INTRODUCTION TO DIFFERENTIAL GEOMETRY

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Preface

These are notes for the lecture course *"Differential Geometry I"* given by the second author at ETH Zürich in the fall semester 2017. They are based on a lecture course[1] given by the first author at the University of Wisconsin–Madison in the fall semester 1983.

One can distinguish *extrinsic differential geometry* and *intrinsic differential geometry*. The former restricts attention to submanifolds of Euclidean space while the latter studies manifolds equipped with a Riemannian metric. The extrinsic theory is more accessible because we can visualize curves and surfaces in $\mathbb{R}^3$, but some topics can best be handled with the intrinsic theory. The definitions in Chapter 2 have been worded in such a way that it is easy to read them either extrinsically or intrinsically and the subsequent chapters are mostly (but not entirely) extrinsic. One can teach a self contained one semester course in extrinsic differential geometry by starting with Chapter 2 and skipping the sections marked with an asterisk like §2.8.

This document is designed to be read either as a .pdf file or as a printed book.

We thank everyone who pointed out errors or typos in earlier versions of this book. In particular, we thank Charel Antony and Samuel Trautwein for many helpful comments.

6 August 2018 *Joel W. Robbin and Dietmar A. Salamon*

[1] Extrinsic Differential Geometry

Contents

Chapter 1

What is Differential Geometry?

1.1 Cartography and Differential Geometry

Carl Friedrich Gauß (1777-1855) is the father of differential geometry. He was (among many other things) a cartographer and many terms in modern differential geometry (chart, atlas, map, coordinate system, geodesic, etc.) reflect these origins. He was led to his *Theorema Egregium* (see 5.3.1) by the question of whether it is possible to draw an accurate map of a portion of our planet. Let us begin by discussing a mathematical formulation of this problem.

Consider the two dimensional sphere S^2 sitting in the three dimensional Euclidean space $\mathbb{R}^3$. It is cut out by the equation

$$x^2 + y^2 + z^2 = 1.$$

A map of a small region $U \subset S^2$ is represented mathematically by a one-one correspondence with a small region in the plane $z = 0$. In this book we will represent this with the notation $\phi : U \to \phi(U) \subset \mathbb{R}^2$ and call such an object a *chart* or a *system of local coordinates.*

What does it mean that ϕ is an "accurate" map? Ideally the user would want to use the map to compute the length of a curve in S^2. The length of a curve γ connecting two points $p, q \in S^2$ is given by the formula

$$L(\gamma) = \int_0^1 |\dot{\gamma}(t)| \, dt, \qquad \gamma(0) = p, \;\; \gamma(1) = q,$$

so the user will want the chart ϕ to satisfy $L(\gamma) = L(\phi \circ \gamma)$ for all curves γ. It is a consequence of the *Theorema Egregium* that there is no such chart.

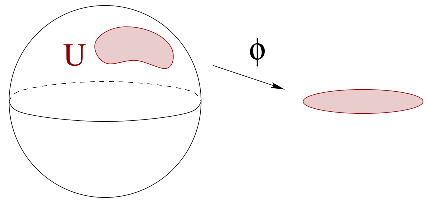


Figure 1.1: A chart

Perhaps the user of such a map will be content to use the map to plot the shortest path between two points p and q in U. This path is called a *geodesic*. Denote this shortest path by γ_{pq}. It satisfies $L(\gamma_{pq}) = d_U(p,q)$ where

$$d_U(p,q) = \inf\{L(\gamma) \,|\, \gamma(t) \in U,\ \gamma(0) = p,\ \ \gamma(1) = q\}$$

so our less demanding user will be content if the chart ϕ satisfies

$$d_U(p,q) = d_E(\phi(p), \phi(q))$$

where $d_E(p,q)$ is the length of the shortest path. It is also a consequence of the *Theorema Egregium* that there is no such chart.

Now suppose our user is content to have a map which makes it easy to navigate along the shortest path connecting the two paths. Ideally the user would use a straight edge, magnetic compass, and protractor to do this. S/he would draw a straight line on the map connecting p and q and steer a course which maintains a constant angle (on the map) between the course and meridians. This *can* be done by the method of stereographic projection. This chart is *conformal* (which means that it preserves angles). According to Wikipedia stereographic projection was known to the ancient Greeks and a map using stereographic projection was constructed in the early 16th century. Exercises 3.7.5, 3.7.12, and 6.4.22 use stereographic projection; the latter exercise deals with the *Poincaré model* of the hyperbolic plane. The hyperbolic plane provides a counter example the Euclid's Parallel Postulate. (See Wikipedia.)

Exercise 1.1.1. It is more or less obvious that for any surface $M \subset \mathbb{R}^3$ there is a unique shortest path in M connecting them if they are sufficiently close. (This will be proved in Theorem 4.4.5.) This shortest path is called the *minimal geodesic* connecting p and q. Use this fact to prove that the minimal geodesic joining two points p and q in S^2 is an arc of the great circle through p and q. (This is the intersection of the sphere with the plane through p, q, and the center of the sphere.) Also prove that the minimal geodesic connecting

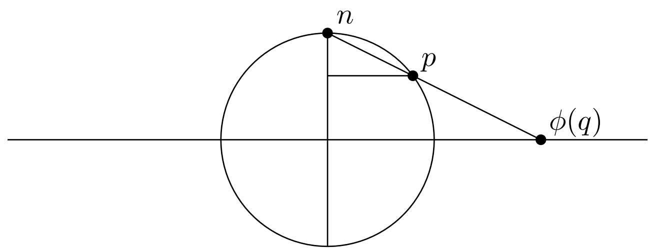


Figure 1.2: Stereographic Projection

two points in a plane is the straight line segment connecting them. **Hint:** Both a great circle in a sphere and a line in a plane are preserved by a reflection. (See also Exercise 4.2.5 below.)

Exercise 1.1.2. Stereographic projection is defined by the condition that for $p \in S^2 \setminus n$ the point $\phi(p)$ lies in the xy-plane $z = 0$ and the three points $n = (0, 0, 1)$, p, and $\phi(p)$ are collinear. Using the formula that the cosine of the angle between two unit vectors is their inner product prove that ϕ is conformal. **Hint:** The plane of p, q, and n intersects the xy-plane in a straight line and the sphere in a circle through n. The plane of n, p, and $\phi(p)$ intersects the sphere in a meridian. A proof that stereographic projection is conformal can be found in [9, page 248]. The proof is elementary in the sense that it doesn't use calculus. An elementary proof can also be found online at `http://people.reed.edu/jerry/311/stereo.pdf`.

Exercise 1.1.3. It may seem fairly obvious that you can't draw an accurate map of a portion of the earth because the sphere is curved. However the cylinder $C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ is also curved, but the map $\psi : \mathbb{R}^2 \to C$ defined by $\psi(s, t) = (\cos t, \sin t, s)$ preserves lengths of curves, i.e. $L(\psi \circ \gamma) = L(\gamma)$ for any curve $\gamma : [a, b] \to \mathbb{R}^2$. Prove this.

1.1.4. Standard Notations. *The standard notations* $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ *denote respectively the natural numbers (= positive integers), the non-negative integers, the integers, the rational numbers, the real numbers, and the complex numbers. We denote the identity map of a set* X *by* id_X *and the* $n \times n$ *identity matrix by* $\mathbb{1}_n$ *or simply* $\mathbb{1}$. *The notation* V^* *is used for the* *dual of a vector space V, but when $\mathbb{K}$ is a field like $\mathbb{R}$ or $\mathbb{C}$ the notation $\mathbb{K}^*$ is sometimes used for the multiplicative group $\mathbb{K} \setminus \{0\}$. The terms* smooth*,* infinitely differentiable*, and C^∞ are all synonymous.*

1.2 Coordinates

The rest of this chapter defines category of smooth manifolds and smooth maps between them. Before giving the precise definitions we will introduce some terminology and give some examples.

Definition 1.2.1. *A* **chart** *on a set M is a pair (ϕ, U) where U is a subset of M and $\phi : U \to \phi(U)$ is a bijection*[1] *from U to an open set $\phi(U)$ in $\mathbb{R}^m$. An* **atlas** *on M is a collection $\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$ of charts such that the domains U_α cover M, i.e.*

$$M = \bigcup_{\alpha \in A} U_\alpha.$$

The idea is that if $\phi(p) = (x_1(p), \ldots, x_m(p))$ for $p \in U$ then the functions x_i form a *system of local coordinates* defined on the subset U of M. The *dimension* of M should be m since it takes m numbers to uniquely specify a point of U. We will soon impose conditions on charts (ϕ, U), however *for the moment we are assuming nothing about the maps ϕ* (other than that they are bijective).

Example 1.2.2. Every open subset $U \subset \mathbb{R}^m$ has an atlas consisting of a single chart, namely $(\phi, U) = (\mathrm{id}_U, U)$ where id_U denotes the identity map of U.

Example 1.2.3. Assume that $W \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open sets, that M is a subset of the product $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$, and $f : W \to V$ is a map whose graph is a subset of M, i.e.

$$\mathrm{graph}(f) := \{(x, y) \in W \times V \mid x \in W, \;\; y = f(x)\} \subset M.$$

Let $U = (W \cap V) \cap \mathrm{graph}(f)$ and let $\phi(x, y) = x$ be the projection of U onto W. Then the pair (ϕ, U) is a chart on M. The inverse map is given by $\phi^{-1}(x) = (x, f(x)))$.

Example 1.2.4. The m-sphere

$$S^m = \left\{p = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1} \mid x_0^2 + \cdots + x_m^2 = 1\right\}$$

[1] See Appendix A.1 for a discussion of the terms injective, surjective, bijective.

has an atlas consisting of the $2m+2$ charts $\phi_{i\pm} : U_{i\pm} \to \mathbb{D}^m$ where $\mathbb{D}^m$ is the open unit disk in $\mathbb{R}^m$, $U_{i\pm} = \{p \in S^m \mid \pm x_i > 0\}$, and $\phi_{i\pm}$ is the projection which discards the ith coordinate. (See Example 2.1.13 below.)

Example 1.2.5. Let $A = A^\mathsf{T} \in \mathbb{R}^{(m+1)\times(m+1)}$ be a symmetric matrix and define a quadratic form $F : \mathbb{R}^{m+1} \to \mathbb{R}$ by

$$F(p) := x^\mathsf{T} Ax, \qquad p = (x_0, \dots, x_m).$$

After a linear change of coordinates the function F has the form

$$F(p) = x_0^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_r^2.$$

(Here r is the rank of the matrix A.) The set $M = F^{-1}(1)$ has an atlas of $2m+1$ charts by the same construction as in Example 1.2.4, in fact S^{m+1} is the special case where $A = \mathbb{1}_n$, the $n \times n$ identity matrix. (See Example 2.1.12 below for another way to construct charts.)

Figure 1.3 enumerates the familiar **quadric surfaces** in $\mathbb{R}^3$. When $W = \mathbb{R}^2$ and $V = \mathbb{R}$ the *paraboloids* are examples of graphs as in Example 1.2.3 and the *ellipsoid* and the two *hyperboloids* are instances of the *quadric surfaces* defined in Example 1.2.5. The sphere is an instance of the ellipsoid ($a = b = c = 1$) and the cylinder is a limit (as $c \to \infty$) of the hyperbolic paraboloid. The pictures were generated by computer using the parameterizations

$$x = a\cos(t)\sin(s), \qquad y = b\sin(t)\sin(s), \qquad z = c\cos(s)$$

for the ellipsoid,

$$x = a\cos(t)\sinh(s), \qquad y = b\sin(t)\sinh(s), \qquad z = c\cosh(s)$$

for the hyperbolic paraboloid, and

$$x = a\cosh(t)\sinh(s), \qquad y = b\sinh(t)\sinh(s), \qquad z = c\cosh(s)$$

for the elliptic paraboloid. These quadric surfaces will be often used in the sequel to illustrate important concepts.

In the following two examples $\mathbb{K}$ denotes either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, $\mathbb{K}^* := \{\lambda \in K \mid \lambda \neq 0\}$ denote the corresponding multiplicative group, and V denotes a vector space over $\mathbb{K}$.

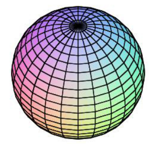
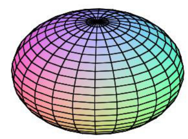
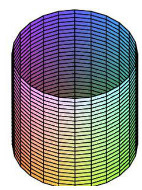
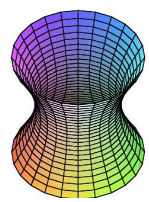
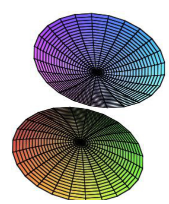
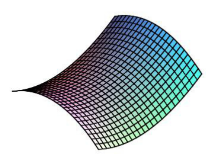
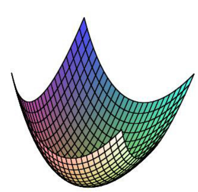
Unit Sphere	$x^2 + y^2 + z^2 = 1$	
Ellipsoid	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$	
Cylinder	$x^2 + y^2 = 1$	
Hyperbolic Hyperboloid (of one sheet)	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$	
Elliptic Hyperboloid (of two sheets)	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = -1$	
Hyperbolic Paraboloid	$z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$	
Elliptic Paraboloid	$z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$	

Figure 1.3: Quadric Surfaces

Example 1.2.6. The **projective space** of V is the set of lines (through the origin) in V. In other words,

$$P(V) = \{\ell \subset V \,|\, \ell \text{ is a 1-dimensional } \mathbb{K}\text{-linear subspace}\}$$

When $\mathbb{K} = \mathbb{R}$ and $V = \mathbb{R}^{m+1}$ this is denoted by $\mathbb{R}\mathrm{P}^m$ and when $\mathbb{K} = \mathbb{C}$ and $\mathbb{C} = \mathbb{R}^{m+1}$ this is denoted by $\mathbb{C}\mathrm{P}^m$. For our purposes we can identify the spaces $\mathbb{C}^{m+1}$ and $\mathbb{R}^{2m+2}$ but the projective spaces $\mathbb{C}\mathrm{P}^m$ and $\mathbb{R}\mathrm{P}^{2m}$ are very different. The various lines $\ell \in P(V)$ intersect in the origin, however, after the harmless identification

$$P(V) = \{[v] \mid v \in V \setminus \{0\}, \quad [v] := \mathbb{K}^* v = \mathbb{K}v \setminus \{0\}$$

the elements of $P(V)$ become disjoint, i.e. $P(V)$ is the set of equivalence classes of an equivalence relation on the open set $V \setminus \{0\}$. Assume that $V = \mathbb{K}^{m+1}$ and define an atlas on $P(V)$ as follows. For each $i = 0, 1, \ldots, m$ let $U_i = \{[v]\,|, v = (x_0, \ldots, x_m)\; x_i \neq 0\}$ and define a bijection $\phi : U_i \to \mathbb{K}^m$ by the formula

$$\phi_i([v]) = \left(\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i} \ldots, \frac{x_m}{x_i} \right).$$

This atlas consists of $m+1$ charts.

Example 1.2.7. For each positive integer k the set

$$G_k(V) := \{\ell \subset V \,|\, \ell \text{ is a } k\text{-dimensional } \mathbb{K}\text{-linear subspace}\}$$

is called the **Grassmann manifold** of k-planes in V. Thus $G_1(V) = P(V)$. Assume that $V = \mathbb{K}^n$ and define an atlas on $G_k(V)$ as follows. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{K}^n$, i.e. e_i is the ith column of the $n \times n$ identity matrix $\mathbb{1}_n$. Each partition $\{1, 2, \ldots, n\} = I \cup J$, $I = \{i_1 < \cdots < i_k\}$, $J = j_1 < \cdots < j_{n-k}$ of the first the first n natural numbers determines a direct sum decomposition

$$\mathbb{K}^n = V = V_I \oplus V_J$$

via the formulas $V_I = \mathbb{K}e_{i_1} + \cdots + \mathbb{K}e_{i_k}$ and $V_J = \mathbb{K}e_{j_1} + \cdots + \mathbb{K}e_{j_{n-k}}$. Let U_I denote the set of $\ell \in G_k(V)$ which are transverse to V_J, i.e. such that $\ell \cap V_J = \{0\}$. The elements of U_I are precisely those k-planes of form $\ell = \mathrm{graph}(A)$ where $A : V_I \to V_J$ is a linear map. Define $\phi_I : U_i \to \mathbb{K}^{k \times (n-k)}$ by the formula

$$\phi_I(\ell) = (a_{rs}), \qquad Ae_{i_r} = \sum_{s=1}^{n-k} a_{sr} e_{j_s}.$$

Exercise 1.2.8. Prove that the set of all pairs (ϕ_I, U_I) as I ranges over the subsets of $\{1, \ldots, n\}$ of cardinality k form an atlas.

1.3 Topological Manifolds*

Definition 1.3.1. *A* **topological manifold** *is a topological space M such that each point $p \in M$ has an open neighborhood U which is homeomorphic to an open subset of a Euclidean space.*

Brouwer's *Invariance of Domain Theorem* asserts that, when $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are nonempty open sets and $\phi : U \to V$ is a homeomorphism, then $m = n$. This means that if M is a connected topological manifold and some point of M has a neighborhood homeomorphic to an open subset of $\mathbb{R}^m$, then every point of M has a neighborhood homeomorphic to an open subset of that same $\mathbb{R}^m$. In this case we say that M has **dimension** m or is m-dimensional or is an m-manifold. Brouwer's theorem is fairly difficult (see [7, p. 126] for example) but if ϕ is a diffeomorphism the result is an easy consequence of the invariance of the rank in linear algebra and the chain rule. (See equation (1.4.1) below.)

By definition, a topological m-manifold M admits an atlas where every chart (ϕ, U) of the atlas is a homeomorphism $\phi : U \to \phi(U)$ from an open set of $U \subset M$ to an open set $\phi(U) \subset \mathbb{R}^m$. The following definition and lemma explains when a given atlas determines a topology on M.

Definition 1.3.2. *Let M be a set. Two charts (ϕ_1, U_1) and (ϕ_2, U_2) are said to be* **topologically compatible** *iff $\phi_1(U_1 \cap U_2)$ is open in $\phi_1(U_1)$, $\phi_2(U_1 \cap U_2)$ is open in $\phi_2(U_2)$, and the* **transition map**

$$\phi_{21} := \phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

is a homeomorphism. An atlas is said to be a **topological atlas** *iff any two charts in this atlas are topologically compatible.*

Lemma 1.3.3. *Let $\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$ be an atlas on a set M. Then*
(a) The collection of all subsets $U \subset M$ such that $\phi_\alpha(U \cap U_\alpha)$ is an open subset of $\mathbb{R}^m$ is a topology on M and M is a topological manifold in this topology.
(b) If M is a topological manifold and $\mathscr{A}$ is an atlas for M such that each ϕ_α is a homeomorphism, then the topology in part (a) coincides with the topology of M.

If M is already a topological manifold, then the collection of all charts (U, ϕ) on M such that ϕ is a homeomorphism is a topological atlas. It is the unique *maximal atlas* in the sense that it contains every other topological atlas. However, we will often need to consider smaller atlases, even finite atlases. Lemma 1.3.3 says that *any* atlas determines the topology of M.

Exercise 1.3.4. Prove Lemma 1.3.3.

Exercise 1.3.5. Equip each example in §1.2 with a topology by showing that the atlas in the example is a topological atlas. Conclude that each of these examples is a topological manifold. **Hint:** The Grassmann Manifold (Example 1.2.7) is tricky because you need an explicit formula for the transition map.

Any subset $S \subset X$ of a topological space X inherits a topology from X, called the **relative topology** of S. A subset $U_0 \subset S$ is called **relatively open** in S (or S**-open** for short) if there is an open set $U \subset X$ such that $U_0 = U \cap S$. A subset $A_0 \subset S$ is called **relatively closed** (or S**-closed** for short) if there is a closed set $A \subset X$ such that $A_0 = A \cap S$. The relative topology on S is the finest topology such that the inclusion map $S \to X$ is continuous.

Exercise 1.3.6. Show that the relative topology satisfies the axioms of a topology (i.e. arbitrary unions and finite intersections of S-open sets are S-open, and the empty set and S itself are S-open). Show that the complement of an S-open set in S is S-closed and vice versa.

Exercise 1.3.7. Each of the sets defined in Exercises 1.2.2, 1.2.3, 1.2.4, and 1.2.5 is a subset of some Euclidean space $\mathbb{R}^k$. Show that the topology in Exercise 1.3.5 is the relative topology inherited from the topology of $\mathbb{R}^k$.[2]

If $\sim$ is an equivalence relation on a topological space X, the **quotient space**

$$Y := X/\!\sim\, := \{[x] \,|\, x \in X\}$$

is the set of all equivalence classes $[x] := \{x' \in X \,|\, x' \sim x\}$. The map

$$\pi : X \to Y$$

defined by $\pi(x) = [x]$ will be called the **obvious projection**. The quotient space inherits the **quotient topology** from Y. Namely, a set $V \subset Y$ is open in this topology iff the preimage $\pi^{-1}(V)$ is open in X. This topology is the coarsest topology on Y such that projection $\pi : X \to Y$ is continuous. Since the operation $V \mapsto \pi^{-1}(V)$ commutes with arbitrary unions and intersections the quotient topology obviously satisfies the axioms of a topology.

Exercise 1.3.8. Show that the atlases for $\mathbb{R}\mathrm{P}^m$ and $\mathbb{C}\mathrm{P}^m$ defined in Exercise 1.2.6 equip $P(V)$ with the quotient topology inherited from the open set $V \setminus \{0\}$. (Recall that in that exercise $V = \mathbb{K}^m$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.)

[2] The topology on $\mathbb{R}^k$ is of course the metric topology defined by the distance function $d(x, y) = |x - y|$.

1.4 Smooth Manifolds Defined*

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. A map $f : U \to V$ is called **smooth** iff it is infinitely differentiable, i.e. iff all its partial derivatives

$$\partial^\alpha f = \frac{\partial^{\alpha_1+\cdots+\alpha_n} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}, \qquad \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}_0^n,$$

exist and are continuous. In later chapters we will sometimes write $C^\infty(U, V)$ for the set of smooth maps from U to V.

Definition 1.4.1. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. For a smooth map $f = (f_1, \ldots, f_m) : U \to V$ and a point $x \in U$ the* **derivative of f at x** *is the linear map $df(x) : \mathbb{R}^n \to \mathbb{R}^m$ defined by*

$$df(x)\xi := \left.\frac{d}{dt}\right|_{t=0} f(x + t\xi) = \lim_{t \to 0} \frac{f(x + t\xi) - f(x)}{t}, \qquad \xi \in \mathbb{R}^n.$$

This linear map is represented by the **Jacobian matrix** *of f at x which will also be denoted by*

$$df(x) := \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix} \in \mathbb{R}^{m \times n}.$$

Note that we use the same notation for the Jacobian matrix and the corresponding linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.

The derivative satisfies the **chain rule**. Namely, if $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, $W \subset \mathbb{R}^\ell$ are open sets and $f : U \to V$ and $g : V \to W$ are smooth maps then $g \circ f : U \to W$ is smooth and

$$d(g \circ f)(x) = dg(f(x)) \circ df(x) : \mathbb{R}^n \to \mathbb{R}^\ell \tag{1.4.1}$$

for every $x \in U$. Moreover the identity map $\mathrm{id}_U : U \to U$ is always smooth and its derivative at every point is the identity map of $\mathbb{R}^n$. This implies that, if $f : U \to V$ is a **diffeomorphism** (i.e. f is bijective and f and f^{-1} are both smooth), then its derivative at every point is an invertible linear map. This is why the Invariance of Domain Theorem (discussed after Definition 1.3.1) is easy for diffeomorphisms: if $f : U \to V$ is a diffeomorphism, then the Jacobian matrix $df(x) \in \mathbb{R}^{m \times n}$ is invertible for every $x \in U$ and so $m = n$. The Inverse Function Theorem (see Theorem A.2.2 in Appendix A.2) is a kind of converse.

Definition 1.4.2 (Smooth Manifold). *Let M be a set. A* **chart** *on M is a pair (ϕ, U) where $U \subset M$ and ϕ is a bijection from U to an open subset $\phi(U) \subset \mathbb{R}^m$ of some Euclidean space. Two charts (ϕ_1, U_1) and (ϕ_2, U_2) are said to be* **smoothly compatible** *iff $\phi_1(U_1 \cap U_2)$ and $\phi_2(U_1 \cap U_2)$ are both open in $\mathbb{R}^m$ and the* **transition map**

$$\phi_{21} = \phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2) \tag{1.4.2}$$

is a diffeomorphism. A **smooth atlas** *on M is a collection $\mathscr{A}$ of charts on M any two of which are smoothly compatible and such that the sets U, as (ϕ, U) ranges over $\mathscr{A}$, cover M (i.e. for every $p \in M$ there is a chart $(\phi, U) \in \mathscr{A}$ with $p \in U$). A* **maximal smooth atlas** *is an atlas which contains every chart which is smoothly compatible with each of its members. A* **smooth manifold** *is a pair consisting of a set M and a maximal atlas $\mathscr{A}$ on M.*

Lemma 1.4.3. *If $\mathscr{A}$ is an atlas, then so is the collection $\overline{\mathscr{A}}$ of all charts compatible with each member of $\mathscr{A}$. The atlas $\overline{\mathscr{A}}$ is obviously maximal. In other words, every atlas extends uniquely to a maximal atlas.*

Proof. Let (ϕ_1, U_1) and (ϕ_2, U_2) be charts in $\overline{\mathscr{A}}$ and let $x \in \phi_1(U_1 \cap U_2)$. Choose a chart $(\phi, U) \in \mathscr{A}$ such that $\phi_1^{-1}(x) \in U$. Then $\phi_1(U \cap U_1 \cap U_2)$ is an open neighborhood of x in $\mathbb{R}^m$ and the transition maps

$$\begin{aligned} &\phi \circ \phi_1^{-1} : \phi_1(U \cap U_1 \cap U_2) \to \phi(U \cap U_1 \cap U_2), \\ &\phi_2 \circ \phi^{-1} : \phi(U \cap U_1 \cap U_2) \to \phi_2(U \cap U_1 \cap U_2) \end{aligned}$$

are smooth by definition of $\overline{\mathscr{A}}$. Hence so is their composition. This shows that the map $\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$ is smooth near x. Since x was chosen arbitrary, this map is smooth. Apply the same argument to its inverse to deduce that it is a diffeomorphism. Thus $\overline{\mathscr{A}}$ is an atlas. □

Definitions 1.4.2 and 1.3.2 are *mutatis mutandis* the same, so every smooth atlas on a set M is *a fortiori* a topological atlas, i.e. every smooth manifold is a topological manifold. (See Lemma 1.3.3.) Moreover the definitions are worded in such a way that it is obvious that every smooth map is continuous.

Exercise 1.4.4. Show that each of the atlases from the examples in §1.2 is a smooth atlas. (You must show that the transition maps from Exercise 1.3.5 are smooth.)

When $\mathscr{A}$ is a smooth atlas on a topological manifold M one says that $\mathscr{A}$ is a **smooth structure** on the (topological) manifold M iff $\mathscr{A} \subset \mathscr{B}$, where $\mathscr{B}$ is the maximal topological atlas on M. *When no confusion can result we generally drop the notation for the maximal smooth atlas as in the following exercise.*

Exercise 1.4.5. Let M, N, and P be smooth manifolds and $f : M \to N$ and $g : N \to P$ be smooth maps. Prove that the identity map id_M is smooth and that the composition $g \circ f : M \to P$ is a smooth map. (This is of course an easy consequence of the chain rule (1.4.1).)

Remark 1.4.6. It is easy to see that a topological manifold can have many distinct smooth structures. For example, $\{(\mathrm{id}_{\mathbb{R}}, \mathbb{R})\}$ and $\{(\phi, \mathbb{R})\}$ where $\phi(x) = x^3$ are atlases on the real numbers which extend to distinct smooth structures but determine the same topology. However these two manifolds are diffeomorphic via the map $x \mapsto x^{1/3}$. In the 1950's it was proved that there are smooth manifolds which are homeomorphic but not diffeomorphic and that there are topological manifolds which admit no smooth structure. In the 1980's it was proved in dimension $m = 4$ that there are uncountably many smooth manifolds that are all homeomorphic to $\mathbb{R}^4$ but no two of them are diffeomorphic to each other. These theorems are very surprising and very deep.

A collection of sets and maps between them is called a *category* if the collection of maps contains the identity map of every set in the collection and the composition of any two maps in the collection is also in the collection. The sets are called the *objects* of the category and the maps are called the *morphisms* of the category. An invertible morphism whose inverse is also in the category is called an *isomorphism.* Some examples are the category of *all* sets and maps, the category of topological spaces and continuous maps (the isomorphisms are the homeomorphisms), the category of topological manifolds and continuous maps between them, and the category of smooth manifolds and smooth maps (the isomorphisms are the diffeomorphisms). Each of the last three categories is a subcategory of the preceding one.

Often categories are enlarged by a kind of "gluing process". For example, the "global" category of smooth manifolds and smooth maps was constructed from the "local" category of open sets in Euclidean space and smooth maps between them via the device of charts and atlases. (The chain rule shows that this local category is in fact a category.) The point of Definition 1.3.2 is to show (via Lemma 1.3.3) that topological manifolds can be defined in an manner analogous to the definition we gave for smooth manifolds in Definition 1.4.2.

Other kinds of manifolds (and hence other kinds of geometry) are defined by choosing other local categories, i.e. by imposing conditions on the transition maps in Equation (1.4.2). For example, a *real analytic manifold* is one where the transition maps are real analytic, a *complex manifold* is one whose coordinate charts take values in $\mathbb{C}^n$ and whose transition maps are holomorphic diffeomorphisms, and a *symplectic manifold* is one whose coordinate charts take values in $\mathbb{R}^{2n}$ and whose transition maps are *canonical transformations* in the sense of classical mechanics. Thus $\mathbb{CP}^n$ is a complex manifold and $\mathbb{RP}^n$ is a real analytic manifold.

1.5 The Master Plan

In studying differential geometry it is best to begin with *extrinsic differential geometry* which is the study of the geometry of submanifolds of Euclidean space as in Examples 1.2.3, and 1.2.5. This is because we can visualize curves and surfaces in $\mathbb{R}^3$. However, there are a few topics in the later chapters which require the more abstract definition 1.4.2 even to say interesting things about extrinsic geometry. There is a generalization to these manifolds involving a structure called a *Riemannian metric*. We will call this generalization *intrinsic differential geometry*. Examples 1.2.6 and 1.2.7 fit into this more general definition so intrinsic differential geometry can be used to study them.

Since an open set in Euclidean space is a smooth manifold the definition of a submanifold of Euclidean space (see §2.1 below) is *mutatis mutandis* the same as the definition of a submanifold of a manifold. The definitions in Chapter 2 are worded in such a way that it is easy to read them either extrinsically or intrinsically and the subsequent chapters are mostly (but not entirely) extrinsic. Those sections which require intrinsic differential geometry (or which translate extrinsic concepts into intrinsic ones) are marked with a *.

Chapter 2

Foundations

This chapter introduces various fundamental concepts that are central to the fields of differential geometry and differential topology. Both fields concern the study of smooth manifolds and their diffeomorphisms. The chapter begins with an introduction to submanifolds of Euclidean space and smooth maps (see §2.1), to tangent spaces and derivatives (see §2.2), and to submanifolds and embeddings (see §2.3). In §2.4 we move on to vector fields and flows and introduce the Lie bracket of two vector fields. Lie groups and their Lie algebras, in the extrinsic setting, are the subject of §2.5. In §2.6 we introduce vector bundles over a manifold as subbundles on a trivial bundle and in §2.7 we prove the theorem of Frobenius. The last two sections of this chapter are concerned with carrying over all these concepts from the extrinsic to the intrinsic setting and can be skipped at first reading (see §2.8 and §2.9).

2.1 Submanifolds of Euclidean Space

To carry out the Master Plan §1.5 we must (as was done in [13]) extend the definition of smooth map to maps $f : X \to Y$ between subsets $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^\ell$ which are not necessarily open. In this case a map $f : X \to Y$ is called **smooth** if for each $x_0 \in X$ there exists an open neighborhood $U \subset \mathbb{R}^k$ of x_0 and a smooth map $F : U \to \mathbb{R}^\ell$ that agrees with f on $U \cap X$. A map $f : X \to Y$ is called a **diffeomorphism** if f is bijective and f and f^{-1} are smooth. When there exists a diffeomorphism $f : X \to Y$ then X and Y are called **diffeomorphic**. *When X and Y are open these definitions coincide with the usage in §1.4.*

Exercise 2.1.1 (Chain Rule). Let $X \subset \mathbb{R}^k$, $Y \subset \mathbb{R}^\ell$, $Z \subset \mathbb{R}^m$ be arbitrary subsets. If $f : X \to Y$ and $g : Y \to Z$ are smooth maps then so is the composition $g \circ f : X \to Z$. The identity map $\mathrm{id} : X \to X$ is smooth.

Exercise 2.1.2. Let $E \subset \mathbb{R}^k$ be an m-dimensional linear subspace and let $v_1, \dots, v_m$ be a basis of E. Then the map $f : \mathbb{R}^m \to E$ defined by $f(x) := \sum_{i=1}^m x_i v_i$ is a diffeomorphism.

Definition 2.1.3. *Let $k, m \in \mathbb{N}_0$. A subset $M \subset \mathbb{R}^k$ is called a* **smooth m-dimensional submanifold of $\mathbb{R}^k$** *iff every point $p \in M$ has an open neighborhood $U \subset \mathbb{R}^k$ such that $U \cap M$ is diffeomorphic to an open subset $\Omega \subset \mathbb{R}^m$. A diffeomorphism*

$$\phi : U \cap M \to \Omega$$

is called a **coordinate chart** *of M and its inverse*

$$\psi := \phi^{-1} : \Omega \to U \cap M$$

is called a **(smooth) parametrization** *of $U \cap M$.*

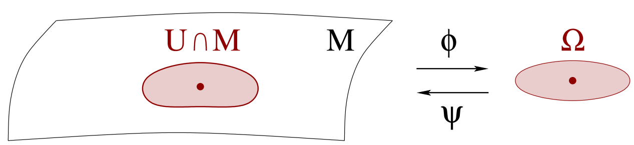


Figure 2.1: A coordinate chart $\phi : U \cap M \to \Omega$.

In Definition 2.1.3 we have used the fact that the domain of a smooth map can be an arbitrary subset of Euclidean space and need not be open (see page 15). The term *m-manifold in $\mathbb{R}^k$* is short for *m-dimensional submanifold of $\mathbb{R}^k$*. In keeping with the Master Plan §1.5 we will sometimes say *manifold* rather than *submanifold of $\mathbb{R}^k$* to indicate that the context holds in both the intrinsic and extrinsic settings.

Lemma 2.1.4. *If $M \subset \mathbb{R}^k$ is a nonempty smooth m-manifold then $m \le k$.*

Proof. Fix an element $p_0 \in M$, choose a coordinate chart $\phi : U \cap M \to \Omega$ with $p_0 \in U$ and values in an open subset $\Omega \subset \mathbb{R}^m$, and denote its inverse by $\psi := \phi^{-1} : \Omega \to U \cap M$. Shrinking U, if necessary, we may assume that ϕ extends to a smooth map $\Phi : U \to \mathbb{R}^m$. This extension satisfies $\Phi(\psi(x)) = \phi(\psi(x)) = x$ and hence $d\Phi(\psi(x))d\psi(x) = \mathrm{id} : \mathbb{R}^m \to \mathbb{R}^m$ for all $x \in \Omega$, by the chain rule. Hence the derivative $d\psi(x) : \mathbb{R}^m \to \mathbb{R}^k$ is injective for all $x \in \Omega$, and hence $m \le k$ because Ω is nonempty. This proves Lemma 2.1.4. $\square$

Example 2.1.5. Consider the 2-sphere

$$M := S^2 = \left\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\right\}$$

depicted in Figure 2.2 and let $U \subset \mathbb{R}^3$ and $\Omega \subset \mathbb{R}^2$ be the open sets

$$U := \left\{(x, y, z) \in \mathbb{R}^3 \,|\, z > 0\right\}, \qquad \Omega := \left\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\right\}.$$

The map $\phi : U \cap M \to \Omega$ given by

$$\phi(x, y, z) := (x, y)$$

is bijective and its inverse $\psi := \phi^{-1} : \Omega \to U \cap M$ is given by

$$\psi(x, y) = (x, y, \sqrt{1 - x^2 - y^2}).$$

Since both ϕ and ψ are smooth, the map ϕ is a coordinate chart on S^2. Similarly, we can use the open sets $z < 0$, $y > 0$, $y < 0$, $x > 0$, $x < 0$ to cover S^2 by six coordinate charts. Hence S^2 is a manifold. A similar argument shows that the unit sphere $S^m \subset \mathbb{R}^{m+1}$ (see Example 2.1.13 below) is a manifold for every integer $m \geq 0$.

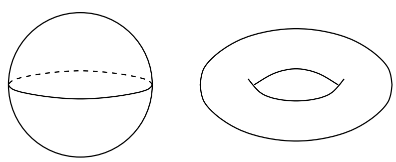


Figure 2.2: The 2-sphere and the 2-torus.

Example 2.1.6. Let $\Omega \subset \mathbb{R}^m$ be an open set and $h : \Omega \to \mathbb{R}^{k-m}$ be a smooth map. Then the graph of h is a smooth submanifold of $\mathbb{R}^m \times \mathbb{R}^{k-m} = \mathbb{R}^k$:

$$M = \mathrm{graph}(h) := \{(x, y) \,|\, x \in \Omega,\, y = h(x)\}.$$

It can be covered by a single coordinate chart $\phi : U \cap M \to V$ where $U := \Omega \times \mathbb{R}^{k-m}$, ϕ is the projection onto Ω, and $\psi := \phi^{-1} : \Omega \to U$ is given by $\psi(x) = (x, h(x))$ for $x \in \Omega$.

Exercise 2.1.7 (The case $m = 0$). Show that a subset $M \subset \mathbb{R}^k$ is a 0-dimensional submanifold if and only if M is discrete, i.e. for every $p \in M$ there is an open set $U \subset \mathbb{R}^k$ such that $U \cap M = \{p\}$.

Exercise 2.1.8 (The case $m = k$). Show that a subset $M \subset \mathbb{R}^m$ is an m-dimensional submanifold if and only if M is open.

Exercise 2.1.9 (Products). If $M_i \subset \mathbb{R}^{k_i}$ is an m_i-manifold for $i = 1, 2$ show that $M_1 \times M_2$ is an $(m_1 + m_2)$-dimensional submanifold of $\mathbb{R}^{k_1+k_2}$. Prove by induction that the n-torus $\mathbb{T}^n$ is a smooth submanifold of $\mathbb{C}^n$.

The next theorem characterizes smooth submanifolds of Euclidean space. In particular condition (iii) will be useful in many cases for verifying the manifold condition. We emphasize that the sets $U_0 := U \cap M$ that appear in Definition 2.1.3 are open subsets of M with respect to the relative topology that M inherits from the ambient space $\mathbb{R}^k$ and that such relatively open sets are also called M-open (see page 9).

Theorem 2.1.10 (Manifolds). *Let m and k be integers with $0 \le m \le k$. Let $M \subset \mathbb{R}^k$ be a set and $p_0 \in M$. Then the following are equivalent.*

(i) *There exists an M-open neighborhood $U_0 \subset M$ of p_0 and a diffeomorphism*

$$\phi_0 : U_0 \to \Omega_0$$

onto an open set $\Omega_0 \subset \mathbb{R}^m$.

(ii) *There exist open sets $U, \Omega \subset \mathbb{R}^k$ and a diffeomorphism $\phi : U \to \Omega$ such that $p_0 \in U$ and*

$$\phi(U \cap M) = \Omega \cap (\mathbb{R}^m \times \{0\}).$$

(iii) *There exists an open set $U \subset \mathbb{R}^k$ and a smooth map $f : U \to \mathbb{R}^{k-m}$ such that $p_0 \in U$, the differential $df(p) : \mathbb{R}^k \to \mathbb{R}^{k-m}$ is surjective for every $p \in U \cap M$, and*

$$U \cap M = f^{-1}(0) = \{q \in U \mid f(q) = 0\}.$$

Moreover, if (i) holds then the diffeomorphism $\phi : U \to \Omega$ in (ii) can be chosen such that $U \cap M \subset U_0$ and $\phi(p) = (\phi_0(p), 0)$ for every $p \in U \cap M$.

Proof. First assume (ii) and denote the diffeomorphism in (ii) by

$$\phi = (\phi_1, \phi_2, \ldots, \phi_k) : U \to \Omega \subset \mathbb{R}^k.$$

Then part (i) holds with $U_0 := U \cap M$, $\Omega_0 := \{x \in \mathbb{R}^m \mid (x, 0) \in \Omega\}$, and

$$\phi_0 := (\phi_1, \ldots, \phi_m)|_{U_0} : U_0 \to \Omega_0,$$

and part (iii) holds with $f := (\phi_{m+1}, \ldots, \phi_k) : U \to \mathbb{R}^{k-m}$. This shows that part (ii) implies both (i) and (iii).

We prove that (i) implies (ii). Let $\phi_0 : U_0 \to \Omega_0$ be the coordinate chart in part (i), let $\psi_0 := \phi_0^{-1} : \Omega_0 \to U_0$ be its inverse, and denote

$$x_0 := \phi_0(p_0) \in \Omega_0.$$

Then, by Lemma 2.1.4, the derivative $d\psi_0(x_0) : \mathbb{R}^m \to \mathbb{R}^k$ is an injective linear map. Hence there exists a matrix $B \in \mathbb{R}^{k\times(k-m)}$ such that

$$\det([d\psi_0(x_0)\, B]) \neq 0.$$

Define the map $\psi : \Omega_0 \times \mathbb{R}^{k-m} \to \mathbb{R}^k$ by

$$\psi(x,y) := \psi_0(x) + By.$$

Then the $k \times k$-matrix $d\psi(x_0, 0) = [d\psi_0(x_0)\, B] \in \mathbb{R}^{k\times k}$ is nonsingular, by choice of B. Hence, by the Inverse Function Theorem A.2.2, there exists an open neighborhood $\widetilde{\Omega} \subset \Omega_0 \times \mathbb{R}^{k-m}$ of $(x_0, 0)$ such that $\widetilde{U} := \psi(\widetilde{\Omega}) \subset \mathbb{R}^k$ is open and $\psi|_{\widetilde{\Omega}} : \widetilde{\Omega} \to \widetilde{U}$ is a diffeomorphism. In particular, the restriction of ψ to $\widetilde{\Omega}$ is injective. Now the set

$$\widetilde{U}_0 := \left\{\psi_0(x)\,\middle|\,x \in \Omega_0,\,(x,0) \in \widetilde{\Omega}\right\} = \left\{p \in U_0\,\middle|\,(\phi_0(p),0) \in \widetilde{\Omega}\right\} \subset M$$

is M-open and contains p_0. Hence, by the definition of the relative topology, there exists an open set $W \subset \mathbb{R}^k$ such that $\widetilde{U}_0 = W \cap M$. Define

$$U := \widetilde{U} \cap W, \qquad \Omega := \widetilde{\Omega} \cap \psi^{-1}(W).$$

Then $U \cap M = \widetilde{U}_0$ and ψ restricts to a diffeomorphism from Ω to U.

Now let $(x,y) \in \Omega$. We claim that

$$\psi(x,y) \in M \qquad \Longleftrightarrow \qquad y = 0. \tag{2.1.1}$$

If $y = 0$ then obviously $\psi(x,y) = \psi_0(x) \in M$. Conversely, let $(x,y) \in \Omega$ and suppose that $p := \psi(x,y) \in M$. Then $p \in U \cap M = \widetilde{U} \cap W \cap M = \widetilde{U}_0 \subset U_0$ and hence $(\phi_0(p),0) \in \widetilde{\Omega}$, by definition of $\widetilde{U}_0$. This implies

$$\psi(\phi_0(p),0) = \psi_0(\phi_0(p)) = p = \psi(x,y).$$

Since the pairs (x,y) and $(\phi_0(p),0)$ both belong to the set $\widetilde{\Omega}$ and the restriction of ψ to $\widetilde{\Omega}$ is injective we obtain $x = \phi_0(p)$ and $y = 0$. This proves (2.1.1). It follows from (2.1.1) that the map $\phi := (\psi|_\Omega)^{-1} : U \to \Omega$ satisfies $\phi(U \cap M) = \{(x,y) \in \Omega\,|\,\psi(x,y) \in M\} = \Omega \cap (\mathbb{R}^m \times \{0\})$. Thus we have proved that (i) implies (ii).

We prove that (iii) implies (ii). Let $f : U \to \mathbb{R}^{k-m}$ be as in part (iii). Then $p_0 \in U$ and the derivative $df(p_0) : \mathbb{R}^k \to \mathbb{R}^{k-m}$ is a surjective linear map. Hence there exists a matrix $A \in \mathbb{R}^{m\times k}$ such that

$$\det \begin{pmatrix} A \\ df(p_0) \end{pmatrix} \neq 0.$$

Define the map $\phi : U \to \mathbb{R}^k$ by

$$\phi(p) := \begin{pmatrix} Ap \\ df(p) \end{pmatrix} \qquad \text{for } p \in U.$$

Then $\det(d\phi(p_0)) \neq 0$. Hence, by the Inverse Function Theorem A.2.2, there exists an open neighborhood $U' \subset U$ of p_0 such that $\Omega' := \phi(U')$ is an open subset of $\mathbb{R}^k$ and the restriction

$$\phi' := \phi|_{U'} : U' \to \Omega'$$

is a diffeomorphism. In particular, the restriction $\phi|_{U'}$ is injective. Moreover, it follows fromn the assumptions on f and the definition of ϕ that

$$U' \cap M = \big\{p \in U' \,\big|\, f(p) = 0\big\} = \big\{p \in U' \,\big|\, \phi(p) \in \mathbb{R}^m \times \{0\}\big\}$$

and hence

$$\phi'(U' \cap M) = \Omega' \cap \big(\mathbb{R}^m \times \{0\}\big).$$

Hence the diffeomorphism $\phi' : U' \to \Omega'$ satisfies the requirements of part (ii). This proves Theorem 2.1.10. □

Definition 2.1.11. *Let $U \subset \mathbb{R}^k$ be an open set and $f : U \to \mathbb{R}^\ell$ be a smooth function. An element $c \in \mathbb{R}^\ell$ is called a* **regular value** *of f if, for all $p \in U$, we have*

$$f(p) = c \qquad \Longrightarrow \qquad df(p) : \mathbb{R}^k \to \mathbb{R}^\ell \text{ is surjective.}$$

Otherwise c is called a **singular value** *of f. Theorem 2.1.10 asserts that, if c is a regular value of f the preimage*

$$M := f^{-1}(c) = \{p \in U \,|\, f(p) = c\}$$

is a smooth $(k - \ell)$-dimensional submanifold of $\mathbb{R}^k$.

Examples and Exercises

Example 2.1.12. Let $A = A^\mathsf{T} \in \mathbb{R}^{k\times k}$ be a symmetric matrix and define the function $f : \mathbb{R}^k \to \mathbb{R}$ by

$$f(x) := x^\mathsf{T} Ax.$$

Then $df(x)\xi = 2x^\mathsf{T} A\xi$ for $x, \xi \in \mathbb{R}^k$ and hence the linear map $df(x) : \mathbb{R}^k \to \mathbb{R}$ is surjective if and only if $Ax \neq 0$. Thus $c = 0$ is the only singular value of f and, for $c \in \mathbb{R} \setminus \{0\}$, the set

$$M := f^{-1}(c) = \left\{ x \in \mathbb{R}^k \,|\, x^\mathsf{T} Ax = c \right\}$$

is a smooth manifold of dimension $m = k - 1$.

Example 2.1.13 (The sphere). As a special case of Example 2.1.12 consider the case $k = m + 1$, $A = \mathbb{1}$, and $c = 1$. Then $f(x) = |x|^2$ and so we have another proof that the unit sphere

$$S^m = \left\{ x \in \mathbb{R}^{m+1} \mid |x|^2 = 1 \right\}$$

in $\mathbb{R}^{m+1}$ is a smooth m-manifold. (See Examples 1.2.4 and 2.1.5.)

Example 2.1.14. Define the map $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y) := |x - y|^2$. This is another special case of Example 2.1.12 and so, for every $r > 0$, the set

$$M := \left\{ (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |x - y| = r \right\}$$

is a smooth 5-manifold.

Example 2.1.15 (The 2-torus). Let $0 < r < 1$ and define $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) := (x^2 + y^2 + r^2 - z^2 - 1)^2 - 4(x^2 + y^2)(r^2 - z^2).$$

This map has zero as a regular value and $M := f^{-1}(0)$ is diffeomorphic to the 2-torus $\mathbb{T}^2 = S^1 \times S^1$. An explicit diffeomorphism is given by

$$(e^{\mathbf{i}s}, e^{\mathbf{i}t}) \mapsto \big((1 + r\cos(s))\cos(t), (1 + r\cos(s))\sin(t), r\sin(s) \big).$$

This example corresponds to the second surface in Figure 2.2.

Exercise: Show that $f(x, y, z) = 0$ if and only if $(\sqrt{x^2 + y^2} - 1)^2 + z^2 = r^2$. Verify that zero is a regular value of f.

Example 2.1.16. The set

$$M := \left\{(x^2, y^2, z^2, yz, zx, xy) \,|\, x, y, z \in \mathbb{R},\ x^2 + y^2 + z^2 = 1\right\}$$

is a smooth 2-manifold in $\mathbb{R}^6$. To see this, define an equivalence relation on the unit sphere $S^2 \subset \mathbb{R}^3$ by $p \sim q$ iff $q = \pm p$. The quotient space (the set of equivalence classes) is called the **real projective plane** and is denoted by

$$\mathbb{R}\mathrm{P}^2 := S^2/\{\pm 1\}.$$

(See Example 1.2.6.) It is equipped with the quotient topology, i.e. a subset $U \subset \mathbb{R}\mathrm{P}^2$ is open, by definition, iff its preimage under the obvious projection $S^2 \to \mathbb{R}\mathrm{P}^2$ is an open subset of S^2. Now the map $f : S^2 \to \mathbb{R}^6$ defined by

$$f(x, y, z) := (x^2, y^2, z^2, yz, zx, xy)$$

descends to a homeomorphism from $\mathbb{R}\mathrm{P}^2$ onto M. The submanifold M is covered by the local smooth parameterizations

$$\begin{aligned}
\Omega \to M &: (x, y) \mapsto f(x, y, \sqrt{1 - x^2 - y^2}),\\
\Omega \to M &: (x, z) \mapsto f(x, \sqrt{1 - x^2 - z^2}, z),\\
\Omega \to M &: (y, z) \mapsto f(\sqrt{1 - y^2 - z^2}, y, z),
\end{aligned}$$

defined on the open unit disc $\Omega \subset \mathbb{R}^2$. We remark the following.

(a) M is *not* the preimage of a regular value under a smooth map $\mathbb{R}^6 \to \mathbb{R}^4$.

(b) M is *not* diffeomorphic to a submanifold of $\mathbb{R}^3$.

(c) The projection $\Sigma := \left\{(yz, zx, xy) \,|\, x, y, z \in \mathbb{R},\ x^2 + y^2 + z^2 = 1\right\}$ of M onto the last three coordinates is called the **Roman surface** and was discovered by Jakob Steiner. The Roman surface can also be represented as the set of solutions $(\xi, \eta, \zeta) \in \mathbb{R}^3$ of the equation $\eta^2\zeta^2 + \zeta^2\xi^2 + \xi^2\eta^2 = \xi\eta\zeta$. It is not a submanifold of $\mathbb{R}^3$.

Exercise: Prove this. Show that M is diffeomorphic to a submanifold of $\mathbb{R}^4$. Show that M is diffeomorphic to $\mathbb{R}\mathrm{P}^2$ as defined in Example 1.2.6.

Exercise 2.1.17. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and define the Hamiltonian function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ (kinetic plus potential energy) by

$$H(x, y) := \frac{1}{2}\,|y|^2 + V(x).$$

Prove that c is a regular value of H if and only if it is a regular value of V.

Exercise 2.1.18. Consider the **general linear group**

$$\mathrm{GL}(n,\mathbb{R}) = \left\{g \in \mathbb{R}^{n\times n} \,|\, \det(g) \neq 0\right\}$$

Prove that the derivative of the function $f = \det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is given by

$$df(g)v = \det(g)\,\mathrm{trace}(g^{-1}v)$$

for every $g \in \mathrm{GL}(n,\mathbb{R})$ and every $v \in \mathbb{R}^{n\times n}$. Deduce that the **special linear group**

$$\mathrm{SL}(n,\mathbb{R}) := \{g \in \mathrm{GL}(n,\mathbb{R}) \,|\, \det(g) = 1\}$$

is a smooth submanifold of $\mathbb{R}^{n\times n}$.

Example 2.1.19. The **orthogonal group**

$$\mathrm{O}(n) := \left\{g \in \mathbb{R}^{n\times n} \,|\, g^\mathsf{T} g = 1\!\!1\right\}$$

is a smooth submanifold of $\mathbb{R}^{n\times n}$. To see this, denote by

$$\mathscr{S}_n := \left\{S \in \mathbb{R}^{n\times n} \,|\, S^\mathsf{T} = S\right\}$$

the vector space of symmetric matrices and define $f : \mathbb{R}^{n\times n} \to \mathscr{S}_n$ by

$$f(g) := g^\mathsf{T} g.$$

Its derivative $df(g) : \mathbb{R}^{n\times n} \to \mathscr{S}_n$ is given by

$$df(g)v = g^\mathsf{T} v + v^\mathsf{T} g.$$

This map is surjective for every $g \in \mathrm{O}(n)$: if $g^\mathsf{T} g = 1\!\!1$ and $S = S^\mathsf{T} \in \mathscr{S}_n$ then the matrix $v := \frac{1}{2} gS$ satisfies

$$df(g)v = \frac{1}{2} g^\mathsf{T} gS + \frac{1}{2}(gS)^\mathsf{T} g = \frac{1}{2}S + \frac{1}{2}S^\mathsf{T} = S.$$

Hence $1\!\!1$ is a regular value of f and so $\mathrm{O}(n)$ is a smooth manifold. It has the dimension

$$\dim \mathrm{O}(n) = n^2 - \dim \mathscr{S}_n = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Exercise 2.1.20. Prove that the set

$$M := \left\{(x,y) \in \mathbb{R}^2 \,|\, xy = 0\right\}$$

is not a submanifold of $\mathbb{R}^2$. **Hint:** If $U \subset \mathbb{R}^2$ is a neighborhood of the origin and $f : U \to \mathbb{R}$ is a smooth map such that $U \cap M = f^{-1}(0)$ then $df(0,0) = 0$.

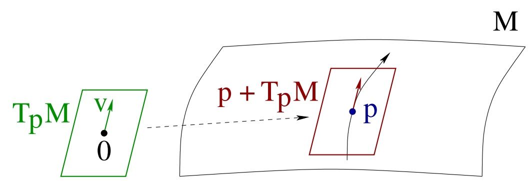


Figure 2.3: The tangent space T_pM and the translated tangent space $p+T_pM$.

2.2 Tangent Spaces and Derivatives

The main reason for first discussing the extrinsic notion of embedded manifolds in Euclidean space as explained in the Master Plan §1.5 is that the concept of a tangent vector is much easier to digest in the embedded case: it is simply the derivative of a curve in M, understood as a vector in the ambient Euclidean space in which M is embedded.

2.2.1 Tangent Space

Definition 2.2.1. *Let $M \subset \mathbb{R}^k$ be a smooth m-dimensional manifold and fix a point $p \in M$. A vector $v \in \mathbb{R}^k$ is called a* **tangent vector** *of M at p if there exists a smooth curve $\gamma : \mathbb{R} \to M$ such that*

$$\gamma(0) = p, \qquad \dot{\gamma}(0) = v.$$

The set

$$T_pM := \{\dot{\gamma}(0) \,|\, \gamma : \mathbb{R} \to M \text{ is smooth, } \gamma(0) = p\}$$

of tangent vectors of M at p is called the **tangent space** *of M at p.*

Theorem 2.2.3 below shows that T_pM is a linear subspace of $\mathbb{R}^k$. As does any linear subspace it contains the origin; it need not actually intersect M. Its translate $p + T_pM$ touches M at p; this is what you should visualize for T_pM (see Figure 2.3).

Remark 2.2.2. Let $p \in M$ be as in Definition 2.2.1 and let $v \in \mathbb{R}^k$. Then

$$v \in T_pM \qquad \Longleftrightarrow \qquad \left\{ \begin{array}{l} \exists \varepsilon > 0 \;\; \exists \gamma : (-\varepsilon, \varepsilon) \to M \text{ such that} \\ \gamma \text{ is smooth, } \gamma(0) = p,\, \dot{\gamma}(0) = v. \end{array} \right.$$

To see this suppose that $\gamma : (-\varepsilon, \varepsilon) \to M$ is a smooth curve with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Define $\widetilde{\gamma} : \mathbb{R} \to M$ by

$$\widetilde{\gamma}(t) := \gamma\left(\frac{\varepsilon t}{\sqrt{\varepsilon^2 + t^2}}\right), \qquad t \in \mathbb{R}.$$

Then $\widetilde{\gamma}$ is smooth and satisfies $\widetilde{\gamma}(0) = p$ and $\dot{\widetilde{\gamma}}(0) = v$. Hence $v \in T_pM$.

Theorem 2.2.3 (Tangent spaces). *Let $M \subset \mathbb{R}^k$ be a smooth m-dimensional manifold and fix a point $p \in M$. Then the following holds.*

(i) *Let $U_0 \subset M$ be an M-open set with $p \in U_0$ and let $\phi_0 : U_0 \to \Omega_0$ be a diffeomorphism onto an open subset $\Omega_0 \subset \mathbb{R}^m$. Let $x_0 := \phi_0(p)$ and let $\psi_0 := \phi_0^{-1} : \Omega_0 \to U_0$ be the inverse map. Then*

$$T_pM = \mathrm{im}\left(d\psi_0(x_0) : \mathbb{R}^m \to \mathbb{R}^k\right).$$

(ii) *Let $U, \Omega \subset \mathbb{R}^k$ be open sets and $\phi : U \to \Omega$ be a diffeomorphism such that $p \in U$ and $\phi(U \cap M) = \Omega \cap (\mathbb{R}^m \times \{0\})$. Then*

$$T_pM = d\phi(p)^{-1}\left(\mathbb{R}^m \times \{0\}\right).$$

(iii) *Let $U \subset \mathbb{R}^k$ be an open neighborhood of p and $f : U \to \mathbb{R}^{k-m}$ be a smooth map such that 0 is a regular value of f and $U \cap M = f^{-1}(0)$. Then*

$$T_pM = \ker df(p).$$

(iv) *T_pM is an m-dimensional linear subspace of $\mathbb{R}^k$.*

Proof. Let $\psi_0 : \Omega_0 \to U_0$ and $x_0 \in \Omega_0$ be as in (i) and let $\phi : U \to \Omega$ be as in (ii). We prove that

$$\mathrm{im}\, d\psi_0(x_0) \subset T_pM \subset d\phi(p)^{-1}\left(\mathbb{R}^m \times \{0\}\right). \tag{2.2.1}$$

To prove the first inclusion in (2.2.1), choose a constant $r > 0$ such that

$$B_r(x_0) := \{x \in \mathbb{R}^m \mid |x - x_0| < r\} \subset \Omega_0.$$

Now let $\xi \in \mathbb{R}^m$ and choose $\varepsilon > 0$ so small that

$$\varepsilon\,|\xi| \le r.$$

Then $x_0 + t\xi \in \Omega_0$ for all $t \in \mathbb{R}$ with $|t| < \varepsilon$. Define $\gamma : (-\varepsilon, \varepsilon) \to M$ by

$$\gamma(t) := \psi_0(x_0 + t\xi) \qquad \text{for } -\varepsilon < t < \varepsilon.$$

Then γ is a smooth curve in M satisfying

$$\gamma(0) = \psi_0(x_0) = p, \qquad \dot\gamma(0) = \left.\frac{d}{dt}\right|_{t=0} \psi_0(x_0 + t\xi) = d\psi_0(x_0)\xi.$$

Hence it follows from Remark 2.2.2 that $d\psi_0(x_0)\xi \in T_pM$, as claimed.

To prove the second inclusion in (2.2.1) we fix a vector $v \in T_pM$. Then, by definition of the tangent space, there exists a smooth curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. Let $U \subset \mathbb{R}^k$ be as in (ii) and choose $\varepsilon > 0$ so small that $\gamma(t) \in U$ for $|t| < \varepsilon$. Then

$$\phi(\gamma(t)) \in \phi(U \cap M) \subset \mathbb{R}^m \times \{0\}$$

for $|t| < \varepsilon$ and hence

$$d\phi(p)v = d\phi(\gamma(0))\dot\gamma(0) = \left.\frac{d}{dt}\right|_{t=0} \phi(\gamma(t)) \in \mathbb{R}^m \times \{0\}.$$

This shows that $v \in d\phi(p)^{-1}\left(\mathbb{R}^m \times \{0\}\right)$ and thus we have proved (2.2.1).

Now the sets $\operatorname{im} d\psi_0(x_0)$ and $d\phi(p)^{-1}\left(\mathbb{R}^m \times \{0\}\right)$ are both m-dimensional linear subspaces of $\mathbb{R}^k$. Hence it follows from (2.2.1) that these subspaces agree and that they both agree with T_pM. Thus we have proved assertions (i), (ii), and (iv).

We prove (iii). If $v \in T_pM$ then there is a smooth curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. For t sufficiently small we have $\gamma(t) \in U$, where $U \subset \mathbb{R}^k$ is the open set in (iii), and $f(\gamma(t)) = 0$. Hence

$$df(p)v = df(\gamma(0))\dot\gamma(0) = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = 0$$

and this implies $T_pM \subset \ker df(p)$. Since T_pM and the kernel of $df(p)$ are both m-dimensional linear subspaces of $\mathbb{R}^k$ we deduce that $T_pM = \ker df(p)$. This proves part (iii) and Theorem 2.2.3. □

Example 2.2.4. Let $A = A^\mathsf{T} \in \mathbb{R}^{k\times k}$ be a nonzero matrix as in Example 2.1.12 and let $c \neq 0$. Then, by Theorem 2.2.3 (iii), the tangent space of the manifold

$$M = \left\{x \in \mathbb{R}^k \,|\, x^\mathsf{T}Ax = c\right\}$$

at a point $x \in M$ is the $k-1$-dimensional linear subspace

$$T_xM = \left\{\xi \in \mathbb{R}^k \,|\, x^\mathsf{T}A\xi = 0\right\}.$$

Example 2.2.5. As a special case of Example 2.2.4 with $A = \mathbb{1}$ and $c = 1$ we find that the tangent space of the unit sphere $S^m \subset \mathbb{R}^{m+1}$ at a point $x \in S^m$ is the orthogonal complement of x:

$$T_xS^m = x^\perp = \left\{\xi \in \mathbb{R}^{m+1} \,|\, \langle x, \xi\rangle = 0\right\}.$$

Here $\langle x, \xi\rangle = \sum_{i=0}^m x_i\xi_i$ denotes the standard inner product on $\mathbb{R}^{m+1}$.

Exercise 2.2.6. What is the tangent space of the 5-manifold

$$M := \left\{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 \,|\, |x-y| = r\right\}$$

at a point $(x,y) \in M$? (See Exercise 2.1.14.)

Example 2.2.7. Let $H(x,y) := \frac{1}{2}|y|^2 + V(x)$ be as in Exercise 2.1.17 and let c be a regular value of H. If $(x,y) \in M := H^{-1}(c)$ Then

$$T_{(x,y)}M = \{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, \langle y,\eta\rangle + \langle \nabla V(x), \xi\rangle = 0\}.$$

Here $\nabla V := (\partial V/\partial x_1, \ldots, \partial V/\partial x_n) : \mathbb{R}^n \to \mathbb{R}^n$ denotes the gradient of V.

Exercise 2.2.8. The tangent space of $\mathrm{SL}(n,\mathbb{R})$ at the identity matrix is the space

$$\mathfrak{sl}(n,\mathbb{R}) := T_{\mathbb{1}}\mathrm{SL}(n,\mathbb{R}) = \left\{\xi \in \mathbb{R}^{n\times n} \,|\, \mathrm{trace}(\xi) = 0\right\}$$

of traceless matrices. (Prove this, using Exercise 2.1.18.)

Example 2.2.9. The tangent space of $\mathrm{O}(n)$ at g is

$$T_g\mathrm{O}(n) = \left\{v \in \mathbb{R}^{n\times n} \,|\, g^{\mathsf{T}}v + v^{\mathsf{T}}g = 0\right\}.$$

In particular, the tangent space of $\mathrm{O}(n)$ at the identity matrix is the space of skew-symmetric matrices

$$\mathfrak{o}(n) := T_{\mathbb{1}}\mathrm{O}(n) = \left\{\xi \in \mathbb{R}^{n\times n} \,|\, \xi^{\mathsf{T}} + \xi = 0\right\}$$

To see this, choose a smooth curve $\mathbb{R} \to \mathrm{O}(n) : t \mapsto g(t)$. Then $g(t)^{\mathsf{T}}g(t) = \mathbb{1}$ for all $t \in \mathbb{R}$ and, differentiating this identity with respect to t, we obtain $g(t)^{\mathsf{T}}\dot{g}(t) + \dot{g}(t)^{\mathsf{T}}g(t) = 0$ for every t. Hence every matrix $v \in T_g\mathrm{O}(n)$ satisfies the equation $g^{\mathsf{T}}v + v^{\mathsf{T}}g = 0$. With this understood, the claim follows from the fact that $g^{\mathsf{T}}v + v^{\mathsf{T}}g = 0$ if and only if the matrix $\xi := g^{-1}v$ is skew-symmetric and that the space of skew-symmetric matrices in $\mathbb{R}^{n\times n}$ has dimension $n(n-1)/2$.

Exercise 2.2.10. Let $\Omega \subset \mathbb{R}^m$ be an open set and $h : \Omega \to \mathbb{R}^{k-m}$ be a smooth map. Prove that the tangent space of the graph of h at a point $(x, h(x))$ is the graph of the differential $dh(x) : \mathbb{R}^m \to \mathbb{R}^{k-m}$:

$$M = \{(x, h(x)) \,|\, x \in \Omega\}, \qquad T_{(x,h(x))}M = \{(\xi, dh(x)\xi) \,|\, \xi \in \mathbb{R}^m\}.$$

Exercise 2.2.11 (Monge coordinates). Let M be a smooth m-manifold in $\mathbb{R}^k$ and suppose that $p \in M$ is such that the projection $T_pM \to \mathbb{R}^m \times \{0\}$ is invertible. Prove that there exists an open set $\Omega \subset \mathbb{R}^m$ and a smooth map $h : \Omega \to \mathbb{R}^{k-m}$ such that the graph of h is an M-open neighborhood of p (see Example 2.1.6). Of course, the projection $T_pM \to \mathbb{R}^m \times \{0\}$ need not be invertible, but it must be invertible for at least one of the $\binom{k}{m}$ choices of the m dimensional coordinate plane. Hence every point of M has an M-open neighborhood which may be expressed as a graph of a function of some of the coordinates in terms of the others as in e.g. Example 2.1.5.

2.2.2 Derivative

A key purpose behind the concept of a smooth manifold is to carry over the notion of a smooth map and its derivatives from the realm of first year analysis to the present geometric setting. Here is the basic definition. It appeals to the notion of a smooth map between arbitrary subsets of Euclidean spaces as introduced on page 15.

Definition 2.2.12. *Let $M \subset \mathbb{R}^k$ be an m-dimensional smooth manifold and*

$$f : M \to \mathbb{R}^\ell$$

be a smooth map. The **derivative** *of f at a point $p \in M$ is the map*

$$df(p) : T_pM \to \mathbb{R}^\ell$$

defined as follows. Given a tangent vector $v \in T_pM$ choose a smooth curve

$$\gamma : \mathbb{R} \to M$$

satisfying

$$\gamma(0) = p, \qquad \dot\gamma(0) = v.$$

Now define the vector

$$df(p)v \in \mathbb{R}^\ell$$

by

$$df(p)v := \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = \lim_{h\to 0} \frac{f(\gamma(h)) - f(p)}{h}. \tag{2.2.2}$$

That the limit on the right in equation (2.2.2) exists follows from our assumptions. We must prove, however, that the derivative is well defined, i.e. that the right hand side of (2.2.2) depends only on the tangent vector v and not on the choice of the curve γ used in the definition. This is the content of the first assertion in the next theorem.

Theorem 2.2.13 (Derivatives). *Let $M \subset \mathbb{R}^k$ be an m-dimensional smooth manifold and $f : M \to \mathbb{R}^\ell$ be a smooth map. Fix a point $p \in M$. Then the following holds.*

(i) *The right hand side of* (2.2.2) *is independent of γ.*

(ii) *The map $df(p) : T_pM \to \mathbb{R}^\ell$ is linear.*

(iii) *If $N \subset \mathbb{R}^\ell$ is a smooth n-manifold and $f(M) \subset N$ then*

$$df(p)T_pM \subset T_{f(p)}N.$$

(iv) (Chain Rule) *Let N be as in (iii), suppose that $f(M) \subset N$, and let $g : N \to \mathbb{R}^d$ be a smooth map. Then*

$$d(g \circ f)(p) = dg(f(p)) \circ df(p) : T_pM \to \mathbb{R}^d.$$

(v) *If $f = \mathrm{id} : M \to M$ then $df(p) = \mathrm{id} : T_pM \to T_pM$.*

Proof. We prove (i). Let $v \in T_pM$ and $\gamma : \mathbb{R} \to M$ be as in Definition 2.2.12. By definition there is an open neighborhood $U \subset \mathbb{R}^k$ of p and a smooth map $F : U \to \mathbb{R}^\ell$ such that

$$F(p') = f(p') \qquad \text{for all } p' \in U \cap M.$$

Let $dF(p) \in \mathbb{R}^{\ell \times k}$ denote the Jacobian matrix (i.e. the matrix of all first partial derivatives) of F at p. Then, since $\gamma(t) \in U \cap M$ for t sufficiently small, we have

$$\begin{aligned} dF(p)v &= dF(\gamma(0))\dot{\gamma}(0) \\ &= \left.\frac{d}{dt}\right|_{t=0} F(\gamma(t)) \\ &= \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)). \end{aligned}$$

The right hand side of this identity is independent of the choice of F while the left hand side is independent of the choice of γ. Hence the right hand side is also independent of the choice of γ and this proves (i). Assertion (ii) follows immediately from the identity

$$df(p)v = dF(p)v$$

just established.

Assertion (iii) follows directly from the definitions. Namely, if γ is as in Definition 2.2.12 then

$$\beta := f \circ \gamma : \mathbb{R} \to N$$

is a smooth curve in N satisfying

$$\beta(0) = f(\gamma(0)) = f(p) =: q, \qquad \dot\beta(0) = df(p)v =: w.$$

Hence $w \in T_qN$. Assertion (iv) also follows directly from the definitions. If $g : N \to \mathbb{R}^d$ is a smooth map and β, q, w are as above then

$$\begin{aligned} d(g \circ f)(p)v &= \left.\frac{d}{dt}\right|_{t=0} g(f(\gamma(t))) \\ &= \left.\frac{d}{dt}\right|_{t=0} g(\beta(t)) \\ &= dg(q)w \\ &= dg(f(p))df(p)v. \end{aligned}$$

and this proves (iv). Assertion (v) follows directly from the definitions and this proves Theorem 2.2.13. □

Corollary 2.2.14 (Diffeomorphisms). *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $N \subset \mathbb{R}^\ell$ be a smooth n-manifold and let $f : M \to N$ be a diffeomorphism. Then $m = n$ and the differential $df(p) : T_pM \to T_{f(p)}N$ is a vector space isomorphism with inverse*

$$df(p)^{-1} = df^{-1}(f(p)) : T_{f(p)}N \to T_pM$$

for all $p \in M$.

Proof. Define $g := f^{-1} : N \to M$ so that

$$g \circ f = \mathrm{id}_M, \qquad f \circ g = \mathrm{id}_N.$$

Then it follows from Theorem 2.2.13 that, for $p \in M$ and $q := f(p) \in N$, we have

$$dg(q) \circ df(p) = \mathrm{id} : T_pM \to T_pM, \qquad df(p) \circ dg(q) = \mathrm{id} : T_qN \to T_qN.$$

Hence $df(p) : T_pM \to T_qN$ is a vector space isomorphism with inverse

$$dg(q) = df(p)^{-1} : T_qN \to T_pM.$$

Hence $m = n$ and this proves Corollary 2.2.14. □

2.2.3 The Inverse Function Theorem

Corollary 2.2.14 is analogous to the corresponding assertion for smooth maps between open subsets of Euclidean space. Likewise, the inverse function theorem for manifolds is a partial converse of Corollary 2.2.14.

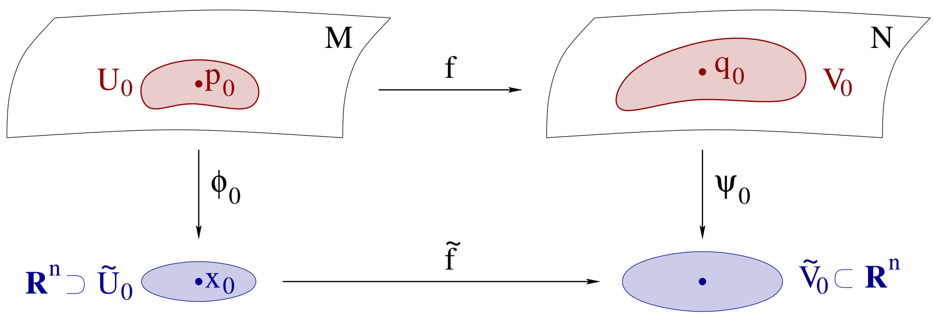


Figure 2.4: The Inverse Function Theorem.

Theorem 2.2.15 (Inverse Function Theorem). *Assume that $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^\ell$ are smooth n-manifolds and $f : M \to N$ is a smooth map. Let $p_0 \in M$ and suppose that the differential $df(p_0) : T_{p_0}M \to T_{f(p_0)}N$ is a vector space isomorphism. Then there is an M-open neighborhood $U \subset M$ of p_0 such that $V := f(U) \subset N$ is an N-open subset of N and the restriction $f|_U : U \to V$ is a diffeomorphism.*

Proof. Choose coordinate charts $\phi_0 : U_0 \to \widetilde{U}_0$, defined on an M-open neighborhood $U_0 \subset M$ of p_0 onto an open set $\widetilde{U}_0 \subset \mathbb{R}^n$, and $\psi_0 : V_0 \to \widetilde{V}_0$, defined on an N-open neighborhood $V_0 \subset N$ of $q_0 := f(p_0)$ onto an open set $\widetilde{V}_0 \subset \mathbb{R}^n$. Shrinking U_0, if necessary, we may assume that $f(U_0) \subset V_0$. Then the map

$$\widetilde{f} := \psi_0 \circ f \circ \phi_0^{-1} : \widetilde{U}_0 \to \widetilde{V}_0$$

(see Figure 2.4) is smooth and its differential $d\widetilde{f}(x_0) : \mathbb{R}^n \to \mathbb{R}^n$ is bijective at $x_0 := \phi_0(p_0)$. Hence the Inverse Function Theorem A.2.2 asserts that there exists an open neighborhood $\widetilde{U} \subset \widetilde{U}_0$ of x_0 such that $\widetilde{V} := \widetilde{f}(\widetilde{U})$ is an open subset of $\widetilde{V}_0$ and the restriction of $\widetilde{f}$ to $\widetilde{U}$ is a diffeomorphism from $\widetilde{U}$ to $\widetilde{V}$. Hence the assertion holds with $U := \phi_0^{-1}(\widetilde{U})$ and $V := \psi_0^{-1}(\widetilde{V})$. This proves Theorem 2.2.15. □

Definition 2.2.16 (Regular value). *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold, let $N \subset \mathbb{R}^\ell$ be a smooth n-manifold, and let $f : M \to N$ be a smooth map. An element $q \in N$ is called a* **regular value** *of f if, for every $p \in M$ with $f(p) = q$, the differential $df(p) : T_pM \to T_{f(p)}N$ is surjective.*

Theorem 2.2.17 (Regular Values). *Let $f : M \to N$ be as in Definition 2.2.16 and let $q \in N$ be a regular value of f. Then the set*

$$P := f^{-1}(q) = \{p \in M \mid f(p) = q\}$$

is a smooth submanifold of $\mathbb{R}^k$ of dimension $m - n$ and, for each point $p \in P$, its tangent space at p is given by

$$T_pP = \ker df(p) = \{v \in T_pM \mid df(p)v = 0\}.$$

Proof. Let $p_0 \in P$ and choose a coordinate chart $\phi_0 : U_0 \to \phi_0(U_0) \subset \mathbb{R}^m$ on an M-open neighborhood $U_0 \subset M$ of p_0. Likewise, choose a coordinate chart $\psi_0 : V_0 \to \psi_0(V_0) \subset \mathbb{R}^n$ on an N-open neighborhood $V_0 \subset N$ of q. Shrinking U_0, if necessary, we may assume that $f(U_0) \subset V_0$. Then the point $c_0 := \psi_0(q)$ is a regular value of the map

$$f_0 := \psi_0 \circ f \circ \phi_0^{-1} : \phi_0(U_0) \to \mathbb{R}^n.$$

Namely, if $x \in \phi_0(U_0)$ satisfies $f_0(x) = c_0$, then $p := \phi_0^{-1}(x) \in U_0 \cap P$, so the maps $d\phi_0^{-1}(x) : \mathbb{R}^m \to T_pM$, $df(p) : T_pM \to T_qN$, and $d\psi_0(q) : T_qN \to \mathbb{R}^n$ are all surjective, hence so is their composition, and by the chain rule this composition is the derivative $df_0(x) : \mathbb{R}^m \to \mathbb{R}^n$. With this understood, it follows from Theorem 2.1.10 that the set

$$f_0^{-1}(c_0) = \left\{x \in \phi_0(U_0) \mid f(\phi_0^{-1}(x)) = q\right\} = \phi_0(U_0 \cap P)$$

is a manifold of of dimension $m-n$ contained in the open set $\phi_0(U_0) \subset \mathbb{R}^m$. Using Definition 2.1.3 and shrinking U_0 further, if necessary, we may assume that the set $\phi_0(U_0 \cap P)$ is diffeomorphic to an open subset of $\mathbb{R}^{m-n}$. Compsing this diffeomorphism with ϕ_0 we find that $U_0 \subset P$ is diffeomorphic to the same open subset of $\mathbb{R}^{m-n}$. Since the set $U_0 \subset M$ is M-open, there exists an open set $U \subset \mathbb{R}^k$ such that $U \cap M = U_0$, hence $U \cap P = U_0 \cap P$, and so $U_0 \cap P$ is a P-open neighborhood of p_0. Thus we have proved that every element $p_0 \in P$ has a P-open neigborhood that is diffeomorphic to an open subset of $\mathbb{R}^{m-n}$. Thus $P \subset \mathbb{R}^k$ is a manifold of dimension $m - n$ (Definition 2.1.3).

Now let $p \in P$ and $v \in T_pP$. Then there exists a smooth curve $\gamma : \mathbb{R} \to P$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. Since $f(\gamma(t)) = 0$ for all t, we have

$$df(p)v = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) = 0$$

and so $v \in \ker df(p)$. Hence $T_pP \subset \ker df(p)$ and equality holds because both T_pP and $\ker df(p)$ are $(m-n)$-dimensional linear subspaces of $\mathbb{R}^k$. This proves Theorem 2.2.17. $\square$

2.3 Submanifolds and Embeddings

This section deals with subsets of a manifold M that are themselves manifolds as in Definition 2.1.3. Such subsets are called submanifolds of M.

Definition 2.3.1 (**Submanifold**). *Let $M \subset \mathbb{R}^k$ be an m-dimensional manifold. A subset $L \subset M$ is called a* **submanifold** *of M of dimension ℓ, if L itself is an ℓ-manifold.*

Definition 2.3.2 (**Embedding**). *Let $M \subset \mathbb{R}^k$ be an m-dimensional manifold and $N \subset \mathbb{R}^\ell$ be an n-dimensional manifold. A smooth map $f : N \to M$ is called an* **immersion** *if its differential $df(q) : T_qN \to T_{f(q)}M$ is injective for every $q \in N$. It is called* **proper** *if, for every compact subset $K \subset f(N)$, the preimage $f^{-1}(K) = \{q \in N \mid f(q) \in K\}$ is compact. The map f is called an* **embedding** *if it is a proper injective immersion.*

Remark 2.3.3. In our definition of proper maps it is important that the compact set K is required to be contained in the image of f. The literature also contains a stronger definition of *proper* which requires that $f^{-1}(K)$ is a compact subset of M for every compact subset $K \subset N$, whether or not K is contained in the image of f. This holds if and only if the map f is proper in the sense of Definition 2.3.2 and has an M-closed image. (Exercise!)

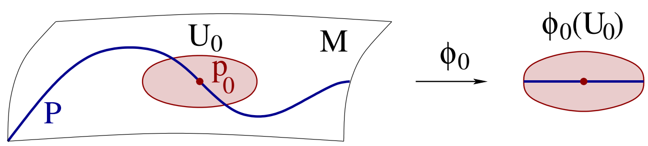


Figure 2.5: A coordinate chart adapted to a submanifold.

Theorem 2.3.4 (**Submanifolds**). *Let $M \subset \mathbb{R}^k$ be an m-dimensional manifold and $N \subset \mathbb{R}^\ell$ be an n-dimensional manifold.*

(i) *If $f : N \to M$ is an embedding then $f(N)$ is a submanifold of M.*

(ii) *If $P \subset M$ is a submanifold then the inclusion $P \to M$ is an embedding.*

(iii) *A subset $P \subset M$ is a submanifold of dimension n if and only if, for every $p_0 \in P$, there exists a coordinate chart $\phi : U \to \mathbb{R}^m$ on an M-open neighborhood U of p_0 such that $\phi(U \cap P) = \phi(U) \cap (\mathbb{R}^n \times \{0\})$ (Figure 2.5).*

(iv) *A subset $P \subset M$ is a submanifold of dimension n if and only if, for every $p_0 \in P$, there exists an M-open neighborhood $U \subset M$ of p_0 and a smooth map $g : U \to \mathbb{R}^{m-n}$ such that 0 is a regular value of g and $U \cap P = g^{-1}(0)$.*

Proof. See page 35. □

Lemma 2.3.5 (Embeddings). *Let M and N be as in Theorem 2.3.4, let $f : N \to M$ be an embedding, let $q_0 \in N$, and define*

$$P := f(N), \qquad p_0 := f(q_0) \in P.$$

Then there exists an M-open neighborhood $U \subset M$ of p_0, an N-open neighborhood $V \subset N$ of q_0, an open neighborhood $W \subset \mathbb{R}^{m-n}$ of the origin, and a diffeomorphism $F : V \times W \to U$ such that, for all $q \in V$ and all $z \in W$,

$$F(q, 0) = f(q), \tag{2.3.1}$$

$$F(q, z) \in P \qquad \Longleftrightarrow \qquad z = 0. \tag{2.3.2}$$

Proof. Choose any coordinate chart $\phi_0 : U_0 \to \mathbb{R}^m$ on an M-open neighborhood $U_0 \subset M$ of p_0. Then $d(\phi_0 \circ f)(q_0) = d\phi_0(f(q_0)) \circ df(q_0) : T_{q_0}N \to \mathbb{R}^m$ is injective. Hence there is a linear map $B : \mathbb{R}^{m-n} \to \mathbb{R}^m$ such that the map

$$T_{q_0}N \times \mathbb{R}^{m-n} \to \mathbb{R}^m : (w, \zeta) \mapsto d(\phi_0 \circ f)(q_0)w + B\zeta \tag{2.3.3}$$

is a vector space isomorphism. Define the set

$$\Omega := \left\{(q, z) \in N \times \mathbb{R}^{m-n} \,|\, f(q) \in U_0,\, \phi_0(f(q)) + Bz \in \phi_0(U_0)\right\}.$$

This is an open subset of $N \times \mathbb{R}^{m-n}$ and we define $F : \Omega \to M$ by

$$F(q, z) := \phi_0^{-1}\left(\phi_0(f(q)) + Bz\right).$$

This map is smooth, it satisfies $F(q, 0) = f(q)$ for all $q \in f^{-1}(U_0)$, and the derivative $dF(q_0, 0) : T_{q_0}N \times \mathbb{R}^{m-n} \to T_{p_0}M$ is the composition of the map (2.3.3) with $d\phi_0(p_0)^{-1} : \mathbb{R}^m \to T_{p_0}M$ and so is a vector space isomorphism. Thus the Inverse Function Theorem 2.2.15 asserts that there is an N-open neighborhood $V_0 \subset N$ of q_0 and an open neighborhood $W_0 \subset \mathbb{R}^{m-n}$ of the origin such that $V_0 \times W_0 \subset \Omega$, the set $U_0 := F(V_0 \times W_0)$ is M-open, and the restriction of F to $V_0 \times W_0$ is a diffeomorphism onto U_0. Thus we have constructed a diffeomorphism $F : V_0 \times W_0 \to U_0$ that satisfies (2.3.1).

We claim that the restriction of F to the product $V \times W$ of sufficiently small open neighborhoods $V \subset N$ of q_0 and $W \subset \mathbb{R}^{m-n}$ of the origin also satisfies (2.3.2). Otherwise, there exist sequences $q_i \in V_0$ converging to q_0 and $z_i \in W_0 \setminus \{0\}$ converging to zero such that $F(q_i, z_i) \in P$. Hence there exists a sequence $q_i' \in N$ such that $F(q_i, z_i) = f(q_i')$. This sequence converges to $f(q_0)$. Since f is proper we may assume, passing to a suitable subsequence if necessary, that q_i' converges to a point $q_0' \in N$. Then

$$f(q_0') = \lim_{i\to\infty} f(q_i') = \lim_{i\to\infty} F(q_i, z_i) = f(q_0).$$

Since f is injective, this implies $q_0' = q_0$. Hence $(q_i', 0) \in V_0 \times W_0$ for i sufficiently large and $F(q_i', 0) = f(q_i') = F(q_i, z_i)$. This contradicts the fact that the map $F : V_0 \times W_0 \to M$ is injective, and proves Lemma 2.3.5. $\square$

Proof of Theorem 2.3.4. We prove (i). Let $q_0 \in N$, denote $p_0 := f(q_0) \in P$, and choose a diffeomorphism $F : V \times W \to U$ as in Lemma 2.3.5. Then set $V \subset N$ is diffeomorphic to an open subset of $\mathbb{R}^n$ (after schrinking V if necessry), the set $U \cap P$ is P-open because $U \subset M$ is M-open, and we have $U \cap P = \{F(q, 0) \,|\, q \in V\} = f(V)$ by (2.3.1) and (2.3.2). Hence the map $f : V \to U \cap P$ is a diffeomorphism whose inverse is the composition of the smooth maps $F^{-1} : U \cap P \to V \times W$ and $V \times W \to V : (q, z) \mapsto q$. Hence a P-open neighborhood of p_0 is diffeomorphic to an open subset of $\mathbb{R}^n$. Since $p_0 \in P$ was chosen arbitrary, this shows that P is an n-dimensional submanifold of M.

We prove (ii). The inclusion $\iota : P \to M$ is obviously smooth and injective (it extends to the identity map on $\mathbb{R}^k$). Moreover, $T_pP \subset T_pM$ for every $p \in P$ and the differential $d\iota(p) : T_pP \to T_pM$ is the obvious inclusion for every $p \in P$. That ι is proper follows immediately from the definition. Hence ι is an embedding.

We prove (iii). If a coordinate chart ϕ_0 as in (iii) exists then the set $U_0 \cap P$ is P-open and is diffeomorphic to an open subset of $\mathbb{R}^n$. Since the point $p_0 \in P$ was chosen arbitrary this proves that P is an n-dimensional submanifold of M. Conversely, suppose that P is an n-dimensional submanifold of M and let $p_0 \in P$. Choose any coordinate chart $\phi_0 : U_0 \to \mathbb{R}^m$ of M defined on an M-open neighborhood $U_0 \subset M$ of p_0. Then $\phi_0(U_0 \cap P)$ is an n-dimensional submanifold of $\mathbb{R}^m$. Hence Theorem 2.1.10 asserts that there are open sets $V, W \subset \mathbb{R}^m$ with $p_0 \in V \subset \phi_0(U_0)$ and a diffeomorphism $\psi : V \to W$ such that

$$\phi_0(p_0) \in V, \qquad \psi(V \cap \phi_0(U_0 \cap P)) = W \cap (\mathbb{R}^n \times \{0\}).$$

Now define $U := \phi_0^{-1}(V) \subset U_0$. Then $p_0 \in U$, the chart ϕ_0 restricts to a diffeomorphism from U to V, the composition $\phi := \psi \circ \phi_0|_U : U \to W$ is a diffeomorphism, and $\phi(U \cap P) = \psi(V \cap \phi_0(U_0 \cap P)) = W \cap (\mathbb{R}^n \times \{0\})$.

We prove (iv). That the condition is sufficient follows directly from Theorem 2.2.17. To prove that it is necessary, assume that $P \subset M$, is a submanifold of dimension n, fix an element $p_0 \in P$, and choose a coordinate chart $\phi : U \to \mathbb{R}^m$ on an M-open neighborhood $U \subset M$ of p_0 as in part (iii). Define the map $g : U \to \mathbb{R}^{m-n}$ by $g(p) := (\phi_{n+1}(p), \ldots, \phi_m(p))$ for $p \in U$. Then 0 is a regular value of g and $g^{-1}(0) = U \cap P$. This proves Theorem 2.3.4. $\square$

Example 2.3.6. Let $S^1 \subset \mathbb{R}^2 \cong \mathbb{C}$ be the unit circle and consider the map $f : S^1 \to \mathbb{R}^2$ given by $f(x, y) := (x, xy)$. This map is a proper immersion but is not injective (the points $(0, 1)$ and $(0, -1)$ have the same image under f). The image $f(S^1)$ is a figure 8 in $\mathbb{R}^2$ and is not a submanifold (Figure 2.6).

