Classical Mechanics

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Preface

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CHAPTER 1

Lagrangian Mechanics

Calculus of Variations

Classical mechanics deals with differential equations originating from extremals of *functionals*, i.e. functions defined on an infinite-dimensional function space. The study of such extremality properties of functionals is known as the *calculus of variations*. To illustrate this fundamental principle, let us consider the *variational formulation* of second order elliptic operators in divergence form based on [Str14, pp. 167–168].
Let $n \in \mathbb{N}$, $n \geq 1$, and $\Omega \subseteq\subseteq \mathbb{R}^n$ such that $\bar{\Omega}$ is a manifold with boundary. Moreover, let $H_0^1(\Omega)$ denote the Sobolev space $W_0^{1,2}(\Omega)$ with inner product

$$\langle u, v\rangle_{H_0^1(\Omega)} = \int_\Omega uv + \int_\Omega \nabla u \nabla v.$$

Suppose $a^{ij} \in C^\infty(\bar{\Omega})$ symmetric, $f \in C^\infty(\bar{\Omega})$ and consider the second order homogenous Dirichlet problem

$$\begin{cases} -\dfrac{\partial}{\partial x^j}\left(a^{ij}\dfrac{\partial u}{\partial x^i}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

Suppose $u \in C^\infty(\bar{\Omega})$ solves (1). Then integration by parts (see [Lee13, p. 436]) yields

$$\int_\Omega fv = -\int_\Omega \frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial u}{\partial x^i}\right)v = -\int_\Omega \operatorname{div}(X)v = \int_\Omega \langle X, \nabla v\rangle = \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^j}$$

for any $v \in C_c^\infty(\Omega)$, where $X := \left(a^{ij}\frac{\partial u}{\partial x^i}\right)_j$. Thus we say that $u \in H_0^1(\Omega)$ is a *weak solution* of (1) iff

$$\forall v \in C_c^\infty(\Omega) : \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^j} = \int_\Omega fv.$$

If $(a^{ij})_{ij}$ is *uniformly elliptic*, i.e. there exists $\lambda > 0$ such that

$$\forall x \in \Omega \forall \xi \in \mathbb{R}^n : a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2,$$

then (1) admits a unique weak solution $u \in H_0^1(\Omega)$ (in fact $u \in C^\infty(\Omega)$ using *regularity theory*, for more details see [Str14, p. 175]). Indeed, observe that

$$\langle\cdot,\cdot\rangle_a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

defined by

$$\langle u, v\rangle_a := \int_\Omega a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} \tag{2}$$

is an inner product on $H_0^1(\Omega)$ with induced norm equivalent to the standard one on $H_0^1(\Omega)$ due to Poincaré's inequality [Str14, p. 107]. Applying the Riesz Representation theorem [Str14, pp. 49–50] yields the result. Moreover, this solution can be characterized by a *variational principle*, i.e. if we define the *energy functional* $E : H_0^1(\Omega) \to \mathbb{R}$

$$E(v) := \frac{1}{2}\|v\|_a^2 - \int_\Omega f v,$$

for any $v \in H_0^1(\Omega)$, where $\|\cdot\|_a$ denotes the norm induced by the inner product (2), then $u \in H_0^1(\Omega)$ solves (1) if and only if

$$E(u) = \inf_{v \in H_0^1(\Omega)} E(v). \tag{3}$$

Indeed, suppose $u \in H_0^1(\Omega)$ is a solution of (1). Let $v \in H_0^1(\Omega)$. Then $u = v + w$ for $w := u - v \in H_0^1(\Omega)$ and we compute

$$E(v) = E(u+w) = \frac{1}{2}\|u\|_a^2 + \langle u, w\rangle_a + \frac{1}{2}\|w\|_a^2 - \int_\Omega f(u+w) = E(u) + \frac{1}{2}\|w\|_a^2 \geq E(u)$$

with equality if and only if $u = v$ a.e. Conversly, suppose the infimum is attained by some $u \in H_0^1(\Omega)$. Thus by elementary calculus

$$0 = \left.\frac{d}{dt}\right|_{t=0} E(u + tv) = \langle u, v\rangle_a - \int_\Omega f v \tag{4}$$

for all $v \in H_0^1(\Omega)$.

