamelessly ripped off the preface of [Lee13])!

Winterthur, September 8, 2018 — Yannis Bähni

# Contents

CHAPTER 1

# Lagrangian Mechanics

## Introduction

Classical mechanics deals with ordinary differential equations originating from extremals of ***functionals***, that is functions defined on an infinite-dimensional function space. The study of such extremality properties of functionals is known as the ***calculus of variations***. To illustrate this fundamental principle, let us consider the *variational formulation* of second order elliptic operators in divergence form based on [Str14, pp. 167–168].
For convention, unless expliciety stated otherwise, we will assume that all manifolds are smooth, that is of class $C^\infty$, finite-dimensional, Hausdorff and paracompact with at most countably many connected components.
Let $n \in \mathbb{N}$, $n \geq 1$, and $\Omega \subseteq\subseteq \mathbb{R}^n$ such that $\bar{\Omega}$ is a smooth manifold with boundary. Moreover, let $H_0^1(\Omega)$ denote the Sobolev space $W_0^{1,2}(\Omega)$ with inner product

$$\langle u, v\rangle_{H_0^1(\Omega)} = \int_\Omega uv + \int_\Omega \nabla u \nabla v.$$

Suppose $a^{ij} \in C^\infty(\bar{\Omega})$ symmetric, $f \in C^\infty(\bar{\Omega})$ and consider the second order homogenous Dirichlet problem

$$\begin{cases} -\dfrac{\partial}{\partial x^j}\left(a^{ij}\dfrac{\partial u}{\partial x^i}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

Suppose $u \in C^\infty(\bar{\Omega})$ solves (1). Then integration by parts (see [Lee13, p. 436]) yields

$$\int_\Omega fv = -\int_\Omega \frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial u}{\partial x^i}\right)v = -\int_\Omega \operatorname{div}(X)v = \int_\Omega \langle X, \nabla v\rangle = \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^j}$$

for any $v \in C_c^\infty(\Omega)$, where $X := \left(a^{ij}\frac{\partial u}{\partial x^i}\right)_j$. Thus we say that $u \in H_0^1(\Omega)$ is a *weak solution* of (1) iff

$$\forall v \in C_c^\infty(\Omega) : \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^j} = \int_\Omega fv.$$

If $(a^{ij})_{ij}$ is *uniformly elliptic*, i.e. there exists $\lambda > 0$ such that

$$\forall x \in \Omega \forall \xi \in \mathbb{R}^n : a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2,$$

then (1) admits a unique weak solution $u \in H_0^1(\Omega)$ (in fact $u \in C^\infty(\Omega)$ using *regularity theory*, for more details see [Str14, p. 175]). Indeed, observe that

$$\langle \cdot, \cdot \rangle_a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

defined by

$$\langle u, v \rangle_a := \int_\Omega a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} \tag{2}$$

is an inner product on $H_0^1(\Omega)$ with induced norm equivalent to the standard one on $H_0^1(\Omega)$ due to Poincaré's inequality [Str14, p. 107]. Applying the Riesz Representation theorem [Str14, pp. 49–50] yields the result. Moreover, this solution can be characterized by a *variational principle*, i.e. if we define the *energy functional* $E : H_0^1(\Omega) \to \mathbb{R}$

$$E(v) := \frac{1}{2}\|v\|_a^2 - \int_\Omega f v,$$

for any $v \in H_0^1(\Omega)$, where $\|\cdot\|_a$ denotes the norm induced by the inner product (2), then $u \in H_0^1(\Omega)$ solves (1) if and only if

$$E(u) = \inf_{v \in H_0^1(\Omega)} E(v). \tag{3}$$

Indeed, suppose $u \in H_0^1(\Omega)$ is a solution of (1). Let $v \in H_0^1(\Omega)$. Then $u = v + w$ for $w := u - v \in H_0^1(\Omega)$ and we compute

$$E(v) = E(u+w) = \frac{1}{2}\|u\|_a^2 + \langle u, w \rangle_a + \frac{1}{2}\|w\|_a^2 - \int_\Omega f(u+w) = E(u) + \frac{1}{2}\|w\|_a^2 \geq E(u)$$

with equality if and only if $u = v$ a.e. Conversly, suppose the infimum is attained by some $u \in H_0^1(\Omega)$. Thus by elementary calculus

$$0 = \left.\frac{d}{dt}\right|_{t=0} E(u + tv) = \langle u, v \rangle_a - \int_\Omega f v \tag{4}$$

for all $v \in H_0^1(\Omega)$.

Suppose now that $u \in C^\infty(\bar{\Omega})$ with $u|_{\partial\Omega} = 0$ solves the variational formulation (3). Then again integration by parts yields

$$\langle u, v \rangle_a - \int_\Omega f v = -\int_\Omega \operatorname{div}(X) v - \int_\Omega f v = \int_\Omega \left( -\frac{\partial}{\partial x^j} \left( a^{ij} \frac{\partial u}{\partial x^i} \right) - f \right) v$$

for all $v \in C_c^\infty(\Omega)$ and where $X := \left( a^{ij} \frac{\partial u}{\partial x^i} \right)_j$. Hence (4) implies

$$\forall v \in C_c^\infty(\Omega) : \int_\Omega \left( -\frac{\partial}{\partial x^j} \left( a^{ij} \frac{\partial u}{\partial x^i} \right) - f \right) v = 0.$$

We might expect that this implies

$$-\frac{\partial}{\partial x^j} \left( a^{ij} \frac{\partial u}{\partial x^i} \right) = f.$$

That this is indeed the case, is guaranteed by a foundational result in the *calculus of variations* (therefore the name).

**Proposition 1.1 (Fundamental Lemma of Calculus of Variations [Str14, p. 40]).** *Let $\Omega \subseteq \mathbb{R}^n$ open and $f \in L^1_{\text{loc}}(\Omega)$. If*

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega f\varphi = 0,$$

*then $f = 0$ a.e.*

Thus we recovered a second order partial differential equation from the variational formulation. In fact, this is exactly the boundary value problem (1) from the beginning of our exposition. This technique, and in particular the fundamental lemma of calculus of variations 1.1 will play an important role in our treatment of classical mechanics. However, since we are concerned with smooth manifolds only, we use a version of the fundamental lemma of calculus of variations 1.1, which is fairly easy to prove and hence really deserves the terminology "lemma".

**Lemma 1.2 (Fundamental Lemma of Calculus of Variations, Smooth Version).** *Let $\Omega \subseteq \mathbb{R}^n$ open and $f \in C^\infty(\Omega)$. If*

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega f\varphi = 0,$$

*then $f = 0$.*

*Proof.* Towards a contradiction, assume that $f \neq 0$ on $\Omega$. Thus there exists $x_0 \in \Omega$, such that $f(x_0) \neq 0$. Without loss of generality, we may assume that $f(x_0) > 0$, since otherwise, consider $-f$ instead of $f$. The smoothness of $f$ implies the continuity of $f$ on $\Omega$. Thus there exists $\delta > 0$, such that $f(x) \in B_{f(x_0)/2}\left(f(x_0)\right)$ holds for all $x \in B_\delta(x_0)$ or equivalently, $f(x) > f(x_0)/2 > 0$ for all $x \in B_\delta(x_0)$. By lemma 2.22 [Lee13, p. 42], there exists a smooth bump function $\varphi$ supported in $B_\delta(x_0)$ and $\varphi = 1$ on $\overline{B}_{\delta/2}(x_0)$. In particular, $\varphi \in C_c^\infty(\Omega)$. Therefore we have

$$0 = \int_\Omega f\varphi = \int_{B_\delta(x_0)} f\varphi \geq \int_{B_{\delta/2}(x_0)} f\varphi > \frac{1}{2} f(x_0)\, |B_{\delta/2}(x_0)| > 0,$$

which is a contradiction. □

**Exercise 1.3.** [1] Let $\Omega \subseteq\subseteq \mathbb{R}^n$, $2 \leq p < \infty$ and define $\mathcal{B} := \{v \in C^\infty(\overline{\Omega}) : v|_{\partial\Omega} = 0\}$. Moreover, define $E_p : \mathcal{B} \to \mathbb{R}$ by $E_p(v) := \int_\Omega |\nabla v|^p$. Derive the partial differential equation satisfied by minimizers $u \in \mathcal{B}$ of the variational problem $E(u) = \inf_{v \in \mathcal{B}} E(v)$.

[1] This is exercise 1.2.($b$) from exercise sheet 1 of the course *Functional Analysis II* taught by *Prof. Dr. A. Carlotto* at ETHZ in the spring of 2018, which can be found here.

**Lagrangian Systems and the Principle of Least Action**

Mechanical systems, for example a pendulum, are modelled using the language of differential geometry. Thus it is necessary to introduce the relevant physical counterparts.

**Definition 1.4 (Configuration Space).** *A **configuration space** is defined to be a finite-dimensional smooth manifold.*

**Definition 1.5 (Motion).** *A **motion in a configuration space $M$** is defined to be a path $\gamma \in C^\infty(J, M)$, where $J \subseteq \mathbb{R}$ is an interval.*

**Definition 1.6 (State).** *A **state of the configuration space** is defined to be an element of the tangent bundle of the configuration space, called the **state space**.*

One should think of a state $(x, v)$ of a configuration space as follows: $x$ gives the position of the mechanical system and $v$ its velocity. The fundamental principle governing motions of mechanical systems is the following.

**Axiom 1 (Newton-Laplace Determinacy Principle).** *A motion in a configuration space is completely determined by a state at some instant of time.*

The Newton-Laplace determinacy principle 1 motivates our main definition of this chapter.

**Definition 1.7 (Lagrangian System).** *A **Lagrangian system** is defined to be a tuple $(M, L)$ consisting of a smooth manifold $M$ and a function $L \in C^\infty(TM \times \mathbb{R})$, called a **Lagrangian function**.*

**Example 1.8.** For a smooth manifold $M$ let $T \in C^\infty(TM \times \mathbb{R})$ and $V \in C^\infty(M \times \mathbb{R})$. Define $L \in C^\infty(TM \times \mathbb{R})$ by $L := T - V$. In this situation, $T$ is called the ***kinetic energy*** and $V$ is called the ***potential energy***.

**Definition 1.9 (Path Space).** *Let $M$ be a smooth manifold, $x_0, x_1 \in M$ and $t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$. Define the **path space of $M$ connecting $(x_0, t_0)$ and $(x_1, t_1)$** to be the set*

$$\mathcal{P}(M)_{x_1,t_1}^{x_0,t_0} := \{\gamma \in C^\infty([t_0, t_1], M) : \gamma(t_0) = x_0 \text{ and } \gamma(t_1) = x_1\}. \tag{5}$$

**Remark 1.10.** For the sake of simplicity, we will just use the terminology *path space* for $\mathcal{P}(M)_{x_1,t_1}^{x_0,t_0}$ and simply write $\mathcal{P}(M)$.

**Definition 1.11 (Variation).** *Let $\mathcal{P}(M)$ be a path space and $\gamma \in \mathcal{P}(M)$. A **variation of $\gamma$** is defined to be a morphism $\Gamma \in C^\infty([t_0, t_1] \times [-\varepsilon_0, \varepsilon_0], M)$ for some $\varepsilon_0 > 0$ and such that*

- *$\Gamma(t, 0) = \gamma$ for all $t \in [t_1, t_0]$.*
- *$\Gamma(t_0, \varepsilon) = x_0$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.*
- *$\Gamma(t_1, \varepsilon) = x_1$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.*

**Remark 1.12.** If $\Gamma$ is a variation of $\gamma \in \mathcal{P}(M)$, we write $\gamma_\varepsilon(\cdot) := \Gamma(\cdot, \varepsilon)$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. With this notation, $\gamma_\varepsilon \in \mathcal{P}(M)$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

**Example 1.13 (Perturbation of a Path along a Single Direction).** Let $M^n$ be a smooth manifold, $(U, \varphi)$ a chart and suppose that $\gamma$ is a path in $U$. With respect to this chart, we can write the coordinate representation of $\gamma$ as

$$\gamma(t) = \left(\gamma^1(t), \ldots, \gamma^n(t)\right)$$

for any $t \in [t_0, t_1]$. Let $f \in C_c^\infty(t_0, t_1)$. Consider the family $\Gamma : [t_0, t_1] \times [-\varepsilon_0, \varepsilon_0] \to M$ defined by

$$\Gamma(t, \varepsilon) := (\iota \circ \varphi^{-1}) \left(\gamma^1(t), \ldots, \gamma^i(t) + \varepsilon f(t), \ldots, \gamma^n(t)\right)$$

where $\iota : U \hookrightarrow M$ denotes inclusion and $\varepsilon_0 > 0$ is to be determined. Suppose $\|f\|_\infty \neq 0$. By exercise 1.14, there exists $\delta > 0$ such that

$$U_\delta := \left\{x \in \mathbb{R}^n : \operatorname{dist}\left(x, \gamma([t_0, t_1])\right) < \delta\right\} \subseteq \varphi(U).$$

Choose $\varepsilon_0 > 0$ such that $0 < \varepsilon_0 < \delta / \|f\|_\infty$. Then in coordinates

$$\operatorname{dist}\left(\gamma_\varepsilon(t), \gamma([t_0, t_1])\right) \leq |\gamma_\varepsilon(t) - \gamma(t)| \leq |\varepsilon| \|f\|_\infty \leq \varepsilon_0 \|f\|_\infty < \delta$$

for all $t \in [t_0, t_1]$. Hence $\gamma_\varepsilon(t) \in U_\delta$ and thus $\gamma_\varepsilon(t) \in \varphi(U)$. Therefore, $\Gamma$ is indeed well-defined. Moreover, it is easy to show that the properties of definition 1.11 holds, therefore, $\Gamma$ is a variation of $\gamma$. In fact, this example shows, that any path $\gamma$ contained in a single chart admits infinitely many variations. An example of such a variation is shown in figure 1.

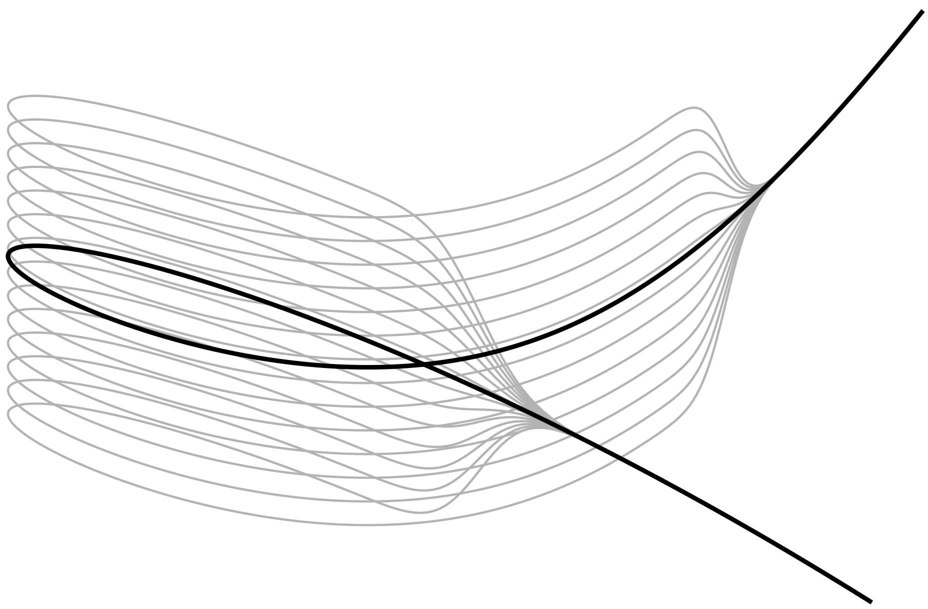

Figure 1. Example of a variation of the path $\gamma(t) = (\gamma^1(t), \gamma^2(t))$ in $\mathbb{R}^2$ defined by $\gamma(t) := (t^2 + \sin(t)\cos(t), t^3 - t)$ for $t \in [-\frac{3}{2}, \frac{3}{2}]$ along the second coordinate using a smooth bump function as in [Lee13, p. 42].

**Exercise 1.14.** Let $(X, d)$ be a metric space and $A \subseteq U \subseteq X$ where $U$ is open in $X$ and $A$ is closed in $X$. Then there exists $\delta > 0$ such that

$$U_\delta := \{x \in X : \operatorname{dist}(x, A) < \delta\} \subseteq U.$$

**Definition 1.15 (Action Functional).** *Let $(M, L)$ be a Lagrangian system and $\mathcal{P}(M)$ be a path space. The morphism $S : \mathcal{P}(M) \to \mathbb{R}$ defined by*

$$S(\gamma) := \int_{t_0}^{t_1} L\left(\gamma(t), \dot{\gamma}(t), t\right) dt$$

*is called the* ***action functional associated to the Lagrangian system*** $(M, L)$.

Motions of Lagrangian systems are characterized by an axiom.

**Axiom 2 (Hamilton's Principle of Least Action).** *Let $(M, L)$ be a Lagrangian system and $\mathcal{P}(M)$ be a path space. A path $\gamma \in C^\infty([t_0, t_1], M)$ describes a motion of $(M, L)$ between $(x_0, t_0)$ and $(x_1, t_1)$ if and only if*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\gamma_\varepsilon) = 0 \tag{6}$$

*for all variations $\gamma_\varepsilon$ of $\gamma$.*

**Definition 1.16 (Extremal).** *A motion of a Lagrangian system between two points is called an* ***extremal of the action functional $S$***.

The Newton-Laplace determinacy principle 1 implies that motions of mechanical systems can be described as solutions of second order ordinary differential equations. That this is indeed the case, is shown by the next theorem. But first, let us fix some notation. Let $M^n$ be a smooth manifold and $(U, \varphi)$ be a chart on $M$ with coordinates $(x^i)$. In what follows, we will use the abbreviation

$$\frac{\partial}{\partial x} := \left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right),$$

where as usual $\frac{\partial}{\partial x^i} : U \to TM$ denotes the *$i$-th coordinate vector field*, that is

$$\frac{\partial f}{\partial x^i}(x) := \left.\frac{\partial}{\partial x^i}\right|_x f = \partial_i (f \circ \varphi^{-1}) \left(\varphi(x)\right),$$

for all $i = 1, \dots, n$, $x \in U$ and $f \in C^\infty(M)$. Also recall, that on this chart

$$df_x = \frac{\partial f}{\partial x^i}(x) dx^i|_x \tag{7}$$

holds for all $x \in U$ (see [Lee13, p. 281]). Additionally, we need the following proposition.

**Proposition 1.17 (Derivative of a Function along a Curve [Lee13, p. 283]).** *Suppose $M$ is a smooth manifold, $J \subseteq \mathbb{R}$ an interval, $\gamma \in C^\infty(J, M)$ a curve on $M$ and $f \in C^\infty(M)$. Then for all $t \in J$ holds*

$$(f \circ \gamma)'(t) = df_{\gamma(t)}\left(\gamma'(t)\right).$$

**Theorem 1.18 (Euler-Lagrange Equations).** *Let $(M^n, L)$ be a Lagrangian system. A path $\gamma \in C^\infty([t_0,t_1], M)$ describes a motion of $(M,L)$ between $(x_0,t_0)$ and $(x_1,t_1)$ if and only if with respect to all charts $(U,x^i)$*

$$\frac{\partial L}{\partial x}\big(\gamma(t),\dot\gamma(t),t\big) - \frac{d}{dt}\frac{\partial L}{\partial v}\big(\gamma(t),\dot\gamma(t),t\big) = 0 \tag{8}$$

*holds, where $(x^i, v^i)$ denotes the standard coordinates on $TM$. The system of equations (8) is referred to as the* ***Euler-Lagrange equations****.*

*Proof.* By Hamilton's principle of least action 2, we may assume that $\gamma$ is an extremal of the action functional $S$. The proof is divided into two steps.
*Step 1: Suppose that $\gamma$ is contained in a chart domain $U$.* Let $t \in [t_0,t_1]$ and abreviate $x_t := (\gamma(t),\dot\gamma(t),t)$. Suppose $\Gamma : [t_0,t_1]\times[-\varepsilon_0,\varepsilon_0] \to M$ is a variation of $\gamma$. Then there exists a rectangle $\mathcal{R}$ such that

$$[t_0,t_1]\times\{0\} \subseteq \mathcal{R} \subseteq [t_0,t_1]\times[-\varepsilon_0,\varepsilon_0]$$

and $\Gamma(\mathcal{R}) \subseteq U$. Indeed, $\Gamma$ is continuous since $\Gamma$ is smooth and so $\Gamma^{-1}(U)$ is open in $[t_0,t_1]\times[-\varepsilon_0,\varepsilon_0]$. Since $\gamma$ is a path in $U$, we get

$$[t_0,t_1]\times\{0\} \subseteq \Gamma^{-1}(U)$$

by the definition of a variation. By exercise 2.4. (c) [Lee11, p. 22], the standard Euclidean metric and the *maximum metric* $|\cdot|_\infty$ generate the same topology, thus for all $t \in [t_0,t_1]$ there exists $r_t > 0$ such that

$$B_{r_t}(t,0) := \big\{(x,\varepsilon) \in [t_0,t_1]\times[-\varepsilon_0,\varepsilon_0] : \max\{|x-t|, |\varepsilon|\} < r_t\big\} \subseteq \Gamma^{-1}(U).$$

Since $[t_0,t_1]\times\{0\}$ is compact in $[t_0,t_1]\times[-\varepsilon_0,\varepsilon_0]$, we find $m \in \mathbb{N}$ such that

$$[t_0,t_1]\times\{0\} \subseteq \bigcup_{i=1}^m B_{r_i}(t_i,0).$$

Set $r := \max_{i=1,\dots,m} r_i$ and define $\mathcal{R} := [t_0,t_1]\times(-r,r)$. Then if $(t,\varepsilon) \in \mathcal{R}$ we get that there exists some index $i$ such that $(t,0) \in B_{r_i}(t_i,0)$. Hence $|t-t_i| < r_i$ and so

$$|(t,\varepsilon)-(t_i,0)|_\infty = \max\{|t-t_i|, |\varepsilon|\} < r_i.$$

Thus $(t,\varepsilon) \subseteq B_{r_i}(t_i,0) \subseteq \Gamma^{-1}(U)$ and so $\Gamma(\mathcal{R}) \subseteq U$. Hence we can write

$$\gamma_\varepsilon(t) = \big(\gamma_\varepsilon^1(t),\dots,\gamma_\varepsilon^n(t)\big)$$

and

$$\dot\gamma_\varepsilon(t) = \big(\dot\gamma_\varepsilon^1(t),\dots,\dot\gamma_\varepsilon^n(t)\big)$$

for all $(x,\varepsilon) \in \mathcal{R}$, where the dot denotes a derivative with respect to time.
Using the formula for the derivative of a function along a curve 1.17, we compute

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} L\big(\gamma_\varepsilon(t),\dot\gamma_\varepsilon(t),t\big) = dL_{x_t}\left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\gamma_\varepsilon(t), \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\dot\gamma_\varepsilon(t), 0\right)$$

$$= dL_{x_t}\left(\frac{d\gamma_\varepsilon^j(t)}{d\varepsilon}(0)\frac{\partial}{\partial x^j}\Big|_{\gamma(t)}, \frac{d\dot\gamma_\varepsilon^j(t)}{d\varepsilon}(0)\frac{\partial}{\partial v^j}\Big|_{\dot\gamma(t)}, 0\right).$$

for all variations $\gamma_\varepsilon$ of $\gamma$ in $U$. Moreover, using the formula for the differential of a function on coordinates (7) yields

$$dL_{x_t} = \frac{\partial L}{\partial x^i}(x_t)dx^i|_{x_t} + \frac{\partial L}{\partial v^i}(x_t)dv^i|_{x_t} + \frac{\partial L}{\partial t}(x_t)dt|_{x_t}.$$

Therefore

$$\begin{aligned}
0 &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(\gamma_\varepsilon)\\
&= \int_{t_0}^{t_1} \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} L\left(\gamma_\varepsilon(t), \dot\gamma_\varepsilon(t), t\right) dt\\
&= \int_{t_0}^{t_1} dL_{x_t}\left(\frac{d\gamma_\varepsilon^j(t)}{d\varepsilon}(0)\frac{\partial}{\partial x^j}\Big|_{\gamma(t)}, \frac{d\dot\gamma_\varepsilon^j(t)}{d\varepsilon}(0)\frac{\partial}{\partial v^j}\Big|_{\dot\gamma(t)}, 0\right)\\
&= \int_{t_0}^{t_1} \frac{\partial L}{\partial x^i}(x_t)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt + \int_{t_0}^{t_1} \frac{\partial L}{\partial v^i}(x_t)\frac{d\dot\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt\\
&= \int_{t_0}^{t_1} \frac{\partial L}{\partial x^i}(x_t)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt + \int_{t_0}^{t_1} \frac{\partial L}{\partial v^i}(x_t)\left(\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)\right)' dt\\
&= \int_{t_0}^{t_1} \frac{\partial L}{\partial x^i}(x_t)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt + \frac{\partial L}{\partial v^i}(x_t)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt}\frac{\partial L}{\partial v^i}(x_t)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt\\
&= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^i}(x_t) - \frac{d}{dt}\frac{\partial L}{\partial v^i}(x_t)\right)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt
\end{aligned}$$

since $\gamma_\varepsilon^i(t_0)$ and $\gamma_\varepsilon^i(t_1)$ are constant by definition of a variation. Let $f \in C_c^\infty(t_0, t_1)$, $j = 1, \dots, n$ and $\gamma_\varepsilon$ be the variation of $\gamma$ defined in example 1.13 along the $j$-th direction. Above computation therefore yields

$$0 = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^j}(x_t) - \frac{d}{dt}\frac{\partial L}{\partial v^j}(x_t)\right) f(t)dt$$

for all $f \in C_c^\infty(t_0, t_1)$. Hence the fundamental lemma of calculus of variations 1.2 implies

$$\frac{\partial L}{\partial x^j}(x_t) - \frac{d}{dt}\frac{\partial L}{\partial v^j}(x_t) = 0$$

for all $j = 1, \dots, n$.