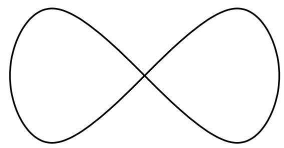


Figure 2.6: A proper immersion.

Example 2.3.7. Consider the restriction of the map f in Example 2.3.6 to the submanifold $N := S^1 \setminus \{(0, -1)\}$. The resulting map $f : N \to \mathbb{R}^2$ is an injective immersion but it is not proper. It has the same image as before and hence $f(N)$ is not a manifold.

Example 2.3.8. The map $f : \mathbb{R} \to \mathbb{R}^2$ given by $f(t) := (t^2, t^3)$ is proper and injective, but is not an embedding (its differential at $x = t$ is not injective). The image of f is the set $f(\mathbb{R}) = C := \{(x, y) \in \mathbb{R}^2 \,|\, x^3 = y^2\}$ (see Figure 2.7) and is not a submanifold. (Prove this!)

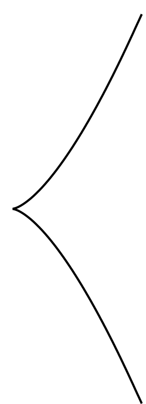


Figure 2.7: A proper injection.

Example 2.3.9. Define the map $f : \mathbb{R} \to \mathbb{R}^2$ by $f(t) := (\cos(t), \sin(t))$. This map is an immersion, but it is neither injective nor proper. However, its image is the unit circle in $\mathbb{R}^2$ and hence is a submanifold of $\mathbb{R}^2$. The map $\mathbb{R} \to \mathbb{R}^2 : t \mapsto f(t^3)$ is not an immersion and is neither injective nor proper, but its image is still the unit circle.

2.4 Vector Fields and Flows

2.4.1 Vector Fields

Definition 2.4.1 (Vector Field). *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold. A* **(smooth) vector field** *on M is a smooth map $X : M \to \mathbb{R}^k$ such that*

$$X(p) \in T_pM$$

for every $p \in M$. The set of smooth vector fields on M will be denoted by

$$\mathrm{Vect}(M) := \left\{ X : M \to \mathbb{R}^k \,|\, X \text{ is smooth, } X(p) \in T_pM \text{ for all } p \in M \right\}.$$

Exercise 2.4.2. Prove that the set of smooth vector fields on M is a real vector space.

Example 2.4.3. Denote the standard cross product on $\mathbb{R}^3$ by

$$x \times y := \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix}$$

For $x, y \in \mathbb{R}^3$. Fix a vector $\xi \in S^2$ and define the maps $X, Y : S^2 \to \mathbb{R}^3$ by

$$X(p) := \xi \times p, \qquad Y(p) := (\xi \times p) \times p.$$

These are vector fields with zeros $\pm\xi$. Their integral curves (see Definition 2.4.6 below) are illustrated in Figure 2.8.

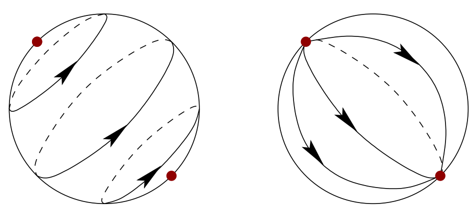


Figure 2.8: Two vector fields on the 2-sphere.

Example 2.4.4. Let $M := \mathbb{R}^2$. A vector field on M is then any smooth map $X : \mathbb{R}^2 \to \mathbb{R}^2$. As an example consider the vector field

$$X(x, y) := (x, -y).$$

This vector field has a single zero at the origin and its integral curves are illustrated in Figure 2.9.

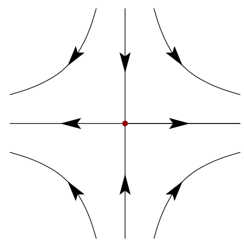


Figure 2.9: A hyperbolic fixed point.

Example 2.4.5. Every smooth function $f : \mathbb{R}^m \to \mathbb{R}$ determines a gradient vector field

$$X = \nabla f := \begin{pmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \vdots \\ \dfrac{\partial f}{\partial x_m} \end{pmatrix} : \mathbb{R}^m \to \mathbb{R}^m.$$

Definition 2.4.6 (Integral curves). *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold, let $X \in \mathrm{Vect}(M)$ be a smooth vector field on M, and let $I \subset \mathbb{R}$ be an open interval. A smooth map $\gamma : I \to M$ is called an* **integral curve** *of X if it satisfies the equation*

$$\dot\gamma(t) = X(\gamma(t))$$

for every $t \in I$.

Theorem 2.4.7. *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $X \in \mathrm{Vect}(M)$ be a smooth vector field on M. Fix a point $p_0 \in M$. Then the following holds.*

(i) *There is an open interval $I \subset \mathbb{R}$ containing 0 and a smooth curve $\gamma : I \to M$ satisfying the equation*

$$\dot\gamma(t) = X(\gamma(t)), \qquad \gamma(0) = p_0 \tag{2.4.1}$$

for every $t \in I$.

(ii) *If $\gamma_1 : I_1 \to M$ and $\gamma_2 : I_2 \to M$ are two solutions of (2.4.1) on open intervals I_1 and I_2 containing 0, then $\gamma_1(t) = \gamma_2(t)$ for every $t \in I_1 \cap I_2$.*

Proof. We prove (i). Let $\phi_0 : U_0 \to \mathbb{R}^m$ be a coordinate chart on M, defined on an M-open neighborhood $U_0 \subset M$ of p_0. The image of ϕ_0 is an open set

$$\Omega := \phi_0(U_0) \subset \mathbb{R}^m$$

and we denote the inverse map by $\psi_0 := \phi_0^{-1} : \Omega \to M$. Then, by Theorem 2.2.3, the differential $d\psi_0(x) : \mathbb{R}^m \to \mathbb{R}^k$ is injective and its image is the tangent space $T_{\psi_0(x)}M$ for every $x \in \Omega$. Define $f : \Omega \to \mathbb{R}^m$ by

$$f(x) := d\psi_0(x)^{-1}X(\psi_0(x)), \qquad x \in \Omega.$$

This map is smooth and hence, by the basic existence and uniqueness theorem for ordinary differential equations in $\mathbb{R}^m$ (see [17]), the equation

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0 := \phi_0(p_0), \tag{2.4.2}$$

has a solution $x : I \to \Omega$ on some open interval $I \subset \mathbb{R}$ containing 0. Hence the function

$$\gamma := \psi_0 \circ x : I \to U_0 \subset M$$

is a smooth solution of (2.4.1). This proves (i).

The local uniqueness theorem asserts that two solutions $\gamma_i : I_i \to M$ of (2.4.1) for $i = 1, 2$ agree on the interval $(-\varepsilon, \varepsilon) \subset I_1 \cap I_2$ for $\varepsilon > 0$ sufficiently small. This follows immediately from the standard uniqueness theorem for the solutions of (2.4.2) in [17] and the fact that $x : I \to \Omega$ is a solution of (2.4.2) if and only if $\gamma := \psi_0 \circ x : I \to U_0$ is a solution of (2.4.1).

To prove (ii) we observe that the set

$$I := I_1 \cap I_2$$

is an open interval containing zero and hence is connected. Now consider the set

$$A := \{t \in I \,|\, \gamma_1(t) = \gamma_2(t)\}.$$

This set is nonempty, because $0 \in A$. It is closed, relative to I, because the maps $\gamma_1 : I \to M$ and $\gamma_2 : I \to M$ are continuous. Namely, if $t_i \in I$ is a sequence converging to $t \in I$ then $\gamma_1(t_i) = \gamma_2(t_i)$ for every i and, taking the limit $i \to \infty$, we obtain $\gamma_1(t) = \gamma_2(t)$ and hence $t \in A$. The set A is also open by the local uniqueness theorem. Since I is connected it follows that $A = I$. This proves (ii) and Theorem 2.4.7. □

2.4.2 The Flow of a Vector Field

Definition 2.4.8 (The flow of a vector field). *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $X \in \mathrm{Vect}(M)$ be a smooth vector field on M. For $p_0 \in M$ the* **maximal existence interval of** p_0 *is the open interval*

$$I(p_0) := \bigcup \left\{ I \,\middle|\, \begin{array}{l} I \subset \mathbb{R} \text{ is an open interval containing } 0 \\ \text{and there is a solution } x : I \to M \text{ of } (2.4.1) \end{array} \right\}.$$

By Theorem 2.4.7 equation (2.4.1) *has a solution $\gamma : I(p_0) \to M$. The* **flow of** X *is the map $\phi : \mathcal{D} \to M$ defined by*

$$\mathcal{D} := \{(t, p_0) \,|\, p_0 \in M,\, t \in I(p_0)\}$$

and $\phi(t, p_0) := \gamma(t)$, where $\gamma : I(p_0) \to M$ is the unique solution of (2.4.1).

Theorem 2.4.9. *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $X \in \mathrm{Vect}(M)$ be a smooth vector field on M. Let $\phi : \mathcal{D} \to M$ be the flow of X. Then the following holds.*

(i) *$\mathcal{D}$ is an open subset of $\mathbb{R} \times M$.*

(ii) *The map $\phi : \mathcal{D} \to M$ is smooth.*

(iii) *Let $p_0 \in M$ and $s \in I(p_0)$. Then*

$$I(\phi(s, p_0)) = I(p_0) - s \tag{2.4.3}$$

and, for every $t \in \mathbb{R}$ with $s + t \in I(p_0)$, we have

$$\phi(s + t, p_0) = \phi(t, \phi(s, p_0)). \tag{2.4.4}$$

Proof. See page 41. □

Lemma 2.4.10. *Let M, X, $\mathcal{D}$, ϕ be as in Theorem 2.4.9 and let $K \subset M$ be a compact set. Then there exists an M-open set $U \subset M$ and an $\varepsilon > 0$ such that $K \subset U$, $(-\varepsilon, \varepsilon) \times U \subset \mathcal{D}$, and ϕ is smooth on $(-\varepsilon, \varepsilon) \times U$.*

Proof. In the case where $M = \Omega$ is an open subset of $\mathbb{R}^m$ this was proved in [18, Thm 4.1.4]. Using local coordinates we deduce (as in the proof of Theorem 2.4.7) that, for every $p \in M$, there exists an M-open neighborhood $U_p \subset M$ of p and an $\varepsilon_p > 0$ such that $(-\varepsilon_p, \varepsilon_p) \times U_p \subset \mathcal{D}$ and the restriction of ϕ to $(-\varepsilon_p, \varepsilon_p) \times U_p$ is smooth. Using this observation for every $p \in K$ (and the axiom of choice) we obtain an M-open cover $K \subset \bigcup_{p \in K} U_p$. Since the set K is compact there exists a finite subcover $K \subset U_{p_1} \cup \cdots \cup U_{p_N} =: U$. Now define $\varepsilon := \min\{\varepsilon_{p_1}, \ldots, \varepsilon_{p_N}\}$ to deduce that $(-\varepsilon, \varepsilon) \times U \subset \mathcal{D}$ and ϕ is smooth on $(-\varepsilon, \varepsilon) \times U$. This proves Lemma 2.4.10. □

Proof of Theorem 2.4.9. We prove (iii). The map $\gamma : I(p_0) - s \to M$ defined by $\gamma(t) := \phi(s+t, p_0)$ is a solution of the initial value problem $\dot\gamma(t) = X(\gamma(t))$ with $\gamma(0) = \phi(s, p_0)$. Hence $I(p_0) - s \subset I(\phi(s, p_0))$ and equation (2.4.4) holds for every $t \in \mathbb{R}$ with $s + t \in I(p_0)$. In particular, with $t = -s$, we have $p_0 = \phi(-s, \phi(s, p_0))$. Thus we obtain equality in equation (2.4.3) by the same argument with the pair $(s.p_0)$ replaced by $(-s, \phi(s, p_0))$.

We prove (i) and (ii). Let $(t_0, p_0) \in \mathcal{D}$ so that $p_0 \in M$ and $t_0 \in I(p_0)$. Suppose $t_0 \geq 0$. Then $K := \{\phi(t, p_0) \,|\, 0 \leq t \leq t_0\}$ is a compact subset of M. (It is the image of the compact interval $[0, t_0]$ under the unique solution $\gamma : I(p_0) \to M$ of (2.4.1).) Hence, by Lemma 2.4.10, there is an M-open set $U \subset M$ and an $\varepsilon > 0$ such that

$$K \subset U, \qquad (-\varepsilon, \varepsilon) \times U \subset \mathcal{D},$$

and ϕ is smooth on $(-\varepsilon, \varepsilon) \times U$. Choose N so large that $t_0/N < \varepsilon$. Define $U_0 := U$ and, for $k = 1, \dots, N$, define the sets $U_k \subset M$ inductively by

$$U_k := \{p \in U \,|\, \phi(t_0/N, p) \in U_{k-1}\}.$$

These sets are open in the relative topology of M.

We prove by induction on k that $(-\varepsilon, kt_0/N + \varepsilon) \times U_k \subset \mathcal{D}$ and ϕ is smooth on $(-\varepsilon, kt_0/N + \varepsilon) \times U_k$. For $k = 0$ this holds by definition of ε and U. If $k \in \{1, \dots, N\}$ and the assertion holds for $k - 1$ then we have

$$\begin{aligned} p \in U_k \quad &\Longrightarrow \quad p \in U,\ \phi(t_0/N, p) \in U_{k-1} \\ &\Longrightarrow \quad (-\varepsilon, \varepsilon) \subset I(p),\ (-\varepsilon, (k-1)t_0/N + \varepsilon) \subset I(\phi(t_0/N, p)) \\ &\Longrightarrow \quad (-\varepsilon, kt_0/N + \varepsilon) \subset I(p). \end{aligned}$$

Here the last implication follows from (2.4.3). Moreover, for $p \in U_k$ and $t_0/N - \varepsilon < t < kt_0/N + \varepsilon$, we have, by (2.4.4), that

$$\phi(t, p) = \phi(t - t_0/N, \phi(t_0/N, p))$$

Since $\phi(t_0/N, p) \in U_{k-1}$ for $p \in U_k$ the right hand side is a smooth map on the open set $(t_0/N - \varepsilon, kt_0/N + \varepsilon) \times U_k$. Since $U_k \subset U$, ϕ is also a smooth map on $(-\varepsilon, \varepsilon) \times U_k$ and hence on $(-\varepsilon, kt_0/N + \varepsilon) \times U_k$. This completes the induction. With $k = N$ we have found an open neighborhood of (t_0, p_0) contained in $\mathcal{D}$, namely the set $(-\varepsilon, t_0 + \varepsilon) \times U_N$, on which ϕ is smooth. The case $t_0 \leq 0$ is treated similarly. This proves (i) and (ii) and Theorem 2.4.9. □

Definition 2.4.11. *A vector field $X \in \mathrm{Vect}(M)$ is called* **complete** *if, for each $p_0 \in M$, there is an integral curve $\gamma : \mathbb{R} \to M$ of X with $\gamma(0) = p_0$.*

Lemma 2.4.12. *Let $M \subset \mathbb{R}^k$ is a compact manifold. Then every vector field on M is complete.*

Proof. Let $X \in \mathrm{Vect}(M)$. It follows from Lemma 2.4.10 with $K = M$ that there exists an $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset I(p)$ for all $p \in M$. By Theorem 2.4.9 this implies $I(p) = \mathbb{R}$ for all $p \in M$. Hence X is complete. □

Let $M \subset \mathbb{R}^k$ be a smooth manifold and $X \in \mathrm{Vect}(M)$. Then

$$X \text{ is complete} \quad \iff \quad I(p) = \mathbb{R}\ \forall\, p \in M \quad \iff \quad \mathcal{D} = \mathbb{R} \times M.$$

Assume X is complete, let $\phi : \mathbb{R} \times M \to M$ be the flow of X, and define the map $\phi^t : M \to M$ by $\phi^t(p) := \phi(t, p)$ for $t \in \mathbb{R}$ and $p \in M$. Then Theorem 2.4.9 asserts that ϕ^t is smooth for every $t \in \mathbb{R}$ and that

$$\phi^{s+t} = \phi^s \circ \phi^t, \qquad \phi^0 = \mathrm{id} \tag{2.4.5}$$

for all $s, t \in \mathbb{R}$. In particular this implies that $\phi^t \circ \phi^{-t} = \phi^{-t} \circ \phi^t = \mathrm{id}$. Hence ϕ^t is bijective and $(\phi^t)^{-1} = \phi^{-t}$, so each ϕ^t is a diffeomorphism.

Exercise 2.4.13. Let $M \subset \mathbb{R}^k$ be a smooth manifold. A vector field X on M is said to have **compact support** if there exists a compact subset $K \subset M$ such that $X(p) = 0$ for every $p \in M \setminus K$. Prove that every vector field with compact support is complete.

We close this subsection with an important observation about incomplete vector fields. The lemma asserts that an integral curve on a finite existence interval must leave every compact subset of M.

Lemma 2.4.14. *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold, let $X \in \mathrm{Vect}(M)$, let $\phi : \mathcal{D} \to M$ be the flow of X, let $K \subset M$ be a compact set, and let $p_0 \in M$ be an element such that*

$$I(p_0) \cap [0, \infty) = [0, b), \qquad 0 < b < \infty.$$

Then there exists a number $0 < t_K < b$ such that

$$t_K < t < b \qquad \Longrightarrow \qquad \phi(t, p_0) \in M \setminus K$$

Proof. By Lemma 2.4.10 there exists an $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset I(p)$ for every $p \in K$. Choose ε so small that $\varepsilon < b$ and define

$$t_K := b - \varepsilon > 0.$$

Choose a real number $t_K < t < b$. Then $I(\phi(t, p_0)) = [0, b - t)$ by equation (2.4.3) in part (ii) of Theorem 2.4.9. Since $0 < b - t < b - t_k = \varepsilon$, this shows that $(-\varepsilon, \varepsilon) \not\subset I(\phi(t, p_0))$ and hence $\phi(t, p_0) \notin K$. This proves Lemma 2.4.14. □

The next corollary is an immediate consequence of Lemma 2.4.14. In this formulation the result will be used in §4.6

Corollary 2.4.15. *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold, let $X \in \mathrm{Vect}(M)$, and let $\gamma : (0,T) \to M$ be an integral curve of X. If there exists a compact set $K \subset M$ that contains the image of γ, then γ extends to an integral curve of X on the interval $(-\rho, T+\rho)$ for some $\rho > 0$.*

Proof. Here is another more direct proof that does not rely on Lemma 2.4.10. Since K is compact, there exists a constant $c > 0$ such that $|X(p)| \le c$ for all $p \in K$. Since $\gamma(t) \in K$ for $0 < t < T$, this implies

$$|\gamma(t) - \gamma(s)| = \left|\int_s^t \dot\gamma(r)\,dr\right| \le \int_s^t |\dot\gamma(r)|\,dr = \int_s^t |X(\gamma(r))|\,dr \le c(t-s)$$

for $0 < s < t < T$. Thus the limit $p_0 := \lim_{t\searrow 0}\gamma(t)$ exists in $\mathbb{R}^k$ and, since K is a closed subset of $\mathbb{R}^k$, we have $p_0 \in K \subset M$. Define $\gamma_0 : [0,T) \to M$ by

$$\gamma_0(t) := \begin{cases} p_0, & \text{for } t = 0, \\ \gamma(t), & \text{for } 0 < t < T. \end{cases}$$

We prove that γ_0 is differentiable at $t = 0$ and $\dot\gamma_0(0) = X(p_0)$. To see this, fix a constant $\varepsilon > 0$. Since the curve $[0,T) \to \mathbb{R}^k : t \mapsto X(\gamma(t))$ is continuous, there exists a constant $\delta > 0$ such that

$$0 < t \le \delta \qquad \Longrightarrow \qquad |X(\gamma(t)) - X(p_0)| \le \varepsilon.$$

Hence, for $0 < s < t \le \delta$, we have

$$\begin{aligned}
|\gamma(t) - \gamma(s) - (t-s)X(p_0)| &= \left|\int_s^t (\dot\gamma(r) - X(p_0))\,dr\right| \\
&= \left|\int_s^t (X(\gamma(r)) - X(p_0))\,dr\right| \\
&\le \int_s^t |X(\gamma(r)) - X(p_0)|\,dr \\
&\le (t-s)\varepsilon.
\end{aligned}$$

Take the limit $s \to 0$ to obtain

$$\left|\frac{\gamma(t) - p_0}{t} - X(p_0)\right| = \lim_{s\to 0}\frac{|\gamma(t) - \gamma(s) - (t-s)X(p_0)|}{t-s} \le \varepsilon$$

for $0 < t \le \delta$. Thus γ_0 is differentiable at $t = 0$ with $\dot\gamma_0(0) = X(p_0)$, as claimed. Hence γ extends to an integral curve $\widetilde\gamma : (-\rho, T) \to M$ of X for some $\rho > 0$ via $\widetilde\gamma(t) := \phi(t, p_0)$ for $-\rho < t \le 0$ and $\widetilde\gamma(t) := \gamma(t)$ for $0 < t < T$. Here ϕ is the flow of X. That γ also extends beyond $t = T$, follows by replacing $\gamma(t)$ with $\gamma(T-t)$ and X with $-X$. This proves Corollary 2.4.15. $\square$

The Group of Diffeomorphisms

Let us denote the space of diffeomorphisms of M by

$$\mathrm{Diff}(M) := \{\phi : M \to M \,|\, \phi \text{ is a diffeomorphism}\}.$$

This is a group. The group operation is composition and the neutral element is the identity. Now equation (2.4.5) asserts that the flow of a complete vector field $X \in \mathrm{Vect}(M)$ is a group homomorphism

$$\mathbb{R} \to \mathrm{Diff}(M) : t \mapsto \phi^t.$$

This homomorphism is smooth and is characterized by the equation

$$\frac{d}{dt}\phi^t(p) = X(\phi^t(p)), \qquad \phi^0(p) = p$$

for all $p \in M$ and $t \in \mathbb{R}$. We will often abbreviate this equation in the form

$$\frac{d}{dt}\phi^t = X \circ \phi^t, \qquad \phi^0 = \mathrm{id}. \tag{2.4.6}$$

Exercise 2.4.16 (Isotopy). Let $M \subset \mathbb{R}^k$ be a compact manifold and $I \subset \mathbb{R}$ be an open interval containing 0. Let

$$I \times M \to \mathbb{R}^k : (t, p) \mapsto X_t(p)$$

be a smooth map such that $X_t \in \mathrm{Vect}(M)$ for every t. Prove that there is a smooth family of diffeomorphisms $I \times M \to M : (t, p) \mapsto \phi_t(p)$ satisfying

$$\frac{d}{dt}\phi_t = X_t \circ \phi_t, \qquad \phi_0 = \mathrm{id} \tag{2.4.7}$$

for every $t \in I$. Such a family of diffeomorphisms

$$I \to \mathrm{Diff}(M) : t \mapsto \phi_t$$

is called an **isotopy** of M. Conversely prove that every smooth isotopy $I \to \mathrm{Diff}(M) : t \mapsto \phi_t$ is generated (uniquely) by a smooth family of vector fields $I \to \mathrm{Vect}(M) : t \mapsto X_t$.

2.4.3 The Lie Bracket

Let $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^\ell$ be smooth m-manifolds and $X \in \mathrm{Vect}(M)$ be smooth vector field on M. If $\psi : N \to M$ is a diffeomorphism, the **pullback** of X under ψ is the vector field on N defined by

$$(\psi^* X)(q) := d\psi(q)^{-1} X(\psi(q)) \tag{2.4.8}$$

for $q \in N$. If $\phi : M \to N$ is a diffeomorphism then the **pushforward** of X under ψ is the vector field on N defined by

$$(\phi_* X)(q) := d\phi(\phi^{-1}(q)) X(\phi^{-1}(q)) \tag{2.4.9}$$

for $q \in N$.

Lemma 2.4.17. *Let $M \subset \mathbb{R}^k$, $N \subset \mathbb{R}^\ell$, and $P \subset \mathbb{R}^n$ be smooth m-dimensional submanifolds and let $X \in \mathrm{Vect}(M)$ and $Z \in \mathrm{Vect}(P)$. Then*

$$\phi_* X = (\phi^{-1})^* X \tag{2.4.10}$$

and

$$(\psi \circ \phi)_* X = \psi_* \phi_* X, \qquad (\psi \circ \phi)^* Z = \phi^* \psi^* Z. \tag{2.4.11}$$

Proof. Equation (2.4.10) follows from the fact that

$$d\phi^{-1}(q) = d\phi(\phi^{-1}(q))^{-1} : T_q N \to T_{\phi^{-1}(q)} M$$

for all $q \in N$ (Corollary 2.2.14) and the equations in (2.4.11) follow directly from the chain rule (Theorem 2.2.13). This proves Lemma 2.4.17. □

We think of a vector field on M as a smooth map

$$X : M \to \mathbb{R}^k$$

that satisfies the condition $X(p) \in T_p M$ for every $p \in M$. Ignoring this condition temporarily, we can differentiate X as a map from M to $\mathbb{R}^k$ and its differential at p is then a linear map

$$dX(p) : T_p M \to \mathbb{R}^k.$$

In general, this differential will no longer take values in the tangent space $T_p M$. However, if we have two vector fields X and Y on M the next lemma shows that the difference of the derivative of X in the direction Y and of Y in the direction X does take values in the tangent spaces of M.

Lemma 2.4.18. *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $X, Y \in \mathrm{Vect}(M)$ be complete vector fields. Denote by*

$$\mathbb{R} \to \mathrm{Diff}(M) : t \mapsto \phi^t, \qquad \mathbb{R} \to \mathrm{Diff}(M) : t \mapsto \psi^t$$

the flows of X and Y, respectively. Fix a point $p \in M$ and consider the smooth map $\gamma : \mathbb{R} \to M$ defined by

$$\gamma(t) := \phi^t \circ \psi^t \circ \phi^{-t} \circ \psi^{-t}(p). \tag{2.4.12}$$

Then $\dot\gamma(0) = 0$ and

$$\begin{aligned}
\frac{1}{2}\ddot\gamma(0) &= \left.\frac{d}{ds}\right|_{s=0} \left((\phi^s)_* Y\right)(p) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\left(\psi^t\right)^* X\right)(p) \\
&= dX(p)Y(p) - dY(p)X(p) \qquad \in \quad T_pM.
\end{aligned}$$

Proof. Define the map $\beta : \mathbb{R}^2 \to M$ by

$$\beta(s,t) := \phi^s \circ \psi^t \circ \phi^{-s} \circ \psi^{-t}(p)$$

for $s, t \in \mathbb{R}$. Then

$$\gamma(t) = \beta(t,t)$$

and

$$\frac{\partial\beta}{\partial s}(0,t) = X(p) - d\psi^t(\psi^{-t}(p))X(\psi^{-t}(p)), \tag{2.4.13}$$

$$\frac{\partial\beta}{\partial t}(s,0) = d\phi^s(\phi^{-s}(p))Y(\phi^{-s}(p)) - Y(p). \tag{2.4.14}$$

for all $s, t \in \mathbb{R}$. Hence

$$\dot\gamma(0) = \frac{\partial\beta}{\partial s}(0,0) + \frac{\partial\beta}{\partial t}(0,0) = 0.$$

Moreover, $\beta(s,0) = \beta(0,t) = p$ for all s and t, hence

$$\frac{\partial^2\beta}{\partial s^2}(0,0) = \frac{\partial^2\beta}{\partial t^2}(0,0) = 0$$

and therefore

$$\ddot\gamma(0) = 2\frac{\partial^2\beta}{\partial s\partial t}(0,0). \tag{2.4.15}$$

Combining equations (2.4.14) and (2.4.15) we find

$$\begin{aligned}
\frac{1}{2}\ddot{\gamma}(0) &= \frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial \beta}{\partial t}(s,0) = \frac{d}{ds}\Big|_{s=0} d\phi^s(\phi^{-s}(p))Y(\phi^{-s}(p)) \\
&= \frac{d}{ds}\Big|_{s=0} \left((\phi^s)_* Y\right)(p)).
\end{aligned}$$

Likewise, combining equations (2.4.13) and (2.4.15) we find

$$\begin{aligned}
\frac{1}{2}\ddot{\gamma}(0) &= \frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial \beta}{\partial s}(0,t) = -\frac{d}{dt}\Big|_{t=0} d\psi^t(\psi^{-t}(p))X(\psi^{-t}(p)) \\
&= \frac{d}{dt}\Big|_{t=0} d\psi^{-t}(\psi^t(p))X(\psi^t(p)) \\
&= \frac{d}{dt}\Big|_{t=0} d\psi^t(p)^{-1}X(\psi^t(p)) \\
&= \frac{d}{dt}\Big|_{t=0} \left((\psi^t)^* X\right)(p)).
\end{aligned}$$

Here the right hand side is the derivative of a smooth curve in the tangent space T_pM and hence is itself an element of T_pM. Moreover, we have

$$\begin{aligned}
\frac{1}{2}\ddot{\gamma}(0) &= \frac{\partial}{\partial s}\Big|_{s=0} d\phi^s(\phi^{-s}(p))Y(\phi^{-s}(p)) \\
&= \frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} \phi^s \circ \psi^t \circ \phi^{-s}(p) \\
&= \frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} \phi^s \circ \psi^t \circ \phi^{-s}(p) \\
&= \frac{\partial}{\partial t}\Big|_{t=0} \big(X(\psi^t(p)) - d\psi^t(p)X(p)\big) \\
&= dX(p)Y(p) - \frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} \psi^t \circ \phi^s(p) \\
&= dX(p)Y(p) - \frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} \psi^t \circ \phi^s(p) \\
&= dX(p)Y(p) - \frac{\partial}{\partial s}\Big|_{s=0} Y(\phi^s(p)) \\
&= dX(p)Y(p) - dY(p)X(p).
\end{aligned}$$

This proves Lemma 2.4.18. $\square$

Definition 2.4.19 (Lie Bracket). *Let $M \subset \mathbb{R}^k$ be a smooth manifold and let $X, Y \in \mathrm{Vect}(M)$ be smooth vector fields on M. The* **Lie bracket** *of X and Y is the vector field $[X, Y] \in \mathrm{Vect}(M)$ defined by*

$$[X, Y](p) := dX(p)Y(p) - dY(p)X(p). \tag{2.4.16}$$

Warning: In the literature on differential geometry the Lie bracket of two vector fields is often (but not always) defined with the opposite sign. The rationale behind the present choice of the sign will be explained in § 2.5.6.

Lemma 2.4.20. *Let $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^\ell$ be smooth manifolds, let X, Y, Z be smooth vector fields on M, and let*

$$\phi : N \to M$$

be a diffeomorphism. Then

$$\phi^*[X, Y] = [\phi^* X, \phi^* Y], \tag{2.4.17}$$

$$[X, Y] + [Y, X] = 0, \tag{2.4.18}$$

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \tag{2.4.19}$$

The last equation is called the **Jacobi identity**

Proof. Let $\mathbb{R} \to \mathrm{Diff}(M) : t \mapsto \psi^t$ be the flow of Y. Then the map

$$\mathbb{R} \to \mathrm{Diff}(N) : t \mapsto \phi^{-1} \circ \psi^t \circ \phi$$

is the flow of the vector field $\phi^* Y$ on N. Hence, by Lemma 2.4.17 and Lemma 2.4.18, we have

$$\begin{aligned} [\phi^* X, \phi^* Y] &= \left.\frac{d}{dt}\right|_{t=0} \left(\phi^{-1} \circ \psi^t \circ \phi\right)^* \phi^* X \\ &= \left.\frac{d}{dt}\right|_{t=0} \phi^* \left(\psi^t\right)^* X \\ &= \phi^*[X, Y]. \end{aligned}$$

This proves (2.4.17). Equation (2.4.18) is obvious. To prove (2.4.19), let ϕ^t be the flow of X. Then by (2.4.17) and (2.4.18) and Lemma 2.4.18 we have

$$\begin{aligned} [[Y, Z], X] &= \left.\frac{d}{dt}\right|_{t=0} (\phi^t)^*[Y, Z] \\ &= \left.\frac{d}{dt}\right|_{t=0} [(\phi^t)^* Y, (\phi^t)^* Z] \\ &= [[Y, X], Z] + [Y, [Z, X]] \\ &= [Z, [X, Y]] + [Y, [Z, X]]. \end{aligned}$$

This proves Lemma 2.4.20. □

Definition 2.4.21. *A* **Lie algebra** *is a real vector space* $\mathfrak{g}$ *equipped with a skew-symmetric bilinear map* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (\xi, \eta) \mapsto [\xi, \eta]$ *that satisfies the Jacobi identity* $[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0$ *for all* $\xi, \eta, \zeta \in \mathfrak{g}$.

Example 2.4.22. The Vector fields on a smooth manifold $M \subset \mathbb{R}^k$ form a Lie algebra with the Lie bracket (2.4.16). The space $\mathfrak{gl}(n, \mathbb{R}) = \mathbb{R}^{n \times n}$ of real $n \times n$-matrices is a Lie algebra with the Lie bracket

$$[\xi, \eta] := \xi\eta - \eta\xi.$$

It is also interesting to consider subspaces of $\mathfrak{gl}(n, \mathbb{R})$ that are invariant under this Lie bracket. An example is the space

$$\mathfrak{o}(n) := \left\{ \xi \in \mathfrak{gl}(n, \mathbb{R}) \,|\, \xi^{\mathsf{T}} + \xi = 0 \right\}$$

of skew-symmetric $n \times n$-matrices. It is a nontrivial fact that every finite-dimensional Lie algebra is isomorphic to a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ for some n. For example, the cross product defines a Lie algebra structure on $\mathbb{R}^3$ and the resulting Lie algebra is isomorphic to $\mathfrak{o}(3)$.

Remark 2.4.23. There is a linear map $\mathbb{R}^{m \times m} \to \mathrm{Vect}(\mathbb{R}^m) : \xi \mapsto X_\xi$ which assigns to a matrix $\xi \in \mathfrak{gl}(m, \mathbb{R})$ the linear vector field $X_\xi : \mathbb{R}^m \to \mathbb{R}^m$ given by $X_\xi(x) := \xi x$ for $x \in \mathbb{R}^m$. This map preserves the Lie bracket, i.e. $[X_\xi, X_\eta] = X_{[\xi,\eta]}$, and hence is a **Lie algebra homomorphism**.

Exercise 2.4.24. Let $\gamma : \mathbb{R} \to \mathbb{R}^k$ be a C^2-curve and assume $\dot\gamma(0) = 0$. Prove that the curve $[0, \infty) \to \mathbb{R}^k : t \mapsto \gamma(\sqrt{t})$ is differentiable at $t = 0$ and $\frac{d}{dt}\big|_{t=0} \gamma(\sqrt{t}) = \frac{1}{2}\ddot\gamma(0)$.

To understand the Lie bracket geometrically, consider again the curve

$$\gamma(t) := \phi^t \circ \psi^t \circ \phi^{-t} \circ \psi^{-t}(p)$$

in Lemma 2.4.18, where ϕ^t and ψ^t are the flows of the vector fields X and Y, respectively. Since $\dot\gamma(0) = 0$, Exercise 2.4.24 asserts that

$$[X, Y](p) = \frac{1}{2}\ddot\gamma(0) = \frac{d}{dt}\bigg|_{t=0} \phi^{\sqrt{t}} \circ \psi^{\sqrt{t}} \circ \phi^{-\sqrt{t}} \circ \psi^{-\sqrt{t}}(p). \qquad (2.4.20)$$

(Compare Equations (2.5.7) and (2.4.20).) Geometrically this means that by following first the backward flow of Y for time ε, then the backward flow of X for time ε, then the forward flow of Y for time ε, and finally the forward flow of X for time ε, we will not, in general, get back to the original point p where we started but approximately obtain an *"error"* $\varepsilon^2[X, Y](p)$. An example of this (which we learned from Donaldson) is the mathematical formulation of parking a car.

Example 2.4.25 (Parking a Car). The configuration space for driving a car in the plane is the manifold $M := \mathbb{C} \times S^1$, where $S^1 \subset \mathbb{C}$ denotes the unit circle. Thus a point in M is a pair $p = (z, \lambda) \in \mathbb{C} \times \mathbb{C}$ with $|\lambda| = 1$. The point $z \in \mathbb{C}$ represents the position of the car and the unit vector $\lambda \in S^1$ represents the direction in which it is pointing. The *left turn* is represented by a vector field X and the *right turn* by a vector field Y on M. These vector field are given by

$$X(z, \lambda) := (\lambda, \mathbf{i}\lambda), \qquad Y(z, \lambda) := (\lambda, -\mathbf{i}\lambda).$$

Their Lie bracket is the vector field

$$[X, Y](z, \lambda) = (-2\mathbf{i}\lambda, 0).$$

This vector field represents a sideways move of the car to the right. And a sideways move by $2\varepsilon^2$ can be achieved by following a backward right turn for time ε, then a backward left turn for time ε, then a forward right turn for time ε, and finally a forward left turn for time ε.

This example can be reformulated by identifying $\mathbb{C}$ with $\mathbb{R}^2$ via $z = x + \mathbf{i}y$ and representing a point in the unit circle by the angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ via $\lambda = e^{\mathbf{i}\theta}$. In this formulation the manifold is $M = \mathbb{R}^2 \times \mathbb{R}/2\pi\mathbb{Z}$, a point in M is represented by a triple $(x, y, \theta) \in \mathbb{R}^3$, the vector fields X and Y are

$$X(x, y, \theta) := (\cos(\theta), \sin(\theta), 1), \qquad Y(x, y, \theta) := (\cos(\theta), \sin(\theta), -1),$$

and their Lie bracket is $[X, Y](x, y, \theta) = 2(\sin(\theta), -\cos(\theta), 0)$.

Lemma 2.4.26. *Let $X, Y \in \mathrm{Vect}(M)$ be complete vector fields on a manifold M and $\phi^t, \psi^t \in \mathrm{Diff}(M)$ be the flows of X and Y, respectively. Then the Lie bracket $[X, Y]$ vanishes if and only if the flows of X and Y commute, i.e. $\phi^s \circ \psi^t = \psi^t \circ \phi^s$ for all $s, t \in \mathbb{R}$.*

Proof. If the flows of X and Y commute then the Lie bracket $[X, Y]$ vanishes by Lemma 2.4.18. Conversely, suppose that $[X, Y] = 0$. Then we have

$$\frac{d}{ds}\,(\phi^s)_*\,Y = (\phi^s)_*\,\frac{d}{dr}\,(\phi^r)_*\,Y = (\phi^s)_*\,[X, Y] = 0$$

for every $s \in \mathbb{R}$ and hence

$$(\phi^s)_* Y = Y. \tag{2.4.21}$$

Fix a real number s and define the curve $\gamma : \mathbb{R} \to M$ by $\gamma(t) := \phi^s(\psi^t(p))$ for $t \in \mathbb{R}$. Then $\gamma(0) = \phi^s(p)$ and

$$\dot\gamma(t) = d\phi^s(\psi^t(p))Y(\psi^t(p)) = ((\phi^s)_*\,Y)\,(\gamma(t)) = Y(\gamma(t))$$

for all t. Here the last equation follows from (2.4.21). Since ψ^t is the flow of Y we obtain $\gamma(t) = \psi^t(\phi^s(p))$ for all $t \in \mathbb{R}$ and this proves Lemma 2.4.26. $\square$

2.5 Lie Groups

Combining the concept of a group and a manifold, it is interesting to consider groups which are also manifolds and have the property that the group operation and the inverse define smooth maps. We shall only consider groups of matrices.

2.5.1 Definition and Examples

Definition 2.5.1 (Lie Group). *A nonempty subset* $\mathrm{G} \subset \mathbb{R}^{n\times n}$ *is called a* **Lie group** *if it is a submanifold of* $\mathbb{R}^{n\times n}$ *and a subgroup of* $\mathrm{GL}(n,\mathbb{R})$*, i.e.*

$$g, h \in \mathrm{G} \qquad \Longrightarrow \qquad gh \in \mathrm{G}$$

(where gh *denotes the product of the matrices* g *and* h*) and*

$$g \in \mathrm{G} \qquad \Longrightarrow \qquad \det(g) \neq 0 \text{ and } g^{-1} \in \mathrm{G}.$$

(Since $\mathrm{G} \neq \emptyset$ *it follows from these conditions that the identity matrix* $\mathbb{1}$ *is an element of* G*.)*

Example 2.5.2. The general linear group $\mathrm{G} = \mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$ and hence is a Lie group. By Exercise 2.1.18 the special linear group

$$\mathrm{SL}(n,\mathbb{R}) = \{g \in \mathrm{GL}(n,\mathbb{R}) \mid \det(g) = 1\}$$

is a Lie group and, by Example 2.1.19, the special orthogonal group

$$\mathrm{SO}(n) := \left\{g \in \mathrm{GL}(n,\mathbb{R}) \,|\, g^{\mathsf{T}} g = \mathbb{1},\, \det(g) = 1\right\}$$

is a Lie group. In fact every orthogonal matrix has determinant ± 1 and so $\mathrm{SO}(n)$ is an open subset of $\mathrm{O}(n)$ (in the relative topology).

In a similar vein the group $\mathrm{GL}(n,\mathbb{C}) := \{g \in \mathbb{C}^{n\times n} \mid \det(g) \neq 0\}$ of complex matrices with nonzero (complex) determinant is an open subset of $\mathbb{C}^{n\times n}$ and hence is a Lie group. As in the real case, the subgroups

$$\mathrm{SL}(n,\mathbb{C}) := \{g \in \mathrm{GL}(n,\mathbb{C}) \mid \det(g) = 1\},$$

$$\mathrm{U}(n) := \{g \in \mathrm{GL}(n,\mathbb{C}) \,|\, g^* g = \mathbb{1}\},$$

$$\mathrm{SU}(n) := \{g \in \mathrm{GL}(n,\mathbb{C}) \,|\, g^* g = \mathbb{1},\, \det(g) = 1\}$$

are submanifolds of $\mathrm{GL}(n,\mathbb{C})$ and hence are Lie groups. Here $g^* := \bar{g}^{\mathsf{T}}$ denotes the conjugate transpose of a complex matrix.

Exercise 2.5.3. Prove that $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{U}(n)$, and $\mathrm{SU}(n)$ are Lie groups. Prove that $\mathrm{SO}(n)$ is connected and that $\mathrm{O}(n)$ has two connected components.

Exercise 2.5.4. Prove that $\mathrm{GL}(n, \mathbb{C})$ can be identified with the group

$$\mathrm{G} := \{\Phi \in \mathrm{GL}(2n, \mathbb{R}) \,|\, \Phi J_0 = J_0 \Phi\}, \qquad J_0 := \begin{pmatrix} 0 & -1\!\!1 \\ 1\!\!1 & 0 \end{pmatrix}.$$

Hint: Use the isomorphism $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}^n : (x, y) \mapsto x + \mathbf{i}y$. Show that a matrix $\Phi \in \mathbb{R}^{2n \times 2n}$ commutes with J_0 if and only if it has the form

$$\Phi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}, \qquad X, Y \in \mathbb{R}^{n \times n}.$$

What is the relation between the real determinant of Φ and the complex determinant of $X + \mathbf{i}Y$?

Exercise 2.5.5. Let J_0 be as in Exercise 2.5.4 and define

$$\mathrm{Sp}(2n) := \left\{\Psi \in \mathrm{GL}(2n, \mathbb{R}) \,|\, \Psi^{\mathsf{T}} J_0 \Psi = J_0\right\}.$$

This is the **symplectic linear group**. Prove that $\mathrm{Sp}(2n)$ is a Lie group. **Hint:** See [12, Lemma 1.1.12].

Example 2.5.6 (Unit Quaternions). The **Quaternions** form a four-dimensional associative unital algebra $\mathbb{H}$, equipped with a basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$. The elements of $\mathbb{H}$ are vectors of the form

$$x = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3 \qquad x_0, x_1, x_2, x_3 \in \mathbb{R}. \tag{2.5.1}$$

The product structure is the bilinear map $\mathbb{H} \times \mathbb{H} \to \mathbb{H} : (x, y) \mapsto xy$, determined by the relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

This product structure is associative but not commutative. The quaternions are equipped with an involution $\mathbb{H} \to \mathbb{H} : x \mapsto \bar{x}$, which assigns to a quaternion x of the form (2.5.1) its **conjugate** $\bar{x} := x_0 - \mathbf{i}x_1 - \mathbf{j}x_2 - \mathbf{k}x_3$. This involution satisfies the conditions

$$\overline{x + y} = \bar{x} + \bar{y}, \qquad \overline{xy} = \bar{y}\bar{x}, \qquad x\bar{x} = |x|^2, \qquad |xy| = |x|\,|y|$$

for $x, y \in \mathbb{H}$, where $|x| := \sqrt{x_0^2 + x_2^2 + x_2^2 + x_3^2}$ denotes the Euclidean norm of the quaternion (2.5.1). Thus the **unit quaternions** form a group

$$\mathrm{Sp}(1) := \{x \in \mathbb{H} \,|\, |x| = 1\}$$

with the inverse map $x \mapsto \bar{x}$. Note that the group $\mathrm{Sp}(1)$ is diffeomorphic to the 3-sphere $S^3 \subset \mathbb{R}^4$ under the isomorphism $\mathbb{H} \cong \mathbb{R}^4$. **Warning:** The unit quaternions (a compact Lie group) are not to be confused with the symplectic linear group in Exercise 2.5.5 (a noncompact Lie group) despite the similarity in notation.

Let $\mathrm{G} \subset \mathrm{GL}(n,\mathbb{R})$ be a Lie group. Then the maps

$$\mathrm{G}\times\mathrm{G} \to \mathrm{G} : (g,h) \mapsto gh, \qquad \mathrm{G}\to\mathrm{G} : g \mapsto g^{-1}$$

are smooth (see [18]). Fixing an element $h \in \mathrm{G}$ we find that the derivative of the map $\mathrm{G}\to\mathrm{G} : g \mapsto gh$ at $g \in \mathrm{G}$ is given by the linear map

$$T_g\mathrm{G} \to T_{gh}\mathrm{G} : \widehat{g} \mapsto \widehat{g}h. \tag{2.5.2}$$

Here $\widehat{g}$ and h are both matrices in $\mathbb{R}^{n\times n}$ and $\widehat{g}h$ denotes the matrix product. In fact, if $\widehat{g} \in T_g\mathrm{G}$ then, since G is a manifold, there exists a smooth curve $\gamma : \mathbb{R} \to \mathrm{G}$ with $\gamma(0) = g$ and $\dot\gamma(0) = \widehat{g}$. Since G is a group we obtain a smooth curve $\beta : \mathbb{R} \to \mathrm{G}$ given by $\beta(t) := \gamma(t)h$. It satisfies $\beta(0) = gh$ and so $\widehat{g}h = \dot\beta(0) \in T_{gh}\mathrm{G}$.

The linear map (2.5.2) is obviously a vector space isomorphism whose inverse is given by right multiplication with h^{-1}. It is sometimes convenient to define the map $R_h : \mathrm{G}\to\mathrm{G}$ by

$$R_h(g) := gh$$

for $g \in \mathrm{G}$ (*right multiplication* by h). This is a diffeomorphism and the linear map (2.5.2) is the derivative of R_h at g, so

$$dR_h(g)\widehat{g} = \widehat{g}h \qquad \text{for } \widehat{g} \in T_g\mathrm{G}.$$

Similarly, each element $g \in \mathrm{G}$ determines a diffeomorphism $L_g : \mathrm{G}\to\mathrm{G}$, given by

$$L_g(h) := gh$$

for $h \in \mathrm{G}$ (*left multiplication* by g). Its derivative at $h \in \mathrm{G}$ is again given by matrix multiplication, i.e. the linear map $dL_g(h) : T_h\mathrm{G} \to T_{gh}\mathrm{G}$ is given by

$$dL_g(h)\widehat{h} = g\widehat{h} \qquad \text{for } \widehat{h} \in T_h\mathrm{G}. \tag{2.5.3}$$

Since L_g is a diffeomorphism its differential $dL_g(h) : T_h\mathrm{G} \to T_{gh}\mathrm{G}$ is again a vector space isomorphism for every $h \in \mathrm{G}$.

Exercise 2.5.7. Prove that the map $\mathrm{G}\to\mathrm{G} : g \mapsto g^{-1}$ is a diffeomorphism and that its derivative at $g \in \mathrm{G}$ is the vector space isomorphism

$$T_g\mathrm{G} \to T_{g^{-1}}\mathrm{G} : v \mapsto -g^{-1}vg^{-1}.$$

Hint: Use [18] or any textbook on first year analysis.

2.5.2 The Lie Algebra of a Lie Group

Let

$$\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$$

be a Lie group. Its tangent space at the identity matrix $\mathbb{1} \in \mathrm{G}$ is called the **Lie algebra** of G and will be denoted by

$$\mathfrak{g} = \mathrm{Lie}(\mathrm{G}) := T_{\mathbb{1}}\mathrm{G}.$$

This terminology is justified by the fact that $\mathfrak{g}$ is in fact a Lie algebra, i.e. it is invariant under the standard Lie bracket operation

$$[\xi, \eta] := \xi\eta - \eta\xi$$

on the space $\mathbb{R}^{n\times n}$ of square matrices (see Lemma 2.5.9 below). The proof requires the notion of the **exponential matrix**. For $\xi \in \mathbb{R}^{n\times n}$ and $t \in \mathbb{R}$ we define

$$\exp(t\xi) := \sum_{k=0}^{\infty} \frac{t^k \xi^k}{k!}. \tag{2.5.4}$$

A standard result in first year analysis asserts that this series converges absolutely (and uniformly on compact t-intervals), that the map

$$\mathbb{R} \to \mathbb{R}^{n\times n} : t \mapsto \exp(t\xi)$$

is smooth and satisfies the differential equation

$$\frac{d}{dt}\exp(t\xi) = \xi \exp(t\xi) = \exp(t\xi)\xi, \tag{2.5.5}$$

and that

$$\exp((s+t)\xi) = \exp(s\xi)\exp(t\xi), \qquad \exp(0\xi) = \mathbb{1} \tag{2.5.6}$$

for all $s, t \in \mathbb{R}$. This shows that the matrix $\exp(t\xi)$ is invertible for each t and that the map $\mathbb{R} \to \mathrm{GL}(n, \mathbb{R}) : t \mapsto \exp(t\xi)$ is a group homomorphism.

Exercise 2.5.8. Prove the following analogue of (2.4.12). For $\xi, \eta \in \mathfrak{g}$

$$\left.\frac{d}{dt}\right|_{t=0} \exp(\sqrt{t}\xi)\exp(\sqrt{t}\eta)\exp(-\sqrt{t}\xi)\exp(-\sqrt{t}\eta) = [\xi, \eta] \tag{2.5.7}$$

In other words, the infinitesimal Lie group commutator is the matrix commutator. (Compare Equations (2.5.7) and (2.4.20).)

Lemma 2.5.9. *Let* $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ *be a Lie group and denote by* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ *its Lie algebra. Then the following holds.*

(i) *If* $\xi \in \mathfrak{g}$ *then* $\exp(t\xi) \in \mathrm{G}$ *for every* $t \in \mathbb{R}$.

(ii) *If* $g \in \mathrm{G}$ *and* $\eta \in \mathfrak{g}$ *then* $g\eta g^{-1} \in \mathfrak{g}$.

(iii) *If* $\xi, \eta \in \mathfrak{g}$ *then* $[\xi, \eta] = \xi\eta - \eta\xi \in \mathfrak{g}$.

Proof. We prove (i). For every $g \in \mathrm{G}$ we have a vector space isomorphism $\mathfrak{g} = T_{\mathbb{1}}\mathrm{G} \to T_g\mathrm{G} : \xi \mapsto \xi g$ as in (2.5.2). Hence each element $\xi \in \mathfrak{g}$ determines a vector field $X_\xi \in \mathrm{Vect}(\mathrm{G})$, defined by

$$X_\xi(g) := \xi g \in T_g\mathrm{G}, \qquad g \in \mathrm{G}. \tag{2.5.8}$$

By Theorem 2.4.7 there is an integral curve $\gamma : (-\varepsilon, \varepsilon) \to \mathrm{G}$ satisfying

$$\dot\gamma(t) = X_\xi(\gamma(t)) = \xi\gamma(t), \qquad \gamma(0) = \mathbb{1}.$$

By (2.5.5), the curve $(-\varepsilon, \varepsilon) \to \mathbb{R}^{n\times n} : t \mapsto \exp(t\xi)$ satisfies the same initial value problem and hence, by uniqueness, we have $\exp(t\xi) = \gamma(t) \in \mathrm{G}$ for all $t \in \mathbb{R}$ with $|t| < \varepsilon$. Now let $t \in \mathbb{R}$ and choose $N \in \mathbb{N}$ such that $\left|\frac{t}{N}\right| < \varepsilon$. Then $\exp(\frac{t}{N}\xi) \in \mathrm{G}$ and hence it follows from (2.5.6) that

$$\exp(t\xi) = \exp\left(\frac{t}{N}\xi\right)^N \in \mathrm{G}.$$

This proves (i).

We prove (ii). Consider the smooth curve $\gamma : \mathbb{R} \to \mathbb{R}^{n\times n}$ defined by

$$\gamma(t) := g\exp(t\eta)g^{-1}.$$

By (i) we have $\gamma(t) \in \mathrm{G}$ for every $t \in \mathbb{R}$. Since $\gamma(0) = \mathbb{1}$ we have

$$g\eta g^{-1} = \dot\gamma(0) \in \mathfrak{g}.$$

This proves (ii).

We prove (iii). Define the smooth map $\eta : \mathbb{R} \to \mathbb{R}^{n\times n}$ by

$$\eta(t) := \exp(t\xi)\eta\exp(-t\xi).$$

By (i) we have $\exp(t\xi) \in \mathrm{G}$ and, by (ii), we have $\eta(t) \in \mathfrak{g}$ for every $t \in \mathbb{R}$. Hence $[\xi, \eta] = \dot\eta(0) \in \mathfrak{g}$. This proves (iii) and Lemma 2.5.9. □

By Lemma 2.5.9 the curve $\gamma : \mathbb{R} \to \mathrm{G}$ defined by $\gamma(t) := \exp(t\xi)g$ is the integral curve of the vector field X_ξ in (2.5.8) with initial condition $\gamma(0) = g$. Thus X_ξ is complete for every $\xi \in \mathfrak{g}$.

Lemma 2.5.10. *If $\xi \in \mathfrak{g}$ and $\gamma : \mathbb{R} \to \mathrm{G}$ is a smooth curve satisfying*

$$\gamma(s+t) = \gamma(s)\gamma(t), \qquad \gamma(0) = \mathbb{1}, \qquad \dot\gamma(0) = \xi, \tag{2.5.9}$$

then $\gamma(t) = \exp(t\xi)$ for every $t \in \mathbb{R}$.

Proof. For every $t \in \mathbb{R}$ we have

$$\dot\gamma(t) = \left.\frac{d}{ds}\right|_{s=0} \gamma(s+t) = \left.\frac{d}{ds}\right|_{s=0} \gamma(s)\gamma(t) = \dot\gamma(0)\gamma(t) = \xi\gamma(t).$$

Hence γ is the integral curve of the vector field X_ξ in (2.5.8) with $\gamma(0) = \mathbb{1}$. This implies $\gamma(t) = \exp(t\xi)$ for every $t \in \mathbb{R}$, as claimed. □

Example 2.5.11. Since the general linear group $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$ its Lie algebra is the space of all real $n \times n$-matrices

$$\mathfrak{gl}(n,\mathbb{R}) := \mathrm{Lie}(\mathrm{GL}(n,\mathbb{R})) = \mathbb{R}^{n\times n}.$$

The Lie algebra of the special linear group is

$$\mathfrak{sl}(n,\mathbb{R}) := \mathrm{Lie}(\mathrm{SL}(n,\mathbb{R})) = \{\xi \in \mathfrak{gl}(n,\mathbb{R}) \,|\, \mathrm{trace}(\xi) = 0\}$$

(see Exercise 2.2.8) and the Lie algebra of the special orthogonal group is

$$\mathfrak{so}(n) := \mathrm{Lie}(\mathrm{SO}(n)) = \left\{\xi \in \mathfrak{gl}(n,\mathbb{R}) \,|\, \xi^\mathsf{T} + \xi = 0\right\} = \mathfrak{o}(n)$$

(see Example 2.2.9).

Exercise 2.5.12. Prove that the Lie algebras of the general linear group over $\mathbb{C}$, the special linear group over $\mathbb{C}$, the unitary group, and the special unitary group are given by

$$\mathfrak{gl}(n,\mathbb{C}) := \mathrm{Lie}(\mathrm{GL}(n,\mathbb{C})) = \mathbb{C}^{n\times n},$$

$$\mathfrak{sl}(n,\mathbb{C}) := \mathrm{Lie}(\mathrm{SL}(n,\mathbb{C})) = \{\xi \in \mathfrak{gl}(n,\mathbb{C}) \,|\, \mathrm{trace}(\xi) = 0\},$$

$$\mathfrak{u}(n) := \mathrm{Lie}(\mathrm{U}(n)) = \{\xi \in \mathfrak{gl}(n,\mathbb{R}) \,|\, \xi^* + \xi = 0\},$$

$$\mathfrak{su}(n) := \mathrm{Lie}(\mathrm{SU}(n)) = \{\xi \in \mathfrak{gl}(n,\mathbb{C}) \,|\, \xi^* + \xi = 0,\ \mathrm{trace}(\xi) = 0\}.$$

These are vector spaces over the reals. Determine their real dimensions. Which of these are also complex vector spaces?

Exercise 2.5.13. Let $\mathrm{G} \subset \mathrm{GL}(n,\mathbb{R})$ be a subgroup. Prove that G is a Lie group if and only if it is a closed subset of $\mathrm{GL}(n,\mathbb{R})$ in the relative topology.

2.5.3 Lie Group Homomorphisms

Let G and H be Lie groups and $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. A **Lie group homomorphism** from G to H is a smooth map $\rho : \mathrm{G} \to \mathrm{H}$ that is a group homomorphism. A **Lie group isomorphism** is a bijective Lie group homomorphism whose inverse is also a Lie group homomorphism. A **Lie algebra homomorphism** from $\mathfrak{g}$ to $\mathfrak{h}$ is a linear map that preserves the Lie bracket.

Lemma 2.5.14. *Let* G *and* H *be Lie groups and denote their Lie algebras by* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ *and* $\mathfrak{h} := \mathrm{Lie}(\mathrm{H})$. *Let* $\rho : \mathrm{G} \to \mathrm{H}$ *be a Lie group homomorphism and denote its derivative at* $\mathbb{1} \in \mathrm{G}$ *by*

$$\dot\rho := d\rho(\mathbb{1}) : \mathfrak{g} \to \mathfrak{h}.$$

Then $\dot\rho$ *is a Lie algebra homomorphism.*

Proof. The proof has three steps.

Step 1. *For all* $\xi \in \mathfrak{g}$ *and* $t \in \mathbb{R}$ *we have* $\rho(\exp(t\xi)) = \exp(t\dot\rho(\xi))$.

Fix an element $\xi \in \mathfrak{g}$. Then, by Lemma 2.5.9, we have $\exp(t\xi) \in \mathrm{G}$ for every $t \in \mathbb{R}$. Thus we can define a map $\gamma : \mathbb{R} \to \mathrm{H}$ by $\gamma(t) := \rho(\exp(t\xi))$. Since ρ is smooth, this is a smooth curve in H and, since ρ is a group homomorphism and the exponential map satisfies (2.5.6), our curve γ satisfies the conditions

$$\gamma(s+t) = \gamma(s)\gamma(t), \qquad \gamma(0) = \mathbb{1}, \qquad \dot\gamma(0) = d\rho(\mathbb{1})\xi = \dot\rho(\xi).$$

Hence it follows from Lemma 2.5.10 that $\gamma(t) = \exp(t\dot\rho(\xi))$. This proves Step 1.

Step 2. *For all* $g \in \mathrm{G}$ *and* $\eta \in \mathfrak{g}$ *we have* $\dot\rho(g\eta g^{-1}) = \rho(g)\dot\rho(\eta)\rho(g^{-1})$.

Define the smooth curve $\gamma : \mathbb{R} \to \mathrm{G}$ by $\gamma(t) := g\exp(t\eta)g^{-1}$. This curve takes values in G by Lemma 2.5.9. By Step 1 we have

$$\rho(\gamma(t)) = \rho(g)\rho(\exp(t\eta))\rho(g)^{-1} = \rho(g)\exp(t\dot\rho(\eta))\rho(g)^{-1}$$

for every t. Since $\gamma(0) = \mathbb{1}$ and $\dot\gamma(0) = g\eta g^{-1}$ we obtain

$$\begin{aligned}
\dot\rho(g\eta g^{-1}) &= d\rho(\gamma(0))\dot\gamma(0) \\
&= \left.\frac{d}{dt}\right|_{t=0} \rho(\gamma(t)) \\
&= \left.\frac{d}{dt}\right|_{t=0} \rho(g)\exp(t\dot\rho(\eta))\rho(g^{-1}) \\
&= \rho(g)\dot\rho(\eta)\rho(g^{-1}).
\end{aligned}$$

This proves Step 2.

Step 3. *For all $\xi, \eta \in \mathfrak{g}$ we have*

$$\dot\rho([\xi,\eta]) = [\dot\rho(\xi), \dot\rho(\eta)].$$

Define the curve $\eta : \mathbb{R} \to \mathfrak{g}$ by

$$\eta(t) := \exp(t\xi)\eta\exp(-t\xi)$$

for $t \in \mathbb{R}$. By Lemma 2.5.9 this curve takes values in the Lie algebra of G and

$$\dot\eta(0) = [\xi,\eta].$$

Hence

$$\begin{aligned}
\dot\rho([\xi,\eta]) &= \left.\frac{d}{dt}\right|_{t=0} \dot\rho\left(\exp(t\xi)\eta\exp(-t\xi)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \rho\left(\exp(t\xi)\right)\dot\rho(\eta)\rho\left(\exp(-t\xi)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \exp\left(t\dot\rho(\xi)\right)\dot\rho(\eta)\exp\left(-t\dot\rho(\xi)\right) \\
&= \left[\dot\rho(\xi), \dot\rho(\eta)\right].
\end{aligned}$$

Here the first equation follows from the fact that $\dot\rho$ is linear, the second equation follows from Step 2 with $g = \exp(t\xi)$, and the third equation follows from Step 1. This proves Step 3 and Lemma 2.5.14. □

Example 2.5.15. The complex determinant defines a Lie group homomorphism $\det : \mathrm{U}(n) \to S^1$. The associated Lie algebra homomorphism is

$$\mathrm{trace} = \dot{\det} : \mathfrak{u}(n) \to \mathbf{i}\mathbb{R} = \mathrm{Lie}(S^1).$$

Example 2.5.16 (Unit Quaternions and SU(2)). The Lie group SU(2) is diffeomorphic to the 3-sphere. Every matrix in SU(2) can be written as

$$g = \begin{pmatrix} x_0 + \mathbf{i}x_1 & x_2 + \mathbf{i}x_3 \\ -x_2 + \mathbf{i}x_3 & x_0 - \mathbf{i}x_1 \end{pmatrix}, \qquad x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1. \tag{2.5.10}$$

Here the x_i are real numbers. They can be interpreted as the coordinates of a unit quaternion $x = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3 \in \mathrm{Sp}(1)$ (see Example 2.5.6). The reader may verify that the map $\mathrm{Sp}(1) \to \mathrm{SU}(2) : x \mapsto g$ in (2.5.10) is a Lie group isomorphism.

Exercise 2.5.17 (The double cover of $\mathrm{SO}(3)$**).** Identify the imaginary part of $\mathbb{H}$ with $\mathbb{R}^3$ and write a vector $\xi \in \mathbb{R}^3 = \mathrm{Im}(\mathbb{H})$ as a purely imaginary quaternion $\xi = \mathbf{i}\xi_1 + \mathbf{j}\xi_1 + \mathbf{k}\xi_3$. Prove that if $\xi \in \mathrm{Im}(\mathbb{H})$ and $x \in \mathrm{Sp}(1)$ then $x\xi\bar{x} \in \mathrm{Im}(\mathbb{H})$. Define the map $\rho : \mathrm{Sp}(1) \to \mathrm{SO}(3)$ by

$$\rho(x)\xi := x\xi\bar{x}$$

for $x \in \mathrm{Sp}(1)$ and $\xi \in \mathrm{Im}(\mathbb{H})$. Prove that the linear map $\rho(x) : \mathbb{R}^3 \to \mathbb{R}^3$ is represented by the 3×3-matrix

$$\rho(x) = \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1x_2 - x_0x_3) & 2(x_1x_3 + x_0x_2) \\ 2(x_1x_2 + x_0x_3) & x_0^2 + x_2^2 - x_3^2 - x_1^2 & 2(x_2x_3 - x_0x_1) \\ 2(x_1x_3 - x_0x_2) & 2(x_2x_3 + x_0x_1) & x_0^2 + x_3^2 - x_1^2 - x_2^2 \end{pmatrix}.$$

Show that ρ is a Lie group homomorphism. Find a formula for the map

$$\dot{\rho} := d\rho(\mathbb{1}) : \mathfrak{sp}(1) \to \mathfrak{so}(3)$$

and show that it is a Lie algebra isomorphism. For $x, y \in \mathrm{Sp}(1)$ prove that $\rho(x) = \rho(y)$ if and only if $y = \pm x$.

Example 2.5.18. Consider the map

$$\mathrm{GL}(n, \mathbb{R}) \to \mathrm{Diff}(\mathbb{R}^n) : g \mapsto \phi_g$$

which assigns to every nonsingular matrix $g \in \mathrm{GL}(n, \mathbb{R})$ the linear diffeomorphism $\phi_g : \mathbb{R}^n \to \mathbb{R}^n$ given by $\phi_g(x) := gx$ for $x \in \mathbb{R}^n$. This map $g \mapsto \phi_g$ is a group homomorphism. The group $\mathrm{Diff}(\mathbb{R}^n)$ is infinite dimensional and thus cannot be a Lie group. However, it has many properties in common with Lie groups. For example one can define what is meant by a smooth path in $\mathrm{Diff}(\mathbb{R}^n)$ and extend formally the notion of a tangent vector (as the derivative of a path through a given element of $\mathrm{Diff}(\mathbb{R}^n)$) to this setting. In particular, the tangent space of $\mathrm{Diff}(\mathbb{R}^n)$ at the identity can then be identified with the space of vector fields

$$T_{\mathrm{id}}\mathrm{Diff}(\mathbb{R}^n) = \mathrm{Vect}(\mathbb{R}^n).$$

Differentiating the map $g \mapsto \phi_g$, one then obtains a linear map

$$\mathfrak{gl}(n, \mathbb{R}) \to \mathrm{Vect}(\mathbb{R}^n) : \xi \mapsto X_\xi$$

which assigns to every matrix $\xi \in \mathfrak{gl}(n, \mathbb{R})$ the vector field $X_\xi : \mathbb{R}^n \to \mathbb{R}^n$ given by $X_\xi(x) := \xi x$ for $x \in \mathbb{R}^n$. We have already seen in Remark 2.4.23 that this map is a Lie algebra homomorphism.

Example 2.5.19. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then the set

$$\mathrm{Aut}(\mathfrak{g}) := \left\{ \Phi : \mathfrak{g} \to \mathfrak{g} \;\middle|\; \begin{array}{l} \Phi \text{ is a bijective linear map,} \\ \Phi[\xi,\eta] = [\Phi\xi, \Phi\eta] \;\forall\, \xi,\eta \in \mathfrak{g} \end{array} \right\}$$

of **Lie algebra automorphisms** of $\mathfrak{g}$ is a Lie group. Its Lie algebra is the space of **derivations** on $\mathfrak{g}$ denoted by

$$\mathrm{Der}(\mathfrak{g}) := \left\{ A : \mathfrak{g} \to \mathfrak{g} \;\middle|\; \begin{array}{l} A \text{ is a linear map,} \\ A[\xi,\eta] = [A\xi,\eta] + [\xi, A\eta] \;\forall\, \xi,\eta \in \mathfrak{g} \end{array} \right\}.$$

Now suppose that $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ is the Lie algebra of a Lie group G. Then there is a map

$$\mathrm{ad} : \mathrm{G} \to \mathrm{Aut}(\mathfrak{g}), \qquad \mathrm{ad}(g)\eta := g\eta g^{-1}, \tag{2.5.11}$$

for $g \in \mathrm{G}$ and $\eta \in \mathfrak{g}$. Lemma 2.5.9 (ii) asserts that $\mathrm{ad}(g)$ is indeed a linear map from $\mathfrak{g}$ to itself for every $g \in \mathrm{G}$. The reader may verify that the map

$$\mathrm{ad}(g) : \mathfrak{g} \to \mathfrak{g}$$

is a Lie algebra automorphism for every $g \in \mathrm{G}$ and that the map $\mathrm{ad} : \mathrm{G} \to \mathrm{Aut}(\mathfrak{g})$ is a Lie group homomorphism. The associated Lie algebra homomorphism is the map

$$\mathrm{Ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g}), \qquad \mathrm{Ad}(\xi)\eta := [\xi,\eta], \tag{2.5.12}$$

for $\xi,\eta \in \mathfrak{g}$. To verify the claim $\mathrm{Ad} = \dot{\mathrm{ad}}$ we compute

$$\dot{\mathrm{ad}}(\xi)\eta = \frac{d}{dt}\bigg|_{t=0} \mathrm{ad}(\exp(t\xi))\eta = \frac{d}{dt}\bigg|_{t=0} \exp(t\xi)\eta\exp(-t\xi) = [\xi,\eta].$$

Exercise 2.5.20. Let $\mathfrak{g}$ be any Lie algebra and define the map

$$\mathrm{Ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$

by (2.5.12). Prove that the endomorphism

$$\mathrm{Ad}(\xi) : \mathfrak{g} \to \mathfrak{g}$$

is a derivation for every $\xi \in \mathfrak{g}$ and that $\mathrm{Ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is a Lie algebra homomorphism. If $\mathfrak{g}$ is finite dimensional, prove that $\mathrm{Aut}(\mathfrak{g})$ is a Lie group with Lie algebra $\mathrm{Der}(\mathfrak{g})$.

2.5.4 Lie Groups and Diffeomorphisms

There is a natural correspondence between Lie groups and Lie algebras on the one hand and diffeomorphisms and vector fields on the other hand. We summarize this correspondence in the following table

Lie groups	**Diffeomorphisms**
$\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$	$\mathrm{Diff}(M)$
$\mathfrak{g} = \mathrm{Lie}(\mathrm{G}) = T_{\mathbb{1}}\mathrm{G}$	$\mathrm{Vect}(M) = T_{\mathrm{id}}\mathrm{Diff}(M)$
exponential map	flow of a vector field
$t \mapsto \exp(t\xi)$	$t \mapsto \phi^t = \text{“}\exp(tX)\text{''}$
adjoint representation	pushforward
$\xi \mapsto g\xi g^{-1}$	$X \mapsto \phi_* X$
Lie bracket on $\mathfrak{g}$	Lie bracket of vector fields
$[\xi, \eta] = \xi\eta - \eta\xi$	$[X, Y] = dX \cdot Y - dY \cdot X.$

To understand the correspondence between the exponential map and the flow of a vector field compare equation (2.4.6) with equation (2.5.5). To understand the correspondence between the adjoint representation and pushforward observe that

$$\phi_* Y = \left.\frac{d}{dt}\right|_{t=0} \phi \circ \psi^t \circ \phi^{-1}, \qquad g\eta g^{-1} = \left.\frac{d}{dt}\right|_{t=0} g \exp(t\eta) g^{-1},$$

where ψ^t denotes the flow of Y. To understand the correspondence between the Lie brackets recall that

$$[X, Y] = \left.\frac{d}{dt}\right|_{t=0} (\phi^t)_* Y, \qquad [\xi, \eta] = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)\eta\exp(-t\xi),$$

where ϕ^t denotes the flow of X. We emphasize that the analogy between Lie groups and Diffeomorphisms only works well when the manifold M is compact so that every vector field on M is complete. The next exercise gives another parallel between the Lie bracket on the Lie algebra of a Lie group and the Lie bracket of two vector fields.

Exercise 2.5.21. Let $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $\xi, \eta \in \mathfrak{g}$. Define the smooth curve $\gamma : \mathbb{R} \to \mathrm{G}$ by

$$\gamma(t) := \exp(t\xi)\exp(t\eta)\exp(-t\xi)\exp(-t\eta).$$

Prove that $\dot\gamma(0) = 0$ and $\frac{1}{2}\ddot\gamma(0) = [\xi, \eta]$. Compare this with Lemma 2.4.18.

Exercise 2.5.22. Let $\mathrm{G} \subset \mathrm{GL}(n, \mathbb{R})$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $\xi, \eta \in \mathfrak{g}$. Show that $[\xi, \eta] = 0$ if and only if the exponential maps commute, i.e. $\exp(s\xi)\exp(t\eta) = \exp(t\eta)\exp(s\xi)$ for all $s, t \in \mathbb{R}$. How can this observation be deduced from Lemma 2.4.26?

2.5.5 Smooth Maps and Algebra Homomorphisms

Let M be a smooth submanifold of $\mathbb{R}^k$. Denote by $\mathscr{F}(M) := C^\infty(M,\mathbb{R})$ the space of smooth real valued functions $f : M \to \mathbb{R}$. Then $\mathscr{F}(M)$ is a commutative unital algebra. Each $p \in M$ determines a unital algebra homomorphism $\varepsilon_p : \mathscr{F}(M) \to \mathbb{R}$ defined by $\varepsilon_p(f) = f(p)$ for $p \in M$.

Theorem 2.5.23. *Every unital algebra homomorphism $\varepsilon : \mathscr{F}(M) \to \mathbb{R}$ has the form $\varepsilon = \varepsilon_p$ for some $p \in M$.*

Proof. Assume that $\varepsilon : \mathscr{F}(M) \to \mathbb{R}$ is an algebra homomorphism.

Claim. *For all $f, g \in \mathscr{F}(M)$ we have $\varepsilon(g) = 0 \implies \varepsilon(f) \in f(g^{-1}(0))$.*

Indeed, the function $f - \varepsilon(f) \cdot 1$ lies in the kernel of ε and so the function $h := (f - \varepsilon(f) \cdot 1)^2 + g^2$ also lies in the kernel of ε. There must be at least one point $p \in M$ where $h(p) = 0$ for otherwise $1 = \varepsilon(h)\varepsilon(1/h) = 0$. For this point p we have $f(p) = \varepsilon(p)$ and $g(p) = 0$, hence $p \in g^{-1}(0)$, and therefore $\varepsilon(f) = f(p) \in f(g^{-1}(0))$. This proves the claim.

The theorem asserts that there exists a $p \in M$ such that every $f \in \mathscr{F}(M)$ satisfies $\varepsilon(f) = f(p)$. Assume, by contradiction, that this is false. Then for every $p \in M$ there exists a function $f \in \mathscr{F}(M)$ such that $f(p) \neq \varepsilon(f)$. Replace f by $f - \varepsilon(f)$ to obtain $f(p) \neq 0 = \varepsilon(f)$. Now use the axiom of choice to obtain a family of functions $f_p \in \mathscr{F}(M)$, one for every $p \in M$, such that $f_p(p) \neq 0 = \varepsilon(f_p)$ for all $p \in M$. Then the set $U_p := f_p^{-1}(\mathbb{R} \setminus \{0\})$ is an M-open neighborhood of p for every $p \in M$. Choose a sequence of compact sets $K_n \subset M$ such that $K_n \subset \mathrm{int}_M(K_{n+1})$ for all n and $M = \bigcup_n K_n$. Then, for each n, there is a $g_n \in \mathscr{F}(M)$ (a finite sum of the form $\sum_i f_{p_i}^2$) such that $\varepsilon(g_n) = 0$ and $g_n(q) > 0$ for all $q \in K_n$. If M is compact, this is already a contradiction because a positive function cannot belong to the kernel of ε. Otherwise, choose $f \in \mathscr{F}(M)$ such that $f(q) \geq n$ for all $q \in M \setminus K_n$ and all $n \in \mathbb{N}$. Then $\varepsilon(f) \in f(g_n^{-1}(0)) \subset f(M \setminus K_n) \subset [n, \infty)$ by the claim and so $\varepsilon(f) \geq n$ for all n. This is a contradiction and proves Theorem 2.5.23. $\square$

Now let N be another smooth submanifold (say of $\mathbb{R}^\ell$) and let $C^\infty(M,N)$ denote the space of smooth maps from M to N. A **homomorphism** from $\mathscr{F}(N)$ to $\mathscr{F}(M)$ is a (real) linear map $\Phi : \mathscr{F}(N) \to \mathscr{F}(M)$ that satisfies

$$\Phi(fg) = \Phi(f)\Phi(g), \qquad \Phi(1) = 1.$$

An **automorphism** of the algebra $\mathscr{F}(M)$ is a bijective homomorphism $\Phi : \mathscr{F}(M) \to \mathscr{F}(M)$. Let $\mathrm{Hom}(\mathscr{F}(N), \mathscr{F}(M))$ denote the space of homomorphisms from $\mathscr{F}(N)$ to $\mathscr{F}(M)$. The automorphisms of $\mathscr{F}(M)$ form a group denoted by $\mathrm{Aut}(\mathscr{F}(M))$.

Corollary 2.5.24. *The pullback operation*

$$C^\infty(M,N) \to \mathrm{Hom}(\mathscr{F}(N),\mathscr{F}(M)) : \phi \mapsto \phi^*$$

is bijective. In particular, the map $\mathrm{Diff}(M) \to \mathrm{Aut}(\mathscr{F}(M)) : \phi \mapsto \phi^*$ *is an anti-isomorphism of groups.*

Proof. This is an exercise with hint. Let $\Phi : \mathscr{F}(N) \to \mathscr{F}(M)$ be a unital algebra homomorphism. By Theorem 2.5.23 there exists a map $\phi : M \to N$ such that $\varepsilon_p \circ \Phi = \varepsilon_{\phi(p)}$ for all $p \in M$. Prove that $f \circ \phi : M \to \mathbb{R}$ is smooth for every smooth map $f : N \to \mathbb{R}$ and deduce that ϕ is smooth. □

Remark 2.5.25. The pullback operation is **functorial**, i.e.

$$(\psi \circ \phi)^* = \psi^* \circ \phi^*, \qquad \mathrm{id}_M^* = \mathrm{id}_{\mathscr{F}(M)}.$$

for $\phi \in C^\infty(M,N)$ and $\psi \in C^\infty(N,P)$. Here id denotes the identity map of the space indicated in the subscript. Hence Corollary 2.5.24 may be summarized by saying that the category of smooth manifolds and smooth maps is anti-isomorphic to a subcategory of the category of commutative unital algebras and unital algebra homomorphisms.

Exercise 2.5.26. If M is compact, then there is a slightly different way to prove Theorem 2.5.23. An **ideal** in $\mathscr{F}(M)$ is a linear subspace $\mathscr{J} \subset \mathscr{F}(M)$ satisfying the condition $f \in \mathscr{F}(M),\ g \in \mathscr{J} \implies fg \in \mathscr{J}$. A **maximal ideal** in $\mathscr{F}(M)$ is an ideal $\mathscr{J} \subsetneq \mathscr{F}(M)$ such that every ideal $\mathscr{J}' \subsetneq \mathscr{F}(M)$ containing $\mathscr{J}$ is equal to $\mathscr{J}$. Prove that, if M is compact and $\mathscr{J} \subset \mathscr{F}(M)$ is an ideal with the property that for every $p \in M$ there is an $f \in \mathscr{J}$ with $f(p) \neq 0$, then $\mathscr{J} = \mathscr{F}(M)$. Deduce that each maximal ideal in $\mathscr{F}(M)$ has the form $\mathscr{J}_p := \{f \in \mathscr{F}(M) \mid f(p) = 0\}$ for some $p \in M$.

Exercise 2.5.27. If M is compact, give another proof of Corollary 2.5.24 as follows. The set $\Phi^{-1}(\mathscr{J}_p)$ is a maximal ideal in $\mathscr{F}(N)$ for each $p \in M$. Use Exercise 2.5.26 to deduce that there is a unique map $\phi : M \to N$ such that $\Phi^{-1}(\mathscr{J}_p) = \mathscr{J}_{\phi(p)}$ for all $p \in M$. Show that ϕ is smooth and $\phi^* = \Phi$.

Exercise 2.5.28. It is a theorem of ring theory that, when $I \subset R$ is an ideal in a ring R, the quotient ring R/I is a field if and only if the ideal I is maximal. Show that the kernel of the ring homomorphism $\varepsilon_p : \mathscr{F}(M) \to \mathbb{R}$ of Theorem 2.5.23 is the ideal $\mathscr{J}_p$ of Exercise 2.5.26. Conclude that M is compact if and only if every maximal ideal $\mathscr{J}$ in $\mathscr{F}(M)$ is of the form $\mathscr{J} = \mathscr{J}_p$ for some $p \in M$. **Hint:** The functions of compact support form an ideal. It can be shown that if M is not compact and $\mathscr{J}$ is a maximal ideal containing all functions of compact support then the quotient field $\mathscr{F}(M)/\mathscr{J}$ is a non-Archimedean ordered field which properly contains $\mathbb{R}$.

2.5.6 Vector Fields and Derivations

A **derivation** of $\mathscr{F}(M)$ is a linear map $\delta : \mathscr{F}(M) \to \mathscr{F}(M)$ that satisfies

$$\delta(fg) = \delta(f)g + f\delta(g).$$

and the derivations form a Lie algebra denoted by $\mathrm{Der}(\mathscr{F}(M))$. We may think of $\mathrm{Der}(\mathscr{F}(M))$ as the Lie algebra of $\mathrm{Aut}(\mathscr{F}(M))$ with the Lie bracket given by the commutator. By Theorem 2.5.23 the pullback operation

$$\mathrm{Diff}(M) \to \mathrm{Aut}(\mathscr{F}(M)) : \phi \mapsto \phi^* \tag{2.5.13}$$

can be thought of as a Lie group anti-isomorphism. Differentiating it at the identity $\phi = \mathrm{id}$ gives a linear map

$$\mathrm{Vect}(M) \to \mathrm{Der}(\mathscr{F}(M)) : X \mapsto \mathcal{L}_X. \tag{2.5.14}$$

Here the operator $\mathcal{L}_X : \mathscr{F}(M) \to \mathscr{F}(M)$ is given by the derivative of a function f in the direction of the vector field X, i.e.

$$\mathcal{L}_X f := df \cdot X = \left.\frac{d}{dt}\right|_{t=0} f \circ \phi^t,$$

where ϕ^t denotes the flow of X. Since the map (2.5.14) is the derivative of the "Lie group" anti-homomorphism (2.5.13) we expect it to be a Lie algebra anti-homomorphism. Indeed, one can show that

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_Y\mathcal{L}_X - \mathcal{L}_X\mathcal{L}_Y = -[\mathcal{L}_X, \mathcal{L}_Y] \tag{2.5.15}$$

for $X, Y \in \mathrm{Vect}(M)$. This confirms that our sign in the definition of the Lie bracket is consistent with the standard conventions in the theory of Lie groups. In the literature the difference between a vector field and the associated derivation $\mathcal{L}_X$ is sometimes neglected in the notation and many authors write $Xf := df \cdot X = \mathcal{L}_X f$, thus thinking of a vector field on a manifold M as an operator on the space of functions. With this notation one obtains the equation $[X, Y]f = Y(Xf) - X(Yf)$ and here lies the origin for the use of the opposite sign for the Lie bracket in many books on differential geometry.

Exercise 2.5.29. Prove that the map (2.5.14) is bijective. **Hint:** Fix a derivation $\delta \in \mathrm{Der}(\mathscr{F}(M))$ and prove the following. **Fact 1:** If $U \subset M$ is an open set and $f \in \mathscr{F}(M)$ vanishes on U then $\delta(f)$ vanishes on U. **Fact 2:** If $p \in M$ and the derivative $df(p) : T_pM \to \mathbb{R}$ is zero then $(\delta(f))(p) = 0$. (By Fact 1, the proof of Fact 2 can be reduced to an argument in local coordinates.)

Exercise 2.5.30. Verify the formula (2.5.15).

2.6 Vector Bundles and Submersions

2.6.1 Submersions

Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and $N \subset \mathbb{R}^\ell$ be a smooth n-manifold. A smooth map $f : N \to M$ is called a **submersion** if its derivative

$$df(q) : T_q N \to T_{f(q)} M$$

is surjective for every $q \in N$.

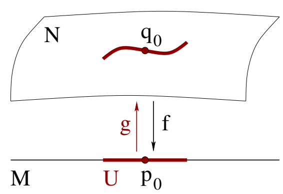


Figure 2.10: A local right inverse of a submersion.

Lemma 2.6.1. *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold, $N \subset \mathbb{R}^\ell$ be a smooth n-manifold, and $f : N \to M$ be a smooth map. The following are equivalent.*

(i) *f is a submersion.*

(ii) *For every $q_0 \in N$ there is an M-open neighborhood U of $p_0 := f(q_0)$ and a smooth map $g : U \to N$ such that $g(f(q_0)) = q_0$ and $f \circ g = \mathrm{id} : U \to U$. Thus f has a local right inverse near every point in N (see Figure 2.10).*

Proof. We prove that (i) implies (ii). Since the derivative

$$df(q_0) : T_{q_0} N \to T_{p_0} M$$

is surjective we have $n \geq m$ and

$$\dim \ker df(q_0) = n - m.$$

Hence there is a linear map $A : \mathbb{R}^\ell \to \mathbb{R}^{n-m}$ whose restriction to the kernel of $df(q_0)$ is bijective. Now define the map $\psi : N \to M \times \mathbb{R}^{n-m}$ by

$$\psi(q) := (f(q), A(q - q_0))$$

for $q \in N$. Then $\psi(q_0) = (p_0, 0)$ and its derivative

$$d\psi(q_0) : T_{q_0} N \to T_{p_0} M \times \mathbb{R}^{n-m}$$

sends $w \in T_{q_0}N$ to $(df(q_0)w, Aw)$ and is therefore bijective. Hence it follows from the inverse function theorem for manifolds (Theorem 2.2.15) that there is an N-open neighborhood $V \subset N$ of q_0 such that the set

$$W := \psi(N) \subset M \times \mathbb{R}^{n-m}$$

is an open neighborhood of $(p_0, 0)$ and $\psi|_V : V \to W$ is a diffeomorphism. Let

$$U := \{p \in M \,|\, (p, 0) \in W\}$$

and define the map $g : U \to N$ by

$$g(p) := \psi^{-1}(p, 0).$$

Then $p_0 \in U$, g is smooth and

$$(p, 0) = \psi(g(p)) = (f(g(p)), A(g(p) - q_0)).$$

Hence $f(g(p)) = p$ for all $p \in U$ and

$$g(p_0) = \psi^{-1}(p_0, 0) = q_0.$$

This shows that (i) implies (ii). The converse is an easy consequence of the chain rule and is left to the reader. This proves Lemma 2.6.1 □

Corollary 2.6.2. *The image of a submersion $f : N \to M$ is open.*

Proof. If $p_0 = f(q_0) \in f(N)$ then the neighborhood $U \subset M$ of p_0 in Lemma 2.6.1 (ii) is contained in the image of f. □

Corollary 2.6.3. *If N is a nonempty compact manifold, M is a connected manifold, and $f : N \to M$ is a submersion then f is surjective and M is compact.*

Proof. The image $f(M)$ is an open subset of M by Corollary 2.6.2, it is a relatively closed subset of M because N is compact, and it is nonempty because N is nonempty. Since M is connected this implies that $f(N) = M$. In particular, M is compact. □

Exercise 2.6.4. Let $f : N \to M$ be a smooth map. Prove that the sets $\{q \in N \,|\, df(q) \text{ is injective}\}$ and $\{q \in N \,|\, df(q) \text{ is surjective}\}$ are open (in the relative topology of N).

2.6.2 Vector Bundles

Let $M \subset \mathbb{R}^k$ be an m-dimensional smooth manifold. A **(smooth) vector bundle (over M of rank n)** is a smooth submanifold $E \subset M \times \mathbb{R}^\ell$ of dimension $m + n$ such that, for every $p \in M$, the set

$$E_p := \left\{ v \in \mathbb{R}^\ell \,|\, (p, v) \in E \right\}$$

is an n-dimensional linear subspace of $\mathbb{R}^\ell$ (called the **fiber of E over p**). If $E \subset M \times \mathbb{R}^\ell$ is a vector bundle then a **(smooth) section of E** is smooth map $s : M \to \mathbb{R}^\ell$ such that $s(p) \in E_p$ for every $p \in M$. A vector bundle E over M is equipped with a smooth map

$$\pi : E \to M$$

defined by $\pi(p, v) := p$ This map is called the **canonical projection** of E. A section $s : M \to \mathbb{R}^\ell$ of E determines a smooth map $\sigma : M \to E$ which sends the point $p \in M$ to the pair $(p, s(p)) \in E$. This map satisfies

$$\pi \circ \sigma = \mathrm{id}.$$

It is sometimes convenient to abuse notation and eliminate the distinction between s and σ. Thus we will sometimes use the same letter s for the map from M to $\mathbb{R}^\ell$ and the map from M to E.

Example 2.6.5. Let $M \subset \mathbb{R}^k$ be a smooth m-dimensional submanifold. The set

$$TM := \{(p, v) \,|\, p \in M,\, v \in T_pM\}$$

is called the **tangent bundle** of M. This is a subset of $M \times \mathbb{R}^k$ and, for every $p \in M$, its fiber T_pM is an m-dimensional linear subspace of $\mathbb{R}^k$ by Theorem 2.2.3. However, it is not immediately clear from the definition that TM is a submanifold of $M \times \mathbb{R}^k$. This will be proved below. The sections of TM are the vector fields on M.

Exercise 2.6.6. Let $f : M \to N$ be a smooth map between manifolds. Prove that the tangent map

$$TM \to TN : (p, v) \mapsto (f(p), df(p)v)$$

is smooth.