Suppose now that $u \in C^\infty(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$ solves the variational formulation (3). Then again integration by parts yields

$$\langle u, v\rangle_a - \int_\Omega f v = -\int_\Omega \operatorname{div}(X) v - \int_\Omega f v = \int_\Omega \left(-\frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial u}{\partial x^i}\right) - f\right) v$$

for all $v \in C_c^\infty(\Omega)$ and where $X := \left(a^{ij}\frac{\partial u}{\partial x^i}\right)_j$. Hence (4) implies

$$\forall v \in C_c^\infty(\Omega) : \int_\Omega \left(-\frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial u}{\partial x^i}\right) - f\right) v = 0.$$

We might expect that this implies

$$-\frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial u}{\partial x^i}\right) = f.$$

That this is indeed the case, is guaranteed by a foundational result in the *calculus of variations* (therefore the name).

Proposition 1.1 (Fundamental Lemma of Calculus of Variations). *Let $\Omega \subseteq \mathbb{R}^n$ open and $f \in L^1_{\text{loc}}(\Omega)$. If*

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega f\varphi = 0,$$

then $f = 0$ a.e.

Proof. See [Str14, p. 40]. □

Thus we recovered a second order partial differential equation from the variational formulation. In fact, this is exactly the boundary value problem (1) from the beginning of our exposition. This technique, and in particular the fundamental lemma of calculus of variations 1.1 will play an important role in our treatment of classical mechanics.

Exercise 1.2. [1] Let $\Omega \subseteq\subseteq \mathbb{R}^n, 2 \leq p < \infty$ and define $\mathcal{B} := \{v \in C^\infty(\bar{\Omega}) : v|_{\partial\Omega} = 0\}$. Moreover, define $E_p : \mathcal{B} \to \mathbb{R}$ by $E_p(v) := \int_\Omega |\nabla v|^p$. Derive the partial differential equation satisfied by minimizers $u \in \mathcal{B}$ of the variational problem $E(u) = \inf_{v\in\mathcal{B}} E(v)$.

Lagrangian Systems and the Principle of Least Action

Mechanical systems, i.e. a pendulum, are modelled using the language of differential geometry. Thus it is necessary to introduce the relevant physical terminology.

Definition 1.3 (Configuration Space). *A finite-dimensional manifold in* Diff *is said to be a* ***configuration space****.*

As easily imaginable in the three dimensional Euclidean space $\mathbb{R}^3$, the motion of a pendulum covered over time is the image of a path in a configuration space.

Definition 1.4 (Motion). *A motion of a configuration space M is defined to be a path $\gamma \in \mathsf{Diff}(J, M)$, where $J \subseteq \mathbb{R}$ is an interval.*

Definition 1.5 (State). *A* ***state in the configuration space*** *is defined to be an element of the tangent bundle of the configuration space.*

Also in classical mechanics, one has to rely on basic principles, which are to some extent experimentally verified.

Axiom 1 (Newton-Laplace Determinacy Principle). *A motion in a configuration space is completely determined by a state at some instant.*

The Newton-Laplace determinacy principle 1 motivates our main definition of this chapter.

Definition 1.6 (Lagrangian System). *A* ***Lagrangian system*** *is defined to be a tuple (M, L) consisting of an object $M \in \mathsf{Diff}$ and a morphism $L \in \mathsf{Diff}(TM \times \mathbb{R}, \mathbb{R})$, called a* ***Lagrangian function****.*

[1] This is exercise 1.2.(b) from exercise sheet 1 of the course *Functional Analysis II* taught by *Prof. Dr. A. Carlotto* at ETHZ in the spring of 2018, which can be found here.

Definition 1.7 (Path Space). *Let $M \in \mathsf{Diff}$, $q_0, q_1 \in M$ and $t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$. Define the* ***path space of M connecting (q_0, t_0) and (q_1, t_1)*** *to be the set*

$$\mathcal{P}(M)^{q_0,t_0}_{q_1,t_1} := \{\gamma \in \mathsf{Diff}([t_0, t_1], M) : \gamma(t_0) = q_0 \text{ and } \gamma(t_1) = q_1\}. \tag{5}$$

Remark 1.8. For the sake of simplicity, we will just use the terminology *path space* for $\mathcal{P}(M)^{q_0,t_0}_{q_1,t_1}$ and simply write $\mathcal{P}(M)$. We implicitely assume the conditions of definition 1.7, however.