Conversly, if we assume that the Euler-Lagrange equations (8) hold, above computation yields

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(\gamma_\varepsilon) = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^i}(x_t) - \frac{d}{dt}\frac{\partial L}{\partial v^i}(x_t)\right)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt = 0$$

for every variation $\gamma_\varepsilon$ of $\gamma$.
*Step 2: Suppose that $\gamma$ is an arbitrary extremal of $S$.* The key technical result used here is the following lemma.

**Lemma 1.19 (Lebesgue Number Lemma [Lee11, p. 194]).** *Every open cover of a compact metric space admits a Lebesgue number, i.e. a number $\delta > 0$ such that every subset of the metric space with diameter less than $\delta$ is contained in a member of the family.*

Let $(U_\alpha)_{\alpha\in A}$ be the smooth structure on $M$, i.e. the maximal smooth atlas. Since $\gamma$ is continuous, $\left(\gamma^{-1}(U_\alpha)\right)_{\alpha\in A}$ is an open cover for $[t_0, t_1]$. By the Lebesgue number lemma 1.19, this open cover admits a Lebesgue number $\delta > 0$. Let $N \in \mathbb{N}$ such that $(t_1 - t_0)/N < \delta$ and define

$$t_i := \frac{i}{N}(t_1 - t_0) + t_0$$

for all $i = 0, \ldots, N$. Then for all $i = 1, \ldots, N$, $\gamma|_{[t_{i-1},t_i]}$ is contained in $U_\alpha$ for some $\alpha \in A$. Let us extend the construction of example 1.13. Suppose $f \in C_c^\infty(t_{i-1}, t_i)$. Then we can define a variation $\Gamma : [t_0, t_1] \times [-\varepsilon_0, \varepsilon_0] \to M$ as follows: Define

$$\Gamma : ([t_0, t_1] \setminus \operatorname{supp} f) \times [-\varepsilon_0, \varepsilon_0] \to M$$

by $\Gamma(t, \varepsilon) := \gamma(t)$, and $\Gamma : (t_{i-1}, t_i) \times [-\varepsilon_0, \varepsilon_0] \to M$ to be the map defined in example 1.13. Since both definitions agree on the overlap $(t_{i-1}, t_i) \setminus \operatorname{supp} f$, an application of the gluing lemma for smooth maps [Lee13, p. 35] yields the existence of a variation $\Gamma$ of $\gamma$ on $M$. Therefore, step 1 implies the Euler-Lagrange equations (8). The converse direction is content of problem 1-1. □

Due to the Newton-Laplace Determinacy Principle 1, the motions on a Lagrangian system are inherently characterized by the Lagrangian function and locally by the Euler-Lagrange equations (8). Hence any motion satisfies locally a system of second order ordinary differential equations. This system bears its own name.

**Definition 1.20 (Equations of Motion).** *The Euler-Lagrange equations* (8) *of a Lagrangian system are called the* ***equations of motion****.*

**Example 1.21 (Motions on Riemannian Manifolds).** Let $(M^n, g)$ be a Riemannian manifold and consider the Lagrangian $L$ on $M$ defined in example 1.8 with kinetic energy

$$T(x, v, t) := \frac{1}{2} g_x(v, v) = \frac{1}{2}|v|_g^2$$

and potential energy $V(x, t) := 0$ for $x \in M$, $v \in T_x M$ and $t \in \mathbb{R}$. Let $(U, x^i)$ be a chart on $M$. We compute

$$\begin{aligned} L(x, v, t) &= \frac{1}{2} g_x(v, v) \\ &= \frac{1}{2} g_x\left( v^i \left.\frac{\partial}{\partial x^i}\right|_x, v^j \left.\frac{\partial}{\partial x^j}\right|_x \right) \end{aligned}$$

$$= \frac{1}{2} g_x \left( \left.\frac{\partial}{\partial x^i}\right|_x, \left.\frac{\partial}{\partial x^j}\right|_x \right) v^i v^j$$
$$= \frac{1}{2} g_{ij}(x) v^i v^j,$$

where $g_{ij}(x) := g_x \left( \left.\frac{\partial}{\partial x^i}\right|_x, \left.\frac{\partial}{\partial x^j}\right|_x \right)$. Thus

$$\frac{\partial L}{\partial x^l}(x, v, t) = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^l}(x) v^i v^j$$

and in particular

$$\frac{\partial L}{\partial x^l} \left(\gamma(t), \dot\gamma(t), t\right) = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^l} \left(\gamma(t)\right) \dot\gamma^i(t) \dot\gamma^j(t),$$

for all $l = 1, \dots, n$. Moreover

$$\frac{\partial L}{\partial v^l}(x, v, t) = \frac{1}{2} g_{ij}(x) \delta^i_l v^j + \frac{1}{2} g_{ij}(x) v^i \delta^j_l = \frac{1}{2} g_{lj}(x) v^j + \frac{1}{2} g_{il}(x) v^i$$

implies

$$\begin{aligned}
\frac{d}{dt} \frac{\partial L}{\partial v^l} \left(\gamma(t), \dot\gamma(t), t\right) &= \frac{1}{2} \frac{d}{dt} g_{lj}(\gamma) \dot\gamma^j + \frac{1}{2} g_{lj}(\gamma) \ddot\gamma^j + \frac{1}{2} \frac{d}{dt} g_{il}(\gamma) \dot\gamma^i + \frac{1}{2} g_{il}(\gamma) \ddot\gamma^i \\
&= \frac{1}{2} d g_{lj}(\dot\gamma) \dot\gamma^j + \frac{1}{2} g_{lj}(\gamma) \ddot\gamma^j + \frac{1}{2} d g_{il}(\dot\gamma) \dot\gamma^i + \frac{1}{2} g_{il}(\gamma) \ddot\gamma^i \\
&= \frac{1}{2} \frac{\partial g_{lj}}{\partial x^k} \dot\gamma^k \dot\gamma^j + \frac{1}{2} g_{lj}(\gamma) \ddot\gamma^j + \frac{1}{2} \frac{\partial g_{il}}{\partial x^k} \dot\gamma^k \dot\gamma^i + \frac{1}{2} g_{il}(\gamma) \ddot\gamma^i \\
&= \frac{1}{2} \frac{\partial g_{jl}}{\partial x^k} \dot\gamma^k \dot\gamma^j + \frac{1}{2} g_{jl}(\gamma) \ddot\gamma^j + \frac{1}{2} \frac{\partial g_{il}}{\partial x^k} \dot\gamma^k \dot\gamma^i + \frac{1}{2} g_{il}(\gamma) \ddot\gamma^i \\
&= g_{il} \ddot\gamma^i + \frac{1}{2} \frac{\partial g_{jl}}{\partial x^i} \dot\gamma^i \dot\gamma^j + \frac{1}{2} \frac{\partial g_{il}}{\partial x^j} \dot\gamma^i \dot\gamma^j.
\end{aligned}$$

Therefore the Euler-Lagrange equations (8) read

$$0 = \frac{d}{dt} \frac{\partial L}{\partial v^l} - \frac{\partial L}{\partial x^l} = g_{il} \ddot\gamma^i + \frac{1}{2} \left( \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) \dot\gamma^i \dot\gamma^j,$$

for all $l = 1, \dots, n$. Multiplying both sides by $g^{kl}$ yields

$$\ddot\gamma^k + \Gamma^k_{ij} \dot\gamma^i \dot\gamma^j = 0, \tag{9}$$

for all $k = 1, \dots, n$, where

$$\Gamma^k_{ij} := \frac{1}{2} g^{kl} \left( \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

are the ***Christoffel symbols*** with respect to the choosen chart (see [Lee97, p. 70]). The system of equations (9) is called ***geodesic equations*** (see [Lee97, p. 58]). Hence extremals

$\gamma$ of the action functional satisfy the geodesic equation and are therefore geodesics on the Riemannian manifold $M$.

**Lemma 1.22.** *Let $(M, L)$ be a Lagrangian system and define $L + df \in C^\infty(TM \times \mathbb{R})$ by*

$$(L + df)(x, v, t) := L(x, v, t) + df_x(v)$$

*for any $f \in C^\infty(M)$. Then $(M, L)$ and $(M, L + df)$ admit the same equations of motion.*

*Proof.* Let us denote the action function corresponding to $L + df$ by $\widetilde{S}$ and suppose $\gamma_\varepsilon$ is a variation of $\gamma$ in $M$. Using the formula for the derivative of a function along a curve [Lee13, p. 283] we compute

$$\begin{aligned}
\widetilde{S}(\gamma_\varepsilon) &= \int_{t_0}^{t_1} L(\gamma_\varepsilon(t), \dot{\gamma}_\varepsilon(t), t)dt + \int_{t_0}^{t_1} df_{\gamma_\varepsilon(t)}\left(\dot{\gamma}_\varepsilon(t)\right) dt \\
&= S(\gamma_\varepsilon) + \int_{t_0}^{t_1} (f \circ \gamma_\varepsilon)'(t)dt \\
&= S(\gamma_\varepsilon) + f\left(\gamma_\varepsilon(t_1)\right) - f\left(\gamma_\varepsilon(t_0)\right) \\
&= S(\gamma_\varepsilon) + f(x_1) - f(x_0).
\end{aligned}$$

In particular

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \widetilde{S}(\gamma_\varepsilon) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\gamma_\varepsilon).$$

□

**Remark 1.23.** Lemma 1.22 implies, that the Lagrangian of a mechanical system can only be determined up to differentials of smooth functions. Actually, in coordinates, also up to total time derivatives. Hence a *law of motion*, that is a Lagrangian describing a certain mechanical system, is in fact an equivalence class of Lagrangian functions.

## Legendre Transform

In this section we *dualize* the notion of a Lagrangian function, that is, to each Lagrangian function $L \in C^\infty(TM)$ we will associate a *dual function* $L^* \in C^\infty(T^*M)$. It turns out, that in this dual formulation, the equations of motion take a very symmetric form. To simplify the notation and illuminating the main concept, we consider Lagrangian functions of a special type.

**Definition 1.24 (Autonomous System).** *An **autonomous Lagrangian system** is defined to be a tuple $(M, L)$ consitsing of a smooth manifold $M$ and a function $L \in C^\infty(M)$.*

Let $(M^n, L)$ be an autonomous Lagrangian system and $(U, x^i)$ a chart on $M$. Moreover, let $(x^i, v^i)$ denote standard coordinates on $TM$, that is $v^i := dx^i$ for all $i = 1, \dots, n$. Expanding the Euler-Lagrange equations (8) yields

$$\frac{\partial L}{\partial x^j}\left(\gamma(t), \dot{\gamma}(t)\right) = \frac{d}{dt}\frac{\partial L}{\partial v^j}\left(\gamma(t), \dot{\gamma}(t)\right)$$

$$= \frac{\partial^2 L}{\partial x^i \partial v^j}\big(\gamma(t), \dot\gamma(t)\big)\,\dot\gamma^i(t) + \frac{\partial^2 L}{\partial v^i \partial v^j}\big(\gamma(t), \dot\gamma(t)\big)\,\ddot\gamma^i(t)$$

for all $j = 1, \ldots, n$. In order to solve above system of second order ordinary differential equations for $\ddot\gamma^i(t)$ and all initial conditions in the chart on $TU$, the matrix $\mathcal{H}_L(x, v)$ defined by

$$\mathcal{H}_L(x, v) := \left(\frac{\partial^2 L}{\partial v^i \partial v^j}(x, v)\right)^i_j \tag{10}$$

must be invertible on $TU$.

**Definition 1.25 (Nondegenrate System).** *An autonomous Lagrangian system $(M, L)$ is said to be* ***nondegenerate****, iff for all coordinate charts $U$ on $M$, $\det\mathcal{H}_L(x, v) \neq 0$ holds on $TU$.*

**Example 1.26 (Nondegenrate System on a Riemannian Manifold).** Let $(M, g)$ be a Riemannian manifold. Consider the Lagrangian $T - V$ with kinetic energy $T \in C^\infty(TM)$ defined by $T(v) := \frac{1}{2}|v|^2$ and potential energy $V \in C^\infty(M)$. Then the computation performed in example 1.21 yields

$$\mathcal{H}_{T-V}(x, v) = \big(g_{ij}(x)\big)^i_j$$

on every chart since $\frac{\partial V}{\partial v^i} = 0$ for every $i$, and so this Lagrangian system is nondegenerate.

The nondegeneracy of an autonomous Lagrangian system is intrinsically connected to a certain differential form in $\Omega^1(TM)$, which we will construct now. For every $(x, v) \in TM$ we can define a covector $D^{\mathcal{F}}_{(x,v)}L \in T^*_x M$ by setting

$$D^{\mathcal{F}} L_{(x,v)} := \left.\frac{\partial}{\partial v^i}\right|_{(x,v)} (L) dx^i|_x. \tag{11}$$

Let $(\widetilde U, \widetilde x^i)$ be another chart on $M$ such that $U \cap \widetilde U \neq \varnothing$. Denote the induced coordinates on $TM$ by $(\widetilde x^i, \widetilde v^i)$. Then on $U \cap \widetilde U$ we have that

$$\frac{\partial}{\partial \widetilde v^i} = \frac{\partial x^j}{\partial \widetilde v^i}\frac{\partial}{\partial x^j} + \frac{\partial v^j}{\partial \widetilde v^i}\frac{\partial}{\partial v^j} = \frac{\partial v^j}{\partial \widetilde v^i}\frac{\partial}{\partial v^j}.$$

Moreover

$$\frac{\partial}{\partial x^j} = \frac{\partial \widetilde x^k}{\partial x^j}\frac{\partial}{\partial \widetilde x^k}$$

which implies

$$d\widetilde x^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial \widetilde x^k}{\partial x^j} d\widetilde x^i\left(\frac{\partial}{\partial \widetilde x^k}\right) = \frac{\partial \widetilde x^k}{\partial x^j}\delta^i_k = \frac{\partial \widetilde x^i}{\partial x^j}.$$

Thus

$$d\widetilde x^i = \frac{\partial \widetilde x^i}{\partial x^j} dx^j$$

or equivalently

$$v^j = \frac{\partial x^j}{\partial \tilde{x}^i} \tilde{v}^i,$$

and so we compute

$$D^{\mathcal{F}} L = \frac{\partial L}{\partial \tilde{v}^i} d\tilde{x}^i = \frac{\partial v^j}{\partial \tilde{v}^i} \frac{\partial L}{\partial v^j} \frac{\partial \tilde{x}^i}{\partial x^k} dx^k = \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\partial L}{\partial v^j} \frac{\partial \tilde{x}^i}{\partial x^k} dx^k = \frac{\partial L}{\partial v^j} \delta^j_k dx^k = \frac{\partial L}{\partial v^j} dx^j.$$

Therefore, $D^{\mathcal{F}} L$ is independent of the choice of coordinates.

**Definition 1.27 (Fibrewise Differential[2]).** *Let $(M, L)$ be an autonomous Lagrangian system. The form $D^{\mathcal{F}} L \in \Omega^1(TM)$ defined on a chart $(U, x^i)$ of $M$ by*

$$D^{\mathcal{F}} L := \frac{\partial L}{\partial v^i} dx^i \tag{12}$$

*where $(x^i, v^i)$ denotes the induced standard coordinates on $TM$, is called the **fibrewise differential of $L$**.*

**Remark 1.28.** The preceeding discussion showed, that the fibrewise differential $D^{\mathcal{F}} L$ is well-defined.

**Example 1.29 (Fibrewise Differential on a Riemannian Manifold).** Consider the autonomous Lagrangian system as defined in example 1.26. Then the computation performed in example 1.21 yields

$$D^{\mathcal{F}}(T - V)_{(x,v)} = g_{ij}(x) v^i dx^j$$

on every chart since $\frac{\partial V}{\partial v^j} = 0$ for all $j$.

Recall, that a 2-covector on a finite-dimensional real vector space is said to be *nondegenrate*, iff the matrix representation with respect to some basis is invertible. Moreover, a *nondegenerate 2-form* on a smooth manifold $M$ is defined to be a 2-form $\omega$, such that $\omega_x$ is a nondegenrate 2-covector for all $x \in M$ (see [Lee13, pp. 565,567]).

**Proposition 1.30.** *An autonomous Lagrangian system $(M, L)$ is nondegenerate if and only if $d(D^{\mathcal{F}} L)$ is nondegenerate.*

*Proof.* Using the computation performed in [Lee13, p. 363], we get

$$d(D^{\mathcal{F}} L) = d\left(\frac{\partial L}{\partial v^j} dx^j\right) = \frac{\partial^2 L}{\partial x^i \partial v^j} dx^i \wedge dx^j + \frac{\partial^2 L}{\partial v^i \partial v^j} dv^i \wedge dx^j.$$

---

[2]This terminology is adapted from exercise C.3. on problem sheet C of the lecture *Differential geometry I* taught by *Will J. Merry* at *ETH Zürich* in the autumn semester 2018, which can be found here. See also [Maz12, p. 2].

Moreover, using part (e) of properties of the wedge product [Lee13, p. 356], we compute

$$\begin{aligned}
d(D^{\mathcal{F}}L)\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^l}\right) =& \frac{\partial^2 L}{\partial x^i \partial v^j}\det\begin{pmatrix} dx^i\left(\frac{\partial}{\partial x^k}\right) & dx^j\left(\frac{\partial}{\partial x^k}\right) \\ dx^i\left(\frac{\partial}{\partial x^l}\right) & dx^j\left(\frac{\partial}{\partial x^l}\right)\end{pmatrix} \\
&+ \frac{\partial^2 L}{\partial v^i \partial v^j}\det\begin{pmatrix} dv^i\left(\frac{\partial}{\partial x^k}\right) & dx^j\left(\frac{\partial}{\partial x^k}\right) \\ dv^i\left(\frac{\partial}{\partial x^l}\right) & dx^j\left(\frac{\partial}{\partial x^l}\right)\end{pmatrix} \\
=& \frac{\partial^2 L}{\partial x^i \partial v^j}(\delta^i_k\delta^j_l - \delta^i_l\delta^j_k) \\
=& \frac{\partial^2 L}{\partial x^k \partial v^l} - \frac{\partial^2 L}{\partial x^l \partial v^k}
\end{aligned}$$

for all $k, l = 1, \ldots, n$. Similarly, we compute

$$d(D^{\mathcal{F}}L)\left(\frac{\partial}{\partial v^k},\frac{\partial}{\partial x^l}\right) = \frac{\partial^2 L}{\partial v^k \partial v^l} \qquad \text{and} \qquad d(D^{\mathcal{F}}L)\left(\frac{\partial}{\partial v^k},\frac{\partial}{\partial v^l}\right) = 0,$$

and using skew-symmetry, we also deduce

$$d(D^{\mathcal{F}}L)\left(\frac{\partial}{\partial x^k},\frac{\partial}{\partial v^l}\right) = -\frac{\partial^2 L}{\partial v^k \partial v^l}.$$

Therefore, the matrix representing $d(D^{\mathcal{F}}L)$ with respect to the standard basis is given by the block matrix

$$d(D^{\mathcal{F}}L) = \left(\begin{array}{c|c} * & -\mathcal{H}_L \\ \hline \mathcal{H}_L & 0 \end{array}\right),$$

where $\mathcal{H}_L$ is the matrix defined in (10). Thus

$$\det\left(d(D^{\mathcal{F}}L)\right) = (-1)^n(\det\mathcal{H}_L)^2$$

Hence the matrix representation of $d(D^{\mathcal{F}}L)$ is invertible if and only if $\mathcal{H}_L$ is invertible, and the conclusion follows. □

So far, we have associated to each Lagrangian system $(M, L)$ a 1-form on $TM$, the fibrewise differential $D^{\mathcal{F}}L$. In order to get closer to our goal of dualizing the concept of a Lagrangian function, we need also a 1-form on $T^*M$. Suppose $(U, x^i)$ is a chart on $M$. The induced standard coordinates on the cotangent bundle $T^*M$ of $M$ are given by $(x^i, \xi_i)$, where $\xi_i := \frac{\partial}{\partial x^i}$, considered as an element of the double dual $T^{**}U$. On this chart, define a one 1-form $\alpha$ by $\alpha := \xi_i dx^i$. Suppose $(\widetilde{x}^i, \widetilde{\xi}_i)$ are other coordinates. Then from the computations performed at the beginning of the previous section, we have that

$$\widetilde{\xi}_i = \frac{\partial x^j}{\partial \widetilde{x}^i}\xi_j \qquad \text{and} \qquad d\widetilde{x}^i = \frac{\partial \widetilde{x}^i}{\partial x^k}dx^k.$$

Thus

$$\alpha = \widetilde{\xi}_i d\widetilde{x}^i = \frac{\partial x^j}{\partial \widetilde{x}^i}\xi_j \frac{\partial \widetilde{x}^i}{\partial x^k}dx^k = \xi_j \delta^j_k dx^k = \xi_j dx^j,$$

and so, $\alpha$ is independen of the choice of coordinates.

**Definition 1.31 (Tautological Form).** *Let $M$ be a smooth manifold. The* ***tautological form on $T^*M$****, denoted by $\alpha$, is the form $\alpha \in \Omega^1(T^*M)$ defined locally by*

$$\alpha := \xi_i dx^i,$$

*where $(x^i, \xi_i)$ denotes the standard coordinates on $T^*M$.*

**Remark 1.32.** The preceeding discussion showed, that the tautological form $\alpha$ is well-defined.

**Remark 1.33.** The tautological form $\alpha$ as well as the fibrewise derivative $D^{\mathcal{F}}L$ on an autonomous Lagrangian system $(M, L)$ admit invariant definitions, that is a coordinate free definition. For the invariant definition of $\alpha$ see [Lee13, p. 569] or [Sil08, pp. 10–11], and for the invariant definition of $D^{\mathcal{F}}L$ see [Tak08, p. 31].

Recall, that if $F \in C^\infty(M, N)$ for some smooth manifolds $M$ and $N$, and $l \in \mathbb{N}$, we can define a mapping $F^* : \Gamma\left(T^{(0,l)}TN\right) \to \Gamma\left(T^{(0,l)}TM\right)$, called the *pullback by $F$*, by

$$(F^*A)_x(v_1, \ldots, v_l) := A_{F(x)}\left(dF_x(v_1), \ldots, dF_x(v_l)\right)$$

for all $x \in M$ and $v_1, \ldots, v_l \in T_xM$ (see [Lee13, p. 320]).