Exercise 2.6.7. Let $V \subset \mathbb{R}^\ell$ be an n-dimensional linear subspace. The **orthogonal projection** of $\mathbb{R}^\ell$ onto V is the matrix $\Pi \in \mathbb{R}^{\ell\times\ell}$ that satisfies

$$\Pi = \Pi^2 = \Pi^\mathsf{T}, \qquad \operatorname{im}\Pi = V. \tag{2.6.1}$$

Prove that there is a unique matrix $\Pi \in \mathbb{R}^{\ell\times\ell}$ satisfying (2.6.1). Prove that, for every symmetric matrix $S = S^\mathsf{T} \in \mathbb{R}^{\ell\times\ell}$, the kernel of S is the orthogonal complement of the image of S. If $D \in \mathbb{R}^{\ell\times n}$ is any injective matrix whose image is V, prove that $\det(D^\mathsf{T}D) \neq 0$ and

$$\Pi = D(D^\mathsf{T}D)^{-1}D^\mathsf{T}. \tag{2.6.2}$$

Theorem 2.6.8 (Vector Bundles). *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold and let $E \subset M \times \mathbb{R}^\ell$ be a subset. Assume that, for every $p \in M$, the set*

$$E_p := \left\{ v \in \mathbb{R}^\ell \,|\, (p, v) \in E \right\} \tag{2.6.3}$$

is an n-dimensional linear subspace of $\mathbb{R}^\ell$. Let $\Pi : M \to \mathbb{R}^{\ell\times\ell}$ be the map that assigns to each $p \in M$ the orthogonal projection of $\mathbb{R}^\ell$ onto E_p, i.e.

$$\Pi(p) = \Pi(p)^2 = \Pi(p)^\mathsf{T}, \qquad \operatorname{im}\Pi(p) = E_p. \tag{2.6.4}$$

Then the following are equivalent.

(i) *E is a vector bundle.*

(ii) *For every $p_0 \in M$ and every $v_0 \in E_{p_0}$ there is a smooth map $s : M \to \mathbb{R}^\ell$ such that $s(p_0) = v_0$ and $s(p) \in E_p$ for all $p \in M$.*

(iii) *The map $\Pi : M \to \mathbb{R}^{\ell\times\ell}$ is smooth.*

(iv) *For every $p_0 \in M$ there is an open neighborhood $U \subset M$ of p_0 and a diffeomorphism $\pi^{-1}(U) \to U \times \mathbb{R}^n : (p, v) \mapsto \Phi(p, v) = (p, \Phi_p(v))$ such that the map $\Phi_p : E_p \to \mathbb{R}^n$ is an isometric isomorphism for all $p \in U$.*

(v) *For every $p_0 \in M$ there is an open neighborhood $U \subset M$ of p_0 and a diffeomorphism $\pi^{-1}(U) \to U \times \mathbb{R}^n : (p, v) \mapsto \Phi(p, v) = (p, \Phi_p(v))$ such that the map $\Phi_p : E_p \to \mathbb{R}^n$ is a vector space isomorphism for all $p \in U$.*

Condition (i) implies that the projection $\pi : E \to M$ is a submersion. In (ii) the section s can be chosen to have compact support, i.e. there is a compact subset $K \subset M$ such that $s(p) = 0$ for $p \notin K$.

Proof. See page 70. $\square$

Definition 2.6.9. *The maps $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ in Theorem 2.6.8 are called* **local trivializations** *of E. They fit into commutative diagrams*

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^n \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd} .$$

Corollary 2.6.10. *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold. Then TM is a vector bundle over M and hence is a smooth $2m$-manifold in $\mathbb{R}^k \times \mathbb{R}^k$.*

Proof. Let $\phi : U \to \Omega$ be a coordinate chart on an M-open set $U \subset M$ with values in an open subset $\Omega \subset \mathbb{R}^m$. Denote its inverse by $\psi := \phi^{-1} : \Omega \to M$. By Theorem 2.2.3 the linear map $d\psi(x) : \mathbb{R}^m \to \mathbb{R}^k$ is injective and its image is $T_{\psi(x)}M$ for every $x \in \Omega$. Hence the map $D : U \to \mathbb{R}^{k\times m}$ defined by

$$D(p) := d\psi(\phi(p)) \in \mathbb{R}^{k\times m}$$

is smooth and, for every $p \in U$, the linear map $D(p) : \mathbb{R}^m \to \mathbb{R}^k$ is injective and its image is T_pM. Thus the function $\Pi^{TM} : M \to \mathbb{R}^{k\times k}$ defined by (2.6.4) with $E_p = T_pM$ is given by

$$\Pi^{TM}(p) = D(p)\left(D(p)^{\mathsf{T}} D(p)\right)^{-1} D(p)^{\mathsf{T}} \qquad \text{for } p \in U.$$

Hence Π^{TM} is smooth and so TM is a vector bundle by Theorem 2.6.8. □

Let $M \subset \mathbb{R}^k$ be an m-manifold, $N \subset \mathbb{R}^\ell$ be an n-manifold, $f : N \to M$ be a smooth map, and $E \subset M \times \mathbb{R}^d$ be a vector bundle. The **pullback bundle** is the vector bundle $f^*E \to N$ defined by

$$f^*E := \left\{(q, v) \in N \times \mathbb{R}^d \,|\, v \in E_{f(q)}\right\}$$

and the **normal bundle of** E is the vector bundle $E^\perp \to M$ defined by

$$E^\perp := \left\{(p, w) \in M \times \mathbb{R}^d \,|\, \langle v, w\rangle = 0 \;\forall\, v \in E_p\right\}.$$

Corollary 2.6.11. *The pullback and normal bundles are vector bundles.*

Proof. Let $\Pi = \Pi^E : M \to \mathbb{R}^{d\times d}$ be the map defined by (2.6.4). This map is smooth by Theorem 2.6.8. Moreover, the corresponding maps for f^*E and $E^\perp$ are given by

$$\Pi^{f^*E} = \Pi^E \circ f : N \to \mathbb{R}^{d\times d}, \qquad \Pi^{E^\perp} = \mathbb{1} - \Pi^E : M \to \mathbb{R}^{d\times d}.$$

These maps are smooth and hence it follows from Theorem 2.6.8 that f^*E and $E^\perp$ are vector bundles. □

Proof of Theorem 2.6.8. We first assume that E is a vector bundle and prove that $\pi : E \to M$ is a submersion. Let $\sigma : M \to E$ denote the zero section given by $\sigma(p) := (p, 0)$. Then $\pi \circ \sigma = \mathrm{id}$ and hence it follows from the chain rule that the derivative $d\pi(p, 0) : T_{(p,0)}E \to T_pM$ is surjective. Now it follows from Exercise 2.6.4 that for every $p \in M$ there is an $\varepsilon > 0$ such that the derivative $d\pi(p, v) : T_{(p,v)}E \to T_pM$ is surjective for every $v \in E_p$ with $|v| < \varepsilon$. Consider the map $f_\lambda : E \to E$ defined by

$$f_\lambda(p, v) := (p, \lambda v).$$

This map is a diffeomorphism for every $\lambda > 0$. It satisfies

$$\pi = \pi \circ f_\lambda$$

and hence

$$d\pi(p, v) = d\pi(p, \lambda v) \circ df_\lambda(p, v) : T_{(p,v)}E \to T_pM.$$

Since $df_\lambda(p, v)$ is bijective and $d\pi(p, \lambda v)$ is surjective for $\lambda < \varepsilon/|v|$ it follows that $d\pi(p, v)$ is surjective for every $p \in M$ and every $v \in E_p$. Thus the projection $\pi : E \to M$ is a submersion for every vector bundle E over M.

We prove that (i) implies (ii). Let $p_0 \in M$ and $v_0 \in E_{p_0}$. We have already proved that π is a submersion. Hence it follows from Lemma 2.6.1 that there exists an M-open neighborhood $U \subset M$ of p_0 and a smooth map

$$\sigma_0 : U \to E$$

such that

$$\pi \circ \sigma_0 = \mathrm{id} : U \to U, \qquad \sigma_0(p_0) = (p_0, v_0).$$

Define the map $s_0 : U \to \mathbb{R}^\ell$ by

$$(p, s_0(p)) := \sigma_0(p) \qquad \text{for } p \in U.$$

Then $s_0(p_0) = v_0$ and $s_0(p) \in E_p$ for all $p \in U$. Now choose $\varepsilon > 0$ such that

$$\{p \in M \mid |p - p_0| < \varepsilon\} \subset U$$

and choose a smooth cutoff function $\beta : \mathbb{R}^k \to [0, 1]$ such that $\beta(p_0) = 1$ and $\beta(p) = 0$ for $|p - p_0| \geq \varepsilon$. Define $s : M \to \mathbb{R}^\ell$ by

$$s(p) := \begin{cases} \beta(p)s_0(p), & \text{if } p \in U, \\ 0, & \text{if } p \notin U. \end{cases}$$

This map satisfies the requirements of (ii).

We prove that (ii) implies (iii). Thus we assume that E satisfies (ii). Choose $p_0 \in M$ and a basis $v_1, \dots, v_n$ of E_{p_0}. By (ii) there exists smooth sections $s_1, \dots, s_n : M \to \mathbb{R}^\ell$ of E such that $s_i(p_0) = v_i$ for $i = 1, \dots, n$. Now there exists an M-open neighborhood $U \subset M$ of p_0 such that the vectors $s_1(p), \dots, s_n(p)$ are linearly independent, and hence form a basis of E_p for every $p \in U$. Hence, for every $p \in U$, we have

$$E_p = \mathrm{im} D(p), \qquad D(p) := [s_1(p) \cdots s_n(p)] \in \mathbb{R}^{\ell \times n}.$$

By Exercise 2.6.7, this implies $\Pi(p) = D(p)(D(p)^\mathsf{T} D(p))^{-1} D(p)^\mathsf{T}$ for every $p \in U$. Thus every $p_0 \in M$ has a neighborhood U such that the restriction of Π to U is smooth. This shows that (ii) implies (iii).

We prove that (iii) implies (iv). Fix a point $p_0 \in M$ and choose a basis $v_1, \dots, v_n$ of E_{p_0}. For $p \in M$ define

$$D(p) := [\Pi(p)v_1 \cdots \Pi(p)v_n] \in \mathbb{R}^{\ell \times n}$$

Then $D : M \to \mathbb{R}^{\ell \times n}$ is a smooth map and $D(p_0)$ has rank n. Hence the set

$$U := \{p \in M \,|\, \mathrm{rank} D(p) = n\} \subset M$$

is an open neighborhood of p_0 and $E_p = \mathrm{im} D(p)$ for all $p \in U$. Thus

$$\pi^{-1}(U) = \{(p, v) \in E \,|\, p \in U\} \subset E$$

is an open set containing $\pi^{-1}(p_0)$. Define the map $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by

$$\Phi(p, v) := \big(p, \Phi_p(v)\big), \qquad \Phi_p(v) := \Big(D(p)^\mathsf{T} D(p)\Big)^{-1/2} D(p)^\mathsf{T} v$$

for $p \in U$ and $v \in E_p$. This map is bijective and its inverse is given by

$$\Phi^{-1}(p, \xi) = \big(p, \Phi_p^{-1}(\xi)\big), \qquad \Phi_p^{-1}(\xi) = D(p) \Big(D(p)^\mathsf{T} D(p)\Big)^{-1/2} \xi$$

for $p \in U$ and $\xi \in \mathbb{R}^n$. Thus Φ is a diffeomorphism and $|\Phi_p(v)| = |v|$ for all $p \in U$ and all $v \in E_p$. This shows that (iii) implies (iv).

That (iv) implies (v) is obvious.

We prove that (v) implies (i). Shrinking U if necessary, we may assume that there exists a coordinate chart $\phi : U \to \Omega$ with values in an open set $\Omega \subset \mathbb{R}^m$. Then the composition $(\phi \times \mathrm{id}) \circ \Phi : \pi^{-1}(U) \to \Omega \times \mathbb{R}^n$ is a diffeomorphism. Thus $E \subset \mathbb{R}^k \times \mathbb{R}^\ell$ is a manifold of dimension $m + n$ and this proves Theorem 2.6.8. □

Exercise 2.6.12. Construct a vector bundle $E \subset S^1 \times \mathbb{R}^2$ of rank 1 that does not admit a *global trivialization*, i.e. that is not isomorphic to the trivial bundle $S^1 \times \mathbb{R}$. Such a vector bundle is called a **Möbius strip**. Define the notion of an isomorphism between two vector bundles E and F over M.

2.6.3 The Implicit Function Theorem

In this subsection we carry over the Implicit Function Theorem in Corollary A.2.6 to smooth maps on vector bundles.

Theorem 2.6.13 (Implicit Function Theorem). *Let $M \subset \mathbb{R}^k$ be a smooth m-manifold, let $N \subset \mathbb{R}^k$ be a smooth n-manifold, let $E \subset M \times \mathbb{R}^\ell$ be a smooth vector bundle of rank n, let $W \subset E$ be open, and let $f : W \to N$ be a smooth map. For $p \in M$ define $f_p : W_p \to N$ by*

$$W_p := \{v \in E_p \,|\, (p, v) \in W\}, \qquad f_p(v) := f(p, v).$$

Let $p_0 \in M$ such that $0 \in W_{p_0}$ and $df_{p_0}(0) : T_{p_0}M \to T_{q_0}N$ is bijective, where $q_0 := f(p_0, 0) \in N$. Then there exists a constant $\varepsilon > 0$, open neighborhoods $U_0 \subset M$ of p_0 and $V_0 \subset N$ of q_0, and a smooth map $h : U_0 \times V_0 \to \mathbb{R}^\ell$ such that $\{(p, v) \in E \,|\, p \in U_0,\, |v| < \varepsilon\} \subset W$ and

$$h(p, q) \in E_p, \qquad |h(p, q)| < \varepsilon \tag{2.6.5}$$

for all $(p, q) \in U_0 \times V_0$ and

$$f_p(v) = 0 \qquad \iff \qquad v = h(p, q) \tag{2.6.6}$$

for all $(p, q) \in U_0 \times V_0$, and all $v \in E_p$ with $|v| < \varepsilon$.

Proof. Choose a coordinate chart $\psi : V \to \mathbb{R}^n$ on an open set $V \subset N$ containing q_0. Choose an open neighborhood $U \subset M$ of p_0 such that $(p, 0) \in W$ and $f(p, 0) \in V$ for all $p \in \overline{U}$, there is a coordinate chart $\phi : U \to \Omega \subset \mathbb{R}^m$, and there is a local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ as in Theorem 2.6.8 with $|\Phi_p(v)| = |v|$ for $p \in U$ and $v \in E_p$. Define $B_r := \{\xi \in \mathbb{R}^n \mid |\xi| < r\}$ and choose $r > 0$ so small that $\Phi^{-1}(U \times B_r) \subset W$ and $f \circ \Phi^{-1}(U \times B_r) \subset V$. Define the map $F : \Omega \times \mathbb{R}^n \times B_r \to \mathbb{R}^n$ by

$$F(x, y, \xi) := \psi \circ f \circ \Phi^{-1}\left(\phi^{-1}(x), \xi\right) - y$$

for $(x, y) \in \Omega \times \mathbb{R}^n$ and $\xi \in B_r$. Let $x_0 := \phi(p_0)$ and $y_0 := \psi(q_0)$. Then we have $F(x_0, y_0, 0) = 0$ and the derivative $d_3F(x_0, y_0, 0) : \mathbb{R}^n \to \mathbb{R}^n$ of F with respect to ξ at $(x_0, y_0, 0)$ is bijective. Hence Corollary A.2.6 asserts that there exist open neighborhoods $U_0 \subset U$ of p_0 and $V_0 \subset V$ of q_0, a constant $0 < \varepsilon < r$, and a smooth map $g : \phi(U_0) \times \psi(V_0) \to B_\varepsilon$ such that

$$F(x, y, \xi) = 0 \qquad \iff \qquad g(x, y) = \xi$$

for all $(x, y) \in \phi(U_0) \times \psi(V_0)$ and all $\xi \in B_\varepsilon$. Thus the map

$$h : U_0 \times V_0 \to \mathbb{R}^\ell, \qquad h(p, q) := \Phi_p^{-1}\big(g(\phi(p), \psi(q))\big),$$

satisfies the requirements of Theorem 2.6.13. $\square$

2.7 The Theorem of Frobenius

Let $M \subset \mathbb{R}^k$ be an m-dimensional manifold and n be a nonnegative integer. A **subbundle of rank** n of the tangent bundle TM is a subset $E \subset TM$ that is itself a vector bundle of rank n over M, i.e. it is a submanifold of TM and the fiber $E_p = \{v \in T_pM \,|\, (p,v) \in E\}$ is an n-dimensional linear subspace of T_pM for every $p \in M$. Note that the rank n of a subbundle is necessarily less than or equal to m. In the literature a subbundle of the tangent bundle is sometimes called a *distribution* on M. We shall, however, not use this terminology in order to avoid confusion with the concept of a distribution in the functional analytic setting.

Definition 2.7.1. *Let $M \subset \mathbb{R}^k$ be an m-dimensional manifold and $E \subset TM$ be a subbundle of rank n. The subbundle E is called* **involutive** *if, for any two vector fields $X, Y \in \mathrm{Vect}(M)$, we have*

$$X(p), Y(p) \in E_p \;\forall\, p \in M \qquad \Longrightarrow \qquad [X,Y](p) \in E_p \;\forall\, p \in M. \tag{2.7.1}$$

The subbundle E is called **integrable** *if, for every $p_0 \in M$, there exists a submanifold $N \subset M$ such that $p_0 \in N$ and $T_pN = E_p$ for every $p \in N$. A* **foliation box for** *E (see Figure 2.11) is a coordinate chart $\phi : U \to \Omega$ on an M-open subset $U \subset M$ with values in an open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$ such that the set $\Omega \cap (\mathbb{R}^n \times \{y\})$ is connected for every $y \in \mathbb{R}^{m-n}$ and, for every $p \in U$ and every $v \in T_pM$, we have*

$$v \in E_p \qquad \Longleftrightarrow \qquad d\phi(p)v \in \mathbb{R}^n \times \{0\}.$$

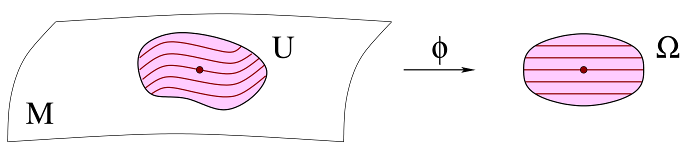


Figure 2.11: A foliation box.

Theorem 2.7.2 (**Frobenius**)**.** *Let $M \subset \mathbb{R}^k$ be an m-dimensional manifold and $E \subset TM$ be a subbundle of rank n. Then the following are equivalent.*

(i) *E is involutive.*

(ii) *E is integrable.*

(iii) *For every $p_0 \in M$ there is a foliation box $\phi : U \to \Omega$ with $p_0 \in U$.*

Proof. See page 74. □

It is easy to show that (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) (see below). The hard part of the theorem is to prove that (i) $\Longrightarrow$ (iii). This requires the following lemma.

Lemma 2.7.3. *Let $E \subset TM$ be an involutive subbundle and $X \in \mathrm{Vect}(M)$ be a complete vector field such that $X(p) \in E_p$ for every $p \in M$. Denote by*

$$\mathbb{R} \to \mathrm{Diff}(M) : t \mapsto \phi^t$$

the flow of X. Then, for all $t \in \mathbb{R}$ and $p \in M$, we have

$$d\phi^t(p)E_p = E_{\phi^t(p)}. \tag{2.7.2}$$

Proof. See page 76. □

Lemma 2.7.3 implies Theorem 2.7.2. We prove first that (iii) implies (ii). Let $p_0 \in M$, choose a foliation box $\phi : U \to \Omega$ for E with $p_0 \in U$, and define

$$N := (p \in U \,|\, \phi(p) \in \mathbb{R}^n \times \{y_0\}\}$$

where $(x_0, y_0) := \phi(p_0) \in \Omega$. Then N satisfies the requirements of (ii).

We prove that (ii) implies (i). Choose two vector fields $X, Y \in \mathrm{Vect}(M)$ that satisfy $X(p), Y(p) \in E_p$ for all $p \in M$ and fix a point $p_0 \in M$. Then, by (ii), there exists a submanifold $N \subset M$ containing p_0 such that $T_pN = E_p$ for every $p \in N$. Hence the restrictions $X|_N$ and $Y|_N$ are vector fields on N and so is the restriction of the Lie bracket $[X, Y]$ to N. Thus we have $[X, Y](p_0) \in T_{p_0}N = E_{p_0}$ as claimed.

We prove that (i) implies (iii). Thus we assume that E is an involutive subbundle of TM and fix a point $p_0 \in M$. By Theorem 2.6.8 there exist vector fields $X_1, \dots, X_n \in \mathrm{Vect}(M)$ such that $X_i(p) \in E_p$ for all i and p and the vectors $X_1(p_0), \dots, X_n(p_0)$ form a basis of E_{p_0}. Using Theorem 2.6.8 again we find vector fields $Y_1, \dots, Y_{m-n} \in \mathrm{Vect}(M)$ such that the vectors

$$X_1(p_0), \dots, X_n(p_0), Y_1(p_0), \dots, Y_{m-n}(p_0)$$

form a basis of $T_{p_0}M$. Using cutoff functions as in the proof of Theorem 2.6.8 we may assume without loss of generality that the vector fields X_i and Y_j have compact support and hence are complete (see Exercise 2.4.13). Denote by $\phi_1^t, \dots, \phi_n^t$ the flows of the vector fields $X_1, \dots, X_n$, respectively, and by $\psi_1^t, \dots, \psi_{m-n}^t$ the flows of the vector fields $Y_1, \dots, Y_{m-n}$. Define the map

$$\psi : \mathbb{R}^n \times \mathbb{R}^{m-n} \to M$$

by

$$\psi(x, y) := \phi_1^{x_1} \circ \cdots \circ \phi_n^{x_n} \circ \psi_1^{y_1} \circ \cdots \circ \psi_{m-n}^{y_{m-n}}(p_0).$$

By Lemma 2.7.3, this map satisfies

$$\frac{\partial \psi}{\partial x_i}(x, y) \in E_{\psi(x,y)} \tag{2.7.3}$$

for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^{m-n}$. Moreover,

$$\frac{\partial \psi}{\partial x_i}(0,0) = X_i(p_0), \qquad \frac{\partial \psi}{\partial y_j}(0,0) = Y_j(p_0),$$

and so the derivative

$$d\psi(0,0) : \mathbb{R}^n \times \mathbb{R}^{m-n} \to T_{p_0}M$$

is bijective. Hence, by the Inverse Function Theorem 2.2.15, there is an open neighborhood $\Omega \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$ of the origin such that the set

$$U := \psi(\Omega) \subset M$$

is an M-open neighborhood of p_0 and $\psi|_\Omega : \Omega \to U$ is a diffeomorphism. Thus the vectors $\partial\psi/\partial x_i(x, y)$ are linearly independent for every $(x, y) \in \Omega$ and, by (2.7.3), form a basis of $E_{\psi(x,y)}$. Hence

$$\phi := (\psi|_\Omega)^{-1} : U \to \Omega$$

is a foliation box and this proves Theorem 2.7.2. □

To complete the proof of the Frobenius theorem it remains to prove Lemma 2.7.3. This requires the following result.

Lemma 2.7.4. *Let $E \subset TM$ be an involutive subbundle. If $\beta : \mathbb{R}^2 \to M$ is a smooth map such that*

$$\frac{\partial \beta}{\partial s}(s, 0) \in E_{\beta(s,0)}, \qquad \frac{\partial \beta}{\partial t}(s, t) \in E_{\beta(s,t)}, \tag{2.7.4}$$

for all $s, t \in \mathbb{R}$ then

$$\frac{\partial \beta}{\partial s}(s, t) \in E_{\beta(s,t)}, \tag{2.7.5}$$

for all $s, t \in \mathbb{R}$.

Proof. See page 76. □

Lemma 2.7.4 implies Lemma 2.7.3. Let $X \in \mathrm{Vect}(M)$ be a complete vector field satisfying $X(p) \in E_p$ for every $p \in M$ and let ϕ^t be the flow of X. Choose a point $p_0 \in M$ and a vector $v_0 \in E_{p_0}$. By Theorem 2.6.8 there is a vector field $Y \in \mathrm{Vect}(M)$ with values in E such that $Y(p_0) = v_0$. Moreover this vector field may be chosen to have compact support and hence it is complete (see Exercise 2.4.13). Thus there is a solution $\gamma : \mathbb{R} \to M$ of the initial value problem

$$\dot\gamma(s) = Y(\gamma(s)), \qquad \gamma(0) = p_0.$$

Define $\beta : \mathbb{R}^2 \to M$ by

$$\beta(s,t) := \phi^t(\gamma(s))$$

for $s, t \in \mathbb{R}$. Then

$$\begin{aligned} \frac{\partial\beta}{\partial s}(s,0) &= \dot\gamma(s) = Y(\gamma(s)) \in E_{\beta(s,0)}, \\ \frac{\partial\beta}{\partial t}(s,t) &= X(\beta(s,t)) \in E_{\beta(s,t)} \end{aligned}$$

for all $s, t \in \mathbb{R}$. Hence it follows from Lemma 2.7.4 that

$$d\phi^t(p_0)v_0 = d\phi^t(\gamma(0))\dot\gamma(0) = \frac{\partial\beta}{\partial s}(0,t) \in E_{\phi^t(p_0)}$$

for every $t \in \mathbb{R}$. This proves Lemma 2.7.3. □

Proof of Lemma 2.7.4. Given any point $p_0 \in M$ we choose a coordinate chart $\phi : U \to \Omega$, defined on an M-open set $U \subset M$ with values in an open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$, such that $p_0 \in U$ and $d\phi(p_0)E_{p_0} = \mathbb{R}^n \times \{0\}$. Shrinking U, if necessary, we obtain that $d\phi(p)E_p$ is the graph of a matrix $A \in \mathbb{R}^{(m-n)\times n}$ for every $p \in U$. Thus there is a map $A : \Omega \to \mathbb{R}^{(m-n)\times n}$ such that, for every $p \in U$, we have

$$d\phi(p)E_p = \{(\xi, A(x,y)\xi) \,|\, \xi \in \mathbb{R}^n\}, \qquad (x,y) := \phi(p) \in \Omega. \tag{2.7.6}$$

For $(x,y) \in \Omega$ we define the linear maps

$$\frac{\partial A}{\partial x}(x,y) : \mathbb{R}^n \to \mathbb{R}^{(m-n)\times n}, \qquad \frac{\partial A}{\partial y}(x,y) : \mathbb{R}^{m-n} \to \mathbb{R}^{(m-n)\times n}$$

by

$$\frac{\partial A}{\partial x}(x,y)\cdot\xi := \sum_{i=1}^n \xi_i \frac{\partial A}{\partial x_i}(x,y), \qquad \frac{\partial A}{\partial y}(x,y)\cdot\eta := \sum_{j=1}^{m-n} \eta_j \frac{\partial A}{\partial y_j}(x,y),$$

for $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ and $\eta = (\eta_1, \dots, \eta_{m-n}) \in \mathbb{R}^{m-n}$. We prove the following.

Claim 1. *Let $(x,y) \in \Omega$, $\xi, \xi' \in \mathbb{R}^n$ and define $\eta, \eta' \in \mathbb{R}^{m-n}$ by $\eta := A(x,y)\xi$ and $\eta' := A(x,y)\xi'$. Then*

$$\left(\frac{\partial A}{\partial x}(x,y)\cdot\xi + \frac{\partial A}{\partial y}(x,y)\cdot\eta\right)\xi' = \left(\frac{\partial A}{\partial x}(x,y)\cdot\xi' + \frac{\partial A}{\partial y}(x,y)\cdot\eta'\right)\xi.$$

The graphs of the matrices $A(z)$ determine a subbundle $\widetilde{E} \subset \Omega \times \mathbb{R}^m$ with fibers

$$\widetilde{E}_z := \left\{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^{m-n} \,|\, \eta = A(x,y)\xi\right\}$$

for $z = (x,y) \in \Omega$. This subbundle is the image of

$$E|_U := \{(p,v) \,|\, p \in U,\, v \in E_p\}$$

under the diffeomorphism $TM|_U \to \Omega \times \mathbb{R}^m : (p,v) \mapsto (\phi(p), d\phi(p)v)$ and hence it is involutive. Now define the vector fields $\zeta, \zeta' : \Omega \to \mathbb{R}^m$ by

$$\zeta(z) := (\xi, A(z)\xi), \qquad \zeta'(z) := (\xi', A(z)\xi'), \qquad z \in \Omega.$$

Then ζ and ζ' are sections of $\widetilde{E}$ and their Lie bracket $[\zeta,\zeta']$ is given by

$$[\zeta,\zeta'](z) = \big(0, \big(dA(z)\zeta'(z)\big)\,\xi(z) - \left(dA(z)\zeta(z)\right)\xi'(z)\big)\,.$$

Since $\widetilde{E}$ is involutive the Lie bracket $[\zeta,\zeta']$ must take values in the graph of A. Hence the right hand side vanishes and this proves Claim 1.

Claim 2. *Let $I, J \subset \mathbb{R}$ be open intervals and $z = (x,y) : I^2 \to \Omega$ be a smooth map. Fix two points $s_0 \in I$ and $t_0 \in J$ and assume that*

$$\frac{\partial y}{\partial s}(s_0,t_0) = A\big(x(s_0,t_0), y(s_0,t_0)\big)\frac{\partial x}{\partial s}(s_0,t_0), \tag{2.7.7}$$

$$\frac{\partial y}{\partial t}(s,t) = A\big(x(s,t), y(s,t)\big)\frac{\partial x}{\partial t}(s,t) \tag{2.7.8}$$

for all $s \in I$ and $t \in J$. Then

$$\frac{\partial y}{\partial s}(s_0,t) = A\big(x(s_0,t), y(s_0,t)\big)\frac{\partial x}{\partial s}(s_0,t) \tag{2.7.9}$$

for all $t \in J$.

Equation (2.7.9) holds by assumption for $t = t_0$. Moreover, dropping the argument $z(s_0,t) = z = (x,y)$ for notational convenience we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial s} - A\cdot\frac{\partial x}{\partial s}\right) &= \frac{\partial^2 y}{\partial s\partial t} - A\frac{\partial^2 x}{\partial s\partial t} - \left(\frac{\partial A}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial A}{\partial y}\cdot\frac{\partial y}{\partial t}\right)\frac{\partial x}{\partial s} \\
&= \frac{\partial^2 y}{\partial s\partial t} - A\frac{\partial^2 x}{\partial s\partial t} - \left(\frac{\partial A}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial A}{\partial y}\cdot\left(A\frac{\partial x}{\partial t}\right)\right)\frac{\partial x}{\partial s} \\
&= \frac{\partial^2 y}{\partial s\partial t} - A\frac{\partial^2 x}{\partial s\partial t} - \left(\frac{\partial A}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial A}{\partial y}\cdot\left(A\frac{\partial x}{\partial s}\right)\right)\frac{\partial x}{\partial t} \\
&= \frac{\partial^2 y}{\partial s\partial t} - A\frac{\partial^2 x}{\partial s\partial t} - \left(\frac{\partial A}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial A}{\partial y}\cdot\frac{\partial y}{\partial s}\right)\frac{\partial x}{\partial t} \\
&\quad + \left(\frac{\partial A}{\partial y}\cdot\left(\frac{\partial y}{\partial s} - A\frac{\partial x}{\partial s}\right)\right)\frac{\partial x}{\partial t} \\
&= \left(\frac{\partial A}{\partial y}\cdot\left(\frac{\partial y}{\partial s} - A\frac{\partial x}{\partial s}\right)\right)\frac{\partial x}{\partial t}
\end{aligned}$$

Here the second step follows from (2.7.8), the third equation follows from Claim 1, and the last step follows by differentiating equation (2.7.8) with respect to s. Define $\eta : J \to \mathbb{R}^{m-n}$ by

$$\eta(t) := \frac{\partial y}{\partial s}(s_0,t) - A\big(x(s_0,t), y(s_0,t)\big)\frac{\partial x}{\partial s}(s_0,t).$$

By (2.7.7) and what we have just proved, the function η satisfies the linear differential equation

$$\dot\eta(t) = \left(\frac{\partial A}{\partial y}\big(x(s_0,t), y(s_0,t)\big)\cdot\eta(t)\right)\frac{\partial x}{\partial t}(s_0,t), \qquad \eta(t_0) = 0.$$

Hence $\eta(t) = 0$ for all $t \in J$. This proves (2.7.9) and Claim 2.

Now let $\beta : \mathbb{R}^2 \to M$ be a smooth map satisfying (2.7.4) and fix a real number s_0. Consider the set $W := \{t \in \mathbb{R} \,|\, \partial_s\beta(s_0,t) \in E_{\beta(s_0,t)}\}$. By going to local coordinates, we obtain from Claim 2 that W is open. Moreover, W is obviously closed, and $W \ne \emptyset$ because $0 \in W$ by (2.7.4). Hence $W = \mathbb{R}$. Since $s_0 \in \mathbb{R}$ was chosen arbitrarily, this proves (2.7.5) and Lemma 2.7.4. □

Any subbundle $E \subset TM$ determines an equivalence relation on M via

$$p_0 \sim p_1 \iff \begin{array}{l}\text{there is a smooth curve } \gamma : [0,1] \to M \\ \text{such that } \gamma(0) = p_0,\ \gamma(1) = p_1,\ \dot\gamma(t) \in E_{\gamma(t)}\ \forall\, t\end{array} \tag{2.7.10}$$

If E is integrable this equivalence relation is called a **foliation** and the equivalence class of $p_0 \in M$ is called the **leaf** of the foliation through p_0. The next example shows that the leaves do not need to be submanifolds

Example 2.7.5. Consider the torus $M := S^1 \times S^1 \subset \mathbb{C}^2$ with the tangent bundle

$$TM = \left\{(z_1, z_2, \mathbf{i}\lambda_1 z_1, \mathbf{i}\lambda_2 z_2) \in \mathbb{C}^4 \,|\, |z_1| = |z_2| = 1,\, \lambda_1, \lambda_2 \in \mathbb{R}\right\}.$$

Let ω_1, ω_2 be real numbers and consider the subbundle

$$E := \left\{(z_1, z_2, \mathbf{i}t\omega_1 z_1, \mathbf{i}t\omega_2 z_2) \in \mathbb{C}^4 \,|\, |z_1| = |z_2| = 1,\, t \in \mathbb{R}\right\}.$$

The leaf of this subbundle through $z = (z_1, z_2) \in \mathbb{T}^2$ is given by

$$L = \left\{\left(e^{\mathbf{i}t\omega_1} z_1, e^{\mathbf{i}t\omega_2} z_2\right) \Big|\, t \in \mathbb{R}\right\}.$$

It is a submanifold if and only if the quotient ω_1/ω_2 is a rational number (or $\omega_2 = 0$). Otherwise each leaf is a dense subset of $\mathbb{T}^2$.

Exercise 2.7.6. Prove that (2.7.10) defines an equivalence relation for every subbundle $E \subset TM$.

Exercise 2.7.7. Each subbundle $E \subset TM$ of rank 1 is integrable.

Exercise 2.7.8. Consider the manifold $M = \mathbb{R}^3$. Prove that the subbundle $E \subset TM = \mathbb{R}^3 \times \mathbb{R}^3$ with fiber $E_p = \left\{(\xi, \eta, \zeta) \in \mathbb{R}^3 \,|\, \zeta - y\xi = 0\right\}$ over $p = (x, y, z) \in \mathbb{R}^3$ is not integrable and that any two points in $\mathbb{R}^3$ can be connected by a path tangent to E.

Exercise 2.7.9. Consider the manifold $M = S^3 \subset \mathbb{R}^4 = \mathbb{C}^2$ and define

$$E := \left\{(z, \zeta) \in \mathbb{C}^2 \times \mathbb{C}^2 \,|\, |z| = 1,\, \zeta \perp z,\, \mathbf{i}\zeta \perp z\right\}.$$

Thus the fiber

$$E_z \subset T_z S^3 = z^\perp$$

is the maximal complex linear subspace of $T_z S^3$. Prove that E has real rank 2 and is not integrable.

Exercise 2.7.10. Let $E \subset TM$ be an involutive subbundle of rank n and let $L \subset M$ be a leaf of the foliation determined by E. A subset $V \subset L$ is called **L-open** if it can be written as a union of submanifolds N of M with tangent spaces $T_pN = E_p$ for $p \in N$. Prove that the L-open sets form a topology on L (called the **intrinsic topology**). Prove that the obvious inclusion $\iota : L \to M$ is continuous with respect to the intrinsic topology on L. Prove that the inclusion $\iota : L \to M$ is proper if and only if the intrinsic topology on L agrees with the relative topology inherited from M (called the **extrinsic topology**).

Remark 2.7.11. It is surprisingly difficult to prove that each closed leaf L of a foliation is a submanifold of M. A proof due to David Epstein [5] is sketched in §2.9.4 below.

2.8 The Intrinsic Definition of a Manifold*

It is somewhat restrictive to only consider manifolds that are embedded in some Euclidean space. Although we shall see that (at least) every compact manifold admits an embedding into a Euclidean space, such an embedding is in many cases not a natural part of the structure of a manifold. In particular, we encounter manifolds that are described as quotient spaces and there are manifolds that are embedded in certain infinite dimensional Hilbert spaces. For this reason it is convenient, at this point, to introduce a more general *intrinisc* definition of a manifold.[1] This requires some background from point set topology that is not covered in the first year analysis courses. We shall then see that all the definitions and results of this chapter carry over in a natural manner to the intrinsic setting. We begin by recalling the intrinsing definition of a smooth manifold in §1.4.

2.8.1 Definition and Examples

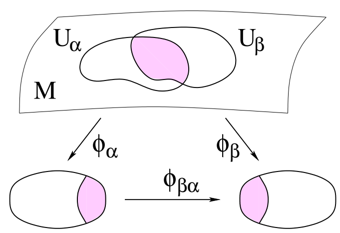


Figure 2.12: Coordinate charts and transition maps.

Definition 2.8.1 (Smooth m-Manifold). *Let $m \in \mathbb{N}_0$ and M be a set. A* **chart** *on M is a pair (ϕ, U) where $U \subset M$ and ϕ is a bijection from U to an open set $\phi(U) \subset \mathbb{R}^m$. Two charts (ϕ_1, U_1), (ϕ_2, U_2) are called* **compatible** *iff $\phi_1(U_1 \cap U_2)$ and $\phi_2(U_1 \cap U_2)$ are open and the* **transition map**

$$\phi_{21} = \phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2) \tag{2.8.1}$$

is a diffeomorphism. A **smooth atlas** *on M is a collection $\mathscr{A}$ of charts on M any two of which are compatible and such that the sets U, as (ϕ, U) ranges over $\mathscr{A}$, cover M (i.e. for every $p \in M$ there is a chart $(\phi, U) \in \mathscr{A}$ with $p \in U$). A* **maximal smooth atlas** *is an atlas which contains every chart which is compatible with each of its members. A* **smooth m-manifold** *is a pair consisting of a set M and a maximal atlas $\mathscr{A}$ on M.*

[1] See Chapter 1 for an overview.

In Lemma 1.4.3 it was shown that, if $\mathscr{A}$ is an atlas, then so is the collection $\overline{\mathscr{A}}$ of all charts compatible with each member of $\mathscr{A}$. Moreover, the atlas $\overline{\mathscr{A}}$ is maximal, so every atlas extends uniquely to a maximal atlas. For this reason, a manifold is usually specified by giving its underlying set M and some atlas on M. Generally, the notation for the atlas is suppressed and the manifold is denoted simply by M. The members of the atlas are called **coordinate charts** or simply **charts** on M. By Lemma 1.3.3 a smooth m-manifold admits a unique topology such that, for each chart (ϕ, U) of the smooth atlas, the set

$$U \subset M$$

is open and the bijection

$$\phi : U \to \phi(U)$$

is a homeomorphism onto the open set $\phi(U) \subset \mathbb{R}^m$. This topology is called the intrinsic topology of M and is described in the following definition.

Definition 2.8.2. *Let M be a smooth m-manifold. The* **intrinsic topology** *on the set M is the topology induced by the charts, i.e. a subset*

$$W \subset M$$

is open in the intrinsic topology iff $\phi(U \cap W)$ is an open subset of $\mathbb{R}^m$ for every chart (ϕ, U) on M.

Remark 2.8.3. Let $M \subset \mathbb{R}^k$ be smooth m-dimensional submanifold of $\mathbb{R}^k$ as in Definition 2.1.3. Then the set of all diffeomorphisms $(\phi, U \cap M)$ as in Definition 2.1.3 form a smooth atlas as in Definition 2.8.1. The intrinsic topology on the resulting smooth manifold is the same as the relative topology defined in §1.3.

Remark 2.8.4. A **topological manifold** is a topological space such that each point has a neighborhood U homeomorphic to an open subset of $\mathbb{R}^m$. Thus a smooth manifold (with the intrinsic topology) is a topological manifold and its maximal smooth atlas $\mathscr{A}$ is a subset of the set $\mathscr{A}_0$ of all pairs (ϕ, U) where $U \subset M$ is an open set and ϕ is a homeomorphism from U to an open subset of $\mathbb{R}^m$. One says that the maximal smooth atlas $\mathscr{A}$ is a **smooth structure** on the topological manifold M if the topology of M is the intrinsic topology of the smooth structure and every chart of the smooth structure is a homeomorphism. As explained in §1.4 a topological manifold can have many distinct smooth structures (see Remark 1.4.6). However, it is a deep theorem beyond the scope of this book that there are topological manifolds which do not admit any smooth structure.

Example 2.8.5. The **complex projective space** $\mathbb{C}\mathrm{P}^n$ is the set

$$\mathbb{C}\mathrm{P}^n = \left\{ \ell \subset \mathbb{C}^{n+1} \,|\, \ell \text{ is a 1-dimensional complex subspace} \right\}$$

of complex lines in $\mathbb{C}^{n+1}$. It can be identified with the quotient space

$$\mathbb{C}\mathrm{P}^n = \left(\mathbb{C}^{n+1} \setminus \{0\} \right) / \mathbb{C}^*$$

of nonzero vectors in $\mathbb{C}^{n+1}$ modulo the action of the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of nonzero complex numbers. The equivalence class of a nonzero vector $z = (z_0, \dots, z_n) \in \mathbb{C}^{n+1}$ will be denoted by

$$[z] = [z_0 : z_1 : \cdots : z_n] := \{\lambda z \,|\, \lambda \in \mathbb{C}^*\}$$

and the associated line is $\ell = \mathbb{C}z$. An atlas on $\mathbb{C}\mathrm{P}^n$ is given by the open cover $U_i := \{[z_0 : \cdots : z_n] \,|\, z_i \neq 0\}$ for $i = 0, 1, \dots, n$ and the coordinate charts $\phi_i : U_i \to \mathbb{C}^n$ are

$$\phi_i([z_0 : \cdots : z_n]) := \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i} \right). \tag{2.8.2}$$

Exercise: Prove that each ϕ_i is a homeomorphism and the transition maps are holomorphic. Prove that the manifold topology is the quotient topology, i.e. if $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}\mathrm{P}^n$ denotes the obvious projection, then a subset $U \subset \mathbb{C}\mathrm{P}^n$ is open if and only if $\pi^{-1}(U)$ is an open subset of $\mathbb{C}^{n+1} \setminus \{0\}$.

Example 2.8.6. The **real projective space** $\mathbb{R}\mathrm{P}^n$ is the set

$$\mathbb{R}\mathrm{P}^n = \left\{ \ell \subset \mathbb{R}^{n+1} \,|\, \ell \text{ is a 1-dimensional linear subspace} \right\}$$

of real lines in $\mathbb{R}^{n+1}$. It can again be identified with the quotient space

$$\mathbb{R}\mathrm{P}^n = \left(\mathbb{R}^{n+1} \setminus \{0\} \right) / \mathbb{R}^*$$

of nonzero vectors in $\mathbb{R}^{n+1}$ modulo the action of the multiplicative group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ of nonzero real numbers, and the equivalence class of a nonzero vector $x = (x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ will be denoted by

$$[x] = [x_0 : x_1 : \cdots : x_n] := \{\lambda x \,|\, \lambda \in \mathbb{R}^*\}.$$

An atlas on $\mathbb{R}\mathrm{P}^n$ is given by the open cover

$$U_i := \{[x_0 : \cdots : x_n] \,|\, x_i \neq 0\}$$

and the coordinate charts $\phi_i : U_i \to \mathbb{R}^n$ are again given by (2.8.2), with z_j replaced by x_j. The arguments in Example 2.8.5 show that these coordinate charts form an atlas and the manifold topology is the quotient topology. The transition maps are real analytic diffeomorphisms.

Example 2.8.7. The **real n-torus** is the topological space

$$\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$$

equipped with the quotient topology. Thus two vectors $x, y \in \mathbb{R}^n$ are equivalent if their difference $x - y \in \mathbb{Z}^n$ is an integer vector and we denote by $\pi : \mathbb{R}^n \to \mathbb{T}^n$ the obvious projection which assigns to each vector $x \in \mathbb{R}^n$ its equivalence class

$$\pi(x) := [x] := x + \mathbb{Z}^n.$$

Then a set $U \subset \mathbb{T}^n$ is open if and only if the set $\pi^{-1}(U)$ is an open subset of $\mathbb{R}^n$. An atlas on $\mathbb{T}^n$ is given by the open cover

$$U_\alpha := \{[x] \,|\, x \in \mathbb{R}^n,\, |x - \alpha| < 1/2\},$$

parametrized by vectors $\alpha \in \mathbb{R}^n$, and the coordinate charts $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ defined by $\phi_\alpha([x]) := x$ for $x \in \mathbb{R}^n$ with $|x - \alpha| < 1/2$. **Exercise:** Show that each transition map for this atlas is a translation by an integer vector.

Example 2.8.8. Consider the **complex Grassmannian**

$$\mathrm{G}_k(\mathbb{C}^n) := \{V \subset \mathbb{C}^n \,|\, v \text{ is a } k\text{-dimensional complex linear subspace}\}.$$

This set can again be described as a quotient space $\mathrm{G}_k(\mathbb{C}^n) \cong \mathcal{F}_k(\mathbb{C}^n)/\mathrm{U}(k)$. Here

$$\mathcal{F}_k(\mathbb{C}^n) := \left\{D \in \mathbb{C}^{n\times k} \,|\, D^*D = \mathbb{1}\right\}$$

denotes the set of unitary k-frames in $\mathbb{C}^n$ and the group $\mathrm{U}(k)$ acts on $\mathcal{F}_k(\mathbb{C}^n)$ contravariantly by $D \mapsto Dg$ for $g \in \mathrm{U}(k)$. The projection

$$\pi : \mathcal{F}_k(\mathbb{C}^n) \to \mathrm{G}_k(\mathbb{C}^n)$$

sends a matrix $D \in \mathcal{F}_k(\mathbb{C}^n)$ to its image $V := \pi(D) := \operatorname{im} D$. A subset $U \subset \mathrm{G}_k(\mathbb{C}^n)$ is open if and only if $\pi^{-1}(U)$ is an open subset of $\mathcal{F}_k(\mathbb{C}^n)$. Given a k-dimensional subspace $V \subset \mathbb{C}^n$ we can define an open set $U_V \subset \mathrm{G}_k(\mathbb{C}^n)$ as the set of all k-dimensional subspaces of $\mathbb{C}^n$ that can be represented as graphs of linear maps from V to $V^\perp$. This set of graphs can be identified with the complex vector space $\mathrm{Hom}^{\mathbb{C}}(V, V^\perp)$ of complex linear maps from V to $V^\perp$ and hence with $\mathbb{C}^{(n-k)\times k}$. This leads to an atlas on $\mathrm{G}_k(\mathbb{C}^n)$ with holomorphic transition maps and shows that $\mathrm{G}_k(\mathbb{C}^n)$ is a manifold of complex dimension $kn - k^2$. **Exercise:** Verify the details of this construction. Find explicit formulas for the coordinate charts and their transition maps. Carry this over to the real setting. Show that $\mathbb{C}\mathrm{P}^n$ and $\mathbb{R}\mathrm{P}^n$ are special cases.

Example 2.8.9 (The real line with two zeros). A topological space M is called **Hausdorff** if any two points in M can be separated by disjoint open neighborhoods. This example shows that a manifold need not be a Hausdorff space. Consider the quotient space

$$M := \mathbb{R} \times \{0,1\}/\equiv$$

where $[x,0] \equiv [x,1]$ for $x \neq 0$. An atlas on M consists of two coordinate charts $\phi_0 : U_0 \to \mathbb{R}$ and $\phi_1 : U_1 \to \mathbb{R}$ where

$$U_i := \{[x,i] \,|\, x \in \mathbb{R}\}, \qquad \phi_i([x,i]) := x$$

for $i = 0, 1$. Thus M is a 1-manifold. But the topology on M is not Hausdorff, because the points $[0,0]$ and $[0,1]$ cannot be separated by disjoint open neighborhoods.

Example 2.8.10 (A 2-manifold without a countable atlas). Consider the vector space $X = \mathbb{R} \times \mathbb{R}^2$ with the equivalence relation

$$[t_1, x_1, y_2] \equiv [t_2, x_2, y_2] \iff \begin{array}{l} \text{either } y_1 = y_2 \neq 0,\ t_1 + x_1 y_1 = t_2 + x_2 y_2 \\ \text{or } y_1 = y_2 = 0,\ t_1 = t_2,\ x_1 = x_2. \end{array}$$

For $y \neq 0$ we have $[0,x,y] \equiv [t, x - t/y, y]$, however, each point $(x,0)$ on the x-axis gets replaced by the uncountable set $\mathbb{R} \times \{(x,0)\}$. Our manifold is the quotient space $M := X/\equiv$. This time we do not use the quotient topology but the topology induced by our atlas Definition 2.8.2. The coordinate charts are parametrized by the reals: for $t \in \mathbb{R}$ the set $U_t \subset M$ and the coordinate chart $\phi_t : U_t \to \mathbb{R}^2$ are given by

$$U_t := \{[t,x,y] \,|\, x, y \in \mathbb{R}\}, \qquad \phi_t([t,x,y]) := (x,y).$$

A subset $U \subset M$ is open, by definition, if $\phi_t(U \cap U_t)$ is an open subset of $\mathbb{R}^2$ for every $t \in \mathbb{R}$. With this topology each ϕ_t is a homeomorphism from U_t onto $\mathbb{R}^2$ and M admits a countable dense subset $S := \{[0,x,y] \,|\, x, y \in \mathbb{Q}\}$. However, there is no atlas on M consisting of countably many charts. (Each coordinate chart can contain at most countably many of the points $[t,0,0]$.) The function $f : M \to \mathbb{R}$ given by $f([t,x,y]) := t + xy$ is smooth and each point $[t,0,0]$ is a critical point of f with value t. Thus f has no regular value. **Exercise:** Show that M is a path-connected Hausdorff space.

In Theorem 2.9.12 we will show that smooth manifolds whose topology is Hausdorff and second countable are precisely those that can be embedded in Euclidean space. Most authors tacitly assume that manifolds are Hausdorff and second countable and so will we after the end of the present chapter. However before §2.9.1 there is no need to impose these hypotheses.

2.8.2 Smooth Maps and Diffeomorphisms

Our next goal is to carry over all the definitions from embedded manifolds in Euclidean space to the intrinsic setting.

Definition 2.8.11. *Let*

$$(M, \{(\phi_\alpha, U_\alpha)\}_{\alpha\in A}), \qquad (N, \{(\psi_\beta, V_\beta)\}_{\beta\in B})$$

be smooth manifolds. A map $f : M \to N$ is called **smooth** *if it is continuous and the map*

$$f_{\beta\alpha} := \psi_\beta \circ f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \to \psi_\beta(V_\beta) \tag{2.8.3}$$

is smooth for every $\alpha \in A$ and every $\beta \in B$. It is called a **diffeomorphism** *if it is bijective and f and f^{-1} are smooth. The manifolds M and N are called* **diffeomorphic** *if there exists a diffeomorphism $f : M \to N$.*

The reader may check that the notion of a smooth map is independent of the atlas used in the definition, that compositions of smooth maps are smooth, and that sums and products of smooth maps from M to $\mathbb{R}$ are smooth.

Exercise 2.8.12. Let M be a smooth m-dimensional manifold with an atlas

$$\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha\in A}.$$

Consider the quotient space

$$\widetilde{M} := \bigcup_{\alpha\in A} \{\alpha\} \times \phi_\alpha(U_\alpha)/\sim$$

where

$$(\alpha, x) \sim (\beta, y) \qquad \stackrel{\text{def}}{\Longleftrightarrow} \qquad \phi_\alpha^{-1}(x) = \phi_\beta^{-1}(y).$$

for $\alpha, \beta \in A$, $x \in \phi_\alpha(U_\alpha)$, and $y \in \phi_\beta(U_\beta)$. Define an atlas on $\widetilde{M}$ by

$$\widetilde{U}_\alpha := \{[\alpha, x] \,|\, x \in \phi_\alpha(U_\alpha)\}, \qquad \widetilde{\phi}_\alpha([\alpha, x]) := x.$$

Prove that $\widetilde{M}$ is a smooth m-manifold and that it is diffeomorphic to M.

Exercise 2.8.13. Prove that $\mathbb{C}\mathrm{P}^1$ is diffeomorphic to S^2. **Hint:** Stereographic projection.

2.8.3 Tangent Spaces and Derivatives

If M is a submanifold of Euclidean space and $p \in M$ we have defined the tangent space of M at p as the set of all derivatives $\dot{\gamma}(0)$ of smooth curves $\gamma : \mathbb{R} \to M$ that pass through $p = \gamma(0)$. We cannot do this for manifolds in the intrinsic sense, as the derivative of a curve has yet to be defined. In fact, the purpose of introducing a tangent space of M is precisely to allow us to define what we mean by the derivative of a smooth map. There are two approaches. One is to introduce an appropriate equivalence relation on the set of curves through p and the other is to use local coordinates.

Definition 2.8.14. *Let M be a manifold with an atlas $\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$ and let $p \in M$. Two smooth curves $\gamma_0, \gamma_1 : \mathbb{R} \to M$ with $\gamma_0(0) = \gamma_1(0) = p$ are called p-**equivalent** if for some (and hence every) $\alpha \in A$ with $p \in U_\alpha$ we have*

$$\left.\frac{d}{dt}\right|_{t=0} \phi_\alpha(\gamma_0(t)) = \left.\frac{d}{dt}\right|_{t=0} \phi_\alpha(\gamma_1(t)).$$

We write $\gamma_0 \overset{p}{\sim} \gamma_1$ if γ_0 is p-equivalent to γ_1 and denote the equivalence class of a smooth curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$ by $[\gamma]_p$. The **tangent space** *of M at p is the set of equivalence classes*

$$T_pM := \{[\gamma]_p \,|\, \gamma : \mathbb{R} \to M \text{ is smooth and } \gamma(0) = p\}. \tag{2.8.4}$$

Definition 2.8.15. *Let M be a manifold with an atlas $\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$ and let $p \in M$. The* $\mathscr{A}$**-tangent space** *of M at p is the quotient space*

$$T_p^{\mathscr{A}} M := \bigcup_{p \in U_\alpha} \{\alpha\} \times \mathbb{R}^m / \overset{p}{\sim}, \tag{2.8.5}$$

where the union runs over all $\alpha \in A$ with $p \in U_\alpha$ and

$$(\alpha, \xi) \overset{p}{\sim} (\beta, \eta) \qquad \Longleftrightarrow \qquad d\left(\phi_\beta \circ \phi_\alpha^{-1}\right)(x)\xi = \eta, \quad x := \phi_\alpha(p).$$

The equivalence class will be denoted by $[\alpha, \xi]_p$.

In Definition 2.8.14 it is not immediately obvious that the set T_pM in (2.8.4) is a vector space. However, the quotient space $T_p^{\mathscr{A}} M$ in (2.8.5) is obviously a vector space of dimension m and there is a natural bijection

$$T_pM \to T_p^{\mathscr{A}} M : [\gamma]_p \mapsto \left[\alpha, \left.\frac{d}{dt}\right|_{t=0} \phi_\alpha(\gamma(t))\right]_p. \tag{2.8.6}$$

This bijection induces a vector space structure on the set T_pM. In other words, the set T_pM in (2.8.4) admits a unique vector space structure such that the map $T_pM \to T_p^{\mathscr{A}} M$ in (2.8.6) is a vector space isomorphism.

Exercise 2.8.16. Verify the phrase *"and hence every"* in Definition 2.8.14 and deduce that the map $T_pM \to T_p^{\mathscr{A}}M$ in (2.8.6) is well defined. Show that it is bijective.

From now on we will use either Definition 2.8.14 or Definition 2.8.15 or both, whichever way is most convenient, and drop the superscript $\mathscr{A}$.

Definition 2.8.17. *For each smooth curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$ we define the derivative $\dot\gamma(0) \in T_pM$ as the equivalence class*

$$\dot\gamma(0) := [\gamma]_p \cong \left[\alpha, \left.\frac{d}{dt}\right|_{t=0} \phi_\alpha(\gamma(t))\right]_p \in T_pM.$$

Definition 2.8.18. *If $f : M \to N$ is a smooth map between two manifolds $(M, \{(\phi_\alpha, U_\alpha)\}_{\alpha\in A})$ and $(N, \{(\psi_\beta, V_\beta)\}_{\beta\in B})$ we define the derivative*

$$df(p) : T_pM \to T_{f(p)}N$$

by the formula

$$df(p)[\gamma]_p := [f \circ \gamma]_{f(p)} \tag{2.8.7}$$

for each smooth curve $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$. Here we use (2.8.4). *Under the isomorphism* (2.8.6) *this corresponds to the linear map*

$$df(p)[\alpha, \xi]_p := [\beta, df_{\beta\alpha}(x)\xi]_{f(p)}, \qquad x := \phi_\alpha(p), \tag{2.8.8}$$

for $\alpha \in A$ with $p \in U_\alpha$ and $\beta \in B$ with $f(p) \in V_\beta$, where $f_{\beta\alpha}$ is given by (2.8.3).

Remark 2.8.19. Think of $N = \mathbb{R}^n$ as a manifold with a single coordinate chart, namely the identity map $\psi_\beta = \mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$. For every $q \in N = \mathbb{R}^n$ the tangent space T_qN is then canonically isomorphic to $\mathbb{R}^n$ via (2.8.5). Thus for every smooth map $f : M \to \mathbb{R}^n$ the derivative of f at $p \in M$ is a linear map $df(p) : T_pM \to \mathbb{R}^n$, and the formula (2.8.8) reads

$$df(p)[\alpha, \xi]_p = d(f \circ \phi_\alpha^{-1})(x)\xi, \qquad x := \phi_\alpha(p).$$

This formula also applies to maps defined on some open subset of M. In particular, with $f = \phi_\alpha : U_\alpha \to \mathbb{R}^m$ we have

$$d\phi_\alpha(p)[\alpha, \xi]_p = \xi.$$

Thus the map $d\phi_\alpha(p) : T_pM \to \mathbb{R}^m$ is the canonical vector space isomorphism determined by α.

With these definitions the derivative of f at p is a linear map and we have the chain rule for the composition of two smooth maps as in Theorem 2.2.13.

2.8.4 Submanifolds and Embeddings

Definition 2.8.20. *Let M be a smooth m-manifold and $n \in \{0, 1, \dots, m\}$. A subset*

$$N \subset M$$

is called an n-dimensional **submanifold** *of M if, for every element $p \in N$, there exists a local coordinate chart*

$$\phi : U \to \Omega$$

for M, defined on an an open neighborhood $U \subset M$ of p and with values in an open set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$, such that

$$\phi(U \cap N) = \Omega \cap (\mathbb{R}^n \times \{0\}).$$

By Theorem 2.1.10 an m-manifold

$$M \subset \mathbb{R}^k$$

in the sense of Definition 2.1.3 is a submanifold of $\mathbb{R}^k$ in the sense of Definition 2.8.20. By Theorem 2.3.4 the notion of a submanifold $N \subset M$ of a manifold $M \subset \mathbb{R}^k$ in Definition 2.3.1 agrees with the notion of a submanifold in Definition 2.8.20.

Exercise 2.8.21. Let N be a submanifold of M. Show that if M is Hausdorff so is N, and if M is paracompact so is N.

Exercise 2.8.22. Let N be a submanifold of M and let P be a submanifold of N. Prove that P is a submanifold of M. **Hint:** Use Theorem 2.1.10.

Exercise 2.8.23. Let N be a submanifold of M. Prove that there exists an open set $U \subset M$ such that $N \subset U$ and N is closed in the relative topology of U.

All the theorems we have proved for embedded manifolds and their proofs carry over almost word for word to the present setting. For example we have the inverse function theorem, the notion of a regular value, the implicit function theorem, the notion of an immersion, the notion of an embedding as a proper injective immersion, and the fact from Theorem 2.3.4 that a subset $P \subset M$ is a submanifold if and only if it is the image of an embedding.

Example 2.8.24 (Veronese embedding). The map

$$\mathbb{C}\mathrm{P}^2 \to \mathbb{C}\mathrm{P}^5 : [z_0 : z_1 : z_2] \mapsto [z_0^2 : z_1^2 : z_2^2 : z_1 z_2 : z_2 z_0 : z_0 z_1]$$

is an embedding. (**Exercise:** Prove this.) It restricts to an embedding of the real projective plane into $\mathbb{R}\mathrm{P}^5$ and also gives rise to embeddings of $\mathbb{R}\mathrm{P}^2$ into $\mathbb{R}^4$ as well as to the Roman surface: an immersion of $\mathbb{R}\mathrm{P}^2$ into $\mathbb{R}^3$. (See Example 2.1.16.) There are similar embeddings

$$\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^{N-1}, \qquad N := \binom{n+d}{d},$$

for all n and d, defined in terms of monomials of degree d in $n+1$ variables. These are the **Veronese embeddings**.

Example 2.8.25 (Plücker embedding). The Grassmannian $\mathrm{G}_2(\mathbb{R}^4)$ of 2-planes in $\mathbb{R}^4$ is a smooth 4-manifold and can be expressed as the quotient of the space $\mathcal{F}_2(\mathbb{R}^4)$ of orthonormal 2-frames in $\mathbb{R}^4$ by the orthogonal group $\mathrm{O}(2)$. (See Example 2.8.8.) Write an orthonormal 2-frame in $\mathbb{R}^4$ as a matrix

$$D = \begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}, \qquad D^{\mathsf{T}} D = \mathbb{1}.$$

Then the map $f : \mathrm{G}_2(\mathbb{R}^4) \to \mathbb{R}\mathrm{P}^5$, defined by

$$f([D]) := [p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}], \qquad p_{ij} := x_i y_j - x_j y_i.,$$

is an embedding and its image is the quadric

$$X := f(\mathrm{G}_2(\mathbb{R}^4)) = \left\{ p \in \mathbb{R}\mathrm{P}^5 \,|\, p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} = 0 \right\}.$$

(**Exercise:** Prove this.) There are analogous embeddings

$$f : \mathrm{G}_k(\mathbb{R}^n) \to \mathbb{R}\mathrm{P}^{N-1}, \qquad N := \binom{n}{k},$$

for all k and n, defined in terms of the $k \times k$-minors of the (orthonormal) frames. These are the **Plücker embeddings**.

2.8.5 Tangent Bundle and Vector Fields

Let M be a m-manifold with an atlas $\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha\in A}$. The **tangent bundle** of M is defined as the disjoint union of the tangent spaces, i.e.

$$TM := \bigcup_{p\in M} \{p\} \times T_pM = \{(p,v)\,|\,p \in M,\, v \in T_pM\}\,.$$

Denote by $\pi : TM \to M$ the projection given by $\pi(p,v) := p$. Recall the notion of a submersion as a smooth map between smooth manifolds, whose derivative is surjective at each point.

Lemma 2.8.26. *The tangent bundle of M is a smooth $2m$-manifold with coordinate charts*

$$\widetilde{\phi}_\alpha : \widetilde{U}_\alpha := \pi^{-1}(U_\alpha) \to \phi_\alpha(U_\alpha) \times \mathbb{R}^m, \qquad \widetilde{\phi}_\alpha(p,v) := (\phi_\alpha(p), d\phi_\alpha(p)v)\,.$$

The projection $\pi : TM \to M$ is a surjective submersion If M is second countable and Hausdorff so is TM.

Proof. For each pair $\alpha, \beta \in A$ the set

$$\widetilde{\phi}_\alpha(\widetilde{U}_\alpha \cap \widetilde{U}_\beta) = \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^m$$

is open in $\mathbb{R}^m \times \mathbb{R}^m$ and the transition map

$$\widetilde{\phi}_{\beta\alpha} := \widetilde{\phi}_\beta \circ \widetilde{\phi}_\alpha^{-1} : \widetilde{\phi}_\alpha(\widetilde{U}_\alpha \cap \widetilde{U}_\beta) \to \widetilde{\phi}_\beta(\widetilde{U}_\alpha \cap \widetilde{U}_\beta)$$

is given by

$$\widetilde{\phi}_{\beta\alpha}(x,\xi) = (\phi_{\beta\alpha}(x), d\phi_{\beta\alpha}(x)\xi)$$

for $x \in \phi_\alpha(U_\alpha \cap U_\beta)$ and $\xi \in \mathbb{R}^m$ where

$$\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1}.$$

Thus the transition maps are all diffeomorphisms and so the coordinate charts $\widetilde{\phi}_\alpha$ define an atlas on TM. The topology on TM is determined by this atlas via Definition 2.8.2. If M has a countable atlas so does TM. The remaining assertions are easy exercises. □

Definition 2.8.27. *Let M be a smooth m-manifold. A* **(smooth) vector field** *on M is a collection of tangent vectors $X(p) \in T_pM$, one for each point $p \in M$, such that the map $M \to TM : p \mapsto (p, X(p))$ is smooth. The set of smooth vector fields on M will be denoted by* $\mathrm{Vect}(M)$.

Associated to a vector field is a smooth map $M \to TM$ whose composition with the projection $\pi : TM \to M$ is the identity map on M. Strictly speaking this map should be denoted by a symbol other than X, for example by $\widetilde{X}$. However, it is convenient at this point, and common practice, to slightly abuse notation and denote the map from M to TM also by X. Thus a vector field can be defined as a smooth map

$$X : M \to TM$$

such that

$$\pi \circ X = \mathrm{id} : M \to M.$$

Such a map is also called a *section of the tangent bundle.*

Now suppose $\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$ is an atlas on M and $X : M \to TM$ is a vector field on M. Then X determines a collection of smooth maps

$$X_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^m$$

given by

$$X_\alpha(x) := d\phi_\alpha(p)X(p), \qquad p := \phi_\alpha^{-1}(x), \tag{2.8.9}$$

for $x \in \phi_\alpha(U_\alpha)$. We can think of each X_α as a vector field on the open set $\phi_\alpha(U_\alpha) \subset \mathbb{R}^m$, representing the vector field X on the coordinate patch U_α. These local vector fields X_α satisfy the condition

$$X_\beta(\phi_{\beta\alpha}(x)) = d\phi_{\beta\alpha}(x)X_\alpha(x) \tag{2.8.10}$$

for $x \in \phi_\alpha(U_\alpha \cap U_\beta)$. This equation can also be expressed in the form

$$X_\alpha|_{\phi_\alpha(U_\alpha \cap U_\beta)} = \phi_{\beta\alpha}^* X_\beta|_{\phi_\beta(U_\alpha \cap U_\beta)}. \tag{2.8.11}$$

Conversely, any collection of smooth maps $X_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^m$ satisfying (2.8.10) determines a unique vectorfield X on M via (2.8.9). Thus we can define the Lie bracket of two vector fields $X, Y \in \mathrm{Vect}(M)$ by

$$[X, Y]_\alpha(x) := [X_\alpha, Y_\alpha](x) = dX_\alpha(x)Y_\alpha(x) - dY_\alpha(x)X_\alpha(x) \tag{2.8.12}$$

for $\alpha \in A$ and $x \in \phi_\alpha(U_\alpha)$. It follows from equation (2.4.17) in Lemma 2.4.20 that the local vector fields

$$[X, Y]_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^m$$

satisfy (2.8.11) and hence determine a unique vector field $[X, Y]$ on M via

$$[X, Y](p) := d\phi_\alpha(p)^{-1}[X_\alpha, Y_\alpha](\phi_\alpha(p)), \qquad p \in U_\alpha. \tag{2.8.13}$$

Thus the **Lie bracket** of X and Y is defined on U_α as the pullback of the Lie bracket of the vector fields X_α and Y_α under the coordinate chart ϕ_α. With this understood all the results in §2.4 about vector fields and flows along with their proofs carry over word for word to the intrinsic setting whenever M is a Hausdorff space. This includes the existence and uniquess result for integral curves in Theorem 2.4.7, the concept of the flow of a vector field in Definition 2.4.8 and its properties in Theorem 2.4.9, the notion of completeness of a vector field (that the integral curves exist for all time), and the various properties of the Lie bracket such as the Jacobi identity (2.4.19), the formulas in Lemma 2.4.18, and the fact that the Lie bracket of two vector fields vanishes if and only if the corresponding flows commute (see Lemma 2.4.26). One can also carry over the notion of a **subbundle** $E \subset TM$ **of rank** n to the intrinsic setting by the condition that E is a smooth submanifold of TM and intersects each fiber T_pM in an n-dimensional linear subspace

$$E_p := \{v \in T_pM \,|\, (p,v) \in E\}.$$

Then the characterization of subbundles in Theorem 2.6.8 and the theorem of Frobenius 2.7.2 including their proofs also carry over to the intrinsic setting whenever M is a Hausdorff space.

2.8.6 Coordinate Notation

Fix a coordinate chart $\phi_\alpha : U_\alpha \to \mathbb{R}^m$ on an m-manifold M. The components of ϕ_α are smooth real valued functions on the open subset U_α of M and it is customary to denote them by

$$x^1, \ldots, x^m : U_\alpha \to \mathbb{R}.$$

The derivatives of these functions at $p \in U_\alpha$ are linear functionals

$$dx^i(p) : T_pM \to \mathbb{R}, \qquad i = 1, \ldots, m. \tag{2.8.14}$$

They form a basis of the dual space

$$T_p^*M := \mathrm{Hom}(T_pM, \mathbb{R}).$$

(A coordinate chart on M can in fact be characterized as an m-tuple of real valued functions on an open subset of M whose derivatives are everywhere linearly independent and which, taken together, form an injective map.) The dual basis of T_pM will be denoted by

$$\frac{\partial}{\partial x^1}(p), \ldots, \frac{\partial}{\partial x^m}(p) \in T_pM. \tag{2.8.15}$$

Thus
$$dx^i(p)\frac{\partial}{\partial x^j}(p) = \delta^i_j := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$
for $i, j = 1, \dots, m$ and $\partial/\partial x^i$ is a vector field on the coordinate patch U_α. For each $p \in U_\alpha$ it is the canonical basis of T_pM determined by ϕ_α. In the notation of (2.8.5) and Remark 2.8.19 we have
$$\frac{\partial}{\partial x^i}(p) = [\alpha, e_i]_p = d\phi_\alpha(p)^{-1}e_i$$
where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ (with 1 in the ith place) denotes the standard basis vector of $\mathbb{R}^m$. In other words, for all $\xi = (\xi^1, \dots, \xi^m) \in \mathbb{R}^m$ and all $p \in U_\alpha$, the tangent vector
$$v := d\phi_\alpha(p)^{-1}\xi \in T_pM$$
is given by
$$v = [\alpha, \xi]_p = \sum_{i=1}^m \xi^i \frac{\partial}{\partial x^i}(p). \tag{2.8.16}$$
Thus the restriction of a vector field $X \in \mathrm{Vect}(M)$ to U_α has the form
$$X|_{U_\alpha} = \sum_{i=1}^m \xi^i \frac{\partial}{\partial x^i}$$
where $\xi^1, \dots, \xi^m : U_\alpha \to \mathbb{R}$ are smooth real valued functions. If the map
$$X_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^m$$
is defined by (2.8.9) then
$$X_\alpha \circ \phi_\alpha^{-1} = (\xi^1, \dots, \xi^m).$$
The above notation is motivated by the observation that the derivative of a smooth function $f : M \to \mathbb{R}$ in the direction of a vector field X on a coordinate patch U_α is given by
$$\mathcal{L}_X f|_{U_\alpha} = \sum_{i=1}^m \xi^i \frac{\partial f}{\partial x^i}.$$
Here the term $\partial f/\partial x^i$ is understood as first writing f as a function of $x^1, \dots, x^m$, then taking the partial derivative, and afterwards expressing this partial derivative again as a function of p. Thus $\partial f/\partial x^i$ is the shorthand notation for the function $\left(\frac{\partial}{\partial x^i}(f \circ \phi_\alpha^{-1})\right) \circ \phi_\alpha : U_\alpha \to \mathbb{R}$.

2.9 Consequences of Paracompactness*

In geometry it is often necessary to turn a construction in local coordinates into a global geometric object. A key technical tool for such *"local to global"* constructions is an existence theorem for partitions of unity.

2.9.1 Paracompactness

The existence of a countable atlas is of fundamental importance for almost everything we will prove about manifolds. The next two remarks describe several equivalent conditions.

Remark 2.9.1. Let M be a smooth manifold and denote by

$$\mathscr{U} \subset 2^M$$

the topology induced by the atlas as in Definition 2.8.2. Then the following are equivalent.

(a) M admits a countable atlas.

(b) M is **σ-compact**, i.e. there is a sequence of compact subsets $K_i \subset M$ such that $K_i \subset \mathrm{int}(K_{i+1})$ for every $i \in \mathbb{N}$ and $M = \bigcup_{i=1}^{\infty} K_i$.

(c) Every open cover of M has a countable subcover.

(d) M is **second countable**, i.e. there is a countable collection of open sets $\mathscr{B} \subset \mathscr{U}$ such that every open set $U \in \mathscr{U}$ is a union of open sets from the collection $\mathscr{B}$. ($\mathscr{B}$ is then called a **countable base** for the topology of M.)

That (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) $\Longrightarrow$ (a) and (a) $\Longrightarrow$ (d) follows directly from the definitions. The proof that (d) implies (a) requires the construction of a countable refinement and the axiom of choice. (A **refinement** of an open cover $\{U_i\}_{i\in I}$ is an open cover $\{V_j\}_{j\in J}$ such that each set V_j is contained in one of the sets U_i.)

Remark 2.9.2. Let M and $\mathscr{U}$ be as in Remark 2.9.1 and suppose in addition that M is a connected Hausdorff space. Then the existence of a countable atlas is also equivalent to each of the following conditions.

(e) M is **metrizable**, i.e. there is a distance function $d : M \times M \to [0, \infty)$ such that $\mathscr{U}$ is the topology induced by d.

(f) M is **paracompact**, i.e. every open cover of M has a locally finite refinement. (An open cover $\{V_j\}_{j\in J}$ is called **locally finite** if every $p \in M$ has a neighborhood that intersects only finitely many V_j.)

That (a) implies (e) follows from the **Urysohn Metrization Theorem** which asserts (in its original form) that every normal second countable topological space is metrizable [14, Theorem 34.1]. A topological space M is called **normal** if points are closed and, for any two disjoint closed sets $A, B \subset M$, there are disjoint open sets $U, V \subset M$ such that $A \subset U$ and $B \subset V$. It is called **regular** if points are closed and, for every closed set $A \subset M$ and every $b \in M \setminus A$, there are disjoint open sets $U, V \subset M$ such that $A \subset U$ and $b \in V$. It is called **locally compact** if, for every open set $U \subset M$ and every $p \in U$, there is a compact neighborhood of p contained in U. It is easy to show that every manifold is locally compact and every locally compact Hausdorff space is regular. **Tychonoff's Lemma** asserts that a regular topological space with a countable base is normal [14, Theorem 32.1]. Hence it follows from the Urysohn Metrization Theorem that every Hausdorff manifold with a countable base is metrizable. That (e) implies (f) follows from a more general theorem which asserts that every metric space is paracompact (see [14, Theorem 41.4] and [16]). Conversely, the **Smirnov Metrization Theorem** asserts that a paracompact Hausdorff space is metrizable if and only it is locally metrizable, i.e. every point has a metrizable neighborhood (see [14, Theorem 42.1]). Since every manifold is locally metrizable this shows that (f) implies (e). Thus we have (a) $\Longrightarrow$ (e) $\Longleftrightarrow$ (f) for every Hausdorff manifold.

The proof that (f) implies (a) does not require the Hausdorff property but we do need the assumption that M is connected. (A manifold with uncountably many connected components, each of which is paracompact, is itself paracompact but does not admit a countable atlas.) Here is a sketch. If M is a paracompact manifold then there is a locally finite open cover $\{U_\alpha\}_{\alpha \in A}$ by coordinate charts. Since each set U_α has a countable dense subset, the set $\{\alpha \in A \,|\, U_\alpha \cap U_{\alpha_0} \neq \emptyset\}$ is at most countable for each $\alpha_0 \in A$. Since M is connected we can reach each point from U_{α_0} through a finite sequence of sets $U_{\alpha_1}, \ldots, U_{\alpha_\ell}$ with $U_{\alpha_{i-1}} \cap U_{\alpha_i} \neq \emptyset$. This implies that the index set A is countable and hence M admits a countable atlas.

Remark 2.9.3. A **Riemann surface** is a 1-dimensional complex manifold (i.e. the coordinate charts take values in $\mathbb{C}$ and the transition maps are holomorphic) with a Hausdorff topology. It is a deep theorem in the theory of Riemann surfaces that every connected Riemann surface is necessarily second countable (see [2]). Thus pathological examples of the type discussed in Example 2.8.10 cannot be constructed with holomorphic transition maps.

Exercise 2.9.4. Prove that every manifold is locally compact. Find an example of a manifold M and a point $p_0 \in M$ such that every closed neighborhood of p_0 is non-compact. **Hint:** The example is necessarily non-Hausdorff.

Exercise 2.9.5. Prove that a manifold M admits a countable atlas if and only if it is σ-compact if and only if every open cover of M has a countable subcover if and only if it is second countable. **Hint:** The topology of $\mathbb{R}^m$ is second countable and every open subset of $\mathbb{R}^m$ is σ-compact.

Exercise 2.9.6. Prove that every submanifold $M \subset \mathbb{R}^k$ (Definition 2.1.3) is second countable.

Exercise 2.9.7. Prove that every connected component of a manifold M is an open subset of M and is path-connected.

2.9.2 Partitions of Unity

Definition 2.9.8. *Let M be a smooth manifold. A* **partition of unity** *on M is a collection of smooth functions*

$$\theta_\alpha : M \to [0,1], \qquad \alpha \in A,$$

such that each point $p \in M$ has an open neighborhood $V \subset M$ on which only finitely many θ_α do not vanish, i.e.

$$\#\{\alpha \in A \,|\, \theta_\alpha|_V \not\equiv 0\} < \infty, \tag{2.9.1}$$

and, for every $p \in M$, we have

$$\sum_{\alpha \in A} \theta_\alpha(p) = 1. \tag{2.9.2}$$

If $\{U_\alpha\}_{\alpha\in A}$ is an open cover of M then a partition of unity $\{\theta_\alpha\}_{\alpha\in A}$ (indexed by the same set A) is called **subordinate to the cover** *if each θ_α is supported in U_α, i.e.*

$$\mathrm{supp}(\theta_\alpha) := \overline{\{p \in M \,|\, \theta_\alpha(p) \neq 0\}} \subset U_\alpha.$$

Theorem 2.9.9 (**Partitions of unity**)**.** *Let M be a smooth manifold whose topology is paracompact and Hausdorff. Then, for every open cover of M, there exists a partition of unity subordinate to that cover.*

Proof. See page 98. □

Lemma 2.9.10. *Let M be a smooth manifold with a Hausdorff topology. Then, for every open set $V \subset M$ and every compact set $K \subset V$, there exists a smooth function $\kappa : M \to [0, \infty)$ with compact support such that* $\mathrm{supp}(\kappa) \subset V$ *and $\kappa(p) > 0$ for every $p \in K$.*

Proof. Assume first that $K = \{p_0\}$ is a single point. Since M is a manifold it is locally compact. Hence there is a compact neighborhood $C \subset V$ of p_0. Since M is Hausdorff C is closed and hence the set $U := \mathrm{int}(C)$ is a neighborhood of p_0 whose closure $\overline{U} \subset C$ is compact and contained in V. Shrinking U, if necessary, we may assume that there is a coordinate chart $\phi : U \to \Omega$ with values in some open neighborhood $\Omega \subset \mathbb{R}^m$ of the origin such that $\phi(p_0) = 0$. (Here m is the dimension of M.) Now choose a smooth function $\kappa_0 : \Omega \to [0, \infty)$ with compact support such that $\kappa_0(0) > 0$. Then the function $\kappa : M \to [0, 1]$, defined by $\kappa|_U := \kappa_0 \circ \phi$ and $\kappa(p) := 0$ for $p \in M \setminus U$ is supported in V and satisfies $\kappa(p_0) > 0$. This proves the lemma in the case where K is a point.

Now let K be any compact subset of V. Then, by the first part of the proof, there is a collection of smooth functions $\kappa_p : M \to [0, \infty)$, one for every $p \in K$, such that $\kappa_p(p) > 0$ and $\mathrm{supp}(\kappa_p) \subset V$. Since K is compact there are finitely many points $p_1, \dots, p_k \in K$ such that the sets $\left\{p \in M \,|\, \kappa_{p_j}(p) > 0\right\}$ cover K. Hence the function $\kappa := \sum_j \kappa_{p_j}$ is supported in V and is everywhere positive on K. This proves Lemma 2.9.10. □

Lemma 2.9.11. *Let M be a topological space. If $\{V_i\}_{i \in I}$ is a locally finite collection of open sets in M then*

$$\overline{\bigcup_{i \in I_0} V_i} = \bigcup_{i \in I_0} \overline{V}_i$$

for every subset $I_0 \subset I$.

Proof. The set $\bigcup_{i \in I_0} \overline{V}_i$ is obviously contained in the closure of $\bigcup_{i \in I_0} V_i$. To prove the converse choose a point $p_0 \in M \setminus \bigcup_{i \in I_0} \overline{V}_i$. Since the collection $\{V_i\}_{i \in I}$ is locally finite, there exists an open neighborhood U of p_0 such that the set $I_1 := \{i \in I \,|\, V_i \cap U \neq \emptyset\}$ is finite. Hence the set

$$U_0 := U \setminus \bigcup_{i \in I_0 \cap I_1} \overline{V}_i$$

is an open neighborhood of p_0 and we have $U_0 \cap V_i = \emptyset$ for every $i \in I_0$. Hence $p_0 \notin \overline{\bigcup_{i \in I_0} V_i}$. This proves Lemma 2.9.11. □

Proof of Theorem 2.9.9. Let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of M. We prove in four steps that there is a partition of unity subordinate to this cover. The proofs of steps one and two are taken from [14, Lemma 41.6].

Step 1. *There is a locally finite open cover $\{V_i\}_{i\in I}$ of M such that, for every $i \in I$, the closure $\overline{V}_i$ is compact and contained in one of the sets U_α.*

Denote by $\mathscr{V} \subset 2^M$ the set of all open sets $V \subset M$ such that $\overline{V}$ is compact and $\overline{V} \subset U_\alpha$ for some $\alpha \in A$. Since M is a locally compact Hausdorff space the collection $\mathscr{V}$ is an open cover of M. (If $p \in M$ then there is an $\alpha \in A$ such that $p \in U_\alpha$; since M is locally compact there is a compact neighborhood $K \subset U_\alpha$ of p; since M is Hausdorff K is closed and thus $V := \mathrm{int}(K)$ is an open neighborhood of p with $\overline{V} \subset K \subset U_\alpha$.) Since M is paracompact the open cover $\mathscr{V}$ has a locally finite refinement $\{V_i\}_{i\in I}$. This cover satisfies the requirements of Step 1.

Step 2. *There is a collection of compact sets $K_i \subset V_i$, one for each $i \in I$, such that $M = \bigcup_{i\in I} K_i$.*

Denote by $\mathscr{W} \subset 2^M$ the set of all open sets $W \subset M$ such that $\overline{W} \subset V_i$ for some i. Since M is a locally compact Hausdorff space, the collection $\mathscr{W}$ is an open cover of M. Since M is paracompact $\mathscr{W}$ has a locally finite refinement $\{W_j\}_{j\in J}$. By the axiom of choice there is a map

$$J \to I : j \mapsto i_j$$

such that

$$\overline{W}_j \subset V_{i_j} \qquad \forall\, j \in J.$$

Since the collection $\{W_j\}_{j\in J}$ is locally finite, we have

$$K_i := \overline{\bigcup_{i_j=i} W_j} = \bigcup_{i_j=i} \overline{W}_j \subset V_i$$

by Lemma 2.9.11. Since $\overline{V}_i$ is compact so is K_i.

Step 3. *There is a partition of unity subordinate to the cover $\{V_i\}_{i\in I}$.*

Choose a collection of compact sets $K_i \subset V_i$ for $i \in I$ as in Step 2. Then, by Lemma 2.9.10 and the axiom of choice, there is a collection of smooth functions $\kappa_i : M \to [0,\infty)$ with compact support such that

$$\mathrm{supp}(\kappa_i) \subset V_i, \qquad \kappa_i|_{K_i} > 0 \qquad \forall\, i \in I.$$

Since the cover $\{V_i\}_{i\in I}$ is locally finite the sum

$$\kappa := \sum_{i\in I} \kappa_i : M \to \mathbb{R}$$

is **locally finite** (i.e. each point in M has a neighborhood in which only finitely many terms do not vanish) and thus defines a smooth function on M. This function is everywhere positive, because each summand is nonnegative and, for each $p \in M$, there is an $i \in I$ with $p \in K_i$ so that $\kappa_i(p) > 0$. Thus the funtions $\chi_i := \kappa_i/\kappa$ define a partition of unity satisfying $\mathrm{supp}(\chi_i) \subset V_i$ for every $i \in I$ as required.

Step 4. *There is a partition of unity subordinate to the cover $\{U_\alpha\}_{\alpha\in A}$.*

Let $\{\chi_i\}_{i\in I}$ be the partition of unity constructed in Step 3. By the axiom of choice there is a map $I \to A : i \mapsto \alpha_i$ such that $V_i \subset U_{\alpha_i}$ for every $i \in I$. For $\alpha \in A$ define $\theta_\alpha : M \to [0,1]$ by

$$\theta_\alpha := \sum_{\alpha_i=\alpha} \chi_i.$$

Here the sum runs over all indices $i \in I$ with $\alpha_i = \alpha$. This sum is locally finite and hence is a smooth function on M. Moreover, each point in M has an open neighborhood in which only finitely many of the θ_α do not vanish. Hence the sum of the θ_α is a well defined function on M and

$$\sum_{\alpha\in A} \theta_\alpha = \sum_{\alpha\in A} \sum_{\alpha_i=\alpha} \chi_i = \sum_{i\in I} \chi_i \equiv 1.$$

This shows that the functions θ_α form a partition of unity. To prove the inclusion $\mathrm{supp}(\theta_\alpha) \subset U_\alpha$ we consider the open sets

$$W_i := \{p \in M \,|\, \chi_i(p) > 0\}$$

for $i \in I$. Since $W_i \subset V_i$ this collection is locally finite. Hence, by Lemma 2.9.11, we have

$$\mathrm{supp}(\theta_\alpha) = \overline{\bigcup_{\alpha_i=\alpha} W_i} = \bigcup_{\alpha_i=\alpha} \overline{W}_i = \bigcup_{\alpha_i=\alpha} \mathrm{supp}(\chi_i) \subset \bigcup_{\alpha_i=\alpha} V_i \subset U_\alpha.$$

This proves Theorem 2.9.9. $\square$

2.9.3 Embedding in Euclidean Space

Theorem 2.9.12. *Let M be a second countable smooth m-manifold with a Hausdorff topology. Then there exists an embedding $f : M \to \mathbb{R}^{2m+1}$ with a closed image.*

Proof. The proof has five steps.

Step 1. *Let $U \subset M$ be an open set and let $K \subset U$ be a compact set. Then there exists an integer $k \in \mathbb{N}$, a smooth map $f : M \to \mathbb{R}^k$, and an open set $V \subset M$, such that $K \subset V \subset U$, the restriction $f|_V : V \to \mathbb{R}^k$ is an injective immersion, and $f(p) = 0$ for all $p \in M \setminus U$.*

Choose a smooth atlas $\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha\in A}$ on M such that, for each $\alpha \in A$, either $U_\alpha \subset U$ or $U_\alpha \cap K = \emptyset$. Since M is a paracompact Hausdorff manifold, Theorem 2.9.9 asserts that there exists a partition of unity $\{\theta_\alpha\}_{\alpha\in A}$ subordinate to the open cover $\{U_\alpha\}_{\alpha\in A}$ of M. Since the sets U_α with $U_\alpha \subset U$ form an open cover of K and K is a compact subset of M, there exist finitely many indices $\alpha_1, \dots, \alpha_\ell \in A$ such that

$$K \subset U_{\alpha_1} \cup \cdots \cup U_{\alpha_\ell} =: V \subset U.$$

Let $k := \ell(m + 1)$ and, for $i = 1, \dots, \ell$, abbreviate

$$\phi_i := \phi_{\alpha_i}, \qquad \theta_i := \theta_{\alpha_i}.$$

Define the smooth map $f : M \to \mathbb{R}^k$ by

$$f(p) := \begin{pmatrix} \theta_1(p) \\ \theta_1(p)\phi_1(p) \\ \vdots \\ \theta_\ell(p) \\ \theta_\ell(p)\phi_\ell(p) \end{pmatrix} \qquad \text{for } p \in M.$$

Then the restriction $f|_V : V \to \mathbb{R}^k$ is injective. Namely, if $p_0, p_1 \in V$ satisfy

$$f(p_0) = f(p_1)$$

then

$$I := \{i \,|\, \theta_i(p_0) > 0\} = \{i \,|\, \theta_i(p_1) > 0\} \neq \emptyset$$

and, for $i \in I$, we have $\theta_i(p_0) = \theta_i(p_1)$, hence $\phi_i(p_0) = \phi_i(p_1)$, and so $p_0 = p_1$. Moreover, for every $p \in K$ the derivative $df(p) : T_pM \to \mathbb{R}^k$ is injective, and this proves Step 1.