Definition 1.9 (Variation). *Let $\mathcal{P}(M)$ be a path space and $\gamma \in \mathcal{P}(M)$. A* ***variation of γ*** *is defined to be a morphism $\Gamma \in \mathsf{Diff}([t_0, t_1] \times [-\varepsilon_0, \varepsilon_0], M)$ for some $\varepsilon_0 > 0$ and such that*

- *$\Gamma(t, 0) = \gamma$ for all $t \in [t_1, t_0]$.*
- *$\Gamma(t_0, \varepsilon) = q_0$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.*
- *$\Gamma(t_1, \varepsilon) = q_1$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.*

Remark 1.10. If Γ is a variation of $\gamma \in \mathcal{P}(M)$, we write $\gamma_\varepsilon(-) := \Gamma(-, \varepsilon)$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

Example 1.11 (Perturbation of a Path along a Single Direction). Let $M \in \mathsf{Diff}$ of dimension n, (U, φ) a chart and suppose that γ is a path in U. With respect to this chart, we can write the coordinate representation of γ as

$$\gamma(t) = \left(\gamma^1(t), \ldots, \gamma^n(t)\right)$$

for any $t \in [t_0, t_1]$. Let $f \in C^\infty_c(t_0, t_1)$. Consider the family $\Gamma : [t_0, t_1] \times [-\varepsilon_0, \varepsilon_0] \to M$ defined by

$$\Gamma(t, \varepsilon) := (\iota \circ \varphi^{-1}) \left(\gamma^1(t), \ldots, \gamma^i(t) + \varepsilon f(t), \ldots, \gamma^n(t)\right)$$

where $\iota : U \hookrightarrow M$ denotes inclusion and $\varepsilon_0 > 0$ is to be determined. By exercise 1.12, there exists $\delta > 0$ such that

$$U_\delta := \{x \in \mathbb{R}^n : \mathrm{dist}(x, \gamma([t_0, t_1])) < \delta\} \subseteq \varphi(U).$$

Choose $\varepsilon_0 > 0$ such that $0 < \varepsilon_0 < \delta / \|f\|_\infty$. Then in coordinates

$$\mathrm{dist}\left(\gamma_\varepsilon(t), \gamma([t_0, t_1])\right) \leq |\gamma_\varepsilon(t) - \gamma(t)| = |\varepsilon| \|f\|_\infty \leq \varepsilon_0 \|f\|_\infty < \delta$$

for all $t \in [t_0, t_1]$. Hence $\gamma_\varepsilon(t) \in U_\delta$ and thus $\gamma_\varepsilon(t) \in \varphi(U)$. Therefore, Γ is indeed well-defined. Moreover, it is easy to show that the properties of definition 1.9 holds, therefore, Γ is a variation of γ. In fact, this example shows, that any path γ contained in a single chart admits infinitely many variations. An example of such a variation is shown in figure 1.

Exercise 1.12. Let $U \subseteq \mathbb{R}^n$ open and $A \subseteq U$ closed. Then there exists $\delta > 0$ such that

$$U_\delta := \{x \in \mathbb{R}^n : \mathrm{dist}(x, A) < \delta\} \subseteq U.$$