**Definition 1.34 (Legendre Transform).** *A* ***Legendre transform of an autonomous Lagrangian system $(M, L)$*** *is defined to be a fibrewise mapping $\tau_L \in C^\infty(TM, T^*M)$ such that*

$$D^{\mathcal{F}}L = \tau_L^*(\alpha).$$

**Example 1.35 (Legendre Transform on a Riemannian Manifold).** Let $(M, L)$ be a Lagrangian system. Then the morphism $\tau_L : TM \to T^*M$ defined by

$$\tau_L(x, v) := \left(x, D^{\mathcal{F}}L_{(x,v)}\right) \tag{13}$$

is a Legendre transform. In particular, if we consider the Lagrangian system defined in example 1.26, we get that the above defined Legendre transform is a diffeomorphism. Indeed, suppose that $\tau_{T-V}(x, v) = \tau_{T-V}(\widetilde{x}, \widetilde{v})$. Then $x = \widetilde{x}$ and

$$g_{ij}(x)v^i dx^j = g_{ij}(x)\widetilde{v}^i dx^j$$

using example 1.29. So we must have

$$g_{ij}(x)v^i = g_{ij}(x)\widetilde{v}^i$$

for all $j$. Multiplying both sides by $g^{kj}(x)$ yields $v^k = \tilde{v}^k$ for every $k$ and hence $v = \tilde{v}$. Thus $\tau_{T-V}$ is injective. Let $\xi \in T_x^*M$ be given by $\xi_i dx^i|_x$. Then $\tau_{T-V}(x, v) = (x, \xi)$, where $v$ is given in coordinates by $v^k := g^{ki}(x)\xi_i$.

Since the nondegenracy of a Lagragian system $(M, L)$ is inherently connected to the nondegenracy of the form $d(D^{\mathcal{F}}L)$ and the definition of the Legendre transform invokes the form $D^{\mathcal{F}}L$, one would expect a connection between the nondegeneracy of the Lagrangian system and a local property of Legendre transform.

**Lemma 1.36.** *A Legendre transform on a Lagrangian system is a local diffeomorphism if and only if the Lagrangian system is nondegenrate.*

*Proof.* Denote the Lagrangian system by $(M, L)$. Let $(U, x^i)$ be a chart on $M$ and denote by $(x^i, v^i)$ and $(x^i, \xi_i)$ the induced standard coordinates on $TM$ and $T^*M$, respectively. Then we compute

$$\tau_L^*(\alpha) = \tau_\alpha^*(\xi_j dx^j) = (\xi_j \circ \tau_L)d\left(x^j \circ \tau_L\right),$$

which must coincide with

$$D^{\mathcal{F}}L = \frac{\partial L}{\partial v^j}dx^j.$$

Thus in coordinates

$$\tau_L(x, v) = \left(x, \frac{\partial L}{\partial v}\right) \tag{14}$$

and so

$$D\tau_L|_{(x,v)} = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & \mathcal{H}_L \end{array}\right)$$

at every $(x, v) \in TM$. Hence

$$\det\left(D\tau_L|_{(x,v)}\right) = \det\mathcal{H}_L.$$

If $\tau_L$ is a local diffeomorphism, by definition, we have that some restriction of $\tau_L$ to some neighbourhood of $(x, v)$ is a diffeomorphism, and so, by properties of differentials (d) [Lee13, p. 55], we have that $D\tau_L|_{(x,v)}$ is an isomorphism. Conversly, if the Lagrangian system is nondegenerate, we conclude using the inverse function theorem for manifolds [Lee13, p. 79], that $\tau_L$ is a local diffeomorphism. ☐

**Definition 1.37 (Energy).** *The* ***energy of an autonomous Lagrangian system*** $(M, L)$ *is defined to be the function* $E_L \in C^\infty(TM)$ *given by*

$$E_L(x, v) := D^{\mathcal{F}}L_{(x,v)}(v) - L(x, v),$$

*in standard coordinates* $(x^i, v^i)$ *of* $TM$.

**Example 1.38 (Energy on a Riemannian Manifold).** Consider the Lagrangian system defined in example 1.26. Then the computation performed in example 1.29 yields

$$E_{T-V}(x, v) = \frac{\partial T}{\partial v^k}v^k - \frac{\partial V}{\partial v^k}v^k - T(v) + V(x)$$

$$
\begin{aligned}
&= \frac{1}{2} g_{ij} \delta^i_k v^j v^k + \frac{1}{2} g_{ij} v^i \delta^j_k v^k - T(v) + V(x) \\
&= g_{ij} v^i v^j - T(v) + V(x) \\
&= T(v) + V(x)
\end{aligned}
$$

for every $(x, v) \in TM$. Hence the energy of this Lagrangian system is given by *kinetic energy plus potential energy*.

**Definition 1.39 (Hamiltonian Function).** *Let $(M, L)$ be an autonomous Lagrangian system and $\tau_L$ a diffeomorphic Legendre transform. The morphism $H_L \in C^\infty(T^*M)$ defined by*

$$H_L := E_L \circ \tau_L^{-1}$$

*is called the* ***Hamiltonian function associated to the Lagrangian function $L$***.

**Example 1.40 (Hamiltonian function on a Riemannian Manifold).** Consider the Lagrangian system defined in example 1.26. By example 1.35 the Legendre transform $\tau_{T-V}$ is a diffeomorphism. Using example 1.38, we compute

$$
\begin{aligned}
H_{T-V}(x, \xi) &= E_{T-V}\left(\tau^{-1}_{T-V}(x, \xi)\right) \\
&= E_{T-V}\left(x, v\right) \\
&= T(v) + V(x) \\
&= \frac{1}{2} g_{ij}(x) v^i v^j + V(x) \\
&= \frac{1}{2} g_{ij}(x) g^{ik}(x) \xi_k g^{jl}(x) \xi_l + V(x) \\
&= \frac{1}{2} \delta^k_j \xi_j g^{jl}(x) \xi_l + V(x) \\
&= \frac{1}{2} g^{kl}(x) \xi_k \xi_l + V(x)
\end{aligned}
$$

where $v = \left(g^{ki}\right)^k_i \xi$.

**Theorem 1.41 (Hamilton's Equations).** *Let $\gamma$ be a motion on an autonomous Lagrangian system $(M^n, L)$ and suppose that $\tau_L$ is a diffeomorphic Legendre transform. Then $\gamma$ satisfies the Euler-Lagrange equations in every chart if and only if the path*

$$\left(\gamma(t), \xi(t)\right) := \tau_L\left(\gamma(t), \dot{\gamma}(t)\right)$$

*satisfies the following system of first order ordinary differential equations in every chart:*

$$\dot{\gamma}(t) = \frac{\partial H_L}{\partial \xi}\left(\gamma(t), \xi(t)\right) \qquad \text{and} \qquad \dot{\xi}(t) = -\frac{\partial H_L}{\partial x}\left(\gamma(t), \xi(t)\right) \tag{15}$$

*The equations (15) are called* ***Hamilton's equations***.

*Proof.* First we compute $H_L$ in standard coordinates $(x^i, \xi_i)$ on $T^*M$. By (14), the Legendre transform is given by

$$\tau_L(x, v) = \left(x, \frac{\partial L}{\partial v}(x, v)\right) \tag{16}$$

in standard coordinates on $TM$. Since $\tau_L$ is a diffeomorphism by assumption, in particular it is a local diffeomorphism (see [Lee13, p. 80]). Hence by lemma 1.36, the Lagrangian system $(M, L)$ is nondegenerate. So considering $\tau_L^{-1}(x, \xi)$, we can apply the implicit function theorem [Lee13, p. 661] to obtain $v$ implicitely from the equation

$$\xi = \frac{\partial L}{\partial v}(x, v).$$

Hence in coordinates

$$H_L(x, \xi) = \left(\frac{\partial L}{\partial v^i} v^i - L(x, v)\right)\bigg|_{\xi = \frac{\partial L}{\partial v}}.$$

Therefore

$$\frac{\partial H_L}{\partial \xi^j} = \frac{\partial}{\partial \xi_j}\left(\xi_i v^i - L(x, v)\right)\big|_{\xi = \frac{\partial L}{\partial v}} = \delta_i^j v^i = v^j.$$

Hence

$$\frac{\partial H_L}{\partial \xi^j}\left(\gamma(t), \xi(t)\right) = \dot{\gamma}^j(t),$$

for all $j = 1, \ldots, n$. Moreover, we have that

$$\frac{\partial H_L}{\partial x^j} = \frac{\partial}{\partial x^j}\left(\frac{\partial L}{\partial v^i} v^i - L(x, v)\right)\bigg|_{\xi = \frac{\partial L}{\partial v}} = -\frac{\partial L}{\partial x^j}(x, v)\bigg|_{\xi = \frac{\partial L}{\partial v}},$$

and so

$$\frac{\partial H_L}{\partial x^j}\left(\gamma(t), \xi(t)\right) = -\frac{\partial L}{\partial x^j}\left(\gamma(t), \dot{\gamma}(t)\right),$$

for all $j = 1, \ldots, n$. If the Euler-Lagrange equations (8) hold, then we get

$$\frac{\partial H_L}{\partial x^j}\left(\gamma(t), \xi(t)\right) = -\frac{d}{dt}\frac{\partial L}{\partial v^j}\left(\gamma(t), \dot{\gamma}(t)\right) = -\dot{\xi}_j(t),$$

and thus the Hamilton's equations (15) hold. Conversly, if we suppose that Hamilton's equations (15) hold, we get that

$$-\frac{d}{dt}\frac{\partial L}{\partial v^j}\left(\gamma(t), \dot{\gamma}(t)\right) = -\dot{\xi}_j(t) = \frac{\partial H_L}{\partial x^j}\left(\gamma(t), \xi(t)\right) = -\frac{\partial L}{\partial x^j}\left(\gamma(t), \dot{\gamma}(t)\right),$$

and so the Euler-Lagrange equations (8) are satisfied. □

**Remark 1.42.** Under some reasonable assumptions on the Lagrangian system it can be shown that the Legendre transform (13) defined in example 1.35 is always a diffeomorphism. For more details see [Maz12, p. 8].

## Conservation Laws and Noether's Theorem

**Definition 1.43 (Conservation Law).** *A* ***conservation law for a Lagrangian system*** $(\boldsymbol{M}, \boldsymbol{L})$ *is defined to be a function* $I \in C^\infty(TM)$ *such that*

$$\frac{d}{dt} I\left(\gamma(t), \dot{\gamma}(t)\right) = 0$$

*for all extremals of the action functional* (1.15).

**Proposition 1.44 (Conservation of Energy).** *The energy of an autonomous Lagrangian system is a conservation law.*

*Proof.* By definition of the fibrewise differential 1.27 we have that

$$D^{\mathcal{F}} L_{(\gamma,\dot{\gamma})}(\dot{\gamma}) = \frac{\partial L}{\partial v^i}(\gamma,\dot{\gamma})\, dx^i \left(\dot{\gamma}^j \frac{\partial}{\partial x^j}\right) = \frac{\partial L}{\partial v^i}(\gamma,\dot{\gamma})\,\dot{\gamma}^j \delta^i_j = \frac{\partial L}{\partial v^i}(\gamma,\dot{\gamma})\,\dot{\gamma}^i.$$

Thus by definition of the energy 1.37 and the Euler-Lagrange equations 1.18 we compute

$$\begin{aligned}
\frac{d}{dt} E(\gamma,\dot{\gamma}) &= \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(\gamma,\dot{\gamma})\,\dot{\gamma}^i\right) - \frac{d}{dt} L(\gamma,\dot{\gamma}) \\
&= \frac{d}{dt}\frac{\partial L}{\partial v^i}(\gamma,\dot{\gamma})\,\dot{\gamma}^i + \frac{\partial L}{\partial v^i}(\gamma,\dot{\gamma})\,\ddot{\gamma}^i - \frac{\partial L}{\partial x^i}(\gamma,\dot{\gamma})\dot{\gamma}^i - \frac{\partial L}{\partial v^i}(\gamma,\dot{\gamma})\ddot{\gamma}^i \\
&= \frac{d}{dt}\frac{\partial L}{\partial v^i}(\gamma,\dot{\gamma})\,\dot{\gamma}^i - \frac{\partial L}{\partial x^i}(\gamma,\dot{\gamma})\dot{\gamma}^i \\
&= \frac{\partial L}{\partial x^i}(\gamma,\dot{\gamma})\,\dot{\gamma}^i - \frac{\partial L}{\partial x^i}(\gamma,\dot{\gamma})\dot{\gamma}^i \\
&= 0.
\end{aligned}$$

□

Recall, that for a smooth manifold $M$, we define the *set of diffeomorphisms on $M$* by

$$\mathrm{Diff}(M) := \{\varphi \in C^\infty(M,M) : \varphi \text{ is a diffeomorphism}\}.$$

In fact $\mathrm{Diff}(M)$ constitutes a group under ordinary composition of maps. Thus we define a *one-parameter group of diffeomorphisms of $M$* to be a group homomorphisms

$$(\mathbb{R},+) \to \mathrm{Diff}(M)$$

Explicitely, given any one-parameter group $\theta : (\mathbb{R},+) \to \mathrm{Diff}(M)$, we define $\theta_s := \theta(s)$ for all $s \in \mathbb{R}$ and we can therefore write $(\theta_s)_{s\in\mathbb{R}}$ for the one-parameter group $\theta$ of diffeomorphisms of $M$. Since $\theta$ is a homomorphism of groups, we have that

$$\theta_{s+t} = \theta_s \circ \theta_t \qquad \text{and} \qquad \theta_0 = \mathrm{id}_M$$

for all $s,t \in \mathbb{R}$. We say that the one-parameter group $(\theta_s)_{s\in\mathbb{R}}$ of diffeomorphisms of $M$ is smooth, iff the corresponding map $\theta : \mathbb{R} \times M \to M$ defined by $(s,x) \mapsto \theta_s(x)$ is smooth. If $F \in C^\infty(M,N)$ for two smooth manifolds $M$ and $N$, for $x \in M$ we define the *differential of $F$ at $x$* to be the mapping $DF_x : T_xM \to T_{F(x)}N$, given by $DF_x(v)(f) := v(f \circ F)$ for all $f \in C^\infty(N)$. These fibrewise mappings can be assembled to the *global differential of* $F$, defined to be the mapping $DF : TM \to TN$ given by $DF(x, v) := \big(F(x), DF_x(v)\big)$. The global differential is a smooth map (see [Lee13, p. 68]) and has the following properties.

**Proposition 1.45 (Properties of the Global Differential [Lee13, p. 68]).** *Let $M, N, P$ be smooth manifolds, $F \in C^\infty(M, N)$ and $G \in C^\infty(N, P)$. Then:*

(a) $D(G \circ F) = DG \circ DF$.
(b) $D(\mathrm{id}_M) = \mathrm{id}_{TM}$.
(c) *If $F$ is a diffeomorphism, then $DF$ is a diffeomorphism with $(DF)^{-1} = D\left(F^{-1}\right)$.*

**Remark 1.46.** In a more sophisticated language, proposition 1.45 says that the global differential is a functor $D : \mathsf{Man} \to \mathsf{Man}$, where $\mathsf{Man}$ denotes the category of finite-dimensional smooth manifolds.

**Lemma 1.47.** *Let $(\theta_s)_{s\in\mathbb{R}}$ be a smooth one-parameter group of diffeomorphisms of a smooth manifold $M$. Then $(D\theta_s)_{s\in\mathbb{R}}$ is a smooth one-prameter group of diffeomorphisms of $TM$.*

*Proof.* Part (c) of the properties of the global differential 1.45 implies that $D\theta_s$ is a diffeomorphism for all $s \in \mathbb{R}$. Moreover, by part (c) of the properties of the global differential 1.45 we compute

$$D\theta_{s+t} = D(\theta_s \circ \theta_t) = D\theta_s \circ D\theta_t$$

for all $s, t \in \mathbb{R}$. Lastly, part (b) of the properties of the global differential 1.45 implies

$$D\theta_0 = D(\mathrm{id}_M) = \mathrm{id}_{TM} .$$

□

Given a one-parameter group $(\theta_s)_{s\in\mathbb{R}}$ of diffeomorphisms of a smooth manifold $M$, we can define a vector field $V$ by

$$V_x := \left.\frac{d}{ds}\right|_{s=0} \theta_s(x)$$

for all $x \in M$. This vector field is actually smooth by [Lee13, p. 210] and is called the *infinitesimal generator of* $\theta$.

**Definition 1.48 (Symmetry).** *A* ***symmetry of an autonomous Lagrangian system*** $(M, L)$ *is defined to be a diffeomorphism $F \in \mathrm{Diff}(M)$, such that*

$$L \circ DF = L.$$

Recall, that if $k \in \mathbb{N}$ and $X \in \mathfrak{X}(M)$ for a smooth manifold $M$, we can define a mapping $i_X : \Omega^{k+1}(M) \to \Omega^k(M)$, called *interior multiplication*, by

$$(i_X\omega)_x(v_1, \ldots, v_k) := \omega_x\left(X|_x, v_1, \ldots, v_k\right)$$

for all $x \in M$ and $v_1, \ldots, v_k \in T_x M$. One-parameter groups of symmetries of autonomous Lagrangian systems give rise to conservation laws.

**Theorem 1.49 (Noether's Theorem, Lagrangian Version).** *Let $(\theta_s)_{s\in\mathbb{R}}$ be a smooth one-parameter group of symmetries of an autonomous Lagrangian system. Then $i_V\left(D^{\mathcal{F}}L\right)$ is a conservation law, where $V$ denotes the infinitesimal generator of the one-parameter group $(D\theta_s)_{s\in\mathbb{R}}$ of diffeomorphisms of $TM$. The conservation law $i_V\left(D^{\mathcal{F}}L\right)$ is called the* ***Noether integral****.*

*Proof.* Let $\left(TU, (x^i, v^i)\right)$ be a chart on $TM$. First we compute the infinitesimal generator $V$ of the one-parameter group $(\theta_s)_{s\in\mathbb{R}}$ in the chart $\left(U, (x^i)\right)$. Let $x \in U$. Then

$$V_x = \left.\frac{d}{ds}\right|_{s=0} \theta_s(x) = \frac{d\theta_s^i(x)}{ds}(0)\left.\frac{\partial}{\partial x^i}\right|_{\theta_0(x)} = \frac{d\theta_s^i(x)}{ds}(0)\left.\frac{\partial}{\partial x^i}\right|_{x}.$$

Thus

$$V_x = V^i(x)\left.\frac{\partial}{\partial x^i}\right|_{x}$$

where $V^i : U \to \mathbb{R}$ are given by

$$V^i(x) := \frac{d\theta_s^i(x)}{ds}(0).$$

Next consider the infinitesimal generator $V$ of the one-parameter group $(D\theta_s)_{s\in\mathbb{R}}$. For $(x, v) \in TU$, where $v = v^i\frac{\partial}{\partial x^i}$, we compute

$$\begin{aligned}
V_{(x,v)} &= \left.\frac{d}{ds}\right|_{s=0} \left(\theta_s(x), D\theta_s|_x(v)\right) \\
&= \left.\frac{d}{ds}\right|_{s=0} \left(\theta_s(x), v^j\frac{\partial\theta_s^i}{\partial x^j}(x)\left.\frac{\partial}{\partial x^i}\right|_{\theta_s(x)}\right) \\
&= \frac{d\theta_s^i(x)}{ds}(0)\left.\frac{\partial}{\partial x^i}\right|_{(x,v)} + v^j\frac{\partial^2\theta^i}{\partial s\partial x^j}(0,x)\left.\frac{\partial}{\partial v^i}\right|_{(x,v)} \\
&= V^i(x)\left.\frac{\partial}{\partial x^i}\right|_{(x,v)} + v^j\frac{\partial^2\theta^i}{\partial x^j\partial s}(0,x)\left.\frac{\partial}{\partial v^i}\right|_{(x,v)} \\
&= V^i(x)\left.\frac{\partial}{\partial x^i}\right|_{(x,v)} + v^j\frac{\partial}{\partial x^j}\frac{d\theta_s^i(x)}{ds}(0)\left.\frac{\partial}{\partial v^i}\right|_{(x,v)} \\
&= V^i(x)\left.\frac{\partial}{\partial x^i}\right|_{(x,v)} + v^j\frac{\partial V^i}{\partial x^j}(x)\left.\frac{\partial}{\partial v^i}\right|_{(x,v)}.
\end{aligned}$$

Therefore

$$i_V\left(D^{\mathcal{F}}L\right)(x,v) = D^{\mathcal{F}}L_{(x,v)}\left(V_{(x,v)}\right)$$

$$
\begin{aligned}
&= \frac{\partial L}{\partial v^i}(x,v)dx^i|_{(x,v)}\left(V_{(x,v)}\right) \\
&= \frac{\partial L}{\partial v^i}(x,v)V^i(x).
\end{aligned}
$$

For $(x,v) \in TM$ set $\gamma(s) := d\theta_s(x,v)$. If $f \in C^\infty(TM)$, the definition of the velocity of a curve and of the differential yields

$$
(Vf)(x,v) = V_{(x,v)}f = \left(\frac{d}{ds}\bigg|_{s=0}\gamma(s)\right)f = D\gamma\left(\frac{d}{ds}\bigg|_{s=0}\right)f = \frac{d}{ds}\bigg|_{s=0}(f\circ\gamma).
$$

So using the Euler-Lagrange equations 1.18 and the assumption that $\theta_s$ is a symmetry of $(M,L)$ for all $s \in \mathbb{R}$, we get

$$
\begin{aligned}
\frac{d}{dt}i_V\left(D^{\mathcal{F}}L\right)(\gamma,\dot\gamma) &= \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(\gamma,\dot\gamma)V^i(\gamma)\right) \\
&= \frac{d}{dt}\frac{\partial L}{\partial v^i}(\gamma,\dot\gamma)V^i(\gamma) + \frac{\partial L}{\partial v^i}(\gamma,\dot\gamma)\frac{d}{dt}V^i(\gamma) \\
&= \frac{\partial L}{\partial x^i}(\gamma,\dot\gamma)V^i(\gamma) + \frac{\partial L}{\partial v^i}(\gamma,\dot\gamma)\frac{d}{dt}V^i(\gamma) \\
&= \frac{\partial L}{\partial x^i}(\gamma,\dot\gamma)V^i(\gamma) + \frac{\partial L}{\partial v^i}(\gamma,\dot\gamma)\frac{\partial V^i}{\partial x^j}(\gamma)\dot\gamma^j \\
&= V_{(\gamma,\dot\gamma)}L \\
&= \frac{d}{ds}\bigg|_{s=0}\left(L\circ D\theta_s\right)(\gamma,\dot\gamma) \\
&= \frac{d}{ds}\bigg|_{s=0}L(\gamma,\dot\gamma) \\
&= 0.
\end{aligned}
$$

$\square$

## Problems

1-1. Adopt the theory developed in the section on the *Legendre Transform* to the non-autonomous case, that is to the case of a Lagrangian system where the Lagrangian function can depend on time.

1-2. Complete the proof of theorem 1.18 about the Euler-Lagrange equations. *Hint:* Use the generalized notion of a *fibrewise differential* established in problem 1-1.

CHAPTER 2

# Hamiltonian Mechanics

## Symplectic Geometry

A profound difference between the tangent bundle $TM$ and the cotangent bundle $T^*M$ of a smooth manifold $M$ is that on the latter there exists a natural 1-form, the tautological form $\alpha$ defined in definition 1.31.

**The Category of Symplectic Manifolds.** Recall that a form $\omega$ on a smooth manifold $M$ is said to be *closed*, iff $d\omega = 0$.

**Definition 2.1 (Symplectic Manifold).** *A* ***symplectic manifold*** *is defined to be a tuple $(M, \omega)$ consisting of a smooth manifold $M$ and a closed nondegenerate 2-form $\omega \in \Omega^2(M)$, called a* ***symplectic form on $M$****.*

**Example 2.2 (The Cotangent Bundle).** Let $M$ be a smooth manifold and consider the tautological form $\alpha \in \Omega^1(T^*M)$ defined by $\alpha := \xi_i dx^i$ on a chart $\big(T^*U, (x^i, \xi^i)\big)$ on $T^*M$. Define $\omega \in \Omega^2(T^*M)$ by $\omega := -d\alpha$. It is immediate that $\omega$ is closed since $d\omega = -(d \circ d)(\alpha) = 0$. Moreover, we compute locally

$$\omega = -d(\xi_i dx^i) = -\frac{\partial \xi_i}{\partial x^j} dx^j \wedge dx^i - \frac{\partial \xi_i}{\partial \xi_j} d\xi_j \wedge dx^i = \delta_i^j dx^i \wedge d\xi_j = \sum_i dx^i \wedge d\xi_i.$$

Thus $\omega$ is nondegenerate.