Step 2. *Let $f : M \to \mathbb{R}^k$ be an injective immersion and let $\mathcal{A} \subset \mathbb{R}^{(2m+1)\times k}$ be a nonempty open set. Then there exists a matrix $A \in \mathcal{A}$ such that the map $Af : M \to \mathbb{R}^{2m+1}$ is an injective immersion.*

The proof of Step 2 uses the Theorem of Sard (see [1, 13]). The sets

$$\begin{aligned} W_0 &:= \left\{(p,q) \in M \times M \,\middle|\, p \neq q\right\}, \\ W_1 &:= \left\{(p,v) \in TM \,\middle|\, v \neq 0\right\} \end{aligned}$$

are open subsets of smooth second countable Hausdorff $2m$-manifolds and the maps

$$F_0 : \mathcal{A} \times W_0 \to \mathbb{R}^{2m+1}, \qquad F_1 : \mathcal{A} \times W_1 \to \mathbb{R}^{2m+1},$$

defined by

$$F_0(A,p,q) := A(f(p) - f(q)), \qquad F_1(A,p,v) := Adf(p)v$$

for $A \in \mathcal{A}$, $(p,q) \in W_0$, and $(p,v) \in W_1$, are smooth. Moreover, the zero vector in $\mathbb{R}^{2m+1}$ is a regular value of F_0 because f is injective and of F_1 because f is an immersion. Hence it follows from the intrinsic analogue of Theorem 2.2.17 that the sets

$$\begin{aligned} \mathcal{M}_0 &:= F_0^{-1}(0) = \left\{(A,p,q) \in \mathcal{A} \times W_0 \,\middle|\, Af(p) = Af(q)\right\}, \\ \mathcal{M}_1 &:= F_1^{-1}(0) = \left\{(A,p,v) \in \mathcal{A} \times W_1 \,\middle|\, Adf(p)v = 0\right\} \end{aligned}$$

are smooth manifolds of dimension

$$\dim \mathcal{M}_0 = \dim \mathcal{M}_1 = (2m+1)k - 1.$$

Since M is a second countable Hausdorff manifold, so are $\mathcal{M}_0$ and $\mathcal{M}_1$. Hence the Theorem of Sard asserts that the canonical projections

$$\begin{aligned} \mathcal{M}_0 \to \mathcal{A} : (A,p,q) \mapsto A =: \pi_0(A,p,q), \\ \mathcal{M}_1 \to \mathcal{A} : (A,p,v) \mapsto A =: \pi_1(A,p,v), \end{aligned}$$

have a common regular value $A \in \mathcal{A}$. Since

$$\dim \mathcal{M}_0 = \dim \mathcal{M}_1 < \dim \mathcal{A},$$

this implies

$$A \in \mathcal{A} \setminus (\pi_0(\mathcal{M}_0) \cup \pi_1(\mathcal{M}_1)).$$

Hence $Af : M \to \mathbb{R}^{2m+1}$ is an injective immersion and this proves Step 2.

If M is compact, the result follows from Steps 1 and 2 with $K = U = M$. In the noncompact case the proof requires two more steps to construct an embedding into $\mathbb{R}^{4m+4}$ and a further step to reduce the dimension to $2m+1$.

Step 3. *Assume M is not compact. Then there exists a sequence of open sets $U_i \subset M$, a sequence of smooth functions $\rho_i : M \to [0,1]$, and a sequence of compact sets $K_i \subset U_i$ such that*

$$\mathrm{supp}(\rho_i) \subset U_i, \qquad K_i = \rho_i^{-1}(1) \subset U_i, \qquad U_i \cap U_j = \emptyset$$

for all $i, j \in \mathbb{N}$ with $|i-j| \geq 2$ and $M = \bigcup_{i=1}^{\infty} K_i$.

Since M is second countable, there exists a sequence of compact sets $C_i \subset M$ such that $C_i \subset \mathrm{int}(C_{i+1})$ for all $i \in \mathbb{N}$ and $M = \bigcup_{i \in N} C_i$ (Remark 2.9.1). Define the compact sets $B_i \subset M$ by $C_0 := \emptyset$ and

$$B_i := \overline{C_i \setminus C_{i-1}} \qquad \text{for } i \in \mathbb{N}.$$

Then $M = \bigcup_{i \in \mathbb{N}} B_i$ and, for all $i, j \in \mathbb{N}$ with $j \geq i+2$, we have

$$B_i \subset C_i \subset \mathrm{int}(C_{j-1}), \qquad B_j \subset C_j \setminus \mathrm{int}(C_{j-1})$$

and so $B_i \cap B_j = \emptyset$. Since M is metrizable by Remark 2.9.2, there exists a distance function $d : M \times M \to [0, \infty)$ that induces the intrinsic topology on M. Define

$$A_i := \bigcup_{j \in \mathbb{N} \setminus \{i-1, i, i+1\}} B_j, \qquad \varepsilon_i := d(A_i, B_i) = \inf_{p \in A_i, q \in B_i} d(p,q).$$

Then A_i is a closed subset of M, because any convergent sequence in M must belong to a finite union of the B_j. Since $A_i \cap B_i = \emptyset$, this implies $\varepsilon_i > 0$. For $i \in \mathbb{N}$ define the set $U_i \subset M$ by

$$U_i := \left\{ p \in M \,\middle|\, \text{there exists a } q \in B_i \text{ with } d(p,q) < \varepsilon_i/3 \right\}.$$

Then $\{U_i\}_{i \in \mathbb{N}}$ is a sequence of open subsets of M such that $B_i \subset U_i \subset C_{i+1}$ for all $i \in \mathbb{N}$ and $U_i \cap U_j = \emptyset$ for $|i-j| \geq 2$. In particular, each set U_i has a compact closure.

For each i there exists of a partition of unity subordinate to the open cover $M = U_i \cup (M \setminus B_i)$ and hence a smooth function $\rho_i : M \to [0,1]$ such that $\mathrm{supp}(\rho_i) \subset U_i$ and $\rho_i|_{B_i} \equiv 1$. Define $K_i := \rho_i^{-1}(1) = \{p \in U_i \,|\, \rho_i(p) = 1\}$ for $i \in \mathbb{N}$. Then K_i is a compact set and $B_i \subset K_i \subset U_i$ for each $i \in \mathbb{N}$. Hence $M = \bigcup_{i \in \mathbb{N}} K_i$ and this proves Step 3.

Step 4. *Assume M is not compact. Then there exists an embedding*

$$f : M \to \mathbb{R}^{4m+4}$$

with a closed image and a pair of orthonormal vectors $x, y \in \mathbb{R}^{4m+4}$ such that, for every $\varepsilon > 0$, there exists a compact set $K \subset M$ with

$$\sup_{p\in M\setminus K} \inf_{s,t\in\mathbb{R}} \left| \frac{f(p)}{|f(p)|} - sx - ty \right| < \varepsilon. \tag{2.9.3}$$

Assume M is not compact and let K_i, U_i, ρ_i be as in Step 3. Then, by Steps 1 and 2, there exists a sequence of smooth maps $g_i : M \to \mathbb{R}^{2m+1}$ such that $g_i|_{M\setminus U_i} \equiv 0$, the restriction $g_i|_{K_i} : K_i \to \mathbb{R}^{2m+1}$ is injective, and the derivative $dg_i(p) : T_pM \to \mathbb{R}^{2m+1}$ is injective for all $p \in K_i$ and all $i \in \mathbb{N}$. Let $\xi \in \mathbb{R}^{2m+1}$ be a unit vector and define the maps $f_i : M \to \mathbb{R}^{2m+1}$ by

$$f_i(p) := \rho_i(p) \left(i\xi + \frac{g_i(p)}{\sqrt{1 + |g_i(p)|^2}} \right) \tag{2.9.4}$$

for $p \in M$ and $i \in \mathbb{N}$. Then the restriction $f_i|_{K_i} : K_i \to \mathbb{R}^{2m+1}$ is injective, the derivative $df_i(p) : T_pM \to \mathbb{R}^{2m+1}$ is injective for all $p \in K_i$, and

$$\mathrm{supp}(f_i) \subset U_i, \qquad f_i(K_i) \subset B_1(i\xi), \qquad f_i(M) \subset B_{i+1}(0).$$

Define the maps $f^{\mathrm{odd}}, f^{\mathrm{ev}} : M \to \mathbb{R}^{2m+1}$ and $\rho^{\mathrm{odd}}, \rho^{\mathrm{ev}} : M \to \mathbb{R}$ by

$$\rho^{\mathrm{odd}}(p) := \begin{cases} \rho_{2i-1}(p), & \text{if } i \in \mathbb{N} \text{ and } p \in U_{2i-1}, \\ 0, & \text{if } p \in M \setminus \bigcup_{i\in\mathbb{N}} U_{2i-1}, \end{cases}$$

$$f^{\mathrm{odd}}(p) := \begin{cases} f_{2i-1}(p), & \text{if } i \in \mathbb{N} \text{ and } p \in U_{2i-1}, \\ 0, & \text{if } p \in M \setminus \bigcup_{i\in\mathbb{N}} U_{2i-1}, \end{cases}$$

$$\rho^{\mathrm{ev}}(p) := \begin{cases} \rho_{2i}(p), & \text{if } i \in \mathbb{N} \text{ and } p \in U_{2i}, \\ 0, & \text{if } p \in M \setminus \bigcup_{i\in\mathbb{N}} U_{2i}, \end{cases}$$

$$f^{\mathrm{ev}}(p) := \begin{cases} f_{2i}(p), & \text{if } i \in \mathbb{N} \text{ and } p \in U_{2i}, \\ 0, & \text{if } p \in M \setminus \bigcup_{i\in\mathbb{N}} U_{2i}, \end{cases}$$

and define the map $f : M \to \mathbb{R}^{4m+4}$ by

$$f(p) := \left(\rho^{\mathrm{odd}}(p), f^{\mathrm{odd}}(p), \rho^{\mathrm{ev}}(p), f^{\mathrm{ev}}(p) \right)$$

for $p \in M$.

We prove that f is injective. To see this, note that

$$\begin{aligned} p \in K_{2i-1} \quad &\Longrightarrow \quad \begin{cases} 2i-2 < \left|f^{\mathrm{odd}}(p)\right| < 2i, \\ |f^{\mathrm{ev}}(p)| < 2i+1, \end{cases} \\ p \in K_{2i} \quad &\Longrightarrow \quad \begin{cases} 2i-1 < |f^{\mathrm{ev}}(p)| < 2i+1, \\ \left|f^{\mathrm{odd}}(p)\right| < 2i+2, \end{cases} \end{aligned} \tag{2.9.5}$$

Now let $p_0, p_1 \in M$ such that $f(p_0) = f(p_1)$. Assume first that $p_0 \in K_{2i-1}$. Then $\rho^{\mathrm{odd}}(p_1) = \rho^{\mathrm{odd}}(p_0) = 1$ and hence $p_1 \in \bigcup_{j\in\mathbb{N}} K_{2j-1}$. By (2.9.5), we also have $2i-2 < |f^{\mathrm{odd}}(p_1)| = |f^{\mathrm{odd}}(p_0)| < 2i$ and hence $p_1 \in K_{2i-1}$. This implies $f_{2i-1}(p_1) = f^{\mathrm{odd}}(p_1) = f^{\mathrm{odd}}(p_0) = f_{2i-1}(p_0)$ and so $p_0 = p_1$. Now assume $p_0 \in K_{2i}$. Then $\rho^{\mathrm{ev}}(p_1) = \rho^{\mathrm{ev}}(p_0) = 1$ and hence $p_1 \in \bigcup_{j\in\mathbb{N}} K_{2j}$. By (2.9.5), we also have $2i-1 < |f^{\mathrm{ev}}(p_1)| = |f^{\mathrm{ev}}(p_0)| < 2i+1$, so $p_1 \in K_{2i}$, which implies $f_{2i}(p_1) = f^{\mathrm{ev}}(p_1) = f^{\mathrm{ev}}(p_0) = f_{2i}(p_0)$, and so again $p_0 = p_1$. This shows that f is injective. That f is an immersion follows from the fact that the derivative $df_i(p)$ is injective for all $p \in K_i$ and all $i \in \mathbb{N}$.

We prove that f is proper and has a closed image. Let $(p_\nu)_{\nu\in\mathbb{N}}$ be a sequence in M such that the sequence $(f(p_\nu))_{\nu\in\mathbb{N}}$ in $\mathbb{R}^{4m+4}$ is bounded. Choose $i \in \mathbb{N}$ such that $|f^{\mathrm{odd}}(p_\nu)| < 2i$ and $|f^{\mathrm{ev}}(p_\nu)| < 2i+1$ for all $\nu \in \mathbb{N}$. Then $p_\nu \in \bigcup_{j=1}^{2i} K_j$ for all $\nu \in \mathbb{N}$ by (2.9.5). Hence $(p_\nu)_{\nu\in\mathbb{N}}$ has a convergent subsequence. Thus $f : M \to \mathbb{R}^{4m+4}$ is an embedding with a closed image.

Next consider the pair of orthonormal vectors

$$x := (0, \xi, 0, 0), \qquad y := (0, 0, 0, \xi)$$

in $\mathbb{R}^{4m+4} = \mathbb{R} \times \mathbb{R}^{2m+1} \times \mathbb{R} \times \mathbb{R}^{2m+1}$. Let $(p_\nu)_{\nu\in\mathbb{N}}$ be a sequence in M that does not have a convergent subsequence and choose a sequence $i_\nu \in \mathbb{N}$ such that $p_\nu \in K_{2i_\nu-1} \cup K_{2i_\nu}$ for all $\nu \in \mathbb{N}$. Then i_ν tends to infinity. If $p_\nu \in K_{2i_\nu-1}$ for all ν, then we have $\limsup_{\nu\to\infty} |f^{\mathrm{odd}}(p_\nu)|^{-1}|f^{\mathrm{ev}}(p_\nu)| \le 1$ by (2.9.5). Passing to a subsequence, still denoted by $(p_\nu)_{\nu\in\mathbb{N}}$, we may assume that the limit $\lambda := \lim_{\nu\to\infty} |f^{\mathrm{odd}}(f_\nu)|^{-1}|f^{\mathrm{ev}}(p_\nu)|$ exists. Then

$$0 \le \lambda \le 1, \quad \lim_{\nu\to\infty} \frac{\left|f^{\mathrm{odd}}(p_\nu)\right|}{|f(p_\nu)|} = \frac{1}{\sqrt{1+\lambda^2}}, \quad \lim_{\nu\to\infty} \frac{|f^{\mathrm{ev}}(p_\nu)|}{|f(p_\nu)|} = \frac{\lambda}{\sqrt{1+\lambda^2}},$$

and it follows from (2.9.4) that

$$\lim_{\nu\to\infty} \frac{f^{\mathrm{odd}}(p_\nu)}{|f^{\mathrm{odd}}(p_\nu)|} = \xi, \qquad \lim_{\nu\to\infty} \frac{f^{\mathrm{ev}}(p_\nu)}{|f^{\mathrm{odd}}(p_\nu)|} = \lambda\xi.$$

This implies

$$\lim_{\nu\to\infty} \frac{f(p_\nu)}{|f(p_\nu)|} = \left(0, \frac{\xi}{\sqrt{1+\lambda^2}}, 0, \frac{\lambda\xi}{\sqrt{1+\lambda^2}}\right) = \frac{1}{\sqrt{1+\lambda^2}}x + \frac{\lambda}{\sqrt{1+\lambda^2}}y.$$

Similarly, if $p_\nu \in K_{2i_\nu}$ for all ν, there exists a subsequence such that the limit $\lambda := \lim_{\nu\to\infty}|f^{\mathrm{ev}}(p_\nu)|^{-1}|f^{\mathrm{odd}}(p_\nu)|$ exists and, by (2.9.4), this implies

$$\lim_{\nu\to\infty}\frac{f(p_\nu)}{|f(p_\nu)|} = \left(0, \frac{\lambda\xi}{\sqrt{1+\lambda^2}}, 0, \frac{\xi}{\sqrt{1+\lambda^2}}\right) = \frac{\lambda}{\sqrt{1+\lambda^2}}x + \frac{1}{\sqrt{1+\lambda^2}}y.$$

This shows that the vectors x and y satisfy the requirements of Step 4.

Step 5. *There exists an embedding $f : M \to \mathbb{R}^{2m+1}$ with a closed image.*

For compact manifolds the result was proved in Steps 1 and 2 and for $m = 0$ the assertion is obvious, because then M is a finite or countable set with the discrete topology. Thus assume that M is not compact and $m \geq 1$. Choose $f : M \to \mathbb{R}^{4m+4}$ and $x, y \in \mathbb{R}^{4m+4}$ as in Step 4 and define

$$\mathcal{A} := \left\{ A \in \mathbb{R}^{(2m+1)\times(4m+4)} \,\middle|\, \begin{array}{l}\text{the vectors } Ax \text{ and } Ay\\ \text{are linearly independent}\end{array} \right\}.$$

Since $m \geq 1$, this is a nonempty open subset of $\mathbb{R}^{(2m+1)\times(4m+4)}$. We prove that the map $Af : M \to \mathbb{R}^{2m+1}$ is proper and has a closed image for every $A \in \mathcal{A}$. To see this, fix a matrix $A \in \mathcal{A}$. Let $(p_\nu)_{\nu\in\mathbb{N}}$ be a sequence in M that does not have a convergent subsequence. Then by Step 4 there exists a subsequence, still denoted by $(p_\nu)_{\nu\in\mathbb{N}}$, and real numbers $s, t \in \mathbb{R}$ such that

$$s^2 + t^2 = 1, \qquad \lim_{\nu\to\infty}\frac{f(p_\nu)}{|f(p_\nu)|} = sx + ty, \qquad \lim_{\nu\to\infty}|f(p_\nu)| = \infty.$$

This implies

$$\lim_{\nu\to\infty}\frac{Af(p_\nu)}{|f(p_\nu)|} = sAx + tAy \neq 0$$

and hence $\lim_{\nu\to\infty}|Af(p_\nu)| = \infty$. Thus the preimage of every compact subset of $\mathbb{R}^{2m+1}$ under the map $Af : M \to \mathbb{R}^{2m+1}$ is a compact subset of M, and hence Af is proper and has a closed image (Remark 2.3.3).

Now it follows from Step 2 that there exists a matrix $A \in \mathcal{A}$ such that the map $Af : M \to \mathbb{R}^{2m+1}$ is an injective immersion. Hence it is an embedding with a closed image. This proves Step 5 and Theorem 2.9.12. □

The **Whitney Embedding Theorem** asserts that every second countable Hausdorff m-manifold M admits an embedding $f : M \to \mathbb{R}^{2m}$. The proof is based on the *Whitney Trick* and goes beyond the scope of this book. The next exercise shows that Whitney's theorem is sharp.

Remark 2.9.13. The manifold $\mathbb{R}\mathrm{P}^2$ cannot be embedded into $\mathbb{R}^3$. The same is true for the **Klein bottle** $K := \mathbb{R}^2/\equiv$ where the equivalence relation is given by $[x, y] \equiv [x + k, \ell - y]$ for $x, y \in \mathbb{R}$ and $k, \ell \in \mathbb{Z}$.

2.9.4 Leaves of a Foliation

Let M be an m-dimensional paracompact Hausdorff manifold and $E \subset TM$ be an integrable subbundle of rank n. Let $L \subset M$ be a closed leaf of the foliation determined by E. Then L is a smooth n-dimensional submanifold of M. Here is a sketch of David Epstein's proof of this fact in [5].

(a) *The space L with the intrinsic topology admits the structure of a manifold such that the obvious inclusion $\iota : L \to M$ is an injective immersion.* This is an easy exercise. For the definition of the intrinsic topology see Exercise 2.7.10. The dimension of L is n.

(b) *If $f : X \to Y$ is a continuous map between topological spaces such that Y is paracompact and there is an open cover $\{V_j\}_{j\in J}$ of Y such that $f^{-1}(V_j)$ is paracompact for each j, then X is paracompact.* To see this, we may assume that the cover $\{V_j\}_{j\in J}$ is locally finite. Now let $\{U_\alpha\}_{\alpha\in A}$ be an open cover of X. Then the sets $U_\alpha \cap f^{-1}(V_j)$ define an open cover of $f^{-1}(V_j)$. Choose a locally finite refinement $\{W_{ij}\}_{i\in I_j}$ of this cover. Then the open cover $\{W_{ij}\}_{j\in J, i\in I_j}$ of M is a locally finite refinement of $\{U_\alpha\}_{\alpha\in A}$.

(c) *The intrinsic topology of L is paracompact.* This follows from (b) and the fact that the intersection of L with every foliation box is paracompact in the intrinsic topology.

(d) *The intrinsic topology of L is second countable.* This follows from (a) and (c) and the fact that every connected paracompact manifold is second countable (see Remark 2.9.2).

(e) *The intersection of L with a foliation box consists of at most countably many connected components.* This follows immediately from (d).

(f) *If L is a closed subset of M then the intersection of L with a foliation box has only finitely many connected components.* To see this, we choose a transverse slice of the foliation at $p_0 \in L$, i.e. a connected submanifold $T \subset M$ through p_0, diffeomorphic to an open ball in $\mathbb{R}^{m-n}$, whose tangent space at each point $p \in T$ is a complement of E_p. By (d) we have that $T \cap L$ is at most countable. If this set is not finite, even after shrinking T, there must be a sequence $p_i \in (T \cap L) \setminus \{p_0\}$ converging to p_0. Using the holonomy of the leaf (obtained by transporting transverse slices along a curve via a lifting argument) we find that every point $p \in T \cap L$ is the limit point of a sequence in $(T \cap L) \setminus \{p\}$. Hence the one-point set $\{p\}$ has empty interior in the relative topology of $T \cap L$ for each $p \in T \cap L$. Thus $T \cap L$ is a countable union of closed subsets with empty interior. Since $T \cap L$ admits the structure of a complete metric space, this contradicts the Baire category theorem.

(g) It follows immediately from (f) that L is a submanifold of M.

2.9.5 Principal Bundles

An interesting class of foliations arises from smooth Lie group actions. Let $\mathrm{G} \subset \mathrm{GL}(N, \mathbb{R})$ be a compact Lie group and let P be a smooth m-manifold whose topology is Hausdorff and second countable. A **smooth (contravariant)** G**-action on** P is a smooth map

$$P \times \mathrm{G} \to P : (p, g) \mapsto pg \tag{2.9.6}$$

that satisfies the conditions

$$(pg)h = p(gh), \qquad p\mathbb{1} = p \tag{2.9.7}$$

for all $p \in P$ and all $g, h \in \mathrm{G}$. Fix any such group action. Then every group element $g \in \mathrm{G}$ determines a diffeomorphism $P \to P : p \mapsto pg$, whose differential at $p \in P$ is denoted by $T_pP \to T_{pg}P : v \mapsto vg$. Every Lie algebra element $\xi \in \mathfrak{g} := \mathrm{Lie}(\mathrm{G}) = T_{\mathbb{1}}\mathrm{G}$ determines a vector field $X_\xi \in \mathrm{Vect}(P)$ which assigns to each $p \in P$ the tangent vector

$$X_\xi(p) := p\xi := \left.\frac{d}{dt}\right|_{t=0} p\exp(t\xi) \in T_pP. \tag{2.9.8}$$

The linear map $\mathfrak{g} \to \mathrm{Vect}(P) : \xi \mapsto X_\xi$ is called the **infinitesimal action**. It is a Lie algebra anti-homomorphism because the group action is contravariant. (Exercise: Prove that $[X_\xi, X_\eta] = -X_{[\xi,\eta]}$ for $\xi, \eta \in \mathfrak{g}$.) The group action (2.9.6) is said to be with **finite isotropy** if the **isotropy subgroup**

$$\mathrm{G}_p := \{g \in \mathrm{G} \,|\, pg = p\}$$

is finite for all $p \in P$. The isotropy subgroup G_p is a Lie subgroup of G with Lie algebra $\mathfrak{g}_p := \mathrm{Lie}(\mathrm{G}_p) = \{\xi \in \mathfrak{g} \,|\, X_\xi(p) = 0\}$. Since G is compact, this shows that G_p is a finite subgroup of G if and only if $\mathfrak{g}_p = \{0\}$ or, equivalently, the map $\mathfrak{g} \to T_pP : \xi \mapsto X_\xi(p) = p\xi$ is injective. Thus, in the finite isotropy case, the group action determines an involutive subbundle $E \subset TP$ with the fibers $E_p := p\mathfrak{g} = \{X_\xi(p) \,|\, \xi \in \mathfrak{g}\}$ for $p \in P$. When G is connected, the leaves of the corresponding foliation are the group orbits $p\mathrm{G} := \{pg \,|\, g \in \mathrm{G}\}$. These are the elements of the **orbit space**

$$P/\mathrm{G} := \{p\mathrm{G} \,|\, p \in P\}.$$

There is a natural projection $\pi : P \to P/\mathrm{G}$ defined by $\pi(p) := p\mathrm{G}$ for $p \in P$ and the orbit space P/G is equipped with the quotient topology (a subset $U \subset P/\mathrm{G}$ is open if and only if $\pi^{-1}(U)$ is an open subset of P). The group action is called **free** if $\mathrm{G}_p = \{\mathbb{1}\}$ for all $p \in P$. The next theorem shows that, in the case of a free action, the quotient space admits a unique smooth structure such that the projection $\pi : P \to P/\mathrm{G}$ is a submersion.

Theorem 2.9.14 (Principal Bundle). *Let P be a smooth m-manifold whose topology is Hausdorff and second countable. Suppose P is equipped with a smooth contravariant action of a compact Lie group* G *and assume the group action is free. Then* $\dim(\mathrm{G}) \le m$ *and* $B := P/\mathrm{G}$ *admits a unique smooth structure such that the projection* $\pi : P \to B$ *is a submersion. The intrinsic topology of B, induced by the smooth structure, agrees with the quotient topology, and it is Hausdorff and second countable.*

Proof. For each $p \in P$ the map $\mathrm{G} \to P : g \mapsto pg$ is an embedding and this implies $k := \dim(\mathrm{G}) \le \dim(P) = m$. Define $n := m - k$. A **local slice** of the group action is a smooth map $\iota : \Omega \to P$, defined on an open set $\Omega \subset \mathbb{R}^n$, such that the map $\Omega \times \mathrm{G} \to P : (x, g) \mapsto \iota(x)g$ is an embedding. With this understood, we prove the assertions in five steps.

Step 1. *For every $p_0 \in P$ there exists a local slice $\iota_0 : \Omega_0 \to P$, defined on an open neighborhood $\Omega_0 \subset \mathbb{R}^n$ of the origin, such that $\iota_0(0) = p_0$.*

Choose a coordinate chart $\phi : V \to \mathbb{R}^m$ on an open neighborhood $V \subset P$ of p_0 such that $\phi(p_0) = 0$ and $\phi(V) = \mathbb{R}^m$. Define $v_1, \dots, v_m \in T_{p_0}P$ by

$$d\phi(p_0)v_i := e_i \qquad \text{for } i = 1, \dots, m,$$

where $e_1, \dots, e_m$ is the standard basis of $\mathbb{R}^m$. Reorder the coordinates on $\mathbb{R}^m$, if necessary, such that the vectors $v_1, \dots, v_n$ project to a basis of the quotient space $T_{p_0}P/p_0\mathfrak{g}$. Define $\iota : \mathbb{R}^n \to P$ by

$$\iota(x_1, \dots, x_n) := \phi^{-1}(x_1, \dots, x_n, 0, \dots, 0)$$

and define the map $\psi : \mathbb{R}^n \times \mathrm{G} \to P$ by

$$\psi(x, g) := \iota(x)g \qquad \text{for } x \in \mathbb{R}^n \text{ and } g \in \mathrm{G}.$$

Then ψ is smooth and its differential $d\psi(0, \mathbb{1}) : \mathbb{R}^n \times \mathfrak{g} \to T_{p_0}P$ is given by

$$d\psi(0, \mathbb{1})(\widehat{x}, \xi) = \sum_{i=1}^{n} \widehat{x}_i v_i + p_0\xi$$

for $\widehat{x} = (\widehat{x}_1, \dots, \widehat{x}_n) \in \mathbb{R}^n$ and $\xi \in \mathfrak{g}$. Hence $d\psi(0, \mathbb{1})$ is bijective and so it follows from the Inverse Function Theorem 2.2.15 that there exist open neighborhoods $\Omega \subset \mathbb{R}^n$ of 0, $\Omega_1 \subset \mathrm{G}$ of $\mathbb{1}$, and $W \subset P$ of p_0 such that the restricted map

$$\psi_1 := \psi|_{\Omega \times \Omega_1} : \Omega \times \Omega_1 \to W$$

is a diffeomorphism.

Next we prove that there exists an open neigborhood $\Omega_0 \subset \Omega$ of the origin such that the restricted map

$$\psi_0 := \psi|_{\Omega_0 \times \mathrm{G}} : \Omega_0 \times \mathrm{G} \to P$$

is injective. Suppose otherwise that no such neighborhood Ω_0 exists. Then there exist sequences $(x_i, g_i), (x_i', g_i') \in \Omega \times \mathrm{G}$ such that $(x_i, g_i) \neq (x_i', g_i')$ and $\psi(x_i, g_i) = \psi(x_i', g_i')$ for all i and the sequences $(x_i)_{i\in\mathbb{N}}$ and $(x_i')_{i\in\mathbb{N}}$ in Ω converge to the origin. Since G is compact we may assume, by passing to a subsequence if necessary, that the sequences $(g_i)_{i\in\mathbb{N}}$ and $(g_i')_{i\in\mathbb{N}}$ converge. Denote the limits by

$$g := \lim_{i\to\infty} g_i \in \mathrm{G}, \qquad g' := \lim_{i\to\infty} g_i' \in \mathrm{G}.$$

Then

$$p_0 g = \lim_{i\to\infty} \iota(x_i) g_i = \lim_{i\to\infty} \iota(x_i') g_i' = p_0 g'$$

and so $g = g'$ because the group action is free. Thus the sequence $(g_i' g_i^{-1})_{i\in\mathbb{N}}$ in G converges to $\mathbb{1}$ and hence belongs to the set Ω_1 for i sufficiently large. Since

$$\psi_1(x_i, \mathbb{1}) = \iota(x_i) = \iota(x_i') g_i' g_i^{-1} = \psi_1(x_i', g_i' g_i^{-1})$$

for all i, this contradicts the injectivity of ψ_1. Thus we have proved that the map $\psi_0 : \Omega_0 \times \mathrm{G} \to P$ is injective for a suitable neighborhood $\Omega_0 \subset \Omega$ of the origin. That it is an immersion is a direct consequence of the formula

$$d\psi_0(x,g)(\widehat{x}, \widehat{g}) = \big(d\iota(x)\widehat{x} + \iota(x)(\widehat{g} g^{-1})\big) g = \big(d\psi_0(x, \mathbb{1})(\widehat{x}, \widehat{g} g^{-1})\big) g$$

for all $x \in \Omega_0$, $\widehat{x} \in \mathbb{R}^n$, $g \in \mathrm{G}$, and $\widehat{g} \in T_g\mathrm{G}$, and the fact that the differential $d\psi_0(x, \mathbb{1})$ is bijective for all $x \in \Omega_0$ (even for all $x \in \Omega$).

Thus we have proved that $\psi_0 : \Omega_0 \times \mathrm{G} \to P$ is an injective immersion. Shrinking Ω_0 further, if necessary, we may assume that Ω_0 has a compact closure and that ψ is injective on $\overline{\Omega}_0 \times \mathrm{G}$. This implies that ψ_0 is proper. Namely, if $(x_i, g_i)_{i\in\mathbb{N}}$ is a sequence in $\Omega_0 \times \mathrm{G}$ and $(x, g) \in \Omega_0 \times \mathrm{G}$ such that $\psi_0(x, g) = \lim_{i\to\infty} \psi_0(x_i, g_i)$, then there is a subsequence $(x_{i_\nu}, g_{i_\nu})_{\nu\in\mathbb{N}}$ that converges to a pair $(x', g') \in \overline{\Omega}_0 \times \mathrm{G}$. This subsequence satisfies

$$\psi(x', g') = \lim_{\nu\to\infty} \psi_0(x_{i_\nu}, g_{i_\nu}) = \psi(x, g).$$

Since ψ is injective on $\overline{\Omega}_0 \times \mathrm{G}$, this implies $x = x'$ and $g = g'$. Thus every subsequence of $(x_i, g_i)_{i\in\mathbb{N}}$ has a further subsequence that converges to (x, g) and so the sequence $(x_i, g_i)_{i\in\mathbb{N}}$ itself converges to (x, g). Thus the map $\psi_0 : \Omega_0 \times \mathrm{G} \to P$ is a proper injective immersion and this proves Step 1.

Step 2. *Let $\iota : \Omega \to P$ be a local slice. Then the set $U := \pi(\iota(\Omega)) \subset B$ is open in the quotient topology and the map $\pi \circ \iota : \Omega \to U$ is a homeomorphism with respect to the quotient topology on U.*

The map $\psi : \Omega \times \mathrm{G} \to P$, defined by $\psi(x, g) := \iota(x)g$ for $x \in \Omega$ and $g \in \mathrm{G}$, is an embedding. Hence $W := \psi(\Omega \times \mathrm{G})$ is an open G-invariant subset of P and $\psi : \Omega \times \mathrm{G} \to W$ is a G-equivariant homeomorphism. Moreover, for every element $p \in P$, we have $\pi(p) \in U$ if and only if there exists an element $x \in \Omega$ and an element $g \in \mathrm{G}$ such that $p = \iota(x)g = \psi(x, g)$. Thus $\pi^{-1}(U) = \psi(\Omega \times \mathrm{G}) = W$ is an open subset of P, and so U is an open subset of $B = P/\mathrm{G}$ with respect to the quotient topology. The continuity of $\pi \circ \iota : \Omega \to U$ follows directly from the definition. Moreover, if $\Omega' \subset \Omega$ is an open set and $U' := \pi(\iota(\Omega'))$, then $\pi^{-1}(U') = \psi(\Omega' \times \mathrm{G})$ is open by the same argument, and so $U' \subset B$ is open with respect to the quotient topology. Thus $\pi \circ \iota : \Omega \to U$ is a homeomorphism and this proves Step 2.

Step 3. *By Step 1 there exists a collection $\iota_\alpha : \Omega_\alpha \to P$, $\alpha \in A$, of local slices such that the sets $U_\alpha := \pi(\iota_\alpha(\Omega_\alpha))$ cover the orbit space $B = P/\mathrm{G}$. For $\alpha \in A$ define*

$$\phi_\alpha := (\pi \circ \iota_\alpha)^{-1} : U_\alpha \to \Omega_\alpha.$$

Then $\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$ is a smooth structure on B which renders the canonical projection $\pi : P \to B$ into a submersion. Moreover, this smooth structure is compatible with the quotient topology on B.

For $\alpha, \beta \in A$ define $\Omega_{\alpha\beta} := \phi_\alpha(U_\alpha \cap U_\beta)$ and $\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1} : \Omega_{\alpha\beta} \to \Omega_{\beta\alpha}$. We must prove that $\phi_{\beta\alpha}$ is smooth. To see this, define $\psi_\alpha : \Omega_\alpha \times \mathrm{G} \to P$ by $\psi_\alpha(x, g) := \iota_\alpha(x)g$ for $\alpha \in A$, $x \in \Omega_\alpha$, and $g \in \mathrm{G}$. Then ψ_α is a diffeomorphism onto its image and $\psi_\alpha(\Omega_{\alpha\beta} \times \mathrm{G}) = \psi_\beta(\Omega_{\beta\alpha} \times \mathrm{G}) = \pi^{-1}(U_\alpha \cap U_\beta)$. For $x \in \Omega_{\alpha\beta}$ the element $\phi_{\beta\alpha}(x) \in \Omega_{\beta\alpha}$ is the projection of $\psi_\beta^{-1} \circ \psi_\alpha(x, \mathbb{1})$ onto the first factor. Thus $\phi_{\beta\alpha}$ is smooth and so is its inverse $\phi_{\alpha\beta}$. This shows that $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ is a smooth structure on B. Second, π is a submersion with respect to this smooth structure, because $\phi_\alpha \circ \pi \circ \psi_\alpha(x, g) = x$ for all $\alpha \in A$, all $x \in \Omega_\alpha$, and all $g \in \mathrm{G}$. Third, this smooth structure is compatible with the quotient topology by Step 2. This proves Step 3.

Step 4. *There is only one smooth structure on B with respect to which the projection $\pi : P \to B$ is a submersion.*

Fix any smooth structure on B for which the projection $\pi : P \to B$ is a submersion. Then the dimension of B is $n = \dim(P) - \dim(\mathrm{G})$, and so the smooth structure consists of bijections $\phi_\alpha : U_\alpha \to \Omega_\alpha$ from subsets $U_\alpha \subset B$ onto open sets $\Omega_\alpha \subset \mathbb{R}^n$ such that the sets U_α cover B and the transition maps are diffeomorphisms between open subsets of $\mathbb{R}^n$.

We prove that the intrinsic topology on B agrees with the quotient topology. To see this, fix a subset $U \subset B$. Then the following are equivalent.

(a) U is open with respect to the intrinsic topology on B.

(b) $\phi_\alpha(U \cap U_\alpha)$ is open in $\mathbb{R}^n$ for all $\alpha \in A$.

(c) $\pi^{-1}(U \cap U_\alpha)$ is open in P for all $\alpha \in A$.

(d) $\pi^{-1}(U)$ is open in P.

(e) U is open with respect to the quotient topology on B.

The equivalence of (a) and (b) follows from the definition of the intrinsic topology. That (b) implies (c) follows from the three observations that the set $\pi^{-1}(U_\alpha)$ is open in P, the map $\phi_\alpha \circ \pi : \pi^{-1}(U_\alpha) \to \Omega_\alpha$ is continuous, and $(\phi_\alpha \circ \pi)^{-1}(\phi_\alpha(U \cap U_\alpha)) = \pi^{-1}(U \cap U_\alpha)$. That (c) implies (b) follows from the fact that the map $\phi_\alpha \circ \pi : \pi^{-1}(U_\alpha) \to \Omega_\alpha$ is a submersion and hence maps the open set $\pi^{-1}(U \cap U_\alpha)$ onto an open subset of Ω_α (Corollary 2.6.2). The equivalence of (c) and (d) follows from the fact that the map $\pi : P \to B$ is continuous and $U_\alpha \subset B$ is open (both with respect to the intrinsic topology on B) and so $\pi^{-1}(U_\alpha)$ is open in P for all $\alpha \in A$. The equivalence of (d) and (e) follows from the definition of the quotient topology on B.

Now let $\iota : \Omega \to P$ be a local slice and define the set $U := \pi(\iota(\Omega)) \subset B$ and the map $\phi := (\pi \circ \iota)^{-1} : U \to \Omega$. Then the composition

$$\phi_\alpha \circ \phi^{-1} = \phi_\alpha \circ \pi \circ \iota : \phi(U \cap U_\alpha) \to \phi_\alpha(U \cap U_\alpha)$$

is a homeomorphism between open subsets of $\mathbb{R}^n$. Moreover, $\phi_\alpha \circ \phi^{-1}$ is the composition of the smooth maps $\iota : \{x \in \Omega \,|\, \pi(\iota(x)) \in U_\alpha\} \to \pi^{-1}(U \cap U_\alpha)$, $\pi : \pi^{-1}(U \cap U_\alpha) \to U \cap U_\alpha$, and $\phi_\alpha : U \cap U_\alpha \to \phi_\alpha(U \cap U_\alpha)$. So $\phi_\alpha \circ \phi^{-1}$ is smooth and its differential is everywhere bijective because π is a submersion and the kernel of $d\pi(\iota(x))$ is transverse to the image of $d\iota(x)$. Thus $\phi_\alpha \circ \phi^{-1}$ is a diffeomorphism by the Inverse Function Theorem and this proves Step 4.

Step 5. *The quotient topology on B is a Hausdorff and second countable.*

Let $\iota_\alpha : \Omega_\alpha \to P$ for $\alpha \in A$ be a collection of local slices such that the sets $U_\alpha := \pi(\iota_\alpha(\Omega_\alpha))$ cover B. Then the open sets $\pi^{-1}(U_\alpha)$ form an open cover of P and so there is a countable subcover. Thus B is second countable. To prove that B is Hausdorff, fix two distinct elements $b_0, b_1 \in B$ and choose $p_0, p_1 \in P$ such that $\pi(p_0) = b_0$ and $\pi(p_1) = b_1$. Then $p_0\mathrm{G}$ and $p_1\mathrm{G}$ are disjoint compact subsets of P and hence can be separated by disjoint open subsets $U_0, U_1 \subset P$, because P is a Hausdorff space. Now for $i = 0, 1$ the set $V_i := \{p \in P \,|\, p\mathrm{G} \subset U_i\}$ is open (exercise) and contains the orbit $p_i\mathrm{G}$. Hence $W_0 := \pi(V_0)$ and $W_1 := \pi(V_1)$ are disjoint open subsets of B such that $b_0 \in W_0$ and $b_1 \in W_1$. This proves Step 5 and Theorem 2.9.14. $\square$

Example 2.9.15. There are many important examples of free group actions and principal bundles. A class of examples arises from orthonormal frame bundles (§3.4). The complex projective space $B = \mathbb{C}\mathrm{P}^n$ arises from the action of the circle $\mathrm{G} = S^1$ on the unit sphere $P = S^{2n+1} \subset \mathbb{C}^{n+1}$ (Example 2.8.5). The real projective space $B = \mathbb{R}\mathrm{P}^n$ arises from the action of the finite group $\mathrm{G} = \mathbb{Z}/2\mathbb{Z}$ on the unit sphere $P = S^n \subset \mathbb{R}^{n+1}$ (Example 2.8.6). The complex Grassmannian $B = \mathrm{G}_k(\mathbb{C}^n)$ arises from the action of $\mathrm{G} = \mathrm{U}(k)$ on the space $P = \mathcal{F}_k(\mathbb{C}^n)$ of unitary k-frames in $\mathbb{C}^n$ (Example 3.7.6). If G is a Lie group and $\mathrm{K} \subset \mathrm{G}$ is a compact subgroup then, by Theorem 2.9.14, the **homogeneous space** G/K admits a unique smooth structure such that the projection $\pi : \mathrm{G} \to \mathrm{G}/\mathrm{K}$ is a submersion. The example $\mathrm{SL}(2, \mathbb{C})/\mathrm{SU}(2)$ can be identified with hyperbolic 3-space (§6.4.4), the example $\mathrm{U}(n)/\mathrm{O}(n)$ can be identified with the space of Lagrangian subspaces of a symplectic vector space [12, Lemma 2.3.2], the example $\mathrm{Sp}(2n)/\mathrm{U}(n)$ can be identified with Siegel upper half space or the space of compatible linear complex structures on a symplectic vector space [12, Lemma 2.5.12], and the example $\mathrm{G}_2/\mathrm{SO}(4)$ can be identified with the associative Grassmannian [19, Remark 8.4]. The last three examples go beyond the scope of the present book.

Standing Assumption

We have seen that all the results in the present chapter carry over to the intrinsic setting, assuming that the topology of M is Hausdorff and paracompact. In fact, in many cases it is enough to assume the Hausdorff property. However, these results mainly deal with introducing the basic concepts like smooth maps, embeddings, submersions, vector fields, flows, and verifying their elementary properties, i.e. with setting up the language for differential geometry and topology. When it comes to the substance of the subject we shall deal with Riemannian metrics and they only exist on paracompact Hausdorff manifolds. Another central ingredient in differential topology is the theorem of Sard and that requires second countability. To quote Moe Hirsch [10]: "Manifolds that are not paracompact are amusing, but they never occur naturally and it is difficult to prove anything about them." Thus we will set the following convention for the remaining chapters.

We assume from now on that each intrinsic manifold M is Hausdorff and second countable and hence is also paracompact.

For most of this text we will in fact continue to develop the theory for submanifolds of Euclidean space and indicate, wherever necessary, how to extend the definitions, theorems, and proofs to the intrinsic setting.

Chapter 3

The Levi-Civita Connection

For a submanifold of Euclidean space the inner product on the ambient space determines an inner product on each tangent space, the *first fundamental form*. The *second fundamental form* is obtained by differentiating the map which assigns to each point in $M \subset \mathbb{R}^n$ the orthogonal projection onto the tangent space (§3.1). The covariant derivative of a vector field along a curve is the orthogonal projection of the derivative in the ambient space onto the tangent space (§3.2). We will show how the covariant derive gives rise to parallel transport (§3.3), examine the frame bundle (§3.4), discuss motions without “sliding, twisting, and wobbling”, and prove the development theorem (§3.5).

In §3.6 we will see that the covariant derivative is determined by the Christoffel symbols in local coordinates and thus carries over to the intrinsic setting. The intrinsic setting of Riemannian manifolds is explained in §3.7. The covariant derivative takes the form of a family of linear operators $\nabla : \mathrm{Vect}(\gamma) \to \mathrm{Vect}(\gamma)$, one for every smooth curve $\gamma : I \to M$, and these operators are uniquely characterized by the axioms of Theorem 3.7.8. This family of linear operators is the *Levi-Civita connection*.

3.1 Second Fundamental Form

Let $M \subset \mathbb{R}^n$ be a smooth m-manifold. Then each tangent space of M is an m-dimensional real vector space and hence is isomorphic to $\mathbb{R}^m$. Thus any two tangent spaces T_pM and T_qM are of course isomorphic to each other. While there is no canonical isomorphism from T_pM to T_qM we shall see that every smooth curve γ in M connecting p to q induces an isomorphism between the tangent spaces via parallel transport of tangent vectors along γ.

Throughout we use the standard inner product on $\mathbb{R}^n$ given by

$$\langle v, w\rangle = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

for $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ and $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$. The associated Euclidean norm will be denoted by

$$|v| = \sqrt{\langle v, v\rangle} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

for $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$. When $M \subset \mathbb{R}^n$ is a smooth m-dimensional submanifold, a first observation is that each tangent space of M inherits an inner product from the ambient space $\mathbb{R}^n$. The resulting *field of inner products* is called the first fundamental form.

Definition 3.1.1. *Let $M \subset \mathbb{R}^n$ be a smooth m-dimensional submanifold. The* **first fundamental form on** M *is the field which assigns to each $p \in M$ the bilinear map*

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

defined by

$$g_p(v, w) = \langle v, w\rangle \tag{3.1.1}$$

for $v, w \in T_pM$.

A second observation is that the inner product on the ambient space also determines an orthogonal projection of $\mathbb{R}^n$ onto the tangent space T_pM for each $p \in M$. This projection can be represented by the matrix $\Pi(p) \in \mathbb{R}^{n\times n}$ which is uniquely determined by the conditions

$$\Pi(p) = \Pi(p)^2 = \Pi(p)^{\mathsf{T}}, \tag{3.1.2}$$

and

$$\Pi(p)v = v \qquad \Longleftrightarrow \qquad v \in T_pM \tag{3.1.3}$$

for $p \in M$ and $v \in \mathbb{R}^n$ (see Exercise 2.6.7).

Lemma 3.1.2. *The map $\Pi : M \to \mathbb{R}^{n\times n}$ defined by* (3.1.2) *and* (3.1.3) *is smooth.*

Proof. This follows directly from Theorem 2.6.8 and Corollary 2.6.10. More explicitly, if $U \subset M$ is an open set and $\phi : U \to \Omega$ is a coordinate chart onto an open subset $\Omega \subset \mathbb{R}^m$ with the inverse $\psi := \phi^{-1} : \Omega \to U$, then

$$\Pi(p) = d\psi(\phi(p))\Big(d\psi(\phi(p))^{\mathsf{T}} d\psi(\phi(p))\Big)^{-1} d\psi(\phi(p))^{\mathsf{T}}$$

for $p \in U$ and this proves Lemma 3.1.2. $\square$

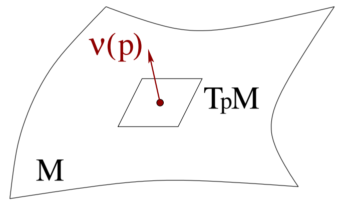


Figure 3.1: A unit normal vector field.

Example 3.1.3 (**Gauß Map**)**.** Let $M \subset \mathbb{R}^{m+1}$ be a submanifold of codimension one. Then $TM^\perp$ is a vector bundle of rank one (Corollary 2.6.11), and so each fiber $T_pM^\perp$ is spanned by a unit vector $\nu(p) \in \mathbb{R}^m$, determined by T_pM up to a sign. By Theorem 2.6.8 each $p_0 \in M$ has an open neighborhood $U \subset M$ on which there exists a smooth map

$$\nu : U \to \mathbb{R}^{m+1}$$

satisfying

$$\nu(p) \perp T_pM, \qquad |\nu(p)| = 1 \tag{3.1.4}$$

for all $p \in U$ (see Figure 3.1). Such a map ν is called a **Gauß map**. The function $\Pi : M \to \mathbb{R}^{n\times n}$ is in this case given by

$$\Pi(p) = \mathbb{1} - \nu(p)\nu(p)^\mathsf{T} \tag{3.1.5}$$

for $p \in U$.

Example 3.1.4. Let $M = S^2 \subset \mathbb{R}^3$. Then $\nu(p) = p$ and so

$$\Pi(p) = \mathbb{1} - pp^\mathsf{T} = \begin{pmatrix} 1-x^2 & -xy & -xz \\ -yx & 1-y^2 & -yz \\ -zx & -zy & 1-z^2 \end{pmatrix}$$

for $p = (x, y, z) \in S^2$.

Example 3.1.5 (**Möbius Strip**)**.** Consider the submanifold

$$M := \left\{ (x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x = (1 + r\cos(\theta/2))\cos(\theta), \\ y = (1 + r\cos(\theta/2))\sin(\theta), \\ z = r\sin(\theta/2),\ r, \theta \in \mathbb{R},\ |r| < \varepsilon \end{array} \right\}$$

for $\varepsilon > 0$ sufficiently small. Show that there does not exist a global smooth function $\nu : M \to \mathbb{R}^3$ satisfying (3.1.4).

Example 3.1.6. Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^{n-m}$ be a smooth function such that $0 \in \mathbb{R}^{n-m}$ is a regular value of f and $U \cap M = f^{-1}(0)$. Then $T_pM = \ker df(p)$ and

$$\Pi(p) = \mathbb{1} - df(p)^\mathsf{T} \left(df(p) df(p)^\mathsf{T} \right)^{-1} df(p)$$

for every $p \in U \cap M$.

Example 3.1.7. Let $\Omega \subset \mathbb{R}^m$ be an open set and $\psi : \Omega \to M$ be a smooth embedding. Then $T_{\psi(x)}M = \operatorname{im} d\psi(x)$ and

$$\Pi(\psi(x)) = d\psi(x) \left(d\psi(x)^\mathsf{T} d\psi(x) \right)^{-1} d\psi(x)^\mathsf{T}$$

for every $x \in \Omega$.

Next we differentiate the map $\Pi : M \to \mathbb{R}^{n \times n}$ in Lemma 3.1.2. The derivative at $p \in M$ takes the form of a linear map

$$d\Pi(p) : T_pM \to \mathbb{R}^{n \times n}$$

which, as usual, is defined by

$$d\Pi(p)v := \left. \frac{d}{dt} \right|_{t=0} \Pi(\gamma(t)) \in \mathbb{R}^{n \times n}$$

for $v \in T_pM$, where $\gamma : \mathbb{R} \to M$ is chosen such that $\gamma(0) = p$ and $\dot\gamma(0) = v$ (see Definition 2.2.12). We emphasize that the expression $d\Pi(p)v$ is a matrix and can therefore be multiplied by a vector in $\mathbb{R}^n$.

Lemma 3.1.8. *For all $p \in M$ and $v, w \in T_pM$ we have*

$$\big(d\Pi(p)v\big)w = \big(d\Pi(p)w\big)v \in T_pM^\perp.$$

Proof. Choose a smooth path $\gamma : \mathbb{R} \to M$ and a vector field $X : \mathbb{R} \to \mathbb{R}^n$ along γ such that

$$\gamma(0) = p, \qquad \dot\gamma(0) = v, \qquad X(0) = w.$$

For example, we can choose $X(t) := \Pi(\gamma(t))w$. Then

$$X(t) = \Pi(\gamma(t))X(t)$$

for every $t \in \mathbb{R}$. Differentiate this equation to obtain

$$\dot X(t) = \Pi(\gamma(t))\dot X(t) + \big(d\Pi(\gamma(t))\dot\gamma(t)\big)X(t). \tag{3.1.6}$$

Hence

$$\big(d\Pi(\gamma(t))\dot\gamma(t)\big)X(t) = \big(\mathbb{1} - \Pi(\gamma(t))\big)\dot X(t) \in T_{\gamma(t)}M^\perp \tag{3.1.7}$$

for every $t \in \mathbb{R}$ and, with $t = 0$, we obtain $(d\Pi(p)v)\, w \in T_pM^\perp$.

Now choose a smooth map

$$\mathbb{R}^2 \to M : (s,t) \mapsto \gamma(s,t)$$

satisfying

$$\gamma(0,0) = p, \qquad \frac{\partial \gamma}{\partial s}(0,0) = v, \qquad \frac{\partial \gamma}{\partial t}(0,0) = w,$$

(for example by doing this in local coordinates) and denote

$$X(s,t) := \frac{\partial \gamma}{\partial s}(s,t) \in T_{\gamma(s,t)}M, \qquad Y(s,t) := \frac{\partial \gamma}{\partial t}(s,t) \in T_{\gamma(s,t)}M.$$

Then

$$\frac{\partial Y}{\partial s} = \frac{\partial^2 \gamma}{\partial s \partial t} = \frac{\partial X}{\partial t}$$

and hence, using (3.1.7), we obtain

$$\begin{aligned}
\left(d\Pi(\gamma)\frac{\partial \gamma}{\partial t}\right)\frac{\partial \gamma}{\partial s} &= \left(d\Pi(\gamma)\frac{\partial \gamma}{\partial t}\right) X \\
&= \bigl(\mathbb{1} - \Pi(\gamma)\bigr)\frac{\partial X}{\partial t} \\
&= \bigl(\mathbb{1} - \Pi(\gamma)\bigr)\frac{\partial Y}{\partial s} \\
&= \left(d\Pi(\gamma)\frac{\partial \gamma}{\partial s}\right) Y \\
&= \left(d\Pi(\gamma)\frac{\partial \gamma}{\partial s}\right)\frac{\partial \gamma}{\partial t}.
\end{aligned}$$

With $s = t = 0$ we obtain

$$\bigl(d\Pi(p)w\bigr)v = \bigl(d\Pi(p)v\bigr)w \in T_pM^{\perp}$$

and this proves Lemma 3.1.8. □

Definition 3.1.9. *The collection of symmetric bilinear maps*

$$h_p : T_pM \times T_pM \to T_pM^{\perp},$$

defined by

$$h_p(v,w) := (d\Pi(p)v)w = (d\Pi(p)w)v \tag{3.1.8}$$

for $p \in M$ and $v, w \in T_pM$ is called the **second fundamental form** *on M.*

Example 3.1.10. Let $M \subset \mathbb{R}^{m+1}$ be an m-manifold and $\nu : M \to S^m$ be a Gauß map so that $T_pM = \nu(p)^\perp$ for every $p \in M$ (see Example 3.1.3). Then $\Pi(p) = \mathbb{1} - \nu(p)\nu(p)^\mathsf{T}$ and hence

$$h_p(v, w) = -\nu(p)\langle d\nu(p)v, w\rangle$$

for $p \in M$ and $v, w \in T_pM$.

Exercise 3.1.11. Choose a splitting $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ and write the elements of $\mathbb{R}^n$ as tuples $(x, y) = (x_1, \dots, x_m, y_1, \dots, y_{n-m})$ Let $M \subset \mathbb{R}^n$ be a smooth m-dimensional submanifold such that $p = 0 \in M$ and

$$T_0M = \mathbb{R}^m \times \{0\}, \qquad T_0M^\perp = \{0\} \times \mathbb{R}^{n-m}.$$

By the implicit function theorem, there are open neighborhoods $\Omega \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^{n-m}$ of zero and a smooth map $f : \Omega \to V$ such that

$$M \cap (\Omega \times V) = \mathrm{graph}(f) = \{(x, f(x)) \,|\, x \in \Omega\}\,.$$

Thus $f(0) = 0$ and $df(0) = 0$. Prove that the second fundamental form $h_p : T_pM \times T_pM \to T_pM^\perp$ is given by the second derivatives of f, i.e.

$$h_p(v, w) = \left(0, \sum_{i,j=1}^m \frac{\partial^2 f}{\partial x_i \partial x_j}(0) v_i w_j\right)$$

for $v, w \in T_pM = \mathbb{R}^m \times \{0\}$.

Exercise 3.1.12. Let $M \subset \mathbb{R}^n$ be an m-manifold. Fix a point $p \in M$ and a unit tangent vector $v \in T_pM$ so that $|v| = 1$ and define

$$L := \{p + tv + w \,|\, t \in \mathbb{R},\, w \perp T_pM\}\,.$$

Let $\gamma : (-\varepsilon, \varepsilon) \to M \cap L$ be a smooth curve such that $\gamma(0) = p$, $\dot\gamma(0) = v$, and $|\dot\gamma(t)| = 1$ for all t. Prove that

$$\ddot\gamma(0) = h_p(v, v).$$

Draw a picture of M and L in the case $n = 3$ and $m = 2$.

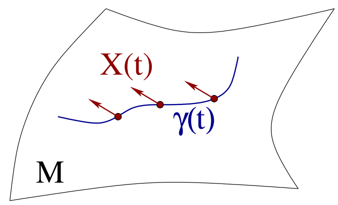


Figure 3.2: A vector field along a curve.

3.2 Covariant Derivative

Definition 3.2.1. *Let $I \subset \mathbb{R}$ be an open interval and let $\gamma : I \to M$ be a smooth curve. A* **vector field along** γ *is a smooth map $X : I \to \mathbb{R}^n$ such that $X(t) \in T_{\gamma(t)}M$ for every $t \in I$ (see Figure 3.2). The set of smooth vector fields along γ is a real vector space and will be denoted by*

$$\mathrm{Vect}(\gamma) := \left\{ X : I \to \mathbb{R}^n \,|\, X \textit{ is smooth and } X(t) \in T_{\gamma(t)}M \;\forall\, t \in I \right\}.$$

The first derivative $\dot{X}(t)$ of a vector field along γ at $t \in I$ will, in general, not be tangent to M. We may decompose it as a sum of a tangent vector and a normal vector in the form

$$\dot{X}(t) = \Pi(\gamma(t))\dot{X}(t) + \big(\mathbb{1} - \Pi(\gamma(t))\big)\dot{X}(t),$$

where $\Pi : M \to \mathbb{R}^{n\times n}$ is defined by (3.1.2) and (3.1.3). The tangential component of this decomposition plays an important geometric role. It is called the covariant derivative of X at t.

Definition 3.2.2 (Covariant Derivative). *Let $I \subset \mathbb{R}$ be an open interval, let $\gamma : I \to M$ be a smooth curve, and let $X \in \mathrm{Vect}(\gamma)$. The* **covariant derivative of** X *is the vector field $\nabla X \in \mathrm{Vect}(\gamma)$, defined by*

$$\nabla X(t) := \Pi(\gamma(t))\dot{X}(t) \in T_{\gamma(t)}M \tag{3.2.1}$$

for $t \in I$.

Lemma 3.2.3 (Gauß–Weingarten Formula). *The derivative of a vector field X along a curve γ is given by*

$$\dot{X}(t) = \nabla X(t) + h_{\gamma(t)}(\dot{\gamma}(t), X(t)). \tag{3.2.2}$$

Here the first summand is tangent to M and the second summand is orthogonal to the tangent space of M at $\gamma(t)$.

Proof. This is equation (3.1.6) in the proof of Lemma 3.1.8. $\square$

It follows directly from the definition that the covariant derivative along a curve $\gamma : I \to M$ is a linear operator $\nabla : \mathrm{Vect}(\gamma) \to \mathrm{Vect}(\gamma)$. The following lemma summarizes the basic properties of this operator.

Lemma 3.2.4 (**Covariant Derivative).** *The covariant derivative satisfies the following axioms for any two open intervals $I, J \subset \mathbb{R}$.*

(i) *Let $\gamma : I \to M$ be a smooth curve, let $\lambda : I \to \mathbb{R}$ be a smooth function, and let $X \in \mathrm{Vect}(\gamma)$. Then*

$$\nabla(\lambda X) = \dot{\lambda} X + \lambda \nabla X. \tag{3.2.3}$$

(ii) *Let $\gamma : I \to M$ be a smooth curve, let $\sigma : J \to I$ be a smooth function and let $X \in \mathrm{Vect}(\gamma)$. Then*

$$\nabla(X \circ \sigma) = \dot{\sigma}(\nabla X \circ \sigma). \tag{3.2.4}$$

(iii) *Let $\gamma : I \to M$ be a smooth curve and let $X, Y \in \mathrm{Vect}(\gamma)$. Then*

$$\frac{d}{dt}\langle X, Y\rangle = \langle \nabla X, Y\rangle + \langle X, \nabla Y\rangle. \tag{3.2.5}$$

(iv) *Let $\gamma : I \times J \to M$ be a smooth map, denote by ∇_s the covariant derivative along the curve $s \mapsto \gamma(s,t)$ (with t fixed), and denote by ∇_t the covariant derivative along the curve $t \mapsto \gamma(s,t)$ (with s fixed). Then*

$$\nabla_s \partial_t \gamma = \nabla_t \partial_s \gamma. \tag{3.2.6}$$

Proof. Part (i) follows from the Leibniz rule $\frac{d}{dt}(\lambda X) = \dot{\lambda} X + \lambda \dot{X}$ and (ii) follows from the chain rule $\frac{d}{dt}(X \circ \sigma) = \dot{\sigma}(\dot{X} \circ \sigma)$. To prove part (iii), use the orthogonal projections $\Pi(\gamma(t)) : \mathbb{R}^n \to T_{\gamma(t)}M$ to obtain

$$\begin{aligned}
\frac{d}{dt}\langle X, Y\rangle &= \langle \dot{X}, Y\rangle + \langle X, \dot{Y}\rangle \\
&= \langle \dot{X}, \Pi(\gamma)Y\rangle + \langle \Pi(\gamma)X, \dot{Y}\rangle \\
&= \langle \Pi(\gamma)\dot{X}, Y\rangle + \langle X, \Pi(\gamma)\dot{Y}\rangle \\
&= \langle \nabla X, Y\rangle + \langle X, \nabla Y\rangle
\end{aligned}$$

Part (iv) holds because the second derivatives commute and this proves Lemma 3.2.4. □

Part (i) in Lemma 3.2.4 asserts that the operator ∇ is what is called a *connection*, part (iii) asserts that it is compatible with the first fundamental form, and part (iv) asserts that it is *torsion-free*. Theorem 3.7.8 below asserts that these conditions (together with an extended chain rule) determine the covariant derivative uniquely.

3.3 Parallel Transport

Definition 3.3.1 (Parallel Vector Field). *Let $I \subset \mathbb{R}$ be an interval and let $\gamma : I \to M$ be a smooth curve. A vector field X along γ is called* **parallel** *if*

$$\nabla X(t) = 0$$

for all $t \in I$.

Example 3.3.2. Assume $m = k$ so that $M \subset \mathbb{R}^m$ is an open set. Then a vector field along a smooth curve $\gamma : I \to M$ is a smooth map $X : I \to \mathbb{R}^m$. Its covariant derivative is equal to the ordinary derivative $\nabla X(t) = \dot{X}(t)$ and hence X is is parallel if and only it is constant.

Remark 3.3.3. For every $X \in \mathrm{Vect}(\gamma)$ and every $t \in I$ we have

$$\nabla X(t) = 0 \qquad \Longleftrightarrow \qquad \dot{X}(t) \perp T_{\gamma(t)}M.$$

In particular, $\dot{\gamma}$ is a vector field along γ and $\nabla\dot{\gamma}(t) = \Pi(\gamma(t))\ddot{\gamma}(t)$. Hence $\dot{\gamma}$ is a parallel vector field along γ if and only if $\ddot{\gamma}(t) \perp T_{\gamma(t)}M$ for all $t \in I$. We will return to this observation in Chapter 4.

In general, a vector field X along a smooth curve $\gamma : I \to M$ is parallel if and only if $\dot{X}(t)$ is orthogonal to $T_{\gamma(t)}M$ for every t and, by the Gauß–Weingarten formula (3.2.2), we have

$$\nabla X = 0 \qquad \Longleftrightarrow \qquad \dot{X} = h_\gamma(\dot{\gamma}, X).$$

The next theorem shows that any given tangent vector $v_0 \in T_{\gamma(t_0)}M$ extends uniquely to a parallel vector field along γ.

Theorem 3.3.4 (Existence and Uniqueness). *Let $I \subset \mathbb{R}$ be an interval and $\gamma : I \to M$ be a smooth curve. Let $t_0 \in I$ and $v_0 \in T_{\gamma(t_0)}M$ be given. Then there is a unique parallel vector field $X \in \mathrm{Vect}(\gamma)$ such that $X(t_0) = v_0$.*

Proof. Choose a basis $e_1, \dots, e_m$ of the tangent space $T_{\gamma(t_0)}M$ and let

$$X_1, \dots, X_m \in \mathrm{Vect}(\gamma)$$

be vector fields along γ such that

$$X_i(t_0) = e_i, \qquad i = 1, \dots, m.$$

(For example choose $X_i(t) := \Pi(\gamma(t))e_i$.) Then the vectors $X_i(t_0)$ are linearly independent. Since linear independence is an open condition there is a constant $\varepsilon > 0$ such that the vectors $X_1(t), \dots, X_m(t) \in T_{\gamma(t)}M$ are linearly independent for every $t \in I_0 := (t_0 - \varepsilon, t_0 + \varepsilon) \cap I$. Since $T_{\gamma(t)}M$ is an m-dimensional real vector space this implies that the vectors $X_i(t)$ form a basis of $T_{\gamma(t)}M$ for every $t \in I_0$. We express the vector $\nabla X_i(t) \in T_{\gamma(t)}M$ in this basis and denote the coefficients by $a_i^k(t)$ so that

$$\nabla X_i(t) = \sum_{k=1}^m a_i^k(t) X_k(t).$$

The resulting functions $a_i^k : I_0 \to \mathbb{R}$ are smooth. Likewise, if $X : I \to \mathbb{R}^n$ is any vector field along γ then there are smooth functions $\xi^i : I_0 \to \mathbb{R}$ such that

$$X(t) = \sum_{i=1}^m \xi^i(t) X_i(t) \qquad \text{for all } t \in I_0.$$

The derivative of X is given by

$$\dot{X}(t) = \sum_{i=1}^m \left(\dot{\xi}^i(t) X_i(t) + \xi^i(t) \dot{X}_i(t) \right)$$

and the covariant derivative by

$$\begin{aligned}
\nabla X(t) &= \sum_{i=1}^m \left(\dot{\xi}^i(t) X_i(t) + \xi^i(t) \nabla X_i(t) \right) \\
&= \sum_{i=1}^m \dot{\xi}^i(t) X_i(t) + \sum_{i=1}^m \xi^i(t) \sum_{k=1}^m a_i^k(t) X_k(t) \\
&= \sum_{k=1}^m \left(\dot{\xi}^k(t) + \sum_{i=1}^m a_i^k(t) \xi^i(t) \right) X_k(t)
\end{aligned}$$

for $t \in I_0$. Hence $\nabla X(t) = 0$ if and only if

$$\dot{\xi}(t) + A(t)\xi(t) = 0, \qquad A(t) := \begin{pmatrix} a_1^1(t) & \cdots & a_m^1(t) \\ \vdots & & \vdots \\ a_1^m(t) & \cdots & a_m^m(t) \end{pmatrix}.$$

Thus we have translated the equation $\nabla X = 0$ over the interval I_0 into a time dependent linear ordinary differential equation. By a theorem in Analysis II (see [18, Lemma 4.4.3]), this equation has a unique solution for any initial condition at any point in I_0. Thus we have proved that every $t_0 \in I$ is contained in an interval $I_0 \subset I$, open in the relative topology of I, such that, for every $t_1 \in I_0$ and every $v_1 \in T_{\gamma(t_1)}M$, there exists a unique parallel vector field $X : I_0 \to \mathbb{R}^n$ along $\gamma|_{I_0}$ satisfying $X(t_1) = v_1$. We formulate this condition on the interval I_0 as a logical formula:

$$\begin{array}{l}\forall\, t_1 \in I_0 \;\; \forall\, v_1 \in T_{\gamma(t_1)}M \;\; \exists!\, X \in \mathrm{Vect}(\gamma|_{I_0}) \\ \text{such that } \nabla X = 0 \text{ and } X(t_1) = v_1.\end{array} \tag{3.3.1}$$

If two I-open intervals $I_0, I_1 \subset I$ satisfy this condition and have nonempty intersection, then their union $I_0 \cup I_1$ also satisfies (3.3.1). (Prove this!) Now define

$$J := \bigcup \{ I_0 \subset I \mid I_0 \text{ is an } I\text{-open interval, } I_0 \text{ satisfies (3.3.1), } t_0 \in I_0 \}.$$

This interval J satisfies (3.3.1). Moreover, it is nonempty and, by definition, it is open in the relative topology of I. We prove that it is also closed in the relative topology of I. Thus let $(t_i)_{i\in\mathbb{N}}$ be a sequence in J converging to a point $t^* \in I$. By what we have proved above, there exists a constant $\varepsilon > 0$ such that the interval $I^* := (t^* - \varepsilon, t^* + \varepsilon) \cap I$ satisfies (3.3.1). Since the sequence $(t_i)_{i\in\mathbb{N}}$ converges to t^*, there exists an $i \in \mathbb{N}$ such that $t_i \in I^*$. Since $t_i \in J$ there exists an interval $I_0 \subset I$, open in the relative topology of I, that contains t_0 and t_i and satisfies (3.3.1). Hence the interval $I_0 \cup I^*$ is open in the relative topology of I, contains t_0 and t^*, and satisfies (3.3.1). This shows that $t^* \in J$. Thus we have proved that the interval J is nonempty, and open and closed in the relative topology of I. Hence $J = I$ and this proves Theorem 3.3.4. $\square$

Definition 3.3.5 (Parallel Transport). *Let $I \subset \mathbb{R}$ be an interval and let $\gamma : I \to M$ be a smooth curve. For $t_0, t \in I$ we define the map*

$$\Phi_\gamma(t, t_0) : T_{\gamma(t_0)}M \to T_{\gamma(t)}M$$

by $\Phi_\gamma(t, t_0)v_0 := X(t)$ where $X \in \mathrm{Vect}(\gamma)$ is the unique parallel vector field along γ satisfying $X(t_0) = v_0$. The collection of maps $\Phi_\gamma(t, t_0)$ for $t, t_0 \in I$ is called **parallel transport along** *γ.*

Recall the notation

$$\gamma^*TM = \big\{ (s, v) \mid s \in I,\, v \in T_{\gamma(s)}M \big\}$$

for the pullback tangent bundle. This set is a smooth submanifold of $I \times \mathbb{R}^n$. (See Theorem 2.6.8 and Corollary 2.6.11.) The next theorem summarizes the properties of parallel transport. In particular, the last assertion shows that the covariant derivative can be recovered from the parallel transport maps.

Theorem 3.3.6 (Parallel Transport). *Let $\gamma : I \to M$ be a smooth curve on an interval $I \subset \mathbb{R}$.*

(i) *The map $\Phi_\gamma(t,s) : T_{\gamma(s)}M \to T_{\gamma(t)}M$ is linear for all $s,t \in I$.*

(ii) *For all $r,s,t \in I$ we have*

$$\Phi_\gamma(t,s) \circ \Phi_\gamma(s,r) = \Phi_\gamma(t,r), \qquad \Phi_\gamma(t,t) = \mathrm{id}.$$

(iii) *For all $s,t \in I$ and all $v,w \in T_{\gamma(s)}M$ we have*

$$\langle \Phi_\gamma(t,s)v, \Phi_\gamma(t,s)w \rangle = \langle v,w \rangle.$$

Thus $\Phi_\gamma(t,s) : T_{\gamma(s)}M \to T_{\gamma(t)}M$ is an orthogonal transformation.

(iv) *If $J \subset \mathbb{R}$ is an interval and $\sigma : J \to I$ is a smooth map then*

$$\Phi_{\gamma\circ\sigma}(t,s) = \Phi_\gamma(\sigma(t),\sigma(s)).$$

for all $s,t \in J$.

(v) *The map*

$$I \times \gamma^*TM \to \gamma^*TM : (t,(s,v)) \mapsto (t, \Phi_\gamma(t,s)v)$$

is smooth.

(vi) *For all $X \in \mathrm{Vect}(\gamma)$ and $t, t_0 \in I$ we have*

$$\frac{d}{dt}\Phi_\gamma(t_0,t)X(t) = \Phi_\gamma(t_0,t)\nabla X(t).$$

Proof. Assertion (i) holds because the sum of two parallel vector fields along γ is again parallel and the product of a parallel vector field with a constant real number is again parallel. Assertion (ii) follows directly from the uniqueness statement in Theorem 3.3.4.

We prove (iii). Fix a number $s \in I$ and two tangent vectors

$$v, w \in T_{\gamma(s)}M.$$

Define the vector fields $X, Y \in \mathrm{Vect}(\gamma)$ along γ by

$$X(t) := \Phi_\gamma(t,s)v, \qquad Y(t) := \Phi_\gamma(t,s)w.$$

These vector fields are parallel. Thus, by equation (3.2.5) in Lemma 3.2.4, we have

$$\frac{d}{dt}\langle X,Y \rangle = \langle \nabla X, Y \rangle + \langle X, \nabla Y \rangle = 0.$$

Hence the function $I \to \mathbb{R} : t \mapsto \langle X(t), Y(t) \rangle$ is constant and this proves (iii).

We prove (iv). Fix an element $s \in J$ and a tangent vector $v \in T_{\gamma(\sigma(s))}M$. Define the vector field X along γ by

$$X(t) := \Phi_\gamma(t, \sigma(s))v$$

for $t \in I$. Thus X is the unique parallel vector field along γ that satisfies

$$X(\alpha(s)) = v.$$

Denote

$$\widetilde{\gamma} := \gamma \circ \sigma : J \to M, \qquad \widetilde{X} := X \circ \sigma : I \to \mathbb{R}^n$$

Then $\widetilde{X}$ is a vector field along $\widetilde{\gamma}$ and, by the chain rule, we have

$$\frac{d}{dt}\widetilde{X}(t) = \frac{d}{dt}X(\sigma(t)) = \dot{\sigma}(t)\dot{X}(\sigma(t)).$$

Projecting orthogonally onto the tangent space $T_{\gamma(\sigma(t))}M$ we obtain

$$\nabla\widetilde{X}(t) = \dot{\sigma}(t)\nabla X(\sigma(t)) = 0$$

for every $t \in J$. Hence $\widetilde{X}$ is the unique parallel vector field along $\widetilde{\gamma}$ that satisfies $\widetilde{X}(s) = v$. Thus

$$\Phi_{\widetilde{\gamma}}(t, s)v = \widetilde{X}(t) = X(\sigma(t)) = \Phi_\gamma(\sigma(t), \sigma(s))v.$$

This proves (iv).

We prove (v). Fix a point $t_0 \in I$, choose an orthonormal basis $e_1, \dots, e_m$ of $T_{\gamma(t_0)}M$, and define $X_i(t) := \Phi_\gamma(t, t_0)e_i$ for $t \in I$ and $i = 1, \dots, m$. Thus $X_i \in \mathrm{Vect}(\gamma)$ is the unique parallel vector field along γ such that $X_i(t_0) = e_i$. Then by (iii) we have

$$\langle X_i(t), X_j(t)\rangle = \delta_{ij}$$

for all $i, j \in \{1, \dots, m\}$ and all $t \in I$. Hence the vectors $X_1(t), \dots, X_m(t)$ form an orthonormal basis of $T_{\gamma(t)}M$ for every $t \in I$. This implies that, for each $s \in I$ and each tangent vector $v \in T_{\gamma(s)}M$, we have

$$v = \sum_{i=1}^m \langle X_i(s), v\rangle X_i(s).$$

Since each vector field X_i is parallel it satisfies $X_i(t) = \Phi_\gamma(t, s)X_i(s)$. Hence

$$\Phi_\gamma(t, s)v = \sum_{i=1}^m \langle X_i(s), v\rangle X_i(t) \tag{3.3.2}$$

for all $s, t \in I$ and $v \in T_{\gamma(s)}M$. This proves (v)

We prove (vi). Let $X_1, \dots, X_m \in \mathrm{Vect}(\gamma)$ be as in the proof of (v). Thus every vector field X along γ can be written in the form

$$X(t) = \sum_{i=1}^m \xi^i(t) X_i(t), \qquad \xi^i(t) := \langle X_i(t), X(t) \rangle.$$

Since the vector fields X_i are parallel we have

$$\nabla X(t) = \sum_{i=1}^m \dot{\xi}^i(t) X_i(t)$$

for all $t \in I$. Hence

$$\Phi_\gamma(t_0, t) X(t) = \sum_{i=1}^m \xi^i(t) X_i(t_0), \qquad \Phi_\gamma(t_0, t) \nabla X(t) = \sum_{i=1}^m \dot{\xi}^i(t) X_i(t_0).$$

Evidently, the derivative of the first sum with respect to t is equal to the second sum. This proves (vi) and the theorem. $\square$

Remark 3.3.7. For $s, t \in I$ we can think of the linear map

$$\Phi_\gamma(t, s) \Pi(\gamma(s)) : \mathbb{R}^n \to T_{\gamma(t)} M \subset \mathbb{R}^n$$

as a real $n \times n$ matrix. The formula (3.3.2) in the proof of (v) shows that this matrix can be expressed in the form

$$\Phi_\gamma(t, s) \Pi(\gamma(s)) = \sum_{i=1}^m X_i(t) X_i(s)^{\mathsf{T}} \in \mathbb{R}^{n \times n}.$$

The right hand side defines a smooth matrix valued function on $I \times I$ and this is equivalent to the assertion in (v).

Remark 3.3.8. It follows from assertions (ii) and (iii) in Theorem 3.3.6 that

$$\Phi_\gamma(t, s)^{-1} = \Phi_\gamma(s, t) = \Phi_\gamma(t, s)^*$$

for all $s, t \in I$. Here the linear map $\Phi_\gamma(t, s)^* : T_{\gamma(t)} M \to T_{\gamma(s)} M$ is understood as the adjoint operator of $\Phi_\gamma(t, s) : T_{\gamma(s)} M \to T_{\gamma(t)} M$ with respect to the inner products on the two subspaces of $\mathbb{R}^n$ inherited from the Euclidean inner product on the ambient space.

The two theorems in this section carry over verbatim to any smooth vector bundle $E \subset M \times \mathbb{R}^n$ over a manifold. As in the case of the tangent bundle one can define the covariant derivative of a section of E along γ as the orthogonal projection of the ordinary derivative in the ambient space $\mathbb{R}^n$ onto the fiber $E_{\gamma(t)}$. Instead of *parallel vector fields* one then speaks about *horizontal sections* and one proves as in Theorem 3.3.4 that there is a unique horizontal section along γ through any point in any of the fibers $E_{\gamma(t_0)}$. This gives parallel transport maps from $E_{\gamma(s)}$ to $E_{\gamma(t)}$ for any pair $s, t \in I$ and Theorem 3.3.6 carries over verbatim to all vector bundles $E \subset M \times \mathbb{R}^n$. We spell this out in more detail in the case where $E = TM^\perp \subset M \times \mathbb{R}^n$ is the normal bundle of M.

Let $\gamma : I \to M$ be a smooth curve. A **normal vector field along** γ is a smooth map $Y : I \to \mathbb{R}^n$ such that $Y(t) \perp T_{\gamma(t)}M$ for every $t \in I$. The set of normal vector fields along γ will be denoted by

$$\mathrm{Vect}^\perp(\gamma) := \left\{ Y : I \to \mathbb{R}^n \,|\, Y \text{ is smooth and } Y(t) \perp T_{\gamma(t)}M \text{ for all } t \in I \right\}.$$

This is again a real vector space. The **covariant derivative** of a normal vector field $Y \in \mathrm{Vect}^\perp(\gamma)$ at $t \in I$ is defined as the orthogonal projection of the ordinary derivative onto the orthogonal complement of $T_{\gamma(t)}M$ and will be denoted by

$$\nabla^\perp Y(t) := \big(\mathbb{1} - \Pi(\gamma(t))\big)\dot{Y}(t). \tag{3.3.3}$$

Thus the covariant derivative defines a linear operator

$$\nabla^\perp : \mathrm{Vect}^\perp(\gamma) \to \mathrm{Vect}^\perp(\gamma).$$

There is a version of the Gauß–Weingarten formula for the covariant derivative of a normal vector field. This is the content of the next lemma.

Lemma 3.3.9. *Let $M \subset \mathbb{R}^n$ be a smooth m-manifold. For $p \in M$ and $u \in T_pM$ define the linear map $h_p(u) : T_pM \to T_pM^\perp$ by*

$$h_p(u)v := h_p(u, v) = \big(d\Pi(p)u\big)v \tag{3.3.4}$$

for $v \in T_pM$. Then the following holds.

(i) *The adjoint operator $h_p(u)^* : T_pM^\perp \to T_pM$ is given by*

$$h_p(u)^*w = \big(d\Pi(p)u\big)w, \qquad w \in T_pM^\perp. \tag{3.3.5}$$

(ii) *If $I \subset \mathbb{R}$ is an interval, $\gamma : I \to M$ is a smooth curve, and $Y \in \mathrm{Vect}^\perp(\gamma)$ then the derivative of Y satisfies the* **Gauß–Weingarten formula**

$$\dot{Y}(t) = \nabla^\perp Y(t) - h_{\gamma(t)}(\dot{\gamma}(t))^*Y(t). \tag{3.3.6}$$

Proof. Since $\Pi(p) \in \mathbb{R}^{n\times n}$ is a symmetric matrix for every $p \in M$ so is the matrix $d\Pi(p)u$ for every $p \in M$ and every $u \in T_pM$. Hence

$$\begin{aligned} \langle v, h_p(u)^* w\rangle &= \langle h_p(u)v, w\rangle \\ &= \big\langle \big(d\Pi(p)u\big)v, w\big\rangle \\ &= \big\langle v, \big(d\Pi(p)u\big)w\big\rangle \end{aligned}$$

for every $v \in T_pM$ and every $w \in T_pM^\perp$. This proves (i).

To prove (ii) we observe that, for $Y \in \mathrm{Vect}^\perp(\gamma)$ and $t \in I$, we have

$$\Pi(\gamma(t))Y(t) = 0.$$

Differentiating this identity we obtain

$$\Pi(\gamma(t))\dot{Y}(t) + \big(d\Pi(\gamma(t))\dot\gamma(t)\big)Y(t) = 0$$

and hence

$$\begin{aligned} \dot{Y}(t) &= \dot{Y}(t) - \Pi(\gamma(t))\dot{Y}(t) - \big(d\Pi(\gamma(t))\dot\gamma(t)\big)Y(t) \\ &= \nabla^\perp Y(t) - h_{\gamma(t)}(\dot\gamma(t))^* Y(t) \end{aligned}$$

for $t \in I$. Here the last equation follows from (i) and the definition of $\nabla^\perp$. This proves Lemma 3.3.9. □

Theorem 3.3.4 and its proof carry over to the normal bundle $TM^\perp$. Thus, if $\gamma : I \to M$ is a smooth curve then, for all $s \in I$ and $w \in T_{\gamma(s)}M^\perp$, there is a unique normal vector field $Y \in \mathrm{Vect}^\perp(\gamma)$ such that

$$\nabla^\perp Y \equiv 0, \qquad Y(s) = w.$$

This gives rise to parallel transport maps

$$\Phi^\perp_\gamma(t,s) : T_{\gamma(s)}M^\perp \to T_{\gamma(t)}M^\perp$$

defined by

$$\Phi^\perp_\gamma(t,s)w := Y(t)$$

for $s, t \in I$ and $w \in T_{\gamma(s)}M^\perp$, where Y is the unique normal vector field along γ satisfying $\nabla^\perp Y \equiv 0$ and $Y(s) = w$. These parallel transport maps satisfy exactly the same conditions that have been spelled out in Theorem 3.3.6 for the tangent bundle and the proof carries over verbatim to the present setting.

3.4 Frame Bundle

Frames of a Vector Space

Let V be an m-dimensional real vector space. A **frame of** V is a basis $e_1, \dots, e_m$ of V. It determines a vector space isomorphism $e : \mathbb{R}^m \to V$ via

$$e\xi := \sum_{i=1}^{m} \xi^i e_i, \qquad \xi = (\xi^1, \dots, \xi^m) \in \mathbb{R}^m.$$

Conversely, each isomorphism $e : \mathbb{R}^m \to V$ determines a basis $e_1, \dots, e_m$ of V via $e_i = e(0, \dots, 0, 1, 0 \dots, 0)$, where the coordinate 1 appears in the ith place. The set of vector space isomorphisms from $\mathbb{R}^m$ to V will be denoted by

$$\mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V) := \{e : \mathbb{R}^m \to V \,|\, e \text{ is a vector space isomorphism}\}.$$

The general linear group $\mathrm{GL}(m) = \mathrm{GL}(m, \mathbb{R})$ (of nonsingular real $m \times m$-matrices) acts on this space by composition on the right via

$$\mathrm{GL}(m) \times \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V) \to \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V) : (a, e) \mapsto a^*e := e \circ a.$$

This is a **contravariant group action** in that

$$a^*b^*e = (ba)^*e, \qquad \mathbb{1}^*e = e$$

for $a, b \in \mathrm{GL}(m)$ and $e \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V)$. Moreover, the action is **free**, i.e. for all $a \in \mathrm{GL}(m)$ and $e \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V)$, we have

$$a^*e = e \qquad \Longleftrightarrow \qquad a = \mathbb{1}.$$

It is **transitive** in that for all $e, e' \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V)$ there is a group element $a \in \mathrm{GL}(m)$ such that $e' = a^*e$. Thus we can identify the space $\mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V)$ with the group $\mathrm{GL}(m)$ via the bijection

$$\mathrm{GL}(m) \to \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V) : a \mapsto a^*e_0$$

induced by a fixed element $e_0 \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V)$. This identification is not canonical; it depends on the choice of e_0. The space $\mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, V)$ admits a bijection to a group but is not itself a group.

Frame Bundle

Definition 3.4.1. *Let $M \subset \mathbb{R}^n$ be a smooth m-dimensional submanifold. The* **frame bundle** *of M is the set*

$$\mathcal{F}(M) := \{(p,e) \mid p \in M,\ e \in \mathcal{F}(M)_p\}, \tag{3.4.1}$$

where $\mathcal{F}(M)_p$ is the space of frames of the tangent space at p, i.e.

$$\mathcal{F}(M)_p := \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_pM).$$

Define a right action of $\mathrm{GL}(m)$ on $\mathcal{F}(M)$ by

$$a^*(p,e) := (p, a^*e) = (p, e \circ a) \tag{3.4.2}$$

for $a \in \mathrm{GL}(m)$ and $(p,e) \in \mathcal{F}(M)$.

One can think of a frame $e \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_pM)$ as a linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$ whose image is T_pM and hence as an $n \times m$-matrix of rank m. The basis of T_pM associated to this frame is given by the columns of the matrix $e \in \mathbb{R}^{n\times m}$. Thus the frame bundle $\mathcal{F}(M)$ of an embedded manifold $M \subset \mathbb{R}^n$ is a subset of the Euclidean space $\mathbb{R}^n \times \mathbb{R}^{n\times m}$.

Lemma 3.4.2. *The frame bundle*

$$\mathcal{F}(M) \subset \mathbb{R}^n \times \mathbb{R}^{n\times m}$$

is a smooth manifold of dimension $m + m^2$, the group action

$$\mathrm{GL}(m) \times \mathcal{F}(M) \to \mathcal{F}(M) : (a,(p,e)) \mapsto a^*(p,e)$$

is smooth, and the projection

$$\pi : \mathcal{F}(M) \to M$$

defined by $\pi(p,e) := p$ for $(p,e) \in \mathcal{F}(M)$ is a surjective submersion. The orbits of the $\mathrm{GL}(m)$-action on $\mathcal{F}(M)$ are the fibers of this projection, i.e.

$$\mathrm{GL}(m)^*(p,e) = \pi^{-1}(p) \cong \mathcal{F}(M)_p$$

for $(p,e) \in \mathcal{F}(M)$, and the group $\mathrm{GL}(m)$ acts freely and transitively on each of these fibers.