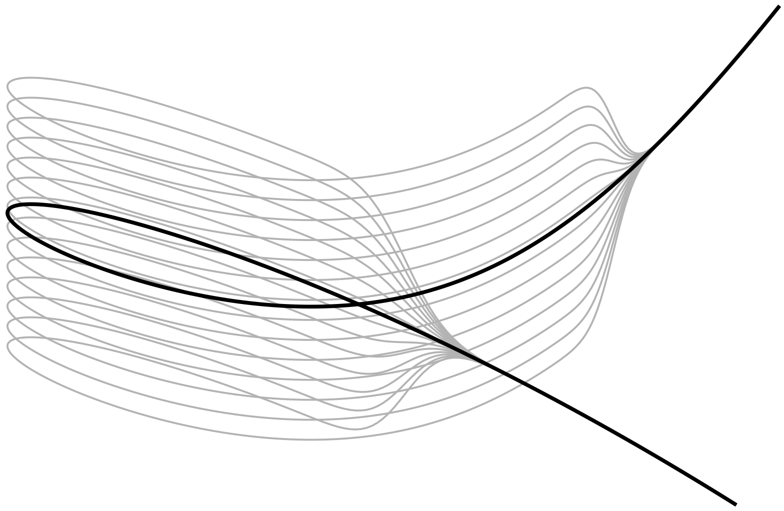

Figure 1. Example of a variation along the second coordinate using a smooth bump function as in [Lee13, p. 42].

Definition 1.13 (Action Functional). *Let (M, L) be a Lagrangian system and $\mathcal{P}(M)$ be a path space. The morphism $S : \mathcal{P}(M) \to \mathbb{R}$ defined by*

$$S(\gamma) := \int_{t_0}^{t_1} L\left(\gamma(t), \dot{\gamma}(t), t\right) dt$$

is called the ***action functional****.*

Definition 1.14 (Motion of a Lagrangian System). *A* ***motion of a Lagrangian system*** $(\boldsymbol{M}, \boldsymbol{L})$ ***between*** $(\boldsymbol{q_0}, \boldsymbol{t_0})$ ***and*** $(\boldsymbol{q_1}, \boldsymbol{t_1})$ *is defined to be a morphism $\gamma \in \mathsf{Diff}([t_0, t_1], M)$.*

Motions of Lagrangian systems are characterized by an axiom.

Axiom 2 (Hamilton's Principle of Least Action). *Let (M, L) be a Lagrangian system and $\mathcal{P}(M)$ be a path space. A path $\gamma \in \mathsf{Diff}([t_0, t_1], M)$ describes a motion of (M, L) between (q_0, t_0) and (q_1, t_1) if and only if*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\gamma_\varepsilon) = 0 \tag{6}$$

for all variations γ_ε of γ.

Definition 1.15 (Extremal). *A motion of a Lagrangian system between two points is called an* ***extremal of the action functional*** $\boldsymbol{S}$*.*

The Newton-Laplace determinacy principle 1 implies that motions of mechanical systems can be described as solutions of second order differential equations. That this is indeed the case, is shown by the next theorem.

Theorem 1.16 (Euler-Lagrange Equations). *Let (M, L) be a Lagrangian system. If a path $\gamma \in \mathsf{Diff}([t_0, t_1], M)$ describes a motion of (M, L) between (q_0, t_0) and (q_1, t_1) then for all charts (U, q^i)*

$$\frac{\partial L}{\partial q}\big(\gamma(t), \dot{\gamma}(t), t\big) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\big(\gamma(t), \dot{\gamma}(t), t\big) = 0 \tag{7}$$

*holds, where $(q, \dot{q})$ denotes the standard coordinates on TM. The system of equations (7) is referred to as the **Euler-Lagrange equations**.*

Proof. By Hamilton's principle of least action 2, we may assume that γ is an extremal of the action functional S. The proof is divided into two steps.

Step 1: Suppose that the extremal γ of S is conatined in a chart domain U. Let $t \in [t_0, t_1]$ and abreviate $p := (\gamma(t), \dot{\gamma}(t), t)$ Using the formula for the derivative of a function along a curve [Lee13, p. 283], we compute

$$\begin{aligned}
\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L\left(\gamma_\varepsilon(t), \dot{\gamma}_\varepsilon(t), t\right) &= dL_p\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\gamma_\varepsilon(t), \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\dot{\gamma}_\varepsilon(t), 0\right)\\
&= dL_p\left(\frac{d\gamma_\varepsilon^j(t)}{d\varepsilon}(0)\left.\frac{\partial}{\partial q^j}\right|_{\gamma(t)}, \frac{d\dot{\gamma}_\varepsilon^j(t)}{d\varepsilon}(0)\left.\frac{\partial}{\partial \dot{q}^j}\right|_{\dot{\gamma}(t)}, 0\right).
\end{aligned}$$