**Definition 2.3.** *A morphism $F : (M, \omega) \to (\widetilde{M}, \widetilde{\omega})$ between two symplectic manifolds $(M, \omega)$ and $(\widetilde{M}, \widetilde{\omega})$ is defined to be a morphism $F \in C^\infty(M, \widetilde{M})$ such that $F^*\widetilde{\omega} = \omega$.*

**Exercise 2.4.** Consider as objects symplectic manifolds and as morphisms the ones from definition 2.3. Show that they do form a category, the ***category of symplectic manifolds***.

**Definition 2.5 (Symplectomorphism).** *A* ***symplectomorphism*** *is defined to be an isomorphism in the category of symplectic manifolds.*

**The Tangent-Cotangent Bundle Isomorphism.** As in Riemannian geometry, one very important feature of a symplectic manifold $(M, \omega)$ is that there is a canonical identification of the tangent bundle $TM$ and the cotangent bundle $T^*M$ (for the Riemannian case see [Lee13, p. 341]). But first we recall some basic facts from the tensor calculus on smooth manifolds.

**Lemma 2.6 (Vector Bundle Chart Lemma [Lee13, p. 253]).** *Let $M$ be a smooth manifold, $k \in \mathbb{N}$ and suppose that for all $x \in M$ we are given a real vector space $E_x$ of dimension $k$. Let $E := \coprod_{x\in M} E_x$ and let $\pi : E \to M$ be given by $\pi(x, v) := x$. Moreover, suppose that we are given the following data:*

(i) *An open cover $(U_\alpha)_{\alpha\in A}$ of $M$.*
(ii) *For all $\alpha \in A$ a bijection $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ such that the restriction $\Phi_\alpha|_{E_x} : E_x \to \{x\} \times \mathbb{R}^k \cong \mathbb{R}^k$ is an isomorphism of vector spaces for all $x \in M$.*
(iii) *For all $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \varnothing$, a smooth mapping $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ such that the mapping $\Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ is of the form $\Phi_\alpha \circ \Phi_\beta^{-1}(x, v) = \big(x, \tau_{\alpha\beta}(x)v\big)$.*

*Then $E$ admits a unique topology and a smooth structure making it into a smooth manifold and a smooth vector bundle $\pi : E \to M$ of rank $k$ with local trivializations $(U_\alpha, \Phi_\alpha)_{\alpha\in A}$.*

Let $M^n$ be a smooth manifold and let $k, l \in \mathbb{N}$. For all $x \in M$ define the space of ***mixed tensors of type $(k, l)$ on $T_xM$*** by

$$T^{(k,l)}(T_xM) := \underbrace{T_xM \otimes \cdots \otimes T_xM}_{k} \otimes \underbrace{T_x^*M \otimes \cdots \otimes T_x^*M}_{l}.$$

By proposition 12.10 [Lee13, p. 311] we have that

$$T^{(k,l)}(T_xM) \cong \mathrm{L}\big(\underbrace{T_x^*M, \dots, T_x^*M}_{k}, \underbrace{T_xM, \dots, T_xM}_{l}; \mathbb{R}\big)$$

since $(T_x^*M)^* \cong T_xM$ canonically ($T_xM$ is finite-dimensional) where the latter denotes the space of all $\mathbb{R}$-valued multilinear forms on

$$\underbrace{T_x^*M \times \cdots \times T_x^*M}_{k} \times \underbrace{T_xM \times \cdots \times T_xM}_{l}.$$

We will always think of mixed tensors as multilinear forms. Let $(U, x^i)$ be a chart about $x$. Then using corollary 12.12 [Lee13, p. 313] we get that a basis for $T^{(k,l)}(T_xM)$ is given by all elements

$$\left.\frac{\partial}{\partial x^{i_1}}\right|_x \otimes \cdots \otimes \left.\frac{\partial}{\partial x^{i_k}}\right|_x \otimes dx^{j_1}|_x \otimes \cdots \otimes dx^{j_l}|_x$$

for all $1 \le i_1, \dots, i_k, j_1, \dots, j_l \le n$. Consequently, $\dim T^{(k,l)}(T_xM) = n^{k+l}$ and a particular tensor $A \in T^{(k,l)}(T_xM)$ expressed in this basis is given by

$$A = A^{i_1\dots i_k}_{j_1\dots j_l} \left.\frac{\partial}{\partial x^{i_1}}\right|_x \otimes \cdots \otimes \left.\frac{\partial}{\partial x^{i_k}}\right|_x \otimes dx^{j_1}|_x \otimes \cdots \otimes dx^{j_l}|_x \tag{1}$$

where

$$A^{i_1\dots i_k}_{j_1\dots j_l} := A\left(dx^{i_1}|_x, \dots, dx^{i_k}|_x, \left.\frac{\partial}{\partial x^{j_1}}\right|_x, \dots \left.\frac{\partial}{\partial x^{j_l}}\right|_x\right). \tag{2}$$

Next we want to "glue" together the different spaces of mixed tensors.

**Proposition 2.7.** *Let $M$ be a smooth manifold and let $k, l \in \mathbb{N}$. Then*

$$T^{(k,l)}TM := \coprod_{x \in M} T^{(k,l)}(T_x M)$$

*admits a unique topology and a smooth structure making it into a smooth manifold and a smooth vector bundle $\pi : T^{(k,l)}TM \to M$ of rank $n^{k+l}$. This smooth vector bundle is called the* ***bundle of mixed tensors of type $(k, l)$ on $M$****.*

*Proof.* This is an application of the vector bundle chart lemma 2.6. For all $x \in M$ define $E_x := T^{(k,l)}(T_x M)$. By the preceeding discussion, $\dim E_x = n^{k+l}$. Let $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ denote the smooth structure on $M$. Then clearly $(U_\alpha)_{\alpha \in A}$ is an open cover for $M$. For each $\alpha \in A$, define

$$\Phi_\alpha : \begin{cases} \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{n^{k+l}} \\ (x, A) \mapsto \big(x, (A^{i_1 \dots i_k}_{j_1 \dots j_l})\big) \end{cases}$$

where we expressed $A$ as in (1). Observe, that this map strongly depends on the coordinate functions. Clearly, the inverse is given by

$$\Phi_\alpha^{-1} : \begin{cases} U_\alpha \times \mathbb{R}^{n^{k+l}} \to \pi^{-1}(U_\alpha) \\ \big(x, (A^{i_1 \dots i_k}_{j_1 \dots j_l})\big) \mapsto (x, A) \end{cases} .$$

Hence each $\Phi_\alpha$ is bijective. Now we have to check, that $\Phi_\alpha|_{E_x}$ is an isomorphism for all $x \in M$. By elementary linear algebra it is enough to show that $\Phi_\alpha$ is linear. So let $\lambda \in \mathbb{R}$ and $A, B \in E_x$. Then

$$\begin{aligned} \Phi_\alpha|_{E_x}(x, A + \lambda B) &= \big(x, (A + \lambda B)^{i_1 \dots i_k}_{j_1 \dots j_l})\big) \\ &= \big(x, (A^{i_1 \dots i_k}_{j_1 \dots j_l}) + \lambda (B^{i_1 \dots i_k}_{j_1 \dots j_l})\big) \\ &= \Phi_\alpha|_{E_x}(x, A) + \lambda \Phi_\alpha|_{E_x}(x, B). \end{aligned}$$

Lastly, let $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$ and coordinates $(x^i_\alpha)$ and $(x^i_\beta)$, respectively. Then for $x \in U_\alpha \cap U_\beta$ we have that

$$\left.\frac{\partial}{\partial x^i_\alpha}\right|_x = \frac{\partial x^j_\beta}{\partial x^i_\alpha}(x) \left.\frac{\partial}{\partial x^j_\beta}\right|_x \qquad \text{and} \qquad dx^i_\alpha|_x = \frac{\partial x^i_\alpha}{\partial x^j_\beta}(x) dx^j_\beta|_x.$$

So if $A^{i_1 \dots i_k}_{j_1 \dots j_l}$ are coordinates of a mixed tensor with respect to the basis induced by $(x^i_\alpha)$, we compute

$$\begin{aligned} A^{i_1 \dots i_k}_{j_1 \dots j_l} &= A\left(dx^{i_1}_\alpha|_x, \dots, dx^{i_k}_\alpha|_x, \left.\frac{\partial}{\partial x^{j_1}_\alpha}\right|_x, \dots \left.\frac{\partial}{\partial x^{j_l}_\alpha}\right|_x\right) \\ &= \frac{\partial x^{i_1}_\alpha}{\partial x^{p_1}_\beta}(x) \cdots \frac{\partial x^{i_k}_\alpha}{\partial x^{p_k}_\beta}(x) \frac{\partial x^{q_1}_\beta}{\partial x^{j_1}_\alpha}(x) \cdots \frac{\partial x^{q_l}_\beta}{\partial x^{j_l}_\alpha}(x) A^{p_1 \dots p_k}_{q_1 \dots q_l} \end{aligned}$$

Thus define $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n^{k+l}, \mathbb{R})$ by

$$\tau_{\alpha\beta}(x) := \left( \frac{\partial x_\alpha^{i_1}}{\partial x_\beta^{p_1}}(x) \cdots \frac{\partial x_\alpha^{i_k}}{\partial x_\beta^{p_k}}(x) \frac{\partial x_\beta^{q_1}}{\partial x_\alpha^{j_1}}(x) \cdots \frac{\partial x_\beta^{q_l}}{\partial x_\alpha^{j_l}}(x) \right).$$

Then $\tau_{\alpha\beta}$ is clearly smooth and moreover

$$\Phi_\alpha \circ \Phi_\beta^{-1} \left(x, (A^{p_1 \dots p_k}_{q_1 \dots q_l})\right) = \left(x, (A^{i_1 \dots i_k}_{j_1 \dots j_l})\right) = \left(x, \tau_{\alpha\beta}(x)(A^{p_1 \dots p_k}_{q_1 \dots q_l})\right).$$

Therefore, conditions (i)-(iii) in the vector bundle chart lemma 2.6 are satisfied and the statement follows. □

**Remark 2.8.** There is a much more abstract approach for constructing vector bundles[1] than the explicit one used for the bundle of mixed tensors in proposition 2.7. Let us first formulate a *metatheorem*:

> "Anything one can do with vector spaces, one can also do with vector bundles."

We make this precise now. Let $\mathsf{Vect}$ denote the category of finite-dimensional real vector spaces. A functor

$$\mathcal{F} : \underbrace{\mathsf{Vect} \times \cdots \times \mathsf{Vect}}_{k} \to \mathsf{Vect}$$

which is either contravariant or covariant in its arguments, is said to be *smooth*, iff for all vector spaces $V_1, \dots, V_k, W_1, \dots, W_k \in \mathsf{Vect}$ the induced map

$$\bigoplus_{i=1}^{k} \widetilde{\mathrm{L}}(V_i, W_i) \to \mathrm{L}\left(\mathcal{F}(V_1, \dots, V_k), \mathcal{F}(W_1, \dots, W_k)\right)$$

where

$$\widetilde{\mathrm{L}}(V_i, W_i) := \begin{cases} \mathrm{L}(V_i, W_i) & \mathcal{F} \text{ is covariant in the } i\text{-th argument,} \\ \mathrm{L}(W_i, V_i) & \mathcal{F} \text{ is contravariant in the } i\text{-th argument,} \end{cases}$$

is a smooth map. The formal statement of the metatheorem can now be phrased as follows. If $\mathcal{F} : \mathsf{Vect} \times \cdots \times \mathsf{Vect} \to \mathsf{Vect}$ is a smooth functor as above and $\pi_i : E_i \to M$ are $k$ vector bundles, then $\pi : \mathcal{F}(E_1, \dots, E_k) \to M$ is a vector bundle where

$$\mathcal{F}(E_1, \dots, E_k) := \coprod_{x \in M} \mathcal{F}(E_1|_x, \dots, E_k|_x)$$

and $\pi(x, v) := x$.

Recall, that in a category $\mathscr{C}$, a *section* of a morphism $f : X \to Y$ is a morphism $\sigma : Y \to X$ such that $f \circ \sigma = \mathrm{id}_Y$.

[1]See lecture 14 from the lecture notes of the course *Differential Geometry I* taught by *Will J. Merry* at the *ETH Zurich* in the autumn semester 2018.

**Definition 2.9 (Tensor Field).** *Let $M$ be a smooth manifold and $k, l \in \mathbb{N}$. A **smooth tensor field of type $(k, l)$ on $M$** is defined to be a section of $\pi : T^{(k,l)}TM \to M$. The space of all smooth tensor fields of type $(k, l)$ on $M$ is denoted by $\mathcal{T}^{k,l}(M) := \Gamma\left(T^{(k,l)}TM\right)$.*

**Example 2.10 (Vector Field and Covector Field).** Let $M$ be a smooth manifold. Of particular importance are the tensor fields such that $k + l = 1$. If $k = 1$, such tensor fields are called ***vector fields*** and we write $\mathfrak{X}(M) := \Gamma\left(T^{(1,0)}TM\right)$. Likewise, if $l = 1$, we call such tensor fields ***covector fields*** and write $\mathfrak{X}^*(M) := \Gamma\left(T^{(0,1)}TM\right)$.

Let $\left(U, (x^i)\right)$ be a chart on $M$ and $A : M \to T^{(k,l)}TM$ such that $A_x \in T^{(k,l)}(T_xM)$ for all $x \in M$. From (1) we get that

$$A_x = A^{i_1 \dots i_k}_{j_1 \dots j_l}(x) \frac{\partial}{\partial x^{i_1}}\bigg|_x \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}}\bigg|_x \otimes dx^{j_1}|_x \otimes \cdots \otimes dx^{j_l}|_x$$

for all $x \in U$ where $A^{i_1 \dots i_k}_{j_1 \dots j_l} : U \to \mathbb{R}$ are given as in (2). We will call these functions the ***component functions of $A$***. Recall, that a map $F : M \to N$ between two smooth manifolds $M$ and $N$ is said to be *smooth*, iff for every $x \in M$ there exists a chart $(U, \varphi)$ about $x$ on $M$ and a chart $(V, \psi)$ about $F(x)$ on $N$ such that $U \cap F^{-1}(V)$ is open in $M$ and $\psi \circ F \circ \varphi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \to \psi(V)$ is smooth. Moreover, if $A \subseteq U \subseteq M$, where $U$ is open and $A$ is closed in $M$, a function $\psi \in C^\infty(M)$ is said to be a *smooth bump function for $A$ supported in $U$*, iff $0 \le \psi \le 1$, $\psi|_A = 1$ and $\operatorname{supp} \psi \subseteq U$. The paracompactness condition guarantees that smooth bump functions exist in great abundance.

**Proposition 2.11 (Existence of Smooth Bump Functions [Lee13, p. 44]).** *Let $M$ be a smooth manifold and $A \subseteq U \subseteq M$, where $U$ is open and $A$ is closed in $M$. Then there exists a smooth bump function for $A$ supported in $U$.*

**Proposition 2.12 (Smoothness Criteria for Tensor Fields [Lee13, p. 317]).** *Let $M$ be smooth manifold, $k, l \in \mathbb{N}$ and $A : M \to T^{(k,l)}TM$ such that $A_x \in T^{(k,l)}T_xM$ for all $x \in M$. Then the following conditions are equivalent:*

(a) $A \in \Gamma\left(T^{(k,l)}TM\right)$.
(b) *In every smooth coordinate chart, the component functions of $A$ are smooth.*
(c) *Each point of $M$ is contained in a chart in which $A$ has smooth component functions.*
(d) *For all $\omega^1, \dots, \omega^k \in \mathfrak{X}^*(M)$ and $X_1, \dots, X_l \in \mathfrak{X}(M)$, the function*

$$\mathcal{A}(\omega^1, \dots, \omega^k, X_1, \dots, X_l) : M \to \mathbb{R}$$

*defined by*

$$\mathcal{A}\left(\omega^1, \dots, \omega^k, X_1, \dots, X_l\right)(x) := A_x\left(\omega^1_x, \dots, \omega^k_x, X_1|_x, \dots, X_l|_x\right) \tag{3}$$

*is smooth.*
(e) *Let $U \subseteq M$ be open. If $\omega^1, \dots, \omega^k \in \mathfrak{X}^*(U)$ and $X_1, \dots, X_l \in \mathfrak{X}(U)$, then $\mathcal{A}$ defined by (3) belongs to $C^\infty(U)$.*

*Proof.* We prove (a) $\Leftrightarrow$ (b) and (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (b).
To prove (a) $\Leftrightarrow$ (b), let $x \in M$ and $\big(U, (x^i)\big)$ be a smooth chart on $M$ about $x$. Proposition 2.7 yields a map $\Phi_U : \pi^{-1}(U) \to U \times \mathbb{R}^{n^{k+l}}$, and the proof of the vector bundle chart lemma implies, that the corresponding chart on $T^{(k,l)}TM$ is given by $\big(\pi^{-1}(U), \widetilde{\varphi}\big)$, where

$$\widetilde{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^{n^{k+l}}$$

is defined by

$$\widetilde{\varphi} := \left(\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}}\right) \circ \Phi_U.$$

Since $A_x \in T^{(k,l)}T_xM$ for all $x \in M$, we have that

$$A^{-1}\left(\pi^{-1}(U)\right) = (\pi \circ A)^{-1}(U) = \mathrm{id}_M(U) = U.$$

Hence $U \cap A^{-1}\left(\pi^{-1}(U)\right) = U$, which is open in $M$, and

$$\widetilde{\varphi} \circ A \circ \varphi^{-1} : \varphi(U) \to \widetilde{\varphi}\left(\pi^{-1}(U)\right)$$

is given by

$$\begin{aligned}\left(\widetilde{\varphi} \circ A \circ \varphi^{-1}\right)\left(\varphi(y)\right) &= \left(\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}}\right)\left(\Phi_U(A_y)\right)\\ &= \left(\varphi(y), (A^{i_1\ldots i_k}_{j_1\ldots j_l})(y)\right)\\ &= \left(\varphi(y), \left((A^{i_1\ldots i_k}_{j_1\ldots j_l}) \circ \varphi^{-1}\right)\left(\varphi(y)\right)\right)\end{aligned}$$

for all $y \in U$. Thus $\widetilde{\varphi} \circ A \circ \varphi^{-1}$ is smooth if and only if $(A^{i_1\ldots i_k}_{j_1\ldots j_l}) \circ \varphi^{-1}$ is smooth, which is equivalent to $A^{i_1\ldots i_k}_{j_1\ldots j_l}$ being smooth.
The implication (b) $\Rightarrow$ (c) is immediate.
To prove (c) $\Rightarrow$ (d), suppose $x \in M$ and let $(U, (x^i))$ be a chart about $x$ such that the component functions of $A$ are smooth. By example 2.10 and the equivalence (a) $\Leftrightarrow$ (b) we have

$$\omega^i = \omega^i_j dx^j \qquad \text{and} \qquad X_i = X_i^j \frac{\partial}{\partial x^j}$$

on $U$ for smooth functions $\omega^i_j$ and $X_i^j$. Thus for any $y \in U$ we compute

$$\begin{aligned}\mathcal{A}\left(\omega^1, \ldots, \omega^k, X_1, \ldots, X_l\right)(y) &= A_x\left(\omega^1_x, \ldots, \omega^k_x, X_1|_x, \ldots, X_l|_x\right)\\ &= \omega^1_{i_1}(y)\cdots\omega^k_{i_k}(y)X_1^{j_1}(y)\cdots X_l^{j_l}(y)A^{i_1\ldots i_k}_{j_1\ldots j_l}(y)\end{aligned}$$

and so $\mathcal{A}\left(\omega^1, \ldots, \omega^k, X_1, \ldots, X_l\right)$ is smooth.
To prove (d) $\Rightarrow$ (e), we use the fact that smoothness is a local property. Let $x \in U$ and suppose $(V, \varphi)$ is a chart on $U$ centered at $x$. Then $\varphi(V) \subseteq \mathbb{R}^n$ is open and so we find $\varepsilon > 0$ such that $B_\varepsilon(0) \subseteq \varphi(V)$. Set $A := \varphi^{-1}\left(\overline{B}_{\varepsilon/2}(0)\right) \subseteq U$. Then $A$ is closed in $U$ and

by proposition 2.11 there exists a smooth bump function $\psi \in C^\infty(U)$ for $A$ supported in $U$. Define $\widetilde{\omega}^i : M \to T^*M$ and $\widetilde{X}_i : M \to TM$ by

$$\widetilde{\omega}^i_x := \begin{cases} \psi(x)\omega^i_x & x \in U, \\ 0_x & x \in M \setminus \operatorname{supp}\psi, \end{cases} \quad \text{and} \quad \widetilde{X}_i|_x := \begin{cases} \psi(x)X_i|_x & x \in U, \\ 0_x & x \in M \setminus \operatorname{supp}\psi. \end{cases}$$

Then $\widetilde{\omega}^i \in \mathfrak{X}^*(M)$ and $\widetilde{X}_i \in \mathfrak{X}(M)$ by the gluing lemma for smooth maps (see [Lee13, p. 35]). Moreover, on $\varphi^{-1}\left(B_{\varepsilon/2}(0)\right)$ we have that $\widetilde{\omega}^i = \omega^i$ and $\widetilde{X}_i = X_i$. But then also

$$\mathcal{A}\left(\widetilde{\omega}^1, \dots, \widetilde{\omega}^k, \widetilde{X}_1, \dots, \widetilde{X}_l\right) = \mathcal{A}\left(\omega^1, \dots, \omega^k, X_1, \dots, X_l\right)$$

on this neighbourhood, and so since the former is smooth by assumption, so is the latter. Finally, to prove (e) $\Rightarrow$ (b), let $(U, (x^i))$ be a chart about $x \in M$. Consider $\omega^i \in \mathfrak{X}^*(U)$ and $X_i \in \mathfrak{X}(U)$ defined by

$$\omega^i := \delta^i_j dx^j \qquad \text{and} \qquad X_i := \delta^j_i \frac{\partial}{\partial x^j}.$$

Then it is easy to verify that

$$\mathcal{A}\left(\omega^{i_1}, \dots, \omega^{i_k}, X_{j_1}, \dots, X_{j_l}\right) = A^{i_1 \dots i_k}_{j_1 \dots j_l}$$

holds on $U$. Thus by assumption, each component function is smooth. $\square$

Part (d) of the smoothness criteria for tensor fields 2.12 implies that for any tensor field $A \in \Gamma\left(T^{(k,l)}TM\right)$ there is a mapping

$$\mathcal{A} : \underbrace{\mathfrak{X}^*(M) \times \cdots \times \mathfrak{X}^*(M)}_{k} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l} \to C^\infty(M)$$

defined by

$$\left(\omega^1, \dots, \omega^k, X_1, \dots, X_l\right) \mapsto \mathcal{A}\left(\omega^1, \dots, \omega^k, X_1, \dots, X_l\right).$$

We will call this mapping the ***map induced by the tensor field*** $A$.

**Proposition 2.13 (Tensor Field Characterisation Lemma [Lee13, p. 318]).** *Let $M$ be a smooth manifold and $k, l \in \mathbb{N}$. A mapping*

$$\mathcal{A} : \underbrace{\mathfrak{X}^*(M) \times \cdots \times \mathfrak{X}^*(M)}_{k} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l} \to C^\infty(M)$$

*is induced by a $(k, l)$-tensor field if and only if $\mathcal{A}$ is multilinear over $C^\infty(M)$.*

*Proof.* Suppose $\mathcal{A}$ is induced by a $(k, l)$-tensor field $A$. Let $\omega^1, \dots, \omega^k, \widetilde{\omega}^i \in \mathfrak{X}^*(M)$ and $X_1, \dots, X_l \in \mathfrak{X}(M)$ as well as $f \in C^\infty(M)$. Then for any $x \in M$ we compute

$$\begin{aligned} \mathcal{A}\left(\dots, \omega^i + f\widetilde{\omega}^i, \dots\right)(x) &= A_x\left(\dots, \omega^i_x + f(x)\widetilde{\omega}^i_x, \dots\right) \\ &= A_x\left(\dots, \omega^i_x, \dots\right) + f(x)A_x\left(\dots, \widetilde{\omega}^i_x, \dots\right) \\ &= \mathcal{A}\left(\dots, \omega^i, \dots\right)(x) + f(x)\mathcal{A}\left(\dots, \widetilde{\omega}^i, \dots\right)(x) \end{aligned}$$

$$= \left(\mathcal{A}\left(\dots,\omega^i,\dots\right) + f\mathcal{A}\left(\dots,\widetilde{\omega}^i,\dots\right)\right)(x).$$

Thus $\mathcal{A}$ is $C^\infty(M)$-multilinear with respect to the first $k$ arguments. Similarly, $\mathcal{A}$ is $C^\infty(M)$-multilinear with repect to the last $l$ arguments.