Proof. Let $U \subset M$ be an M-open set. A **moving frame** over U is a finite sequence of smooth vector fields $E_1, \dots, E_m \in \mathrm{Vect}(U)$ of m vector fields on U such that the vectors $E_1(p), \dots, E_m(p)$ form a basis of T_pM for each $p \in U$. Any such moving frame gives a bijection

$$U \times \mathrm{GL}(m) \to \mathcal{F}(U) : (p, a) \mapsto a^*(p, E(p)) = (p, E(p) \circ a),$$

where

$$E(p) := (E_1(p), \dots, E_m(p)) \in \mathcal{F}(M)_p$$

for $p \in U$. This bijection (when composed with a parametrization of U) gives a parametrization of the open set $\mathcal{F}(U)$ in $\mathcal{F}(M)$. The assertions of the lemma then follow from the fact that the diagram

$$\begin{tikzcd}
U \times \mathrm{GL}(m) \arrow[rr] \arrow[dr, "\mathrm{pr}_1"'] & & \mathcal{F}(U) \arrow[dl, "\pi"] \\
& U &
\end{tikzcd}$$

commutes. More precisely, suppose that there exists a coordinate chart

$$\phi : U \to \Omega$$

with values in an open set $\Omega \subset \mathbb{R}^m$, and denote its inverse by

$$\psi := \phi^{-1} : \Omega \to U.$$

Then the open set

$$\mathcal{F}(U) = \pi^{-1}(U) = \{(p, e) \in \mathcal{F}(M) \,|\, p \in U\} = (U \times \mathbb{R}^{n \times m}) \cap \mathcal{F}(M)$$

is parametrized by the map

$$\Omega \times \mathrm{GL}(m) \to \mathcal{F}(U) : (x, a) \mapsto \big(\psi(x), d\psi(x) \circ a\big).$$

This map is amooth and so is its inverse

$$\mathcal{F}(U) \to \Omega \times \mathrm{GL}(m) : (p, e) \mapsto \big(\phi(p), d\phi(p) \circ e\big).$$

These are the desired coordinate chart on $\mathcal{F}(M)$. Thus $\mathcal{F}(M)$ is a smooth manifold of dimension $m + m^2$. Moreover, in these coordinates the projection $\pi : \mathcal{F}(U) \to U$ is the map $\Omega \times \mathrm{GL}(m) \to \Omega : (x, a) \mapsto x$ and so π is a submersion. The remaining assertions follow directly from the definitions and this proves Lemma 3.4.2. $\square$

The frame bundle $\mathcal{F}(M)$ is a **principal bundle** over M with **structure group** $\mathrm{GL}(m)$. More generally, a principal bundle over a manifold B with structure group G is a smooth manifold P equipped with a surjective submersion $\pi : P \to B$ and a smooth contravariant action

$$\mathrm{G} \times P \to P : (g, p) \mapsto pg$$

by a Lie group G such that $\pi(pg) = \pi(p)$ for all $p \in P$ and $g \in \mathrm{G}$ and such that the group G acts freely and transitively on the fiber $P_b := \pi^{-1}(b)$ for each $b \in B$. In this book we shall mostly be concerned with the frame bundle of a manifold M and the orthonormal frame bundle.

Definition 3.4.3. *The* **orthonormal frame bundle of** M *is the set*

$$\mathcal{O}(M) := \left\{ (p, e) \in \mathbb{R}^n \times \mathbb{R}^{n \times m} \,|\, p \in M,\, \mathrm{im}\, e = T_pM,\, e^{\mathsf{T}} e = \mathbb{1}_{m \times m} \right\}.$$

If we denote by $e_i := e(0, \dots, 0, 1, 0, \dots, 0)$ (with 1 as the ith argument) the basis of T_pM induced by the isomorphism $e : \mathbb{R}^m \to T_pM$ then we have

$$e^{\mathsf{T}} e = \mathbb{1} \quad \Longleftrightarrow \quad \langle e_i, e_j \rangle = \delta_{ij} \quad \Longleftrightarrow \quad \begin{array}{l} e_1, \dots, e_m \text{ is an} \\ \text{orthonormal basis.} \end{array}$$

Thus $\mathcal{O}(M)$ is the bundle of orthonormal frames of the tangent spaces T_pM or the bundle of orthogonal isomorphisms $e : \mathbb{R}^m \to T_pM$. It is a principal bundle over M with structure group $\mathrm{O}(m)$.

Exercise 3.4.4. Prove that $\mathcal{O}(M)$ is a submanifold of $\mathcal{F}(M)$ and that the obvious projection $\pi : \mathcal{O}(M) \to M$ is a submersion. Prove that the action of $\mathrm{GL}(m)$ on $\mathcal{F}(M)$ restricts to an action of the orthogonal group $\mathrm{O}(m)$ on $\mathcal{O}(M)$ whose orbits are the fibers

$$\begin{aligned} \mathcal{O}(M)_p &:= \left\{ e \in \mathbb{R}^{n \times m} \,\middle|\, (p, e) \in \mathcal{O}(M) \right\} \\ &= \left\{ e \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_pM) \,\middle|\, e^{\mathsf{T}} e = \mathbb{1} \right\}. \end{aligned}$$

Hint: If $\phi : U \to \Omega$ is a coordinate chart on M with inverse $\psi : \Omega \to U$ then

$$e_x := d\psi(x)(d\psi(x)^{\mathsf{T}} d\psi(x))^{-1/2} : \mathbb{R}^m \to T_{\psi(x)}M$$

is an orthonormal frame of the tangent space $T_{\psi(x)}M$ for every $x \in \Omega$.

Horizontal Lifts

We have seen in Lemma 3.4.2 that the frame bundle $\mathcal{F}(M)$ is a smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^{n\times m}$. Next we examine the tangent space of $\mathcal{F}(M)$ at a point $(p,e) \in \mathcal{F}(M)$. By Definition 2.2.1, this tangent space is given by

$$T_{(p,e)}\mathcal{F}(M) = \left\{ (\dot\gamma(0), \dot e(0)) \,\middle|\, \begin{array}{l} \mathbb{R} \to \mathcal{F}(M) : t \mapsto (\gamma(t), e(t)) \\ \text{is a smooth curve satisfying} \\ \gamma(0) = p \text{ and } e(0) = e \end{array} \right\}.$$

It is convenient to consider two kinds of curves in $\mathcal{F}(M)$, namely vertical curves with constant projections to M and horizontal lifts of curves in M. We denote by $\mathcal{L}(\mathbb{R}^m, T_pM)$ the space of linear maps from $\mathbb{R}^m$ to T_pM.

Definition 3.4.5 (**Horizontal Lift**). *Let $\gamma : \mathbb{R} \to M$ be a smooth curve. A smooth curve $\beta : \mathbb{R} \to \mathcal{F}(M)$ is called a* **lift of** *γ if*

$$\pi \circ \beta = \gamma.$$

Any such lift has the form $\beta(t) = (\gamma(t), e(t))$ with $e(t) \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_{\gamma(t)}M)$. The associated curve of frames $e(t)$ of the tangent spaces $T_{\gamma(t)}M$ is called a **moving frame along** *γ. A curve $\beta(t) = (\gamma(t), e(t)) \in \mathcal{F}(M)$ is called* **horizontal** *or a* **horizontal lift of** *γ if the vector field $X(t) := e(t)\xi$ along γ is parallel for every $\xi \in \mathbb{R}^m$. Thus a horizontal lift of γ has the form*

$$\beta(t) = (\gamma(t), \Phi_\gamma(t,0)e) \tag{3.4.3}$$

for some $e \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_{\gamma(0)}M)$.

Lemma 3.4.6. **(i)** *The tangent space of $\mathcal{F}(M)$ at $(p,e) \in \mathcal{F}(M)$ is the direct sum*

$$T_{(p,e)}\mathcal{F}(M) = H_{(p,e)} \oplus V_{(p,e)}$$

of the **horizontal space**

$$H_{(p,e)} := \bigl\{ (v, h_p(v)e) \,\big|\, v \in T_pM \bigr\} \tag{3.4.4}$$

and the **vertical space**

$$V_{(p,e)} := \{0\} \times \mathcal{L}(\mathbb{R}^m, T_pM). \tag{3.4.5}$$

(ii) *The vertical space $V_{(p,e)}$ at $(p,e) \in \mathcal{F}(M)$ is the kernel of the linear map*

$$d\pi(p,e) : T_{(p,e)}\mathcal{F}(M) \to T_pM.$$

(iii) *A curve $\beta : \mathbb{R} \to \mathcal{F}(M)$ is horizontal if and only if it is tangent to the horizontal spaces, i.e. $\dot\beta(t) \in H_{\beta(t)}$ for every $t \in \mathbb{R}$.*

(iv) *If $\beta : \mathbb{R} \to \mathcal{F}(M)$ is a horizontal curve so is $a^*\beta$ for every $a \in \mathrm{GL}(m)$.*

Proof. The proof has four steps.

Step 1. *Let $(p,e) \in \mathcal{F}(M)$. Then $V_{(p,e)} = \ker d\pi(p,e) \subset T_{(p,e)}\mathcal{F}(M)$.*

Since π is a submersion, the fiber $\pi^{-1}(p)$ is a submanifold of $\mathcal{F}(M)$ by Theorem 2.2.17 and $T_{(p,e)}\pi^{-1}(p) = \ker d\pi(p,e)$. Now let $(\widehat{p},\widehat{e}) \in \ker d\pi(p,e)$. Then there exists a vertical curve $\beta : \mathbb{R} \to \mathcal{F}(M)$ with $\pi \circ \beta \equiv p$ such that

$$\beta(0) = (p,e), \qquad \dot\beta(0) = (\widehat{p},\widehat{e}).$$

Any such curve has the form $\beta(t) := (p, e(t))$ where $e(t) \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_pM)$. Hence $\widehat{p} = 0$ and $\widehat{e} = \dot e(0) \in \mathcal{L}(\mathbb{R}^m, T_pM)$. This shows that

$$\ker d\pi(p,e) \subset V_{(p,e)}. \tag{3.4.6}$$

Conversely, for every $\widehat{e} \in \mathcal{L}(\mathbb{R}^m, T_pM)$, the curve

$$\mathbb{R} \to \mathcal{L}(\mathbb{R}^m, T_pM) : t \mapsto e(t) := e + t\widehat{e}$$

takes values in the open set $\mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_pM)$ for t sufficiently small and hence $\beta(t) := (p, e(t))$ is a vertical curve with $\dot\beta(0) = (0, \widehat{e})$. Thus

$$V_{(p,e)} \subset \ker d\pi(p,e) \subset T_{(p,e)}\mathcal{F}(M). \tag{3.4.7}$$

Combining (3.4.6) and (3.4.7) we obtain Step 1 and part (ii).

Step 2. *Let $(p,e) \in \mathcal{F}(M)$. Then $H_{(p,e)} \subset T_{(p,e)}\mathcal{F}(M)$. Moreover, every horizontal curve $\beta : \mathbb{R} \to \mathcal{F}(M)$ satisfies $\dot\beta(t) \in H_{\beta(t)}$ for all $t \in \mathbb{R}$.*

Fix a tangent vector $v \in T_pM$, let $\gamma : \mathbb{R} \to M$ be a smooth curve satisfying $\gamma(0) = p$ and $\dot\gamma(0) = v$, and let $\beta : \mathbb{R} \to \mathcal{F}(M)$ be the horizontal lift of γ with $\beta(0) = (p,e)$. Then

$$\beta(t) = (\gamma(t), e(t)), \qquad e(t) := \Phi_\gamma(t,0)e.$$

Fix a vector $\xi \in \mathbb{R}^m$ and consider the vector field

$$X(t) := e(t)\xi = \Phi_\gamma(t,0)e\xi$$

along γ. This vector field is parallel and hence, by the Gauß–Weingarten formula, it satisfies

$$\dot e(0)\xi = \dot X(0) = h_{\gamma(0)}(\dot\gamma(0), X(0)) = h_p(v)e\xi.$$

Here we have used (3.3.4). Thus

$$(v, h_p(v)e) = (\dot\gamma(0), \dot e(0)) = \dot\beta(0) \in T_{\beta(0)}\mathcal{F}(M) = T_{(p,e)}\mathcal{F}(M).$$

and so $H_{(p,e)} \subset T_{(p,e)}\mathcal{F}(M)$. Moreover, $\dot\beta(0) = (v, h_p(v)e) \in H_{(p,e)} = H_{\beta(0)}$ and this proves Step 2.

Step 3. *We prove part (i).*

We have $V_{(p,e)} \subset T_{(p,e)}\mathcal{F}(M)$ by Step 1 and $H_{(p,e)} \subset T_{(p,e)}\mathcal{F}(M)$ by Step 2. Moreover $H_{(p,e)} \cap V_{(p,e)} = \{0\}$ and so $T_{(p,e)}\mathcal{F}(M) = H_{(p,e)} \oplus V_{(p,e)}$ for dimensional reasons. This proves Step 3.

Step 4. *We prove parts (iii) and (iv).*

By Step 2 every horizontal curve $\beta : \mathbb{R} \to \mathcal{F}(M)$ satisfies $\dot\beta(t) \in H_{\beta(t)}$. Conversely, let $\mathbb{R} \to \mathcal{F}(M) : t \mapsto \beta(t) = (\gamma(t), e(t))$ be a smooth curve satisfying $\dot\beta(t) \in H_{\beta(t)}$ for all t. Then $\dot e(t) = h_{\gamma(t)}(\dot\gamma(t))e(t)$ for all t. By the Gauß–Weingarten formula (3.2.2) this implies that the vector field $X(t) = e(t)\xi$ along γ is parallel for every $\xi \in \mathbb{R}^m$, so β is horizontal. This proves part (iii). Part (iv) follows from (iii) and the fact that the horizontal tangent bundle $H \subset T\mathcal{F}(M)$ is invariant under the induced action of the group $\mathrm{GL}(m)$ on $T\mathcal{F}(M)$. This proves Lemma 3.4.6. $\square$

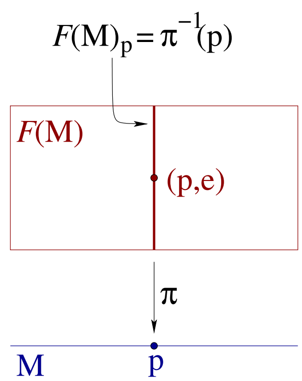


Figure 3.3: The frame bundle.

The reason for the terminology introduced in Definition 3.4.5 is that one draws the extremely crude picture of the frame bundle displayed in Figure 3.3. One thinks of $\mathcal{F}(M)$ as "lying over" M. One would then represent the equation $\gamma = \pi \circ \beta$ by the following commutative diagram:

$$\begin{tikzcd} & \mathcal{F}(M) \arrow[d, "\pi"] \\ \mathbb{R} \arrow[ur, "\beta"] \arrow[r, "\gamma"'] & M \end{tikzcd}\ ;$$

hence the word "lift". The vertical space is tangent to the vertical line in Figure 3.3 while the horizontal space is transverse to the vertical space. This crude imagery can be extremely helpful.

Exercise 3.4.7. The group $\mathrm{GL}(m)$ acts on $\mathcal{F}(M)$ by diffeomorphisms. Thus for each $a \in \mathrm{GL}(m)$ the map

$$\mathcal{F}(M) \to \mathcal{F}(M) : (p,e) \mapsto a^*(p,e) = (p, e \circ a)$$

is a diffeomorphism of $\mathcal{F}(M)$. The derivative of this diffeomorphism is a diffeomorphism of the tangent bundle $T\mathcal{F}(M)$ and this is called the induced action of $\mathrm{GL}(m)$ on $T\mathcal{F}(M)$. Prove that the horizontal and vertical subbundles are invariant under the induced action of $\mathrm{GL}(m)$ on $T\mathcal{F}(M)$.

Exercise 3.4.8. Prove that $H_{(p,e)} \subset T_{(p,e)}\mathcal{O}(M)$ and that

$$T_{(p,e)}\mathcal{O}(M) = H_{(p,e)} \oplus V'_{(p,e)}, \qquad V'_{(p,e)} := V_{(p,e)} \cap T_{(p,e)}\mathcal{O}(M),$$

for every $(p,e) \in \mathcal{O}(M)$.

The following definition introduces an important class of vector fields on the frame bundle that will play a central role in Section 3.5. They will be used to prove the Development Theorem 3.5.21 in §3.5.4 below.

Definition 3.4.9 (Basic Vector Field). *Every vector $\xi \in \mathbb{R}^m$ determines a vector field $B_\xi \in \mathrm{Vect}(\mathcal{F}(M))$ defined by*

$$B_\xi(p,e) := \bigl(e\xi, h_p(e\xi)e\bigr) \tag{3.4.8}$$

for $(p,e) \in \mathcal{F}(M)$. This vector field is horizontal, i.e.

$$B_\xi(p,e) \in H_{(p,e)},$$

and projects to $e\xi$, i.e.

$$d\pi(p,e)B_\xi(p,e) = e\xi$$

for all $(p,e) \in \mathcal{F}(M)$. These two conditions determine the vector field B_ξ uniquely. It is called the **basic vector field** *corresponding to ξ.*

Exercise 3.4.10. (i) Prove that every basic vector field $B_\xi \in \mathrm{Vect}(\mathcal{F}(M))$ is tangent to the orthonormal frame bundle $\mathcal{O}(M)$.

(ii) Let $\mathbb{R} \to \mathcal{F}(M) : t \mapsto (\gamma(t), e(t))$ be an integral curve of the vector field B_ξ and $a \in \mathrm{GL}(m)$. Prove that $\mathbb{R} \to \mathcal{F}(M) : t \mapsto a^*\beta(t) = (\gamma(t), a^*e(t))$ is an integral curve of $B_{a^{-1}\xi}$.

(iii) Prove that the vector field $B_\xi \in \mathrm{Vect}(\mathcal{F}(M))$ is complete for all $\xi \in \mathbb{R}^m$ if and only if the restricted vector field $B_\xi|_{\mathcal{O}(M)} \in \mathrm{Vect}(\mathcal{O}(M))$ on the orthonormal frame bundle is complete for all $\xi \in \mathbb{R}^m$.

Definition 3.4.11. *The manifold M is called* **complete** *if, for every smooth curve $\xi : \mathbb{R} \to \mathbb{R}^m$ and every element $(p_0, e_0) \in \mathcal{F}(M)$, there exists a smooth curve $\beta : \mathbb{R} \to \mathcal{F}(M)$ such that $\beta(0) = (p_0, e_0)$ and $\dot\beta(t) = B_{\xi(t)}(\beta(t))$ for all $t \in \mathbb{R}$.*

3.5 Motions and Developments

Our aim in this sections is to define motion without sliding, twisting, or wobbling. This is the motion that results when a heavy object is rolled, with a minimum of friction, along the floor. It is also the motion of the large snowball a child creates as it rolls it into the bottom part of a snowman.

We shall eventually justify mathematically the physical intuition that either of the curves of contact in such ideal rolling may be specified arbitrarily; the other is then determined uniquely. Thus for example the heavy object may be rolled along an arbitrary curve on the floor; if that curve is marked in wet ink another curve will be traced in the object. Conversely if a curve is marked in wet ink on the object, the object may be rolled so as to trace a curve on the floor. However, if both curves are prescribed, it will be necessary to slide the object as it is being rolled if one wants to keep the curves in contact.

We assume throughout this section that M and M' are two m-dimensional submanifolds of $\mathbb{R}^n$. Objects on M' will be denoted by the same letter as the corresponding objects on M with primes affixed. Thus for example $\Pi'(p') \in \mathbb{R}^{n\times n}$ denotes the orthogonal projection of $\mathbb{R}^n$ onto the tangent space $T_{p'}M'$, ∇' denotes the covariant derivative of a vector field along a curve in M', and $\Phi'_{\gamma'}$ denotes parallel transport along a curve in M'.

3.5.1 Motion

Definition 3.5.1. *A* **motion of M along M' (on an interval $I \subset \mathbb{R}$)** *is a triple (Ψ, γ, γ') of smooth maps*

$$\Psi : I \to \mathrm{O}(n), \qquad \gamma : I \to M, \qquad \gamma' : I \to M'$$

such that

$$\Psi(t)T_{\gamma(t)}M = T_{\gamma'(t)}M' \qquad \forall\, t \in I.$$

Note that a motion also matches normal vectors, i.e.

$$\Psi(t)T_{\gamma(t)}M^\perp = T_{\gamma'(t)}M'^\perp \qquad \forall\, t \in I.$$

Remark 3.5.2. Associated to a motion (Ψ, γ, γ') of M along M' is a family of (affine) isometries $\psi_t : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\psi_t(p) := \gamma'(t) + \Psi(t)\big(p - \gamma(t)\big) \tag{3.5.1}$$

for $t \in I$ and $p \in \mathbb{R}^n$. These isometries satisfy

$$\psi_t(\gamma(t)) = \gamma'(t), \qquad d\psi_t(\gamma(t))T_{\gamma(t)}M = T_{\gamma'(t)}M' \qquad \forall\, t \in I.$$

Remark 3.5.3. There are three operations on motions.

Reparametrization. If (Ψ, γ, γ') is a motion of M along M' on an interval $I \subset \mathbb{R}$ and $\sigma : J \to I$ is a smooth map between intervals then the triple

$$(\Psi \circ \sigma, \gamma \circ \sigma, \gamma' \circ \sigma)$$

is a motion of M along M' on the interval J.

Inversion. If (Ψ, γ, γ') is a motion of M along M' then

$$(\Psi^{-1}, \gamma', \gamma)$$

is a motion of M' along M.

Composition. If (Ψ, γ, γ') is a motion of M along M' on an interval I and $(\Psi', \gamma', \gamma'')$ is a motion of M' along M'' on the same interval, then

$$(\Psi'\Psi, \gamma, \gamma'')$$

is a motion of M along M''.

We now give the three simplest examples of “bad” motions; i.e. motions which do not satisfy the concepts we are about to define. In all three of these examples, p is a point of M and M' is the affine tangent space to M at p:

$$M' := p + T_pM = \{p + v \,|\, v \in T_pM\}.$$

Example 3.5.4 (Pure Sliding). Take a nonzero tangent vector $v \in T_pM$ and let

$$\gamma(t) := p, \qquad \gamma'(t) = p + tv, \qquad \Psi(t) := \mathbb{1}.$$

Then $\dot\gamma(t) = 0$, $\dot\gamma'(t) = v \neq 0$, and so

$$\Psi(t)\dot\gamma(t) \neq \dot\gamma'(t).$$

(See Figure 3.4.)

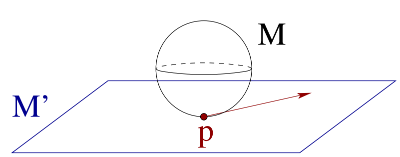


Figure 3.4: Pure sliding.

Example 3.5.5 (Pure Twisting). Let γ and γ' be the constant curves

$$\gamma(t) = \gamma'(t) = p$$

and take $\Psi(t)$ to be the identity on $T_pM^\perp$ and any curve of rotations on the tangent space T_pM. As a concrete example with $m = 2$ and $n = 3$ one can take M to be the sphere of radius one centered at the point $(0, 1, 0)$ and p to be the origin:

$$M := \left\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + (y-1)^2 + z^2 = 1\right\}, \qquad p := (0, 0, 0).$$

Then M' is the (x, z)-plane and $A(t)$ is any curve of rotations in the (x, z)-plane, i.e. about the y-axis $T_pM^\perp$. (See Figure 3.5.)

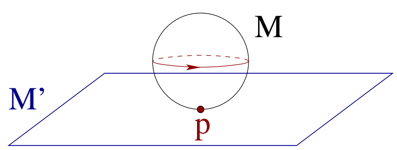


Figure 3.5: Pure twisting.

Example 3.5.6 (Pure Wobbling). This is the same as pure twisting except that $\Psi(t)$ is the identity on T_pM and any curve of rotations on $T_pM^\perp$. As a concrete example with $m = 1$ and $n = 3$ one can take M to be the circle of radius one in the (x, y)-plane centered at the point $(0, 1, 0)$ and p to be the origin:

$$M := \left\{(x, y, 0) \in \mathbb{R}^3 \,|\, x^2 + (y-1)^2 = 1\right\}, \qquad p := (0, 0, 0).$$

Then M' is the x-axis and $\Psi(t)$ is any curve of rotations in the (y, z)-plane, i.e. about the axis M'. (See Figure 3.6.)

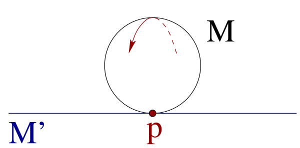


Figure 3.6: Pure wobbling.

3.5.2 Sliding

When a train slides on the track (e.g. in the process of stopping suddenly), there is a terrific screech. Since we usually do not hear a screech, this means that the wheel moves along without sliding. In other words the velocity of the point of contact in the train wheel M equals the velocity of the point of contact in the track M'. But the track is not moving; hence the point of contact in the wheel is not moving. One may explain the paradox this way: the train is moving forward and the wheel is rotating around the axle. The velocity of a point on the wheel is the sum of these two velocities. When the point is on the bottom of the wheel, the two velocities cancel.

Definition 3.5.7. *A motion (Ψ, γ, γ') is said to be* **without sliding** *if it satisfies $\Psi(t)\dot{\gamma}(t) = \dot{\gamma}'(t)$ for every t.*

Here is the geometric picture of the no sliding condition. As explained in Remark 3.5.2 we can view a motion as a smooth family of isometries

$$\psi_t(p) := \gamma'(t) + \Psi(t)\bigl(p - \gamma(t)\bigr)$$

acting on the manifold M with $\gamma(t) \in M$ being the point of contact with M'. Differentiating the curve $t \mapsto \psi_t(p)$ which describes the motion of the point $p \in M$ in the space $\mathbb{R}^n$ we obtain

$$\frac{d}{dt}\psi_t(p) = \dot{\gamma}'(t) - \Psi(t)\dot{\gamma}(t) + \dot{\Psi}(t)\bigl(p - \gamma(t)\bigr).$$

Taking $p = \gamma(t_0)$ we find

$$\left.\frac{d}{dt}\right|_{t=t_0} \psi_t(\gamma(t_0)) = \dot{\gamma}'(t_0) - \Psi(t_0)\dot{\gamma}(t_0).$$

This expression vanishes under the no sliding condition. In general the curve $t \mapsto \psi_t(\gamma(t_0))$ will be non-constant, but (when the motion is without sliding) its velocity will vanish at the instant $t = t_0$; i.e. at the instant when it becomes the point of contact. In other words *the motion is without sliding if and only if the point of contact is motionless.*

We remark that if the motion is without sliding we have:

$$\left|\dot{\gamma}'(t)\right| = |\Psi(t)\dot{\gamma}(t)| = |\dot{\gamma}(t)|$$

so that the curves γ and γ' have the same arclength:

$$\int_{t_0}^{t_1} \left|\dot{\gamma}'(t)\right| \, dt = \int_{t_0}^{t_1} |\dot{\gamma}(t)| \, dt$$

on any interval $[t_0, t_1] \subset I$. Hence any motion with $\dot{\gamma} = 0$ and $\dot{\gamma}' \neq 0$ is not without sliding (such as the example of pure sliding above).

Exercise 3.5.8. Give an example of a motion where $|\dot{\gamma}(t)| = |\dot{\gamma}'(t)|$ for every t but which is not without sliding.

Example 3.5.9. We describe mathematically the motion of the train wheel. Let the center of the wheel move right parallel to the x-axis at height one and the wheel have radius one and make one revolution in 2π units of time. Then the track M' is the x-axis and we take

$$M := \left\{ (x, y) \in \mathbb{R}^2 \,|\, x^2 + (y-1)^2 = 1 \right\}.$$

Choose

$$\begin{aligned} \gamma(t) &:= (\cos(t - \pi/2), 1 + \sin(t - \pi/2)) \\ &= (\sin(t), 1 - \cos(t)), \\ \gamma'(t) &:= (t, 0), \end{aligned}$$

and define $\Psi(t) \in \mathrm{GL}(2)$ by

$$\Psi(t) := \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}.$$

The reader can easily verify that this is a motion without sliding. A fixed point p_0 on M, say $p_0 = (0, 0)$, sweeps out a cycloid with parametric equations

$$x = t - \sin(t), \qquad y = 1 - \cos(t).$$

(Check that $(\dot{x}, \dot{y}) = (0, 0)$ when $y = 0$; i.e. for $t = 2n\pi$.)

Remark 3.5.10. These same formulas give a motion of a sphere M rolling without sliding along a straight line in a plane M'. Namely in coordinates (x, y, z) the sphere has equation

$$x^2 + (y-1)^2 + z^2 = 1,$$

the plane is $y = 0$ and the line is the x-axis. The z-coordinate of a point is unaffected by the motion. Note that the curve γ' traces out a straight line in the plane M' and the curve γ traces out a great circle on the sphere M.

Exercise 3.5.11. The operations of reparametrization, inversion, and composition respect motion without sliding; i.e. if (Ψ, γ, γ') and $(\Psi', \gamma', \gamma'')$ are motions without sliding on an interval I and $\sigma : J \to I$ is a smooth map between intervals, then the motions $(\Psi \circ \sigma, \gamma \circ \sigma, \gamma' \circ \sigma)$, $(\Psi^{-1}, \gamma', \gamma)$, and $(\Psi'\Psi, \gamma, \gamma'')$ are also without sliding.

3.5.3 Twisting and Wobbling

A motion (Ψ, γ, γ') on an intervall $I \subset \mathbb{R}$ transforms vector fields along γ into vector fields along γ' by the formula

$$X'(t) = (\Psi X)(t) := \Psi(t)X(t) \in T_{\gamma'(t)}M'$$

for $t \in I$ and $X \in \mathrm{Vect}(\gamma)$; so $X' \in \mathrm{Vect}(\gamma')$.

Lemma 3.5.12. *Let (Ψ, γ, γ') be a motion of M along M' on an interval $I \subset \mathbb{R}$. Then the following are equivalent.*

(i) *The instantaneous velocity of each tangent vector is normal, i.e. for $t \in I$*

$$\dot{\Psi}(t)T_{\gamma(t)}M \subset T_{\gamma'(t)}M'^{\perp}.$$

(ii) *Ψ intertwines covariant differentiation, i.e. for $X \in \mathrm{Vect}(\gamma)$*

$$\nabla'(\Psi X) = \Psi \nabla X.$$

(iii) *Ψ transforms parallel vector fields along γ into parallel vector fields along γ', i.e. for $X \in \mathrm{Vect}(\gamma)$*

$$\nabla X = 0 \qquad \Longrightarrow \qquad \nabla'(\Psi X) = 0.$$

(iv) *Ψ intertwines parallel transport, i.e. for $s, t \in I$ and $v \in T_{\gamma(s)}M$*

$$\Psi(t)\Phi_\gamma(t, s)v = \Phi'_{\gamma'}(t, s)\Psi(s)v.$$

A motion that satisfies these conditions is called **without twisting***.*

Proof. We prove that (i) is equivalent to (ii). A motion satisfies the equation

$$\Psi(t)\Pi(\gamma(t)) = \Pi'(\gamma'(t))\Psi(t)$$

for every $t \in I$. This restates the condition that $\Psi(t)$ maps tangent vectors of M to tangent vectors of M' and normal vectors of M to normal vectors of M'. Differentiating the equation $X'(t) = \Psi(t)X(t)$ we obtain

$$\dot{X}'(t) = \Psi(t)\dot{X}(t) + \dot{\Psi}(t)X(t).$$

Applying $\Pi'(\gamma'(t))$ this gives

$$\nabla' X' = \Psi \nabla X + \Pi'(\gamma')\dot{\Psi}X.$$

Hence (ii) holds if and only if $\Pi'(\gamma'(t))\dot{\Psi}(t) = 0$ for every $t \in I$. Thus we have proved that (i) is equivalent to (ii). That (ii) implies (iii) is obvious.

We prove that (iii) implies (iv). Let $t_0 \in I$ and $v_0 \in T_{\gamma(t_0)}M$. Define the vector field $X \in \mathrm{Vect}(\gamma)$ by $X(t) := \Phi_\gamma(t, t_0)v_0$ for $t \in I$ and let $X' := \Psi X$. Then $\nabla X = 0$, hence $\nabla' X' = 0$ by (iii), and hence

$$X'(t) = \Phi'_{\gamma'}(t, t_0)X'(t_0) = \Phi'_{\gamma'}(t, t_0)\Psi(t_0)v_0$$

for all $t \in I$. Since $X'(t) = \Psi(t)X(t) = \Psi(t)\Phi_\gamma(t, t_0)v_0$, this implies (iv).

We prove that (iv) implies (ii). Let $X \in \mathrm{Vect}(\gamma)$ and $X' := \Psi X$. By (iv) we have

$$\Phi'_{\gamma'}(t_0, t)X'(t) = \Psi(t_0)\Phi_\gamma(t_0, t)X(t).$$

Differentiating this equation with respect to t at $t = t_0$ and using Theorem 3.3.6, we obtain $\nabla' X'(t_0) = \Psi(t_0)\nabla X(t_0)$. This proves the lemma. $\square$

Lemma 3.5.13. *Let (Ψ, γ, γ') be a motion of M along M' on an interval $I \subset \mathbb{R}$. Then the following are equivalent.*

(i) *The instantaneous velocity of each normal vector is tangent, i.e. for $t \in I$*

$$\dot\Psi(t)T_{\gamma(t)}M^\perp \subset T_{\gamma'(t)}M'.$$

(ii) *Ψ intertwines normal covariant differentiation, i.e. for $Y \in \mathrm{Vect}^\perp(\gamma)$*

$$\nabla'^\perp(\Psi Y) = \Psi\nabla^\perp Y.$$

(iii) *Ψ transforms parallel normal vector fields along γ into parallel normal vector fields along γ', i.e. for $Y \in \mathrm{Vect}^\perp(\gamma)$*

$$\nabla^\perp Y = 0 \qquad \Longrightarrow \qquad \nabla'^\perp(\Psi Y) = 0.$$

(iv) *Ψ intertwines parallel transport of normal vector fields, i.e. for $s, t \in I$ and $w \in T_{\gamma(s)}M^\perp$*

$$\Psi(t)\Phi^\perp_\gamma(t, s)w = \Phi'^\perp_{\gamma'}(t, s)\Psi(s)w.$$

A motion that satisfies these conditions is called **without wobbling**.

The proof that the four conditions in Lemma 3.5.13 are equivalent is word for word analogous to the proof of Lemma 3.5.12 and will be omitted.

In summary a *motion is without twisting iff tangent vectors at the point of contact are rotating towards the normal space and it is without wobbling iff normal vectors at the point of contact are rotating towards the tangent space.* In case $m = 2$ and $n = 3$ motion without twisting means that the instantaneous axis of rotation is parallel to the tangent plane.

Remark 3.5.14. The operations of reparametrization, inversion, and composition respect motion without twisting, respectively without wobbling; i.e. if (Ψ, γ, γ') and $(\Psi', \gamma', \gamma'')$ are motions without twisting, respectively without wobbling, on an interval I and $\sigma : J \to I$ is a smooth map between intervals, then the motions $(\Psi \circ \sigma, \gamma \circ \sigma, \gamma' \circ \sigma)$, $(\Psi^{-1}, \gamma', \gamma)$, and $(\Psi'\Psi, \gamma, \gamma'')$ are also without twisting, respectively without wobbling.

Remark 3.5.15. Let $I \subset \mathbb{R}$ be an interval and $t_0 \in I$. Given curves $\gamma : I \to M$ and $\gamma' : I \to M'$ and an orthogonal matrix $\Psi_0 \in \mathrm{O}(n)$ such that

$$\Psi_0 T_{\gamma(t_0)} M = T_{\gamma'(t_0)} M'$$

there is a unique motion (Ψ, γ, γ') of M along M' (with the given γ and γ') **without twisting or wobbling** satisfying the initial condition:

$$\Psi(t_0) = \Psi_0.$$

Indeed, the path of matrices $\Psi : I \to \mathrm{O}(n)$ is uniquely determined by the conditions (iv) in Lemma 3.5.12 and Lemma 3.5.13. It is given by the explicit formula

$$\begin{aligned} \Psi(t)v &= \Phi'_{\gamma'}(t, t_0)\Psi_0\Phi_\gamma(t_0, t)\Pi(\gamma(t))v \\ &\quad + \Phi'^{\perp}_{\gamma'}(t, t_0)\Psi_0\Phi^{\perp}_\gamma(t_0, t)\big(v - \Pi(\gamma(t))v\big) \end{aligned} \tag{3.5.2}$$

for $t \in I$ and $v \in \mathbb{R}^n$. We prove below a somewhat harder result where the motion is without twisting, wobbling, or sliding. It is in this situation that γ and γ' determine one another (up to an initial condition).

Remark 3.5.16. We can now give another interpretation of parallel transport. Given $\gamma : \mathbb{R} \to M$ and $v_0 \in T_{\gamma(t_0)}M$ take M' to be an affine subspace of the same dimension as M. Let (Ψ, γ, γ') be a motion of M along M' without twisting (and, if you like, without sliding or wobbling). Let $X' \in \mathrm{Vect}(\gamma')$ be the constant vector field along γ' (so that $\nabla' X' = 0$) with value

$$X'(t) = \Psi_0 v_0, \qquad \Psi_0 := \Psi(t_0).$$

Let $X \in \mathrm{Vect}(\gamma)$ be the corresponding vector field along γ so that

$$\Psi(t)X(t) = \Psi_0 v_0$$

Then $X(t) = \Phi_\gamma(t, t_0)v_0$. To put it another way, imagine that M is a ball. To define parallel transport along a given curve γ roll the ball (without sliding) along a plane M' keeping the curve γ in contact with the plane M'. Let γ' be the curve traced out in M'. If a constant vector field in the plane M' is drawn in wet ink along the curve γ' it will mark off a (covariant) parallel vector field along γ in M.

Exercise 3.5.17. Describe parallel transport along a great circle in a sphere.

3.5.4 Development

A development is an intrinsic version of motion without sliding or twisting.

Definition 3.5.18. *A* **development of** M **along** M' **(on an interval** I**)** *is a triple* (Φ, γ, γ') *where* $\gamma : I \to M$ *and* $\gamma' : I \to M'$ *are smooth paths and* Φ *is a family of orthogonal isomorphisms*

$$\Phi(t) : T_{\gamma(t)}M \to T_{\gamma'(t)}M'$$

parametrized by $t \in I$*, such that*

$$\Phi(t)\dot{\gamma}(t) = \dot{\gamma}'(t) \tag{3.5.3}$$

for all $t \in I$ *and* Φ *intertwines parallel transport, i.e.*

$$\Phi(t)\Phi_\gamma(t, s) = \Phi'_{\gamma'}(t, s)\Phi(s) \tag{3.5.4}$$

for all $s, t \in I$*. In particular, the family* Φ *of isomorphisms is smooth, i.e. if* X *is a smooth vector field along* γ *then the formula* $X'(t) := \Phi(t)X(t)$ *defines a smooth vector field along* γ'*.*

Lemma 3.5.19. *Let* $I \subset \mathbb{R}$ *be an interval,* $\gamma : I \to M$ *and* $\gamma' : I \to M'$ *be smooth curves, and* $\Phi(t) : T_{\gamma(t)}M \to T_{\gamma'(t)}M'$ *be a family of orthogonal isomorphisms parametrized by* $t \in I$*. Then the following are equivalent.*

(i) (Φ, γ, γ') *is a development.*

(ii) Φ *satisfies* (3.5.3) *and*

$$\nabla'(\Phi X) = \Phi\nabla X \tag{3.5.5}$$

for all $X \in \mathrm{Vect}(\gamma)$*.*

(iii) *There exists a motion* (Ψ, γ, γ') *without sliding and twisting such that*

$$\Phi(t) = \Psi(t)|_{T_{\gamma(t)}M} \qquad \text{for all } t \in I. \tag{3.5.6}$$

(iv) *There exists a motion* (Ψ, γ, γ') *of* M *along* M' *without sliding, twisting, and wobbling that satisfies* (3.5.6)*.*

Proof. That (3.5.4) is equivalent to (3.5.5) was proved in Lemma 3.5.12. This (i) is equivalent to (ii). That (iv) implies (iii) and (iii) implies (i) is obvious. To prove that (i) implies (iv) choose any $t_0 \in I$ and any orthogonal matrix $\Psi_0 \in \mathrm{O}(n)$ such that $\Psi_0|_{T_{\gamma(t_0)}M} = \Phi(t_0)$ and define $\Psi(t) : \mathbb{R}^n \to \mathbb{R}^n$ by (3.5.2). This proves Lemma 3.5.19. $\square$

Remark 3.5.20. The operations of reparametrization, inversion, and composition yield developments when applied to developments; i.e. if (Φ, γ, γ') is a development of M along M', on an interval I, $(\Phi', \gamma', \gamma'')$ is a development of M' along M'' on the same interval I, and $\sigma : J \to I$ is a smooth map of intervals, then the triples

$$(\Phi \circ \sigma, \gamma \circ \sigma, \gamma' \circ \sigma), \qquad (\Phi^{-1}, \gamma', \gamma)), \qquad (\Phi'\Phi, \gamma, \gamma'')$$

are all developments.

Theorem 3.5.21 (Development Theorem). *Let $p_0 \in M$ and $t_0 \in \mathbb{R}$, let $\gamma' : \mathbb{R} \to M'$ be a smooth curve, and let*

$$\Phi_0 : T_{p_0}M \to T_{\gamma'(t_0)}M'$$

be an orthogonal isomorphism. Then the following holds.

(i) *There exists a development $(\Phi, \gamma, \gamma'|_I)$ on some open interval $I \subset \mathbb{R}$ containing t_0 that satisfies the initial condition*

$$\gamma(t_0) = p_0, \qquad \Phi(t_0) = \Phi_0. \tag{3.5.7}$$

(ii) *Any two developments $(\Phi_1, \gamma_1, \gamma'|_{I_1})$ and $(\Phi_2, \gamma_2, \gamma'|_{I_2})$ as in (i) on two intervals I_1 and I_2 agree on the intersection $I_1 \cap I_2$, i.e.*

$$\gamma_1(t) = \gamma_2(t), \qquad \Phi_1(t) = \Phi_2(t)$$

for every $t \in I_1 \cap I_2$.

(iii) *If M is complete then (i) holds with $I = \mathbb{R}$.*

Proof. Let $\gamma : \mathbb{R} \to M$ be any smooth curve such that

$$\gamma(t_0) = p_0$$

and, for $t \in \mathbb{R}$, define the linear map

$$\Phi(t) : T_{\gamma(t)}M \to T_{\gamma'(t)}M'$$

by

$$\Phi(t) := \Phi'_{\gamma'}(t, t_0)\Phi_0\Phi_\gamma(t_0, t). \tag{3.5.8}$$

This is an orthogonal transformation for every t and it intertwines parallel transport. However, in general $\Phi(t)\dot\gamma(t)$ will not be equal to $\dot\gamma'(t)$.

To construct a development that satisfies (3.5.3), we choose an orthonormal frame $e_0 : \mathbb{R}^m \to T_{p_0}M$ and, for $t \in \mathbb{R}$, define $e(t) : \mathbb{R}^m \to T_{\gamma(t)}M$ by

$$e(t) := \Phi_\gamma(t, t_0)e_0. \tag{3.5.9}$$

We can think of $e(t)$ as a real $n \times m$-matrix and the map

$$\mathbb{R} \to \mathbb{R}^{n\times m} : t \mapsto e(t)$$

is smooth. In fact, the map $t \mapsto (\gamma(t), e(t))$ is a smooth path in the frame bundle $\mathcal{F}(M)$. Define the smooth map $\xi : \mathbb{R} \to \mathbb{R}^m$ by

$$\dot\gamma'(t) = \Phi'_{\gamma'}(t, t_0)\Phi_0 e_0 \xi(t). \tag{3.5.10}$$

We prove the following.

Claim: *The triple (Φ, γ, γ') is a development on an interval $I \subset \mathbb{R}$ if and only if the path $t \mapsto (\gamma(t), e(t))$ satisfies the differential equation*

$$(\dot\gamma(t), \dot e(t)) = B_{\xi(t)}(\gamma(t), e(t)) \tag{3.5.11}$$

for every $t \in I$, where

$$B_{\xi(t)} \in \mathrm{Vect}(\mathcal{F}(M))$$

denotes the basic vector field associated to $\xi(t) \in \mathbb{R}^m$ (see equation (3.4.8)).

The triple (Φ, γ, γ') is a development on I if and only if

$$\Phi(t)\dot\gamma(t) = \dot\gamma'(t)$$

for every $t \in I$. By (3.5.8) and (3.5.10) this is equivalent to the condition

$$\Phi'_{\gamma'}(t, t_0)\Phi_0\Phi_\gamma(t_0, t)\dot\gamma(t) = \dot\gamma'(t) = \Phi'_{\gamma'}(t, t_0)\Phi_0 e_0 \xi(t),$$

hence to

$$\Phi_\gamma(t_0, t)\dot\gamma(t) = e_0\xi(t),$$

and hence to

$$\dot\gamma(t) = \Phi_\gamma(t, t_0)e_0\xi(t) = e(t)\xi(t) \tag{3.5.12}$$

for every $t \in I$. By (3.5.9) and the Gauß–Weingarten formula, we have

$$\dot e(t) = h_{\gamma(t)}(\dot\gamma(t))e(t)$$

for every $t \in \mathbb{R}$. Hence it follows from (3.4.8) that (3.5.12) is equivalent to (3.5.11). This proves the claim.

Parts (i) and (ii) follow directly from the claim. Part (iii) follows from the claim and Definition 3.4.11. This proves Theorem 3.5.21. □

Remark 3.5.22. As any two developments $(\Phi_1, \gamma_1, \gamma'|_{I_1})$ and $(\Phi_2, \gamma_2, \gamma'|_{I_2})$ on two intervals I_1 and I_2 that satisfy the initial condition (3.5.7) agree on $I_1 \cap I_2$ there is a development defined on $I_1 \cup I_2$. Hence there is a unique *maximally defined development* $(\Phi, \gamma, \gamma'|_I)$, defined on a maximal interval I, associated to γ', p_0, Φ_0.

Remark 3.5.23. The statement of Theorem 3.5.21 is essentially symmetric in M and M' as the operation of inversion carries developments to developments. Hence given

$$\gamma : \mathbb{R} \to M, \qquad p_0' \in M', \qquad t_0 \in \mathbb{R}, \qquad \Phi_0 : T_{\gamma(t_0)}M \to T_{p_0'}M',$$

we may speak of the development (Φ, γ, γ') corresponding to γ with initial conditions $\gamma'(t_0) = p_0'$ and $\Phi(t_0) = \Phi_0$.

Corollary 3.5.24 (Motions). *Let $p_0 \in M$ and $t_0 \in \mathbb{R}$, let $\gamma' : \mathbb{R} \to M'$ be a smooth curve, and let $\Psi_0 \in \mathrm{O}(n)$ be a matrix such that*

$$\Psi_0 T_{p_0}M = T_{\gamma'(t_0)}M'.$$

Then the following holds.

(i) *There exists a motion $(\Psi, \gamma, \gamma'|_I)$ without sliding, twisting and wobbling on some open interval $I \subset \mathbb{R}$ containing t_0 that satisfies the initial condition $\gamma(t_0) = p_0$ and $\Psi(t_0) = \Psi_0$.*

(ii) *Any two motions as in (i) on two intervals I_1 and I_2 agree on the intersection $I_1 \cap I_2$.*

(iii) *If M is complete then (i) holds with $I = \mathbb{R}$.*

Proof. Theorem 3.5.21 and Remark 3.5.15. □

Corollary 3.5.25 (Completeness). *The following are equivalent.*

(i) *M is complete, i.e. for every smooth curve $\xi : \mathbb{R} \to \mathbb{R}^m$ and every element $(p_0, e_0) \in \mathcal{F}(M)$, there exists a smooth curve $\beta : \mathbb{R} \to \mathcal{F}(M)$ such that $\dot\beta(t) = B_{\xi(t)}(\beta(t))$ for all $t \in \mathbb{R}$ and $\beta(0) = (p_0, e_0)$ (Definition 3.4.11).*

(ii) *For every smooth curve $\xi : \mathbb{R} \to \mathbb{R}^m$ and every element $(p_0, e_0) \in \mathcal{O}(M)$, there is a smooth curve $\alpha : \mathbb{R} \to \mathcal{O}(M)$ such that $\dot\alpha(t) = B_{\xi(t)}(\alpha(t))$ for every $t \in \mathbb{R}$ and $\alpha(0) = (p_0, e_0)$.*

(iii) *For every smooth curve $\gamma' : \mathbb{R} \to \mathbb{R}^m$, every $p_0 \in M$, and every orthogonal isomorphism $\Phi_0 : T_{p_0}M \to \mathbb{R}^m$ there exists a development (Φ, γ, γ') of M along $M' = \mathbb{R}^m$ on all of $\mathbb{R}$ that satisfies $\gamma(0) = p_0$ and $\Phi(0) = \Phi_0$.*

Proof. We have already noted that the basic vector fields are all all tangent to the orthonormal frame bundle $\mathcal{O}(M) \subset \mathcal{F}(M)$. Now note that if a smooth curve $I \to \mathcal{F}(M) : t \mapsto \beta(t) = (\gamma(t), e(t))$ on an interval $I \subset \mathbb{R}$ satisfies the differential equation $\dot\beta(t) = B_{\xi(t)}(\beta(t))$ for all t then so does the curve

$$I \to \mathcal{F}(M) \mapsto a^*\beta(t) = (\gamma(t), e(t) \circ a)$$

for every $a \in \mathrm{GL}(m, \mathbb{R})$. Since any frame $e_0 : \mathbb{R}^m \to T_{p_0}M$ can be carried to any other (in particular an orthonormal one) by a suitable matrix $a \in \mathrm{GL}(m, \mathbb{R})$, this shows that (i) is equivalent to (ii).

That (i) implies (iii) was proved in Theorem 3.5.21.

We prove that (iii) implies (ii). Fix a smooth map $\xi : \mathbb{R} \to \mathbb{R}^m$ and an element $(p_0, e_0) \in \mathcal{O}(M)$. Define $\Phi_0 := e_0^{-1} : T_{p_0}M \to \mathbb{R}^m$ and

$$\gamma'(t) := \int_0^t \xi(s)\, ds \in \mathbb{R}^m \qquad \text{for } t \in \mathbb{R}.$$

By (ii) there exists a development (Φ, γ, γ') of M along $\mathbb{R}^m$ on all of $\mathbb{R}$ that satisfies the initial conditions $\gamma(0) = p_0$ and $\Phi(0) = \Phi_0$. Then

$$\Phi(t) = \Phi_0\Phi_\gamma(0, t) : T_{\gamma(t)}M \to \mathbb{R}^m, \qquad \Phi(t)\dot\gamma(t) = \dot\gamma'(t) = \xi(t)$$

for all $t \in \mathbb{R}$ by Definition 3.5.18. Define

$$e(t) := \Phi_\gamma(t, 0)e_0 = \Phi(t)^{-1} : \mathbb{R}^m \to T_{\gamma(t)}M$$

for $t \in \mathbb{R}$. Then $(\gamma, e) : \mathbb{R} \to \mathcal{F}(M)$ is a smooth curve that satisfies the initial condition $(\gamma(0), e(0)) = (p_0, e_0)$ and the differential equation

$$\begin{aligned}\dot\gamma(t) &= \Phi(t)^{-1}\xi(t) = e(t)\xi(t),\\ \dot e(t) &= h_{\gamma(t)}(\dot\gamma(t))e(t) = h_{\gamma(t)}(e(t)\xi(t))e(t)\end{aligned}$$

by the Gauß–Weingarten formula. Thus $(\dot\gamma(t), \dot e(t)) = B_{\xi(t)}(\gamma(t), e(t))$ for all $t \in \mathbb{R}$. This proves Corollary 3.5.25. □

It is of course easy to to give an example of a manifold which is not complete; e.g. if (Φ, γ, γ') is any development of M along M' then $M \setminus \{\gamma(t_1)\}$ is not complete as the given development is only defined for $t \neq t_1$. In Section 4.6 we give equivalent characterizations of completeness. In particular, we will see that any compact submanifold of $\mathbb{R}^n$ is complete.

Exercise 3.5.26. An **affine subspace** of $\mathbb{R}^n$ is a subset of the form

$$E = p + \mathbb{E} = \{p + v \,|\, v \in \mathbb{E}\}$$

where $\mathbb{E} \subset \mathbb{R}^n$ is a linear subspace and $p \in \mathbb{R}^n$. Prove that every affine subspace of $\mathbb{R}^n$ is a complete submanifold.

3.6 Christoffel Symbols

The goal of this subsection is to examine the covariant derivative in local coordinates on an embedded manifold $M \subset \mathbb{R}^n$ of dimension m. Let

$$\phi : U \to \Omega$$

be a coordinate chart, defined on an M-open subset $U \subset M$ with values in an open set $\Omega \subset \mathbb{R}^m$, and denote its inverse by

$$\psi := \phi^{-1} : \Omega \to U \subset M.$$

At this point it is convenient to use superscripts for the coordinates of a vector $x \in \Omega$. Thus we write

$$x = (x^1, \dots, x^m) \in \Omega.$$

If $p = \psi(x) \in U$ is the corresponding element of M then the tangent space of M at p is the image of the linear map $d\psi(x) : \mathbb{R}^m \to \mathbb{R}^n$ (Theorem 2.2.3) and thus two tangent vectors $v, w \in T_pM$ can be written in the form

$$\begin{aligned} v = d\psi(x)\xi &= \sum_{i=1}^m \xi^i \frac{\partial \psi}{\partial x^i}(x), \\ w = d\psi(x)\eta &= \sum_{j=1}^m \eta^j \frac{\partial \psi}{\partial x^j}(x) \end{aligned} \tag{3.6.1}$$

for $\xi = (\xi^1, \dots, \xi^m) \in \mathbb{R}^m$ and $\eta = (\eta^1, \dots, \eta^m) \in \mathbb{R}^m$. Recall that the restriction of the inner product in the ambient space $\mathbb{R}^n$ to the tangent space is the first fundamental form $g_p : T_pM \times T_pM \to \mathbb{R}$ (Definition 3.1.1). Thus

$$g_p(v, w) = \langle v, w \rangle = \sum_{i,j=1}^m \xi^i g_{ij}(x) \eta^j, \tag{3.6.2}$$

where the functions $g_{ij} : \Omega \to \mathbb{R}$ are defined by

$$g_{ij}(x) := \left\langle \frac{\partial \psi}{\partial x^i}(x), \frac{\partial \psi}{\partial x^j}(x) \right\rangle \qquad \text{for } x \in \Omega. \tag{3.6.3}$$

In other words, the first fundamental form is in local coordinates represented by the matrix valued function $g = (g_{ij})_{i,j=1}^m : \Omega \to \mathbb{R}^{m \times m}$.

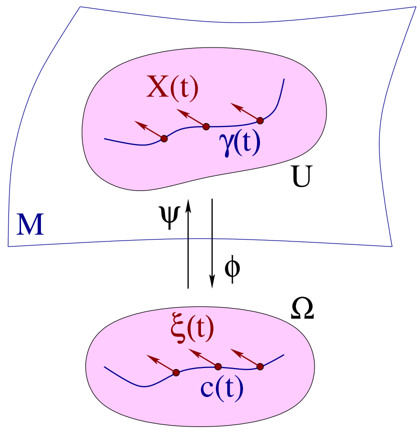


Figure 3.7: A vector field along a curve in local coordinates.

Now let $c = (c^1, \ldots, c^m) : I \to \Omega$ be a smooth curve in Ω, defined on an interval $I \subset \mathbb{R}$, and consider the curve

$$\gamma = \psi \circ c : I \to M$$

(see Figure 3.7). Our goal is to describe the operator $X \mapsto \nabla X$ on the space of vector fields along γ in local coordinates. Let $X : I \to \mathbb{R}^n$ be a vector field along γ. Then

$$X(t) \in T_{\gamma(t)}M = T_{\psi(c(t))}M = \operatorname{im}\big(d\psi(c(t)) : \mathbb{R}^m \to \mathbb{R}^n\big)$$

for every $t \in I$ and hence there exists a unique smooth function

$$\xi = (\xi^1, \ldots, \xi^m) : I \to \mathbb{R}^m$$

such that

$$X(t) = d\psi(c(t))\xi(t) = \sum_{i=1}^m \xi^i(t) \frac{\partial \psi}{\partial x^i}(c(t)). \tag{3.6.4}$$

Differentiate this identity to obtain

$$\dot{X}(t) = \sum_{i=1}^m \dot{\xi}^i(t) \frac{\partial \psi}{\partial x^i}(c(t)) + \sum_{i,j=1}^m \xi^i(t)\dot{c}^j(t) \frac{\partial^2 \psi}{\partial x^i \partial x^j}(c(t)). \tag{3.6.5}$$

We examine the projection $\nabla X(t) = \Pi(\gamma(t))\dot{X}(t)$ of this vector onto the tangent space of M at $\gamma(t)$. The first summand on the right in (3.6.5) is already tangent to M. For the second summand we simply observe that the vector $\Pi(\psi(x))\partial^2\psi/\partial x^i\partial x^j$ lies in tangent space $T_{\psi(x)}M$ and can therefore be expressed as a linear combination of the basis vectors $\partial\psi/\partial x^1,\dots,\partial\psi/\partial x^m$. The coefficients will be denoted by $\Gamma_{ij}^k(x)$. Thus there exist smooth functions $\Gamma_{ij}^k:\Omega\to\mathbb{R}$ for $i,j,k=1,\dots,m$ such that

$$\Pi(\psi(x))\frac{\partial^2\psi}{\partial x^i\partial x^j}(x)=\sum_{k=1}^m\Gamma_{ij}^k(x)\frac{\partial\psi}{\partial x^k}(x) \tag{3.6.6}$$

for all $x\in\Omega$ and all $i,j\in\{1,\dots,m\}$. The coefficients $\Gamma_{ij}^k:\Omega\to\mathbb{R}$ are called the **Christoffel symbols** associated to the coordinate chart $\phi:U\to\Omega$. To sum up we have proved the following.

Lemma 3.6.1. *Let $c:I\to\Omega$ be a smooth curve and define*

$$\gamma:=\psi\circ c:I\to M.$$

If $\xi:I\to\mathbb{R}^m$ is a smooth map and $X\in\mathrm{Vect}(\gamma)$ is given by (3.6.4) *then its covariant derivative at time $t\in I$ is given by*

$$\nabla X(t)=\sum_{k=1}^m\left(\dot\xi^k(t)+\sum_{i,j=1}^m\Gamma_{ij}^k(c(t))\xi^i(t)\dot c^j(t)\right)\frac{\partial\psi}{\partial x^k}(c(t)), \tag{3.6.7}$$

where the Γ_{ij}^k are the Christoffel symbols defined by (3.6.6).

Our next goal is to understand how the Christoffel symbols are determined by the metric in local coordinates. Recall from equation (3.6.2) that the inner products on the tangent spaces inherited from the standard Euclidean inner product on the ambient space $\mathbb{R}^n$ are in local coordinates represented by the matrix valued function

$$g=(g_{ij})_{i,j=1}^m:\Omega\to\mathbb{R}^{m\times m}$$

given by

$$g_{ij}:=\left\langle\frac{\partial\psi}{\partial x^i},\frac{\partial\psi}{\partial x^j}\right\rangle_{\mathbb{R}^n}. \tag{3.6.8}$$

We shall see that the Christoffel symbols are completely determined by the functions $g_{ij}:\Omega\to\mathbb{R}$. Here are first some elementary observations.

Remark 3.6.2. The matrix $g(x) \in \mathbb{R}^{m\times m}$ is symmetric and positive definite for every $x \in \Omega$. This follows from the fact that the matrix $d\psi(x) \in \mathbb{R}^{n\times m}$ has rank m and the matrix $g(x)$ is given by

$$g(x) = d\psi(x)^{\mathsf{T}} d\psi(x)$$

Thus $\xi^{\mathsf{T}} g(x)\xi = |d\psi(x)\xi|^2 > 0$ for all $\xi \in \mathbb{R}^m \setminus \{0\}$.

Remark 3.6.3. For $x \in \Omega$ we have $\det(g(x)) > 0$ by Remark 3.6.2 and so the matrix $g(x)$ is invertible. Denote the entries of the inverse matrix $g(x)^{-1} \in \mathbb{R}^{m\times m}$ by $g^{k\ell}(x)$. They are determined by the condition

$$\sum_{j=1}^{m} g_{ij}(x) g^{jk}(x) = \delta_i^k = \begin{cases} 1, & \text{if } i = k, \\ 0, & \text{if } i \neq k. \end{cases}$$

Since $g(x)$ is symmetric and positive definite, so is its inverse matrix $g(x)^{-1}$. In particular, we have $g^{k\ell}(x) = g^{\ell k}(x)$ for all $x \in \Omega$ and all $k, \ell \in \{1, \dots, m\}$.

Remark 3.6.4. Suppose that $X, Y \in \mathrm{Vect}(\gamma)$ are vector fields along our curve $\gamma = \psi \circ c : I \to M$ and $\xi, \eta : I \to \mathbb{R}^m$ are defined by

$$X(t) = \sum_{i=1}^{m} \xi^i(t) \frac{\partial \psi}{\partial x^i}(c(t)), \qquad Y(t) = \sum_{j=1}^{m} \eta^j(t) \frac{\partial \psi}{\partial x^j}(c(t)).$$

Then the inner product of $X(t)$ and $Y(t)$ is given by

$$\langle X(t), Y(t)\rangle = \sum_{i,j=1}^{m} \xi^i(t) g_{ij}(c(t)) \eta^j(t).$$

Lemma 3.6.5 (Christoffel Symbols). *Let $\Omega \subset \mathbb{R}^m$ be an open set and let $g_{ij} : \Omega \to \mathbb{R}$ for $i, j = 1, \dots, m$ be smooth functions such that each matrix $(g_{ij}(x))_{i,j=1}^m$ is symmetric and positive definite. Let $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ be smooth functions for $i, j, k = 1, \dots, m$. Then the Γ_{ij}^k satisfy the conditions*

$$\Gamma_{ij}^k = \Gamma_{ji}^k, \qquad \frac{\partial g_{ij}}{\partial x^\ell} = \sum_{k=1}^{m} \left(g_{ik}\Gamma_{j\ell}^k + g_{jk}\Gamma_{i\ell}^k \right) \tag{3.6.9}$$

for $i, j, k, \ell = 1, \dots, m$ if and only if they are given by

$$\Gamma_{ij}^k = \sum_{\ell=1}^{m} g^{k\ell} \frac{1}{2} \left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} \right). \tag{3.6.10}$$

If the Γ_{ij}^k are defined by (3.6.6) *and the g_{ij} by* (3.6.8), *then the Γ_{ij}^k satisfy* (3.6.9) *and hence are given by* (3.6.10).

Proof. Suppose that the Γ_{ij}^k are given by (3.6.6) and the g_{ij} by (3.6.8). Let

$$c : I \to \Omega, \qquad \xi, \eta : I \to \mathbb{R}^m$$

be smooth functions and suppose that the vector fields X, Y along the curve

$$\gamma := \psi \circ c : I \to M$$

are given by

$$X(t) := \sum_{i=1}^m \xi^i(t) \frac{\partial \psi}{\partial x^i}(c(t)), \qquad Y(t) := \sum_{j=1}^m \eta^j(t) \frac{\partial \psi}{\partial x^j}(c(t)).$$

Dropping the argument t in each term, we obtain from Remark 3.6.4 and Lemma 3.6.1 that

$$\begin{aligned}
\langle X, Y \rangle &= \sum_{i,j} g_{ij}(c) \xi^i \eta^j, \\
\langle X, \nabla Y \rangle &= \sum_{i,k} g_{ik}(c) \xi^i \left(\dot{\eta}^k + \sum_{j,\ell} \Gamma_{j\ell}^k(c) \eta^j \dot{c}^\ell \right), \\
\langle \nabla X, Y \rangle &= \sum_{j,k} g_{kj}(c) \left(\dot{\xi}^k + \sum_{i,\ell} \Gamma_{i\ell}^k(c) \xi^i \dot{c}^\ell \right) \eta^j.
\end{aligned}$$

Hence it follows from equation (3.2.5) in Lemma 3.2.4 that

$$\begin{aligned}
0 &= \frac{d}{dt} \langle X, Y \rangle - \langle X, \nabla Y \rangle - \langle \nabla X, Y \rangle \\
&= \sum_{i,j} \left(g_{ij}(c) \dot{\xi}^i \eta^j + g_{ij}(c) \xi^i \dot{\eta}^j + \sum_\ell \frac{\partial g_{ij}}{\partial x^\ell}(c) \xi^i \eta^j \dot{c}^\ell \right) \\
&\quad - \sum_{i,k} g_{ik}(c) \xi^i \dot{\eta}^k - \sum_{i,j,k,\ell} g_{ik}(c) \Gamma_{j\ell}^k(c) \xi^i \eta^j \dot{c}^\ell \\
&\quad - \sum_{j,k} g_{kj}(c) \dot{\xi}^k \eta^j - \sum_{i,j,k,\ell} g_{kj}(c) \Gamma_{i\ell}^k(c) \xi^i \eta^j \dot{c}^\ell \\
&= \sum_{i,j,\ell} \left(\frac{\partial g_{ij}}{\partial x^\ell}(c) - \sum_k g_{ik}(c) \Gamma_{j\ell}^k(c) - \sum_k g_{jk}(c) \Gamma_{i\ell}^k(c) \right) \xi^i \eta^j \dot{c}^\ell.
\end{aligned}$$

This holds for all smooth maps $c : I \to \Omega$ and $\xi, \eta : I \to \mathbb{R}^m$, so the Γ_{ij}^k satisfy the second equation in (3.6.9). That they are symmetric in i and j is obvious.

To prove that (3.6.9) is equivalent to (3.6.10), define

$$\Gamma_{\ell ij} := \sum_{k=1}^{m} g_{\ell k}\Gamma_{ij}^{k}. \tag{3.6.11}$$

Then (3.6.9) is equivalent to

$$\Gamma_{\ell ij} = \Gamma_{\ell ji}, \qquad \frac{\partial g_{ij}}{\partial x^\ell} = \Gamma_{ij\ell} + \Gamma_{ji\ell}. \tag{3.6.12}$$

and (3.6.10) is equivalent to

$$\Gamma_{\ell ij} = \frac{1}{2}\left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell}\right). \tag{3.6.13}$$

If the $\Gamma_{\ell ij}$ are given by (3.6.13) then they satisfy

$$\Gamma_{\ell ij} = \Gamma_{\ell ji}$$

and

$$\begin{aligned} 2\Gamma_{ij\ell} + 2\Gamma_{ji\ell} &= \frac{\partial g_{ij}}{\partial x^\ell} + \frac{\partial g_{i\ell}}{\partial x^j} - \frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial g_{ji}}{\partial x^\ell} + \frac{\partial g_{j\ell}}{\partial x^i} - \frac{\partial g_{i\ell}}{\partial x^j} \\ &= 2\frac{\partial g_{ij}}{\partial x^\ell} \end{aligned}$$

for all i, j, ℓ. Conversely, if the $\Gamma_{\ell ij}$ satisfy (3.6.12) then

$$\begin{aligned} \frac{\partial g_{ij}}{\partial x^\ell} &= \Gamma_{ij\ell} + \Gamma_{ji\ell}, \\ \frac{\partial g_{\ell i}}{\partial x^j} &= \Gamma_{\ell ij} + \Gamma_{i\ell j} = \Gamma_{\ell ij} + \Gamma_{ij\ell}, \\ \frac{\partial g_{\ell j}}{\partial x^i} &= \Gamma_{\ell ji} + \Gamma_{j\ell i} = \Gamma_{\ell ij} + \Gamma_{ji\ell}. \end{aligned}$$

Take the sum of the last two minus the first of these equations to obtain

$$\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} = 2\Gamma_{\ell ij}.$$

Thus (3.6.12) is equivalent to (3.6.13) and so (3.6.9) is equivalent to (3.6.10). This proves Lemma 3.6.5. $\square$

3.7 Riemannian Metrics*

We wish to carry over the fundamental notions of differential geometry to the intrinsic setting. First we need an inner product on the tangent spaces to replace the first fundamental form in Definition 3.1.1. This is the content of Definition 3.7.1 and Lemma 3.7.4 below. Second we must introduce the covariant derivative of a vector field along a curve. With this understood all the definitions, theorems, and proofs in this chapter carry over in an almost word by word fashion to the intrinsic setting.

3.7.1 Existence of Riemannian Metrics

We will always consider norms that are induced by inner products. But in general there is no ambient space that can induce an inner product on each tangent space. This leads to the following definition.

Definition 3.7.1. *Let M be a smooth m-manifold. A* **Riemannian metric** *on M is a collection of inner products*

$$T_pM \times T_pM \to \mathbb{R} : (v, w) \mapsto g_p(v, w),$$

one for every $p \in M$, such that the map

$$M \to \mathbb{R} : p \mapsto g_p(X(p), Y(p))$$

is smooth for every pair of vector fields $X, Y \in \mathrm{Vect}(M)$. We will also denote the inner product by $\langle v, w\rangle_p$ and drop the subscript p if the base point is understood from the context. A smooth manifold equipped with a Riemannian metric is called a **Riemannian manifold***.*

Example 3.7.2. If $M \subset \mathbb{R}^n$ is a smooth submanifold then a Riemannian metric on M is given by restricting the standard inner product on $\mathbb{R}^n$ to the tangent spaces $T_pM \subset \mathbb{R}^n$. This is the first fundamental form of an embedded manifold (see Definition 3.1.1).

More generally, assume that M is a Riemannian m-manifold in the intrinsic sense of Definition 3.7.1 with an atlas $\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$. Then the Riemannian metric g determines a collection of smooth functions

$$g_\alpha = (g_{\alpha,ij})_{i,j=1}^m : \phi_\alpha(U_\alpha) \to \mathbb{R}^{m\times m}, \tag{3.7.1}$$

one for each $\alpha \in A$, defined by

$$\xi^{\mathsf{T}} g_\alpha(x)\eta := g_p(v, w), \quad \phi_\alpha(p) = x, \quad d\phi_\alpha(p)v = \xi, \quad d\phi_\alpha(p)w = \eta, \tag{3.7.2}$$

for $x \in \phi_\alpha(U_\alpha)$ and $\xi, \eta \in \mathbb{R}^m$.

Each matrix $g_\alpha(x)$ is symmetrix and positive definite. Note that the tangent vectors v and w in (3.7.2) can also be written in the form

$$v = [\alpha, \xi]_p, \qquad w = [\alpha, \eta]_p.$$

Choosing standard basis vectors

$$\xi = e_i, \qquad \eta = e_j$$

in $\mathbb{R}^m$ we obtain

$$[\alpha, e_i]_p = d\phi_\alpha(p)^{-1} e_i =: \frac{\partial}{\partial x^i}(p)$$

and hence

$$g_{\alpha,ij}(x) = \left\langle \frac{\partial}{\partial x^i}(\phi_\alpha^{-1}(x)), \frac{\partial}{\partial x^j}(\phi_\alpha^{-1}(x)) \right\rangle. \tag{3.7.3}$$

For different coordinate charts the maps g_α and g_β are related through the transition map

$$\phi_{\beta\alpha} := \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

via

$$g_\alpha(x) = d\phi_{\beta\alpha}(x)^{\mathsf{T}} g_\beta(\phi_{\beta\alpha}(x)) d\phi_{\beta\alpha}(x) \tag{3.7.4}$$

for $x \in \phi_\alpha(U_\alpha \cap U_\beta)$. Equation (3.7.4) can also be written in the shorthand notation

$$g_\alpha = \phi_{\beta\alpha}^* g_\beta$$

for $\alpha, \beta \in A$.

Exercise 3.7.3. Every collection of smooth maps

$$g_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^{m\times m}$$

with values in the set of positive definite symmetric matrices that satisfies (3.7.4) for all $\alpha, \beta \in A$ determines a global Riemannian metric via (3.7.2).

In this intrinsic setting there is no canonical metric on M (such as the metric induced by $\mathbb{R}^n$ on an embedded manifold). In fact, it is not completely obvious that a manifold admits a Riemannian metric and this is the content of the next lemma.

Lemma 3.7.4. *Every paracompact Hausdorff manifold admits a Riemannian metric.*

Proof. Let m be the dimension of M and let $\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha\in A}$ be an atlas on M. By Theorem 2.9.9 there is a partition of unity $\{\theta_\alpha\}_{\alpha\in A}$ subordinate to the open cover $\{U_\alpha\}_{\alpha\in A}$. Now there are two equivalent ways to construct a Riemannian metric on M.

The first method is to carry over the standard inner product on $\mathbb{R}^m$ to the tangent spaces T_pM for $p \in U_\alpha$ via the coordinate chart ϕ_α, multiply the resulting Riemannian metric on U_α by the compactly supported function θ_α, extend it by zero to all of M, and then take the sum over all α. This leads to the following formula. The inner product of two tangent vectors $v, w \in T_pM$ is defined by

$$\langle v, w\rangle_p := \sum_{p\in U_\alpha} \theta_\alpha(p)\langle d\phi_\alpha(p)v, d\phi_\alpha(p)w\rangle, \tag{3.7.5}$$

where the sum runs over all $\alpha \in A$ with $p \in U_\alpha$ and the inner product is the standard inner product on $\mathbb{R}^m$. Since $\mathrm{supp}(\theta_\alpha) \subset U_\alpha$ for each α and the sum is locally finite we find that the function

$$M \to \mathbb{R} : p \mapsto \langle X(p), Y(p)\rangle_p$$

is smooth for every pair of vector fields $X, Y \in \mathrm{Vect}(M)$. Moreover, the right hand side of (3.7.5) is symmetric in v and w and is positive for $v = w \neq 0$ because each summand is nonnegative and each summand with $\theta_\alpha(p) > 0$ is positive. Thus equation (3.7.5) defines a Riemannian metric on M.

The second method is to define the functions

$$g_\alpha : \phi_\alpha(U_\alpha) \to \mathbb{R}^{m\times m}$$

by

$$g_\alpha(x) := \sum_{\gamma\in A} \theta_\gamma(\phi_\alpha^{-1}(x)) d\phi_{\gamma\alpha}(x)^{\mathsf{T}} d\phi_{\gamma\alpha}(x) \tag{3.7.6}$$

for $x \in \phi_\alpha(U_\alpha)$ where each summand is defined on $\phi_\alpha(U_\alpha \cap U_\gamma)$ and is understood to be zero for $x \notin \phi_\alpha(U_\alpha \cap U_\gamma)$. We leave it to the reader to verify that these functions are smooth and satisfy the condition (3.7.4) for all $\alpha, \beta \in A$. Moreover, the formulas (3.7.5) and (3.7.6) determine the same Riemannian metric on M. (Prove this!) This proves Lemma 3.7.4. $\square$

Example 3.7.5 (**Fubini–Study Metric).** The complex projective space carries a natural Riemannian metric, defined as follows. Identify $\mathbb{C}\mathrm{P}^n$ with the quotient of the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ by the diagonal action of the circle S^1, i.e. $\mathbb{C}\mathrm{P}^n = S^{2n+1}/S^1$. Then the tangent space of $\mathbb{C}\mathrm{P}^n$ at the equivalence class

$$[z] = [z_0 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n$$

of a point $z = (z_0, \ldots, z_n) \in S^{2n+1}$ can be identified with the orthogonal complement of $\mathbb{C}z$ in $\mathbb{C}^{n+1}$. Now choose the inner product on $T_{[z]}\mathbb{C}\mathrm{P}^n$ to be the one inherited from the standard inner product on $\mathbb{C}^{n+1}$ via this identification. The resulting metric on $\mathbb{C}\mathrm{P}^n$ is called the **Fubini–Study metric**. **Exercise:** Prove that the action of $\mathrm{U}(n+1)$ on $\mathbb{C}^{n+1}$ induces a transitive action of the quotient group

$$\mathrm{PSU}(n+1) := \mathrm{U}(n+1)/S^1$$

by isometries. If $z \in S^{2n+1}$, prove that the unitary matrix

$$g := 2zz^* - \mathbb{1}$$

descends to an isometry ϕ on $\mathbb{C}\mathrm{P}^n$ with fixed point $p := [z]$ and $d\phi(p) = -\mathrm{id}$. Show that, in the case $n = 1$, the pullback of the Fubini–Study metric on $\mathbb{C}\mathrm{P}^1$ under the stereographic projection

$$S^2 \setminus \{(0,0,1)\} \to \mathbb{C}\mathrm{P}^1 \setminus \{[0:1]\} : (x_1, x_2, x_3) \mapsto \left[1 : \frac{x_1 + \mathbf{i}x_2}{1 - x_3}\right]$$

is one quarter of the standard metric on S^2.

Example 3.7.6. Think of the complex Grassmannian $\mathrm{G}_k(\mathbb{C}^n)$ of k-planes in $\mathbb{C}^n$ as a quotient of the space

$$\mathcal{F}_k(\mathbb{C}^n) := \left\{ D \in \mathbb{C}^{n\times k} \,|\, D^*D = \mathbb{1} \right\}$$

of unitary k-frames in $\mathbb{C}^n$ by the right action of the unitary group $\mathrm{U}(k)$. The space $\mathcal{F}_k(\mathbb{C}^n)$ inherits a Riemannian metric from the ambient Euclidean space $\mathbb{C}^{n\times k}$. Show that the tangent space of of $\mathrm{G}_k(\mathbb{C}^n)$ at a point $\Lambda = \mathrm{im}D$, with $D \in \mathcal{F}_k(\mathbb{C}^n)$ can be identified with the space

$$T_\Lambda \mathrm{G}_k(\mathbb{C}^n) = \left\{ \widehat{D} \in \mathbb{C}^{n\times k} \,|\, D^*\widehat{D} = 0 \right\}.$$

Define the inner product on this tangent space to be the restriction of the standard inner product on $\mathbb{C}^{n\times k}$ to this subspace. **Exercise:** Prove that the unitary group $\mathrm{U}(n)$ acts on $\mathrm{G}_k(\mathbb{C}^n)$ by isometries.

3.7.2 The Levi-Civita Connection

A subtle point in this discussion is how to extend the notion of *covariant derivative* to general Riemannian manifolds. In this case the idea of projecting the derivative in the ambient space orthogonally onto the tangent space has no obvious analogue. Instead we shall see how the covariant derivatives of vector fields along curves can be characterized by several axioms and these can be used to define the covariant derivative in the intrinsic setting. An alternative, but somewhat less satisfactory, approach is to carry over the formula for the covariant derivative in local coordinates to the intrinsic setting and show that the result is independent of the choice of the coordinate chart. Of course, these approaches are equivalent and lead to the same result. We formulate them as a series of exercises. The details are straightforward.

Assume throughout that M be a Riemannian m-manifold with an atlas

$$\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha \in A}$$

and suppose that the Riemannian metric is in local coordinates given by

$$g_\alpha = (g_{\alpha,ij})_{i,j=1}^m : \phi_\alpha(U_\alpha) \to \mathbb{R}^{m\times m}$$

for $\alpha \in A$. These functions satisfy (3.7.4) for all $\alpha, \beta \in A$.

Definition 3.7.7. *Let $f : N \to M$ be a smooth map between manifolds. A* **vector field along** f *is a collection of tangent vectors*

$$X(q) \in T_{f(q)}M,$$

one for each $q \in N$, such that the map

$$N \to TM : q \mapsto (f(q), X(q))$$

is smooth. The space of vector fields along f will be denoted by $\mathrm{Vect}(f)$.

As before we will not distinguish in notation between the collection of tangent vectors $X(q) \in T_{f(q)}M$ and the associated map $N \to TM$ and denote them both by X. The following theorem introduces the Levi-Civita connection as a collection of linear operators $\nabla : \mathrm{Vect}(\gamma) \to \mathrm{Vect}(\gamma)$, one for each smooth curve $\gamma : I \to M$.

Theorem 3.7.8 (Levi-Civita Connection). There exists a unique collection of linear operators

$$\nabla : \mathrm{Vect}(\gamma) \to \mathrm{Vect}(\gamma)$$

(called the **covariant derivative**), one for every smooth curve $\gamma : I \to M$ on an open interval $I \subset \mathbb{R}$, satisfying the following axioms.

(Leibniz Rule) For every smooth curve $\gamma : I \to M$, every smooth function $\lambda : I \to \mathbb{R}$, and every vector field $X \in \mathrm{Vect}(\gamma)$, we have

$$\nabla(\lambda X) = \dot{\lambda} X + \lambda \nabla X. \tag{3.7.7}$$

(Chain Rule) Let $\Omega \subset \mathbb{R}^n$ be an open set, let $c : I \to \Omega$ be a smooth curve, let $\gamma : \Omega \to M$ be a smooth map, and let X be a smooth vector field along γ. Denote by $\nabla_i X$ the covariant derivative of X along the curve $x^i \mapsto \gamma(x)$ (with the other coordinates fixed). Then $\nabla_i X$ is a smooth vector field along γ and the covariant derivative of the vector field $X \circ c \in \mathrm{Vect}(\gamma \circ c)$ is

$$\nabla(X \circ c) = \sum_{j=1}^{n} \dot{c}^j(t) \nabla_j X(c(t)). \tag{3.7.8}$$

(Riemannian) For any two vector fields $X, Y \in \mathrm{Vect}(\gamma)$ we have

$$\frac{d}{dt}\langle X, Y\rangle = \langle \nabla X, Y\rangle + \langle X, \nabla Y\rangle. \tag{3.7.9}$$

(Torsion-free) Let $I, J \subset \mathbb{R}$ be open intervals and $\gamma : I \times J \to M$ be a smooth map. Denote by ∇_s the covariant derivative along the curve $s \mapsto \gamma(s,t)$ (with t fixed) and by ∇_t the covariant derivative along the curve $t \mapsto \gamma(s,t)$ (with s fixed). Then

$$\nabla_s \partial_t \gamma = \nabla_t \partial_s \gamma. \tag{3.7.10}$$

Proof. The proof is based on a reformulation of the axioms in local coordinates. The (Leibniz Rule) and (Chain Rule) axioms assert that the covariant derivative is in local coordinates given by Christoffel symbols Γ_{ij}^k as in equation (3.6.7) in Lemma 3.6.1. The (Riemannian) and (Torsion-free) axioms assert that the Christoffel symbols satisfy the equations in (3.6.9) and hence, by Lemma 3.6.5, are given by (3.6.10). (See also Exercise 3.7.10.) This proves Theorem 3.7.8. □

Exercise 3.7.9. The **Christoffel symbols** of the Riemannian metric are the functions $\Gamma^k_{\alpha,ij} : \phi_\alpha(U_\alpha) \to \mathbb{R}$. defined by

$$\Gamma^k_{\alpha,ij} := \sum_{\ell=1}^m g_\alpha^{k\ell} \frac{1}{2} \left(\frac{\partial g_{\alpha,\ell i}}{\partial x^j} + \frac{\partial g_{\alpha,\ell j}}{\partial x^i} - \frac{\partial g_{\alpha,ij}}{\partial x^\ell} \right)$$

(see Lemma 3.6.5). Prove that they are related by the equation

$$\sum_k \frac{\partial \phi^{k'}_{\beta\alpha}}{\partial x^k} \Gamma^k_{\alpha,ij} = \frac{\partial^2 \phi^{k'}_{\beta\alpha}}{\partial x^i \partial x^j} + \sum_{i',j'} \left(\Gamma^{k'}_{\beta,i'j'} \circ \phi_{\beta\alpha} \right) \frac{\partial \phi^{i'}_{\beta\alpha}}{\partial x^i} \frac{\partial \phi^{j'}_{\beta\alpha}}{\partial x^j}.$$

for all $\alpha, \beta \in A$.

Exercise 3.7.10. Denote $\psi_\alpha := \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to M$. Prove that the covariant derivative of a vector field

$$X(t) = \sum_{i=1}^m \xi^i_\alpha(t) \frac{\partial \psi_\alpha}{\partial x^i}(c_\alpha(t))$$

along $\gamma = \psi_\alpha \circ c_\alpha : I \to M$ is given by

$$\nabla X(t) = \sum_{k=1}^m \left(\dot\xi^k_\alpha(t) + \sum_{i,j=1}^m \Gamma^k_{\alpha,ij}(c(t)) \xi^i_\alpha(t) \dot c^j_\alpha(t) \right) \frac{\partial \psi_\alpha}{\partial x^k}(c_\alpha(t)). \tag{3.7.11}$$

Prove that ∇X is independent of the choice of the coordinate chart.

Exercise 3.7.11. Let $\Omega \subset \mathbb{R}^2$ be open and $\lambda : \Omega \to (0, \infty)$ be a smooth function. Let $g : \Omega \to \mathbb{R}^{2\times 2}$ be given by

$$g(x) = \begin{pmatrix} \lambda(x) & 0 \\ 0 & \lambda(x) \end{pmatrix}.$$

Compute the Christoffel symbols Γ^k_{ij} via (3.6.10).

Exercise 3.7.12. Let $\phi : S^2 \setminus \{(0,0,1)\} \to \mathbb{C}$ be the stereographic projection, given by

$$\phi(p) := \left(\frac{p_1}{1 - p_3}, \frac{p_2}{1 - p_3} \right)$$

Prove that the metric $g : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ has the form $g(x) = \lambda(x)\mathbb{1}$ where

$$\lambda(x) := \frac{4}{(1 + |x|^2)^2} \qquad \text{for } x = (x^1, x^2) \in \mathbb{R}^2.$$

Exercise 3.7.13 (Basic Vector Fields: Intrinsic Setting). Let M be a Riemannian m-manifold with an atlas $\mathscr{A} = \{(\phi_\alpha, U_\alpha)\}_{\alpha\in A}$. Prove that the frame bundle (3.4.1) admits the structure of a smooth manifold with the open cover $\widetilde{U}_\alpha := \pi^{-1}(U_\alpha)$ and coordinate charts

$$\widetilde{\phi}_\alpha : \widetilde{U}_\alpha \to \phi_\alpha(U_\alpha) \times \mathrm{GL}(m)$$

given by

$$\widetilde{\phi}_\alpha(p, e) := (\phi_\alpha(p), d\phi_\alpha(p)e)\,.$$

Prove that the derivatives of the horizontal curves in Definition 3.4.5 form a horizontal subbundle $H \subset \mathcal{F}(M)$ whose fibers $H_{(p,e)}$ can in local coordinates be described as follows. Let

$$x := \phi_\alpha(p), \qquad a := d\phi_\alpha(p)e \in \mathrm{GL}(m).$$

and $(\hat{x}, \hat{a}) \in \mathbb{R}^m \times \mathbb{R}^{m\times m}$. This pair has the form

$$(\hat{x}, \hat{a}) = d\widetilde{\phi}_\alpha(p, e)(\hat{p}, \hat{e}), \qquad (\hat{p}, \hat{e}) \in H_{(p,e)},$$

if and only if

$$\hat{a}^k_\ell = -\sum_{i,j=1}^m \Gamma^k_{\alpha,ij}(x)\hat{x}^i a^j_\ell$$

for $k, \ell = 1, \ldots, m$, where the functions $\Gamma^k_{\alpha,ij} : \phi_\alpha(U_\alpha) \to \mathbb{R}$ are the Christoffel symbols. Show that, for every $\xi \in \mathbb{R}^m$ there is a unique horizontal vector field $B_\xi \in \mathrm{Vect}(\mathcal{F}(M))$ such that $d\pi(p, e)B_\xi(p, e) = e\xi$ for $(p, e) \in \mathcal{F}(M)$.

Exercise 3.7.14. Carry over the proofs of Theorem 3.3.4, Theorem 3.3.6, and Theorem 3.5.21 to the intrinsic setting.

Chapter 4

Geodesics

This chapter introduces geodesics in Riemannian manifolds. It begins in §4.1 by introducing geodesics as extremals of the energy and length functionals and characterizing them as solutions of a second order differential equation. In §4.2 we show that minimizing the length with fixed endpoints gives rise to an intrinsic distance function $d : M \times M \to \mathbb{R}$ which induces the topology M inherits from the ambient space $\mathbb{R}^n$. §4.3 introduces the exponential map, §4.5 shows that geodesics minimize the length on short time intervals, §4.4 establishes the existence of geodesically convex neighborhoods, and §4.6 shows that the geodesic flow is complete if and only if (M, d) is a complete metric space, and that in the complete case any two points are joined by a minimal geodesic. §4.7 discusses geodesics in the intrinsic setting.

4.1 Length and Energy

The concept of a geodesic in a manifold generalizes that of a straight line in Euclidean space. A straight line has parametrizations of form $t \mapsto p + \sigma(t)v$ where $\sigma : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism and $p, v \in \mathbb{R}^n$. Different choices of σ yield different parametrizations of the same line. Certain parametrizations are preferred, for example those parametrizations which are "proportional to the arclength", i.e. where $\sigma(t) = at + b$ for constants $a, b \in \mathbb{R}$, so that the tangent vector $\dot{\sigma}(t)v$ has constant length. The same distinctions can be made for geodesics. Some authors use the term geodesic to include all parametrizations of a geodesic while others restrict the term to cover only geodesics parametrized proportional to arclength. We follow the latter course, referring to the more general concept as a "reparametrized geodesic". Thus a reparametrized geodesic need not be a geodesic.

We assume throughout that $M \subset \mathbb{R}^n$ is a smooth m-manifold.

Definition 4.1.1 (Length and Energy). *Let $I = [a,b] \subset \mathbb{R}$ be a compact interval with $a < b$ and let $\gamma : I \to M$ be a smooth curve in M. The* **length** *$L(\gamma)$ and the* **energy** *$E(\gamma)$ are defined by*

$$L(\gamma) := \int_a^b |\dot{\gamma}(t)| \, dt, \tag{4.1.1}$$

$$E(\gamma) := \frac{1}{2}\int_a^b |\dot{\gamma}(t)|^2 \, dt. \tag{4.1.2}$$

A **variation of** *γ is a family of smooth curves $\gamma_s : I \to M$, where s ranges over the reals, such that the map $\mathbb{R} \times I \to M : (s,t) \mapsto \gamma_s(t)$ is smooth and*

$$\gamma_0 = \gamma.$$

The variation $\{\gamma_s\}_{s\in\mathbb{R}}$ is said to have **fixed endpoints** *if $\gamma_s(a) = \gamma(a)$ and $\gamma_s(b) = \gamma(b)$ for all $s \in \mathbb{R}$.*

Remark 4.1.2. The length of a continuous function $\gamma : [a,b] \to \mathbb{R}^n$ can be defined as the supremum of the numbers $\sum_{i=1}^N |\gamma(t_i) - \gamma(t_{i-1})|$ over all partitions $a = t_0 < t_1 < \cdots < t_N = b$ of the interval $[a,b]$. By a theorem in first year analysis [18] this supremum is finite whenever γ is continuously differentiable and is given by (4.1.1).

We shall sometimes suppress the notation for the endpoints of $a, b \in I$. When $\gamma(a) = p$ and $\gamma(b) = q$ we say that γ **is a curve from** p **to** q. One can always compose γ with an affine reparametrization $t' = a + (b-a)t$ to obtain a new curve $\gamma'(t) := \gamma(t')$ on the unit interval $0 \le t \le 1$. This new curve satisfies $L(\gamma') = L(\gamma)$ and $E(\gamma') = (b-a)E(\gamma)$. More generally, the length $L(\gamma)$, but not the energy $E(\gamma)$, is invariant under reparametrization.