Using the formula for the differential of a function in coordinates yields

$$dL_p = \frac{\partial L}{\partial q^i}(p)dq^i|_p + \frac{\partial L}{\partial \dot{q}^i}(p)d\dot{q}^i|_p + \frac{\partial L}{\partial t}(p)dt|_p.$$

So

$$\begin{aligned}
0 &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\gamma_\varepsilon)\\
&= \int_{t_0}^{t_1}\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L\left(\gamma_\varepsilon(t), \dot{\gamma}_\varepsilon(t), t\right)dt\\
&= \int_{t_0}^{t_1} dL_p\left(\frac{d\gamma_\varepsilon^j(t)}{d\varepsilon}(0)\left.\frac{\partial}{\partial q^j}\right|_{\gamma(t)}, \frac{d\dot{\gamma}_\varepsilon^j(t)}{d\varepsilon}(0)\left.\frac{\partial}{\partial \dot{q}^j}\right|_{\dot{\gamma}(t)}, 0\right)\\
&= \int_{t_0}^{t_1}\frac{\partial L}{\partial q^i}(p)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt + \int_{t_0}^{t_1}\frac{\partial L}{\partial \dot{q}^i}(p)\frac{d\dot{\gamma}_\varepsilon^i(t)}{d\varepsilon}(0)dt\\
&= \int_{t_0}^{t_1}\frac{\partial L}{\partial q^i}(p)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt + \int_{t_0}^{t_1}\frac{\partial L}{\partial \dot{q}^i}(p)\left(\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)\right)'dt\\
&= \int_{t_0}^{t_1}\frac{\partial L}{\partial q^i}(p)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt + \left.\frac{\partial L}{\partial \dot{q}^i}(p)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)\right|_{t_0}^{t_1} - \int_{t_0}^{t_1}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}(p)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt.
\end{aligned}$$

Let $f \in C_c^\infty(t_0, t_1)$, $j = 1, \ldots, n$ and γ_ε be the variation of γ defined in example 1.11 along the j-th direction. Above computation therefore yields

$$\begin{aligned} 0 &= \int_{t_0}^{t_1} \frac{\partial L}{\partial q^j}(p) f(t) dt + \frac{\partial L}{\partial \dot{q}^j}(p) f(t)\bigg|_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^j}(p) f(t) dt \\ &= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^j}(p) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^j}(p) \right) f(t) dt \end{aligned}$$

for all $f \in C_c^\infty(t_0, t_1)$. Hence the fundamental lemma of calculus of variations 1.1 implies

$$\frac{\partial L}{\partial q^j}(p) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^j}(p) = 0$$

for all $j = 1, \ldots, n$.

Step 2: Suppose that γ is an arbitrary extremal of S. The key technical result used here is the following lemma.

Lemma 1.17 (Lebesgue Number Lemma). *Every open cover of a compact metric space admits a Lebesgue number, i.e. a number $\delta > 0$ such that every subset of the metric space with diameter less than δ is contained in one member of the family.*

Proof. See [Lee11, p. 194]. □

Let $(U_\alpha)_{\alpha \in A}$ be the smooth structure on M, i.e. the maximal smooth atlas. Since γ is continuous, $\left(\gamma^{-1}(U_\alpha)\right)_{\alpha \in A}$ is an open cover for $[t_0, t_1]$. By the Lebesgue number lemma 1.17, this open cover admits a Lebesgue number $\delta > 0$. Let $k \in \mathbb{N}$ such that $(t_1 - t_0)/k < \delta$ and define

$$x_i := \frac{i}{k}(t_1 - t_0) + t_0$$

for all $i = 0, \ldots, k$. Then for all $i = 1, \ldots, k$, $\gamma|_{[x_{i-1}, x_i]}$ is contained in U_α for some $\alpha \in A$. Hence applying step 1 yields the result.

□

Definition 1.18 (Equations of Motion). *The Euler-Lagrange equations of a Lagrangian system are also called* ***equations of motion****.*

Newtonian Mechanics

Newtonian mechanics studies the motion of a system of point masses in three-dimensional Euclidean space. The basic principles of Newtonian mechanics heavily rely on experimental facts. Newtonian mechanics is modelled on the configuration space $\mathbb{R}^3$.

Axiom 3 (Galileo's Relativity Principle).

CHAPTER 2

Hamiltonian Mechanics

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