Conversly, suppose that

$$\mathcal{A} : \underbrace{\mathfrak{X}^*(M)\times\dots\times\mathfrak{X}^*(M)}_{k} \times \underbrace{\mathfrak{X}(M)\times\dots\times\mathfrak{X}(M)}_{l} \to C^\infty(M)$$

is $C^\infty(M)$-multilinear. We wish to define a $(k,l)$-tensor field $A$ that induces $\mathcal{A}$. That this is indeed possible, is the observation that $\mathcal{A}\left(\omega^1,\dots,\omega^k,X_1,\dots,X_l\right)(x)$ only depends on $\omega^1_x,\dots,\omega^k_x, X_1|_x,\dots,X_l|_x$. Thus we divide the remaining proof into three steps.

*Step 1:* $\mathcal{A}\left(\omega^1,\dots,\omega^k,X_1,\dots,X_l\right)$ *acts locally.* That is, if either some $\omega^i$ or $X_i$ vanish on an open set $U$, then so does $\mathcal{A}\left(\omega^1,\dots,\omega^k,X_1,\dots,X_l\right)$. Let $x\in U$ and $\psi\in C^\infty(M)$ be a smooth bump function for $\{x\}$ supported in $U$. Then $\psi\omega^i = 0$ on $M$ and by $C^\infty(M)$-multilinearity

$$0 = \mathcal{A}\left(\dots,\psi\omega^i,\dots\right) = \psi(x)\mathcal{A}\left(\dots,\omega^i,\dots\right)(x) = \mathcal{A}\left(\dots,\omega^i,\dots\right)(x).$$

An analogous argument works if some $X_i$ vanishes on $U$.

*Step 2:* $\mathcal{A}\left(\omega^1,\dots,\omega^k,X_1,\dots,X_l\right)$ *acts pointwise.* Thats is, if $\omega^i_x$ or $X_i|_x$ vanish for some $x\in M$, then so does $\mathcal{A}\left(\omega^1,\dots,\omega^k,X_1,\dots,X_l\right)$. Let $(U,(x^i))$ be a chart about $x$. Then $\omega^i = \omega^i_j dx^j$ on $U$. Let $\psi\in C^\infty(U)$ denote the smooth bump function used in the proof of part (d) $\Rightarrow$ (e) of the smoothness criteria for tensor fields 2.12. Define

$$\varepsilon^j := \begin{cases} \psi(x)dx^j|_x & x\in U,\\ 0_x & x\in M\setminus\operatorname{supp}\psi,\end{cases} \qquad\text{and}\qquad f^i_j := \begin{cases} \psi(x)\omega^i_j(x) & x\in U,\\ 0_x & x\in M\setminus\operatorname{supp}\psi.\end{cases}$$

Then $\omega^i = f^i_j\varepsilon^j$ on a neighbourhood of $x$ and so by multilinearity and step 1, we have that

$$\mathcal{A}\left(\dots,\omega^i,\dots\right) = f^i_j\mathcal{A}\left(\dots,\varepsilon^j,\dots\right)$$

on a neighbourhood of $x$. But since $\omega^i_x$ vanishes so does each $\omega^i_j(x)$. Hence

$$\mathcal{A}\left(\dots,\omega^i,\dots\right)(x) = f^i_j(x)\mathcal{A}\left(\dots,\varepsilon^j,\dots\right)(x) = \omega^i_j(x)\mathcal{A}\left(\dots,\varepsilon^j,\dots\right)(x) = 0.$$

An analogous argument works if some $X_i|_x$.

*Step 3: Definition of the* $(k,l)$*-tensor field $A$ inducing* $\mathcal{A}$. Let $x\in M$, $\omega^1,\dots,\omega^k\in T^*_xM$ and $v_1,\dots,v_l\in T_xM$. Suppose that $\widetilde{\omega}^1,\dots,\widetilde{\omega}^k\in\mathfrak{X}^*(M)$ and $\widetilde{X}_1,\dots,\widetilde{X}_l\in\mathfrak{X}(M)$ are any extensions, respectively. That is, $\widetilde{\omega}^i_x = \omega^i$ and $\widetilde{X}_i|_x = v_i$. They do always exist, since in a chart $(U,(x^i))$ we may write

$$\omega^i = \omega^i_j dx^j|_x \qquad\text{and}\qquad v_i = v^j_i\left.\frac{\partial}{\partial x^j}\right|_x$$

and so using a smooth bump function for $\{x\}$ supported in $U$ we can construct global maps as in step 2 if we consider the components as constant functions. Now define

$$A_x\left(\omega^1,\dots,\omega^k,v_1,\dots,v_l\right) := \mathcal{A}\left(\widetilde{\omega}^1,\dots,\widetilde{\omega}^k,\widetilde{X}_1,\dots,\widetilde{X}_l\right)(x). \tag{4}$$

This is well-defined by step 2. Now if $\omega^1,\dots,\omega^k \in \mathfrak{X}^*(M)$ and $X_1,\dots,X_l \in \mathfrak{X}(M)$, we have that

$$\mathcal{A}\left(\omega^1,\dots,\omega^k,X_1,\dots,X_l\right)(x) = A_x\left(\omega^1_x,\dots,\omega^k_x,X_1|_x,\dots,X_l|_x\right),$$

since $\omega^i$ and $X_i$ are extensions of $\omega^i_x$ and $X_i|_x$, respectively, for all $x \in M$. So the assumption that $\mathcal{A}$ takes values in the space of smooth functions $C^\infty(M)$ together with part (d) of the smoothness criteria for tensor fields 2.12 yields that $A$ is a smooth $(k,l)$-tensor field which moreover induces $\mathcal{A}$. □

**Proposition 2.14 (Bundle Homomorphism Characterisation Lemma [Lee13, p. 262]).** *Let $\pi : E \to M$ and $\widetilde{\pi} : \widetilde{E} \to M$ be smooth vector bundles over a smooth manifold $M$. A map $\mathcal{F} : \Gamma(E) \to \Gamma(\widetilde{E})$ is linear over $C^\infty(M)$ if and only if there exists a smooth bundle homomorphism $F : E \to \widetilde{E}$ over $M$ such that $\mathcal{F}(\sigma) = F \circ \sigma$ for all $\sigma \in \Gamma(E)$.*

**Theorem 2.15 (Tangent-Cotangent Bundle Isomorphism).** *Let $(M,\omega)$ be a symplectic manifold. Define $\Omega : TM \to T^*M$ by*

$$\Omega(v)(w) := \omega_x(v,w) \tag{5}$$

*for all $x \in M$ and $v, w \in T_xM$. Then $\Omega$ is a well-defined smooth bundle isomorphism. The morphism $\Omega$ is called the* ***tangent-cotangent bundle isomorphism****.*

*Proof.* Using the tensor field characterisation lemma 2.13, $\omega$ induces a map

$$\omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$$

which is $C^\infty(M)$-multilinear. Thus for $X \in \mathfrak{X}(M)$ we define $\Omega_X : \mathfrak{X}(M) \to C^\infty(M)$ by

$$\Omega_X(Y) := \omega(X,Y).$$

Since $\omega$ is multilinear over $C^\infty(M)$, so is $\Omega_X$, and thus again by the tensor field characterisation lemma 2.13, $\Omega_X$ belongs to $\mathfrak{X}^*(M)$. Hence we get a map $\Omega : \mathfrak{X}(M) \to \mathfrak{X}^*(M)$ by $\Omega(X) := \Omega_X$ which is also multilinear over $C^\infty(M)$. Finally, by the bundle homomorphism characterisation lemma 2.14, there exists a smooth vector bundle homomorphism $\Omega : TM \to T^*M$ such that $\Omega_X = \Omega \circ X$ for all $X \in \mathfrak{X}(M)$. Let $x \in M$, $v, w \in T_xM$ and $V, W \in \mathfrak{X}(M)$ be extensions of $v$ and $w$, respectively (see step 3 in the proof of the tensor field characterisation lemma 2.13). We compute

$$\Omega_V|_x(w) = \Omega_V(W)(x) = \omega(V,W)(x) = \omega_x(V|_x, W|_x) = \omega_x(v,w)$$

and since $(\Omega \circ V)|_x(w) = \Omega(V|_x)(w) = \Omega(v)(w)$, we have that $\Omega$ coincides with the map defined in (5). Next we show that $\Omega$ is injective. Let $v, \widetilde{v} \in TM$ such that $\Omega(v) = \Omega(\widetilde{v})$. Since $\Omega$ is a fibrewise mapping, we must have that $v, \widetilde{v} \in T_xM$ for some

$x \in M$. Moreover, by definition we have that $\omega_x(v - \widetilde{v}, w) = 0$ for every $w \in T_x M$. By nondegeneracy, it follows that $v = \widetilde{v}$. Moreover, since $T_x M$ is finite-dimensional, we get that $\Omega$ is also surjective, thus bijective. Since any bijective smooth bundle homomorphism over $M$ is automatically a smooth bundle isomorphism by [Lee13, p. 262], $\Omega$ is a smooth bundle isomorphism. □

**Remark 2.16.** In what follows, we will denote both the smooth bundle isomorphism $\Omega : TM \to T^*M$ as well as the induced $C^\infty(M)$-linear morphism $\Omega : \mathfrak{X}(M) \to \mathfrak{X}^*(M)$ by the same letter $\Omega$. However, as a subtle distinction between those two maps, we will write $\Omega_X$ for the evaluation of the latter at some $X \in \mathfrak{X}(M)$.

## Hamiltonian Systems

If the Legendre transform 1.34 is a diffeomorphism, we can define an associated Hamiltonian function by 1.39, that is a smooth function $H$ on $T^*M$, where $M$ is a smooth manifold. By example 2.2, we know that the cotangent bundle $T^*M$ admits a canonical symplectic structure in terms of the tautological form 1.31. The tuple $(T^*M, H)$ turns out to be the prototype of a much more general structure.

**Definition 2.17 (Hamiltonian System).** *A* ***Hamiltonian system*** *is defined to be a tuple* $\big((M, \omega), H\big)$ *consisting of a symplectic manifold* $(M, \omega)$*, called a* ***phase space****, and a function* $H \in C^\infty(M)$*, called a* ***Hamiltonian function****.*

**Remark 2.18.** In what follows, we will write simply $(M, \omega, H)$ for a Hamiltonian system instead of the more cumbersome $\big((M, \omega), H\big)$. The latter was choosen in the definition to emphasize the similarity to the definition of a Lagrangian system 1.7.

**Hamiltonian Vector Fields.** As in Riemannian geometry, a main advantage of the symplectic structure is to reinstate the definition of the gradient of a smooth function as a vector field instead of a covector field using the tangent-cotangent bundle isomorphism (for the Riemannian case see [Lee13, pp. 342–343]).

**Definition 2.19 (Hamiltonian Vector Field).** *Let* $(M, \omega, H)$ *be a Hamiltonian system and denote by* $\Omega : \mathfrak{X}(M) \to \mathfrak{X}^*(M)$ *the tangent-cotangent bundle isomorphism from proposition 2.15. The vector field* $X_H$ *defined by*

$$X_H := \Omega^{-1}(dH) \tag{6}$$

*is called the* ***Hamiltonian vector field associated to the Hamiltonian system****.*

**Lemma 2.20.** *Let* $(M, \omega, H)$ *be a Hamiltonian system. Then* $i_{X_H}\omega = dH$.

*Proof.* By definition of the Hamiltonian vector field (6) we have that $\Omega_{X_H} = dH$. Thus for any $x \in M$ and $v \in T_x M$ we compute

$$dH_x(v) = \big(\Omega_{X_H}\big)_x(v) = \Omega(X_H|_x)(v) = \omega_x\left(X_H|_x, v\right) = (i_{X_H})_x(v).$$

□

**Definition 2.21 (Invariance).** *Let $M$ be a smooth manifold, $X \in \mathfrak{X}(M)$ a complete vector field with global flow $\theta : \mathbb{R} \times M \to M$ and $l \in \mathbb{N}$. A tensor field $A \in \Gamma\left(T^{(0,l)}TM\right)$ is said to be* ***invariant under the flow $\theta$ of $X$****, iff*

$$\theta_t^* A = A$$

*for all $t \in \mathbb{R}$.*

A useful characterisation of invariance under flows can be given in terms of a special derivative. Recall, that in the setting of definition 2.21, the *Lie derivative of $A$ with respect to $X$*, written $\mathcal{L}_X A$, is defined to be the tensor field $\mathcal{L}_X A \in \Gamma\left(T^{(0,l)}TM\right)$ given by

$$(\mathcal{L}_X A)_x := \left.\frac{d}{dt}\right|_{t=0} (\theta_t^* A)_x$$

for all $x \in M$. By [Lee13, p. 324], we have that $A$ is invariant under the flow of $X$ if and only if $\mathcal{L}_X A = 0$. The next proposition is a prime example why we require a symplectic structure to be both closed and nondegenerate. For the proof, we need one more preliminary result from the calculus of differential forms.

**Proposition 2.22 (Cartan's Magic Formula [Lee13, p. 372]).** *Let $M$ be a smooth manifold, $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^l(M)$ for some $l \in \mathbb{N}$. Then*

$$\mathcal{L}_X \omega = i_X(d\omega) + d(i_X \omega).$$

**Proposition 2.23.** *Let $(M, \omega, H)$ be a Hamiltonian system such that the Hamiltonian vector field is complete. Then the symplectic form is invariant under the flow of the Hamiltonian vector field.*

*Proof.* By the previous discussion it is enough to show that $\mathcal{L}_{X_H}\omega = 0$. Using Cartan's magic formula 2.22, closedness of $\omega$ together with lemma 2.20 we compute

$$\mathcal{L}_{X_H}\omega = i_{X_H}(d\omega) + d(i_{X_H}\omega) = d(i_{X_H}\omega) = (d \circ d)H = 0.$$

□

**Poisson Brackets.**

**Definition 2.24 (Poisson Bracket).** *Let $(M, \omega)$ be a symplectic manifold. Define a mapping*

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

*by*

$$\{f, g\} := \omega(X_f, X_g)$$

*where $X_f$ and $X_g$ are Hamiltonian vector fields associated to the Hamiltonian systems $(M, \omega, f)$ and $(M, \omega, g)$, respectively. The mapping $\{\cdot,\cdot\}$ is called the* ***Poisson bracket on $C^\infty(M)$****.*

Recall, that if $f \in C^\infty(M)$ for a smooth manifold $M$, the *differential of $f$* is defined to be the covector field given by $df_x(v) := vf$ for $x \in M$ and $v \in T_xM$. This is indeed a smooth covector field by part (d) of the smoothness criteria for tensor fields 2.12 since

$$df(X)(x) = df_x(X|_x) = X|_x f = (Xf)(x) \tag{7}$$

for any $X \in \mathfrak{X}(M)$ and $x \in M$, and $Xf$ is smooth by [Lee13, p. 180] (proving this is analogous to the proof of the smoothness criteria for tensor fields 2.12).

**Lemma 2.25.** *Let $(M, \omega)$ be a symplectic manifold. Then $\{f, g\} = X_g f$ holds for all $f, g \in C^\infty(M)$.*

*Proof.* Using lemma 2.20 and equation (7), we compute

$$\{f, g\} = \omega(X_f, X_g) = (i_{X_f}\omega)(X_g) = df(X_g) = X_g f.$$

□

**Definition 2.26 (Integral of Motion).** *Let $(M, \omega, H)$ be a Hamiltonian system. A function $f \in C^\infty(M)$ is said to be an **integral of motion for the Hamiltonian system $(M, \omega, H)$**, iff $\{H, f\} = 0$.*

**Lie Group Actions and Noether's Theorem.** Let us recall some basic facts from the theory of Lie groups and Lie algebras. A *Lie group* is defined to be a group $(G, \cdot)$, such that $G$ is a smooth manifold and the multiplication $\cdot$ as well as the inversion map $\cdot^{-1} : G \to G$ defined by $g \mapsto g^{-1}$ are smooth. If $G$ is a Lie group, we can associate to $G$ its *Lie algebra* $\mathfrak{g}$ defined to be $\mathfrak{g} := T_eG$, where $e$ denotes the neutral element of $G$. It can be shown that $\mathfrak{g} \cong \mathfrak{X}_L(G)$ as real vector spaces, where $\mathfrak{X}_L(G) \subseteq \mathfrak{X}(G)$ denotes the space of *left invariant vector fields on $G$*, that is, the vector fields $X \in \mathfrak{X}(G)$ satisfying $(L_g)_*X = X$, where $L_g$ is the diffeomorphism $L_g : G \to G$ defined by $L_g(h) := gh$ and $(L_g)_*$ is the *pushforward of $X$* defined to be the vector field $\big((L_g)_*X\big)_h := d(L_g)_{g^{-1}h}X|_{g^{-1}h}$ for $h \in G$. Most importantly, any left invariant vector field on $G$ is complete and so we can define the *exponential map* $\exp : \mathfrak{g} \to G$ by

$$\exp v := \gamma(1),$$

where $\gamma \in C^\infty(\mathbb{R}, G)$ is the integral curve of the *left invariant vector field $X_v$ associated to $v$ on $G$*, that is $X_v|_g := d(L_g)_e(v)$, with starting point $\gamma(0) = e$. Then we have that $\gamma(t) = \exp tv$ and $(\exp tv)^{-1} = \exp(-tv)$ for all $v \in \mathfrak{g}$ and $t \in \mathbb{R}$.

The most important applications of Lie groups to smooth manifold theory involve actions by Lie groups on manifolds. Let $G$ be a Lie group and $M$ be a smooth manifold. A map in $C^\infty(G \times M, M)$ given by $(g, x) \mapsto g \cdot x$, is said to be a *left action of $G$ on $M$* iff

$$g \cdot (h \cdot x) = (gh) \cdot x \qquad \text{and} \qquad e \cdot x = x$$

holds for all $g, h \in G$ and $x \in M$. Similarly, a *right action of $G$ on $M$* is defined to be a map in $C^\infty(M \times G, M)$ given by $(x, g) \mapsto x \cdot g$ satisfying

$$(x \cdot g) \cdot h = x \cdot (gh) \qquad \text{and} \qquad x \cdot e = x$$

for all $g, h \in G$ and $x \in M$. Note that any left action of $G$ on $M$ can be transformed into a right action of $G$ on $M$ by defining $x \cdot g := g^{-1} \cdot x$ for all $g \in G$ and $x \in M$, and similarly every right action of $G$ on $M$ can be transformed into a left action of $G$ on $M$.

Suppose we are given a right action of a Lie group $G$ on a smooth manifold $M$. Then each element $v \in \mathfrak{g}$ determines a global flow on $M$ by

$$(t, x) \mapsto x \cdot \exp tv.$$

Define $\widehat{v} \in \mathfrak{X}(M)$ by

$$\widehat{v}_x := \frac{d}{dt}\bigg|_{t=0} x \cdot \exp tv$$

for all $x \in M$. This is the *infinitesimal generator* associated to the above flow (see [Lee13, p. 210]). Hence we get a map $\mathfrak{g} \to \mathfrak{X}(M)$ defined by $v \mapsto \widehat{v}$. By [Lee13, p. 526], this map is actually a *Lie algebra homomorphism*. This is the main reason we are working with right actions rather than left actions.

**Lemma 2.27 (Computing the Differential Using a Velocity Vector [Lee13, p. 70]).** *Let $F \in C^\infty(M, N)$ for two smooth manifolds $M$ and $N$, $x \in M$ and $v \in T_xM$. Then*

$$dF_x(v) = (F \circ \gamma)'(0)$$

*for any path $\gamma \in C^\infty(J, M)$, where $J \subseteq \mathbb{R}$ is an interval such that $0 \in J$, $\gamma(0) = x$ and $\gamma'(0) = v$.*

**Proposition 2.28.** *Suppose we are given a right action of a Lie group $G$ on a smooth manifold $M$. Then for each $v \in \mathfrak{g}$, the infinitesimal generator $\widehat{v}$ associated to the flow generated by $v$ satisfies*

$$(\widehat{v} f)(x) = \frac{d}{dt}\bigg|_{t=0} f(x \cdot \exp tv)$$

*for all $x \in M$ and $f \in C^\infty(M)$.*

*Proof.* Let $x \in M$ and denote by $\theta : M \times G \to M$ the right action of $G$ on $M$. Define $\theta^x : G \to M$ by $\theta^x(g) := x \cdot g$. Then $\theta^x$ is smooth since

$$\theta^x\colon \quad G \cong \{x\} \times G \hookrightarrow M \times G \xrightarrow{\theta} M$$

where the first two maps steem from [Lee13, p. 100]. Set $\gamma(t) := \exp tv$ for all $t \in \mathbb{R}$. Then it is immediate, that

$$x \cdot \exp tv = \theta^x\left(\gamma(t)\right).$$

Thus we compute

$$\begin{aligned}
(\widehat{v} f)(x) &= \widehat{v}_x f \\
&= \frac{d}{dt}\bigg|_{t=0} \theta^x\left(\gamma(t)\right) f \\
&= d(\theta^x)_e(v) f && \text{(by lemma 2.27)}
\end{aligned}$$

$$
\begin{aligned}
&= v(f \circ \theta^x) && \text{(by definition of } d\theta^x\text{)}\\
&= d(f \circ \theta^x)_e(v) && \text{(by definition of } d(f \circ \theta^x)\text{)}\\
&= (f \circ \theta^x \circ \gamma)'(0) && \text{(by lemma 2.27)}\\
&= \left.\frac{d}{dt}\right|_{t=0} f(x \cdot \exp tv).
\end{aligned}
$$

□

**Remark 2.29.** From now on, we will consider left actions of Lie groups $G$ on smooth manifolds $M$ only instead of right actions, since they are more common. This is however no drawback, since any left action can be converted into a right action. Hence if $v \in \mathfrak{g}$, the corresponding infinitesimal generator $V$ is given by

$$\widehat{v}_x = \left.\frac{d}{dt}\right|_{t=0} \exp(-tv) \cdot x.$$

Let $\theta : G \times M \to M$ be a left action of a Lie group $G$ on a symplectic manifold $(M, \omega)$. We say that ***G* acts on *M* by symplectomorphisms**, iff for all $g \in G$, the map $\theta_g : M \to M$ defined by $\theta_g(x) := g \cdot x$ is a symplectomorphism. We adapt the terminology provided in [MS17, p. 203].

**Definition 2.30 (Weakly Hamiltonian Action and Hamiltonian Action).** *A left action of a Lie group G on a symplectic manifold $(M, \omega)$ by symplectomorphisms is said to be a **Hamiltonian action of G on $(M, \omega)$**, iff for each $v \in \mathfrak{g}$, there exists a Hamiltonian system $(M, \omega, H_v)$, such that $X_{H_v} = \widehat{v}$. If additionally the induced mapping $\mathfrak{g} \to C^\infty(M)$ defined by $v \mapsto H_v$*

**Definition 2.31 (Symmetry Group).** *A Lie group G is said to be a **symmetry group of a Hamiltonian system $(M, \omega, H)$**, iff there exists a weakly Hamiltonian action of G on $(M, \omega)$, such that*

$$H(g \cdot x) = H(x)$$

*holds for all $g \in G$ and $x \in M$.*

**Theorem 2.32 (Noether's Theorem, Hamiltonian Version).** *Let G be a symmetry group of a Hamiltonian system $(M, \omega, H)$. Then for each $v \in \mathfrak{g}$, the function $H_v \in C^\infty(M)$ such that $X_{H_v} = \widehat{v}$ is an integral of motion.*

*Proof.* Let $x \in M$. We compute

$$
\begin{aligned}
\{H, H_v\}(x) &= \big(X_{H_v} H\big)(x) && \text{(by lemma 2.25)}\\
&= (\widehat{v} H)(x)\\
&= \left.\frac{d}{dt}\right|_{t=0} H\big(\exp(-tv) \cdot x\big) && \text{(by proposition 2.28)}
\end{aligned}
$$

$$= \frac{d}{dt}\bigg|_{t=0} H(x)$$
$$= 0.$$

□

APPENDIX A

# Basic Category Theory

## Categories

**Definition A.1 (Category).** *A **category** $\mathcal{C}$ consists of*

- *A class* $\mathrm{ob}(\mathcal{C})$*, called the **objects of** $\mathcal{C}$.*
- *A class* $\mathrm{mor}(\mathcal{C})$*, called the **morphisms of** $\mathcal{C}$.*
- *Two functions* $\mathrm{dom} : \mathrm{mor}(\mathcal{C}) \to \mathrm{ob}(\mathcal{C})$ *and* $\mathrm{cod} : \mathrm{mor}(\mathcal{C}) \to \mathrm{ob}(\mathcal{C})$*, which assign to each morphism $f$ in $\mathcal{C}$ its **domain** and **codomain**, respectively.*
- *For each $X \in \mathrm{ob}(\mathcal{C})$ a function* $\mathrm{ob}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C})$ *which assigns a morphism* $\mathrm{id}_X$ *such that* $\mathrm{dom}\,\mathrm{id}_X = \mathrm{cod}\,\mathrm{id}_X = X$.
- *A function*
$$\circ : \{(g, f) \in \mathrm{mor}(\mathcal{C}) \times \mathrm{mor}(\mathcal{C}) : \mathrm{dom}\, g = \mathrm{cod}\, f\} \to \mathrm{mor}(\mathcal{C}) \tag{A.1}$$
*mapping $(g, f)$ to $g \circ f$, called **composition**, such that* $\mathrm{dom}(g \circ f) = \mathrm{dom}\, f$ *and* $\mathrm{cod}(g \circ f) = \mathrm{cod}\, g$.