Remark 4.1.3 (Reparametrization). Let $I = [a,b]$ and $I' = [a',b']$ be compact intevrals. If $\gamma : I \to \mathbb{R}^n$ is a smooth curve and $\sigma : I' \to I$ is a smooth function such that $\sigma(a') = a$, $\sigma(b') = b$, and $\dot{\sigma}(t) \ge 0$ for all $t \in I'$, then

$$L(\gamma \circ \sigma) = L(\gamma). \tag{4.1.3}$$

To see this, we compute

$$L(\gamma\circ\sigma) = \int_{a'}^{b'} \left|\frac{d}{dt'}\gamma(\sigma(t'))\right| dt' = \int_{a'}^{b'} \left|\dot{\gamma}(\sigma(t'))\right| \dot{\sigma}(t')\, dt' = L(\gamma).$$

Here second equation follows from the chain rule and the fact that $\dot{\sigma}(t') \ge 0$ for all $t' \in [a',b']$, and the third equation follows from the change of variables formula for the Riemann integral. This proves equation (4.1.3).

Theorem 4.1.4 (Characterization of Geodesics). *Let $I = [a,b] \subset \mathbb{R}$ be a compact interval and let $\gamma : I \to M$ be a smooth curve. Then the following are equivalent.*

(i) *γ is an* **extremal of the energy functional**, *i.e.*

$$\left.\frac{d}{ds}\right|_{s=0} E(\gamma_s) = 0$$

for every variation $\{\gamma_s\}_{s\in\mathbb{R}}$ of γ with fixed endpoints.

(ii) *γ is* **parametrized proportional to the arclength**, *i.e. the velocity $|\dot\gamma(t)| \equiv c \geq 0$ is constant, and either γ is constant, i.e. $\gamma(t) = p = q$ for all $t \in I$, or $c > 0$ and γ is an* **extremal of the length functional**, *i.e.*

$$\left.\frac{d}{ds}\right|_{s=0} L(\gamma_s) = 0$$

for every variation $\{\gamma_s\}_{s\in\mathbb{R}}$ of γ with fixed endpoints.

(iii) *The velocity vector of γ is parallel, i.e. $\nabla\dot\gamma(t) = 0$ for all $t \in I$.*

(iv) *The acceleration of γ is normal to M, i.e. $\ddot\gamma(t) \perp T_{\gamma(t)}M$ for all $t \in I$.*

(v) *If (Φ, γ, γ') is a development of M along $M' = \mathbb{R}^m$, then $\gamma' : I \to \mathbb{R}^m$ is a straight line parametrized proportional to the arclength, i.e. $\ddot\gamma' \equiv 0$.*

Proof. See page 170. □

Definition 4.1.5 (Geodesic). *A smooth curve $\gamma : I \to M$ on an interval I is called a* **geodesic** *if its restriction to each compact subinterval satisfies the equivalent conditions of Theorem 4.1.4. So γ is a geodesic if and only if*

$$\nabla\dot\gamma(t) = 0 \qquad \text{for all } t \in I. \tag{4.1.4}$$

By the Gauß–Weingarten formula (3.2.2) with $X = \dot\gamma$ this is equivalent to

$$\ddot\gamma(t) = h_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t)) \qquad \text{for all } t \in I. \tag{4.1.5}$$

Remark 4.1.6. (i) The conditions (i) and (ii) in Theorem 4.1.4 are meaningless when I is not compact because then the curve has at most one endpoint and the length and energy integrals may be infinite. However, the conditions (iii), (iv), and (v) in Theorem 4.1.4 are equivalent for smooth curves on any interval, compact or not.

(ii) The function $s \mapsto E(\gamma_s)$ associated to a smooth variation is always smooth and so condition (i) in Theorem 4.1.4 is meaningful. However, more care has to be taken in part (ii) because the function $s \mapsto L(\gamma_s)$ need not be differentiable. However, it is differentiable at $s = 0$ whenever $\dot\gamma(t) \neq 0$ for all $t \in I$.

The Space of Paths

Fix two points $p, q \in M$ and a compact interval $I = [a, b]$ and denote by

$$\Omega_{p,q} := \Omega_{p,q}(I) := \big\{\gamma : I \to M \,\big|\, \gamma \text{ is smooth and } \gamma(a) = p,\, \gamma(b) = q\big\}$$

the space of smooth curves in M from p to q, defined on the interval I. Then the length and energy are functionals $L, E : \Omega_{p,q} \to \mathbb{R}$ and their extremal points can be understood as *critical points* as we now explain.

We may think of the space $\Omega_{p,q}$ as a kind of *"infinite dimensional manifold"*. This is to be understood in a heuristic sense and we use these terms here to emphasize an analogy. Of course, the space $\Omega_{p,q}$ is not a manifold in the strict sense of the word. To begin with it is not embedded in any finite dimensional Euclidean space. However, it has many features in common with manifolds. The first is that we can speak of *smooth curves in* $\Omega_{p,q}$. Of course $\Omega_{p,q}$ is itself a space of curves in M. Thus a smooth curve in $\Omega_{p,q}$ would then be a curve of curves, namly a map $\mathbb{R} \to \Omega_{p,q} : s \mapsto \gamma_s$ that assigns to each real number s a smooth curve $\gamma_s : I \to M$ satisfying $\gamma_s(a) = p$ and $\gamma_s(b) = q$. We shall call such a curve of curves **smooth** if the associated map $\mathbb{R} \times I \to M : (s, t) \mapsto \gamma_s(t)$ is smooth. Thus smooth curves in $\Omega_{p,q}$ are the variations of γ with fixed endpoints introduced in Definition 4.1.1.

Having defined what we mean by a smooth curve in $\Omega_{p,q}$ we can also differentiate such a curve with respect to s. Here we can simply recall that, since $M \subset \mathbb{R}^n$, we have a smooth map $\mathbb{R} \times I \to \mathbb{R}^n$ and the derivative of the curve $s \mapsto \gamma_s$ in $\Omega_{p,q}$ can simply be understood as the partial derivative of the map $(s, t) \mapsto \gamma_s(t)$ with respect to s. Thus, in analogy with embedded manifolds, we define the **tangent space** of the space of curves $\Omega_{p,q}$ at γ as the set of all derivatives of smooth curves $\mathbb{R} \to \Omega_{p,q} : s \mapsto \gamma_s$ passing through γ, i.e.

$$T_\gamma \Omega_{p,q} := \left\{ \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_s \,\middle|\, \mathbb{R} \to \Omega_{p,q} : s \mapsto \gamma_s \text{ is smooth and } \gamma_0 = \gamma \right\}.$$

Let us denote such a partial derivative by $X(t) := \frac{\partial}{\partial s}\big|_{s=0} \gamma_s(t) \in T_{\gamma(t)}M$. Thus we obtain a smooth vector field along γ. Since $\gamma_s(a) = p$ and $\gamma_s(b) = q$ for all s, this vector field must vanish at $t = a, b$. This suggests the formula

$$T_\gamma \Omega_{p,q} = \{X \in \mathrm{Vect}(\gamma) \,|\, X(a) = 0,\, X(b) = 0\}. \tag{4.1.6}$$

That every tangent vector of the path space $\Omega_{p,q}$ at γ is a vector field along γ vanishing at the endpoints follows from the above discussion. The converse inclusion is the content of the next lemma.

Lemma 4.1.7. *Let $p, q \in M$, $\gamma \in \Omega_{p,q}$, and $X \in \mathrm{Vect}(\gamma)$ with $X(a) = 0$ and $X(b) = 0$. Then there exists a smooth map $\mathbb{R} \to \Omega_{p,q} : s \mapsto \gamma_s$ such that*

$$\gamma_0(t) = \gamma(t), \qquad \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_s(t) = X(t) \qquad \textit{for all } t \in I. \tag{4.1.7}$$

Proof. The proof has two steps.

Step 1. *There exists smooth map $M \times I \to \mathbb{R}^n : (r,t) \mapsto Y_t(r)$ with compact support such that $Y_t(r) \in T_rM$ for all $t \in I$ and $r \in M$, $Y_t(r) = 0$ for all $t \in I$ and all $r \in M \setminus K$, and $Y_a(r) = Y_b(r) = 0$ for all $r \in M$.*

Define $Z_t(r) := \Pi(r)X(t)$ for $t \in I$ and $r \in M$. Choose an open set $U \subset \mathbb{R}^n$ such that $\gamma(I) \subset U$ and $\overline{U} \cap M$ is compact (e.g. take $U := \bigcup_{a \le t \le b} B_\varepsilon(\gamma(t))$ for $\varepsilon > 0$ sufficiently small). Now let $\beta : \mathbb{R}^n \to [0,1]$ be a smooth cutoff function with support in the unit ball such that $\beta(0) = 1$ and define the vector fields Y_t by $Y_t(r) := \beta(\varepsilon^{-1}(r - \gamma(t)))Z_t(r)$ for $t \in I$ and $r \in M$.

Step 2. *We prove the lemma.*

The vector field $Y_t : M \to TM$ in Step 1 is complete for each t. Thus there exists a unique smooth map $\mathbb{R} \times I \to M : (s,t) \mapsto \gamma_s(t)$ such that, for each $t \in I$, the curve $\mathbb{R} \to M : s \mapsto \gamma_s(t)$ is the unique solution of the differential equation $\frac{\partial}{\partial s}\gamma_s(t) = Y_t(\gamma_s(t))$ with $\gamma_0(t) = \gamma(t)$. These maps γ_s satisfies (4.1.7) by Step 1. □

We can now define the **derivative of the energy functional** E at γ in the direction of a tangent vector $X \in T_\gamma \Omega_{p,q}$ by

$$dE(\gamma)X := \left.\frac{d}{ds}\right|_{s=0} E(\gamma_s), \tag{4.1.8}$$

where $s \mapsto \gamma_s$ is as in Lemma 4.1.7. Similarly, the **derivative of the length functional** L at γ in the direction of $X \in T_\gamma \Omega_{p,q}$ is defined by

$$dL(\gamma)X := \left.\frac{d}{ds}\right|_{s=0} L(\gamma_s). \tag{4.1.9}$$

To define (4.1.8) and (4.1.9) the functions $s \mapsto E(\gamma_s)$ and $s \mapsto L(\gamma_s)$ must be differentiable at $s = 0$. This is true for E but it only holds for L when $\dot\gamma(t) \neq 0$ for all $t \in I$. Second, we must show that the right hand sides of (4.1.8) and (4.1.9) depend only on X and not on the choice of $\{\gamma_s\}_{s\in\mathbb{R}}$. Third, we must verify that $dE(\gamma) : T_\gamma\Omega_{p,q} \to \mathbb{R}$ and $dL(\gamma) : \Omega_{p,q} \to \mathbb{R}$ are linear maps. This is an exercise in first year analysis (see also the proof of Theorem 4.1.4). A curve $\gamma \in \Omega_{p,q}$ is is then an extremal point of E (respectively L when $\dot\gamma(t) \neq 0$ for all t) if and only if $dE(\gamma) = 0$ (respectively $dL(\gamma) = 0$). Such a curve is also called a **critical point** of E (respectively L).

Characterization of Geodesics

Proof of Theorem 4.1.4. The equivalence of (iii) and (iv) follows directly from the equations $\nabla\dot\gamma(t) = \Pi(\gamma(t))\ddot\gamma(t)$ and $\ker(\Pi(\gamma(t))) = T_{\gamma(t)}M^\perp$.

We prove that (i) is equivalent to (iii) and (iv). Let $X \in T_\gamma\Omega_{p,q}$ and choose a smooth curve of curves $\mathbb{R} \to \Omega_{p,q} : s \mapsto \gamma_s$ satisfying (4.1.7). Then the function $(s,t) \mapsto |\dot\gamma_s(t)|^2$ is smooth and hence

$$\begin{aligned} dE(\gamma)X &= \left.\frac{d}{ds}\right|_{s=0} E(\gamma_s) \\ &= \left.\frac{d}{ds}\right|_{s=0} \frac{1}{2}\int_a^b |\dot\gamma_s(t)|^2\,dt \\ &= \frac{1}{2}\int_a^b \left.\frac{\partial}{\partial s}\right|_{s=0} |\dot\gamma_s(t)|^2\,dt \\ &= \int_a^b \left\langle \dot\gamma(t), \left.\frac{\partial}{\partial s}\right|_{s=0} \dot\gamma_s(t)\right\rangle dt \\ &= \int_a^b \left\langle \dot\gamma(t), \dot X(t)\right\rangle dt \\ &= -\int_a^b \langle \ddot\gamma(t), X(t)\rangle\,dt. \end{aligned} \tag{4.1.10}$$

That (iii) implies (i) follows directly from this identity. To prove that (i) implies (iv) we argue indirectly and assume that there exists a point $t_0 \in [0,1]$ such that $\ddot\gamma(t_0)$ is not orthogonal to $T_{\gamma(t_0)}M$. Then there exists a vector $v_0 \in T_{\gamma(t_0)}M$ such that $\langle\ddot\gamma(t_0), v_0\rangle > 0$. We may assume without loss of generality that $a < t_0 < b$. Then there exists a constant $\varepsilon > 0$ such that $a < t_0 - \varepsilon < t_0 + \varepsilon < b$ and

$$t_0 - \varepsilon < t < t_0 + \varepsilon \qquad \Longrightarrow \qquad \langle\ddot\gamma(t), \Pi(\gamma(t))v_0\rangle > 0.$$

Now choose a smooth cutoff function $\beta : I \to [0,1]$ such that $\beta(t) = 0$ for all $t \in I$ with $|t - t_0| \geq \varepsilon$ and $\beta(t_0) = 1$. Define $X \in T_\gamma\Omega_{p,q}$ by

$$X(t) := \beta(t)\Pi(\gamma(t))v_0 \qquad \text{for } t \in I.$$

Then $\langle\ddot\gamma(t), X(t)\rangle \geq 0$ for all t and $\langle\ddot\gamma(t_0), X(t_0)\rangle > 0$. Hence

$$dE(\gamma)X = -\int_a^b \langle\ddot\gamma(t), X(t)\rangle\,dt < 0$$

and so γ does not satisfy (i). Thus (i) is equivalent to (iii) and (iv).

We prove that (i) is equivalent to (ii). Assume first that γ satisfies (i). Then γ also satisfies (iv) and hence $\ddot{\gamma}(t) \perp T_{\gamma(t)}M$ for all $t \in I$. This implies

$$0 = \langle \ddot{\gamma}(t), \dot{\gamma}(t) \rangle = \frac{1}{2}\frac{d}{dt}\left|\dot{\gamma}(t)\right|^2.$$

Hence the function $I \to \mathbb{R} : t \mapsto |\dot{\gamma}(t)|^2$ is constant. Choose $c \geq 0$ such that $|\dot{\gamma}(t)| \equiv c$. If $c = 0$ then $\gamma(t)$ is constant and so $\gamma(t) \equiv p = q$. If $c > 0$ then

$$\begin{aligned} dL(\gamma)X &= \left.\frac{d}{ds}\right|_{s=0} \int_a^b |\dot{\gamma}_s(t)|\, dt \\ &= \int_a^b \left.\frac{\partial}{\partial s}\right|_{s=0} |\dot{\gamma}_s(t)|\, dt \\ &= \int_a^b |\dot{\gamma}(t)|^{-1} \left\langle \dot{\gamma}(t), \left.\frac{\partial}{\partial s}\right|_{s=0} \dot{\gamma}_s(t) \right\rangle dt \\ &= \frac{1}{c}\int_a^b \left\langle \dot{\gamma}(t), \dot{X}(t) \right\rangle dt \\ &= \frac{1}{c} dE(\gamma)X. \end{aligned}$$

Thus, in the case $c > 0$, γ is an extremal point of E if and only if it is an extremal point of L. Hence (i) is equivalent to (ii).

We prove that (iii) is equivalent to (v). Let (Φ, γ, γ') be a development of M along $M' = \mathbb{R}^m$. Then $\dot{\gamma}'(t) = \Phi(t)\dot{\gamma}(t)$ and $\frac{d}{dt}\Phi(t)X(t) = \Phi(t)\nabla X(t)$ for all $X \in \mathrm{Vect}(\gamma)$ and all $t \in I$. Take $X = \dot{\gamma}$ to obtain $\ddot{\gamma}'(t) = \Phi(t)\nabla\dot{\gamma}(t)$ for all $t \in I$. Thus $\nabla\dot{\gamma} \equiv 0$ if and only if $\ddot{\gamma}' \equiv 0$. This proves Theorem 4.1.4. □

Remark 4.1.3 shows that reparametrization by a nundecreasing surjective map $\sigma : I' \to I$ gives rise to map

$$\Omega_{p,q}(I) \to \Omega_{p,q}(I') : \gamma \mapsto \gamma \circ \sigma$$

which preserves the length functional, i.e.

$$L(\gamma \circ \sigma) = L(\gamma)$$

for all $\gamma \in \Omega_{p,q}(I)$. Thus the chain rule in infinite dimensions should assert that if $\gamma\circ\sigma$ is an extremal (i.e. critical) point of L, then γ is an extremal point of L. moreover, if σ is a diffeomorphism the map $\gamma \mapsto \gamma \circ \sigma$ is bijective and should give rise to a bijective correspondence between the extremal points of L on $\Omega_{p,q}(I)$ and those on $\Omega_{p,q}(I')$. Finally, if the tangent vector field $\dot{\gamma}$ vanishes nowhere, then γ can be parametrized by the arclength. This is spelled out in more detail in the next exercise.

Exercise 4.1.8. Let $\gamma : I = [a,b] \to M$ be a smooth curve such that

$$\dot\gamma(t) \neq 0$$

for all $t \in I$ and define

$$T := L(\gamma) = \int_a^b |\dot\gamma(t)| \, dt.$$

(i) Prove that there exists a unique diffeomorphism $\sigma : [0,T] \to I$ such that

$$\sigma(t') = t \qquad \Longleftrightarrow \qquad t' = \int_a^t |\dot\gamma(s)| \, ds$$

for all $t' \in [0,T]$ and all $t \in [a,b]$. Prove that $\gamma' := \gamma \circ \sigma : [0,T] \to M$ is **parametrized by the arclength**, i.e. $|\dot\gamma'(t')| = 1$ for all $t' \in [0,T]$.
(ii) Prove that

$$dL(\gamma)X = -\int_a^b \langle \dot V(t), X(t)\rangle \, dt, \qquad V(t) := |\dot\gamma(t)|^{-1} \dot\gamma(t). \tag{4.1.11}$$

Hint: See the relevant formula in the proof of Theorem 4.1.4.
(iii) Prove that γ is an extremal point of L if and only if the curve γ' in part (i) is a geodesic.
(iv) Prove that γ is an extremal point of L if and only if there exists a geodesic $\gamma' : I' \to M$ and a diffeomorphism $\sigma : I' \to I$ such that $\gamma' = \gamma \circ \sigma$.

Next we generalize this exercise to cover the case where $\dot\gamma$ is allowed to vanish. Recall from Remark 4.1.6 that the function $s \mapsto L(\gamma_s)$ need not be differentiable. As an example consider the case where $\gamma = \gamma_0$ is constant (see also Exercise 4.5.12 below).

Exercise 4.1.9. Let $\gamma : I \to M$ be a smooth curve and let $X \in T_\gamma\Omega_{p,q}(I)$. Choose a smooth curve of curves $\mathbb{R} \to \Omega_{p,q}(I) : s \mapsto \gamma_s$ that satisfies (4.1.7). Prove that the one-sided derivatives of the function $s \mapsto L(\gamma_s)$ exist at $s = 0$ and satisfy the inequalities

$$-\int_I \left|\dot X(t)\right| \, dt \le \frac{d}{ds} L(\gamma_s)\bigg|_{s=0} \le \int_I \left|\dot X(t)\right| \, dt.$$

Exercise 4.1.10. Let (Φ, γ, γ') be a development of M along M'. Show that γ is a geodesic in M if and only if γ' is a geodesic in M'.

4.2 Distance

Assume that $M \subset \mathbb{R}^n$ is a connected smooth m-dimensional submanifold. Two point $p, q \in M$ are of distance $|p - q|$ apart in the ambient Euclidean space $\mathbb{R}^n$. In this section we define a distance function which is more intimately tied to M by minimizing the length functional over the space of curves in M with fixed endpoints. Thus it may happen that two points in M have a very short distance in $\mathbb{R}^n$ but can only be joined by very long curves in M (see Figure 4.1). This leads to the *intrinsic distance in M*. Throughout we denote by $I = [0, 1]$ the unit interval and, for $p, q \in M$, by

$$\Omega_{p,q} := \{\gamma : [0,1] \to M \mid \gamma \text{ is smooth and } \gamma(0) = p,\, \gamma(1) = q\} \qquad (4.2.1)$$

the space of smooth paths on the unit interval joining p to q. Since M is connected the set $\Omega_{p,q}$ is nonempty for all $p, q \in M$. (Prove this!)

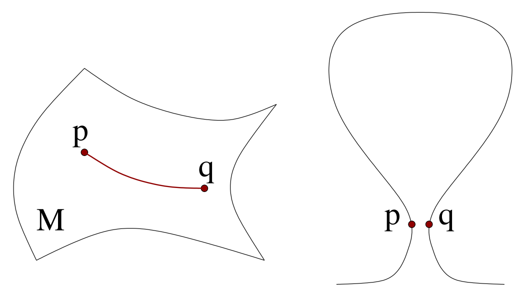


Figure 4.1: Curves in M.

Definition 4.2.1. *The* **intrinsic distance** *between two points $p, q \in M$ is the real number $d(p, q) \geq 0$ defined by*

$$d(p, q) := \inf_{\gamma \in \Omega_{p,q}} L(\gamma). \qquad (4.2.2)$$

The inequality $d(p, q) \geq 0$ holds because each curve has nonnegative length and the inequality $d(p, q) < \infty$ holds because $\Omega_{p,q} \neq \emptyset$.

Remark 4.2.2. Every smooth curve $\gamma : [0, 1] \to \mathbb{R}^n$ with endpoints $\gamma(0) = p$ and $\gamma(1) = q$ satisfies the inequality

$$L(\gamma) = \int_0^1 |\dot{\gamma}(t)|\; dt \geq \left| \int_0^1 \dot{\gamma}(t)\, dt \right| = |p - q|\,.$$

Thus $d(p, q) \geq |p - q|$. For $\gamma(t) := p + t(q - p)$ we have equality and hence the straight lines minimize the length among all curves from p to q.

Lemma 4.2.3. *The function $d : M \times M \to [0, \infty)$ defines a metric on M:*

(i) *If $p, q \in M$ satisfy $d(p, q) = 0$ then $p = q$.*

(ii) *For all $p, q \in M$ we have $d(p, q) = d(q, p)$.*

(iii) *For all $p, q, r \in M$ we have $d(p, r) \le d(p, q) + d(q, r)$.*

Proof. By Remark 4.2.2 we have $d(p, q) \ge |p - q|$ for all $p, q \in M$ and this proves part (i). Part (ii) follows from the fact that the curve $\widetilde{\gamma}(t) := \gamma(1 - t)$ has the same length as γ and belongs to $\Omega_{q,p}$ whenever $\gamma \in \Omega_{p,q}$. To prove part (iii) fix a constant $\varepsilon > 0$ and choose curves $\gamma_0 \in \Omega_{p,q}$ and $\gamma_1 \in \Omega_{q,r}$ such that $L(\gamma_0) < d(p, q) + \varepsilon$ and $L(\gamma_1) < d(q, r) + \varepsilon$. By Remark 4.1.3 we may assume without loss of generality that $\gamma_0(1 - t) = \gamma_1(t) = q$ for $t > 0$ sufficiently small. Under this assumption the curve

$$\gamma(t) := \begin{cases} \gamma_0(2t), & \text{for } 0 \le t \le 1/2, \\ \gamma_1(2t - 1), & \text{for } 1/2 < t \le 1 \end{cases}$$

is smooth. Moreover, $\gamma(0) = p$ and $\gamma(1) = r$ and so $\gamma \in \Omega_{p,r}$. Thus

$$d(p, r) \le L(\gamma) = L(\gamma_0) + L(\gamma_1) < d(p, q) + d(q, r) + 2\varepsilon.$$

Hence $d(p, r) < d(p, q) + d(q, r) + 2\varepsilon$ for every $\varepsilon > 0$. This proves part (iii) and Lemma 4.2.3. □

Remark 4.2.4. It is natural to ask if the infimum in (4.2.2) is always attained. This is easily seen not to be the case in general. For example, let M result from the Euclidean space $\mathbb{R}^m$ by removing a point p_0. Then the distance $d(p, q) = |p - q|$ is equal to the length of the line segment from p to q and any other curve from p to q is longer. Hence if p_0 is in the interior of this line segment the infimum is not attained. We shall prove below that the infimum is attained whenever M is complete.

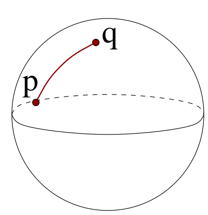


Figure 4.2: A geodesic on the 2-sphere.

Example 4.2.5. Let

$$M := S^2 = \{p \in \mathbb{R}^3 \mid |p| = 1\}$$

be the unit sphere in $\mathbb{R}^3$ and fix two points $p, q \in S^2$. Then $d(p,q)$ is the length of the shortest curve on the 2-sphere connecting p and q. Such a curve is a segment on a great circle through p and q (see Figure 4.2) and its length is

$$d(p,q) = \cos^{-1}(\langle p, q\rangle), \tag{4.2.3}$$

where $\langle p, q\rangle$ denotes the standard inner product, and we have

$$0 \le d(p,q) \le \pi.$$

(See Example 4.3.11 below for details.) By Lemma 4.2.3 this defines a metric on S^2. **Exercise:** Prove directly that (4.2.3) is a distance function on S^2.

We now have two topologies on our manifold $M \subset \mathbb{R}^n$, namely the topology determined by the metric d in Lemma 4.2.3 and the relative topology inherited from $\mathbb{R}^n$. The latter is also determined by a distance function, namely the *extrinsic distance function* defined as the restriction of the Euclidean distance function on $\mathbb{R}^n$ to the subset M. We denote it by

$$d_0 : M \times M \to [0, \infty), \qquad d_0(p,q) := |p - q|. \tag{4.2.4}$$

A natural question is if these two metrics d and d_0 induce the same topology on M. In other words is a subset $U \subset M$ open with respect to d_0 if and only if it is open with respect to d? Or, equivalently, does a sequence $p_\nu \in M$ converge to $p_0 \in M$ with respect to d if and only if it converges to p_0 with respect to d_0? Lemma 4.2.7 answers this question in the affirmative.

Exercise 4.2.6. Prove that every translation of $\mathbb{R}^n$ and every orthogonal transformation preserves the lengths of curves.

Lemma 4.2.7. *For every $p_0 \in M$ we have*

$$\lim_{p,q\to p_0} \frac{d(p,q)}{|p-q|} = 1.$$

Proof. See page 177. □

Lemma 4.2.8. *Let $p_0 \in M$ and let $\phi_0 : U_0 \to \Omega_0$ be a coordinate chart onto an open subset of $\mathbb{R}^m$ such that its derivative $d\phi_0(p_0) : T_{p_0}M \to \mathbb{R}^m$ is an orthogonal transformation. Then*

$$\lim_{p,q\to p_0} \frac{d(p,q)}{|\phi_0(p) - \phi_0(q)|} = 1.$$

Proof. See page 178. □

The lemmas imply that the topology M inherits as a subset of $\mathbb{R}^m$, the topology on M determined by the metric d, and the topology on M induced by the local coordinate systems on M are all the same.

Corollary 4.2.9. *For every subset $U \subset M$ the following are equivalent.*

(i) *U is open with respect to the metric d in* (4.2.2).

(ii) *U is open with respect to the metric d_0 in* (4.2.4).

(iii) *For every coordinate chart $\phi_0 : U_0 \to \Omega_0$ of M onto an open subset $\Omega_0 \subset \mathbb{R}^m$ the set $\phi_0(U_0 \cap U)$ is an open subset of $\mathbb{R}^m$.*

Proof. By Remark 4.2.2 we have

$$|p - q| \le d(p, q) \tag{4.2.5}$$

for all $p, q \in M$. Thus the identity $\mathrm{id}_M : (M, d) \to (M, d_0)$ is Lipschitz continuous with Lipschitz constant one and so every d_0-open subset of M is d-open. Conversely, let $U \subset M$ be a d-open subset of M and let $p_0 \in U$ and $\varepsilon > 0$. Then, by Lemma 4.2.7, there exists a constant $\delta > 0$ such that all $p, q \in M$ with $|p - p_0| < \delta$ and $|q - p_0| < \delta$ satisfy

$$d(p, q) \le (1 + \varepsilon)|p - q|.$$

Since U is d-open, there exists a constant $\rho > 0$ such that

$$B_\rho(p_0, d) \subset U.$$

With

$$\rho_0 := \min\left\{\delta, \frac{\rho}{1+\varepsilon}\right\}$$

this implies $B_{\rho_0}(p_0, d_0) \subset U$. Namely, if $p \in M$ satisfies

$$|p - p_0| < \rho_0 \le \delta$$

then

$$d(p, p_0) \le (1 + \varepsilon)|p - p_0| < (1 + \varepsilon)\rho_0 \le \rho$$

and so $p \in U$. Thus U is d_0-open and this proves that (i) is equivalent to (ii).

That (ii) implies (iii) follows from the fact that each coordinate chart ϕ_0 is a homeomorphism. To prove that (iii) implies (i), we argue indirectly and assume that U is not d-open. Then there exists a sequence $p_\nu \in M \setminus U$ that converges to an element $p_0 \in U$. Let $\phi_0 : U_0 \to \Omega_0$ be a coordinate chart with $p_0 \in U_0$. Then $\lim_{\nu\to\infty} |\phi_0(p_\nu) - \phi_0(p_0)| = 0$ by Lemma 4.2.8. Thus $\phi_0(U_0 \cap U)$ is not open and so U does not satisfy (iii). □

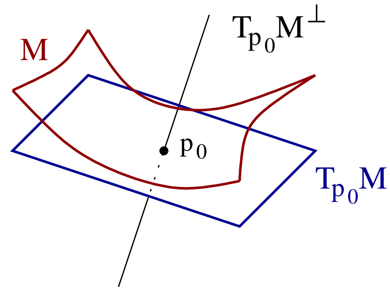


Figure 4.3: Locally, M is the graph of f.

Proof of Lemma 4.2.7. By Remark 4.2.2 the estimate $|p-q| \le d(p,q)$ holds for all $p,q \in M$. The lemma asserts that, for all $p_0 \in M$ and all $\varepsilon > 0$, there exists a d_0-open neighborhood $U_0 \subset M$ of p_0 such that all $p,q \in U_0$ satisfy

$$|p-q| \le d(p,q) \le (1+\varepsilon)|p-q|. \tag{4.2.6}$$

Let $p_0 \in M$ and $\varepsilon > 0$, and define $x : \mathbb{R}^n \to T_{p_0}M$ and $y : \mathbb{R}^n \to T_{p_0}M^{\perp}$ by

$$x(p) := \Pi(p_0)(p-p_0), \qquad y(p) := (\mathbb{1} - \Pi(p_0))\,(p-p_0),$$

where $\Pi(p_0) : \mathbb{R}^n \to T_{p_0}M$ denotes the orthogonal projection as usual. Then the derivative of the map $x|_M : M \to T_{p_0}M$ at $p = p_0$ is the identity on $T_{p_0}M$. Hence the Inverse Function Theorem 2.2.15 asserts that the map $x|_M : M \to T_{p_0}M$ is locally invertible near p_0. Extending this inverse to a smooth map from $T_{p_0}M$ to $\mathbb{R}^n$ and composing it with the map $y : M \to T_{p_0}M^{\perp}$, we obtain a smooth map

$$f : T_{p_0}M \to T_{p_0}M^{\perp}$$

and an open neighborhood $W \subset \mathbb{R}^n$ of p_0 such that

$$p \in M \qquad \Longleftrightarrow \qquad y(p) = f(x(p))$$

for all $p \in W$ (see Figure 4.3). Moreover, by definition the map f satisfies

$$f(0) = 0 \in T_{p_0}M^{\perp}, \qquad df(0) = 0 : T_{p_0}M \to T_{p_0}M^{\perp}.$$

Hence there exists a constant $\delta > 0$ such that, for every $x \in T_{p_0}M$, we have

$$|x| < \delta \quad \Longrightarrow \quad x + f(x) \in W \text{ and } \|df(x)\| = \sup_{0 \ne \widehat{x} \in T_{p_0}M} \frac{|df(x)\widehat{x}|}{|\widehat{x}|} < \varepsilon.$$

Define

$$U_0 := \{p \in M \cap W \mid |x(p)| < \delta\}.$$

Given $p, q \in U_0$ let $\gamma : [0,1] \to M$ be the curve whose projection to the x-axis is the straight line joining $x(p)$ to $x(q)$, i.e.

$$\begin{aligned} x(\gamma(t)) &= x(p) + t(x(q) - x(p)) =: x(t), \\ y(\gamma(t)) &= f(x(\gamma(t))) = f(x(t)) =: y(t). \end{aligned}$$

Then $\gamma(t) \in U_0$ for all $t \in [0,1]$ and

$$\begin{aligned} L(\gamma) &= \int_0^t |\dot{x}(t) + \dot{y}(t)|\; dt \\ &= \int_0^t |\dot{x}(t) + df(x(t))\dot{x}(t)|\; dt \\ &\le \int_0^t \bigl(1 + \|df(x(t))\|\bigr) |\dot{x}(t)|\; dt \\ &\le (1+\varepsilon) \int_0^t |\dot{x}(t)|\; dt \\ &= (1+\varepsilon) |x(p) - x(q)| \\ &= (1+\varepsilon) |\Pi(p_0)(p-q)| \\ &\le (1+\varepsilon) |p-q|. \end{aligned}$$

Hence $d(p,q) \le L(\gamma) \le (1+\varepsilon)|p-q|$ and this proves Lemma 4.2.7. □

Proof of Lemma 4.2.8. By assumption we have

$$|d\phi_0(p_0)v| = |v|$$

for all $v \in T_{p_0}M$. Fix a constant $\varepsilon > 0$. Then, by continuity of the derivative, there exists a d_0-open neighborhood $M_0 \subset M$ of p_0 such that for all $p \in M_0$ and all $v \in T_pM$ we have

$$(1-\varepsilon)|d\phi_0(p)v| \le |v| \le (1+\varepsilon)|d\phi_0(p)v|.$$

Thus for every curve $\gamma : [0,1] \to M_0$ we have

$$(1-\varepsilon)L(\phi_0 \circ \gamma)) \le L(\gamma) \le (1+\varepsilon)L(\phi_0 \circ \gamma).$$

One is tempted to take the infimum over all curves $\gamma : [0,1] \to M_0$ joining two pints $p, q \in M_0$ to obtain the inequality

$$(1-\varepsilon)|\phi_0(p) - \phi_0(q)| \le d(p,q) \le (1+\varepsilon)|\phi_0(p) - \phi_0(q)|. \tag{4.2.7}$$

However, we must justify these inequalities by showing that the infimum over all curves in M_0 agrees with the infimum over all curves in M joining the points p and q.

It suffices to show that the inequalities hold on a smaller heighborhood $M_1 \subset M_0$ of p_0. Choose such a smaller neighborhood M_1 such that the open set $\phi_0(M_1)$ is a convex subset of Ω_0. Then the right inequality in (4.2.7) follows by taking the curve $\gamma : [0,1] \to M_1$ from $\gamma(0) = p$ to $\gamma(1) = q$ such that $\phi_0 \circ \gamma : [0,1] \to \phi_0(M_1)$ is a straight line. To prove the left inequality in (4.2.7) we use the fact that M_0 is d-open by Lemma 4.2.7. Hence, after shrinking M_1 if necessary, there exists a constant $r > 0$ such that

$$p_0 \in M_1 \subset B_r(p_0, d) \subset B_{3r}(p_0, d) \subset M_0.$$

Then, for $p, q \in M_1$ we have $d(p,q) \le 2r$ while $L(\gamma) \ge 4r$ for any curve γ from p to q which leaves M_0. Hence the distance $d(p,q)$ of $p, q \in M_1$ is the infimum of the lengths $L(\gamma)$ over all curves $\gamma : [0,1] \to M_0$ that join $\gamma(0) = p$ to $\gamma(1) = q$. This proves the left inequality in (4.2.7) and Lemma 4.2.8. $\square$

A next question one might ask is the following. Can we choose a coordinate chart $\phi : U \to \Omega$ on M with values in an open set $\Omega \subset \mathbb{R}^m$ so that the length of each smooth curve $\gamma : [0,1] \to U$ is equal to the length of the curve $c := \phi \circ \gamma : [0,1] \to \Omega$? We examine this question by considering the inverse map $\psi := \phi^{-1} : \Omega \to U$. Denote the components of x and $\psi(x)$ by

$$x = (x^1, \dots, x^m) \in \Omega, \qquad \psi(x) = (\psi^1(x), \dots, \psi^n(x)) \in U.$$

Given a smooth curve $[0,1] \to \Omega : t \mapsto c(t) = (c^1(t), \dots, c^m(t))$ we can write the length of the composition $\gamma = \psi \circ c : [0,1] \to M$ in the form

$$\begin{aligned}
L(\psi \circ c) &= \int_0^1 \left| \frac{d}{dt} \psi(c(t)) \right| dt \\
&= \int_0^1 \sqrt{\sum_{\nu=1}^n \left(\frac{d}{dt} \psi^\nu(c(t)) \right)^2} \, dt \\
&= \int_0^1 \sqrt{\sum_{\nu=1}^n \left(\sum_{i=1}^m \frac{\partial \psi^\nu}{\partial x^i}(c(t)) \dot{c}^i(t) \right)^2} \, dt \\
&= \int_0^1 \sqrt{\sum_{\nu=1}^n \sum_{i,j=1}^m \frac{\partial \psi^\nu}{\partial x^i}(c(t)) \frac{\partial \psi^\nu}{\partial x^j}(c(t)) \dot{c}^i(t) \dot{c}^j(t)} \, dt \\
&= \int_0^1 \sqrt{\sum_{i,j=1}^m \dot{c}^i(t) g_{ij}(c(t)) \dot{c}^j(t)} \, dt.
\end{aligned}$$

Here the functions $g_{ij} : \Omega \to \mathbb{R}$ are defined by

$$g_{ij}(x) := \sum_{\nu=1}^{n} \frac{\partial \psi^\nu}{\partial x^i}(x) \frac{\partial \psi^\nu}{\partial x^j}(x) = \left\langle \frac{\partial \psi}{\partial x^i}(x), \frac{\partial \psi}{\partial x^j}(x) \right\rangle. \tag{4.2.8}$$

Thus we have a smooth function $g = (g_{ij}) : \Omega \to \mathbb{R}^{m \times m}$ with values in the positive definite matrices given by $g(x) = d\psi(x)^\mathsf{T} d\psi(x)$ such that

$$L(\psi \circ c) = \int_0^1 \sqrt{\dot{c}(t)^\mathsf{T} g(c(t)) \dot{c}(t)}\, dt \tag{4.2.9}$$

for every smooth curve $c : [0,1] \to \Omega$. Thus the condition $L(\psi \circ c) = L(c)$ for every such curve is equivalent to

$$g_{ij}(x) = \delta_{ij}$$

for all $x \in \Omega$ or, equivalently,

$$d\psi(x)^\mathsf{T} d\psi(x) = \mathbb{1}. \tag{4.2.10}$$

This means that ψ preserves angles and areas. The next example shows that for $M = S^2$ it is impossible to find such coordinates.

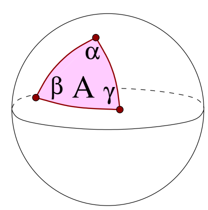


Figure 4.4: A spherical triangle.

Example 4.2.10. Consider the manifold $M = S^2$. If there is a diffeomorphism $\psi : \Omega \to U$ from an open set $\Omega \subset \mathbb{R}^2$ onto an open set $U \subset S^2$ that satisfies (4.2.10) it has to map straight lines onto arcs of great circles and it preserves the area. However, the area A of a spherical triangle bounded by three arcs on great circles satisfies the angle sum formula

$$\alpha + \beta + \gamma = \pi + A.$$

(See Figure 4.4.) Hence there can be no such map ψ.

4.3 Exponential Map

Geodesic Spray

The tangent bundle TM is a smooth $2m$-dimensional manifold in $\mathbb{R}^n \times \mathbb{R}^n$ by Corollary 2.6.10. The next lemma characterizes the tangent bundle of the tangent bundle.

Lemma 4.3.1. *The tangent space of TM at $(p,v) \in TM$ is given by*

$$T_{(p,v)}TM = \left\{ (\hat{p}, \hat{v}) \in \mathbb{R}^n \times \mathbb{R}^n \,\middle|\, \begin{array}{l} \hat{p} \in T_pM \text{ and} \\ \big(\mathbb{1} - \Pi(p)\big)\hat{v} = h_p(\hat{p}, v) \end{array} \right\}. \qquad (4.3.1)$$

Proof. We prove the inclusion "$\subset$" in (4.3.1). Let $(\widehat{p}, \widehat{v}) \in T_{(p,v)}TM$ and choose a smooth curve $\mathbb{R} \to TM : t \mapsto (\gamma(t), X(t))$ such that

$$\gamma(0) = p, \qquad X(0) = v, \qquad \dot{\gamma}(0) = \widehat{p}, \qquad \dot{X}(0) = \widehat{v}.$$

Then the Gauß–Weingarten formula (3.2.2) asserts that

$$\dot{X}(t) = \nabla X(t) + h_{\gamma(t)}(\dot{\gamma}(t), X(t))$$

and hence $(\mathbb{1} - \Pi(\gamma(t)))\dot{X}(t) = h_{\gamma(t)}(\dot{\gamma}(t), X(t))$ for all $t \in \mathbb{R}$. Take $t = 0$ to obtain $(\mathbb{1} - \Pi(p))\widehat{v} = h_p(\widehat{p}, v)$. This proves the inclusion "$\subset$" in (4.3.1). Equality holds because both sides of the equation are $2m$-dimensional linear subspaces of $\mathbb{R}^n \times \mathbb{R}^n$. $\square$

By Lemma 4.3.1 a smooth map $S = (S_1, S_2) : TM \to \mathbb{R}^n \times \mathbb{R}^n$ is a vector field on TM if and only if

$$S_1(p,v) \in T_pM, \qquad (\mathbb{1} - \Pi(p))S_2(p,v) = h_p(S_1(p,v), v)$$

for all $(p,v) \in TM$. A special case is where $S_1(p,v) = v$. Such vector fields correspond to second order differential equations on M.

Definition 4.3.2 (Spray). *A vector field $S \in \mathrm{Vect}(TM)$ is called a* **spray** *if it has the form $S(p,v) = (v, S_2(p,v))$ where $S_2 : TM \to \mathbb{R}^n$ is a smooth map satisfying*

$$(\mathbb{1} - \Pi(p))S_2(p,v) = h_p(v,v), \qquad S_2(p, \lambda v) = \lambda^2 S_2(p,v) \qquad (4.3.2)$$

for all $(p,v) \in TM$ and $\lambda \in \mathbb{R}$. The vector field $S \in \mathrm{Vect}(TM)$ defined by

$$S(p,v) := (v, h_p(v,v)) \quad \in \quad T_{(p,v)}TM \qquad (4.3.3)$$

for $p \in M$ and $v \in T_pM$ is called the **geodesic spray**.

Exponential Map

Lemma 4.3.3. *Let $\gamma : I \to M$ be a smooth curve on an open interval $I \subset \mathbb{R}$. Then γ is a geodesic if and only if the curve $I \to TM : t \mapsto (\gamma(t), \dot\gamma(t))$ is an integral curve of the geodesic spray S in (4.3.3).*

Proof. A smooth curve $I \to TM : t \mapsto (\gamma(t), X(t))$ is an integral curve of S if and only if

$$\dot\gamma(t) = X(t), \qquad \dot X(t) = h_{\gamma(t)}(X(t), X(t))$$

for all $t \in I$. By equation (4.1.5), this holds if and only if γ is a geodesic and $\dot\gamma = X$. □

Combining Lemma 4.3.3 with Theorem 2.4.7 we obtain the following existence and uniqueness result for geodesics.

Lemma 4.3.4. *Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold.*

(i) *For every $p \in M$ and every $v \in T_pM$ there is an $\varepsilon > 0$ and a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ such that*

$$\nabla\dot\gamma \equiv 0, \qquad \gamma(0) = p, \qquad \dot\gamma(0) = v. \tag{4.3.4}$$

(iI) *If $\gamma_1 : I_1 \to M$ and $\gamma_2 : I_2 \to M$ are geodesics and $t_0 \in I_1 \cap I_2$ with*

$$\gamma_1(t_0) = \gamma_2(t_0), \qquad \dot\gamma_1(t_0) = \dot\gamma_2(t_0)$$

then $\gamma_1(t) = \gamma_2(t)$ for all $t \in I_1 \cap I_2$.

Proof. Lemma 4.3.3 and Theorem 2.4.7. □

Definition 4.3.5 (Exponential Map). *For $p \in M$ and $v \in T_pM$ the interval*

$$I_{p,v} := \bigcup \left\{ I \subset \mathbb{R} \,\middle|\, \begin{array}{l} I \text{ is an open interval containing } 0 \text{ and there is a} \\ \text{geodesic } \gamma : I \to M \text{ satisfying } \gamma(0) = p,\ \dot\gamma(0) = v \end{array} \right\}.$$

is called the **maximal existence interval** *for the geodesic through p in the direction v. For $p \in M$ define the set $V_p \subset T_pM$ by*

$$V_p := \{ v \in T_pM \,|\, 1 \in I_{p,v} \}. \tag{4.3.5}$$

The **exponential map** *at p is the map*

$$\exp_p : V_p \to M$$

that assigns to every tangent vector $v \in V_p$ the point $\exp_p(v) := \gamma(1)$, where $\gamma : I_{p,v} \to M$ is the unique geodesic satisfying $\gamma(0) = p$ and $\dot\gamma(0) = v$.

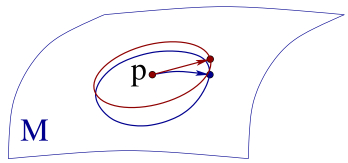


Figure 4.5: The exponential map.

Lemma 4.3.6. **(i)** *The set*

$$V := \{(p, v) \,|\, p \in M,\, v \in V_p\} \subset TM$$

is open and the map $V \to M : (p, v) \mapsto \exp_p(v)$ *is smooth.*
(ii) *If* $p \in M$ *and* $v \in V_p$*, then*

$$I_{p,v} = \{t \in \mathbb{R} \,|\, tv \in V_p\}$$

and the geodesic $\gamma : I_{p,v} \to M$ *with* $\gamma(0) = p$ *and* $\dot\gamma(0) = v$ *is given by*

$$\gamma(t) = \exp_p(tv), \qquad t \in I_{p,v}.$$

Proof. Part (i) follows directly from Lemma 4.3.3 and Theorem 2.4.9. To prove part (ii), fix an element $p \in M$ and a tangent vector $v \in V_p$, and let $\gamma : I_{p,v} \to M$ be the unique geodesic with $\gamma(0) = p$ and $\dot\gamma(0) = v$. Fix a nonzero real number λ and define the map $\gamma_\lambda : \lambda^{-1} I_{p,v} \to M$ by

$$\gamma_\lambda(t) := \gamma(\lambda t) \qquad \text{for } t \in \lambda^{-1} I_{p,v}.$$

Then $\dot\gamma_\lambda(t) = \lambda\dot\gamma(\lambda t)$ ans $\ddot\gamma_\lambda(t) = \lambda^2\ddot\gamma(\lambda t)$ and hence

$$\nabla\dot\gamma_\lambda(t) = \Pi(\gamma_\lambda(t))\ddot\gamma_\lambda(t) = \lambda^2\Pi(\gamma(\lambda t))\ddot\gamma(\lambda t) = \lambda^2\nabla\dot\gamma(\lambda t) = 0$$

for every $t \in \lambda^{-1} I_{p,v}$. This shows that γ_λ is a geodesic with

$$\gamma_\lambda(0) = p, \qquad \dot\gamma_\lambda(0) = \lambda v.$$

In particular, we have $\lambda^{-1} I_{p,v} \subset I_{p,\lambda v}$. Interchanging the roles of v and λv we obtain $\lambda^{-1} I_{p,v} = I_{p,\lambda v}$. Thus

$$\lambda \in I_{p,v} \qquad \Longleftrightarrow \qquad 1 \in I_{p,\lambda v} \qquad \Longleftrightarrow \qquad \lambda v \in V_p$$

and

$$\gamma(\lambda) = \gamma_\lambda(1) = \exp_p(\lambda v)$$

for $\lambda \in I_{p,v}$. This proves Lemma 4.3.6. $\square$

Since $\exp_p(0) = p$ by definition, the derivative of the exponential map at $v = 0$ is a linear map from T_pM to itself. This derivative is the identity map as illustrated in Figure 4.5 and proved in the following corollary.

Corollary 4.3.7. *The map* $\exp_p : V_p \to M$ *is smooth and its derivative at the origin is* $d\exp_p(0) = \mathrm{id} : T_pM \to T_pM$.

Proof. The set V_p is an open subset of the linear subspace $T_pM \subset \mathbb{R}^n$, with respect to the relative topology, and hence is a manifold. The tangent space of V_p at each point is T_pM. By Lemma 4.3.6 the exponential map $\exp_p : V_p \to M$ is smooth and its derivative at the origin is given by

$$d\exp_p(0)v = \left.\frac{d}{dt}\right|_{t=0} \exp_p(tv) = \dot\gamma(0) = v,$$

where $\gamma : I_{p,v} \to M$ is once again the unique geodesic through p in the direction v. This proves Corollary 4.3.7. $\square$

Corollary 4.3.8. *Let* $p \in M$ *and, for* $r > 0$, *denote*

$$B_r(p) := \{v \in T_pM \mid |v| < r\}.$$

If $r > 0$ *is sufficiently small then* $B_r(p) \subset V_p$, *the set*

$$U_r(p) := \exp_p(B_r(p))$$

is an open subset of M, *and the restriction of the exponential map to* $B_r(p)$ *is a diffeomorphism from* $B_r(p)$ *to* $U_r(p)$.

Proof. This follows directly from Corollary 4.3.7 and Theorem 2.2.15. $\square$

Definition 4.3.9 (Injectivity Radius). *Let* $M \subset \mathbb{R}^n$ *be a smooth* m-*manifold. The* **injectivity radius of** M **at** p *is the supremum of all* $r > 0$ *such that the restriction of the exponential map* $\exp_p$ *to* $B_r(p)$ *is a diffeomorphism onto its image*

$$U_r(p) := \exp_p(B_r(p)).$$

It will be denoted by

$$\mathrm{inj}(p) := \mathrm{inj}(p; M) := \sup\left\{ r > 0 \,\middle|\, \begin{array}{c} \exp_p : B_r(p) \to U_r(p) \\ \textit{is a diffeomorphism} \end{array} \right\}.$$

The **injectivity radius of** M *is the infimum of the injectivity radii of* M *at* p *over all* $p \in M$. *It will be denoted by*

$$\mathrm{inj}(M) := \inf_{p \in M} \mathrm{inj}(p; M).$$

Example 4.3.10. The exponential map on $\mathbb{R}^m$ is given by

$$\exp_p(v) = p + v \qquad \text{for } p, v \in \mathbb{R}^m.$$

For every $p \in \mathbb{R}^m$ this map is a diffeomorphism from $T_p\mathbb{R}^m = \mathbb{R}^m$ to $\mathbb{R}^m$ and hence the injectivity radius of $\mathbb{R}^m$ is infinity.

Example 4.3.11. The exponential map on S^m is given by

$$\exp_p(v) = \cos(|v|)p + \frac{\sin(|v|)}{|v|}v$$

for every $p \in S^m$ and every nonzero tangent vector $v \in T_pS^m = p^\perp$. The restriction of this map to the open ball of radius r in T_pM is a diffeomorphism onto its image if and only if $r \le \pi$. Hence the injectivity radius of S^m at every point is π. **Exercise:** Given $p \in S^m$ and $0 \ne v \in T_pS^m = p^\perp$, prove that the geodesic $\gamma : \mathbb{R} \to S^m$ with $\gamma(0) = p$ and $\dot\gamma(0) = v$ is given by $\gamma(t) = \cos(t\,|v|)p + \frac{\sin(t|v|)}{|v|}v$ for $t \in \mathbb{R}$. Show that, in the case $0 \le |v| \le \pi$ there is no shorter curve in S^m connecting p and q and deduce that the intrinsic distance on S^m is given by $d(p,q) = \cos^{-1}(\langle p, q\rangle)$ for $p, q \in S^m$ (see Example 4.2.5 for $m = 2$).

Example 4.3.12. Consider the orthogonal group $\mathrm{O}(n) \subset \mathbb{R}^{n\times n}$ with the standard inner product

$$\langle v, w\rangle := \mathrm{trace}\left(v^\mathsf{T} w\right)$$

on $\mathbb{R}^{n\times n}$. The orthogonal projection $\Pi(g) : \mathbb{R}^{n\times n} \to T_g\mathrm{O}(n)$ is given by

$$\Pi(g)v := \frac{1}{2}\left(v - gv^\mathsf{T} g\right)$$

and the second fundamental form by

$$h_g(v, v) = -gv^\mathsf{T} v.$$

Hence a curve $\gamma : \mathbb{R} \to \mathrm{O}(n)$ is a geodesic if and only if $\gamma^\mathsf{T}\ddot\gamma + \dot\gamma^\mathsf{T}\dot\gamma = 0$ or, equivalently, $\gamma^\mathsf{T}\dot\gamma$ is constant. This shows that geodesics in $\mathrm{O}(n)$ have the form $\gamma(t) = g\exp(t\xi)$ for $g \in \mathrm{O}(n)$ and $\xi \in \mathfrak{o}(n)$. It follows that the exponential map is given by

$$\exp_g(v) = g\exp(g^{-1}v) = \exp(vg^{-1})g$$

for $g \in \mathrm{O}(n)$ and $v \in T_g\mathrm{O}(n)$. In particular, for $g = \mathbb{1}$ the exponential map $\exp_{\mathbb{1}} : \mathfrak{o}(n) \to \mathrm{O}(n)$ agrees with the exponential matrix.

Exercise 4.3.13. What is the injectivity radius of the 2-torus $\mathbb{T}^2 = S^1 \times S^1$, the punctured 2-plane $\mathbb{R}^2 \setminus \{(0, 0)\}$, and the orthogonal group $\mathrm{O}(n)$?

4.4 Convex Neighborhoods

Geodesics in Local Coordinates

Lemma 4.4.1. *Let $M \subset \mathbb{R}^n$ be an m-dimensional manifold and choose a coordinate chart $\phi : U \to \Omega$ with inverse*

$$\psi := \phi^{-1} : \Omega \to U.$$

Let $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ be the Christoffel symbols defined by (3.6.6) *and let $c : I \to \Omega$ be a smooth curve. Then the curve $\gamma := \psi \circ c : I \to M$ is a geodesic if and only if c satisfies the 2nd order differential equation*

$$\ddot{c}^k + \sum_{i,j=1}^m \Gamma_{ij}^k(c)\dot{c}^i\dot{c}^j = 0 \tag{4.4.1}$$

for $k = 1, \ldots, m$.

Proof. This follows immediately from the definition of geodesics and equation (3.6.7) in Lemma 3.6.1 with $X = \dot{\gamma}$ and $\xi = \dot{c}$. □

We remark that Lemma 4.4.1 gives rise to another proof of Lemma 4.3.4 that is based on the existence and uniqueness of solutions of second order differential equations in local coordinates.

Exercise 4.4.2. Let $\Omega \subset \mathbb{R}^m$ be an open set and $g = (g_{ij}) : \Omega \to \mathbb{R}^{m\times m}$ be a smooth map with values in the space of positive definite symmetric matrices. Consider the energy functional

$$E(c) := \int_0^1 L(c(t), \dot{c}(t))\, dt$$

on the space of paths $c : [0, 1] \to \Omega$, where $L : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is defined by

$$L(x, \xi) := \frac{1}{2} \sum_{i,j=1}^m \xi^i g_{ij}(x)\xi^j. \tag{4.4.2}$$

The **Euler–Lagrange equations** of this variational problem have the form

$$\frac{d}{dt}\frac{\partial L}{\partial \xi^k}(c(t), \dot{c}(t)) = \frac{\partial L}{\partial x^k}(c(t), \dot{c}(t)), \qquad k = 1, \ldots, m. \tag{4.4.3}$$

Prove that the Euler–Lagrange equations (4.4.3) are equivalent to the geodesic equations (4.4.1), where the $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ are given by (3.6.10).

Convexity

A subset of an affine space is called convex iff it contains the line segment joining any two of its points. The definition carries over to a submanifold M of Euclidean space (or indeed more generally to any manifold M equipped with a spray) once we reword the definition so as to confront the difficulty that a geodesic joining two points might not exist nor, if it does, need it be unique.

Definition 4.4.3 (**Geodesically Convex Sets**)**.** *Let $M \subset \mathbb{R}^n$ be a smooth m-dimensional manifold. A subset $U \subset M$ is called* **geodesically convex** *if, for all $p_0, p_1 \in U$, there exists a unique geodesic $\gamma : [0,1] \to U$ such that $\gamma(0) = p_0$ and $\gamma(1) = p_1$.*

It is not precluded in Definition 4.4.3 that there be other geodesics from p to q which leave and then re-enter U, and these may even be shorter than the geodesic in U.

Exercise 4.4.4. **(a)** Find a geodesically convex set U in a manifold M and points $p_0, p_1 \in U$ such that the unique geodesic $\gamma : [0,1] \to U$ with $\gamma(0) = p_0$ and $\gamma(1) = p_1$ has length $L(\gamma) > d(p_0, p_1)$. **Hint:** An interval of length bigger than π in S^1.

(b) Find a set U in a manifold M such that any two points in U can be connected by a minimal geodesic in U, but U is not geodesically convex. **Hint:** A closed hemisphere in S^2.

Theorem 4.4.5 (**Convex Neighborhoods**)**.** *Let $M \subset \mathbb{R}^n$ be a smooth m-dimensional submanifold and fix a point $p_0 \in M$. Let $\phi : U \to \Omega$ be any coordinate chart on an open neighborhood $U \subset M$ of p_0 with values in an open set $\Omega \subset \mathbb{R}^m$. Then the set*

$$U_r := \{p \in U \mid |\phi(p) - \phi(p_0)| < r\} \tag{4.4.4}$$

is geodesically convex for $r > 0$ sufficiently small.

Proof. See page 188. □

Corollary 4.4.6. *Let $M \subset \mathbb{R}^n$ be a smooth m-manifold and let $p_0 \in M$. Then, for $r > 0$ sufficiently small, the open ball*

$$U_r(p_0) := \{p \in M \mid d(p_0, p) < r\} \tag{4.4.5}$$

is geodesically convex.

Proof. Choose an orthonormal basis $e_1, \dots, e_m$ of $T_{p_0}M$ and define

$$\begin{aligned} \Omega &:= \{x \in \mathbb{R}^m \mid |x| < \mathrm{inj}(p_0; M)\}, \\ U &:= \{p \in M \mid d(p_0, p) < \mathrm{inj}(p_0; M)\}. \end{aligned} \tag{4.4.6}$$

Define the map $\psi : \Omega \to U$ by

$$\psi(x) := \exp_{p_0}\left(\sum_{i=1}^m x^i e_i\right) \tag{4.4.7}$$

for $x = (x^1, \dots, x^m) \in \Omega$. Then ψ is a diffeomorphism and $d(p_0, \psi(x)) = |x|$ for all $x \in \Omega$ by Theorem 4.5.4. Hence its inverse

$$\phi := \psi^{-1} : U \to \Omega \tag{4.4.8}$$

satisfies $\phi(p_0) = 0$ and $|\phi(p)| = d(p_0, p)$ for all $p \in U$. Thus

$$U_r(p_0) = \{p \in U \mid |\phi(p) - \phi(p_0)| < r\} \qquad \text{for } 0 < r < \mathrm{inj}(p_0; M)$$

and so Corollary 4.4.6 follows from Theorem 4.4.5. □

Definition 4.4.7 (**Geodesically Normal Coordinates**). *The coordinate chart $\phi : U \to \Omega$ in* (4.4.7) *and* (4.4.8) *sends geodesics through p_0 to straight lines through the origin. Its components $x^1, \dots, x^m : U \to \mathbb{R}$ are called* **geodesically normal coordinates** *at p_0.*

Proof of Theorem 4.4.5. Assume without loss of generality that $\phi(p_0) = 0$. Let $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ be the Christoffel symbols of the coordinate chart and, for $x \in \Omega$, define the quadratic function $Q_x : \mathbb{R}^m \to \mathbb{R}$ by

$$Q_x(\xi) := \sum_{k=1}^m \left(\xi^k\right)^2 - \sum_{i,j,k=1}^m x^k \Gamma_{ij}^k(x) \xi^i \xi^j.$$

Shrinking U, if necessary, we may assume that

$$\max_{i,j=1,\dots,m} \left| \sum_{k=1}^m x^k \Gamma_{ij}^k(x) \right| \le \frac{1}{2m} \qquad \text{for all } x \in \Omega.$$

Then, for all $x \in \Omega$ and all $\xi \in \mathbb{R}^m$ we have

$$Q_x(\xi) \ge |\xi|^2 - \frac{1}{2m}\left(\sum_{i=1}^m \left|\xi^i\right|\right)^2 \ge \frac{1}{2}|\xi|^2 \ge 0.$$

Hence Q_x is positive definite for every $x \in \Omega$.

Now let $\gamma : [0,1] \to U$ be a geodesic and define

$$c(t) := \phi(\gamma(t))$$

for $0 \le t \le 1$. Then, by Lemma 4.4.1, c satisfies the differential equation

$$\ddot{c}^k + \sum_{i,j} \Gamma_{ij}^k(c)\dot{c}^i\dot{c}^j = 0.$$

Hence

$$\frac{d^2}{dt^2}\frac{|c|^2}{2} = \frac{d}{dt}\langle \dot{c}, c\rangle = |\dot{c}|^2 + \langle \ddot{c}, c\rangle = Q_c(\dot{c}) \ge \frac{|\dot{c}|^2}{2} \ge 0$$

and so the function $t \mapsto |\phi(\gamma(t))|^2$ is convex. Thus, if $\gamma(0), \gamma(1) \in U_r$ for some $r > 0$, it follows that $\gamma(t) \in U_r$ for all $t \in [0,1]$.

Consider the exponential map

$$V = \{(p,v) \in TM \,|\, v \in V_p\} \to M : (p,v) \mapsto \exp_p(v)$$

in Lemma 4.3.6. Its domain V is open and the exponential map is smooth. Since it sends the pair $(p_0, 0) \in V$ to $\exp_{p_0}(0) = p_0 \in U$, it follows from continuity that there exist constants $\varepsilon > 0$ and $r > 0$ such that

$$p \in U_r, \;\; v \in T_pM, \;\; |v| < \varepsilon \qquad \Longrightarrow \qquad v \in V_p, \;\; \exp_p(v) \in U. \tag{4.4.9}$$

Moreover, we have

$$d\exp_{p_0}(0) = \mathrm{id} : T_{p_0}M \to T_{p_0}M$$

by Corollary 4.3.7. Hence the Implicit Function Theorem 2.6.13 asserts that the constants $\varepsilon > 0$ and $r > 0$ can be chosen such that (4.4.9) holds and there exists a smooth map $h : U_r \times U_r \to \mathbb{R}^n$ that satisfies the conditions

$$h(p,q) \in T_pM, \qquad |h(p,q)| < \varepsilon \tag{4.4.10}$$

for all $p, q \in U_r$ and

$$\exp_p(v) = q \qquad \Longleftrightarrow \qquad v = h(p,q) \tag{4.4.11}$$

for all $p, q \in U_r$ and all $v \in T_pM$ with $|v| < \varepsilon$. In particular, we have

$$h(p_0, p_0) = 0$$

and $\exp_p(h(p,q)) = q$ for all $p, q \in U_r$.

Fix two constants $\varepsilon > 0$ and $r > 0$ and a smooth map $h : U_r \times U_r \to \mathbb{R}^n$ such that (4.4.9), (4.4.10), (4.4.11) are satisfied. We show that any two points $p, q \in U_r$ are joined by a geodesic in U_r. Let $p, q \in U_r$ and define

$$\gamma(t) := \exp_p(th(p,q)) \qquad \text{for } 0 \le t \le 1.$$

This curve $\gamma : [0,1] \to M$ is well defined by (4.4.9) and (4.4.10), it is a geodesic satisfying $\gamma(0) = p \in U_r$ by Lemma 4.3.6, it satisfies $\gamma(1) = q \in U_r$ by (4.4.11), it takes values in U by (4.4.9) and (4.4.10), and so $\gamma([0,1]) \subset U_r$ because the function $[0,1] \to \mathbb{R} : t \mapsto |\phi(\gamma(t))|^2$ is convex.

We show that there exists at most one geodesic in U_r joining p and q. Let $p, q \in U_r$ and let $\gamma : [0,1] \to U_r$ be any geodesic such that $\gamma(0) = p$ and $\gamma(1) = q$. Define $v := \dot\gamma(t) \in T_pM$. Then $\gamma(t) = \exp_p(tv)$ for $0 \le t \le 1$ by Lemma 4.3.6. We claim that $|v| < \varepsilon$. Suppose, by contradiction, that

$$|v| \ge \varepsilon.$$

Then

$$T := \frac{\varepsilon}{|v|} \le 1$$

and, for $0 < t < T$, we have $|tv| < \varepsilon$ and $\exp_p(tv) = \gamma(t) \in U_r$ and so

$$h(p, \gamma(t)) = tv.$$

by (4.4.11). Thus

$$|h(p,\gamma(t))| = t|v| \qquad \text{for } 0 < t < T.$$

Take the limit $t \nearrow T$ to obtain

$$|h(p,\gamma(T))| = T\,|v| = \varepsilon$$

in contradiction to (4.4.10). This contradiction shows that $|v| < \varepsilon$. Since

$$\exp_p(v) = \gamma(1) = q \in U_r$$

it follows from (4.4.11) that $v = h(p,q)$. This proves Theorem 4.4.5. □

Remark 4.4.8. Theorem 4.4.5 and its proof carry over to general sprays (see Definition 4.3.2).

Exercise 4.4.9. Consider the set $U_r(p) = \{q \in M \mid d(p,q) < r\}$ for $p \in M$ and $r > 0$. Corollary 4.4.6 asserts that this set is geodesically convex for r sufficiently small. How large can you choose r in the cases

$$M = S^2, \qquad M = \mathbb{T}^2 = S^1 \times S^1, \qquad M = \mathbb{R}^2, \qquad M = \mathbb{R}^2 \setminus \{0\}.$$

Compare this with the injectivity radius. If the set $U_r(p)$ in these examples is geodesically convex, does it follow that every geodesic in $U_r(p)$ is minimizing?

4.5 Minimal Geodesics

Any straight line segment in Euclidean space is the shortest curve joining its endpoints. The analogous assertion for geodesics in a manifold M is false; consider for example an arc which is more than half of a great circle on a sphere. In this section we consider curves which realize the shortest distance between their endpoints.

Lemma 4.5.1. *Let $I = [a, b]$ be a compact interval, let $\gamma : I \to M$ be a smooth curve, and define $p := \gamma(a)$ and $q := \gamma(b)$. Then the following are equivalent.*

(i) *γ is parametrized proportional to the arclength, i.e. $|\dot\gamma(t)| = c$ is constant, and γ minimizes the length, i.e.*

$$L(\gamma) \le L(\gamma')$$

for every smooth curve γ' in M joining p and q.

(ii) *γ minimizes the energy, i.e.*

$$E(\gamma) \le E(\gamma')$$

for every smooth curve $\gamma' : I \to M$ with $\gamma'(a) = p$ and $\gamma'(b) = q$.

Proof. See page 192. □

Definition 4.5.2 (Minimal Geodesic). *A smooth curve $\gamma : I \to M$ on a compact interval $I \subset \mathbb{R}$ is called a* **minimal geodesic** *if it satisfies the equivalent conditions of Lemma 4.5.1.*

Remark 4.5.3. (i) Condition (i) says that (the velocity $|\dot\gamma|$ is constant and) $L(\gamma) = d(p, q)$, i.e. that γ is a shortest curve from p to q. It is not precluded that there be more than one such γ; consider for example the case where M is a sphere and p and q are antipodal.

(ii) Condition (ii) implies that

$$\left.\frac{d}{ds}\right|_{s=0} E(\gamma_s) = 0$$

for every smooth variation $\mathbb{R} \times I \to M : s \mapsto \gamma_s(t)$ of γ with fixed endpoints. Hence a minimal geodesic is a geodesic.

(iii) Finally, we remark that $L(\gamma)$ (but not $E(\gamma)$) is independent of the parametrization of γ. Hence if γ is a minimal geodesic $L(\gamma) \le L(\gamma')$ for every γ' (from p to q) whereas $E(\gamma) \le E(\gamma')$ for those γ' defined on (an interval the same length as) I.

Proof of Lemma 4.5.1. We prove that (i) implies (ii). Let (c) be the (constant) value of $|\dot\gamma(t)|$. Then

$$L(\gamma) = (b-a)c, \qquad E(\gamma) = \frac{(b-a)c^2}{2}.$$

Then, for every smooth curve $\gamma' : I \to M$ with $\gamma'(a) = p$ and $\gamma'(b) = q$, we have

$$\begin{aligned}
4E(\gamma)^2 &= c^2 L(\gamma)^2 \\
&\le c^2 L(\gamma')^2 \\
&= c^2 \left(\int_a^b \left|\dot\gamma'(t)\right| \, dt \right)^2 \\
&\le c^2 (b-a) \int_a^b \left|\dot\gamma'(t)\right|^2 \, dt \\
&= 2(b-a)c^2 E(\gamma') \\
&= 4E(\gamma)E(\gamma').
\end{aligned}$$

Here the fourth step follows from the Cauchy–Schwarz inequality. Now divide by $4E(\gamma)$ to obtain $E(\gamma) \le E(\gamma')$.

We prove that (ii) implies (i). We have already shown in Remark 4.5.3 that (ii) implies that γ is a geodesic. It is easy to dispose of the case where M is one-dimensional. In that case any γ minimizing $E(\gamma)$ or $L(\gamma)$ must be monotonic onto a subarc; otherwise it could be altered so as to make the integral smaller. Hence suppose M is of dimension at least two. Suppose, by contradiction, that $L(\gamma') < L(\gamma)$ for some curve γ' from p to q. Since the dimension of M is bigger than one, we may approximate γ' by a curve whose tangent vector nowhere vanishes, i.e. we may assume without loss of generality that $\dot\gamma'(t) \ne 0$ for all t. Then we can reparametrize γ' proportional to arclength without changing its length, and by a further transformation we can make its domain equal to I. Thus we may assume without loss of generality that $\gamma' : I \to M$ is a smooth curve with $\gamma'(a) = p$ and $\gamma'(b) = q$ such that $|\gamma'(t)| = c'$ and

$$(b-a)c' = L(\gamma') < L(\gamma) = (b-a)c.$$

This implies $c' < c$ and hence

$$E(\gamma') = \frac{(b-a){c'}^2}{2} < \frac{(b-a)c^2}{2} = E(\gamma).$$

This contradicts (ii) and proves Lemma 4.5.1. □

The next theorem asserts the existence of minimal geodesics.

Theorem 4.5.4 (**Existence of Minimal Geodesics**). *Let $M \subset \mathbb{R}^n$ be a smooth m-manifold, fix a point $p \in M$, and let $r > 0$ be smaller than the injectivity radius of M at p. Let $v \in T_pM$ such that $|v| < r$. Then*

$$d(p,q) = |v|, \qquad q := \exp_p(v),$$

and a curve $\gamma \in \Omega_{p,q}$ has minimal length $L(\gamma) = |v|$ if and only if there is a smooth map $\beta : [0,1] \to [0,1]$ satisfying

$$\beta(0) = 0, \qquad \beta(1) = 1, \qquad \dot{\beta} \geq 0$$

such that $\gamma(t) = \exp_p(\beta(t)v)$ for $0 \leq t \leq 1$.

Proof. See page 195. □

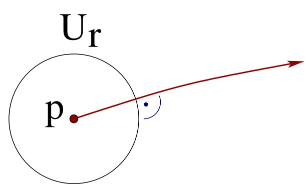


Figure 4.6: The Gauß Lemma.

Lemma 4.5.5 (**Gauß Lemma**). *Let M, p, r be as in Theorem 4.5.4, let $I \subset \mathbb{R}$ be an open interval, and let $w : I \to V_p$ be a smooth curve whose norm*

$$|w(t)| =: r$$

is constant. Define

$$\alpha(s,t) := \exp_p(sw(t))$$

for $(s,t) \in \mathbb{R} \times I$ with $sw(t) \in V_p$. Then

$$\left\langle \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right\rangle \equiv 0.$$

Thus the geodesics through p are orthogonal to the boundaries of the embedded balls $U_r(p)$ in Corollary 4.3.8 (see Figure 4.6).

Proof of Lemma 4.5.5. For every $t \in I$ we have

$$\alpha(0,t) = \exp_p(0) = p$$

and so the assertion holds for $s = 0$, i.e.

$$\left\langle \frac{\partial \alpha}{\partial s}(0,t), \frac{\partial \alpha}{\partial t}(0,t) \right\rangle = 0.$$

Moreover, each curve $s \mapsto \alpha(s,t)$ is a geodesic, i.e.

$$\nabla_s \frac{\partial \alpha}{\partial s} = \Pi(\alpha) \frac{\partial^2 \alpha}{\partial s^2} \equiv 0.$$

By Theorem 4.1.4, the function

$$s \mapsto \left| \frac{\partial \alpha}{\partial s}(s,t) \right|$$

is constant for every t, so that

$$\left| \frac{\partial \alpha}{\partial s}(s,t) \right| = \left| \frac{\partial \alpha}{\partial s}(0,t) \right| = |w(t)| = r \qquad \text{for } (s,t) \in \mathbb{R} \times I.$$

This implies

$$\begin{aligned}
\frac{\partial}{\partial s} \left\langle \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right\rangle &= \left\langle \nabla_s \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \right\rangle + \left\langle \frac{\partial \alpha}{\partial s}, \nabla_s \frac{\partial \alpha}{\partial t} \right\rangle \\
&= \left\langle \frac{\partial \alpha}{\partial s}, \Pi(\alpha) \frac{\partial^2 \alpha}{\partial s \partial t} \right\rangle \\
&= \left\langle \Pi(\alpha) \frac{\partial \alpha}{\partial s}, \frac{\partial^2 \alpha}{\partial s \partial t} \right\rangle \\
&= \left\langle \frac{\partial \alpha}{\partial s}, \frac{\partial^2 \alpha}{\partial s \partial t} \right\rangle \\
&= \frac{1}{2} \frac{\partial}{\partial t} \left| \frac{\partial \alpha}{\partial s} \right|^2 \\
&= 0.
\end{aligned}$$

Since the function $\langle \frac{\partial \alpha}{\partial s}, \frac{\partial \alpha}{\partial t} \rangle$ vanishes for $s = 0$ we obtain

$$\left\langle \frac{\partial \alpha}{\partial s}(s,t), \frac{\partial \alpha}{\partial t}(s,t) \right\rangle = 0$$

for all s and t. This proves Lemma 4.5.5. $\square$

Proof of Theorem 4.5.4. Let $r > 0$ be as in Corollary 4.3.8 and let $v \in T_pM$ such that $0 < |v| =: \varepsilon < r$. Denote $q := \exp_p(v)$ and let $\gamma \in \Omega_{p,q}$. Assume first that

$$\gamma(t) \in \exp_p\left(\overline{B}_\varepsilon(p)\right) = \overline{U}_\varepsilon \qquad \forall\, t \in [0,1].$$

Then there is a unique smooth function $[0,1] \to T_pM : t \mapsto v(t)$ such that $|v(t)| \le \varepsilon$ and $\gamma(t) = \exp_p(v(t))$ for every t. The set

$$I := \{t \in [0,1] \,|\, \gamma(t) \ne p\} = \{t \in [0,1] \,|\, v(t) \ne 0\} \subset (0,1]$$

is open in the relative topology of $(0,1]$. Thus I is a union of open intervals in $(0,1)$ and one half open interval containing 1. Define $\beta : [0,1] \to [0,1]$ and $w : I \to T_pM$ by

$$\beta(t) := \frac{|v(t)|}{\varepsilon}, \qquad w(t) := \varepsilon \frac{v(t)}{|v(t)|}.$$

Then β is continuous, both β and w are smooth on I,

$$\beta(0) = 0, \qquad \beta(1) = 1, \qquad w(1) = v,$$

and

$$|w(t)| = \varepsilon, \qquad \gamma(t) = \exp_p(\beta(t)w(t))$$

for all $t \in I$. We prove that $L(\gamma) \ge \varepsilon$. To see this let $\alpha : [0,1] \times I \to M$ be the map of Lemma 4.5.5, i.e.

$$\alpha(s,t) := \exp_p(sw(t)).$$

Then $\gamma(t) = \alpha(\beta(t), t)$ and hence

$$\dot\gamma(t) = \dot\beta(t)\frac{\partial\alpha}{\partial s}(\beta(t),t) + \frac{\partial\alpha}{\partial t}(\beta(t),t)$$

for every $t > 0$. Hence it follows from Lemma 4.5.5 that

$$|\dot\gamma(t)|^2 = \dot\beta(t)^2\left|\frac{\partial\alpha}{\partial s}(\beta(t),t)\right|^2 + \left|\frac{\partial\alpha}{\partial t}(\beta(t),t)\right|^2 \ge \dot\beta(t)^2\varepsilon^2$$

for every $t \in I$. Hence

$$L(\gamma) = \int_0^1 |\dot\gamma(t)|\,dt = \int_I |\dot\gamma(t)|\,dt \ge \varepsilon\int_I \left|\dot\beta(t)\right|\,dt \ge \varepsilon\int_I \dot\beta(t)\,dt = \varepsilon.$$

Here the last equation follows by applying the fundamental theorem of calculus to each interval in I and using the fact that $\beta(0) = 0$ and $\beta(1) = 1$. If $L(\gamma) = \varepsilon$ we must have

$$\frac{\partial \alpha}{\partial t}(\beta(t), t) = 0, \qquad \dot{\beta}(t) \geq 0 \qquad\qquad \text{for all } t \in I.$$

Thus I is a single half open interval containing 1 and on this interval the condition $\frac{\partial \alpha}{\partial t}(\beta(t), t) = 0$ implies $\dot{w}(t) = 0$. Since $w(1) = v$ we have $w(t) = v$ for every $t \in I$. Hence $\gamma(t) = \exp_p(\beta(t)v)$ for every $t \in [0, 1]$. It follows that β is smooth on the closed interval $[0, 1]$ (and not just on I). Thus we have proved that every $\gamma \in \Omega_{p,q}$ with values in $\overline{U}_\varepsilon$ has length $L(\gamma) \geq \varepsilon$ with equality if and only if γ is a reparametrized geodesic. But if $\gamma \in \Omega_{p,q}$ does not take values only in $\overline{U}_\varepsilon$, there must be a $T \in (0, 1)$ such that $\gamma([0, T]) \subset \overline{U}_\varepsilon$ and $\gamma(T) \in \partial U_\varepsilon$. Then $L(\gamma|_{[0,T]}) \geq \varepsilon$, by what we have just proved, and $L(\gamma|_{[T,1]}) > 0$ because the restriction of γ to $[T, 1]$ cannot be constant; so in this case we have $L(\gamma) > \varepsilon$. This proves Theorem 4.5.4. □

The next corollary gives a partial answer to our problem of finding length minimizing curves. It asserts that geodesics minimize the length *locally.*

Corollary 4.5.6. *Let $M \subset \mathbb{R}^n$ be a smooth m-manifold, let $I \subset \mathbb{R}$ be an open interval, and let $\gamma : I \to M$ be a geodesic. Fix a point $t_0 \in I$. Then there exists a constant $\varepsilon > 0$ such that*

$$t_0 - \varepsilon < s < t < t_0 + \varepsilon \qquad \Longrightarrow \qquad L(\gamma|_{[s,t]}) = d(\gamma(s), \gamma(t)).$$

Proof. Since γ is a geodesic its derivative has constant norm $|\dot{\gamma}(t)| \equiv c$ (see Theorem 4.1.4). Choose $\delta > 0$ so small that the interval $[t_0 - \delta, t_0 + \delta]$ is contained in I. Then there is a constant $r > 0$ such that $r \leq \mathrm{inj}(\gamma(t))$ whenever $|t - t_0| \leq \delta$. Choose $\varepsilon > 0$ such that

$$\varepsilon < \delta, \qquad 2\varepsilon c < r.$$

If $t_0 - \varepsilon < s < t < t_0 + \varepsilon$ then

$$\gamma(t) = \exp_{\gamma(s)}\left((t - s)\dot{\gamma}(s)\right)$$

and

$$|(t - s)\dot{\gamma}(s)| = |t - s|\, c < 2\varepsilon c < r \leq \mathrm{inj}(\gamma(s)).$$

Hence it follows from Theorem 4.5.4 that

$$L(\gamma|_{[s,t]}) = |t - s|\, c = d(\gamma(s), \gamma(t)).$$

This proves Corollary 4.5.6. □

Exercise 4.5.7. How large can the constant ε in Corollary 4.5.6 be chosen in the case $M = S^2$? Compare this with the injectivity radius.

Remark 4.5.8. We conclude from Theorem 4.5.4 that

$$S_r(p) := \big\{q \in M \,\big|\, d(p,q) = r\big\} = \exp_p\big(\big\{v \in T_pM \mid |v| = r\big\}\big) \qquad (4.5.1)$$

for $0 < r < \mathrm{inj}(p;M)$. The Gauß Lemma 4.5.5 shows that the geodesic rays $[0,1] \to M : s \mapsto \exp_p(sv)$ emanating from p are the orthogonal trajectories to the concentric spheres $S_r(p)$.

Exercise 4.5.9. Let

$$M \subset \mathbb{R}^3$$

be of dimension two and suppose that M is invariant under the (orthogonal) reflection about some plane $E \subset \mathbb{R}^3$. Show that E intersects M in a geodesic. (**Hint:** Otherwise there would be points $p, q \in M$ very close to one another joined by two distinct minimal geodesics.) Conclude for example that the coordinate planes intersect the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ in geodesics.

Exercise 4.5.10. Choose geodesic normal coordinates near $p \in M$ via

$$q = \exp_p\left(\sum_{i=1}^m x^i(q)e_i\right),$$

where $e_1, \ldots, e_m$ is an orthonormal basis of T_pM (see Corollary 4.4.6). Then we have $x^i(p) = 0$ and

$$B_r(p) = \{q \in M \mid d(p,q) < r\} = \left\{q \in M \,\middle|\, \sum_{i=1}^m \left|x^i(q)\right|^2 < r^2\right\} \qquad (4.5.2)$$

for $0 < r < \mathrm{inj}(p;M)$. Hence Theorem 4.4.5 asserts that $B_r(p)$ is convex for $r > 0$ sufficiently small.

(i) Show that it can happen that a geodesic in $B_r(p)$ is not minimal. **Hint:** Take M to be the hemisphere $\{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1,\ z > 0\}$ together with the disc $\{(x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 \le 1,\ z = 0\}$, but smooth the corners along the circle $x^2 + y^2 = 1$, $z = 0$. Take $p = (0,0,1)$ and $r = \pi/2$.

(ii) Show that, if $r > 0$ is sufficiently small, then the unique geodesic γ in $B_r(p)$ joining two points $q, q' \in B_r(p)$ is minimal and that in fact any curve γ' from q to q' which is not a reparametrization of γ is strictly longer, i.e. $L(\gamma') > L(\gamma) = d(q,q')$.

Exercise 4.5.11. Let $\gamma : I = [a,b] \to M$ be a smooth curve with endpoints $\gamma(a) = p$ and $\gamma(b) = q$ and nowhere vanishing derivative, i.e. $\gamma(t) \neq 0$ for all $t \in I$. Prove that the following are equivalent.

(i) The curve γ is an **extremal of the length functional**, i.e. every smooth map $\mathbb{R} \times I \to M : (s,t) \mapsto \gamma_s(t)$ with $\gamma_s(a) = p$ and $\gamma_s(b) = q$ for all s satisfies

$$\left.\frac{d}{ds} L(\gamma_s)\right|_{s=0} = 0.$$

(ii) The curve γ is a reparametrized geodesic, i.e. there exists a smooth map $\sigma : [a,b] \to [0,1]$ with $\sigma(a) = 0$, $\sigma(b) = 1$, $\dot\sigma(t) \geq 0$ for all $t \in I$, and a vector $v \in T_pM$ such that

$$q = \exp_p(v), \qquad \gamma(t) = \exp_p(\sigma(t)v)$$

for all $t \in I$. (We remark that the hypothesis $\dot\gamma(t) \neq 0$ implies that σ is actually a diffeomorphism, i.e. $\dot\sigma(t) > 0$ for all $t \in I$.)

(iii) The curve γ minimizes the length functional locally, i.e. there exists an $\varepsilon > 0$ such that $L(\gamma|_{[s,t]}) = d(\gamma(s), \gamma(t))$ for every closed subinterval $[s,t] \subset I$ of length $t - s < \varepsilon$.

It is often convenient to consider curves γ where $\dot\gamma(t)$ is allowed to vanish for some values of t; then γ cannot (in general) be parametrized by arclength. Such a curve $\gamma : I \to M$ can be smooth (as a map) and yet its image may have corners (where $\dot\gamma$ necessarily vanishes). Note that a curve with corners can never minimize the distance, even locally.

Exercise 4.5.12. Show that conditions (ii) and (iii) in Exercise 4.5.11 are equivalent, even without the assumption that $\dot\gamma$ is nowhere vanishing. Deduce that, if $\gamma : I \to M$ is a shortest curve joining p to q, i.e. $L(\gamma) = d(p,q)$, then γ is a reparametrized geodesic.

Show by example that one can have a variation $\{\gamma_s\}_{s\in\mathbb{R}}$ of a reparametrized geodesic $\gamma_0 = \gamma$ for which the map $s \mapsto L(\gamma_s)$ is not even differentiable at $s = 0$. (**Hint:** Take γ to be constant. See also Exercise 4.1.9.)

Show, however, that conditions (i), (ii) and (iii) in Exercise 4.5.11 remain equivalent if the hypothesis that $\dot\gamma$ is nowhere vanishing is weakened to the hypothesis that $\dot\gamma(t) \neq 0$ for all but finitely many $t \in I$. Conclude that a broken geodesic is a reparametrized geodesic if and only if it minimizes arclength locally. (A **broken geodesic** is a continuous map $\gamma : I = [a,b] \to M$ for which there exist $a = t_0 < t_1 < \cdots < t_n = b$ such that $\gamma|_{[t_{i-1},t_i]}$ is a geodesic for $i = 1, \ldots, n$. It is thus a geodesic if and only if $\dot\gamma$ is continuous at the break points, i.e. $\dot\gamma(t_i^-) = \dot\gamma(t_i^+)$ for $i = 1, \ldots, n-1$.)

4.6 Completeness and Hopf–Rinow

For a Riemannian manifold there are different notions of completeness. First, in §3.4 completeness was defined in terms of the completeness of time dependent basic vector fields on the frame bundle (Definition 3.4.9). Second, there is a distance function

$$d : M \times M \to [0, \infty)$$

defined by equation (4.2.2) so that we can speak of completeness of the metric space (M, d) in the sense that every Cauchy sequence converges. Third, there is the question of whether geodesics through any point in any direction exist for all time; if so we call a Riemannian manifold geodesically complete. The remarkable fact is that these three rather different notions of completeness are actually equivalent and that, in the complete case, any two points in M can be joined by a shortest geodesic. This is the content of the Hopf–Rinow theorem. We will spell out the details of the proof for embedded manifolds and leave it to the reader (as a straight forward exercise) to extend the proof to the intrinsic setting.

Definition 4.6.1. *Let $M \subset \mathbb{R}^n$ be an m-dimensional manifold. Given a point $p \in M$ we say that M* **is geodesically complete at** *p if, for every tangent vector $v \in T_pM$, there exists a geodesic $\gamma : \mathbb{R} \to M$ (on the entire real axis) satisfying $\gamma(0) = p$ and $\dot\gamma(0) = v$ (or equivalently $V_p = T_pM$ where $V_p \subset T_pM$ is defined by* (4.3.5)*). The manifold M is called* **geodesically complete** *if it is geodesically complete at every point $p \in M$.*

Definition 4.6.2. *Let (M, d) be a metric space. A subset $A \subset M$ is called* **bounded** *if*

$$\sup_{p \in A} d(p, p_0) < \infty$$

for some (and hence every) point $p_0 \in M$.

Example 4.6.3. A manifold $M \subset \mathbb{R}^n$ can be contained in a bounded subset of $\mathbb{R}^n$ and still not be bounded with respect to the metric (4.2.2). An example is the 1-manifold

$$M = \left\{(x, y) \in \mathbb{R}^2 \,|\, 0 < x < 1,\, y = \sin(1/x)\right\}.$$

Exercise 4.6.4. Let (M, d) be a metric space. Prove that every compact subset $K \subset M$ is closed and bounded. Find an example of a metric space that contains a closed and bounded subset that is not compact.

Theorem 4.6.5 (Completeness). *Let $M \subset \mathbb{R}^n$ be a connected m-dimensional manifold and let $d : M \times M \to [0, \infty)$ be the distance function defined by* (4.1.1), (4.2.1), *and* (4.2.2). *Then the following are equivalent.*

(i) *M is geodesically complete.*

(ii) *There exists a point $p \in M$ such that M is geodesically complete at p.*

(iii) *Every closed and bounded subset of M is compact.*

(iv) *(M, d) is a complete metric space.*

(v) *M is complete, i.e. for every smooth curve $\xi : \mathbb{R} \to \mathbb{R}^m$ and every element $(p_0, e_0) \in \mathcal{F}(M)$ there exists a smooth curve $\beta : \mathbb{R} \to \mathcal{F}(M)$ satisfying*

$$\dot{\beta}(t) = B_{\xi(t)}(\beta(t)), \qquad \beta(0) = (p_0, e_0). \tag{4.6.1}$$

(vi) *The basic vector field $B_\xi \in \mathrm{Vect}(\mathcal{F}(M))$ is complete for every $\xi \in \mathbb{R}^m$.*

(vii) *For every smooth curve $\gamma' : \mathbb{R} \to \mathbb{R}^m$, every $p_0 \in M$, and every orthogonal isomorphism $\Phi_0 : T_{p_0}M \to \mathbb{R}^m$ there exists a development (Φ, γ, γ') of M along $\mathbb{R}^m$ on all of $\mathbb{R}$ that satisfies $\gamma(0) = p_0$ and $\Phi(0) = \Phi_0$.*

Proof. The proof relies on Theorem 4.6.6 below. □

Theorem 4.6.6 (Hopf–Rinow). *Let $M \subset \mathbb{R}^n$ be a connected m-manifold and let $p \in M$. Assume M is geodesically complete at p. Then, for every $q \in M$, there exists a geodesic $\gamma : [0, 1] \to M$ such that*

$$\gamma(0) = p, \qquad \gamma(1) = q, \qquad L(\gamma) = d(p, q).$$

Proof. See page 205. □

Theorem 4.6.6 implies Theorem 4.6.5. That (i) implies (ii) follows directly from the definitions.

