Classical Mechanics

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Preface

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Contents

Preface . . . . **ii**

Chapter 1: Lagrangian Mechanics **1**
Calculus of Variations 1
Lagrangian Systems and the Principle of Least Action 3

Chapter 2: Hamiltonian Mechanics **5**

Bibliography **6**

Index **7**

CHAPTER 1

Lagrangian Mechanics

Calculus of Variations

Let $n \in \mathbb{N}$, $n \geq 1$, $\Omega \subseteq\subseteq \mathbb{R}^n$ and denote by $|\cdot|$ the n-dimensional Lebesgue measure. Suppose $a^{ij} \in L^\infty(\Omega, \mathbb{R}, |\cdot|)$ symmetric, $f \in L^2(\Omega, \mathbb{R}, |\cdot|)$ and consider the second order homogenous Dirichlet problem

$$\begin{cases} -\dfrac{\partial}{\partial x^j}\left(a^{ij}\dfrac{\partial u}{\partial x^i}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

If $(a^{ij})_{ij}$ is *uniformly elliptic*, i.e. there exists $\lambda > 0$ such that

$$\forall x \in \Omega \forall \xi \in \mathbb{R}^n : \ a^{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2,$$

then (1) admits a unique weak solution $u \in H_0^1(\Omega)$, i.e. u satisfies

$$\forall v \in C_c^\infty(\Omega) : \int_\Omega a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} = \int_\Omega f v.$$

Indeed, observe that

$$\langle \cdot, \cdot \rangle_a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

defined by

$$\langle u, v \rangle_a := \int_\Omega a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} \tag{2}$$

is an inner product on $H_0^1(\Omega)$ with induced norm equivalent to the standard one on $H_0^1(\Omega)$ due to Poincaré's inequality. Applying the Riesz Representation theorem yields the result. Moreover, this solution can be characterized by a *variational principle*, i.e. if we define the *energy functional* $E : H_0^1(\Omega) \to \mathbb{R}$

$$E(v) := \frac{1}{2}\|v\|_a^2 - \int_\Omega f v,$$

for any $v \in H_0^1(\Omega)$, where $\|\cdot\|_a$ denotes the norm induced by the inner product (2), then $u \in H_0^1(\Omega)$ solves (1) if and only if

$$E(u) = \inf_{v \in H_0^1(\Omega)} E(v). \tag{3}$$

Indeed, suppose $u \in H_0^1(\Omega)$ is a solution of (1). Let $v \in H_0^1(\Omega)$. Then $u = v + w$ for $w := u - v \in H_0^1(\Omega)$ and we compute

$$E(v) = E(u+w) = \frac{1}{2}\|u\|_a^2 + \langle u, w\rangle_a + \frac{1}{2}\|w\|_a^2 - \int_\Omega f(u+w) = E(u) + \frac{1}{2}\|w\|_a^2 \geq E(u)$$

with equality if and only if $u = v$ a.e. Conversely, suppose the infimum is attained by some $u \in H_0^1(\Omega)$. Thus by elementary calculus

$$0 = \frac{d}{dt}\bigg|_{t=0} E(u + tv) = \langle u, v\rangle_a - \int_\Omega f v \tag{4}$$

for all $v \in H_0^1(\Omega)$.
Suppose now that $u \in C^2(\bar{\Omega})$ with $u|_{\partial\Omega} = 0$ solves the variational formulation (3) and that $a^{ij} \in C^1(\bar{\Omega})$, $f \in C^0(\bar{\Omega})$. Then integration by parts (see [Lee13, p. 436] and use that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega, \mathbb{R}, |\cdot|)$ for any $1 \leq p < \infty$) yields

$$\langle u, v\rangle_a - \int_\Omega f v = -\sum_{i=1}^n \int_\Omega \operatorname{div}(X_i) v - \int_\Omega f v = \int_\Omega \left(-\frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial u}{\partial x^i}\right) - f\right) v$$

for all $v \in C_c^\infty(\Omega)$ and where $X_i := \left(a^{ij}\frac{\partial}{\partial x^j}\right)_i$. Hence (4) implies

$$\forall v \in C_c^\infty(\Omega) : \int_\Omega \left(-\frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial u}{\partial x^i}\right) - f\right) v = 0.$$

We might expect that this implies

$$-\frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial u}{\partial x^i}\right) = f.$$

That this is indeed the case, is guaranteed by a foundational result in the *calculus of variations* (therefore the name).

Proposition 1.1 (Fundamental Lemma of Calculus of Variations). *Let $\Omega \subseteq \mathbb{R}^n$ open and $f \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}, |\cdot|)$, where $|\cdot|$ denotes the n-dimensional Lebesgue measure. If*

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega f\varphi = 0,$$

then $f = 0$ a.e.

Proof. See [Str14, p. 40]. □

Thus we recovered a second order partial differential equation from the variational formulation. In fact, this is exactly the boundary value problem (1) from the beginning of our exposition. This technique, and in particular the fundamental lemma of calculus of variations 1.1 will play an important role in our treatment of classical mechanics.