*Subject to the following axioms:*

- ***(Associativity Axiom)*** *For all $f, g, h \in \mathrm{mor}(\mathcal{C})$ with* $\mathrm{dom}\, h = \mathrm{cod}\, g$ *and* $\mathrm{dom}\, g = \mathrm{cod}\, f$*, we have that*
$$(h \circ g) \circ f = h \circ (g \circ f). \tag{A.2}$$
- ***(Unit Axiom)*** *For all $f \in \mathrm{mor}(\mathcal{C})$ with* $\mathrm{dom}\, f = X$ *and* $\mathrm{cod}\, f = Y$ *we have that*
$$f = f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f. \tag{A.3}$$

**Remark A.2.** Let $\mathcal{C}$ be a category. For $X, Y \in \mathrm{ob}(\mathcal{C})$ we will abreviate
$$\mathcal{C}(X, Y) := \{f \in \mathrm{mor}(\mathcal{C}) : \mathrm{dom}\, f = X \text{ and } \mathrm{cod}\, f = Y\}.$$
Moreover, $f \in \mathcal{C}(X, Y)$ is depicted as
$$f : X \to Y. \tag{A.4}$$

**Example A.3.** Let $*$ be a single, not nearer specified object. Consider as morphisms the class of all cardinal numbers and as composition cardinal addition. By [**halbeisen:set_theory:2012**], cardinal addition is associative and $\varnothing$ serves for the identity $\mathrm{id}_*$.

**Definition A.4 (Locally Small, Hom-Set).** *A category $\mathcal{C}$ is said to be **locally small** if for all $X, Y \in \mathcal{C}$, $\mathcal{C}(X, Y)$ is a set. If $\mathcal{C}$ is locally small, $\mathcal{C}(X, Y)$ is called a **hom-set** for all $X, Y \in \mathcal{C}$.*

**Definition A.5 (Monic).** *Let $\mathcal{C}$ be a category. A morphism $f \in \mathcal{C}(X, Y)$ is said to be* ***monic****, iff for all objects $A \in \mathcal{C}$ and morphisms $g, h \in \mathcal{C}(A, X)$*

$$f \circ g = f \circ h \Rightarrow g = h$$

*holds.*

**Exercise A.6.** In Set, show that a morphism is monic if and only if it is injective.

**Definition A.7 (Epic).** *Let $\mathcal{C}$ be a category. A morphism $f \in \mathcal{C}(X, Y)$ is said to be* ***epic****, iff $f$ is monic in $\mathcal{C}^{\mathrm{op}}$.*

**Exercise A.8.** In Set, show that a morphism is epic if and only if it is surjective.

**Definition A.9 (Isomorphism).** *Let $\mathcal{C}$ be a category. An* ***isomorphism in $\mathcal{C}$*** *is a morphism $f \in \mathcal{C}(X, Y)$, such that there exists a morphism $g \in \mathcal{C}(Y, X)$ with*

$$g \circ f = \mathrm{id}_X \qquad \textit{and} \qquad f \circ g = \mathrm{id}_Y \,.$$

**Exercise A.10.** Let $\mathcal{C}$ be a category. Show that any isomorphism is both monic and epic.

**Exercise A.11.** In Set, show that any monic and epic morphism is an isomorphism.

In the definition of an isomorphism A.9, a morphism is forced to admit a two-sided inverse. However, in reality, often only one-sided inverses do exist. Since they are particularly useful, they get they own terminology.

**Definition A.12 (Section).** *Let $\mathcal{C}$ be a category and $f \in \mathcal{C}(X, Y)$. A morphism $\sigma \in \mathcal{C}(Y, X)$ is called a* ***section of $f$****, iff $f \circ \sigma = \mathrm{id}_Y$.*

**Exercise A.13.** Let $\mathcal{C}$ be a category. Show that any morphism admitting a section is epic.

**Exercise A.14.** In Set, show that any epic morphism admits a section (observe the subtle use of the axiom of choice!).

**Definition A.15 (Retraction).** *Let $\mathcal{C}$ be a category and $f \in \mathcal{C}(X, Y)$. A morphism $\rho \in \mathcal{C}(Y, X)$ is called a* ***retraction of $f$****, iff $\rho \circ f = \mathrm{id}_X$.*

**Exercise A.16.** Let $\mathcal{C}$ be a category. Show that any morphism admitting a retraction is monic.

In algebraic topology, there is a very useful construction on categories.

**Definition A.17 (Congruence).** *Let $\mathcal{C}$ be a category. A* ***congruence on $\mathcal{C}$*** *is an equivalence relation $\sim$ on $\mathrm{mor}(\mathcal{C})$ such that*

*(a) If $f \in \mathcal{C}(X, Y)$ and $f \sim g$, then $g \in \mathcal{C}(X, Y)$.*
*(b) If $f_0 : X \to Y$ and $g_0 : Y \to Z$ such that $f_0 \sim f_1$ and $g_0 \sim g_1$, then $g_0 \circ f_0 \sim g_1 \circ f_1$.*

**Exercise A.18.** Let $\mathcal{C}$ be a category. Show that for any congruence on $\mathcal{C}$, there exists a category $\mathcal{C}'$, called ***quotient category***, with $\mathrm{ob}(\mathcal{C}') = \mathrm{ob}(\mathcal{C})$, for any objects $X, Y \in \mathcal{C}'$

$$\mathcal{C}'(X, Y) = \{[f] : f \in \mathcal{C}(X, Y)\} \,,$$

and pointwise composition.

## Functors

**Definition A.19 (Functor).** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A* ***functor*** $\boldsymbol{F : \mathcal{C} \to \mathcal{D}}$ *is a pair of functions $(F_1, F_2)$, $F_1 : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, called the* ***object function*** *and $F_2 : \mathrm{mor}(\mathcal{C}) \to \mathrm{mor}(\mathcal{D})$, called the* ***morphism function****, such that for every morphism $f : X \to Y$ we have that $F_2(f) : F_1(X) \to F_1(Y)$ and $(F_1, F_2)$ is subject to the following* ***compatibility conditions****:*

- *For all $X \in \mathrm{ob}(\mathcal{C})$, $F_2(\mathrm{id}_X) = \mathrm{id}_{F_1(X)}$.*
- *For all $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ we have that $F_2(g \circ f) = F_2(g) \circ F_2(f)$.*

**Remark A.20.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. It is convenient to denote the components $F_1$ and $F_2$ also with $F$.

APPENDIX B

# Basic Point-Set Topology

APPENDIX C

# Review of Algebraic Topology

APPENDIX D

# Review of Analysis

## Differentiability

**Definition D.1 (Carathéodory Differentiability).** *Let $(V, |\cdot|_V)$ and $(W, |\cdot|_W)$ be finite-dimensional vector spaces, $U \subseteq V$ open and $x_0 \in U$. A map $F : U \to W$ is said to be* ***differentiable at $x_0$****, iff there exists a map $\varphi : U \to \mathrm{L}(V, W)$ such that $\varphi$ is continuous at $x_0$ and*

$$F(x) - F(x_0) = \varphi(x)(x - x_0) \tag{D.1}$$

*holds for all $x \in U$.*

**Example D.2 (Linear Map).** Let $(V, |\cdot|_V)$ and $(W, |\cdot|_W)$ be finite-dimensional vector spaces and $L \in \mathrm{L}(V, W)$. Then $L$ is differentiable at every $x_0 \in V$ since

$$L(x) - L(x_0) = L(x - x_0) = \varphi(x)(x - x_0)$$

holds, where $\varphi : V \to \mathrm{L}(V, W)$ is given by $\varphi(x) := L$.

**Proposition D.3.** *Let $(V, |\cdot|_V)$ and $(W, |\cdot|_W)$ be finite-dimensional vector spaces, $U \subseteq V$ open and $x_0 \in U$. Suppose $\varphi, \psi : U \to \mathrm{L}(V, W)$ are continuous at $x_0$ such that*

$$F(x) - F(x_0) = \varphi(x)(x - x_0) \qquad \textit{and} \qquad F(x) - F(x_0) = \psi(x)(x - x_0)$$

*holds for all $x \in U$. Then $\varphi(x_0) = \psi(x_0)$.*

*Proof.* Definiere $g : U \to M_{mn}(\mathbb{R})$ durch

$$g(x) := \varphi(x) - \psi(x).$$

Dann gilt fÃijr alle $x \in U$

$$g(x)(x - a) = \varphi(x)(x - a) - \psi(x)(x - a) = 0.$$

Somit folgt

$$\|g(a)(x - a)\| = \|\big(g(a) - g(x)\big)(x - a)\| \leq \|g(a) - g(x)\|_{\mathrm{op}} \|x - a\|$$

fÃijr alle $x \in U$ oder Ãďquivalent

$$\left\| g(a) \frac{x - a}{\|x - a\|} \right\| \leq \|g(a) - g(x)\|_{\mathrm{op}} . \tag{D.2}$$

Sei $\varepsilon > 0$. Da $g$ stetig ist in $a$, existiert $r > \delta > 0$, wobei $B_r(a) \subseteq U$, sodass fÃijr alle $x \in \dot{B}_\delta(a)$

$$\|g(a) - g(x)\|_{\mathrm{op}} < \varepsilon$$

gilt. Weiter ist

$$\left\{\|g(a)(x-a)/\,\|x-a\|\| : x \in \dot{B}_\delta(a)\right\} = \{\|g(a)x\| : \|x\| = 1\}.$$

In der Tat, die Inklusion $\subseteq$ ist klar. Angenommen, $\|y\| = 1$. Definiere $x := a + \frac{\delta}{2}y$. Dann gilt

$$\|x - a\| \leq \frac{\delta}{2}\|y\| < \delta$$

und somit $x \in \dot{B}_\delta(a)$. Weiter ist auch

$$g(a)\frac{x-a}{\|x-a\|} = g(a)\frac{\frac{\delta}{2}y}{\frac{\delta}{2}\|y\|} = g(a)y.$$

Daher folgt aus (D.2)

$$\|g(a)\|_{\mathrm{op}} = \sup_{\|x\|=1}\|g(a)x\| = \sup_{x\in\dot{B}_\delta(a)}\left\|g(a)\frac{x-a}{\|x-a\|}\right\| < \varepsilon$$

Da $\varepsilon > 0$ beliebig war, folgt $\|g(a)\|_{\mathrm{op}} = 0$ und somit $g(a) = 0$. Dies impliziert insbesondere $\varphi(a) = \psi(a)$. □

**Definition D.4 (Differential).** *Let $(V, |\cdot|_V)$ and $(W, |\cdot|_W)$ be finite-dimensional vector spaces, $U \subseteq V$ open and $x_0 \in U$. If $F$ is differentiable at $x_0$, define the* ***differential of $F$ at $x_0$****, written $DF_{x_0}$, by*

$$DF_{x_0} := \varphi(x_0)$$

*where $\varphi$ is as in D.1.*

**Lemma D.5.** *Let $U \subseteq \mathbb{R}$ open, $f : U \to \mathbb{R}^n$ and $x_0 \in U$. Then $f$ is differentiable at $x_0$ if and only if*

$$\lim_{x\to x_0, x\in U}\frac{f(x)-f(x_0)}{x-x_0} \in \mathbb{R}. \tag{D.3}$$

*Proof.*

□

**Definition D.6 (Derivative).** *Let $U \subseteq \mathbb{R}$ open and $f : U \to \mathbb{R}^n$ differentiable at $x_0 \in U$. Then the* ***derivative of $f$ at $x_0$****, written $f'(x_0)$, is defined by*

$$f'(x_0) := \lim_{x\to x_0, x\in U}\frac{f(x)-f(x_0)}{x-x_0}.$$

**Definition D.7 (Directional Derivative).** *Let $U \subseteq \mathbb{R}^n$ be open, $F : U \to \mathbb{R}^m$ and $v \in \mathbb{R}^n$. Define the* ***directional derivative of $F$ in direction $v$ at $x_0$****, written $D_v F_{x_0}$, by*

$$D_v F_{x_0} := \lim_{t \to 0, t \in \mathbb{R}} \frac{F(x_0 + tv) - F(x_0)}{t}$$

**Definition D.8 (Partial Derivative).** *Let $U \subseteq \mathbb{R}^n$ open, $F : U \to \mathbb{R}^m$ and $x_0 \in U$. If $F$ is differentiable at $x_0$, then define the* ***$i$-th partial derivative of $F$ at $x_0$****, written $D_i F(x_0)$, by*

$$D_i F(x_0) := D_{e_i} F_{x_0},$$

*where $(e_i)$ denotes the standard basis of $\mathbb{R}^n$.*

**Proposition D.9.** *Let $U \subseteq \mathbb{R}^n$ open, $F : U \to \mathbb{R}^m$ and $x_0 \in U$. If $F$ is differentiable at $x_0$, then*

$$DF_{x_0}(v) = D_v F_{x_0}$$

*for all $v \in \mathbb{R}^n$.*

*Proof.* Consider the composition

$$t \xmapsto{f} x_0 + tv \xmapsto{F} F(x_0 + tv).$$

Then we compute

$$D_v F_{x_0} = (F \circ f)'(0) = D(F \circ f)_0 = DF_{x_0} \circ Df_0 = DF_{x_0} \circ f'(0) = DF_{x_0}(v).$$

□

## The Inverse Function Theorem

**Theorem D.10 (The Inverse Function Theorem).** *Let $U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ smooth and $x \in U$. If $Df_x$ is invertible, then there exists a neighbourhood $V \subseteq U$ of $x$ such that $f : V \to f(V)$ is a diffeomorphism.*

## The Implicit Function Theorem

**Theorem D.11 (The Implicit Function Theorem).** *Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$ be open, $\Phi : U \to \mathbb{R}^k$ smooth, $(x_0, y_0) \in U$ and $c := \Phi(x_0, y_0)$. If*

$$\det \left( D_j \Phi^i_{(x_0, y_0)} \right)^i_{j=n+1,\dots,n+k} \neq 0,$$

*then there exist neighbourhoods $V_0 \subseteq \mathbb{R}^n$ of $x_0$ and $W_0 \subseteq \mathbb{R}^k$ of $y_0$ and a smooth function $F : V_0 \to W_0$ such that $\Phi^{-1}(c) \cap (V_0 \times W_0)$ is the graph of $F$.*

APPENDIX E

# Review of Differential Topology

We follow the treatment as provided by *Will J. Merry* in the year course *Differential Geometry I and II* at the *ETH Zurich* in the autumn semester 2018 and spring semester 2019, respectively. The course notes are available at

https://www.merry.io/differential-geometry/.

Additionally, we rely on [Lee13].

### The Category of Smooth Manifolds

**Definition E.1 (Topological Manifold).** *Let $n \in \mathbb{N}$. A topological space $M$ is said to be a* ***topological manifold of dimension n****, iff*

(i) *$M$ is locally Euclidean of dimension $n$, that is, for every $x \in M$ there exist an open subset $U \subseteq M$ and a function $\varphi : U \to \mathbb{R}^n$ such that $\varphi(U) \subseteq \mathbb{R}^n$ is open and $\varphi : U \to \varphi(U)$ is a homeomorphism. Every such pair $(U, \varphi)$ is called a* ***chart on M about x****.*

(ii) *$M$ is Hausdorff and has at most countably many connected components.*

(iii) *$M$ is paracompact, that is, every open cover of $M$ admits a locally finite open refinement.*

**Definition E.2 (Smooth Atlas).** *A* ***smooth atlas for a topological manifold M*** *is a collection $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ of charts on $M$ such that*

(i) *$(U_\alpha)_{\alpha \in A}$ is an open cover for $M$.*

(ii) *For all $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$, the function*

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

*is smooth. The function $\varphi_\alpha \circ \varphi_\beta^{-1}$ is called a* ***transition function****.*

Let $\mathcal{A}$ and $\mathcal{A}'$ be two smooth atlases on a topological manifold $M$. Define a relation on the set of all smooth atlases on $M$ (this is a subset of the power set $2^M$) by

$$\mathcal{A} \sim \mathcal{A}' \quad :\Leftrightarrow \quad \mathcal{A} \cup \mathcal{A}' \text{ is an atlas for } M.$$

**Exercise E.3.** Show that above relation is actually an equivalence relation on the set of all smooth atlases on a topological manifold $M$.

**Definition E.4 (Smooth Structure).** *A* ***smooth structure on a topological manifold M*** *is an equivalence class $[\mathcal{A}]$ where $\mathcal{A}$ is a smooth atlas for $M$.*

**Definition E.5 (Maximal Smooth Atlas).** *Let $[\mathcal{A}]$ be a smooth structure on a topological manifold $M$. Define the* ***maximal smooth atlas on $M$*** *by $\bigcup_{\mathcal{A}' \in [\mathcal{A}]} \mathcal{A}'$.*

**Definition E.6 (Smooth Manifold).** *Let $n \in \mathbb{N}$. A* ***smooth manifold of dimension $n$*** *is defined to be a pair $(M, \mathcal{A})$, where $M$ is a topological manifold of dimension n and $\mathcal{A}$ is a maximal smooth atlas on $M$.*

**Example E.7 ($n$-Spheres).** Let $n \in \mathbb{N}$. If $n = 0$, we have that $\mathbb{S}^0 = \{\pm 1\}$. It is easily seen that $\mathbb{S}^0$ is a smooth manifold of dimension 0. Let $n \geq 1$. Define $N := e_{n+1}$ and $S := -e_{n+1}$, where $e_{n+1}$ denotes the $n+1$-th standard basis vector of $\mathbb{R}^{n+1}$. Moreover, set

$$U_+ := \mathbb{S}^n \setminus S \qquad \text{and} \qquad U_- := \mathbb{S}^n \setminus N.$$

Then $U_+$ and $U_-$ are open subsets of $\mathbb{S}^n$, the upper and lower hemisphere, respectively. Then the functions $\varphi_\pm : U_\pm \to \mathbb{R}^n$ defined by

$$\varphi_\pm(x) := \frac{1}{1 \pm x_{n+1}}(x_1, \ldots, x_n),$$

are homeomorphisms. Indeed, one can check that $\psi_\pm : \mathbb{R}^n \to U_\pm$ defined by

$$\psi_\pm(x) := \left( \frac{2x}{1+|x|^2}, \frac{\pm(1-|x|^2)}{1+|x|^2} \right)$$

is a continuous inverse for $\varphi_+$ and $\varphi_-$, respectively. We claim that $\{(U_\pm, \varphi_\pm)\}$ is a smooth atlas for $\mathbb{S}^n$. Clearly, $\mathbb{S}^n$ is covered by the two charts. Next we have to calculate the transition functions $\varphi_\mp \circ \varphi_\pm^{-1} = \varphi_\mp \circ \psi_\pm : \varphi_\pm(U_+ \cap U_-) \to \varphi_\mp(U_+ \cap U_-)$. It is easy to see that $\varphi_\pm(U_+ \cap U_-) = \mathbb{R}^n \setminus \{0\}$ and that

$$\varphi_\mp \circ \psi_\pm = \frac{x}{|x|^2},$$

which is smooth. Since $\mathbb{S}^n$ is Hausdorff as a metric space and as a subspace of a second countable space, itself second countable, $\mathbb{S}^n$ equipped with the smooth structure induced by the smooth atlas constructed above, is a smooth manifold of dimension $n$.

**Proposition E.8 (Smooth Manifold Chart Lemma).** *Let $M$ be a set and suppose $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ is a family of subsets $U_\alpha \subseteq M$ and maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, for some fixed $n \in \mathbb{N}$, such that:*

(i) *For all $\alpha \in A$, $\varphi_\alpha(U_\alpha)$ is open and $\varphi : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a bijection.*
(ii) *For all $\alpha, \beta \in A$, $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$.*
(iii) *If $U_\alpha \cap U_\beta \neq \emptyset$, then $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is smooth.*
(iv) *Countably many of the sets $U_\alpha$ cover $M$.*
(v) *If $x, y \in M$ such that $x \neq y$, there either exists some $\alpha \in A$ such that $x, y \in U_\alpha$ or there exists $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta = \emptyset$, $x \in U_\alpha$ and $y \in U_\beta$.*

*Then $M$ admits a unique smooth structure containing the atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$.*

**Definition E.9 (Smooth Map).** *Let $M$ and $N$ be smooth manifolds and $F : M \to N$ a map. We say that $F$ is* ***smooth****, iff for all $x \in M$, there exists a chart $(U, \varphi)$ on $M$ about $x$ and a chart $(V, \psi)$ on $N$ about $F(x)$ such that*

(i) *$U \cap F^{-1}(V)$ is open in $M$.*
(ii) *$\psi \circ F \circ \psi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \to \psi(V)$ is smooth.*

*The* ***set of all smooth maps from $M$ to $N$*** *is denoted by $C^\infty(M, N)$ and the* ***set of all smooth functions on $M$*** *is denoted by $C^\infty(M)$.*

**Exercise E.10.** Let $M$ be a smooth manifold. Show that $C^\infty(M)$ is an $\mathbb{R}$-algebra under pointwise defined operations.

**Example E.11 (Coordinate Functions).** Let $M^n$ be a smooth manifold and $(U, \varphi)$ be a chart about some $x \in M$. Let $\pi^i : \mathbb{R}^n \to \mathbb{R}$ be defined by $\pi^i(x^1, \dots, x^n) := x^i$ for $i = 1, \dots, n$. Define $x^i : U \to \mathbb{R}$ by $x^i := \pi^i \circ \varphi$. Then $x^i \in C^\infty(U)$ and we call $x^i$ a ***coordinate function***. Moreover, we may denote the chart $(U, \varphi)$ by $\left(U, (x^i)\right)$ and say that $(x^i)$ are ***local coordinates about $x$***.

## Tangent Spaces and the Differential

Let $M$ be a smooth manifold and let $x \in M$. Define a binary relation on the set

$$X := \left\{(U, f) : U \subseteq M \text{ neighbourhood of } x,\ f \in C^\infty(U)\right\}$$

by

$$(U, f) \sim (V, g) \quad :\Leftrightarrow \quad \exists W \subseteq U \cap V \text{ neighbourhood of } x \text{, such that } f|_W = g|_W.$$

**Exercise E.12.** Show that the above relation is actually an equivalence relation.

**Definition E.13 (Germ).** *Let $M$ be a smooth manifold and let $x \in M$. The set of* ***germs at $p$****, written $C_x^\infty(M)$ is defined to be $C_x^\infty(M) := X/\sim$.*

**Exercise E.14.** Show that $C_x^\infty(M)$ is an $\mathbb{R}$-algebra under the obvious operations.

**Remark E.15.** Note that if $f \in C^\infty(M)$, then $[(M, f)] \sim [(U, f|_U)]$ for any neighbourhood $U$ of $x$. Thus any germ at $p$ contains a representant which is defined on the whole manifold and we thus may simply write $[f]$ for a germ at $p$.

**Remark E.16.** Let $[f]$ be a germ at $x \in M$. Then $f(x)$ is well-defined. Indeed, if $f|_U = g|_U$ on some neighbourhood of $x$, then in particular $f(x) = g(x)$.