We prove that (ii) implies (iii). Thus assume that M is geodesically complete at the point $p_0 \in M$ and let $K \subset M$ be a closed and bounded subset. Then $r := \sup_{q \in K} d(p_0, q) < \infty$. Hence Theorem 4.6.6 asserts that, for every $q \in K$, there exists a vector $v \in T_{p_0}M$ such that $|v| = d(p_0, q) \leq r$ and $\exp_{p_0}(v) = q$. Thus

$$K \subset \exp_{p_0}(\overline{B}_r(p_0)), \qquad \overline{B}_r(p_0) = \{v \in T_{p_0}M \mid |v| \leq r\}.$$

Then $B := \{v \in T_{p_0}M \mid |v| \leq r, \exp_{p_0}(v) \in K\}$ is a closed and bounded subset of the Euclidean space $T_{p_0}M$. Hence B is compact and $K = \exp_{p_0}(B)$. Since the exponential map $\exp_{p_0} : T_{p_0}M \to M$ is continuous it follows that K is compact. This shows that (ii) implies (iii).

We prove that (iii) implies (iv). Thus assume that every closed and bounded subset of M is compact and choose a Cauchy sequence $p_i \in M$. Choose $i_0 \in \mathbb{N}$ such that $d(p_i, p_j) \le 1$ for all $i, j \in \mathbb{N}$ with $i, j \ge i_0$. Define

$$c := \max_{1 \le i \le i_0} d(p_1, p_i) + 1.$$

Then $d(p_1, p_i) \le d(p_1, p_{i_0}) + d(p_{i_0}, p_i) \le d(p_1, p_{i_0}) + 1 \le c$ for all $i \ge i_0$ and so $d(p_1, p_i) \le c$ for all $i \in \mathbb{N}$. Hence the set $\{p_i \,|\, i \in \mathbb{N}\}$ is bounded and so is its closure. By (iii) this implies that the sequence p_i has a convergent subsequence. Since p_i is a Cauchy sequence, this implies that p_i converges. Thus we have proved that (iii) implies (iv).

We prove that (iv) implies (v). Fix a smooth curve $\xi : \mathbb{R} \to \mathbb{R}^m$ and an element $(p_0, e_0) \in \mathcal{F}(M)$. Assume, by contradiction, that there exists a real number $T > 0$ such that there exists a solution $\beta : [0, T) \to \mathcal{F}(M)$ of equation (4.6.1) that cannot be extended to the interval $[0, T + \varepsilon)$ for any $\varepsilon > 0$. Write $\beta(t) =: (\gamma(t), e(t))$ so that γ and e satisfy the equations

$$\dot\gamma(t) = e(t)\xi(t), \quad \dot e(t) = h_{\gamma(t)}(\dot\gamma(t))e(t), \quad \gamma(0) = p_0, \quad e(0) = e_0.$$

This implies $e(t)\eta \in T_{\gamma(t)}M$ and $\dot e(t)\eta \in T_{\gamma(t)}^\perp M$ for all $\eta \in \mathbb{R}^m$ and therefore

$$\frac{d}{dt}\langle \eta, e(t)^\mathsf{T} e(t)\zeta\rangle = \frac{d}{dt}\langle e(t)\eta, e(t)\zeta\rangle = \langle \dot e(t)\eta, e(t)\zeta\rangle + \langle e(t)\eta, \dot e(t)\zeta\rangle = 0$$

for all $\eta, \zeta \in \mathbb{R}^m$ and all $t \in [0, T)$. Thus the function $t \mapsto e(t)^\mathsf{T} e(t)$ is constant, hence

$$e(t)^\mathsf{T} e(t) = e_0^\mathsf{T} e_0, \qquad \|e(t)\| = \sup_{0 \ne \eta \in \mathbb{R}^m} \frac{|e(t)\eta|}{|\eta|} = \|e_0\| \tag{4.6.2}$$

for $0 \le t < T$, hence

$$|\dot\gamma(t)| = |e(t)\xi(t)| \le \|e_0\|\,|\xi(t)| \le \|e_0\| \sup_{0 \le s \le T} |\xi(s)| =: c_T$$

and so $d(\gamma(s), \gamma(t)) \le L(\gamma|_{[s,t]}) \le (t - s)c_T$ for $0 \le s < t < T$. Since (M, d) is a complete metric space, this shows that the limit $p_1 := \lim_{t \nearrow T} \gamma(t) \in M$ exists. Thus the set $K := \gamma([0, T)) \cup \{p_1\} \subset M$ is compact and so is the set

$$\widetilde K := \left\{(p, e) \in \mathcal{F}(M) \,|\, p \in K,\ e^\mathsf{T} e = e_0^\mathsf{T} e_0\right\} \subset \mathcal{F}(M).$$

By equation (4.6.2) the curve $[0, T) \to \mathbb{R} \times \mathcal{F}(M) : t \mapsto (t, \gamma(t), e(t))$ takes values in the compact set $[0, T] \times \widetilde K$ and is the integral curve of a vector field on the manifold $\mathbb{R} \times \mathcal{F}(M)$. Hence Corollary 2.4.15 asserts that $[0, T)$ cannot be the maximal existence interval of this integral curve, a contradiction. This shows that (iv) implies (v).

That (v) implies (vi) follows by taking $\xi(t) \equiv \xi$ in (v).

We prove that (vi) implies (i). Fix an element $p_0 \in M$ and a tangent vector $v_0 \in T_{p_0}M$. Let $e_0 \in \mathcal{L}_{\mathrm{iso}}(\mathbb{R}^m, T_{p_0}M)$ be any isomorphism and choose $\xi \in \mathbb{R}^m$ such that $e_0\xi = v_0$. By (vi) the vector field B_ξ has a unique integral curve $\mathbb{R} \to \mathcal{F}(M) : t \mapsto \beta(t) = (\gamma(t), e(t))$ with

$$\beta(0) = (p_0, e_0).$$

Thus

$$\dot{\gamma}(t) = e(t)\xi, \qquad \dot{e}(t) = h_{\gamma(t)}(e(t)\xi)e(t),$$

and hence

$$\ddot{\gamma}(t) = \dot{e}(t)\xi = h_{\gamma(t)}(e(t)\xi)e(t)\xi = h_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)).$$

By the Gauß–Weingarten formula, this implies $\nabla\dot{\gamma}(t) = 0$ for every t and hence $\gamma : \mathbb{R} \to M$ is a geodesic with $\gamma(0) = p_0$ and $\dot{\gamma}(0) = e_0\xi = v_0$. Thus M is geodesically complete and this shows that (vi) implies (i).

The equivalence of (v) and (vii) was established in Corollary 3.5.25 and this proves Theorem 4.6.5. □

Lemma 4.6.7. *Let $M \subset \mathbb{R}^n$ be a connected m-manifold and $p \in M$. Suppose $\varepsilon > 0$ is smaller than the injectivity radius of M at p and denote*

$$\Sigma_1(p) := \{v \in T_pM \mid |v| = 1\}, \qquad S_\varepsilon(p) := \left\{p' \in M \mid d(p, p') = \varepsilon\right\}.$$

Then the map $\Sigma_1(p) \to S_\varepsilon(p) : v \mapsto \exp_p(\varepsilon v)$ is a diffeomorphism and, for all $q \in M$, we have

$$d(p,q) > \varepsilon \qquad \Longrightarrow \qquad d(S_\varepsilon(p), q) = d(p,q) - \varepsilon.$$

Proof. By Theorem 4.5.4, we have

$$d(p, \exp_p(v)) = |v| \qquad \text{for all } v \in T_pM \text{ with } |v| \le \varepsilon$$

and

$$d(p,p') > \varepsilon \qquad \text{for all } p' \in M \setminus \left\{\exp_p(v) \mid v \in T_pM,\, |v| \le \varepsilon\right\}.$$

This shows that $S_\varepsilon(p) = \exp_p(\varepsilon\Sigma_1(p))$ and, since ε is smaller than the injectivity radius, the map

$$\Sigma_1(p) \to S_\varepsilon(p) : v \mapsto \exp_p(\varepsilon v)$$

is a diffeomorphism.

To prove the second assertion, let $q \in M$ such that

$$r := d(p, q) > \varepsilon.$$

Fix a constant $\delta > 0$ and choose a smooth curve $\gamma : [0, 1] \to M$ such that

$$\gamma(0) = p, \qquad \gamma(1) = q, \qquad L(\gamma) \leq r + \delta.$$

Choose $t_0 > 0$ such that $\gamma(t_0)$ is the last point of the curve on $S_\varepsilon(p)$, i.e.

$$\gamma(t_0) \in S_\varepsilon(p), \qquad \gamma(t) \notin S_\varepsilon(p) \text{ for } t_0 < t \leq 1.$$

Then

$$\begin{aligned} d(\gamma(t_0), q) &\leq L(\gamma|_{[t_0,1]}) \\ &= L(\gamma) - L(\gamma|_{[0,t_0]}) \\ &\leq L(\gamma) - \varepsilon \\ &\leq r + \delta - \varepsilon. \end{aligned}$$

This shows that $d(S_\varepsilon(p), q) \leq r + \delta - \varepsilon$ for every $\delta > 0$ and therefore

$$d(S_\varepsilon(p), q) \leq r - \varepsilon.$$

Moreover,

$$d(p', q) \geq d(p, q) - d(p, p') = r - \varepsilon$$

for all $p' \in S_\varepsilon(p)$. Thus

$$d(S_\varepsilon(p), q) = r - \varepsilon$$

and this proves Lemma 4.6.7. □

Lemma 4.6.8 (Curve Shortening Lemma). *Let $M \subset \mathbb{R}^n$ be an m-manifold, let $p \in M$, and let ε be a real number such that*

$$0 < \varepsilon < \mathrm{inj}(p; M).$$

Then, for all $v, w \in T_pM$, we have

$$|v| = |w| = \varepsilon, \quad d(\exp_p(v), \exp_p(w)) = 2\varepsilon \qquad \Longrightarrow \qquad v + w = 0.$$

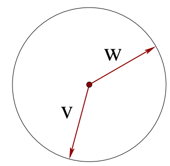


Figure 4.7: Two unit tangent vectors.

Proof. We will prove that, for all $v, w \in T_pM$, we have

$$\lim_{\delta\to 0}\frac{d(\exp_p(\delta v),\exp_p(\delta w))}{\delta} = |v-w|\,. \tag{4.6.3}$$

Assume this holds and suppose, by contradiction, that there exist two tangent vectors $v, w \in T_pM$ such that

$$|v| = |w| = 1, \qquad d(\exp_p(\varepsilon v),\exp_p(\varepsilon w)) = 2\varepsilon, \qquad v + w \neq 0.$$

Then

$$|v-w| < 2$$

(see Figure 4.7). Thus by (4.6.3) there exists a constant $0 < \delta < \varepsilon$ such that

$$d(\exp_p(\delta v),\exp_p(\delta w)) < 2\delta.$$

Then

$$\begin{aligned}
&d(\exp_p(\varepsilon v),\exp_p(\varepsilon w))\\
&\le d(\exp_p(\varepsilon v),\exp_p(\delta v)) + d(\exp_p(\delta v),\exp_p(\delta w)) + d(\exp_p(\delta w),\exp_p(\varepsilon w))\\
&< \varepsilon - \delta + 2\delta + \varepsilon - \delta = 2\varepsilon
\end{aligned}$$

and this contradicts our assumption.

It remains to prove (4.6.3). For this we observe that

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{d(\exp_p(\delta v),\exp_p(\delta w))}{\delta}\\
&= \lim_{\delta\to 0}\frac{d(\exp_p(\delta v),\exp_p(\delta w))}{\left|\exp_p(\delta v)-\exp_p(\delta w)\right|}\frac{\left|\exp_p(\delta v)-\exp_p(\delta w)\right|}{\delta}\\
&= \lim_{\delta\to 0}\frac{\left|\exp_p(\delta v)-\exp_p(\delta w)\right|}{\delta}\\
&= \lim_{\delta\to 0}\left|\frac{\exp_p(\delta v)-p}{\delta} - \frac{\exp_p(\delta w)-p}{\delta}\right|\\
&= |v-w|\,.
\end{aligned}$$

Here the second equality follows from Lemma 4.2.7. □

Proof of Theorem 4.6.6. By assumption $M \subset \mathbb{R}^n$ is a connected submanifold, and $p \in M$ is given such that the exponential map $\exp_p : T_pM \to M$ is defined on the entire tangent space at p. Fix a point $q \in M \setminus \{p\}$ so that

$$0 < r := d(p, q) < \infty.$$

Choose a constant $\varepsilon > 0$ smaller than the injectivity radius of M at p and smaller than r. Then, by Lemma 4.6.7, we have

$$d(S_\varepsilon(p), q) = r - \varepsilon.$$

Hence there exists a tangent vector $v \in T_pM$ such that

$$d(\exp_p(\varepsilon v), q) = r - \varepsilon, \qquad |v| = 1.$$

Define the curve $\gamma : [0, r] \to M$ by

$$\gamma(t) := \exp_p(tv) \qquad \text{for } 0 \le t \le r.$$

By Lemma 4.3.6, this is a geodesic and it satisfies $\gamma(0) = p$. We must prove that $\gamma(r) = q$ and $L(\gamma) = d(p, q)$. Instead we will prove the following stronger statement.

Claim. *For every $t \in [0, r]$ we have*

$$d(\gamma(t), q) = r - t.$$

In particular, $\gamma(r) = q$ and $L(\gamma) = r = d(p, q)$.

Consider the subset

$$I := \{t \in [0, r] \,|\, d(\gamma(t), q) = r - t\} \subset [0, r].$$

This set is nonempty, because $\varepsilon \in I$, it is obviously closed, and

$$t \in I \qquad \Longrightarrow \qquad [0, t] \subset I. \tag{4.6.4}$$

Namely, if $t \in I$ and $0 \le s \le t$ then

$$d(\gamma(s), q) \le d(\gamma(s), \gamma(t)) + d(\gamma(t), q) \le t - s + r - t = r - s$$

and

$$d(\gamma(s), q) \ge d(p, q) - d(p, \gamma(s)) \ge r - s.$$

Hence $d(\gamma(s), q) = r - s$ and hence $s \in I$. This proves (4.6.4).

We prove that I is open (in the relative topology of $[0, r]$). Let $t \in I$ be given with $t < r$. Choose a constant $\varepsilon > 0$ smaller than the injectivity radius of M at $\gamma(t)$ and smaller than $r - t$. Then, by Lemma 4.6.7 with p replaced by $\gamma(t)$, we have

$$d(S_\varepsilon(\gamma(t)), q) = r - t - \varepsilon.$$

Next we choose $w \in T_{\gamma(t)}M$ such that

$$|w| = 1, \qquad d(\exp_{\gamma(t)}(\varepsilon w), q) = r - t - \varepsilon.$$

Then

$$\begin{aligned} d(\gamma(t-\varepsilon), \exp_{\gamma(t)}(\varepsilon w)) &\geq d(\gamma(t-\varepsilon), q) - d(\exp_{\gamma(t)}(\varepsilon w), q) \\ &= (r - t + \varepsilon) - (r - t - \varepsilon) \\ &= 2\varepsilon. \end{aligned}$$

The converse inequality is obvious, because both points have distance ε to $\gamma(t)$ (see Figure 4.8).

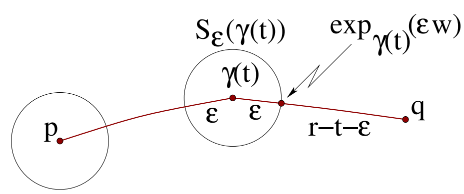


Figure 4.8: The proof of the Hopf–Rinow theorem.

Thus we have proved that

$$d(\gamma(t-\varepsilon), \exp_{\gamma(t)}(\varepsilon w)) = 2\varepsilon.$$

Since

$$\gamma(t-\varepsilon) = \exp_{\gamma(t)}(-\varepsilon\dot\gamma(t)),$$

it follows from Lemma 4.6.8 that

$$w = \dot\gamma(t).$$

Hence $\exp_{\gamma(t)}(sw) = \gamma(t+s)$ and this implies that

$$d(\gamma(t+\varepsilon), q) = r - t - \varepsilon.$$

Thus $t + \varepsilon \in I$ and, by (4.6.4), we have $[0, t+\varepsilon] \in I$. Thus we have proved that I is open. In other words, I is a nonempty subset of $[0, r]$ which is both open and closed, and hence $I = [0, r]$. This proves the claim and Theorem 4.6.6. $\square$

4.7 Geodesics in the Intrinsic Setting*

Intrinsic Distance

Let M be a connected smooth manifold (§2.8.1) equipped with a Riemannian metric (§3.7.1). Then we can define the length of a curve $\gamma : [0,1] \to M$ by the formula (4.1.1) and it is invariant under reparametrization as in Remark 4.1.3. The distance function $d : M \times M \to \mathbb{R}$ is then given by the same formula (4.2.2). We prove that it still defines a metric on M and that this metric induces the same topology as the smooth structure.

Lemma 4.7.1. *Let M be a connected smooth Riemannian manifold and define the function $d : M \times M \to [0,\infty)$ by (4.1.1), (4.2.1), and (4.2.2). Then d is a metric and induces the same topology as the smooth structure.*

Proof. The proof has three steps.

Step 1. *Fix a point $p_0 \in M$ and let $\phi : U \to \Omega$ be a coordinate chart of M onto an open subset $\Omega \subset \mathbb{R}^m$ such that $p_0 \in U$. Then there exists an open neighborhood $V \subset U$ of p_0 and constants $\delta, r > 0$ such that*

$$\delta\,|\phi(p) - \phi(p_0)| \le d(p,p_0) \le \delta^{-1}\,|\phi(p) - \phi(p_0)| \tag{4.7.1}$$

for every $p \in V$ and $d(p,p_0) \ge \delta r$ for every $p \in M \setminus V$.

Denote the inverse of the coordinate chart ϕ by $\psi := \phi^{-1} : \Omega \to M$ and define the map $g = (g_{ij})_{i,j=1}^m : \Omega \to \mathbb{R}^{m\times m}$ by $g_{ij}(x) := \langle \frac{\partial\psi}{\partial x^i}(x), \frac{\partial\psi}{\partial x^j}(x)\rangle_{\psi(x)}$ for $x \in \Omega$. Then a smooth curve $\gamma : [0,1] \to U$ has the length

$$L(\gamma) = \int_0^1 \sqrt{\dot c(t)^\mathsf{T} g(c(t))\dot c(t)}\,dt, \qquad c(t) := \phi(\gamma(t)). \tag{4.7.2}$$

Let $x_0 := \phi(p_0) \in \Omega$ and choose $r > 0$ such that $\overline{B}_r(x_0) \subset \Omega$. Then there is a constant $\delta \in (0,1]$ such that

$$\delta\,|\xi| \le \sqrt{\xi^\mathsf{T} g(x)\xi} \le \delta^{-1}\,|\xi| \tag{4.7.3}$$

for all $x \in B_r(x_0)$ and $\xi, \eta \in \mathbb{R}^m$. Define $V := \phi^{-1}(B_r(x_0)) \subset U$.

Now let $p \in V$ and denote $x := \phi(p) \in B_r(x_0)$. Then, for every smooth curve $\gamma : [0,1] \to V$ with $\gamma(0) = p_0$ and $\gamma(1) = p$, the curve $c := \phi \circ \gamma$ takes values in $B_r(x_0)$ and satisfies $c(0) = x_0$ and $c(1) = x$. Hence, by (4.7.2) and (4.7.3), we have

$$L(\gamma) \ge \delta \int_0^1 |\dot c(t)|\,dt \ge \delta \left| \int_0^1 \dot c(t)\,dt \right| = \delta\,|x - x_0|\,.$$

If $\gamma : [0,1] \to M$ is a smooth curve with endpoints $\gamma(0) = p_0$ and $\gamma(1) = p$ whose image is not entirely contained in V then there is a time $T \in (0,1]$ such that $\gamma(t) \in V$ for $0 \le t < T$ and $\gamma(T) \in \partial V$. Hence $c(t) := \phi(\gamma(t)) \in B_r(x_0)$ for $0 \le t < T$ and $|c(T) - x_0| = r$. Hence, by the above argument, we have

$$L(\gamma) \ge \delta r.$$

This shows that $d(p_0, p) \ge \delta r$ for $p \in M \setminus V$ and $d(p_0, p) \ge \delta\, |\phi(p) - \phi(p_0)|$ for $p \in V$. If $p \in V$, $x := \phi(p)$, and $c(t) := x_0 + t(x - x_0)$ then $\gamma := \psi \circ c$ is a smooth curve in V with $\gamma(0) = p_0$ and $\gamma(1) = p$ and, by (4.7.2) and (4.7.3), we have

$$L(\gamma) \le \delta^{-1} \int_0^1 |\dot{c}(t)|\; dt = \delta^{-1}\, |x - x_0| \, .$$

This proves Step 1.

Step 2. *d is a distance function.*

Step 1 shows that $d(p, p_0) > 0$ for every $p \in M \setminus \{p_0\}$ and hence d satisfies condition (i) in Lemma 4.2.3. The proofs of (ii) and (iii) remain unchanged in the intrinsic setting and this proves Step 2.

Step 3. *The topology on M induced by d agrees with the topology induced by the smooth structure.*

Assume first that $W \subset M$ is open with respect to the manifold topology and let $p_0 \in W$. Let $\phi : U \to \Omega$ be a coordinate chart of M onto an open subset $\Omega \subset \mathbb{R}^m$ such that $p_0 \in U$, and choose an open neighborhood $V \subset U$ of p_0 and constants $\delta, r > 0$ as in Step 1, so that (4.7.1) holds for all $p \in V$ and $d(p, p_0) \ge \delta r$ for every $p \in M \setminus V$. Then $\phi(V \cap W)$ is an open subset of Ω and so there is an $\varepsilon > 0$ such that $B_{\delta^{-1}\varepsilon}(\phi(p_0)) \subset \phi(V \cap W)$ and $\varepsilon < \delta r$. Let $p \in M$ with $d(p, p_0) < \varepsilon$. Then $p \in V$, hence $|\phi(p) - \phi(p_0)| < \delta^{-1} d(p, p_0)$ by (4.7.1), and this implies $\phi(p) \in \phi(V \cap W)$. Thus $B_\varepsilon(p_0, d) \subset W$ and so W is open with respect to d.

Conversely, assume that $W \subset M$ be open with respect to d and choose a coordinate chart $\phi : U \to \Omega$ onto an open set $\Omega \subset \mathbb{R}^m$. We must prove that $\phi(U \cap W)$ is an open subset of Ω. To see this, choose $x_0 \in \phi(U \cap W)$ and let $p_0 := \phi^{-1}(x_0) \in U \cap W$. Now choose $V \subset U$ and $\delta, r > 0$ as in Step 1. Choose $\varepsilon > 0$ such that $B_{\delta^{-1}\varepsilon}(p_0, d) \subset W$ and $B_\varepsilon(x_0) \subset \phi(V)$. Let $x \in \mathbb{R}^n$ such that $|x - x_0| < \varepsilon$. Then $x \in \phi(V)$ and therefore $p := \phi^{-1}(x) \in V$. This implies $d(p, p_0) < \delta^{-1}|\phi(p) - \phi(p_0)| = \delta^{-1}|x - x_0| < \delta^{-1}\varepsilon$, thus $p \in W \cap U$, and so $x = \phi(p) \in \phi(W \cap U)$. Thus $\phi(W \cap U)$ is an open, and so W is open in the manifold topology of M. This proves Step 3 and Lemma 4.7.1. $\square$

Geodesics and the Levi-Civita Connection

With the covariant derivative understood (§3.7.2), we can define geodesics on M as smooth curves $\gamma : I \to M$ that satisfy the equation

$$\nabla \dot{\gamma} = 0,$$

as in Definition 4.1.5. Then all the above results about geodesics, as well as their proofs, carry over almost verbatim to the intrinsic setting. In particular, geodesics are in local coordinates described by equation (4.4.1) (Lemma 4.4.1) and they are the critical points of the energy functional

$$E(\gamma) := \frac{1}{2} \int_0^1 |\dot{\gamma}(t)|^2 \, dt$$

on the space $\Omega_{p,q}$ of all paths $\gamma : [0,1] \to M$ with fixed endpoints $\gamma(0) = p$ and $\gamma(1) = q$. Here we use the fact that Lemma 4.1.7 extends to the intrinsic setting via the Embedding Theorem 2.9.12. So for every vector field $X \in \mathrm{Vect}(\gamma)$ along γ with $X(0) = 0$ and $X(1) = 0$ there exists a curve of curves $\mathbb{R} \to \Omega_{p,q} : s \mapsto \gamma_s$ with $\gamma_0 = \gamma$ and $\partial_s \gamma_s|_{s=0} = X$. Then, by the properties of the Levi-Civita connection, we have

$$\begin{aligned} dE(\gamma)X &= \frac{1}{2} \int_0^1 \partial_s \left|\partial_t \gamma_s(t)\right|^2 \, dt \\ &= \int_0^1 \langle \dot{\gamma}(t), \nabla_t X(t) \rangle \, dt \\ &= -\int_0^1 \langle \nabla_t \dot{\gamma}(t), X(t) \rangle \, dt. \end{aligned}$$

The right hand side vanishes for all X if and only if $\nabla \dot{\gamma} \equiv 0$ (Theorem 4.1.4). With this understood, we find that, for all $p \in M$ and $v \in T_pM$, there exists a unique geodesic $\gamma : I_{p,v} \to M$ on a maximal open interval $I_{p,v} \subset \mathbb{R}$ containing zero that satisfies $\gamma(0) = p$ and $\dot{\gamma}(0) = v$ (Lemma 4.3.4). This gives rise to a smooth exponential map $\exp_p : V_p = \{v \in T_pM \,|\, 1 \in I_{p,v}\} \to M$ as in §4.3 which satisfies $d\exp_p(0) = \mathrm{id} : T_pM \to T_pM$ as in Corollary 4.3.7. This leads directly to the injectivity radius, the Gauß Lemma 4.5.5, the local length minimizing property of geodesics in Theorem 4.5.4, and the Convex Neighborhood Theorem 4.4.5. Also the proof of the equivalence of metric and geodesic completeness in Theorem 4.6.5 and of the Hopf–Rinow Theorem 4.6.6 carry over verbatim to the intrinsic setting of general Riemannian manifolds. The only place where some care must be taken is in the proof of the Curve Shortening Lemma 4.6.8 as is spelled out in Exercise 4.7.2 below.

Examples and Exercises

Exercise 4.7.2. Choose a coordinate chart $\phi : U \to \Omega$ with $\phi(p_0) = 0$ such that the metric in local coordinates satisfies

$$g_{ij}(0) = \delta_{ij}.$$

Refine the estimate (4.7.1) in the proof of Lemma 4.7.1 and show that

$$\lim_{p,q\to p_0} \frac{d(p,q)}{|\phi(p) - \phi(q)|} = 1.$$

This is the intrinsic analogue of Lemma 4.2.8. Use this to prove that equation (4.6.3) continues to hold for all Riemannian manifolds, i.e.

$$\lim_{\delta\to 0} \frac{d(\exp_p(\delta v), \exp_p(\delta w))}{\delta} = |v - w|$$

for $p \in M$ and $v, w \in T_pM$. With this understood, the proof of the Curve Shortening Lemma 4.6.8 carries over verbatim to the intrinsic setting.

Exercise 4.7.3. The real projective space $\mathbb{R}\mathrm{P}^n$ inherits a Riemannian metric from S^n as it is a quotient of S^n by an isometric involution. Prove that each geodesic in S^n with its standard metric descends to a geodesic in $\mathbb{R}\mathrm{P}^n$.

Exercise 4.7.4. Let $f : S^3 \to S^2$ be the **Hopf fibration** defined by

$$f(z, w) = \Big(|z|^2 - |w|^2, 2\mathrm{Re}\,\bar{z}w, 2\mathrm{Im}\,\bar{z}w\Big)$$

Prove that the image of a great circle in S^3 is a nonconstant geodesic in S^2 if and only if it is orthogonal to the fibers of f, which are also great circles. Here we identify S^3 with the unit sphere in $\mathbb{C}^2$. (See also Exercise 2.5.17.)

Exercise 4.7.5. Prove that a nonconstant geodesic $\gamma : \mathbb{R} \to S^{2n+1}$ descends to a nonconstant geodesic in $\mathbb{C}\mathrm{P}^n$ with the Fubini–Study metric (see Example 3.7.5) if and only if $\dot\gamma(t) \perp \mathbb{C}\gamma(t)$ for every $t \in \mathbb{R}$.

Exercise 4.7.6. Consider the manifold

$$\mathcal{F}_k(\mathbb{R}^n) := \Big\{D \in \mathbb{R}^{n\times k} \,\big|\, D^\mathsf{T} D = \mathbb{1}\Big\}$$

of orthonormal k-frames in $\mathbb{R}^n$, equipped with the Riemannian metric inherited from the standard inner product

$$\langle X, Y\rangle := \mathrm{trace}(X^\mathsf{T} Y)$$

on the space of real $n \times k$-matrices.

(a) Prove that

$$T_D\mathcal{F}_k(\mathbb{R}^n) = \left\{X \in \mathbb{R}^{n\times k} \,\middle|\, D^\mathsf{T}X + X^\mathsf{T}D = 0\right\},$$
$$T_D\mathcal{F}_k(\mathbb{R}^n)^\perp = \left\{DA \,\middle|\, A = A^\mathsf{T} \in \mathbb{R}^{k\times k}\right\}.$$

and that the orthogonal projection $\Pi(D) : \mathbb{R}^{n\times k} \to T_D\mathcal{F}_k(\mathbb{R}^n)$ is given by

$$\Pi(D)X = X - \frac{1}{2}D\big(D^\mathsf{T}X + X^\mathsf{T}D\big).$$

(b) Prove that the second fundamental form of $\mathcal{F}_k(\mathbb{R}^n)$ is given by

$$h_D(X)Y = -\frac{1}{2}D\big(X^\mathsf{T}Y + Y^\mathsf{T}X\big)$$

for $D \in \mathcal{F}_k(\mathbb{R}^n)$ and $X, Y \in T_D\mathcal{F}_k(\mathbb{R}^n)$.
(c) Prove that a smooth map $\mathbb{R} \to \mathcal{F}_k(\mathbb{R}^n) : t \mapsto D(t)$ is a geodesic if and only if it satisfies the differential equation

$$\ddot{D} = -D\dot{D}^\mathsf{T}\dot{D}. \tag{4.7.4}$$

Prove that the function $D^\mathsf{T}\dot{D}$ is constant for every geodesic in $\mathcal{F}_k(\mathbb{R}^n)$. Compare this with Example 4.3.12.

Exercise 4.7.7. Let $\mathrm{G}_k(\mathbb{R}^n) = \mathcal{F}_k(\mathbb{R}^n)/\mathrm{O}(k)$ be the real Grassmannian of k-dimensional subspaces in $\mathbb{R}^n$, equipped with a Riemannian metric as in Example 3.7.6. Prove that a geodesics $\mathbb{R} \to \mathcal{F}_k(\mathbb{R}^n) : t \mapsto D(t)$ descends to a nonconstant geodesic in $\mathrm{G}_k(\mathbb{R}^n)$ if and only if $D^\mathsf{T}\dot{D} \equiv 0$ and $\dot{D} \not\equiv 0$. Deduce that the exponential map on $\mathrm{G}_k(\mathbb{R}^n)$ is given by

$$\exp_\Lambda(\widehat{\Lambda}) = \operatorname{im}\left(D\cos\left(\left(\widehat{D}^\mathsf{T}\widehat{D}\right)^{1/2}\right) + \widehat{D}\left(\widehat{D}^\mathsf{T}\widehat{D}\right)^{-1/2}\sin\left(\left(\widehat{D}^\mathsf{T}\widehat{D}\right)^{1/2}\right)\right)$$

for $\Lambda \in \mathcal{F}_k(\mathbb{R}^n)$ and $\widehat{\Lambda} \in T_\Lambda\mathcal{F}_k(\mathbb{R}^n) \setminus \{0\}$. Here we identify the tangent space $T_\Lambda\mathcal{F}_k(\mathbb{R}^n)$ with the space of linear maps from Λ to $\Lambda^\perp$, and choose the matrices $D \in \mathcal{F}_k(\mathbb{R}^n)$ and $\widehat{D} \in \mathbb{R}^{n\times k}$ such that

$$\Lambda = \operatorname{im} D, \qquad D^\mathsf{T}\widehat{D} = 0, \qquad \widehat{\Lambda} \circ D = \widehat{D} : \mathbb{R}^k \to \Lambda^\perp = \ker D^\mathsf{T}.$$

Prove that the group $\mathrm{O}(n)$ acts on $\mathrm{G}_k(\mathbb{R}^n)$ by isometries. Which subgroup acts trivially?

Exercise 4.7.8. Carry over Exercises 4.7.6 and 4.7.7 to the complex Grassmannian $\mathrm{G}_k(\mathbb{C}^n)$. Prove that the group $\mathrm{U}(n)$ acts on $\mathrm{G}_k(\mathbb{C}^n)$ by isometries. Which subgroup acts trivially?

Chapter 5

Curvature

This chapter begins by introducing the notion of an isometry (§5.1). It shows that isometries of embedded manifolds preserve the lengths of curves and can be characterized as diffeomorphisms whose derivatives preserve the inner products. The chapter then moves on to the Riemann curvature tensor (§5.2). The next section is devoted to the generalized Gauß Theorema Egregium which asserts that isometries preserve geodesics, the covariant derivative, and the Riemann curvature tensor (§5.3). That section also shows that isometries form finite dimensional Lie groups. The final section discusses the Riemann curvature tensor in local coordinates and shows how all the definitions and results of the present chapter carry over to the intrinsic setting of Riemannian manifolds (§5.4).

5.1 Isometries

Let M and M' be connected submanifolds of $\mathbb{R}^n$. An isometry is an isomorphism of the intrinsic geometries of M and M'. Recall the definition of the intrinsic distance function

$$d : M \times M \to [0, \infty)$$

in §4.2 by

$$d(p, q) := \inf_{\gamma \in \Omega_{p,q}} L(\gamma), \qquad L(\gamma) = \int_0^1 |\dot{\gamma}(t)|\, dt$$

for $p, q \in M$. Let d' denote the intrinisic distance function on M'.

Theorem 5.1.1 (Isometries). *Let $\phi : M \to M'$ be a bijective map. Then the following are equivalent.*

(i) *ϕ intertwines the distance functions on M and M', i.e.*

$$d'(\phi(p), \phi(q)) = d(p, q)$$

for all $p, q \in M$.

(ii) *ϕ is a diffeomorphism and*

$$d\phi(p) : T_pM \to T_{\phi(p)}M'$$

is an orthogonal isomorphism for every $p \in M$.

(iii) *ϕ is a diffeomorphism and*

$$L(\phi \circ \gamma) = L(\gamma)$$

for every smooth curve $\gamma : [a, b] \to M$.

The bijection ϕ is called an **isometry** *if it satisfies these equivalent conditions. In the case $M = M'$ the isometries $\phi : M \to M$ form a group denoted by $\mathcal{I}(M)$ and called the* **isometry group** *of M.*

Proof. See page 216. □

Lemma 5.1.2. *For every $p \in M$ there exists a constant $\varepsilon > 0$ such that, for all $v, w \in T_pM$ with $0 < |w| < |v| < \varepsilon$, we have*

$$d(\exp_p(w), \exp_p(v)) = |v| - |w| \qquad \Longrightarrow \qquad w = \frac{|w|}{|v|}v. \tag{5.1.1}$$

Proof. See page 215. □

Remark 5.1.3. It follows from the triangle inequality and Theorem 4.5.4 that

$$d(\exp_p(v), \exp_p(w)) \geq d(\exp_p(v), p) - d(\exp_p(w), p) = |v| - |w|$$

whenever $0 < |w| < |v| < \mathrm{inj}(p)$. Lemma 5.1.2 asserts that equality can only hold when w is a positive multiple of v or, to put it differently, that the distance between $\exp_p(v)$ and $\exp_p(w)$ must be strictly bigger that $|v| - |w|$ whenever w is not a positive multiple of v.

Proof of Lemma 5.1.2. As in Corollary 4.3.8 we denote

$$B_\varepsilon(p) := \{v \in T_pM \mid |v| < \varepsilon\},$$
$$U_\varepsilon(p) := \{q \in M \mid d(p,q) < \varepsilon\}.$$

By Theorem 4.5.4 and the definition of the injectivity radius, the exponential map at p is a diffeomorphism $\exp_p : B_\varepsilon(p) \to U_\varepsilon(p)$ for $\varepsilon < \mathrm{inj}(p)$. Choose $0 < r < \mathrm{inj}(p)$. Then the closure of $U_r(p)$ is a compact subset of M. Hence there is a constant $\varepsilon > 0$ such that $\varepsilon < r$ and $\varepsilon < \mathrm{inj}(p')$ for every $p' \in \overline{U_r(p)}$. Since $\varepsilon < r$ we have

$$\varepsilon < \mathrm{inj}(p') \qquad \forall\, p' \in U_\varepsilon(p). \tag{5.1.2}$$

Thus $\exp_{p'} : B_\varepsilon(p') \to U_\varepsilon(p')$ is a diffeomorphism for every $p' \in U_\varepsilon(p)$. Define $p_1 := \exp_p(w)$ and $p_2 := \exp_p(v)$. Then, by assumption, we have $d(p_1,p_2) = |v| - |w| < \varepsilon$. Since $p_1 \in U_\varepsilon(p)$ it follows from our choice of ε that $\varepsilon < \mathrm{inj}(p_1)$. Hence there is a unique tangent vector $v_1 \in T_{p_1}M$ such that

$$|v_1| = d(p_1,p_2) = |v| - |w|, \qquad \exp_{p_1}(v_1) = p_2.$$

Following first the shortest geodesic from p to p_1 and then the shortest geodesic from p_1 to p_2 we obtain (after suitable reparametrization) a smooth $\gamma : [0,2] \to M$ such that

$$\gamma(0) = p, \qquad \gamma(1) = p_1, \qquad \gamma(2) = p_2,$$

and

$$L(\gamma|_{[0,1]}) = d(p,p_1) = |w|, \qquad L(\gamma|_{[1,2]}) = d(p_1,p_2) = |v| - |w|.$$

Thus $L(\gamma) = |v| = d(p,p_2)$. Hence, by Theorem 4.5.4, there is a smooth function $\beta : [0,2] \to [0,1]$ satisfying

$$\beta(0) = 0, \qquad \beta(2) = 1, \qquad \dot\beta(t) \ge 0, \qquad \gamma(t) = \exp_p(\beta(t)v)$$

for every $t \in [0,2]$. This implies

$$\exp_p(w) = p_1 = \gamma(1) = \exp_p(\beta(1)v), \qquad 0 \le \beta(1) \le 1.$$

Since w and $\beta(1)v$ are both elements of $B_\varepsilon(p)$ and $\exp_p$ is injective on $B_\varepsilon(p)$, this implies $w = \beta(1)v$. Since $\beta(1) \ge 0$ we have $\beta(1) = |w| / |v|$. This proves (5.1.1) and Lemma 5.1.2. □

Proof of Theorem 5.1.1. That (ii) implies (iii) follows from the definition of the length of a curve. Namely

$$\begin{aligned} L(\phi\circ\gamma) &= \int_a^b \left|\frac{d}{dt}\phi(\gamma(t))\right| dt \\ &= \int_a^b \left|d\phi(\gamma(t))\dot\gamma(t)\right| dt \\ &= \int_a^b \left|\dot\gamma(t)\right| dt \\ &= L(\gamma). \end{aligned}$$

In the third equation we have used (ii). That (iii) implies (i) follows immediately from the definition of the intrinsic distance functions d and d'.

We prove that (i) implies (ii). Fix a point $p \in M$ and choose $\varepsilon > 0$ so small that $\varepsilon < \mathrm{inj}(p)$ and that the assertion of Lemma 5.1.2 holds for the point $p' := \phi(p) \in M'$. Then there is a unique homeomorphism

$$\Phi_p : B_\varepsilon(p) \to B_\varepsilon(\phi(p))$$

such that the following diagram commutes.

$$\begin{tikzcd} T_pM & \supset & B_\varepsilon(p) \arrow[r, "\Phi_p"] \arrow[d, "\exp_p"'] & B_\varepsilon(\phi(p)) \arrow[d, "\exp'_{\phi(p)}"] & \subset & T_{\phi(p)}M' . \\ M & \supset & U_\varepsilon(p) \arrow[r, "\phi"] & U_\varepsilon(\phi(p)) & \subset & M' \end{tikzcd}$$

Here the vertical maps are diffeomorphisms and $\phi : U_\varepsilon(p) \to U_\varepsilon(\phi(p))$ is a homeomorphism by (i). Hence $\Phi_p : B_\varepsilon(p) \to B_\varepsilon(\phi(p))$ is a homeomorphism.

Claim 1. *The map Φ_p satisfies the following equations for every $v \in B_\varepsilon(p)$ and every $t \in [0,1]$:*

$$\exp'_{\phi(p)}(\Phi_p(v)) = \phi(\exp_p(v)), \tag{5.1.3}$$

$$|\Phi_p(v)| = |v|, \tag{5.1.4}$$

$$\Phi_p(tv) = t\Phi_p(v). \tag{5.1.5}$$

Equation (5.1.3) holds by definition. To prove (5.1.4) we observe that, by Theorem 4.5.4, we have

$$\begin{aligned} |\Phi_p(v)| &= d'(\phi(p), \exp'_{\phi(p)}(\Phi_p(v))) \\ &= d'(\phi(p), \phi(\exp_p(v))) \\ &= d(p, \exp_p(v)) \\ &= |v|. \end{aligned}$$

Here the second equation follows from (5.1.3) and the third equation from (i). Equation (5.1.5) holds for $t = 0$ because $\Phi_p(0) = 0$ and for $t = 1$ it is a tautology. Hence assume $0 < t < 1$. Then

$$\begin{aligned} d'(\exp'_{\phi(p)}(\Phi_p(tv)), \exp'_{\phi(p)}(\Phi_p(v))) &= d'(\phi(\exp_p(tv)), \phi(\exp_p(v))) \\ &= d(\exp_p(tv), \exp_p(v)) \\ &= |v| - |tv| \\ &= |\Phi_p(v)| - |\Phi_p(tv)| \,. \end{aligned}$$

Here the first equation follows from (5.1.3), the second equation from (i), the third equation from Theorem 4.5.4 and the fact that $|v| < \mathrm{inj}(p)$, and the last equation follows from (5.1.4). Since $0 < |\Phi_p(tv)| < |\Phi_p(v)| < \varepsilon$ we can apply Lemma 5.1.2 and obtain

$$\Phi_p(tv) = \frac{|\Phi_p(tv)|}{|\Phi_p(v)|}\Phi_p(v) = t\Phi_p(v).$$

This proves Claim 1.

By Claim 1, Φ_p extends to a bijective map $\Phi_p : T_pM \to T_{\phi(p)}M'$ via

$$\Phi_p(v) := \frac{1}{\delta}\Phi_p(\delta v),$$

where $\delta > 0$ is chosen so small that $\delta\,|v| < \varepsilon$. The right hand side of this equation is independent of the choice of δ. Hence the extension is well defined. It is bijective because the original map Φ_p is a bijection from $B_\varepsilon(p)$ to $B_\varepsilon(\phi(p))$. The reader may verify that the extended map satisfies the conditions (5.1.4) and (5.1.5) for all $v \in T_pM$ and all $t \geq 0$.

Claim 2. *The extended map $\Phi_p : T_pM \to T_{\phi(p)}M'$ is linear and preserves the inner product.*

It follows from the equation (4.6.3) in the proof of Lemma 4.6.8 that

$$\begin{aligned} |v - w| &= \lim_{t\to 0} \frac{d(\exp_p(tv), \exp_p(tw))}{t} \\ &= \lim_{t\to 0} \frac{d'(\phi(\exp_p(tv)), \phi(\exp_p(tw)))}{t} \\ &= \lim_{t\to 0} \frac{d'(\exp'_{\phi(p)}(\Phi_p(tv))), \exp'_{\phi(p)}(\Phi_p(tw))))}{t} \\ &= \lim_{t\to 0} \frac{d'(\exp'_{\phi(p)}(t\Phi_p(v))), \exp'_{\phi(p)}(t\Phi_p(w))))}{t} \\ &= |\Phi_p(v) - \Phi_p(w)| \,. \end{aligned}$$

Here the second equation follows from (i), the third from (5.1.3), the fourth from (5.1.4), and the last equation follows again from (4.6.3). By polarization we obtain

$$\begin{aligned}2\langle v, w\rangle &= |v|^2 + |w|^2 - |v-w|^2 \\ &= |\Phi_p(v)|^2 + |\Phi_p(w)|^2 - |\Phi_p(v) - \Phi_p(w)|^2 \\ &= 2\langle \Phi_p(v), \Phi_p(w)\rangle.\end{aligned}$$

Thus Φ_p preserves the inner product. Hence, for all $v_1, v_2, w \in T_pM$, we have

$$\begin{aligned}\langle \Phi_p(v_1+v_2), \Phi_p(w)\rangle &= \langle v_1+v_2, w\rangle \\ &= \langle v_1, w\rangle + \langle v_2, w\rangle \\ &= \langle \Phi_p(v_1), \Phi_p(w)\rangle + \langle \Phi_p(v_2), \Phi_p(w)\rangle \\ &= \langle \Phi_p(v_1) + \Phi_p(v_2), \Phi_p(w)\rangle.\end{aligned}$$

Since Φ_p is surjective, this implies

$$\Phi_p(v_1+v_2) = \Phi_p(v_1) + \Phi_p(v_2)$$

for all $v_1, v_2 \in T_pM$. With $v_1 = v$ and $v_2 = -v$ we obtain

$$\Phi_p(-v) = -\Phi_p(v)$$

for every $v \in T_pM$ and by (5.1.5) this gives

$$\Phi_p(tv) = t\Phi_p(v)$$

for all $v \in T_pM$ and $t \in \mathbb{R}$. This proves Claim 2.

Claim 3. *ϕ is smooth and $d\phi(p) = \Phi_p$.*

By (5.1.3) we have

$$\phi = \exp'_{\phi(p)} \circ \Phi_p \circ \exp_p^{-1} : U_\varepsilon(p) \to U_\varepsilon(\phi(p)).$$

Since Φ_p is linear, this shows that the restriction of ϕ to the open set $U_\varepsilon(p)$ is smooth. Moreover, for every $v \in T_pM$ we have

$$d\phi(p)v = \left.\frac{d}{dt}\right|_{t=0} \phi(\exp_p(tv)) = \left.\frac{d}{dt}\right|_{t=0} \exp'_{\phi(p)}(t\Phi_p(v)) = \Phi_p(v).$$

Here we have used equations (5.1.3) and (5.1.5) as well as Lemma 4.3.6. This proves Claim 3 and Theorem 5.1.1. □

Exercise 5.1.4. Prove that every isometry $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is an affine map

$$\psi(p) = Ap + b$$

where $A \in \mathrm{O}(n)$ and $b \in \mathbb{R}^n$. Thus ψ is a composition of translation and rotation. **Hint:** Let $e_1, \dots, e_n$ be the standard basis of $\mathbb{R}^m$. Prove that any two vectors $v, w \in \mathbb{R}^n$ that satisfy

$$|v| = |w|$$

and

$$|v - e_i| = |w - e_i| \qquad \text{for } i = 1, \dots, n$$

must be equal.

Remark 5.1.5. If $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry of the ambient Euclidean space with $\psi(M) = M'$ then certainly $\phi := \psi|_M$ is an isometry from M onto M'. On the other hand, if M is a plane manifold

$$M = \{(0, y, z) \in \mathbb{R}^3 \,|\, 0 < y < \pi/2\}$$

and M' is the cylindrical manifold

$$M' = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1,\ x > 0,\ y > 0\}$$

Then the map $\phi : M \to M'$ defined by

$$\phi(0, y, z) := (\cos(y), \sin(y), z)$$

is an isometry which is *not* of the form $\phi = \psi|_M$. Indeed, an isometry of the form $\phi = \psi|_M$ necessarily preserves the second fundamental form (as well as the first) in the sense that

$$d\psi(p) h_p(v, w) = h'_{\psi(p)}(d\psi(p)v, d\psi(p)w)$$

for $v, w \in T_pM$ but in the example h vanishes identically while h' does not.

We may thus distinguish two fundamental question:

I. Given M and M' when are they extrinsically isomorphic, i.e. when is there an ambient isometry $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with $\psi(M) = M'$?

II. Given M and M' when are they intrinsically isomorphic, i.e. when is there an isometry $\phi : M \to M'$ from M onto M'?

As we have noted, both the first and second fundamental forms are preserved by extrinsic isomorphisms while only the first fundamental form need be preserved by an intrinsic isomorphism (i.e. an isometry).

A question which occurred to Gauß (who worked for a while as a cartographer) is this: Can one draw a perfectly accurate map of a portion of the earth? (i.e. a map for which the distance between points on the map is proportional to the distance between the corresponding points on the surface of the earth). We can now pose this question as follows: Is there an isometry from an open subset of a sphere to an open subset of a plane? Gauß answered this question negatively by associating an invariant, the Gaußian curvature

$$K : M \to \mathbb{R},$$

to a surface $M \subset \mathbb{R}^3$. According to his *Theorema Egregium*

$$K' \circ \phi = K$$

for an isometry $\phi : M \to M'$. The sphere has positive curvature; the plane has zero curvature; hence the perfectly accurate map does not exist. Our aim is to explain these ideas.

We shall need a concept slightly more general than that of "isometry".

Definition 5.1.6. *A smooth map $\phi : M \to M'$ is called a* **local isometry** *if its derivative*

$$d\phi(p) : T_pM \to T_{\phi(p)}M'$$

is an orthogonal linear isomorphism for every $p \in M$.

Remark 5.1.7. Let $M \subset \mathbb{R}^n$ and $M' \subset \mathbb{R}^{m'}$ be manifolds and $\phi : M \to M'$ be a map. The following are equivalent.

(i) ϕ is a local isometry.

(ii) For every $p \in M$ there are open neighborhoods $U \subset M$ and $U' \subset M'$ such that the restriction of ϕ to U is an isometry from U onto U'.

That (ii) implies (i) follows immediately from Theorem 5.1.1. On the other hand (i) implies that $d\phi(p)$ is invertible so that (ii) follows from the inverse function theorem.

Example 5.1.8. The map

$$\mathbb{R} \to S^1 : \theta \mapsto e^{i\theta}$$

is a local isometry but not an isometry.

Exercise 5.1.9. Let $M \subset \mathbb{R}^n$ be a compact connected 1-manifold. Prove that M is diffeomorphic to the circle S^1. Define the length of a compact connected Riemannian 1-manifold. Prove that two compact connected 1-manifolds $M, M' \subset \mathbb{R}^n$ are isometric if and only if they have the same length. **Hint:** Let $\gamma : \mathbb{R} \to M$ be a geodesic with $|\dot\gamma(t)| \equiv 1$. Show that γ is not injective; otherwise construct an open cover of M without finite subcover. If $t_0 < t_1$ with $\gamma(t_0) = \gamma(t_1)$ show that $\dot\gamma(t_0) = \dot\gamma(t_1)$; otherwise show that $\gamma(t_0 + t) = \gamma(t_1 - t)$ for all t and find a contradiction.

We close this section with a result which asserts that two local isometries that have the same value and the same derivative at a single point must agree everywhere, provided that the domain is connected.

Lemma 5.1.10. *Let $M \subset \mathbb{R}^n$ and $M' \subset \mathbb{R}^{n'}$ be smooth m-manifolds and assume that M is connected. Let*

$$\phi : M \to M', \qquad \psi : M \to M'$$

be local isometries and let $p_0 \in M$ such that

$$\phi(p_0) = \psi(p_0) =: p_0', \qquad d\phi(p_0) = d\psi(p_0) : T_{p_0}M \to T_{p_0'}M'.$$

Then $\phi(p) = \psi(p)$ for every $p \in M$.

Proof. Define the set

$$M_0 := \{p \in M \,|\, \phi(p) = \psi(p),\, d\phi(p) = d\psi(p)\}.$$

This set is obviously closed. We prove that M_0 is open. Let $p \in M_0$ and choose $U \subset M$ and $U' \subset M'$ as in Remark 5.1.7 (ii). Denote

$$\Phi_p := d\phi(p) = d\psi(p) : T_pM \to T_{p'}M', \qquad p' := \phi(p) = \psi(p)$$

The proof of Theorem 5.1.1 shows that there exists a constant $\varepsilon > 0$ such that $U_\varepsilon(p) \subset U$ and $U_\varepsilon(p') \subset U'$ and

$$q \in U_\varepsilon(p) \qquad \Longrightarrow \qquad \phi(q) = \exp'_{p'} \circ \Phi_p \circ \exp_p^{-1}(q) = \psi(q).$$

Hence $U_\varepsilon(p) \subset M_0$. Thus M_0 is open, closed, and nonempty. Since M is connected it follows that $M_0 = M$ and this proves Lemma 5.1.10. □

5.2 Riemann Curvature Tensor

5.2.1 Definition and Gauß–Codazzi

Let $M \subset \mathbb{R}^n$ be a smooth manifold and $\gamma : \mathbb{R}^2 \to M$ be a smooth map. Denote by (s,t) the coordinates on $\mathbb{R}^2$. Let $Z \in \mathrm{Vect}(\gamma)$ be a smooth vector field along γ, i.e. $Z : \mathbb{R}^2 \to \mathbb{R}^n$ is a smooth map such that $Z(s,t) \in T_{\gamma(s,t)}M$ for all s and t. The **covariant partial derivatives** of Z with respect to the variables s and t are defined by

$$\nabla_s Z := \Pi(\gamma)\frac{\partial Z}{\partial s}, \qquad \nabla_t Z := \Pi(\gamma)\frac{\partial Z}{\partial t}.$$

In particular $\partial_s\gamma = \partial\gamma/\partial s$ and $\partial_t\gamma = \partial\gamma/\partial t$ are vector fields along γ and we have

$$\nabla_s\partial_t\gamma - \nabla_t\partial_s\gamma = 0$$

as both terms on the left are equal to $\Pi(\gamma)\partial_s\partial_t\gamma$. Thus ordinary partial differentiation and covariant partial differentiation commute. The analogous formula (which results on replacing ∂ by ∇ and γ by Z) is in general false. Instead we have the following.

Definition 5.2.1. *The* **Riemann curvature tensor** *assigns to each* $p \in M$ *the bilinear map*

$$R_p : T_pM \times T_pM \to \mathcal{L}(T_pM, T_pM)$$

characterized by the equation

$$R_p(u,v)w = \big(\nabla_s\nabla_t Z - \nabla_t\nabla_s Z\big)(0,0) \tag{5.2.1}$$

for $u, v, w \in T_pM$ *where* $\gamma : \mathbb{R}^2 \to M$ *is a smooth map and* $Z \in \mathrm{Vect}(\gamma)$ *is a smooth vector field along* γ *such that*

$$\gamma(0,0) = p, \qquad \partial_s\gamma(0,0) = u, \qquad \partial_t\gamma(0,0) = v, \qquad Z(0,0) = w. \tag{5.2.2}$$

We must prove that R is well defined, i.e. that the right hand side of equation (5.2.1) is independent of the choice of γ and Z. This follows from the Gauß–Codazzi formula which we prove next. Recall that the second fundamental form can be viewed as a linear map $h_p : T_pM \to \mathcal{L}(T_pM, T_pM^\perp)$ and that, for $u \in T_pM$, the linear map $h_p(u) \in \mathcal{L}(T_pM, T_pM^\perp)$ and its dual $h_p(u)^* \in \mathcal{L}(T_pM^\perp, T_pM)$ are given by

$$h_p(u)v = \big(d\Pi(p)u\big)v, \qquad h_p(u)^*w = \big(d\Pi(p)u\big)w$$

for $v \in T_pM$ and $w \in T_pM^\perp$.

Theorem 5.2.2. *The Riemann curvature tensor is well defined and given by the* **Gauß–Codazzi formula**

$$R_p(u,v) = h_p(u)^*h_p(v) - h_p(v)^*h_p(u) \tag{5.2.3}$$

for $u, v \in T_pM$.

Proof. Let $u, v, w \in T_pM$ and choose a smooth map $\gamma : \mathbb{R}^2 \to M$ and a smooth vector field Z along γ such that (5.2.2) holds. Then, by the Gauß–Weingarten formula (3.2.2), we have

$$\begin{aligned} \nabla_t Z &= \partial_t Z - h_\gamma(\partial_t\gamma)Z \\ &= \partial_t Z - \big(d\Pi(\gamma)\partial_t\gamma\big)Z \\ &= \partial_t Z - \big(\partial_t\big(\Pi\circ\gamma\big)\big)\,Z. \end{aligned}$$

Hence

$$\begin{aligned} \partial_s\nabla_t Z &= \partial_s\partial_t Z - \partial_s\Big(\big(\partial_t\big(\Pi\circ\gamma\big)\big)\,Z\Big) \\ &= \partial_s\partial_t Z - \big(\partial_s\partial_t\big(\Pi\circ\gamma\big)\big)\,Z - \big(\partial_t\big(\Pi\circ\gamma\big)\big)\,\partial_s Z \\ &= \partial_s\partial_t Z - \big(\partial_s\partial_t\big(\Pi\circ\gamma\big)\big)\,Z - \big(d\Pi(\gamma)\partial_t\gamma\big)\big(\nabla_s Z + h_\gamma(\partial_s\gamma)Z\big) \\ &= \partial_s\partial_t Z - \big(\partial_s\partial_t\big(\Pi\circ\gamma\big)\big)\,Z - h_\gamma(\partial_t\gamma)\nabla_s Z - h_\gamma(\partial_t\gamma)^*h_\gamma(\partial_s\gamma)Z. \end{aligned}$$

Interchanging s and t and taking the difference we obtain

$$\begin{aligned} \partial_s\nabla_t Z - \partial_t\nabla_s Z &= h_\gamma(\partial_s\gamma)^*h_\gamma(\partial_t\gamma)Z - h_\gamma(\partial_t\gamma)^*h_\gamma(\partial_s\gamma)Z \\ &\quad + h_\gamma(\partial_s\gamma)\nabla_t Z - h_\gamma(\partial_t\gamma)\nabla_s Z. \end{aligned}$$

Here the first two terms on the right are tangent to M and the last two terms on the right are orthogonal to $T_\gamma M$. Hence

$$\begin{aligned} \nabla_s\nabla_t Z - \nabla_t\nabla_s Z &= \Pi(\gamma)\big(\partial_s\nabla_t Z - \partial_t\nabla_s Z\big) \\ &= h_\gamma(\partial_s\gamma)^*h_\gamma(\partial_t\gamma)Z - h_\gamma(\partial_t\gamma)^*h_\gamma(\partial_s\gamma)Z. \end{aligned}$$

Evaluating the right hand side at $s = t = 0$ we find that

$$\big(\nabla_s\nabla_t Z - \nabla_t\nabla_s Z\big)(0,0) = h_p(u)^*h_p(v)w - h_p(v)^*h_p(u)w.$$

This proves the Gauß–Codazzi equation and shows that the left hand side is independent of the choice of γ and Z. This proves Theorem 5.2.2. $\square$

5.2.2 Covariant Derivative of a Global Vector Field

So far we have only defined the covariant derivatives of vector fields along curves. The same method can be applied to global vector fields. This leads to the following definition.

Definition 5.2.3 (Covariant derivative). *Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold and X be a vector field on M. Fix a point $p \in M$ and a tangent vector $v \in T_pM$. The* **covariant derivative of X at p in the direction v** *is the tangent vector*

$$\nabla_v X(p) := \Pi(p) dX(p) v \in T_pM,$$

where $\Pi(p) \in \mathbb{R}^{n\times n}$ denotes the orthogonal projection onto T_pM.

Remark 5.2.4. If $\gamma : I \to M$ is a smooth curve on an interval $I \subset \mathbb{R}$ and $X \in \mathrm{Vect}(M)$ is a smooth vector field on M then $X \circ \gamma$ is a smooth vector field along γ. The covariant derivative of $X \circ \gamma$ is related to the covariant derivative of X by the formula

$$\nabla(X \circ \gamma)(t) = \nabla_{\dot\gamma(t)} X(\gamma(t)). \tag{5.2.4}$$

Remark 5.2.5 (Gauß–Weingarten formula). Differentiating the equation $X = \Pi X$ (understood as a function from M to $\mathbb{R}^n$) and using the notation $\partial_v X(p) := dX(p)v$ for the derivative of X at p in the direction v we obtain the **Gauß–Weingarten formula** for global vector fields:

$$\partial_v X(p) = \nabla_v X(p) + h_p(v) X(p). \tag{5.2.5}$$

Remark 5.2.6 (Levi-Civita connection). Differentiating a vector field Y on M in the direction of another vector field X we obtain a vector field $\nabla_X Y \in \mathrm{Vect}(M)$ defined by

$$(\nabla_X Y)(p) := \nabla_{X(p)} Y(p)$$

for $p \in M$. This gives rise to a family of linear operators

$$\nabla_X : \mathrm{Vect}(M) \to \mathrm{Vect}(M),$$

one for every vector field $X \in \mathrm{Vect}(M)$, and the assignment

$$\mathrm{Vect}(M) \to \mathcal{L}(\mathrm{Vect}(M), \mathrm{Vect}(M)) : X \mapsto \nabla_X$$

is itself a linear operator. This operator is called the **Levi-Civita connection** on the tangent bundle TM. It satisfies the conditions

$$\nabla_{fX}(Y) = f\nabla_X Y, \tag{5.2.6}$$

$$\nabla_X(fY) = f\nabla_X Y + (\mathcal{L}_X f)\, Y \tag{5.2.7}$$

$$\mathcal{L}_X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle, \tag{5.2.8}$$

$$\nabla_Y X - \nabla_X Y = [X, Y], \tag{5.2.9}$$

for all $X, Y, Z \in \mathrm{Vect}(M)$ and $f \in \mathscr{F}(M)$, where $\mathcal{L}_X f = df \circ X$ and $[X, Y] \in \mathrm{Vect}(M)$ denotes the Lie bracket of the vector fields X and Y. The next lemma asserts that the Levi-Civita connection is uniquely determined by (5.2.8) and (5.2.9).

Lemma 5.2.7 (Uniqueness Lemma). *There is a unique linear operator*

$$\mathrm{Vect}(M) \to \mathcal{L}(\mathrm{Vect}(M), \mathrm{Vect}(M)) : X \mapsto \nabla_X$$

satisfying equations (5.2.8) *and* (5.2.9) *for all* $X, Y, Z \in \mathrm{Vect}(M)$.

Proof. Existence follows from the properties of the Levi-Civita connection. We prove uniqueness. Let $X \mapsto D_X$ be any linear operator from $\mathrm{Vect}(M)$ to $\mathcal{L}(\mathrm{Vect}(M), \mathrm{Vect}(M))$ that satisfies (5.2.8) and (5.2.9). Then we have

$$\begin{aligned}
\mathcal{L}_X\langle Y, Z\rangle &= \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle, \\
\mathcal{L}_Y\langle X, Z\rangle &= \langle D_Y X, Z\rangle + \langle X, D_Y Z\rangle, \\
-\mathcal{L}_Z\langle X, Y\rangle &= -\langle D_Z X, Y\rangle - \langle X, D_Z Y\rangle.
\end{aligned}$$

Adding these three equations we find

$$\begin{aligned}
&\mathcal{L}_X\langle Y, Z\rangle + \mathcal{L}_Y\langle Z, X\rangle - \mathcal{L}_Z\langle X, Y\rangle \\
&= 2\langle D_X Y, Z\rangle + \langle D_Y X - D_X Y, Z\rangle \\
&\quad + \langle X, D_Y Z - D_Z Y\rangle + \langle Y, D_X Z - D_Z X\rangle \\
&= 2\langle D_X Y, Z\rangle + \langle [X, Y], Z\rangle + \langle X, [Z, Y]\rangle + \langle Y, [Z, X]\rangle.
\end{aligned}$$

The same equation holds for the Levi-Civita connection and hence

$$\langle D_X Y, Z\rangle = \langle \nabla_X Y, Z\rangle.$$

This implies $D_X Y = \nabla_X Y$ for all $X, Y \in \mathrm{Vect}(M)$. □

Remark 5.2.8 (The Levi-Civita connection in local coordinates). Let $\phi : U \to \Omega$ be a coordinate chart on an open set $U \subset M$ with values in an open set $\Omega \subset \mathbb{R}^m$. In such a coordinate chart a vector field $X \in \mathrm{Vect}(M)$ is represented by a smooth map

$$\xi = (\xi^1, \dots, \xi^m) : \Omega \to \mathbb{R}^m$$

defined by

$$\xi(\phi(p)) = d\phi(p)X(p)$$

for $p \in U$. If $Y \in \mathrm{Vect}(M)$ is represented by η then $\nabla_X Y$ is represented by the function

$$(\nabla_\xi \eta)^k := \sum_{i=1}^m \frac{\partial \eta^k}{\partial x^i}\xi^i + \sum_{i,j=1}^m \Gamma_{ij}^k \xi^i \eta^j. \tag{5.2.10}$$

Here the $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ are the Christoffel symbols defined by

$$\Gamma_{ij}^k := \sum_{\ell=1}^m g^{k\ell}\frac{1}{2}\left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell}\right), \tag{5.2.11}$$

where g_{ij} is the metric tensor and g^{ij} is the inverse matrix so that

$$\sum_j g_{ij}g^{jk} = \delta_i^k$$

(see Lemma 3.6.5). This formula can be used to prove the existence statement in Lemma 5.2.7 and hence define the Levi-Civita connection in the intrinsic setting.

Exercise 5.2.9. In the proof of Lemma 5.2.7 we did not actually use that the operator $D_X : \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ is linear nor that the operator $X \mapsto D_X$ is linear. Prove directly that if a map

$$D_X : \mathcal{L}(M) \to \mathcal{L}(M)$$

satisfies (5.2.8) for all $Y, Z \in \mathrm{Vect}(M)$ then D_X is linear. Prove that every map

$$\mathrm{Vect}(M) \to \mathcal{L}(\mathrm{Vect}(M), \mathrm{Vect}(M)) : X \mapsto D_X$$

that satisfies (5.2.9) is linear.

5.2.3 A Global Formula

Lemma 5.2.10. *For $X, Y, Z \in \mathrm{Vect}(M)$ we have*

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z. \qquad (5.2.12)$$

Proof. Fix a point $p \in M$. Then the right hand side of equation (5.2.12) at p remains unchanged if we multiply each of the vector fields X, Y, Z by a smooth function $f : M \to [0,1]$ that is equal to one near p. Choosing f with compact support we may therefore assume that the vector fields X and Y are complete. Let ϕ^s denote the flow of X and ψ^t the flow of Y. Define the map $\gamma : \mathbb{R}^2 \to M$ by

$$\gamma(s,t) := \phi^s \circ \psi^t(p), \qquad s, t \in \mathbb{R}.$$

Then

$$\partial_s \gamma = X(\gamma), \qquad \partial_t \gamma = (\phi^s_* Y)(\gamma).$$

Hence, by Remark 5.2.5, we have

$$\nabla_s (Z \circ \gamma) = (\nabla_X Z)(\gamma), \qquad \nabla_t (Z \circ \gamma) = \left(\nabla_{\phi^s_* Y} Z\right)(\gamma).$$

This implies

$$\nabla_s \nabla_t (Z \circ \gamma) = \left(\nabla_{\partial_s \gamma} \nabla_{\phi^s_* Y} Z\right)(\gamma) + \left(\nabla_{\partial_s \phi^s_* Y} Z\right)(\gamma).$$

Since

$$\left.\frac{\partial}{\partial s}\right|_{s=0} \phi^s_* Y = [X,Y]$$

and $\partial_s \gamma = X(\gamma)$ we obtain

$$\begin{aligned}
\nabla_s \nabla_t (Z \circ \gamma)(0,0) &= \nabla_X \nabla_Y Z(p) + \nabla_{[X,Y]} Z(p), \\
\nabla_t \nabla_s (Z \circ \gamma)(0,0) &= \nabla_Y \nabla_X Z(p).
\end{aligned}$$

Hence

$$\begin{aligned}
R_p(X(p), Y(p))Z(p) &= \left(\nabla_s \nabla_t (Z \circ \gamma) - \nabla_t \nabla_s (Z \circ \gamma)\right)(0,0) \\
&= \nabla_X \nabla_Y Z(p) - \nabla_Y \nabla_X Z(p) + \nabla_{[X,Y]} Z(p).
\end{aligned}$$

This proves Lemma 5.2.10. □

Remark 5.2.11. Equation (5.2.12) can be written succinctly as

$$[\nabla_X, \nabla_Y] + \nabla_{[X,Y]} = R(X,Y). \tag{5.2.13}$$

This can be contrasted with the equation

$$[\mathcal{L}_X, \mathcal{L}_Y] + \mathcal{L}_{[X,Y]} = 0 \tag{5.2.14}$$

for the operator $\mathcal{L}_X$ on the space of real valued functions on M.

Remark 5.2.12. Equation (5.2.12) can be used to define the Riemann curvature tensor. To do this one must again prove that the right hand side of equation (5.2.12) at p depends only on the values $X(p), Y(p), Z(p)$ of the vector fields X, Y, Z at the point p. For this it suffices to prove that the map

$$\mathrm{Vect}(M) \times \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M) : (X,Y,Z) \mapsto R(X,Y)Z$$

is linear over the Ring $\mathscr{F}(M)$ of smooth real valued functions on M, i.e.

$$R(fX,Y)Z = R(X,fY)Z = R(X,Y)fZ = fR(X,Y)Z \tag{5.2.15}$$

for $X, Y, Z \in \mathrm{Vect}(M)$ and $f \in \mathscr{F}(M)$. The formula (5.2.15) follows from the equations (5.2.6), (5.2.7), (5.2.14), and $[X, fY] = f[X,Y] - (\mathcal{L}_X f)Y$. It follows from (5.2.15) that the right hand side of (5.2.12) at p depends only on the vectors $X(p)$, $Y(p)$, $Z(p)$. The proof requires two steps. One first shows that if X vanishes near p then the right hand side of (5.2.12) vanishes at p (and similarly for Y and Z). Just multiply X by a smooth function equal to zero at p and equal to one on the support of X; then $fX = X$ and hence the vector field $R(X,Y)Z = R(fX,Y)Z = fR(X,Y)Z$ vanishes at p. Second, we choose a local frame $E_1, \dots, E_m \in \mathrm{Vect}(M)$, i.e. vector fields that form a basis of T_pM for each p in some open set $U \subset M$. Then we may write

$$X = \sum_{i=1}^m \xi^i E_i, \qquad Y = \sum_{j=1}^m \eta^j E_j, \qquad Z = \sum_{k=1}^m \zeta^k E_k$$

in U. Using the first step and the $\mathscr{F}(M)$-multilinearity we obtain

$$R(X,Y)Z = \sum_{i,j,k=1}^m \xi^i \eta^j \zeta^k R(E_i, E_j) E_k$$

in U. If $X'(p) = X(p)$ then $\xi^i(p) = {\xi'}^i(p)$ so if $X(p) = X'(p)$, $Y(p) = Y'(p)$, $Z(p) = Z'(p)$ then $(R(X,Y)Z)(p) = (R(X',Y')Z')(p)$ as required.

5.2.4 Symmetries

Theorem 5.2.13. *The Riemann curvature tensor satisfies*

$$R(Y,X) = -R(X,Y) = R(X,Y)^*, \tag{5.2.16}$$

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \tag{5.2.17}$$

$$\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle, \tag{5.2.18}$$

for $X,Y,Z,W \in \mathrm{Vect}(M)$. *Equation* (5.2.17) *is the* **first Bianchi identity**.

Proof. The first equation in (5.2.16) is obvious from the definition and the second follows from the Gauß–Codazzi formula (5.2.3). Alternatively, choose a smooth map $\gamma : \mathbb{R}^2 \to M$ and two vector fields Z, W along γ. Then

$$\begin{aligned}
0 &= \partial_s\partial_t\langle Z,W\rangle - \partial_t\partial_s\langle Z,W\rangle \\
&= \partial_s\langle \nabla_t Z, W\rangle + \partial_s\langle Z, \nabla_t W\rangle - \partial_t\langle \nabla_s Z, W\rangle - \partial_t\langle Z, \nabla_s W\rangle \\
&= \langle \nabla_s\nabla_t Z, W\rangle + \langle Z, \nabla_s\nabla_t W\rangle - \langle \nabla_t\nabla_s Z, W\rangle - \langle Z, \nabla_t\nabla_s W\rangle \\
&= \langle R(\partial_s\gamma, \partial_t\gamma)Z, W\rangle - \langle Z, R(\partial_s\gamma, \partial_t\gamma)W\rangle.
\end{aligned}$$

This proof has the advantage that it carries over to the intrinsic setting. We prove the first Bianchi identity using (5.2.9) and (5.2.12):

$$\begin{aligned}
&R(X,Y)Z + R(Y,Z)X + R(Z,X)Y \\
&= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z + \nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X + \nabla_{[Y,Z]}X \\
&\quad + \nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y + \nabla_{[Z,X]}Y \\
&= \nabla_{[Y,Z]}X - \nabla_X[Y,Z] + \nabla_{[Z,X]}Y - \nabla_Y[Z,X] + \nabla_{[X,Y]}Z - \nabla_Z[X,Y] \\
&= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]].
\end{aligned}$$

The last term vanishes by the Jacobi identity. We prove (5.2.18) by combining the first Bianchi identity with (5.2.16):

$$\begin{aligned}
&\langle R(X,Y)Z, W\rangle - \langle R(Z,W)X, Y\rangle \\
&= -\langle R(Y,Z)X, W\rangle - \langle R(Z,X)Y, W\rangle - \langle R(Z,W)X, Y\rangle \\
&= \langle R(Y,Z)W, X\rangle + \langle R(Z,X)W, Y\rangle + \langle R(W,Z)X, Y\rangle \\
&= \langle R(Y,Z)W, X\rangle - \langle R(X,W)Z, Y\rangle \\
&= \langle R(Y,Z)W, X\rangle - \langle R(W,X)Y, Z\rangle.
\end{aligned}$$

Note that the first line is related to the last by a cyclic permutation. Repeating this argument we find

$$\langle R(Y,Z)W, X\rangle - \langle R(W,X)Y, Z\rangle = \langle R(Z,W)X, Y\rangle - \langle R(X,Y)Z, W\rangle.$$

Combining the last two identities we obtain (5.2.18). This proves Theorem 5.2.13. □

Remark 5.2.14. We may think of a vector field X on M as a section of the tangent bundle. This is reflected in the alternative notation

$$\Omega^0(M, TM) := \mathrm{Vect}(M).$$

A 1**-form on** M **with values in the tangent bundle** is a collection of linear maps $A(p) : T_pM \to T_pM$, one for every $p \in M$, which is smooth in the sense that for every smooth vector field X and M the assignment $p \mapsto A(p)X(p)$ defines again a smooth vector field on M. We denote by

$$\Omega^1(M, TM)$$

the space of smooth 1-forms on M with values in TM. The covariant derivative of a vector field Y is such a 1-form with values in the tangent bundle which assigns to every $p \in M$ the linear map $T_pM \to T_pM : v \mapsto \nabla_v Y(p)$. Thus we can think of the covariant derivative as a linear operator

$$\nabla : \Omega^0(M, TM) \to \Omega^1(M, TM).$$

The equation (5.2.6) asserts that the operators $X \mapsto \nabla_X$ indeed determine a linear operator from $\Omega^0(M, TM)$ to $\Omega^1(M, TM)$. Equation (5.2.7) asserts that this linear operator ∇ is a **connection** on the tangent bundle of M. Equation (5.2.8) asserts that ∇ is a **Riemannian connection** and equation (5.2.9) asserts that ∇ is **torsion-free**. Thus Lemma 5.2.7 can be restated as asserting that the **Levi-Civita connection** is the unique torsion-free Riemannian connection on the tangent bundle.

Exercise 5.2.15. Extend the notion of a connection to a general vector bundle E, both as a collection of linear operators $\nabla_X : \Omega^0(M, E) \to \Omega^0(M, E)$, one for every vector field $X \in \mathrm{Vect}(M)$, and as a linear operator

$$\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$$

satisfying the analogue of equation (5.2.7). Interpret this equation as a Leibniz rule for the product of a function on M with a section of E. Show that $\nabla^\perp$ is a connection on $TM^\perp$. Extend the notion of curvature to connections on general vector bundles.

Exercise 5.2.16. Show that the field which assigns to each $p \in M$ the multi-linear map $R^\perp_p : T_pM \times T_pM \to \mathcal{L}(T_pM^\perp, T_pM^\perp)$ characterized by

$$R^\perp(\partial_s\gamma, \partial_t\gamma)Y = \nabla^\perp_s\nabla^\perp_t Y - \nabla^\perp_t\nabla^\perp_s Y$$

for $\gamma : \mathbb{R}^2 \to M$ and $Y \in \mathrm{Vect}^\perp(\gamma)$ satisfies the equation

$$R^\perp_p(u, v) = h_p(u)h_p(v)^* - h_p(v)h_p(u)^*$$

for $p \in M$ and $u, v \in T_pM$.

5.2.5 Examples and Exercises

Example 5.2.17. Let $\mathrm{G} \subset \mathrm{O}(n)$ be a **Lie subgroup**, i.e. a subgroup that is also a submanifold. Consider the Riemannian metric on G induced by the inner product

$$\langle v, w \rangle := \mathrm{trace}(v^\mathsf{T} w) \tag{5.2.19}$$

on the ambient space $\mathfrak{gl}(n, \mathbb{R}) = \mathbb{R}^{n\times n}$. Let $\mathfrak{g} := \mathrm{Lie}(\mathrm{G}) = T_{\mathbb{1}}\mathrm{G}$ be the Lie algebra of G. Then the Riemann curvature tensor on G can be expressed in terms of the Lie bracket (see item (d) below).

(a) The maps $g \mapsto ag$, $g \mapsto ga$, $g \mapsto g^{-1}$ are isometries of G for every $a \in \mathrm{G}$.

(b) A smooth map $\gamma : \mathbb{R} \to \mathrm{G}$ is a geodesic if and only if there exist matrices $g \in \mathrm{G}$ and $\xi \in \mathfrak{g}$ such that

$$\gamma(t) = g \exp(t\xi).$$

For $\mathrm{G} = \mathrm{O}(n)$ we have seen this in Example 4.3.12 and the proof in the general case is similar. Hence the exponential map $\exp : \mathfrak{g} \to \mathrm{G}$ defined by the exponential matrix (as in §2.5) agrees with the time-1-map of the geodesic flow (as in §4.3).

(c) Let $\gamma : \mathbb{R} \to \mathrm{G}$ be a smooth curve and $X \in \mathrm{Vect}(\gamma)$ be a smooth vector field along γ. Then the covariant derivative of X is given by

$$\gamma(t)^{-1}\nabla X(t) = \frac{d}{dt}\gamma(t)^{-1}X(t) + \frac{1}{2}\left[\gamma(t)^{-1}\dot\gamma(t), \gamma(t)^{-1}X(t)\right]. \tag{5.2.20}$$

(**Exercise:** Prove equation (5.2.20). **Hint:** Since $\mathfrak{g} \subset \mathfrak{o}(n)$ we have the identity $\mathrm{trace}((\xi\eta + \eta\xi)\zeta) = 0$ for all $\xi, \eta, \zeta \in \mathfrak{g}$.)

(d) The Riemann curvature tensor on G is given by

$$g^{-1}R_g(u, v)w = -\frac{1}{4}[[g^{-1}u, g^{-1}v], g^{-1}w]. \tag{5.2.21}$$

Note that the first Bianchi identity is equivalent to the Jacobi identity. (**Exercise:** Prove equation (5.2.21).)

Exercise 5.2.18. Prove that every Lie subgroup of $\mathrm{O}(n)$ is a closed subset and hence is compact. Show that the inner product (5.2.19) on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(\mathrm{G}) = T_{\mathbb{1}}\mathrm{G}$ of a Lie subgroup $\mathrm{G} \subset \mathrm{O}(n)$ is invariant under conjugation:

$$\langle \xi, \eta \rangle = \langle g\xi g^{-1}, g\eta g^{-1} \rangle$$

for all $g \in \mathrm{G}$ and all $\xi, \eta \in \mathfrak{g}$. Show that

$$\langle [\xi, \eta], \zeta \rangle = \langle \xi, [\eta, \zeta] \rangle$$

for all $\xi, \eta, \zeta \in \mathfrak{g}$.

Example 5.2.19. Let $\mathrm{G} \subset \mathrm{GL}(n,\mathbb{R})$ be any Lie subgroup, not necessarily contained in $\mathrm{O}(n)$, and let

$$\mathfrak{g} := \mathrm{Lie}(\mathrm{G}) = T_{\mathbb{1}}\mathrm{G}$$

be its Lie algebra. Fix any inner product on the Lie algebra $\mathfrak{g}$ (not necessarily invariant under conjugation) and consider the Riemannian metric on G defined by

$$\langle v, w\rangle_g := \langle vg^{-1}, wg^{-1}\rangle$$

for $v, w \in T_g\mathrm{G}$. This metric is called **right invariant**.

(a) Define the linear map $A : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ by

$$\langle A(\xi)\eta, \zeta\rangle = \frac{1}{2}\Big(\langle \xi, [\eta,\zeta]\rangle - \langle \eta, [\zeta,\xi]\rangle - \langle \zeta, [\xi,\eta]\rangle\Big)$$

for $\xi, \eta, \zeta \in \mathfrak{g}$. Then A is the unique linear map that satisfies

$$A(\xi) + A(\xi)^* = 0, \qquad A(\eta)\xi + A(\xi)\eta = [\xi,\eta]$$

for all $\xi, \eta \in \mathfrak{g}$. Here $A(\xi)^*$ denotes the adjoint operator with respect to the given inner product on $\mathfrak{g}$. Note that $A(\xi)\eta = -\frac{1}{2}[\xi,\eta]$ whenever the inner product on $\mathfrak{g}$ is invariant under conjugation.

(b) Let $\gamma : \mathbb{R} \to \mathrm{G}$ be a smooth curve and $X \in \mathrm{Vect}(\gamma)$ be a smooth vector field along γ. Then the covariant derivative of X is given by

$$\nabla X = \left(\frac{d}{dt}(X\gamma^{-1}) + A(\dot\gamma\gamma^{-1})X\gamma^{-1}\right)\gamma.$$

(**Exercise:** Prove this.) Hence a smooth curve $\gamma : \mathbb{R} \to \mathrm{G}$ is a geodesic if and only if it satisfies the equation

$$\frac{d}{dt}(\dot\gamma\gamma^{-1}) + A(\dot\gamma\gamma^{-1})\dot\gamma\gamma^{-1} = 0.$$

(c) The Riemann curvature tensor on G is given by

$$\big(R_g(u,v)w\big)g^{-1} = \Big(A\left(\left[ug^{-1}, vg^{-1}\right]\right) + \left[A\left(ug^{-1}\right), A\left(vg^{-1}\right)\right]\Big)wg^{-1}$$

for $g \in \mathrm{G}$ and $u, v, w \in T_g\mathrm{G}$. (**Exercise:** Prove this.)

5.3 Generalized Theorema Egregium

We will now show that geodesics, covariant differentiation, parallel transport, and the Riemann curvature tensor are all intrinsic, i.e. they are intertwined by isometries. In the extrinsic setting these results are somewhat surprising since these objects are all defined using the second fundamental form, whereas isometries need not preserve the second fundamental form in any sense but only the first fundamental form.

Below we shall give a formula expressing the Gaußian curvature of a surface M^2 in $\mathbb{R}^3$ in terms of the Riemann curvature tensor and the first fundamental form. It follows that the Gaußian curvature is also intrinsic. This fact was called by Gauß the "Theorema Egregium" which explains the title of this section.

5.3.1 Pushforward

We assume throughout this section that $M \subset \mathbb{R}^n$ and $M' \subset \mathbb{R}^{n'}$ are smooth submanifolds of the same dimension m. As in §5.1 we denote objects on M' by the same letters as objects in M with primes affixed. In particular, g' denotes the first fundamental form on M' and R' denotes the Riemann curvature tensor on M'.

Let $\phi : M \to M'$ be a diffeomorphism. Using ϕ we can move objects on M to M'. For example the pushforward of a smooth curve $\gamma : I \to M$ is the curve

$$\phi_*\gamma := \phi \circ \gamma : I \to M',$$

the pushforward of a smooth function $f : M \to \mathbb{R}$ is the function

$$\phi_* f := f \circ \phi^{-1} : M' \to \mathbb{R},$$

the pushforward of a vector field $X \in \mathrm{Vect}(\gamma)$ along a curve $\gamma : I \to M$ is the vector field $\phi_* X \in \mathrm{Vect}(\phi_*\gamma)$ defined by

$$(\phi_* X)(t) := d\phi(\gamma(t))X(t)$$

for $t \in I$, and the pushforward of a global vector field $X \in Vect(M)$ is the vector field $\phi_* X \in \mathrm{Vect}(M')$ defined by

$$(\phi_* X)(\phi(p)) := d\phi(p)X(p)$$

for $p \in M$. Recall that the first fundamental form on M is the Riemannian metric g defined as the restriction of the Euclidean inner product on the ambient space to each tangent space of M. It assigns to each $p \in M$ the bilinear map $g_p \in T_pM \times T_pM \to \mathbb{R}$ given by

$$g_p(u,v) = \langle u, v\rangle, \qquad u,v \in T_pM.$$

Its pushforward is the Riemannian metric which assigns to each $p' \in M'$ the inner product $(\phi_* g)_{p'} : T_{p'}M' \times T_{p'}M' \to \mathbb{R}$ defined by

$$(\phi_* g)_{\phi(p)}(d\phi(p)u, d\phi(p)v) := g_p\,(u,v)$$

for $p := \phi^{-1}(p') \in M$ and $u,v \in T_pM$. The pushforward of the Riemann curvature tensor is the tensor which assigns to each $p' \in M'$ the bilinear map $(\phi_* R)_{p'} : T_{p'}M' \times T_{p'}M' \to \mathcal{L}\left(T_{p'}M', T_{p'}M'\right)$, defined by

$$(\phi_* R)_{\phi(p)}(d\phi(p)u, d\phi(p)v) := d\phi(p)R_p\,(u,v)\,d\phi(p)^{-1}$$

for $p := \phi^{-1}(p') \in M$ and $u,v \in T_pM$.

5.3.2 Theorema Egregium

Theorem 5.3.1 (Theorema Egregium). *The first fundamental form, covariant differentiation, geodesics, parallel transport, and the Riemann curvature tensor are intrinsic. This means that for every isometry $\phi : M \to M'$ the following holds.*

(i) $\phi_* g = g'$.

(ii) *If $X \in \mathrm{Vect}(\gamma)$ is a vector field along a smooth curve $\gamma : I \to M$ then*

$$\nabla'(\phi_* X) = \phi_* \nabla X \tag{5.3.1}$$

and if $X, Y \in \mathrm{Vect}(M)$ are global vector fields then

$$\nabla'_{\phi_* X}\phi_* Y = \phi_*(\nabla_X Y). \tag{5.3.2}$$

(iii) *If $\gamma : I \to M$ is a geodesic then $\phi \circ \gamma : I \to M'$ is a geodesic.*

(iv) *If $\gamma : I \to M$ is a smooth curve then for all $s, t \in I$:*

$$\Phi'_{\phi\circ\gamma}(t,s)d\phi(\gamma(s)) = d\phi(\gamma(t))\Phi_\gamma(t,s). \tag{5.3.3}$$

(v) $\phi_* R = R'$.

Proof. Assertion (i) is simply a restatement of Theorem 5.1.1. To prove (ii) we choose a local smooth parametrization $\psi : \Omega \to U$ of an open set $U \subset M$, defined on an open set $\Omega \subset \mathbb{R}^m$, so that $\psi^{-1} : U \to \Omega$ is a coordinate chart. Suppose without loss of generality that $\gamma(t) \in U$ for all $t \in I$ and define $c : I \to \Omega$ and $\xi : I \to \mathbb{R}^m$ by

$$\gamma(t) = \psi(c(t)), \qquad X(t) = \sum_{i=1}^m \xi^i(t) \frac{\partial \psi}{\partial x^i}(c(t)).$$

Recall from equations (3.6.6) and (3.6.7) that

$$\nabla X(t) = \sum_{k=1}^m \left(\dot\xi^k(t) + \sum_{i,j=1}^m \Gamma_{ij}^k(c(t)) \dot c^i(t) \xi^j(t) \right) \frac{\partial \psi}{\partial x^k}(c(t)),$$

where the Christoffel symbols $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ are defined by

$$\Pi(\psi) \frac{\partial^2 \psi}{\partial x^i \partial x^j} = \sum_{k=1}^m \Gamma_{ij}^k \frac{\partial \psi}{\partial x^k}.$$

Now consider the same formula for $\phi_* X$ using the parametrization

$$\psi' := \phi \circ \psi : \Omega \to U' := \phi(U) \subset M'.$$

The Christoffel symbols ${\Gamma'}_{ij}^k : \Omega \to \mathbb{R}$ associated to this parametrization of U' are defined by the same formula as the Γ_{ij}^k with ψ replaced by ψ'. But the metric tensor for ψ agrees with the metric tensor for ψ':

$$g_{ij} = \left\langle \frac{\partial \psi}{\partial x^i}, \frac{\partial \psi}{\partial x^j} \right\rangle = \left\langle \frac{\partial \psi'}{\partial x^i}, \frac{\partial \psi'}{\partial x^j} \right\rangle.$$

Hence it follows from Lemma 3.6.5 that ${\Gamma'}_{ij}^k = \Gamma_{ij}^k$ for all i, j, k. This implies that the covariant derivative of $\phi_* X$ is given by

$$\begin{aligned}
\nabla'(\phi_* X) &= \sum_{k=1}^m \left(\dot\xi^k + \sum_{i,j=1}^m \Gamma_{ij}^k(c) \dot c^i \xi^j \right) \frac{\partial \psi'}{\partial x^k}(c) \\
&= d\phi(\psi(c)) \sum_{k=1}^m \left(\dot\xi^k + \sum_{i,j=1}^m \Gamma_{ij}^k(c) \dot c^i \xi^j \right) \frac{\partial \psi}{\partial x^k}(c) \\
&= \phi_* \nabla X.
\end{aligned}$$

This proves (5.3.1). Equation (5.3.2) follows immediately from (5.3.1) and Remark 5.2.4.