Exercise 1.2. [1] Let $\Omega \subseteq\subseteq \mathbb{R}^n$, $2 \leq p < \infty$ and define $\mathcal{B} := \{v \in C^2(\bar{\Omega}) : v|_{\partial\Omega} = 0\}$. Moreover, define $E_p : \mathcal{B} \to \mathbb{R}$ by $E_p(v) := \int_\Omega |\nabla v|^p$. Derive the partial differential equation satisfied by minimizers $u \in \mathcal{B}$ of the variational problem $E(u) = \inf_{v\in\mathcal{B}} E(v)$.

Lagrangian Systems and the Principle of Least Action

Definition 1.3 (Lagrangian System). *A **Lagrangian system** is defined to be a tuple (M, L) consisting of an object $M \in \mathsf{Diff}$ and a morphism $L \in \mathsf{Diff}(TM \times \mathbb{R}, \mathbb{R})$, called a **Lagrangian function**.*

Definition 1.4 (Path Space). *Let $M \in \mathsf{Diff}$, $q_0, q_1 \in M$ and $t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$. Define the **path space of M connecting (q_0, t_0) and (q_1, t_1)** to be the set*

$$\mathcal{P}(M)^{q_0,t_0}_{q_1,t_1} := \{\gamma \in \mathsf{Diff}([t_0, t_1], M) : \gamma(t_0) = q_0 \text{ and } \gamma(t_1) = q_1\}. \tag{5}$$

Remark 1.5. For the sake of simplicity, we will just use the terminology *path space* for $\mathcal{P}(M)^{q_0,t_0}_{q_1,t_1}$ and simply write $\mathcal{P}(M)$. We implicitely assume the conditions of definition 1.4, however.

The path space $\mathcal{P}(M)$ is an infinite dimensional real Fréchet manifold. However, we do not need this fact here and any proof would interrupt our exposition. We therefore follow a more heuristical approach as provided in lecture notes [WRS18, pp. 168–169].

Definition 1.6 (Tangent Space of $\mathcal{P}(M)$). *Let $M \in \mathsf{Diff}$ and $\gamma \in \mathcal{P}(M)$. Then we define the **tangent space of $\mathcal{P}(M)$ at γ**, written $T_\gamma\mathcal{P}(M)$ by*

$$T_\gamma\mathcal{P}(M) := \{X \in \mathfrak{X}(\operatorname{im}\gamma) : X(t_0) = X(t_1) = 0\},$$

where $\mathfrak{X}(\operatorname{im}\gamma)$ denotes the space of vector fields along $\operatorname{im}\gamma$.

Definition 1.7 (Variation). *Let $\mathcal{P}(M)$ be a path space and $\gamma \in \mathcal{P}(M)$. A **variation of γ** is defined to be a morphism $\Gamma \in \mathsf{Diff}([t_0, t_1] \times [-\varepsilon_0, \varepsilon_0], M)$ for some $\varepsilon_0 > 0$ and such that*

- *$\Gamma(t, 0) = \gamma$ for all $t \in [t_1, t_0]$.*
- *$\Gamma(t_0, \varepsilon) = q_0$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.*
- *$\Gamma(t_1, \varepsilon) = q_1$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.*

Remark 1.8. If Γ is a variation of $\gamma \in \mathcal{P}(M)$, we write $\gamma_\varepsilon(-) := \Gamma(-, \varepsilon)$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

Definition 1.9 (Action Functional). *Let (M, L) be a Lagrangian system and $\mathcal{P}(M)$ be a path space. The morphism $S : \mathcal{P}(M) \to \mathbb{R}$ defined by*

$$S(\gamma) := \int_{t_0}^{t_1} L(\gamma(t), \gamma'(t), t)dt$$

*is called the **action functional**.*

[1] This is exercise 1.2.(b) from exercise sheet 1 of the course *Functional Analysis II* taught by *Prof. Dr. A. Carlotto* at ETHZ in the spring of 2018, which can be found here.

Axiom 1 (Hamilton's Principle of Least Action). *Let (M, L) be a Lagrangian system and $\mathcal{P}(M)$ be a path space. A path $\gamma \in \mathsf{Diff}([t_0, t_1], M)$ describes a motion of (M, L) between (q_0, t_0) and q_1, t_1 if and only if*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\gamma_\varepsilon) = 0 \tag{6}$$

for all variations γ_ε of γ.

Theorem 1.10 (Euler-Lagrange Equations). *Let (M, L) be a Lagrangian system. A path $\gamma \in \mathsf{Diff}([t_0, t_1], M)$ describes a motion of (M, L) between (q_0, t_0) and (q_1, t_1) if and only if with respect to any chart (U, q^i)*

$$\frac{\partial L}{\partial q}\big(q(t), \dot{q}(t), t\big) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\big(q(t), \dot{q}(t), t\big) = 0 \tag{7}$$

*holds, where q denotes the coordinate representation of γ. The system of equations (7) is referred to as the **Euler-Lagrange equations**.*

Proof.

□

CHAPTER 2

Hamiltonian Mechanics

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Index

Action functional, 2

Euler-Lagrange equations, 2

Hamilton
 's principle of least action, 2

Lagrangian
 function, 1
 system, 1

Path space, 1

Variation, 1