**Definition E.17 (Tangent Space).** *Let $M$ be a smooth manifold and let $x \in M$. The* ***tangent space of $M$ at $x$****, written $T_xM$, is defined to be the vector space $\left(C_x^\infty(M)\right)^*$ such that*

$$v([f]\,[g]) = v\,[f]\,g(x) + f(x)v\,[g]$$

*holds.*

**Lemma E.18.** *Let $M$ be a smooth manifold and $x \in M$. Suppose $\lambda \in C^\infty(M)$ is a constant function. Then $v[\lambda] = 0$ for all $v \in T_xM$.*

*Proof.* This immediately follows from

$$v[\lambda] = v[\lambda \cdot 1] = \lambda v[1] = \lambda v[1 \cdot 1] = 2\lambda v[1] = 2v[\lambda].$$

$\square$

**Definition E.19 (Derivation).** *Let $M$ be a smooth manifold, $x \in M$ and $U$ a neighbourhood of $x$. The* ***space of derivations of $C^\infty(U)$ at $x$****, written $\mathcal{D}_x(U)$, is defined to be the vector space $\left(C^\infty(U)\right)^*$ such that*

$$v(fg) = v(f)g(x) + f(x)v(g)$$

*holds.*

**Proposition E.20.** *Let $M$ be a a smooth manifold, $x \in M$ and $U$ be a neighbourhood of $x$. Then*

$$T_xM \cong \mathcal{D}_x(U).$$

*Proof.* Let $\Phi : T_xM \to \mathcal{D}_x(U)$ be defined by

$$\Phi(v)(f) := v[f]$$

for all $f \in C^\infty(U)$. Clearly $\Phi$ is well-defined and linear. We want to construct an inverse $\Psi : \mathcal{D}_x(U) \to T_xM$ for $\Phi$. This implies, that we should define

$$\Psi(v)[f] = v\left(\tilde{f}\right)$$

where $\tilde{f} \in C^\infty(U)$ such that $\left[\tilde{f}\right] = [f]$.

*Step 1: Existence of $\tilde{f}$.* Let $(V, f)$ be a representant of $[f]$. As in the proof of the smoothness criteria for tensor fields 2.12, we find a neighbourhood $W$ about $x$ such that $\overline{W} \subseteq U \cap V$. Then there exists a smooth bump function $\psi \in C^\infty(U \cap V)$ such that $\psi|_W = 1$ and $\operatorname{supp}\psi \subseteq U \cap V$. Let $\tilde{f} := \psi f$ extended to be zero on $U$. Then clearly $\left[\tilde{f}\right] = [f]$ since $\tilde{f} = f$ on $W$.

*Step 2: $\Psi$ is well-defined.* Suppose that $[f] = [g]$ in $C_x^\infty(M)$. Then $f = g$ on some neighbourhood $V$ of $x$. We claim that $v(f) = v(g)$ on $U \cap V$. Indeed, let $\psi$ be a smooth bump function for $\{x\}$ supported in $U \cap V$. Then $\psi(f-g) = 0$ on $U$ and we compute

$$0 = v\left(\psi(f-g)\right) = v(\psi)(f-g)(x) + \psi(x)v(f-g) = v(f-g).$$

$\square$

**Lemma E.21.** *Let $M$ be a smooth manifold and $U$ a neighbourhood of $x \in M$. Suppose $\lambda \in C^\infty(U)$ is a constant function. Then $v(\lambda) = 0$ for all $v \in \mathcal{D}_x(U)$.*

*Proof.* Using the notation of the proof of proposition E.20, lemma E.18 yields
$$v(\lambda) = (\Phi \circ \Psi)(v)(\lambda) = \Psi(v)[\lambda] = 0.$$
□

**Example E.22 (Coordinate Derivation).** Let $M^n$ be a smooth manifold and $(U, \varphi)$ be a chart on $M$. For every $x \in U$ and every $i = 1, \ldots, n$ define
$$\left.\frac{\partial}{\partial x^i}\right|_x : C^\infty(U) \to \mathbb{R}$$
by
$$\left.\frac{\partial}{\partial x^i}\right|_x (f) := D_i(f \circ \varphi^{-1})\left(\varphi(x)\right).$$
Then clearly $\left.\frac{\partial}{\partial x^i}\right|_x$ is a derivation of $C^\infty(U)$ at $x$. Thus by proposition E.20, $\left.\frac{\partial}{\partial x^i}\right|_x \in T_x M$.

One of the profound features of tangent spaces to a smooth manifold are that they are finite dimensional. In fact, they admit the same dimension as the manifold.

**Lemma E.23.** *Let $\Omega \subseteq \mathbb{R}^n$ be open and star-shaped about $x_0 \in \Omega$. Suppose $f \in C^\infty(\Omega)$. Then there exists $\varphi_1, \ldots, \varphi_n \in C^\infty(\Omega)$ such that $\varphi_i(x_0) = D_i f(x_0)$ and*
$$f(x) = f(x_0) + \pi^i(x - x_0)\varphi_i(x)$$
*holds for all $x \in \Omega$*

*Proof.* For $x \in \Omega$ define $\gamma_x : [0, 1] \to \Omega$ by $\gamma_x(t) := x_0 + t(x - x_0)$ (note that this is only possible since $\Omega$ is assumed to be star-shaped with centre $x_0$). Then
$$\begin{aligned} f(x) - f(x_0) &= \int_0^1 (f \circ \gamma_x)'(t)dt \\ &= \int_0^1 D_i f\left(\gamma_x(t)\right)\dot{\gamma}_x^i(t)dt \\ &= \int_0^1 D_i f\left(\gamma_x(t)\right)\pi^i(x - x_0)dt \\ &= \pi^i(x - x_0)\varphi_i(x) \end{aligned}$$
where
$$\varphi_i(x) := \int_0^1 D_i f\left(\gamma_x(t)\right)dt.$$
□

**Proposition E.24 (Basis for the Tangent Space).** *Let $M^n$ be a smooth manifold and $(U, \varphi)$ a chart on $M$. Then*
$$\left\{\left.\frac{\partial}{\partial x^i}\right|_x : i = 1, \ldots, n\right\}$$

*is a basis for $T_xM$ for all $x \in U$, where $x^i := \pi^i \circ \varphi$.*

*Proof.* Since $\varphi(U) \subseteq \mathbb{R}^n$ is open, there exists $\varepsilon > 0$ such that $B_\varepsilon\left(\varphi(x)\right) \subseteq \varphi(U)$. Set $V := \varphi^{-1}\left(B_\varepsilon\left(\varphi(x)\right)\right)$. Then $V$ is a neighbourhood of $x$ in $M$ and thus by proposition E.20, we have that $T_xM \cong \mathcal{D}_x(V)$. Let $f \in C^\infty(V)$. An application of lemma E.23 to $f \circ \varphi^{-1} \in C^\infty\left(B_\varepsilon\left(\varphi(x)\right)\right)$ yields

$$\begin{aligned}(f \circ \varphi^{-1})(y) &= f(x) + \pi^i\left(y - \varphi(x)\right)\varphi_i(y)\\ &= f(x) + \left(\pi^i(y) - x^i(x)\right)\varphi_i(y)\\ &= f(x) + \left((\pi^i \circ \varphi)\left(\varphi^{-1}(y)\right) - x^i(x)\right)(\varphi_i \circ \varphi)\left(\varphi^{-1}(y)\right).\end{aligned}$$

Thus

$$f = f(x) + \left(x^i - x^i(x)\right)(\varphi_i \circ \varphi)$$

on $V$. Using lemma E.21 we compute

$$v(f) = v\left(\left(x^i - x^i(x)\right)(\varphi_i \circ \varphi)\right) = v(x^i)\varphi_i\left(\varphi(x)\right) = v(x^i)\left.\frac{\partial}{\partial x^i}\right|_x(f). \tag{E.1}$$

Suppose that $\lambda^i \left.\frac{\partial}{\partial x^i}\right|_x = 0$. Then using example D.2 and proposition D.9 we compute

$$0 = \lambda^i \left.\frac{\partial}{\partial x^i}\right|_x(x^j) = \lambda^i D_i \pi^j\left(\varphi(x)\right) = \lambda^i \pi^j(e_i) = \lambda^i \delta_i^j = \lambda^j. \tag{E.2}$$

□

**Definition E.25 (Derivative).** *Let $M$ and $N$ be smooth manifolds and $F \in C^\infty(M, N)$. For $x \in M$, define a map $DF_x : T_xM \to T_{F(x)}N$ by*

$$DF_x(v)(f) := v(f \circ F)$$

*for all $f \in C^\infty(N)$. This map is called the **derivative of $F$ at $x$**.*

**Definition E.26 (Velocity of a Curve).** *Let $J \subseteq \mathbb{R}$ be an open interval and $\gamma \in C^\infty(J, M)$ be a curve in a smooth manifold $M$. For every $t \in J$, define the **velocity vector of $\gamma$ at $t$**, written $\gamma'(t)$, by*

$$\gamma'(t) := D\gamma_t\left(\left.\frac{d}{dt}\right|_t\right) \in T_{\gamma(t)}M.$$

It is immediate from the definition of the velocity vector of a curve E.26, that

$$\gamma'(t)(f) = D\gamma_t\left(\left.\frac{d}{dt}\right|_t\right)(f) = \left.\frac{d}{dt}\right|_t(f \circ \gamma) = (f \circ \gamma)'(t)$$

for all $f \in C^\infty(M)$. Moreover, if $(U, \varphi)$ is a chart on $M$, then equation E.1 yields

$$\gamma'(t) = \gamma'(t)(x^i)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} = (x^i \circ \gamma)'(t)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} = \dot{\gamma}^i(t)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} \tag{E.3}$$

at least sufficiently close to $t$. Velocity vectors to a curve give yet another way to think about the tangent space $T_xM$ to a point $x \in M$ of a smooth maifold $M$. Consider the set

$$X := \left\{\gamma \in C^\infty(J, M) : J \subseteq \mathbb{R} \text{ open interval with } 0 \in J, \gamma(0) = x\right\}.$$

Define a binary relation on $X$ as follows:

$$\gamma_1 \sim \gamma_2 \quad :\Leftrightarrow \quad \exists \text{ chart } (U, \varphi) \text{ about } x \text{ such that } (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

**Exercise E.27.** Show that the above relation is an equivalence relation.

Let $\mathcal{V}_xM := X/\sim$.

**Proposition E.28.** *Let $M$ be a smooth manifold and $x \in M$. Then $T_xM \cong \mathcal{V}_xM$ as sets.*

*Proof.* Define $\Phi : \mathcal{V}_xM \to T_xM$ by $\Phi[\gamma] := \gamma'(0)$. This map is well-defined. Indeed, if $[\gamma_1] = [\gamma_2]$, there exists a chart $(U, \varphi)$ about $x$ such that $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$. This immediately implies that $\dot\gamma_1^i(0) = \dot\gamma_2^i(0)$ for all $i = 1, \ldots, n$. Thus (E.3) yields $\gamma_1'(0) = \gamma_2'(0)$. From this also follows that $\Phi$ is injective. Indeed, if $\gamma_1'(0) = \gamma_2'(0)$, then $\dot\gamma_1^i(0) = \dot\gamma_2^i(0)$ for all $i = 1, \ldots, n$ by (E.3) and proposition E.24. Let $v \in T_xM$. Then in any chart $(U, \varphi)$ about $x$ we have that $v = v^i \frac{\partial}{\partial x^i}\big|_x$. Hence for $\varepsilon > 0$ sufficiently small we can define $\gamma_v : (-\varepsilon, \varepsilon) \to M^n$ by

$$\gamma_v(t) := \varphi^{-1}\left(tv^i, \ldots, tv^n\right).$$

Thus $\Phi$ is surjective. $\square$

We can equip $\mathcal{V}_xM$ with the structure of a vector space by means of the following lemma.

**Lemma E.29.** *Let $V$ be a a finite-dimensional real vector space and $S$ be a set. If there exists a bijection $\varphi : S \to V$, we can equip $V$ with a structure of a real vector space such that $\varphi$ is an isomorphism.*

*Proof.* Just define

$$\lambda x + y := \varphi^{-1}\left(\lambda\varphi(x) + \varphi(y)\right)$$

for all $x, y \in S$ and $\lambda \in \mathbb{R}$. $\square$

**Definition E.30 (Cotangent Space).** *Let $M$ be a smooth manifold. For $x \in M$, define the* ***cotangent space of $M$ at $x$****, written $T_x^*M$, to be*

$$T_x^*M := (T_xM)^*.$$

**Definition E.31 (Differential).** *Let $M$ be a smooth manifold, $U$ a neighbourhood of $x \in M$ and $f \in C^\infty(U)$. Define the* ***differential of $f$ at $x$****, written $df_x$, to be the element $df_x \in T_x^*M$ given by*

$$df_x(v) := v(f).$$

**Lemma E.32 (Basis for the Cotangent Space).** *Let $M^n$ be a smooth manifold and $(U, \varphi)$ a chart on $M$. Then*
$$\left\{dx^i|_x : i = 1, \dots, n\right\}$$
*is a basis for $T_x^* M$ for all $x \in U$, where $x^i := \pi^i \circ \varphi$.*

*Proof.* We only need to note that this is the dual basis of the tangent space basis E.24. This follows from (E.2) since
$$dx^i|_x \left( \left. \frac{\partial}{\partial x^j} \right|_x \right) = \left. \frac{\partial}{\partial x^j} \right|_x (x^i) = \delta^i_j.$$
□

## Submanifolds

**Proposition E.33.** *Let $M^n$ and $N^n$ be smooth manifolds, $F \in C^\infty(M, N)$ and $x \in M$. If $DF_x$ is invertible then there exists a neighbourhood $U$ of $x$ in $M$ such that $F : U \to F(U)$ is a diffeomorphism.*

*Proof.* Let $(V, \varphi)$ be a chart about $x$ and $(W, \psi)$ be a chart about $F(x)$. Then
$$\psi \circ F \circ \varphi^{-1} : \varphi \left( V \cap F^{-1}(W) \right) \to \mathbb{R}^n$$
and using the chain rule yields
$$D \left( \psi \circ F \circ \varphi^{-1} \right)_{\varphi(x)} = D\psi_{F(x)} \circ DF_x \circ D \left( \varphi^{-1} \right)_{\varphi(x)}$$
and thus $D \left( \psi \circ F \circ \varphi^{-1} \right)_{\varphi(x)}$ is invertible. An application of the inverse function theorem D.10 yields a neighbourhood $\tilde{U}$ in $\varphi \left( V \cap F^{-1}(W) \right)$ about $\varphi(x)$ such that the restriction $\psi \circ F \circ \varphi^{-1}|_{\tilde{U}}$ is a diffeomorhism. Set $U := \varphi^{-1}(\tilde{U})$. □

**Proposition E.34.** *Let $U \subseteq \mathbb{R}^n$ be a neighbourhood about $0$ and $f : U \to \mathbb{R}^k$ smooth such that $f(0) = 0$. Then:*

(a) *If $n \leq k$ and the matrix $Df_0$ has maximal rank, then there exists a chart $\psi$ about $0$ on $\mathbb{R}^k$ such that $\psi \circ f = \iota$, where $\iota : \mathbb{R}^n \hookrightarrow \mathbb{R}^k$ denotes the inclusion.*
(b) *If $n \geq k$ and the matrix $Df_0$ has maximal rank, then there exists a chart $\varphi$ about $0$ on $\mathbb{R}^n$ such that $f \circ \varphi = \pi$, where $\pi : \mathbb{R}^n \to \mathbb{R}^k$ denotes the projection.*

**Definition E.35 (Immersion).** *A smooth map $F : M \to N$ is said to be an **immersion**, iff $DF_x$ is injective for all $x \in M$.*

**Definition E.36 (Embedding).** *A smooth map $F : M \to N$ is said to be an **embedding**, iff $F$ is an injective immersion and $F : M \to F(M)$ is a homeomorphism, where $F(M)$ is endowed with the subspace topology.*

**Definition E.37 (Immersed Submanifold).** *Let $M$ and $N$ be smooth manifolds and $M \subseteq N$ as sets. We say that $M$ is an **immersed submanifold of $N$**, iff the inclusion $M \hookrightarrow N$ is an immersion.*

**Definition E.38 (Embedded Submanifold).** *Let $M$ and $N$ be smooth manifolds and $M \subseteq N$ as sets. We say that $M$ is a **embedded submanifold of $N$**, iff the inclusion $M \hookrightarrow N$ is an embedding.*

**Proposition E.39.** *Suppose $F : M^n \to N^k$ is an immersion. Then for any $x \in M$, there exists a chart $U$ of $x$ and a chart $(V, \psi)$ about $F(x)$ such that*

(a) *If $y^i := \pi^i \circ \psi$, then*
$$F(U) \cap V = \left\{ y \in V : y^{n+1}(y) = \cdots = y^k(y) = 0 \right\}.$$
(b) *$F|_U$ is an embedding.*

**Proposition E.40.** *Suppose $F : M^n \to N^k$ is an embedding. Then for any $x \in M$, there exists a chart $U$ of $x$ and a chart $(V, \psi)$ about $F(x)$ such that if $y^i := \pi^i \circ \psi$, then*
$$F(U) \cap V = \left\{ y \in V : y^{n+1}(y) = \cdots = y^k(y) = 0 \right\}.$$

*Proof.* Since $F$ is a homeomorphism onto $F(M)$, we have that $F(U)$ is open in $F(M)$. By definition of the subspace toology, $F(U) = F(M) \cap W$, where $W$ is open in $N$. But then
$$F(U) \cap (W \cap V) = \left\{ y \in V : y^{n+1}(y) = \cdots = y^k(y) = 0 \right\}.$$
□

**Corollary E.41.** *Suppose $M^n \subseteq N^k$ is an embedded submanifold. Then for any $x \in M$, there exists a chart $(U, \varphi)$ about $x$ such that if $x^i := \pi^i \circ \varphi$, then*
$$M \cap U = \left\{ x \in U : x^{n+1}(x) = \cdots = x^k(x) = 0 \right\}.$$
*Every such chart is called a **slice chart for $M$ in $N$**.*

**Definition E.42 (Regular and Critical Point).** *Let $F : M \to N$ be smooth. A point $x \in M$ is said to be a **regular point**, iff* $\operatorname{rank} DF_x = \dim N$. *A point $x \in M$ is said to be a **critical point**, iff $x$ is not a regular point.*

**Definition E.43 (Regular and Critical Value).** *Let $F : M \to N$ be smooth. A point $y \in N$ is said to be a **regular value**, iff $F^{-1}(y)$ consist only of regular points. A point $y \in N$ is said to be a **critical value**, iff $y$ is not a regular value.*

**Theorem E.44 (The Implicit Function Theorem for Manifolds).** *Let $F : M^n \to N^k$ be smooth and suppose that $y \in N$ is a regular value of $F$ such that $F^{-1}(y) \neq \varnothing$. Then $F^{-1}(y)$ is a topological manifold of dimension $n - k$. Moreover, there exists a smooth structure on $F^{-1}(y)$ making it into an embedded submanifold of $M$.*

**Proposition E.45.** *Let $F : M \to N$ be smooth and $y \in N$ a regular value of $F$ such that $F^{-1}(y) \neq \varnothing$. Then*
$$D\iota_x \left( T_x F^{-1}(y) \right) = \ker DF_x$$
*holds for all $x \in F^{-1}(y)$ where $\iota : F^{-1}(y) \hookrightarrow M$ denotes the inclusion.*

**Definition E.46 (Submersion).** *A smooth map $F : M \to N$ is said to be a* ***submersion****, iff every point of $M$ is a regular value.*

The next theorem is the main reason why we require smooth manifolds to admit only countably many connected components.

**Theorem E.47 (Sard's Theorem for Manifolds).** *Let $F : M \to N$ be smooth. Then the set of regular values of $F$ is dense in $N$.*

**Theorem E.48 (The Strong Whitney Embedding Theorem).** *Let $M^n$ be a smooth manifold. Then there exists a proper embedding $M \to \mathbb{R}^{2n}$.*

**Theorem E.49 (The Weak Whitney Embedding Theorem).** *Let $M^n$ be a smooth manifold. Then there exists a proper embedding $M \to \mathbb{R}^{2n+1}$.*

**Theorem E.50 (The Whitney Approximation Theorem).** *Let $F : M \to N$ be a continuous map between two smooth manifolds $M$ and $N$. Then $F$ is homotopic to a smooth map.*

## Vector Fields

**Definition E.51 (Vector Field).** *Let $M$ be a smooth manifold and $U \subseteq M$ open. A* ***vector field on $U$*** *is defined to be a section of the projection $\pi : TU \to U$, where $TU \subseteq TM$. The set of all vector fields on $U$ is denoted by $\mathfrak{X}(U)$.*

**Example E.52 (Coordinate Vector Fields).** Let $M$ be a smooth manifold and $\big(U, (x^i)\big)$ be a chart on $M$. Define $\frac{\partial}{\partial x^i} : U \to TM$ by

$$\frac{\partial}{\partial x^i}(x) := \left.\frac{\partial}{\partial x^i}\right|_x.$$

It immediately follows from the smoothness criteria for tensor fields 2.12 that $\frac{\partial}{\partial x^i} \in \mathfrak{X}(U)$.

**Exercise E.53.** Show that $\mathfrak{X}(U)$ is a $C^\infty(U)$-module. *Hint:* Use the smoothness criteria for tensor fields 2.12.

In contrast to arbitrary tensor fields, vector fields can act on smooth functions.

**Proposition E.54.** *Let $M$ be a smooth manifold and $U \subseteq M$ open. Then $X \in \mathfrak{X}(U)$ if and only if the function $Xf : U \to \mathbb{R}$ defined by $Xf(x) := X(x)f$ is smooth for all $f \in C^\infty(V)$, where $V \subseteq U$ is open.*

*Proof.* Using the smoothness criteria for tensor fields 2.12, we locally write $X = X^i \frac{\partial}{\partial x^i}$, where $X^i$ are smooth functions. Hence

$$Xf = X^i \frac{\partial f}{\partial x^i}$$

which is smooth.
Conversly, suppose that $Xf$ is smooth for any $f \in C^\infty(V)$. Then in particular

$$X(x^j) = X^i \frac{\partial x^j}{\partial x^i} = X^j$$

is smooth. $\square$

We adopt the terminology from [Wei94, p. 218].

**Definition E.55 (Derivation).** *Let $M$ be a smooth manifold and $U \subseteq M$ open. A **derivation of $C^\infty(U)$** is a linear map $D : C^\infty(U) \to C^\infty(U)$ such that*

$$D(fg) = D(f)g + fD(g)$$

*holds for all $f, g \in C^\infty(U)$. Denote the set of all derivations of $C^\infty(U)$ by $\mathrm{Der}(U)$.*

**Exercise E.56.** Show that $\mathrm{Der}(U)$ is a $C^\infty(M)$-module.

**Proposition E.57.** *Let $M$ be a smooth manifold and $U \subseteq M$ be open and non-empty. Then $\mathfrak{X}(U) \cong \mathrm{Der}(U)$ as modules over $C^\infty(U)$.*

*Proof.* Define $\Phi : \mathfrak{X}(U) \to \mathrm{Der}(U)$ by $\Phi(X)(f) := Xf$ using proposition E.54. Moreover, define $\Psi : \mathrm{Der}(U) \to \mathfrak{X}(U)$ by $\Psi(D)(x)(f) := Df$ for all $f \in C^\infty(U)$ again using proposition E.54. $\square$

**Remark E.58.** From now on we will identify vector fields in $\mathfrak{X}(U)$ with derivations $\mathrm{Der}(U)$ by means of proposition E.57.

Proposition E.57 yields a new tool for constructing vector fields.

**Exercise E.59.** Let $M$ be a smooth manifold and $U \subseteq M$ a non-empty open subset. Show that

$$[X, Y] := X \circ Y - Y \circ X \in \mathfrak{X}(U)$$

for any $X, Y \in \mathfrak{X}(U)$.

## Flows

**Definition E.60 (Integral Curve).** *Let $M$ be a smooth manifold and $X \in \mathfrak{X}(M)$. A curve $\gamma \in C^\infty(J, M)$, where $J \subseteq \mathbb{R}$ is an interval, is said to be an **integral curve of $X$**, iff*

$$\gamma'(t) = X_{\gamma(t)}$$

*holds for all $t \in J$.*

**Proposition E.61 (Fundamental Theorem for Autonomous ODEs).** *Let $U \subseteq \mathbb{R}^n$ open and $X \in C^\infty(U, \mathbb{R}^n)$. Consider the initial value problem*

$$\begin{cases} \gamma'(t) = X\left(\gamma^1(t), \dots, \gamma^n(t)\right) \\ \gamma(t_0) = c, \end{cases} \tag{E.4}$$

*for $t_0 \in \mathbb{R}$ and $c \in U$. Then:*

(a) *For any $t_0 \in \mathbb{R}$ and $x_0 \in U$ there exists an open interval $J_0$ containing $t_0$ and an open subset $U_0 \subseteq U$ containing $x_0$ such that for each $c \in U_0$ there is a map $\gamma \in C^1(J_0, U)$ that solves* (E.4).
(b) *Any two solutions to* (E.4) *agree on their common domain.*
(c) *Define*

$$\theta : J_0 \times U_0 \to U$$

*by $\theta(t,x) := \gamma(t)$, where $\gamma(t)$ is the unique solution of* (E.4) *such that $\gamma(t_0) = x$. Then $\theta$ is smooth.*

**Theorem E.62 (Fundamental Theorem of Flows).** *Let $M$ be a smooth manifold and $X \in \mathfrak{X}(M)$. Then there exists a unique open set $\mathcal{D} \subseteq \mathbb{R} \times M$ and a unique smooth map $\theta : \mathcal{D} \to M$, called the **maximal flow associated to $X$**, such that*

(a) *For all $x \in M$ we have that*

$$\mathcal{D} \cap (\mathbb{R} \times \{x\}) = \big(t^-(x), t^+(x)\big) \times \{x\}\,.$$

(b) *$\theta(t,x) = \gamma_x(t)$ for all $(t,x) \in \mathcal{D}$.*

**Definition E.63 (Complete).** *A vector field $X \in \mathfrak{X}(M)$ on a smooth manifold $M$ is said to be **complete**, iff its flow $\theta$ admits the domain $\mathbb{R} \times M$.*

A sufficient condition for completeness is given in the following lemma.

**Lemma E.64.** *Let $M$ be a smooth manifold and $X \in \mathfrak{X}(M)$. Suppose thta there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq \big(t^-(x), t^+(x)\big)$ for all $x \in M$. Then $X$ is complete.*

**Proposition E.65.** *Let $M$ be a smooth manifold and $X \in \mathfrak{X}(M)$ with compact support. Then $X$ is complete.*

**Corollary E.66.** *Every vector field on a compact smooth manifold is complete.*

## Lie groups and Lie algebras

**Definition E.67 (Lie Group).** *A **Lie group** is defined to be a group object in* Man.

**Definition E.68 (Lie Group Homomorphism).** *A map $F \in C^\infty(G, H)$ between two Lie groups $G$ and $H$ is said to be a **Lie group homomorphism**, iff $F : G \to H$ is a homomorphism.*

The group structure of a Lie group induces canonical maps.

**Definition E.69 (Translation).** *Let $G$ be a Lie group and $g \in G$. Define $L_g, R_g \in \mathrm{Diff}(G)$ by*

$$L_g(h) := gh \qquad \text{and} \qquad R_g(h) := hg\,.$$

*These maps are called **left translation** and **right translation by $g$**, respectively.*

**Proposition E.70.** *Every Lie group homomorphism has constant rank.*

**Definition E.71 (Lie Subgroup).** *A **Lie subgroup** of a Lie group $G$ is defined to be a subgroup of $G$, which is itself a Lie group and an immersed submanifold of $G$.*

**Proposition E.72.** *Let$G$ be a Lie group and $H$ be a subgroup of $G$ such that $H$ is an embedded submanifold of $G$. Then $H$ is a Lie subgroup of $G$.*

To every Lie group we can associate an algebraic object.

**Definition E.73 (Lie Algebra).** *A **Lie algebra** is defined to be a real vector space $\mathfrak{g}$, such that there exists a bilinear mapping*

$$[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

*called the **Lie bracket on** $\mathfrak{g}$, such that:*

(i) (***Antisymmetry***) $[x, y] = -[y, x]$,
(ii) (***Jacobi's Identity***) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$,

*holds for all $x, y, z \in \mathfrak{g}$.*

**Example E.74 (Vector Fiels).** Let $M$ be a smooth manifold and $U \subseteq M$ open. Then $\mathfrak{X}(U)$ together with $[\,\cdot\,,\,\cdot\,]$ defined in exercise E.59 is a Lie algebra, called the ***Lie algebra of vector fields on $M$***.

**Definition E.75 (Lie Algebra Homomorphism).** *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. A **Lie algebra homomorphism between $\mathfrak{g}$ and $\mathfrak{h}$** is defined to be a homomorphism $L \in \mathrm{L}(\mathfrak{g}, \mathfrak{h})$ such that*

$$L[x, y] = [Lx, Ly]$$

*holds for all $x, y \in \mathfrak{g}$.*

## Distributions

## Vector Bundles

**Definition E.76 (Fibre Bundle).** *A **fibre bundle** is defined to be a tuple $(E, M, \pi, F)$ consisting of smooth manifolds $E$, $M$ and $F$ together with a surjective map $\pi \in C^\infty(E, M)$ such that there exists an open cover $(U_\alpha)_{\alpha\in A}$ of $M$ and maps $\varphi_\alpha \in C^\infty\big(\pi^{-1}(U_\alpha), F\big)$ for all $\alpha \in A$ such that $(\pi, \varphi_\alpha) : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a diffeomorphism. If $(E, M, \pi, F)$ is a fibre bundle, we call $M$ the **base space**, $E$ the **total space** and $F$ the **fibre**. Moreover, the family $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ is called a **bundle atlas** for $(E, M, \pi, F)$.*

The fibre $F$ of a fibre bundle $(E, M, \pi, F)$ is completely determined by $\pi : E \to M$.

**Proposition E.77.** *Let $(E, M, \pi, F)$ be a fibre bundle. Then $\pi$ is a submersion, $E_x := \pi^{-1}(x)$ is an embedded submanifold of $E$ for all $x \in M$ and $E_x \cong F$ in* $\mathsf{Man}$.

*Proof.* Let $x \in M$. Then there exists a neighbourhood $U_\alpha$ of $x$ such that $\pi = \pi^1 \circ (\pi, \varphi_\alpha)$. But then $\pi$ is a submersion as compositions of submersions. Thus an application of the implicit function theorem for manifolds E.44 yields that $E_x$ is an embedded submanifold of $E$. Now $E_x \cong \{x\} \times F$ by $\varphi_\alpha$, but $\{x\} \times F \cong F$ in $\mathsf{Man}$. ☐

**Example E.78 (Trivial Bundle).** Let $M$ and $F$ be smooth manifolds. Then

$$\pi : M \times F \to M$$

is a fibre bundle.

**Exercise E.79.** Let $M$ and $N$ be smooth manifolds and $G \in C^\infty(M, N)$. Moreover, suppose that $(E, N, \pi)$ is a fibre bundle. Define

$$G^*E := \left\{(x, p) \in M \times E : G(x) = \pi(p)\right\}.$$

Show that $(G^*E, M, \pi^1, F)$ is a fibre bundle. This fibre bundle is called the ***pullback bundle***.

**Definition E.80 (Effective Action).** *A left action $\theta : G \times M \to M$ of a Lie group $G$ on a smooth manfiold $M$ is said to be* ***effective****, iff $\theta_g = \mathrm{id}_M$ if and only if $g = e$.*

**Definition E.81 (Compatibility).** *Let $(E, M, \pi)$ be a fibre bundle and $\theta : G \times F \to F$ an effective Lie group action. Let $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$. We say that $\varphi_\alpha : \pi^{-1}(U_\alpha) \to F$ and $\varphi_\beta : \pi^{-1}(U_\beta) \to F$ are* ***$(G, \theta)$ - compatible****, iff there exists $\widetilde{\rho}_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that*

$$\rho_{\alpha\beta}(x)(y) = \widetilde{\rho}_{\alpha\beta}(x) \cdot y$$

*holds for all $x \in U_\alpha \cap U_\beta$ and $y \in F$, where $\rho_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$ is defined by*

$$\rho_{\alpha\beta}(x) := \varphi_\alpha|_{E_x} \circ \varphi_\beta|_{E_x}^{-1}.$$

**Definition E.82 (Structure Group).** *A* ***structure group*** *of a fibre bundle $(E, M, \pi)$ is a Lie group $G$ such that there exists an effective Lie group action on $F$ and a bundle atlas $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ which is $G$-compatible.*

**Definition E.83 (Vector Bundle).** *Let $k \in \mathbb{N}$. A* ***vector bundle of rank k*** *is defined to be a fibre bundle $(E, M, \pi, \mathbb{R}^k)$ admiting a matrix Lie subgroup of $\mathrm{GL}(k)$ as a structure group.*

As aestetically pleasing the definition of a vector bundle E.83 may be, in practice, it is not that useful. Hence we give an alternative definition.

**Definition E.84 (Vector Bundle).** *Let $E$ and $M$ be smooth manifolds, $\pi \in C^\infty(E, M)$ surjective and $k \in \mathbb{N}$. We say that $(E, M, \pi)$ is a* ***vector bundle of rank k****, iff $E_x$ admits the structure of a $k$-dimensional real vector space and there exists an open cover $(U_\alpha)_{\alpha\in A}$ of $M$ and a map $\varphi_\alpha \in C^\infty\left(\pi^{-1}(U_\alpha), \mathbb{R}^k\right)$ for all $\alpha \in A$ such that*

(i) *$(\pi, \varphi_\alpha) : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ is a diffeomorphism for all $\alpha \in A$.*
(ii) *$\varphi_\alpha|_{E_x} : E_x \to \mathbb{R}^k$ is an isomorphism of vector spaces.*

**Definition E.85 (Vector Bundle Morphism).** *Let $(E, M, \pi)$ and $(E', M', \pi')$ be two vector bundles and $f \in C^\infty(M, M')$. A* ***vector bundle morphism along f*** *is defined to* *be a map $F \in C^\infty(E, E')$ such that*

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \pi \big\downarrow & & \big\downarrow \pi' \\ M & \xrightarrow[f]{} & M' \end{array}$$

*commutes and $F|_{E_x} : E_x \to E'_{f(x)}$ is linear for all $x \in M$.*

**Definition E.86 (Vector Bundle Homomorphism).** *Let $(E, M, \pi)$ and $(E', M, \pi')$ be two vector bundles over the same base space. A **vector bundle homomorphism** is a vector bundle morphism along $\mathrm{id}_M$.*

One particular advantage of studying vector bundles instead of mere fibre bundles is that the set of sections admits an additional structure.

**Lemma E.87.** *Let $(E, M, \pi)$ be a vector bundle. Then for any $U \subseteq M$ open, the set $\Gamma(U, E)$ is a vector space and a $C^\infty(U)$-module.*

## Sheaves

If $M$ is a smooth manifold, so is $U$ for any open subset $U \subseteq M$. Most of the constructions we performed so far also work for this induced smooth structure on $U$. However, it is tedious to explicitely mention this all the time. So we introduce now a foundational notion of a mathematical field called *Algebraic Geometry*.

Let $(X, \mathcal{T}) \in \mathsf{Top}$. Then denote by $\mathcal{O}(X)$ the category of open subsets of $X$, that is the category associated to the poset $(\mathcal{T}, \subseteq)$ (see [Lei16, p. 24]). Recall, that for any two categories $\mathcal{C}$ and $\mathcal{D}$, there exists the functor category $\mathcal{D}^{\mathcal{C}}$ from $\mathcal{C}$ to $\mathcal{D}$ (see [Lei16, p. 30]).

**Definition E.88 (Presheaf).** *Let $X \in \mathsf{Top}$ and $\mathcal{C}$ be a category. A **presheaf of $\mathcal{C}$ on $X$** is defined to be a contravariant functor $\mathcal{O}(X) \to \mathcal{C}$. The **category of presheaves of $\mathcal{C}$ on $X$** is denoted by $\mathsf{PSh}(X; \mathcal{C})$.*

**Remark E.89.** Let $F : \mathcal{O}(X) \to \mathcal{C}$, where $\mathcal{C}$ is the category of a mathematical structure, that is $\mathsf{Grp}$, $\mathsf{Ring}$, $\mathsf{Vect}$, ..., be a presheaf of $\mathcal{C}$ on $X$. Then if $U \subseteq V$ for $U, V \in \mathcal{O}(X)$, we simply write $f|_U$ for $F(U \hookrightarrow V)(f)$, where $f \in F(V)$.

**Example E.90.** Let $(E, M, \pi)$ be a vector bundle. Define $\mathcal{E}_E : \mathcal{O}(X) \to \mathsf{Vect}$ on objects $U \in \mathcal{O}(X)$ by $\mathcal{E}_E(U) := \Gamma(U, E)$ and on morphisms by restriction.

**Definition E.91 (Sheaf).** *Let $X \in \mathsf{Top}$ and $F$ a presheaf of $\mathsf{Set}$ ($\mathsf{Grp}$, $\mathsf{Ring}$, $\mathsf{Vect}$, ...) on $X$. We say that $F$ is a **sheaf on $X$**, iff for all $U \in \mathcal{O}(X)$ the following **gluing condition** is satisfied: Given any open cover $(U_\alpha)_{\alpha \in A}$ for $U$ and $f_\alpha \in F(U_\alpha)$ for all $\alpha \in A$ such that*

$$f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta}$$

*for all $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, then there exists a unique element $f \in F(U)$ with $f|_{U_\alpha} = f_\alpha$ for all $\alpha \in A$.*

From example E.90 we already know that $\mathcal{E}_E$ is a presheaf. In fact, more is true.

**Proposition E.92.** *Let $(E, M, \pi)$ be a vector bundle. Then $\mathcal{E}_E : \mathcal{O}(M) \to \mathsf{Vect}$ is a sheaf.*

**Example E.93 (Tensor Sheaf).** Let $M$ be a smooth manifold. Then tensor fields of type $(k,l)$ can be assembled in a sheaf by proposition E.92. Denote this sheaf by $\mathcal{T}_M^{k,l} := \mathcal{E}_{T^{(k,l)}TM}$. We can assemble these sheaves in a total sheaf $\mathcal{T}_M : \mathcal{O}(M) \to {}_{\mathbb{R}}\mathsf{GAlg}$ by setting

$$\mathcal{T}_M(U) := \bigoplus_{k,l \geq 0} \mathcal{T}_M^{k,l}(U).$$

We call $\mathcal{T}_M$ the ***tensor algebra sheaf on $M$***.

**Example E.94 (Sheaf of Differential Forms).** Let $M^n$ be a smooth manifold and let $0 \leq k \leq n$. Then by E.92, $\mathcal{E}_{\Lambda^k(T^*M)}$ is a sheaf. This sheaf is denoted by $\Omega_M^k$ and called the ***sheaf of differential $k$-forms***. As with tensor fields in example E.93,we can define a sheaf $\Omega_M : \mathcal{O}(M) \to {}_{\mathbb{R}}\mathsf{GSCAlg}$ by

$$\Omega_M(U) := \bigoplus_{0 \leq k \leq n} \Omega_M^k(U).$$

## The Lie Derivative

**Definition E.95 (Pullback).** *Let $l \in \mathbb{N}$ and $F \in C^\infty(M, N)$. Define*

$$F^* : \mathcal{T}^{0,l}(N) \to \mathcal{T}^{0,l}(M)$$

*by*

$$(F^*A)_x(v_1, \ldots, v_l) := A_{F(x)}\left(DF_x(v_1), \ldots, DF_x(v_l)\right)$$

*for all $x \in M$ and $v_1, \ldots, v_l \in T_xM$, if $k \geq 1$ and by $F^*f := f \circ F$ if $k = 0$. We call $F^*A$ the **pullback of $A$ under $F$**.*

To extend the notion of a pullback of a tensor field to arbitrary tensor fields, we must impose an additional contition on the map.

**Definition E.96 (Cotangent Lift).** *Let $F \in C^\infty(M, N)$ be a diffeomorphism. Define a map $DF^\dagger : T^*M \to T^*N$ by*

$$DF^\dagger(x, \xi)(v) := \xi\left((DF_x)^{-1}(v)\right)$$

*for all $v \in T_{F(x)}N$. This map is called the **cotangent lift of the diffeomorphism $F$**.*

**Definition E.97 (Pullback).** *Let $k, l \in \mathbb{N}$ and $f \in C^\infty(M, N)$ a diffeomorphism. Define*

$$F^* : \mathcal{T}^{k,l}(N) \to \mathcal{T}^{k,l}(M)$$

*by*

$$A \mapsto (F^*A)_x \left(\xi^1, \ldots, \xi^k, v_1, \ldots, v_l\right)$$

*for all* $x \in M$, $\xi^1, \ldots, \xi^k \in T_x^* M$ *and* $v_1, \ldots, v_l \in T_x M$, *if* $k \geq 1$, *where the latter is defined to be*

$$A_{F(x)} \left(DF^\dagger(\xi^1), \ldots, DF^\dagger(\xi^k), DF_x(v_1), \ldots, DF_x(v_k)\right)$$

*We call* $F^*A$ *the* ***pullback of A under F***.

**Definition E.98 (Pushforward).** *Let* $k, l \in \mathbb{N}$ *and* $f \in C^\infty(M, N)$ *a diffeomorphism. Define*

$$F_* : \mathcal{T}^{k,l}(M) \to \mathcal{T}^{k,l}(N)$$

*by*

$$F_* A := \left(F^{-1}\right)^* A.$$

**Definition E.99 (Tensor Derivation).** *A* ***tensor derivation on a smooth manifold M*** *is defined to be a sheaf morphism* $\mathcal{D} : \mathcal{T}_M \to \mathcal{T}_M$ *that preserves type and satisfies:*

(i) *For all* $U \in \mathcal{O}(M)$, $\mathcal{D}_U$ *commutes with all contractions of* $\mathcal{T}_M(U)$.
(ii) *For all* $U \in \mathcal{O}(M)$, $\mathcal{D}_U$ *is a derivation, that is*

$$\mathcal{D}_U(A \otimes B) = \mathcal{D}_U A \otimes B + A \otimes \mathcal{D}_U B$$

*holds for all* $A, B \in \mathcal{T}(U)$.

**Proposition E.100.** *Let* $\mathcal{D}$ *and* $\mathcal{D}'$ *be two tensor derivations on a smooth manifold which agree on functions and vector fields. Then* $\mathcal{D} = \mathcal{D}'$.

**Proposition E.101.** *Let* $\mathcal{D}$ *be a sheaf morphism on functions and vector fields. If*

$$\mathcal{D}_U(fg) = \mathcal{D}_U(f)g + f\mathcal{D}_U(g) \quad \text{and} \quad \mathcal{D}_U(fX) = \mathcal{D}_U(f)X + f\mathcal{D}_U(X)$$

*holds for all* $U \in \mathcal{O}(M)$, $f, g \in C^\infty(U)$ *and* $X \in \mathfrak{X}(U)$, *then* $\mathcal{D}$ *extends uniquely to a tensor derivation on* $M$.

**Theorem E.102 (The Lie Derivative).** *Let* $M$ *be a smooth manifold and* $X \in \mathfrak{X}(M)$. *Then there exists a unique tensor derivation*

$$\mathcal{L}_X : \mathcal{T}_M \to \mathcal{T}_M$$

*on* $M$ *such that*

$$\mathcal{L}_X f = Xf \qquad \text{and} \qquad \mathcal{L}_X Y = [X, Y]$$

*for all* $U \in \mathcal{O}(M)$, $f \in C^\infty(U)$ *and* $Y \in \mathfrak{X}(U)$. *This tensor derivation is called the* ***Lie derivative***.

*Proof.* This immediately follows from proposition E.101 since

$$\begin{aligned}\left(\mathcal{L}_X(fY)\right)g &= [X, fY]\,g\\ &= X\left((fY)(g)\right) - fY\left(X(g)\right)\\ &= X\left(fY(g)\right) - fY\left(X(g)\right)\\ &= X(f)Y(g) + f\left(X\left(Y(g)\right)\right) - fY\left(X(g)\right)\\ &= X(f)Y(g) + f\,[X, Y]\,g\end{aligned}$$

implies

$$\mathcal{L}_X(fY) = \mathcal{L}_X(f)Y + f\mathcal{L}_X Y.$$

□

The next proposition shows why the name Lie derivative is appropriate.

**Proposition E.103.** *Let $M$ be a smooth manifold and $X \in \mathfrak{X}(M)$ with flow $\theta$. Then*

$$\mathcal{L}_X A = \left.\frac{d}{dt}\right|_{t=0} \theta_t^*(A)$$

*for any $A \in \mathcal{T}^{k,l}(M)$.*

### Differential Forms

Differential forms are a key technical tool in differential geometry. In contrast to mere tensor fields, they can be both differentiated and integrated.

**Definition E.104.** *Let $M$ be a smooth manifold and $l \in \mathbb{Z}$. A* ***graded derivation of degree $l$ on $M$*** *is defined to be an $\mathbb{R}$-linear sheaf morphism $\mathcal{D} : \Omega_M \to \Omega_M$ satisfying:*

(i) *If $\omega \in \Omega^k(U)$, then $\mathcal{D}_U(\omega) \in \Omega^{k+l}(U)$.*
(ii) *If $\omega \in \Omega^k(U)$ and $\eta \in \Omega(U)$, then*

$$\mathcal{D}_U(\omega \wedge \eta) = \mathcal{D}_U(\omega) \wedge \eta + (-1)^{kl}\omega \wedge \mathcal{D}_U(\eta).$$

**Proposition E.105.** *Let $M$ be a smooth manifold and suppose that $\mathcal{D}$ and $\mathcal{D}'$ are two gradedderivations on $M$ of the same degree which coincide on functions and exact 1-forms. Then $\mathcal{D} = \mathcal{D}'$.*

**Theorem E.106 (The Exterior Differential).** *Let $M$ be a smooth manifold. Then there exists a unique graded derivation $d : \Omega_M \to \Omega$ of degree 1 such that*

$$d_U(f) = df \qquad \text{and} \qquad d \circ d = 0$$

*holds for all $f \in C^\infty(U)$. This graded derivation is called the* ***exterior differential****.*

## Orientability and Orientations

**Definition E.107 (Orientation).** *Let $V$ be a real vector space. An* ***orientation of V*** *is defined to be a choice of one of the two components of $\Lambda^{\dim V}(V) \setminus \{0\}$.*

**Definition E.108 (Determinant Functor).** *Let $k \in \mathbb{N}$. Define a functor*

$$\det : \mathsf{Vect}^{\geq 1} \to \mathsf{Vect}^1$$

*on objects by $\det V := \Lambda^k V$ and on morphisms $L : V \to W$ as follows: If $\dim V = \dim W = k$, then set*

$$\det L(v_1 \wedge \cdots \wedge v_k) := Lv_1 \wedge \cdots \wedge Lv_k$$

*and to be the zero-morphism otherwise.*

**Proposition E.109.** *Let $(E, M, \pi)$ be a vector bundle of rank $k$. The following conditions are equivalent:*

(a) *There exists a nowhere-vanishing section $\sigma \in \Gamma(\det E^*)$.*
(b) *The structure group of $E$ can be reduced to $\mathrm{GL}^+(k)$.*
(c) *The bundle $\det E^* \to M$ is trivial.*

**Definition E.110 (Orientability).** *A vector bundle $(E, M, \pi)$ is said to be* ***orientable****, iff one of the conditions of proposition E.109 is satisfied. A specific choice of a nowhere vanishing section in $\Gamma(\det E^*)$ is called an* ***orientation of E****. A smooth manifold $M$ is said to be* ***orientable****, iff the tangent bundle $\pi : TM \to M$ is orientable.*

**Definition E.111 (Volume Form).** *Let $M^n$ be a smooth manifold. A* ***volume form on M*** *is defined to be a nowhere-vanishing $n$-form.*

**Corollary E.112 (Orientability of Manifolds).** *Let $M$ be a smooth manifold. Then the following conditions are equivalent:*

(a) *$M$ admits a volume form.*
(b) *There exists a smooth atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ on $M$ such that when $U_\alpha \cap U_\beta \neq \varnothing$*

$$\det D(\varphi_\alpha \circ \varphi_\beta^{-1})\left(\varphi_\beta(x)\right) > 0$$

*holds for all $x \in U_\alpha \cap U_\beta$.*
(c) *The bundle $\det E^* \to M$ is trivial.*

APPENDIX F

# Review of Differential Geometry

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# Index