Here is a second proof of (ii). For every vector field $X \in \mathrm{Vect}(M)$ we define the operator $D_X : \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ by

$$D_X Y := \phi^* \left(\nabla_{\phi_* X} \phi_* Y \right).$$

Then, for all $X, Y \in \mathrm{Vect}(M)$, we have

$$D_Y X - D_X Y = \phi^* \left(\nabla_{\phi_* Y} \phi_* X - \nabla_{\phi_* X} \phi_* Y \right) = \phi^* [\phi_* X, \phi_* Y] = [X, Y].$$

Moreover, it follows from (i) that

$$\begin{aligned}
\phi_* \mathcal{L}_X \langle Y, Z \rangle &= \mathcal{L}_{\phi_* X} \langle \phi_* Y, \phi_* Z \rangle \\
&= \langle \nabla_{\phi_* X} \phi_* Y, \phi_* Z \rangle + \langle \phi_* Y, \nabla_{\phi_* X} \phi_* Z \rangle \\
&= \langle \phi_* D_X Y, \phi_* Z \rangle + \langle \phi_* Y, \phi_* D_X Z \rangle \\
&= \phi_* \bigl(\langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle \bigr).
\end{aligned}$$

and hence $\mathcal{L}_X \langle Y, Z \rangle = \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle$ for all $X, Y, Z \in \mathrm{Vect}(M)$. Thus the operator $X \mapsto D_X$ satisfies equations (5.2.8) and (5.2.9) and, by Lemma 5.2.7, it follows that $D_X Y = \nabla_X Y$ for all $X, Y \in \mathrm{Vect}(M)$. This completes the second proof of (ii).

We prove (iii). Since ϕ preserves the first fundamental form it also preserves the energy of curves, namely

$$E(\phi \circ \gamma) = E(\gamma)$$

for every smooth map $\gamma : [0, 1] \to M$. Hence γ is a critical point of the energy functional if and only if $\phi \circ \gamma$ is a critical point of the energy functional. Alternatively it follows from (ii) that

$$\nabla' \left(\frac{d}{dt} \phi \circ \gamma \right) = \nabla' \phi_* \dot{\gamma} = \phi_* \nabla \dot{\gamma}$$

for every smooth curve $\gamma : I \to M$. If γ is a geodesic the last term vanishes and hence $\phi \circ \gamma$ is a geodesic as well. As a third proof we can deduce (iii) from the formula $\phi(\exp_p(v)) = \exp_{\phi(p)}(d\phi(p)v)$ in the proof of Theorem 5.1.1.

We prove (iv). For $t_0 \in I$ and $v_0 \in T_{\gamma(t_0)} M$ define

$$X(t) := \Phi_\gamma(t, t_0) v_0, \qquad X'(t) := \Phi'_{\phi \circ \gamma}(t, t_0) d\phi(\gamma(t_0)) v_0.$$

By (ii) the vector fields X' and $\phi_* X$ along $\phi \circ \gamma$ are both parallel and they agree at $t = t_0$. Hence $X'(t) = \phi_* X(t)$ for all $t \in I$ and this proves (5.3.3).

We prove (v). Fix a smooth map $\gamma : \mathbb{R}^2 \to M$ and a smooth vector field Z along γ, and define $\gamma' = \phi \circ \gamma : \mathbb{R}^2 \to M'$ and $Z' := \phi_* Z \in \mathrm{Vect}(\gamma')$. Then it follows from (ii) that

$$\begin{aligned} R'(\partial_s\gamma', \partial_t\gamma')Z' &= \nabla'_s\nabla'_t Z' - \nabla'_t\nabla'_s Z' \\ &= \phi_* \left(\nabla_s\nabla_t Z - \nabla_t\nabla_s Z\right) \\ &= d\phi(\gamma)R(\partial_s\gamma, \partial_t\gamma)Z \\ &= (\phi_* R)(\partial_s\gamma', \partial_t\gamma')Z'. \end{aligned}$$

This proves (v) and Theorem 5.3.1. □

5.3.3 Gaußian Curvature

As a special case we shall now consider a **hypersurface** $M \subset \mathbb{R}^{m+1}$, i.e. a smooth submanifold of codimension one. We assume that there is a smooth map $\nu : M \to \mathbb{R}^{m+1}$ such that, for every $p \in M$, we have $\nu(p) \perp T_pM$ and $|\nu(p)| = 1$. Such a map always exists locally (see Example 3.1.3). Note that $\nu(p)$ is an element of the unit sphere in $\mathbb{R}^{m+1}$ for every $p \in M$ and hence we can regard ν as a map from M to S^m:

$$\nu : M \to S^m.$$

Such a map is called a **Gauß map** for M. Note that if $\nu : M \to S^2$ is a Gauß map so is $-\nu$, but this is the only ambiguity when M is connected. Differentiating ν at $p \in M$ we obtain a linear map

$$d\nu(p) : T_pM \to T_{\nu(p)}S^m = T_pM$$

Here we use the fact that $T_{\nu(p)}S^m = \nu(p)^\perp$ and, by definition of the Gauß map ν, the tangent space of M at p is also equal to $\nu(p)^\perp$. Thus $d\nu(p)$ is a linear map from the tangent space of M at p to itself.

Definition 5.3.2. *The* **Gaußian curvature** *of the hypersurface M is the real valued function $K : M \to \mathbb{R}$ defined by*

$$K(p) := \det\bigl(d\nu(p) : T_pM \to T_pM\bigr)$$

for $p \in M$. (Replacing ν by $-\nu$ has the effect of replacing K by $(-1)^m K$; so K is independent of the choice of the Gauß map when m is even.)

Remark 5.3.3. Given a subset $B \subset M$ the set $\nu(B) \subset S^m$ is often called the **spherical image** of B. If ν is a diffeomorphism on a neighborhood of B the change of variables formula for an integral gives

$$\int_{\nu(B)} \mu_S = \int_B |K| \mu_M$$

where μ_M and μ_S denote the volume elements on M and S^m, respectively. Introducing the notation $\mathrm{Area}_M(B) := \int_B \mu_M$ we obtain the formula

$$|K(p)| = \lim_{B \to p} \frac{\mathrm{Area}_S(\nu(B))}{\mathrm{Area}_M(B)}.$$

This says that the curvature at p is roughly the ratio of the (m-dimensional) area of the spherical image $\nu(B)$ to the area of B where B is a very small open neighborhood of p in M. The sign of $K(p)$ is positive when the linear map $d\nu(p) : T_pM \to T_pM$ preserves orientation and negative when it reverses orientation.

Remark 5.3.4. We see that the Gaußian curvature is a natural generalization of **Euler's curvature** for a plane curve. Indeed if $M \subset \mathbb{R}^2$ is a 1-manifold and $p \in M$ we can choose a curve $\gamma = (x, y) : (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p$ and $|\dot\gamma(s)| = 1$ for every s. This curve parametrizes M by the arclength and the unit normal vector pointing to the right with respect to the orientation of γ is $\nu(x, y) = (\dot y, -\dot x)$. This is a local Gauß map and its derivative $(\ddot y, -\ddot x)$ is tangent to the curve. The inner product of the latter with the unit tangent vector $\dot\gamma = (\dot x, \dot y)$ is the Gaußian curvature. Thus

$$K := \frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2} = \frac{d\theta}{ds}$$

where s is the arclength parameter and θ is the angle made by the normal (or the tangent) with some constant line. With this convention K is positive at a left turn and negative at a right turn.

Exercise 5.3.5. The Gaußian curvature of a sphere of radius r is constant and has the value r^{-m}.

Exercise 5.3.6. Show that the Gaußian curvature of the surface $z = x^2 - y^2$ is -4 at the origin.

We now restrict to the case of **surfaces**, i.e. of 2-dimensional submanifolds of $\mathbb{R}^3$. Figure 5.1 illustrates the difference between positive and negative Gaußian curvature in dimension two.

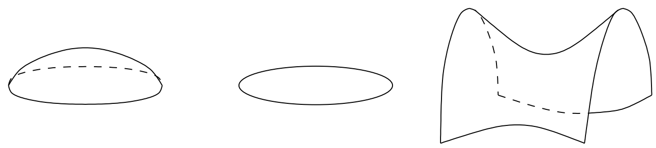


Figure 5.1: Positive and negative Gaußian curvature.

Theorem 5.3.7 (**Gaußian curvature**). *Let $M \subset \mathbb{R}^3$ be a surface and fix a point $p \in M$. If $u, v \in T_pM$ is a basis then*

$$K(p) = \frac{\langle R(u,v)v, u\rangle}{|u|^2|v|^2 - \langle u, v\rangle^2}. \tag{5.3.4}$$

Moreover, for all $u, v, w \in T_pM$, we have

$$R(u,v)w = -K(p)\langle \nu(p), u \times v\rangle \nu(p) \times w. \tag{5.3.5}$$

Proof. The orthogonal projection of $\mathbb{R}^3$ onto the tangent space $T_pM = \nu(p)^\perp$ is given by the 3×3-matrix

$$\Pi(p) = \mathbb{1} - \nu(p)\nu(p)^\mathsf{T}.$$

Hence

$$d\Pi(p)u = -\nu(p)(d\nu(p)u)^\mathsf{T} - \big(d\nu(p)u\big)\nu(p)^\mathsf{T}.$$

Here the first summand is the second fundamental form, which maps T_pM to $T_pM^\perp$, and the second summand is its dual, which maps $T_pM^\perp$ to T_pM. Thus

$$\begin{aligned} h_p(v) &= \nu(p)\big(d\nu(p)v\big)^\mathsf{T} : T_pM \to T_pM^\perp, \\ h_p(u)^* &= \big(d\nu(p)u\big)\nu(p)^\mathsf{T} : T_pM^\perp \to T_pM. \end{aligned}$$

By the Gauß–Codazzi formula this implies

$$\begin{aligned} R_p(u,v)w &= h_p(u)^* h_p(v)w - h_p(v)^* h_p(u)w \\ &= \big(d\nu(p)u\big)\big(d\nu(p)v\big)^\mathsf{T} w - \big(d\nu(p)v\big)\big(d\nu(p)u\big)^\mathsf{T} w \\ &= \langle d\nu(p)v, w\rangle d\nu(p)u - \langle d\nu(p)u, w\rangle d\nu(p)v \end{aligned}$$

and hence

$$\langle R_p(u,v)w, z\rangle = \langle d\nu(p)u, z\rangle\langle d\nu(p)v, w\rangle - \langle d\nu(p)u, w\rangle\langle d\nu(p)v, z\rangle. \tag{5.3.6}$$

Now fix four tangent vectors $u, v, w, z \in T_pM$ and consider the composition

$$\mathbb{R}^3 \xrightarrow{A} \mathbb{R}^3 \xrightarrow{B} \mathbb{R}^3 \xrightarrow{C} \mathbb{R}^3$$

of the linear maps

$$\begin{aligned} A\xi &:= \xi^1\nu(p) + \xi^2 u + \xi^3 v, \\ B\eta &:= \begin{cases} d\nu(p)\eta, & \text{if } \eta \perp \nu(p), \\ \eta, & \text{if } \eta \in \mathbb{R}\nu(p), \end{cases} \\ C\zeta &:= \begin{pmatrix} \langle \zeta, \nu(p) \rangle \\ \langle \zeta, z \rangle \\ \langle \zeta, w \rangle \end{pmatrix}. \end{aligned}$$

This composition is represented by the matrix

$$CBA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \langle d\nu(p)u, z \rangle & \langle d\nu(p)v, z \rangle \\ 0 & \langle d\nu(p)u, w \rangle & \langle d\nu(p)v, w \rangle \end{pmatrix}.$$

Hence, by (5.3.6), we have

$$\begin{aligned} \langle R_p(u,v)w, z \rangle &= \det(CBA) \\ &= \det(A)\det(B)\det(C) \\ &= \langle \nu(p), u \times v \rangle K(p) \langle \nu(p), z \times w \rangle \\ &= -K(p)\langle \nu(p), u \times v \rangle \langle \nu(p) \times w, z \rangle. \end{aligned}$$

This implies (5.3.5) and

$$\begin{aligned} \langle R_p(u,v)v, u \rangle &= K(p)\langle \nu(p), u \times v \rangle^2 \\ &= K(p)\,|u \times v|^2 \\ &= K(p)\left(|u|^2\,|v|^2 - \langle u, v \rangle^2\right). \end{aligned}$$

This proves Theorem 5.3.7. □

Corollary 5.3.8 (**Theorema Egregium of Gauß).** *The Gaußian curvature is intrinsic, i.e. if $\phi : M \to M'$ is an isometry of surfaces in $\mathbb{R}^3$ then*

$$K = K' \circ \phi : M \to \mathbb{R}.$$

Proof. Theorem 5.3.1 and Theorem 5.3.7. □

Exercise 5.3.9. For $m = 1$ the Gaußian curvature is clearly *not* intrinsic as any two curves are locally isometric (parameterized by arclength). Show that the curvature $K(p)$ is intrinsic for even m while its absolute value $|K(p)|$ is intrinsic for odd $m \geq 3$. **Hint:** We still have the equation (5.3.6) which, for $z = u$ and $v = w$, can be written in the form

$$\langle R_p(u,v)v, u\rangle = \det \begin{pmatrix} \langle d\nu(p)u, u\rangle & \langle d\nu(p)u, v\rangle \\ \langle d\nu(p)v, u\rangle & \langle d\nu(p)v, v\rangle \end{pmatrix}.$$

Thus, for an orthonormal basis $v_1, \ldots, v_m$ of T_pM, the 2×2 minors of the matrix

$$\left(\langle d\nu(p)v_i, v_j\rangle\right)_{i,j=1,\ldots,m}$$

are intrinsic. Hence everything reduces to the following assertion.

Lemma. *The determinant of an $m \times m$ matrix is an expression in its 2×2 minors if m is even; the absolute value of the determinant is an expression in the 2×2 minors if m is odd and greater than or equal to 3.*

The lemma is proved by induction on m. For the absolute value, note the formula

$$\det(A)^m = \det(\det(A)I) = \det(AB) = \det(A)\det(B)$$

for an $m \times m$ matrix A where B is the transposed matrix of cofactors.

5.4 Curvature in Local Coordinates*

Riemann

Let $M \subset \mathbb{R}^k$ be an m-dimensional manifold and let

$$\phi = \psi^{-1} : U \to \Omega$$

be a local coordinate chart on an open set $U \subset M$ with values in an open set $\Omega \subset \mathbb{R}^m$. Define the vector fields $E_1, \ldots, E_m$ along ψ by

$$E_i(x) := \frac{\partial \psi}{\partial x^i}(x) \in T_{\psi(x)}M.$$

These vector fields form a basis of $T_{\psi(x)}M$ for every $x \in \Omega$ and the coefficients $g_{ij} : \Omega \to \mathbb{R}$ of the first fundamental form are

$$g_{ij} = \langle E_i, E_j\rangle.$$

Recall from Lemma 3.6.5 that the Christoffel $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ are the coefficients of the Levi-Civita connection, defined by

$$\nabla_i E_j = \sum_{k=1}^m \Gamma_{ij}^k E_k$$

and that they are given by the formula

$$\Gamma_{ij}^k := \sum_{\ell=1}^m g^{k\ell} \frac{1}{2}\big(\partial_i g_{i\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\big).$$

Define the coefficients

$$R_{ijk}^\ell : \Omega \to \mathbb{R}$$

of the curvature tensor by

$$R(E_i, E_j)E_k = \sum_{\ell=1}^m R_{ijk}^\ell E_\ell. \tag{5.4.1}$$

These coefficients are given by

$$R_{ijk}^\ell := \partial_i \Gamma_{jk}^\ell - \partial_j \Gamma_{ik}^\ell + \sum_{\nu=1}^m \left(\Gamma_{i\nu}^\ell \Gamma_{jk}^\nu - \Gamma_{j\nu}^\ell \Gamma_{ik}^\nu \right). \tag{5.4.2}$$

The coefficients of the Riemann curvature tensor have the symmetries

$$R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k} = R_{k\ell ij}, \qquad R_{ijk\ell} := \sum_\nu R_{ijk}^\nu g_{\nu\ell}, \tag{5.4.3}$$

and the first Bianchi identity has the form

$$R_{ijk}^\ell + R_{jki}^\ell + R_{kij}^\ell = 0. \tag{5.4.4}$$

Warning: Care must be taken with the ordering of the indices. Some authors use the notation R_{kij}^ℓ for what we call R_{ijk}^ℓ and $R_{\ell kij}$ for what we call $R_{ijk\ell}$.

Exercise 5.4.1. Prove equations (5.4.2), (5.4.3), and (5.4.4). Use (5.4.2) to give an alternative proof of Theorem 5.3.1.

Gauß

If $M \subset \mathbb{R}^n$ is a 2-manifold (not necessarily embedded in $\mathbb{R}^3$) we can use equation (5.3.4) as the definition of the Gaußian curvature

$$K : M \to \mathbb{R}.$$

Let $\psi : \Omega \to U$ be a local parametrization of an open set $U \subset M$ defined on an open set $\Omega \subset \mathbb{R}^2$. Denote the coordinates in $\mathbb{R}^2$ by (x, y) and define the functions $E, F, G : \Omega \to \mathbb{R}$ by

$$E := |\partial_x \psi|^2, \qquad F := \langle \partial_x \psi, \partial_y \psi \rangle, \qquad G := |\partial_y \psi|^2.$$

We abbreviate

$$D := EG - F^2.$$

Then the composition of the Gaußian curvature $K : M \to \mathbb{R}$ with the parametrization ψ is given by the explicit formula

$$\begin{aligned}
K \circ \psi &= \frac{1}{D^2} \det \begin{pmatrix} E & F & \partial_y F - \frac{1}{2}\partial_x G \\ F & G & \frac{1}{2}\partial_y G \\ \frac{1}{2}\partial_x E & \partial_x F - \frac{1}{2}\partial_y E & -\frac{1}{2}\partial_y^2 E + \partial_x \partial_y F - \frac{1}{2}\partial_x^2 G \end{pmatrix} \\
&\quad - \frac{1}{D^2} \det \begin{pmatrix} E & F & \frac{1}{2}\partial_y E \\ F & G & \frac{1}{2}\partial_x G \\ \frac{1}{2}\partial_y E & \frac{1}{2}\partial_x G & 0 \end{pmatrix} \\
&= -\frac{1}{2\sqrt{D}} \frac{\partial}{\partial x} \left(\frac{E\partial_x G - F\partial_y E}{E\sqrt{D}} \right) \\
&\quad + \frac{1}{2\sqrt{D}} \frac{\partial}{\partial y} \left(\frac{2E\partial_x F - F\partial_x E - E\partial_y E}{E\sqrt{D}} \right).
\end{aligned}$$

This expression simplifies dramatically when $F = 0$ and we get

$$K \circ \psi = -\frac{1}{2\sqrt{EG}} \left(\frac{\partial}{\partial x} \frac{\partial_x G}{\sqrt{EG}} + \frac{\partial}{\partial y} \frac{\partial_y E}{\sqrt{EG}} \right) \tag{5.4.5}$$

Exercise 5.4.2. Prove that the Riemannian metric

$$E = G = \frac{4}{(1 + x^2 + y^2)^2}, \qquad F = 0,$$

on $\mathbb{R}^2$ has constant constant curvature $K = 1$ and the Riemannian metric

$$E = G = \frac{4}{(1 - x^2 - y^2)^2}, \qquad F = 0,$$

on the open unit disc has constant curvature $K = -1$.

Chapter 6

Geometry and Topology

In this chapter we address what might be called the "fundamental problem of intrinsic differential geometry": when are two manifolds isometric? The central tool for addressing this question is the Cartan–Ambrose–Hicks Theorem in Section 6.1. In Sections 6.2, 6.3, and 6.4 we will use this result to examine flat spaces, symmetric spaces, and constant sectional curvature manifolds, respectively. Section 6.5 discusses manifolds of nonpositive sectional curvature and includes a proof of the Cartan Fixed Point Theorem.

6.1 The Cartan–Ambrose–Hicks Theorem

The Cartan–Ambrose–Hicks Theorem answers the question (at least locally) when two manifolds are isometric. In general the equivalent conditions given there are probably more difficult to verify in most examples than the condition that there exist an isometry. However, under additional assumptions it has many important consequences. The section starts with some basic observations about homotopy and simple connectivity.

6.1.1 Homotopy

Definition 6.1.1. *Let M be a manifold and let $I = [a, b]$ be a compact interval. A* **(smooth) homotopy** *of maps from I to M is a smooth map $\gamma : [0,1] \times I \to M$. We often write $\gamma_\lambda(t) = \gamma(\lambda, t)$ for $\lambda \in [0,1]$ and $t \in I$ and call γ a* **(smooth) homotopy between γ_0 and γ_1**. *We say the homotopy has* **fixed endpoints** *if $\gamma_\lambda(a) = \gamma_0(a)$ and $\gamma_\lambda(b) = \gamma_0(b)$ for all $\lambda \in [0,1]$. (See Figure 6.1.)*

We remark that a homotopy and a variation are essentially the same thing, namely a curve of maps (curves). The difference is pedagogical. We used the word "variation" to describe a curve of maps through a given map; when we use this word we are going to differentiate the curve to find a tangent vector (field) to the given map. The word "homotopy" is used to describe a curve joining two maps; it is a global rather than a local (infinitesimal) concept.

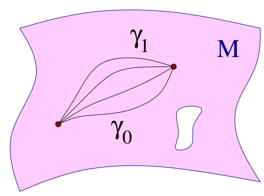


Figure 6.1: A homotopy with fixed endpoints.

Definition 6.1.2. *The manifold M is called* **simply connected** *if for any two curves $\gamma_0, \gamma_1 : [a,b] \to M$ with $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$ there exists a homotopy from γ_0 to γ_1 with endpoints fixed. (The idea is that the space $\Omega_{p,q}$ of curves from p to q is connected.)*

Remark 6.1.3. Two smooth maps $\gamma_0, \gamma_1 : [a,b] \to M$ with the same endpoints can be connected by a continuous homotopy if and only if they can be connected by a smooth homotopy. This follows from the Weierstrass approximation theorem.

Remark 6.1.4. The topological space $\Omega_{p,q}$ of all smooth maps $\gamma : [0,1] \to M$ with the endpoints p and q is connected for some pair of points $p, q \in M$ if and only if it is connected for every pair of points $p, q \in M$. (Prove this!)

Example 6.1.5. The Euclidean space $\mathbb{R}^m$ is simply connected, for any two curves $\gamma_0, \gamma_1 : [a,b] \to \mathbb{R}^m$ with the same endpoints can be joined by the homotopy

$$\gamma_\lambda(t) := \gamma_0(t) + \lambda(\gamma_1(t) - \gamma_0(t)).$$

The punctured plane $\mathbb{C} \setminus \{0\}$ is not simply connected, for the curves

$$\gamma_n(t) := e^{2\pi \mathbf{i} n t}, \qquad 0 \le t \le 1,$$

are not homotopic with fixed endpoints for distinct n.

Exercise 6.1.6. Prove that the m-sphere S^m is simply connected for $m \neq 1$.

6.1.2 The Global C-A-H Theorem

Theorem 6.1.7 (Global C-A-H theorem). *Let $M \subset \mathbb{R}^n$ and $M' \subset \mathbb{R}^{n'}$ be connected, simply connected, complete m-manifolds. Fix two elements $p_0 \in M$ and $p_0' \in M'$ and let $\Phi_0 : T_{p_0}M \to T_{p_0'}M'$ be an orthogonal linear isomorphism. Then the following are equivalent.*

(i) *There is an isometry $\phi : M \to M'$ satisfying*

$$\phi(p_0) = p_0', \qquad d\phi(p_0) = \Phi_0. \tag{6.1.1}$$

(ii) *If (Φ, γ, γ') is a development satisfying the initial condition*

$$\gamma(0) = p_0, \qquad \gamma'(0) = p_0', \qquad \Phi(0) = \Phi_0 \tag{6.1.2}$$

then

$$\gamma(1) = p_0 \qquad \implies \qquad \gamma'(1) = p_0', \quad \Phi(1) = \Phi_0$$

(iii) *If $(\Phi_0, \gamma_0, \gamma_0')$ and $(\Phi_1, \gamma_1, \gamma_1')$ are developments satisfying (6.1.2) then*

$$\gamma_0(1) = \gamma_1(1) \qquad \implies \qquad \gamma_0'(1) = \gamma_1'(1).$$

(iv) *If (Φ, γ, γ') is a development satisfying (6.1.2) then $\Phi_* R_\gamma = R'_{\gamma'}$.*

Proof. See page 248. □

Lemma 6.1.8. *If $\phi : M \to M'$ is a local isometry satisfying (6.1.1) and (Φ, γ, γ') is a development satisfying the initial condition (6.1.2) then*

$$\gamma'(t) = \phi(\gamma(t)), \qquad \Phi(t) = d\phi(\gamma(t)) \qquad \textit{for all } t.$$

Proof. See page 248. □

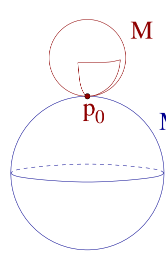


Figure 6.2: Diagram for Example 6.1.9.

Example 6.1.9. Before giving the proof let us interpret the conditions in case M and M' are two-dimensional spheres of radius r and r' respectively in three-dimensional Euclidean space $\mathbb{R}^3$. Imagine that the spheres are tangent at $p_0 = p_0'$. Clearly the spheres will be isometric exactly when $r = r'$. Condition (ii) says that if the spheres are rolled along one another without sliding or twisting then the endpoint $\gamma'(1)$ of one curve of contact depends only on the endpoint $\gamma(1)$ of the other and not on the intervening curve $\gamma(t)$. By Theorem 5.3.7 the Riemann curvature of a 2-manifold at p is determined by the Gaußian curvature $K(p)$; and for spheres we have $K(p) = 1/r^2$.

Exercise 6.1.10. Let γ be the closed curve which bounds an octant as shown in the diagram for Example 6.1.9. Find γ'.

Exercise 6.1.11. Show that in case M is two-dimensional, the condition $\Phi(1) = \Phi_0$ in Theorem 6.1.7 may be dropped from (ii).

Proof of Lemma 6.1.8. Let $I \subset \mathbb{R}$ be an interval containing zero and let $\gamma : I \to M$ be a smooth curve such that $\gamma(0) = p_0$. Define

$$\gamma'(t) := \phi(\gamma(t)), \qquad \Phi(t) := d\phi(\gamma(t))$$

for $t \in I$. Then $\dot{\gamma}' = \Phi\dot{\gamma}$ by the chain rule and, for every vector field X along γ, we have

$$\Phi\nabla X = \nabla'(\Phi X)$$

by Theorem 5.3.1. Hence it follows from Lemma 3.5.12 and Lemma 3.5.19 that (Φ, γ, γ') is a development. Since (Φ, γ, γ') satisfies the initial condition (6.1.2) the assertion follows from the uniqueness result for developments in Theorem 3.5.21. This proves Lemma 6.1.8. $\square$

Proof of Theorem 6.1.7. We first prove a slightly different theorem. Namely, we weaken condition (i) to assert that ϕ is a local isometry (i.e. not necessarily bijective), and prove that this weaker condition is equivalent to (ii), (iii), and (iv) whenever M is connected and simply connected and M' is complete. Thus we drop the hypotheses that M be complete and M' be connected and simply connected.

We prove that (i) implies (ii). Given a development as in (ii) we have, by Lemma 6.1.8,

$$\gamma'(1) = \phi(\gamma(1)) = \phi(p_0) = p_0', \qquad \Phi(1) = d\phi(\gamma(1)) = d\phi(p_0) = \Phi_0,$$

as required.

We prove that (ii) implies (iii) when M' is complete. Choose developments $(\Phi_i, \gamma_i, \gamma_i')$ for $i = 0, 1$ as in (iii). Define a curve $\gamma : [0, 1] \to M$ by "composition"

$$\gamma(t) := \begin{cases} \gamma_0(2t), & 0 \le t \le 1/2, \\ \gamma_1(2-2t), & 1/2 \le t \le 1, \end{cases}$$

so that γ is continuous and piecewise smooth and $\gamma(1) = p_0$. By Theorem 3.5.21 there is a development (Φ, γ, γ') on the interval $[0, 1]$ satisfying (6.1.2) (because M' is complete). Since $\gamma(1) = p_0$ it follows from (ii) that $\gamma'(1) = p_0'$ and $\Phi(1) = \Phi_0$. By the uniqueness of developments and the invariance under reparametrization, we have

$$(\Phi(t), \gamma(t), \gamma'(t)) = \begin{cases} (\Phi_0(2t), \gamma_0(2t), \gamma_0'(2t)), & 0 \le t \le 1/2, \\ (\Phi_1(2-2t), \gamma_1(2-2t), \gamma_1'(2-2t)), & 1/2 \le t \le 1. \end{cases}$$

Hence $\gamma_0'(1) = \gamma'(1/2) = \gamma_1'(1)$ as required.

We prove that (iii) implies (i) when M' is complete and M is connected. Define the map $\phi : M \to M'$ as follows. Fix an element $p \in M$. Since M is connected, there exists a smooth curve $\gamma : [0, 1] \to M$ such that $\gamma(0) = p_0$ and $\gamma(1) = p$. Since M' is complete, there exists a development (Φ, γ, γ') with $\gamma'(0) = p_0'$ and $\Phi(0) = \Phi_0$ (Theorem 3.5.21). Now define $\phi(p) := \gamma'(1)$. By (iii) the endpoint $p' := \gamma'(1)$ is independent of the choice of the curve γ, and so ϕ is well-defined. We prove that this map ϕ satisfies the following

(a) *If (Φ, γ, γ') is a development satisfying $\gamma(0) = p_0$, $\gamma'(0) = p_0'$, $\Phi(0) = \Phi_0$, then $\phi(\gamma(t)) = \gamma'(t)$ for $0 \le t \le 1$.*

(b) *If $p, q \in M$ satisfy $0 < d(p, q) < \mathrm{inj}(p, M)$ and $d(p, q) < \mathrm{inj}(\phi(p), M')$, then $d'(\phi(p), \phi(q)) = d(p, q)$.*

That ϕ satisfies (a) follows directly from the definition and the fact that the triple $(\Phi_t, \gamma_t, \gamma_t')$ defined by $\Phi_t(s) := \Phi(st)$, $\gamma_t(s) := \gamma(st)$, $\gamma_t'(s) := \gamma'(st)$ for $0 \le s \le 1$ is a development. To prove (b), choose a vector $v \in T_pM$ such that $|v| = d(p, q)$ and $\exp_p(v) = q$ (Theorem 4.5.4) and let $\gamma : [0, 1] \to M$ be a smooth curve with $\gamma(0) = p_0$ and $\gamma(t) = \exp_p((2t - 1)v)$ for $\frac{1}{2} \le t \le 1$. Let (Φ, γ, γ') be the unique development of M along M' satisfying $\gamma'(0) = p_0'$ and $\Phi(0) = \Phi_0$ (Theorem 3.5.21). Then $\gamma'(\frac{1}{2}) = \phi(p)$ and $\gamma'(1) = \phi(q)$ by (a) and the restriction of γ' to the interval $[\frac{1}{2}, 1]$ is a geodesic by part (ii) of Lemma 3.5.19 with $X = \dot\gamma$. Hence $\gamma'(t) = \exp_{\phi(p)}((2t - 1)v')$ for $\frac{1}{2} \le t \le 1$, where the tangent vector $v' \in T_{\phi(p)}M'$ is given by $v' := \dot\gamma'(\frac{1}{2}) = \Phi(\frac{1}{2})v$ and hence satisfies $|v'| = |v| = d(p, q) < \mathrm{inj}(\phi(p), M')$. Thus it follows from Theorem 4.5.4 that $d'(\phi(p), \phi(q)) = d'(\phi(p), \exp'_{\phi(p)}(v')) = |v'| = d(p, q)$ and this proves (b). It follows from (b) and Theorem 5.1.1 that ϕ is a local isometry.

We prove that (i) implies (iv). Given a development as in (ii) we have

$$\gamma'(t) = \phi(\gamma(t)), \qquad \Phi(t) = d\phi(\gamma(t))$$

for every t, by Lemma 6.1.8. Hence it follows from Theorem 5.3.1 that

$$\Phi(t)_* R_{\gamma(t)} = (\phi_* R)_{\gamma'(t)} = R'_{\gamma'(t)}$$

for all t as required.

We prove that (iv) implies (iii) when M' is complete and M is simply connected. Choose developments $(\Phi_i, \gamma_i, \gamma_i')$ for $i = 0, 1$ as in (iii). Since M is simply connected there is a homotopy

$$[0,1] \times [0,1] \to M : (\lambda, t) \mapsto \gamma(\lambda, t) = \gamma_\lambda(t)$$

from γ_0 to γ_1 with endpoints fixed. By Theorem 3.5.21 there is, for each λ, a development $(\Phi_\lambda, \gamma_\lambda, \gamma_\lambda')$ on the interval $[0,1]$ with initial conditions

$$\gamma_\lambda'(0) = p_0', \qquad \Phi_\lambda(0) = \Phi_0$$

(because M' is complete). The proof of Theorem 3.5.21 also shows that $\gamma_\lambda(t)$ and $\Phi_\lambda(t)$ depend smoothly on both t and λ. We must prove that

$$\gamma_1'(1) = \gamma_0'(1).$$

To see this we will show that, for each fixed t, the curve

$$\lambda \mapsto (\Phi_\lambda(t), \gamma_\lambda(t), \gamma_\lambda'(t))$$

is a development; then by the definition of development we have that the curve $\lambda \mapsto \gamma_\lambda'(1)$ is smooth and

$$\partial_\lambda \gamma_\lambda'(1) = \Phi_\lambda(1) \partial_\lambda \gamma_\lambda(1) = 0$$

as required.

First choose a basis $e_1, \ldots, e_m$ of $T_{p_0}M$ and extend it to obtain vector fields $E_i \in \mathrm{Vect}(\gamma)$ along the homotopy γ by imposing the conditions that the vector fields $t \mapsto E_i(\lambda, t)$ be parallel, i.e.

$$\nabla_t E_i(\lambda, t) = 0, \qquad E_i(\lambda, 0) = e_i. \tag{6.1.3}$$

Then the vectors $E_1(\lambda, t), \ldots E_m(\lambda, t)$ form a basis of $T_{\gamma_\lambda(t)}M$ for all λ and t. Second, define the vector fields E_i' along γ' by

$$E_i'(\lambda, t) := \Phi_\lambda(t) E_i(\lambda, t) \tag{6.1.4}$$

so that $\nabla_t' E_i' = 0$. Third, define the functions $\xi^1, \ldots, \xi^m : [0,1]^2 \to \mathbb{R}$ by

$$\partial_t \gamma =: \sum_{i=1}^m \xi^i E_i, \qquad \partial_t \gamma' = \sum_{i=1}^m \xi^i E_i'. \tag{6.1.5}$$

Here the second equation follows from (6.1.4) and the fact that $\Phi_\lambda \partial_t \gamma = \partial_t \gamma'$.

Now consider the vector fields

$$X' := \partial_\lambda \gamma', \qquad Y_i' := \nabla'_\lambda E_i' \tag{6.1.6}$$

along γ'. They satisfy the equations

$$\nabla'_t X' = \nabla'_t \partial_\lambda \gamma' = \nabla'_\lambda \partial_t \gamma' = \nabla'_\lambda \big(\sum_{i=1}^m \xi^i E_i'\big) = \sum_{i=1}^m \big(\partial_\lambda \xi^i E_i' + \xi^i Y_i'\big)$$

and

$$\nabla'_t Y_i' = \nabla'_t \nabla'_\lambda E_i' - \nabla'_\lambda \nabla'_t E_i' = R'(\partial_t \gamma', \partial_\lambda \gamma') E_i'.$$

To sum up we have $X'(\lambda, 0) = Y_i'(\lambda, 0) = 0$ and

$$\nabla'_t X' = \sum_{i=1}^m \big(\partial_\lambda \xi^i E_i' + \xi^i Y_i'\big), \qquad \nabla'_t Y_i' = R'(\partial_t \gamma', \partial_\lambda \gamma') E_i'. \tag{6.1.7}$$

On the other hand, the vector fields

$$X' := \Phi_\lambda \partial_\lambda \gamma, \qquad Y_i' := \Phi_\lambda \nabla_\lambda E_i \tag{6.1.8}$$

along γ' satisfy the same equations, namely

$$\begin{aligned}
\nabla'_t X' &= \Phi_\lambda \nabla_t \partial_\lambda \gamma = \Phi_\lambda \nabla_\lambda \partial_t \gamma = \Phi_\lambda \nabla_\lambda \left(\sum_{i=1}^m \xi^i E_i\right) \\
&= \Phi_\lambda \sum_{i=1}^m \left(\partial_\lambda \xi^i E_i + \xi^i \nabla_\lambda E_i\right) = \sum_{i=1}^m \left(\partial_\lambda \xi^i E_i' + \xi^i Y_i'\right), \\
\nabla'_t Y_i' &= \Phi_\lambda \big(\nabla_t \nabla_\lambda E_i - \nabla_\lambda \nabla_t E_i\big) = \Phi_\lambda R(\partial_t \gamma, \partial_\lambda \gamma) E_i \\
&= R'(\Phi_\lambda \partial_t \gamma, \Phi_\lambda \partial_\lambda \gamma) \Phi_\lambda E_i = R'(\partial_t \gamma', X') E_i'.
\end{aligned}$$

Here the last but one equation follows from (iv).

Since the tuples (6.1.6) and (6.1.8) satisfy the same differential equation (6.1.7) and vanish at $t = 0$ they must agree. Hence

$$\partial_\lambda \gamma' = \Phi_\lambda \partial_\lambda \gamma, \qquad \nabla'_\lambda E_i' = \Phi_\lambda \nabla_\lambda E_i$$

for $i = 1, \ldots, m$. This says that $\lambda \mapsto (\Phi_\lambda(t), \gamma_\lambda(t), \gamma'_\lambda(t))$ is a development. For $t = 1$ we obtain $\partial_\lambda \gamma'(\lambda, 1) = 0$ as required.

Now the modified theorem (where ϕ is a local isometry) is proved. The original theorem follows immediately. Condition (iv) is symmetric in M and M'. Thus, if we assume (iv), there are local isometries $\phi : M \to M'$ and $\psi : M' \to M$ satisfying $\phi(p_0) = p_0'$, $d\phi(p_0) = \Phi_0$ and $\psi(p_0') = p_0$, $d\psi(p') = \Phi_0^{-1}$. But then $\psi \circ \phi$ is a local isometry with $\psi \circ \phi(p_0) = p_0$ and $d(\psi \circ \phi)(p_0) = \mathrm{id}$. Hence $\psi \circ \phi$ is the identity. Similarly $\phi \circ \psi$ is the identity so ϕ is bijective (and $\psi = \phi^{-1}$) as required. This proves Theorem 6.1.7. □

Remark 6.1.12. The proof of Theorem 6.1.7 shows that the various implications in the weak version of the theorem (where ϕ is only a local isometry) require the following conditions on M and M':

(i) always implies (ii), (iii), and (iv);

(ii) implies (iii) whenever M' is complete;

(iii) implies (i) whenever M' is complete and M is connected;

(iv) implies (iii) whenever M' is complete and M is simply connected.

Remark 6.1.13. The proof that (iii) implies (i) in Theorem 6.1.7 can be slightly shortened by using the following observation. *Let $\phi : M \to M'$ be a map between smooth manifolds. Assume that $\phi \circ \gamma$ is smooth for every smooth curve $\gamma : [0,1] \to M$. Then ϕ is smooth.*

6.1.3 The Local C-A-H Theorem

Theorem 6.1.14 (**Local C-A-H Theorem**)**.** *Let M and M' be smooth m-manifolds, let $p_0 \in M$ and $p_0' \in M'$, and let $\Phi_0 : T_{p_0}M \to T_{p_0'}M'$ be an orthogonal linear isomorphism. Let $r > 0$ be smaller than the injectvity radii of M at p_0 and of M' at p_0' and define $U_r := \{p \in M \,|\, d(p_0, p) < r\}$ and $U_r' := \{p' \in M' \,|\, d'(p_0', p') < r\}$. Then the following are equivalent.*

(i) *There is an isometry $\phi : U_r \to U_r'$ satisfying* (6.1.1).

(ii) *If (Φ, γ, γ') is a development on an interval $I \subset \mathbb{R}$ with $0 \in I$, satisfying the initial condition* (6.1.2) *as well as $\gamma(I) \subset U_r$ and $\gamma'(I) \subset U_r'$, then*

$$\gamma(1) = p_0 \qquad \Longrightarrow \qquad \gamma'(1) = p_0', \quad \Phi(1) = \Phi_0.$$

(iii) *If $(\Phi_0, \gamma_0, \gamma_0')$ and $(\Phi_1, \gamma_1, \gamma_1')$ are developments as in (ii) then*

$$\gamma_0(1) = \gamma_1(1) \qquad \Longrightarrow \qquad \gamma_0'(1) = \gamma_1'(1).$$

(iv) *If $v \in T_{p_0}M$ with $|v| < r$ and*

$$\gamma(t) := \exp_{p_0}(tv), \quad \gamma'(t) := \exp'_{p_0'}(t\Phi_0 v), \quad \Phi(t) := \Phi'_{\gamma'}(t,0)\Phi_0\Phi_\gamma(0,t),$$

then $\Phi(t)_* R_{\gamma(t)} = R'_{\gamma'(t)}$ for $0 \le t \le 1$.

If these equivalent conditions are satisfied then $\phi(\exp_{p_0}(v)) = \exp'_{p_0'}(\Phi_0 v)$ for all $v \in T_{p_0}M$ with $|v| < r$.

Proof. See page 253. □

Lemma 6.1.15. *Let $p \in M$ and $v, w \in T_pM$ such that $|v| < \mathrm{inj}(p)$. For $0 \le t \le 1$ define*

$$\gamma(t) := \exp(tv), \qquad X(t) := \frac{\partial}{\partial\lambda}\bigg|_{\lambda=0} \exp_p\big(t(v+\lambda w)\big) \in T_{\gamma(t)}M.$$

Then

$$\nabla_t\nabla_t X = R(\dot\gamma, X)\dot\gamma, \qquad X(0) = 0, \qquad \nabla_t X(0) = w. \tag{6.1.9}$$

A vector field along γ satisfying the first equation in (6.1.9) *is called a* **Jacobi field along** γ.

Proof. Define

$$\gamma(\lambda, t) := \exp_p(t(v+\lambda w)), \qquad X(\lambda, t) := \partial_\lambda\gamma(\lambda, t)$$

for all λ and t. Since $\gamma(\lambda, 0) = p$ for all λ we have $X(\lambda, 0) = 0$ and

$$\nabla_t X(\lambda, 0) = \nabla_t\partial_\lambda\gamma(\lambda, 0) = \nabla_\lambda\partial_t\gamma(\lambda, 0) = \frac{d}{d\lambda}\big(v + \lambda w\big) = w.$$

Moreover, $\nabla_t\partial_t\gamma = 0$ and hence

$$\begin{aligned}
\nabla_t\nabla_t X &= \nabla_t\nabla_t\partial_\lambda\gamma \\
&= \nabla_t\nabla_\lambda\partial_t\gamma - \nabla_\lambda\nabla_t\partial_t\gamma \\
&= R(\partial_t\gamma, \partial_\lambda\gamma)\partial_t\gamma \\
&= R(\partial_t\gamma, X)\partial_t\gamma.
\end{aligned}$$

This proves Lemma 6.1.15. □

Proof of Theorem 6.1.14. The proofs (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) $\Longrightarrow$ (iv) are as before; the reader might note that when $L(\gamma) \le r$ we also have $L(\gamma') \le r$ for any development so that there are plenty of developments with $\gamma : [0,1] \to U_r$ and $\gamma' : [0,1] \to U'_r$. The proof that (iv) implies (i) is a little different since (iv) here is somewhat weaker than (iv) of the global theorem: the equation $\Phi_* R = R'$ is only assumed for certain developments.

Hence assume (iv) and define $\phi : U_r \to U'_r$ by

$$\phi := \exp'_{p'_0} \circ \Phi_0 \circ \exp_{p_0}^{-1} : U_r \to U'_r.$$

We must prove that ϕ is an isometry. Thus we fix a point $q \in U_r$ and a tangent vector $u \in T_qM$ and choose $v, w \in T_pM$ with $|v| < r$ such that

$$\exp_{p_0}(v) = q, \qquad d\exp_{p_0}(v)w = u. \tag{6.1.10}$$

Define $\gamma : [0,1] \to U_r$, $\gamma' : [0,1] \to U'_r$, $X \in \mathrm{Vect}(\gamma)$, and $X' \in \mathrm{Vect}(\gamma')$ by

$$\begin{aligned}\gamma(t) &= \exp_{p_0}(tv), \qquad X(t) := \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0} \exp_{p_0}(t(v+\lambda w))\\ \gamma'(t) &= \exp'_{p'_0}(t\Phi_0 v), \qquad X'(t) := \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0} \exp'_{p'_0}(t(\Phi_0 v + \lambda\Phi_0 w)).\end{aligned}$$

Then, by definition of ϕ, we have

$$\gamma' := \phi\circ\gamma, \qquad d\phi(\gamma)X = X'. \tag{6.1.11}$$

By Lemma 6.1.15, X is a solution of (6.1.9) and X' is a solution of

$$\nabla_t\nabla_t X' = R'(\partial_t\gamma', X')\partial_t\gamma', \quad X(\lambda,0)=0, \quad \nabla_t X(\lambda,0) = \Phi_0 w. \tag{6.1.12}$$

Now define $\Phi(t) : T_{\gamma(t)}M \to T_{\gamma'(t)}M'$ by

$$\Phi(t) := \Phi'_{\gamma'}(t,0)\Phi_0\Phi_\gamma(0,t).$$

Then Φ intertwines covariant differentiation. Since $\dot\gamma$ and $\dot\gamma'$ are parallel vector fields with $\dot\gamma'(0) = \Phi_0 v = \Phi(0)\dot\gamma(0)$, we have

$$\Phi(t)\dot\gamma(t) = \dot\gamma'(t)$$

for every t. Moreover, it follows from (iv) that $\Phi_* R_\gamma = R'_{\gamma'}$. Combining this with (6.1.9) we obtain

$$\nabla'_t\nabla'_t(\Phi X) = \Phi\nabla_t\nabla_t X = R'(\Phi\dot\gamma, \Phi X)\Phi\dot\gamma = R'(\dot\gamma', \Phi X)\dot\gamma'.$$

Hence the vector field ΦX along γ' also satisfies the initial value problem (6.1.12) and thus

$$\Phi X = X' = d\phi(\gamma)X.$$

Here we have also used (6.1.11). Using (6.1.10) we find

$$\gamma(1) = \exp_{p_0}(v) = q, \qquad X(1) = d\exp_{p_0}(v)w = u,$$

and so

$$d\phi(q)u = d\phi(\gamma(1))X(1) = X'(1) = \Phi(1)u.$$

Since $\Phi(1) : T_{\gamma(1)}M \to T_{\gamma'(1)}M'$ is an orthogonal transformation this gives

$$|d\phi(q)u| = |\Phi(1)u| = |u|.$$

Hence ϕ is an isometry as claimed. This proves Theorem 6.1.14. $\square$

6.2 Flat Spaces

Our aim in the next few sections is to give applications of the Cartan-Ambrose-Hicks Theorem. It is clear that the hypothesis $\Phi_* R = R'$ for *all* developments will be difficult to verify without drastic hypotheses on the curvature. The most drastic such hypothesis is that the curvature vanishes identically.

Definition 6.2.1. *A Riemannian manifold M is called* **flat** *if the Riemann curvature tensor R vanishes identically.*

Theorem 6.2.2. *Let $M \subset \mathbb{R}^n$ be a smooth m-manifold.*

(i) *M is flat if and only if every point has a neighborhood which is isometric to an open subset of $\mathbb{R}^m$, i.e. at each point $p \in M$ there exist local coordinates $x^1, \dots, x^m$ such that the coordinate vectorfields $E_i = \partial/\partial x^i$ are orthonormal.*

(ii) *Assume M is connected, simply connected, and complete. Then M is flat if and only if there is an isometry $\phi : M \to \mathbb{R}^m$ onto Euclidean space.*

Proof. Assertion (i) follows immediately from Theorem 6.1.14 and (ii) follows immediately from Theorem 6.1.7. □

Exercise 6.2.3. Carry over the Cartan–Ambrose–Hicks theorem and Theorem 6.2.2 to the intrinsic setting.

Exercise 6.2.4. A one-dimensional manifold is always flat.

Exercise 6.2.5. If M_1 and M_2 are flat so is $M = M_1 \times M_2$.

Example 6.2.6. By Exercises 6.2.4 and 6.2.5 the standard torus

$$\mathbb{T}^m = \left\{ z = (z_1, \dots, z_m) \in \mathbb{C}^m \,\middle|\, |z_1| = \cdots = |z_m| = 1 \right\}$$

is flat.

Exercise 6.2.7. For $b \geq a > 0$ and $c \geq 0$ define $M \subset \mathbb{C}^3$ by

$$M := \left\{ (u, v, w) \in \mathbb{C}^3 \mid |u| = a,\, |v| = b,\, w = cuv \right\}.$$

Then M is diffeomorphic to a torus (a product of two circles) and M is flat. If M' is similarly defined from numbers $b' \geq a' > 0$ and $c' \geq 0$ then there is an isometry $\phi : M \to M'$ if and only if $(a, b, c) = (a', b', c')$, i.e. $M = M'$. (**Hint:** Each circle $u = u_0$ is a geodesic as well as each circle $v = v_0$; the numbers a, b, c can be computed from the length of the circle $u = u_0$, the length of the circle $v = v_0$, and the angle between them.)

Exercise 6.2.8 (Developable manifolds). Let $n = m+1$ and let $E(t)$ be a one-parameter family of hyperplanes in $\mathbb{R}^n$. Then there exists a smooth map $u : \mathbb{R} \to \mathbb{R}^n$ such that

$$E(t) = u(t)^\perp, \qquad |u(t)| = 1, \tag{6.2.1}$$

for every t. We assume that $\dot{u}(t) \neq 0$ for every t so that $u(t)$ and $\dot{u}(t)$ are linearly independent. Show that

$$L(t) := u(t)^\perp \cap \dot{u}(t)^\perp = \lim_{s \to t} E(t) \cap E(s). \tag{6.2.2}$$

Thus $L(t)$ is a linear subspace of dimension $m-1$. Now let $\gamma : \mathbb{R} \to \mathbb{R}^n$ be a smooth map such that

$$\langle \dot{\gamma}(t), u(t) \rangle = 0, \qquad \langle \dot{\gamma}(t), \dot{u}(t) \rangle \neq 0 \tag{6.2.3}$$

for all t. This means that $\dot{\gamma}(t) \in E(t)$ and $\dot{\gamma}(t) \notin L(t)$; thus $E(t)$ is spanned by $L(t)$ and $\dot{\gamma}(t)$. For $t \in \mathbb{R}$ and $\varepsilon > 0$ define

$$L(t)_\varepsilon := \{ v \in L(t) \,|\, |v| < \varepsilon \}.$$

Let $I \subset \mathbb{R}$ be a bounded open interval such that the restriction of γ to the closure of I is injective. Prove that, for $\varepsilon > 0$ sufficiently small, the set

$$M_0 := \bigcup_{t \in I} \Big(\gamma(t) + L(t)_\varepsilon \Big)$$

is a smooth manifold of dimension $m = n-1$. A manifold which arises this way is called **developable**. Show that the tangent spaces of M_0 are the original subspaces $E(t)$, i.e.

$$T_p M_0 = E(t) \qquad \text{for} \qquad p \in \gamma(t) + L(t)_\varepsilon.$$

(One therefore calls M_0 the *"envelope"* of the hyperplanes $\gamma(t) + E(t)$.) Show that M_0 is flat (hint: use Gauß–Codazzi). If (Φ, γ, γ') is a development of M_0 along $\mathbb{R}^m$, show that the map $\phi : M_0 \to \mathbb{R}^m$, defined by

$$\phi(\gamma(t) + v) := \gamma'(t) + \Phi(t) v$$

for $v \in L(t)_\varepsilon$, is an isometry onto an open set $M_0' \subset \mathbb{R}^m$. Thus a development *"unrolls"* M_0 onto the Euclidean space $\mathbb{R}^m$. When $n = 3$ and $m = 2$ one can visualize M_0 as a twisted sheet of paper (see Figure 6.3).

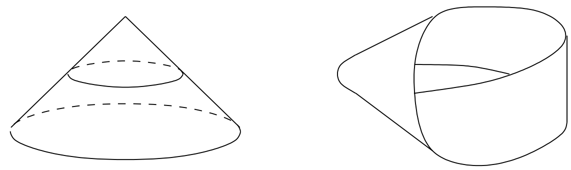


Figure 6.3: Developable surfaces.

Remark 6.2.9. Given a codimension-1 submanifold

$$M \subset \mathbb{R}^{m+1}$$

and a curve $\gamma : \mathbb{R} \to M$ we may form the **osculating developable** M_0 to M along γ by taking

$$E(t) := T_{\gamma(t)}M.$$

This developable has common affine tangent spaces with M along γ as

$$T_{\gamma(t)}M_0 = E(t) = T_{\gamma(t)}M$$

for every t. This gives a nice interpretation of parallel transport: M_0 may be unrolled onto a hyperplane where parallel transport has an obvious meaning and the identification of the tangent spaces thereby defines parallel transport in M. (See Remark 3.5.16.)

Exercise 6.2.10. Each of the following is a developable surface in $\mathbb{R}^3$.

(i) A cone on a plane curve $\Gamma \subset H$, i.e.

$$M = \{tp + (1-t)q \,|\, t > 0, q \in \Gamma\}$$

where $H \subset \mathbb{R}^3$ is an affine hyperplane, $p \in \mathbb{R}^3 \backslash H$, and $\Gamma \subset H$ is a 1-manifold.

(ii) A cylinder on a plane curve Γ, i.e.

$$M = \{q + tv \,|\, q \in \Gamma,\, t \in \mathbb{R}\}$$

where H and Γ are as in (i) and v is a fixed vector not parallel to H. (This is a cone with the cone point p at infinity.)

(iii) The tangent developable to a space curve $\gamma : \mathbb{R} \to \mathbb{R}^3$, i.e.

$$M = \{\gamma(t) + s\dot{\gamma}(t) \,|\, |t - t_0| < \varepsilon,\, 0 < s < \varepsilon\},$$

where $\dot{\gamma}(t_0)$ and $\ddot{\gamma}(t_0)$ are linearly independent and $\varepsilon > 0$ is sufficiently small.

(iv) The paper model of a Möbius strip (see Figure 6.3).

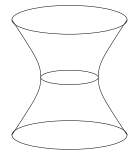


Figure 6.4: A circular one-sheeted hyperboloid.

Remark 6.2.11. A 2-dimensional submanifold $M \subset \mathbb{R}^3$ is called a **ruled surface** if there is a straight line in M through every point. Every developable surface is ruled, however, there are ruled surfaces that are not developable. An example is the manifold $M = \{\gamma(t) + s\ddot{\gamma}(t) \,|\, |t - t_0| < \varepsilon,\, |s| < \varepsilon\}$ where $\gamma : \mathbb{R} \to \mathbb{R}^3$ is a smooth curve with $|\dot{\gamma}| \equiv 1$ and $\ddot{\gamma}(t_0) \neq 0$, and $\varepsilon > 0$ is sufficiently small; this surface is not developable in general. Other examples are the **elliptic hyperboloid**

$$M := \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \right\} \tag{6.2.4}$$

of one sheet depicted in Figure 6.4, the **hyperbolic paraboloid**

$$M := \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, z = \frac{x^2}{a^2} - \frac{y^2}{b^2} \right\}. \tag{6.2.5}$$

(both with two straight lines through every point in M), **Plücker's conoid**

$$M := \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 \neq 0,\, z = \frac{2xy}{x^2 + y^2} \right\}, \tag{6.2.6}$$

the **helicoid**

$$M := \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, \frac{x + \mathbf{i}y}{\sqrt{x^2 + y^2}} = e^{\mathbf{i}\alpha z} \right\}, \tag{6.2.7}$$

and the **Möbius strip**

$$M := \left\{ \begin{pmatrix} \cos(s) \\ \sin(s) \\ 0 \end{pmatrix} + \frac{t}{2} \begin{pmatrix} \cos(s/2)\cos(s) \\ \cos(s/2)\sin(s) \\ \sin(s/2) \end{pmatrix} \,\middle|\, \begin{array}{c} s \in \mathbb{R} \text{ and} \\ -1 < t < 1 \end{array} \right\}. \tag{6.2.8}$$

These five surfaces have negative Gaußian curvature. The Möbius strip in (6.2.8) is not developable, while the paper model of the Möbius strip is. The helicoid in (6.2.7) is a **minimal surface**, i.e. its **mean curvature** (the trace of the second fundamental form) vanishes. A minimal surface which is not ruled is the **catenoid** $M := \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = c^2 \cosh(z/c)\}$. (**Exercise:** Prove all this.)

6.3 Symmetric Spaces

In the last section we applied the Cartan-Ambrose-Hicks Theorem in the flat case; the hypothesis $\Phi_* R = R'$ was easy to verify since both sides vanish. To find more general situations where we can verify this hypothesis note that for any development (Φ, γ, γ') satisfying the initial conditions $\gamma(0) = p_0$, $\gamma'(0) = p'_0$, and $\Phi(0) = \Phi_0$, we have

$$\Phi(t) = \Phi'_{\gamma'}(t,0)\Phi_0\Phi_\gamma(0,t)$$

so that the hypothesis $\Phi_* R = R'$ is certainly implied by the three hypotheses

$$\begin{array}{rcl} \Phi_\gamma(t,0)_* R_{p_0} & = & R_{\gamma(t)} \\ \Phi'_{\gamma'}(t,0)_* R'_{p'_0} & = & R'_{\gamma'(t)} \\ (\Phi_0)_* R_{p_0} & = & R'_{p'_0}. \end{array}$$

The last hypothesis is a condition on the initial linear isomorphism

$$\Phi_0 : T_{p_0}M \to T_{p'_0}M'$$

while the former hypotheses are conditions on M and M' respectively, namely, that the Riemann curvature tensor is invariant by parallel transport. It is rather amazing that this condition is equivalent to a rather simple geometric condition as we now show.

6.3.1 Symmetric Spaces

Definition 6.3.1. *A Riemannian manifold M is called* **symmetric about the point** *$p \in M$ if there is a (necessarily unique) isometry $\phi : M \to M$ satisfying*

$$\phi(p) = p, \qquad d\phi(p) = -\mathrm{id}. \tag{6.3.1}$$

M is called a **symmetric space** *if it is symmetric about each of its points. A Riemannian manifold M is called* **locally symmetric about the point** *$p \in M$ if, for $r > 0$ sufficiently small, there is an isometry*

$$\phi : U_r(p,M) \to U_r(p,M), \qquad U_r(p,M) := \{q \in M \mid d(p,q) < r\},$$

satisfying (6.3.1)*; M is called a* **locally symmetric space** *if it is locally symmetric about each of its points.*

Remark 6.3.2. The proof of Theorem 6.3.4 below will show that, if M is locally symmetric, the isometry $\phi : U_r(p,M) \to U_r(p,M)$ with $\phi(p) = p$ and $d\phi(p) = -\mathrm{id}$ exists whenever $0 < r \le \mathrm{inj}(p)$.

Exercise 6.3.3. Every symmetric space is complete. **Hint:** If $\gamma : I \to M$ is a geodesic and $\phi : M \to M$ is a symmetry about the point $\gamma(t_0)$ for $t_0 \in I$ then

$$\phi(\gamma(t_0 + t)) = \gamma(t_0 - t)$$

for all $t \in \mathbb{R}$ with $t_0 + t, t_0 - t \in I$.

Theorem 6.3.4. *Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold. Then the following are equivalent.*

(i) *M is locally symmetric.*

(ii) *The covariant derivative ∇R (defined below) vanishes identically, i.e.*

$$(\nabla_v R)_p(v_1, v_2)w = 0$$

for all $p \in M$ and $v, v_1, v_2, w \in T_pM$.

(iii) *The curvature tensor R is invariant under parallel transport, i.e.*

$$\Phi_\gamma(t, s)_* R_{\gamma(s)} = R_{\gamma(t)} \tag{6.3.2}$$

for every smooth curve $\gamma : \mathbb{R} \to M$ and all $s, t \in \mathbb{R}$.

Proof. See page 263. □

Corollary 6.3.5. *Let M and M' be locally symmetric spaces and fix two points $p_0 \in M$ and $p_0' \in M'$, and let $\Phi_0 : T_{p_0}M \to T_{p_0'}M'$ be an orthogonal linear isomorphism. Let $r > 0$ be less than the injectivity radius of M at p_0 and the injectivity radius of M' at p_0'. Then the following holds.*

(i) *There is an isometry $\phi : U_r(p_0, M) \to U_r(p_0', M')$ with $\phi(p_0) = p_0'$ and $d\phi(p_0) = \Phi_0$ if and only if Φ_0 intertwines R and R':*

$$(\Phi_0)_* R_{p_0} = R'_{p_0'}. \tag{6.3.3}$$

(ii) *Assume M and M' are connected, simply connected, and complete. Then there is an isometry $\phi : M \to M'$ with $\phi(p_0) = p_0'$ and $d\phi(p_0) = \Phi_0$ if and only if Φ_0 satisfies* (6.3.3)*.*

Proof. In (i) and (ii) the "only if" statement follows from Theorem 5.3.1 (Theorema Egregium) with $\Phi_0 := d\phi(p_0)$. To prove the "if" statement, let (Φ, γ, γ') be a development satisfying $\gamma(0) = p_0$, $\gamma'(0) = p_0'$, and $\Phi(0) = \Phi_0$. Since R and R' are invariant under parallel transport, by Theorem 6.3.4, it follows from the discussion in the beginning of this section that $\Phi_* R = R'$. Hence assertion (i) follows from the local C-A-H Theorem 6.1.14 and (ii) follows from the global C-A-H Theorem 6.1.7 □

Corollary 6.3.6. *A connected, simply connected, complete, locally symmetric space is symmetric.*

Proof. Corollary 6.3.5 (ii) with $M' = M$, $p_0' = p_0$, and $\Phi_0 = -\mathrm{id}$. □

Corollary 6.3.7. *A connected symmetric space M is* **homogeneous**; *i.e. given $p, q \in M$ there exists an isometry $\phi : M \to M$ with $\phi(p) = q$.*

Proof. If M is simply connected the assertion follows from part (ii) of Corollary 6.3.5 with $M = M'$, $p_0 = p$, $p_0' = q$, and $\Phi_0 = \Phi_\gamma(1, 0) : T_pM \to T_qM$, where $\gamma : [0, 1] \to M$ is a curve from p to q. If M is not simply connected we can argue as follows. There is an equivalence relation on M defined by

$$p \sim q \quad :\Longleftrightarrow \quad \exists \text{ isometry } \phi : M \to M \ni \phi(p) = q.$$

Let $p, q \in M$ and suppose that $d(p, q) < \mathrm{inj}(p)$. By Theorem 4.5.4 there is a unique shortest geodesic $\gamma : [0, 1] \to M$ connecting p to q. Since M is symmetric there is an isometry $\phi : M \to M$ such that $\phi(\gamma(1/2)) = \gamma(1/2)$ and $d\phi(\gamma(1/2)) = -\mathrm{id}$. This isometry satisfies $\phi(\gamma(t)) = \gamma(1 - t)$ and hence $\phi(p) = q$. Thus $p \sim q$ whenever $d(p, q) < \mathrm{inj}(p)$. This shows that each equivalence class is open, hence each equivalence class is also closed, and hence there is only one equivalence class because M is connected. This proves Corollary 6.3.7. □

6.3.2 Covariant Derivative of the Curvature

For two vector spaces V, W and an integer $k \geq 1$ we denote by $\mathcal{L}^k(V, W)$ the vector space of multi-linear maps from $V^k = V \times \cdots \times V$ to W. Thus $\mathcal{L}^1(V, W) = \mathcal{L}(V, W)$ is the space of linear maps from V to W.

Definition 6.3.8. The **covariant derivative of the Riemann curvature tensor** assigns to every $p \in M$ a linear map

$$(\nabla R)_p : T_pM \to \mathcal{L}^2(T_pM, \mathcal{L}(T_pM, T_pM))$$

such that

$$\begin{aligned}(\nabla R)(X)(X_1, X_2)Y = \nabla_X\big(R(X_1, X_2)Y\big) &- R(\nabla_X X_1, X_2)Y \\ &- R(X_1, \nabla_X X_2)Y - R(X_1, X_2)\nabla_X Y\end{aligned} \tag{6.3.4}$$

for all $X, X_1, X_2, Y \in \mathrm{Vect}(M)$. We also use the notation

$$(\nabla_v R)_p := (\nabla R)_p(v)$$

for $p \in M$ and $v \in T_pM$ so that

$$(\nabla_X R)(X_1, X_2)Y := (\nabla R)(X)(X_1, X_2)Y$$

for all $X, X_1, X_2, Y \in \mathrm{Vect}(M)$.

Remark 6.3.9. One verifies easily that the map

$$\mathrm{Vect}(M)^4 \to \mathrm{Vect}(M) : (X, X_1, X_2, Y) \mapsto (\nabla_X R)(X_1, X_2)Y,$$

defined by the right hand side of equation (6.3.4), is multi-linear over the ring of functions $\mathscr{F}(M)$. Hence it follows as in Remark 5.2.12 that ∇R is well defined, i.e. that the right hand side of (6.3.4) at $p \in M$ depends only on the tangent vectors $X(p), X_1(p), X_2(p), Y(p)$.

Remark 6.3.10. Let $\gamma : I \to M$ be a smooth curve on an interval $I \subset \mathbb{R}$ and

$$X_1, X_2, Y \in \mathrm{Vect}(\gamma)$$

be smooth vector fields along γ. Then equation (6.3.4) continues to hold with X replaced by $\dot\gamma$ and each ∇_X on the right hand side replaced by the covariant derivative of the respective vector field along γ:

$$\begin{aligned}(\nabla_{\dot\gamma} R)(X_1, X_2)Y = \nabla(R(X_1, X_2)Y) - R(\nabla X_1, X_2)Y \\ - R(X_1, \nabla X_2)Y - R(X_1, X_2)\nabla Y.\end{aligned} \tag{6.3.5}$$

Theorem 6.3.11. **(i)** *If $\gamma : \mathbb{R} \to M$ is a smooth curve such that $\gamma(0) = p$ and $\dot\gamma(0) = v$ then*

$$(\nabla_v R)_p = \left.\frac{d}{dt}\right|_{t=0} \Phi_\gamma(0, t)_* R_{\gamma(t)} \tag{6.3.6}$$

(ii) *The covariant derivative of the Riemann curvature tensor satisfies the* **second Bianchi identity**

$$(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0. \tag{6.3.7}$$

Proof. We prove (i). Let $v_1, v_2, w \in T_pM$ and choose parallel vector fields $X_1, X_2, Y \in \mathrm{Vect}(\gamma)$ along γ satisfying the initial conditions $X_1(0) = v_1$, $X_2(0) = v_2$, $Y(0) = w$. Thus

$$X_1(t) = \Phi_\gamma(t, 0)v_1, \qquad X_2(t) = \Phi_\gamma(t, 0)v_2, \qquad Y(t) = \Phi_\gamma(t, 0)w.$$

Then the last three terms on the right vanish in equation (6.3.5) and hence

$$\begin{aligned}(\nabla_v R)(v_1, v_2)w &= \nabla(R(X_1, X_2)Y)(0) \\ &= \left.\frac{d}{dt}\right|_{t=0} \Phi_{\gamma(0,t)} R_{\gamma(t)}(X_1(t), X_2(t))Y(t) \\ &= \left.\frac{d}{dt}\right|_{t=0} \Phi_{\gamma(0,t)} R_{\gamma(t)}(\Phi_\gamma(t, 0)v_1, \Phi_\gamma(t, 0)v_2)\Phi_\gamma(t, 0)w \\ &= \left.\frac{d}{dt}\right|_{t=0} \left(\Phi_\gamma(0, t)_* R_{\gamma(t)}\right)(v_1, v_2)w.\end{aligned}$$

Here the second equation follows from Theorem 3.3.6. This proves (i).

We prove (ii). Choose a smooth function $\gamma : \mathbb{R}^3 \to M$ and denote by (r,s,t) the coordinates on $\mathbb{R}^3$. If Y is a vector field along γ we have

$$\begin{aligned}(\nabla_{\partial_r\gamma}R)(\partial_s\gamma,\partial_t\gamma)Y &= \nabla_r\big(R(\partial_s\gamma,\partial_t\gamma)Y\big) - R(\partial_s\gamma,\partial_t\gamma)\nabla_r Y \\ &\quad -R(\nabla_r\partial_s\gamma,\partial_t\gamma)Y - R(\partial_s\gamma,\nabla_r\partial_t\gamma)Y \\ &= \nabla_r\left(\nabla_s\nabla_t Y - \nabla_t\nabla_s Y\right) - \left(\nabla_s\nabla_t - \nabla_t\nabla_s\right)\nabla_r Y \\ &\quad +R(\partial_t\gamma,\nabla_r\partial_s\gamma)Y - R(\partial_s\gamma,\nabla_t\partial_r\gamma)Y.\end{aligned}$$

Permuting the variables r,s,t cyclically and taking the sum of the resulting three equations we obtain

$$\begin{aligned}&(\nabla_{\partial_r\gamma}R)(\partial_s\gamma,\partial_t\gamma)Y + (\nabla_{\partial_s\gamma}R)(\partial_t\gamma,\partial_r\gamma)Y + (\nabla_{\partial_t\gamma}R)(\partial_r\gamma,\partial_s\gamma)Y \\ &= \nabla_r\left(\nabla_s\nabla_t Y - \nabla_t\nabla_s Y\right) - \left(\nabla_s\nabla_t - \nabla_t\nabla_s\right)\nabla_r Y \\ &\quad +\nabla_s\left(\nabla_t\nabla_r Y - \nabla_r\nabla_t Y\right) - \left(\nabla_t\nabla_r - \nabla_r\nabla_t\right)\nabla_s Y \\ &\quad +\nabla_t\left(\nabla_r\nabla_s Y - \nabla_s\nabla_r Y\right) - \left(\nabla_r\nabla_s - \nabla_s\nabla_r\right)\nabla_t Y.\end{aligned}$$

The terms on the right cancel out. This proves Theorem 6.3.11. □

Proof of Theorem 6.3.4. We prove that (iii) implies (i). This follows from the local Cartan–Ambrose–Hicks Theorem 6.1.14 with

$$p_0' = p_0 = p, \qquad \Phi_0 = -\mathrm{id} : T_pM \to T_pM.$$

This isomorphism satisfies

$$(\Phi_0)_* R_p = R_p.$$

Hence it follows from the discussion in the beginning of this section that

$$\Phi_* R = R'$$

for every development (Φ,γ,γ') of M along itself satisfying

$$\gamma(0) = \gamma'(0) = p, \qquad \Phi(0) = -\mathrm{id}.$$

Hence, by the local C-A-H Theorem 6.1.14, there is an isometry

$$\phi : U_r(p,M) \to U_r(p,M)$$

satisfying

$$\phi(p) = p, \qquad d\phi(p) = -\mathrm{id}$$

whenever $0 < r < \mathrm{inj}(p;M)$.

We prove that (i) implies (ii). By Theorem 5.3.1 (Theorema Egregium), every isometry $\phi : M \to M'$ preserves the Riemann curvature tensor and covariant differentiation, and hence also the covariant derivative of the Riemann curvature tensor, i.e.

$$\phi_*(\nabla R) = \nabla' R'.$$

Applying this to the local isometry $\phi : U_r(p, M) \to U_r(p, M)$ we obtain

$$\left(\nabla_{d\phi(p)v} R\right)_{\phi(p)} (d\phi(p)v_1, d\phi(p)v_2) = d\phi(p) \left(\nabla_v R\right)(v_1, v_2) d\phi(p)^{-1}$$

for all $v, v_1, v_2 \in T_pM$. Since

$$d\phi(p) = -\mathrm{id}$$

this shows that ∇R vanishes at p.

We prove that (ii) imlies (iii). If ∇R vanishes then equation (6.3.6) in Theorem 6.3.11 shows that the function

$$s \mapsto \Phi_\gamma(t, s)_* R_{\gamma(s)} = \Phi_\gamma(t, 0)_* \Phi_\gamma(0, s)_* R_{\gamma(s)}$$

is constant and hence is everywhere equal to $R_{\gamma(t)}$. This implies (6.3.2) and completes the proof of Theorem 6.3.4. □

Covariant Derivative of the Curvature in Local Coordinates

Let $\phi : U \to \Omega$ be a local coordinate chart on M with values in an open set $\Omega \subset \mathbb{R}^m$, denote its inverse by $\psi := \phi^{-1} : \Omega \to U$, and let

$$E_i(x) := \frac{\partial \psi}{\partial x^i}(x) \in T_{\psi(x)}M, \qquad x \in \Omega, \qquad i = 1, \ldots, m,$$

be the local frame of the tangent bundle determined by this coordinate chart. Let $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ denote the Christoffel symbols and $R_{ijk}^\ell : \Omega \to \mathbb{R}$ the coefficients of the Riemann curvature tensor so that

$$\nabla_i E_j = \sum_k \Gamma_{ij}^k E_k, \qquad R(E_i, E_j)E_k = \sum_\ell R_{ijk}^\ell E_\ell.$$

Given $i, j, k, \ell \in \{1, \ldots, m\}$ we can express the vector field $(\nabla_{E_i} R)(E_j, E_k)E_\ell$ along ψ for each $x \in \Omega$ as a linear combination of the basis vectors $E_i(x)$. This gives rise to functions $\nabla_i R_{jk\ell}^\nu : \Omega \to \mathbb{R}$ defined by

$$(\nabla_{E_i} R)(E_j, E_k)E_\ell =: \sum_\nu \nabla_i R_{jk\ell}^\nu E_\nu. \tag{6.3.8}$$

These functions are given by

$$\begin{aligned}\nabla_i R^\nu_{jk\ell} = \partial_i R^\nu_{jk\ell} + \sum_\mu \Gamma^\nu_{i\mu} R^\mu_{jk\ell} \\ - \sum_\mu \Gamma^\mu_{ij} R^\nu_{\mu k\ell} - \sum_\mu \Gamma^\mu_{ik} R^\nu_{j\mu\ell} - \sum_\mu \Gamma^\mu_{i\ell} R^\nu_{jk\mu}.\end{aligned} \tag{6.3.9}$$

The second Bianchi identity has the form

$$\nabla_i R^\nu_{jk\ell} + \nabla_j R^\nu_{ki\ell} + \nabla_k R^\nu_{ij\ell} = 0. \tag{6.3.10}$$

Exercise: Prove equations (6.3.9) and (6.3.10). **Warning:** As in §5.4, care must be taken with the ordering of the indices. Some authors use the notation $\nabla_i R^\nu_{\ell jk}$ for what we call $\nabla_i R^\nu_{jk\ell}$.

6.3.3 Examples and Exercises

Example 6.3.12. Every flat manifold is locally symmetric.

Example 6.3.13. If M_1 and M_2 are (locally) symmetric, so is $M_1 \times M_2$.

Example 6.3.14. $M = \mathbb{R}^m$ with the standard metric is a symmetric space. Recall that the isometry group $\mathcal{I}(\mathbb{R}^m)$ consists of all affine transformations of the form

$$\phi(x) = Ax + b, \qquad A \in \mathrm{O}(m), \qquad b \in \mathbb{R}^m.$$

(See Exercise 5.1.4.) The isometry with fixed point $p \in \mathbb{R}^m$ and $d\phi(p) = -\mathrm{id}$ is given by $\phi(x) = 2p - x$ for $x \in \mathbb{R}^m$.

Example 6.3.15. The flat tori of Exercise 6.2.7 in the previous section are symmetric (but not simply connected). This shows that the hypothesis of simply connectivity cannot be dropped in the Corollary 6.3.5 (ii).

Example 6.3.16. Below we define manifolds of constant curvature and show that they are locally symmetric. The simplest example, after a flat space, is the unit sphere $S^m = \left\{x \in \mathbb{R}^{m+1} \mid |x| = 1\right\}$. The symmetry ϕ of the sphere about a point $p \in M$ is given by

$$\phi(x) := -x + 2\langle p, x\rangle p$$

for $x \in S^m$. This extends to an orthogonal linear transformation of the ambient space. In fact the group of isometries of S^m is the group $\mathrm{O}(m+1)$ of orthogonal linear transformations of $\mathbb{R}^{m+1}$: see Example 6.4.16 below. In accordance with Corollary 6.3.7 this group acts transitively on S^m.

Example 6.3.17. A compact two-dimensional manifold of constant negative curvature is locally symmetric (as its universal cover is symmetric) but not homogeneous (as closed geodesics of a given period are isolated). Hence it is not symmetric. This shows that the hypothesis that M be simply connected cannot be dropped in the Corollary 6.3.6.

Example 6.3.18. The real projective space $\mathbb{R}\mathrm{P}^n$ with the metric inherited from S^n is a symmetric space and the orthogonal group $\mathrm{O}(n+1)$ acts on it by isometries. The complex projective space $\mathbb{C}\mathrm{P}^n$ with the Fubini–Study metric is a symmetric space and the unitary group $\mathrm{U}(n+1)$ acts on it by isometries: see Example 3.7.5. The complex Grassmannian $\mathrm{G}_k(\mathbb{C}^n)$ is a symmetric space and the unitary group $\mathrm{U}(n)$ acts on it by isometries: see Example 3.7.6. (**Exercise:** Prove this.)

Example 6.3.19. The simplest example of a symmetric space which is not of constant curvature is the orthogonal group $\mathrm{O}(n) = \left\{g \in \mathbb{R}^{n\times n} \,|\, g^{\mathsf{T}}g = \mathbb{1}\right\}$ with the Riemannian metric (5.2.19) of Example 5.2.17. The symmetry ϕ about the point $a \in \mathrm{O}(n)$ is given by $\phi(g) = ag^{-1}a$. This discussion extends to every Lie subgroup $\mathrm{G} \subset \mathrm{O}(n)$. (**Exercise:** Prove this.)

6.4 Constant Curvature

In the §5.3 we saw that the Gaußian curvature of a two-dimensional surface is intrinsic: we gave a formula for it in terms of the Riemann curvature tensor and the first fundamental form. We may use this formula to define the Gaußian curvature for *any* two-dimensional manifold (even if its codimension is greater than one). We make a slightly more general definition.

6.4.1 Sectional Curvature

Definition 6.4.1. *Let $M \subset \mathbb{R}^n$ be a smooth m-dimensional submanifold. Let $p \in M$ and let $E \subset T_pM$ be a 2-dimensional linear subspace of the tangent space. The* **sectional curvature** *of M at (p, E) is the number*

$$K(p,E) = \frac{\langle R_p(u,v)v, u\rangle}{|u|^2|v|^2 - \langle u,v\rangle^2} \tag{6.4.1}$$

where $u, v \in E$ are linearly independent (and hence form a basis of E).

The right hand side of (6.4.1) remains unchanged if we multiply u or v by a nonzero real number or add to one of the vectors a real multiple of the other; hence it depends only on the linear subspace spanned by u ad v.

Example 6.4.2. If $M \subset \mathbb{R}^3$ is a 2-manifold then, by Theorem 5.3.7, the sectional curvature $K(p, T_pM) = K(p)$ is the Gaußian curvature of M at p. More generally, for any 2-manifold $M \subset \mathbb{R}^n$ (whether or not it has codimension one) we *define* the **Gaußian curvature** of M at p by

$$K(p) := K(p, T_pM).$$

Example 6.4.3. If $M \subset \mathbb{R}^{m+1}$ is a submanifold of codimension one and $\nu : M \to S^m$ is a Gauß map then the sectional curvature of a 2-dimensional subspace $E \subset T_pM$ spanned by two linearly independent tangent vectors $u, v \in T_pM$ is given by

$$K(p, E) = \frac{\langle u, d\nu(p)u\rangle\langle v, d\nu(p)v\rangle - \langle u, d\nu(p)v\rangle^2}{|u|^2|v|^2 - \langle u, v\rangle^2}. \tag{6.4.2}$$

This follows from equation (5.3.6) in the proof of Theorem 5.3.7 which holds in all dimensions. In particular, when $M = S^m$, we have $\nu(p) = p$ and hence $K(p, E) = 1$ for all p and E. For a sphere of radius r we have $\nu(p) = p/r$ and hence $K(p, E) = 1/r^2$.

Example 6.4.4. Let $\mathrm{G} \subset \mathrm{O}(n)$ be a Lie subgroup equipped with the Riemannian metric

$$\langle v, w\rangle := \mathrm{trace}(v^{\mathsf{T}}w)$$

for $v, w \in T_g\mathrm{G} \subset \mathbb{R}^{n\times n}$. Then, by Example 5.2.17, the sectional curvature of G at the identity matrix $\mathbb{1}$ is given by

$$K(\mathbb{1}, E) = \frac{1}{4}\left|[\xi, \eta]\right|^2$$

for every 2-dimensional linear subspace $E \subset \mathfrak{g} = \mathrm{Lie}(\mathrm{G}) = T_{\mathbb{1}}\mathrm{G}$ with an orthonormal basis ξ, η.

Exercise 6.4.5. Let $E \subset T_pM$ be a 2-dimensional linear subspace, let $r > 0$ be smaller than the injectivity radius of M at p, and let $N \subset M$ be the 2-dimensional submanifold given by

$$N := \exp_p\left(\{v \in E \mid |v| < r\}\right).$$

Show that the sectional curvature $K(p, E)$ of M at (p, E) agrees with the Gauß curvature of N at p.

Exercise 6.4.6. Let $p \in M \subset \mathbb{R}^n$ and let $E \subset T_pM$ be a 2-dimensional linear subspace. For $r > 0$ let L denote the ball of radius r in the $(n - m + 2)$ dimensional affine subspace of $\mathbb{R}^n$ through p and parallel to the vector subspace $E + T_pM^\perp$:

$$L = \left\{ p + v + w \,|\, v \in E,\, w \in T_pM^\perp,\, |v|^2 + |w|^2 < r^2 \right\}.$$

Show that, for r sufficiently small, $L \cap M$ is a 2-dimensional manifold with Gauß curvature $K_{L\cap M}(p)$ at p given by

$$K_{L\cap M}(p) = K(p, E).$$

6.4.2 Constant Sectional Curvature

Definition 6.4.7. *Let $k \in \mathbb{R}$ and $m \geq 2$ be an integer. An m-manifold $M \subset \mathbb{R}^n$ is said to have* **constant sectional curvature** *k if $K(p, E) = k$ for every $p \in M$ and every 2-dimensional linear subspace $E \subset T_pM$.*

Theorem 6.4.8. *An m-dimensional manifold $M \subset \mathbb{R}^n$ has constant sectional curvature k if and only if*

$$\langle R_p(v_1, v_2)v_3, v_4\rangle = k\Big(\langle v_1, v_4\rangle\langle v_2, v_3\rangle - \langle v_1, v_3\rangle\langle v_2, v_4\rangle\Big) \tag{6.4.3}$$

for all $p \in M$ and all $v_1, v_2, v_3, v_4 \in T_pM$.

Proof. The "only if" statement follows immediately from the definition with $v_1 = v_4 = u$ and $v_2 = v_3 = v$. To prove the converse, we assume that M has constant curvature k. Fix a point $p \in M$ and define the multi-linear map $Q : T_pM^4 \to \mathbb{R}$ by

$$Q(v_1, v_2, v_3, v_4) := \langle R_p(v_1, v_2)v_3, v_4\rangle - k\Big(\langle v_1, v_4\rangle\langle v_2, v_3\rangle - \langle v_1, v_3\rangle\langle v_2, v_4\rangle\Big).$$

Then Q satisfies the equations

$$Q(v_1, v_2, v_3, v_4) + Q(v_2, v_1, v_3, v_4) = 0, \tag{6.4.4}$$

$$Q(v_1, v_2, v_3, v_4) + Q(v_2, v_3, v_1, v_4) + Q(v_3, v_1, v_2, v_4) = 0, \tag{6.4.5}$$

$$Q(v_1, v_2, v_3, v_4) - Q(v_3, v_4, v_1, v_2) = 0, \tag{6.4.6}$$

$$Q(u, v, u, v) = 0 \tag{6.4.7}$$

for all $u, v, v_1, v_2, v_3, v_4 \in T_pM$. Here the first three equations follow from Theorem 5.2.13 and the last follows from the definition of Q and the hypothesis that M have constant sectional curvature k.

We must prove that Q vanishes. Using (6.4.6) and (6.4.7) we find

$$\begin{aligned} 0 &= Q(u, v_1 + v_2, u, v_1 + v_2) \\ &= Q(u, v_1, u, v_2) + Q(u, v_2, u, v_1) \\ &= 2Q(u, v_1, u, v_2) \end{aligned}$$

for all $u, v_1, v_2 \in T_pM$. This implies

$$\begin{aligned} 0 &= Q(u_1 + u_2, v_1, u_1 + u_2, v_2) \\ &= Q(u_1, v_1, u_2, v_2) + Q(u_2, v_1, u_1, v_2) \end{aligned}$$

for all $u_1, u_2, v_1, v_2 \in TpM$. Hence

$$\begin{aligned} Q(v_1, v_2, v_3, v_4) &= -Q(v_3, v_2, v_1, v_4) \\ &= Q(v_2, v_3, v_1, v_4) \\ &= -Q(v_3, v_1, v_2, v_4) - Q(v_1, v_2, v_3, v_4). \end{aligned}$$

Here the second equation follows from (6.4.4) and the last from (6.4.5). Thus

$$Q(v_1, v_2, v_3, v_4) = -\frac{1}{2}Q(v_3, v_1, v_2, v_4) = \frac{1}{2}Q(v_1, v_3, v_2, v_4)$$

for all $v_1, v_2, v_3, v_4 \in T_pM$ and, repeating this argument,

$$Q(v_1, v_2, v_3, v_4) = \frac{1}{4}Q(v_1, v_2, v_3, v_4).$$

Hence $Q \equiv 0$ as claimed. This proves Theorem 6.4.8. □

Remark 6.4.9. The symmetric group S_4 on four symbols acts naturally on the space $\mathcal{L}^4(T_pM, \mathbb{R})$ of multi-linear maps from T_pM^4 to $\mathbb{R}$. The conditions (6.4.4), (6.4.5), (6.4.6), and (6.4.7) say that the four elements

$$\begin{aligned} a &= \mathrm{id} + (12) \\ c &= \mathrm{id} + (123) + (132) \\ b &= \mathrm{id} - (34) \\ d &= \mathrm{id} + (13) + (24) + (13)(24) \end{aligned}$$

of the group ring of S_4 annihilate Q. This suggests an alternate proof of Theorem 6.4.8. A representation of a finite group is completely reducible so one can prove that $Q = 0$ by showing that any vector in any irreducible representation of S_4 which is annihilated by the four elements a, b, c and d must necessarily be zero. This can be checked case by case for each irreducible representation. (The group S_4 has 5 irreducible representations: two of dimension 1, two of dimension 3, and one of dimension 2.)

If M and M' are two m-dimensional manifolds with constant curvature k then every orthogonal isomorphism $\Phi : T_pM \to T_{p'}M'$ intertwines the Riemann curvature tensors by Theorem 6.4.8. Hence by the appropriate version (local or global) of the C-A-H Theorem we have the following corollaries.

Corollary 6.4.10. *Every Riemannian manifold with constant sectional curvature is locally symmetric.*

Proof. Theorem 6.3.4 and Theorem 6.4.8. □

Corollary 6.4.11. *Let M and M' be m-dimensional Riemannian manifolds with constant curvature k and let $p \in M$ and $p' \in M'$. If $r > 0$ is smaller than the injectivity radii of M at p and of M' at p' then, for every orthogonal isomorphism $\Phi : T_pM \to T_{p'}M'$, there exists an isometry*

$$\phi : U_r(p, M) \to U_r(p', M')$$

such that

$$\phi(p) = p', \qquad d\phi(p) = \Phi.$$

Proof. This follows from Corollary 6.3.5 and Corollary 6.4.10. Alternatively one can use Theorem 6.4.8 and the local C-A-H Theorem 6.1.14. □

Corollary 6.4.12. *Any two connected, simply connected, complete Riemannian manifolds with the same constant sectional curvature and the same dimension are isometric.*

Proof. Theorem 6.4.8 and the global C-A-H Theorem 6.1.7. □

Corollary 6.4.13. *Let $M \subset \mathbb{R}^n$ be a connected, simply connected, complete manifold. Then the following are equivalent.*

(i) *M has constant sectional curvature.*

(ii) *For every pair of points $p, q \in M$ and every orthogonal linear isomorphism $\Phi : T_pM \to T_qM$ there is an isometry $\phi : M \to M$ such that*

$$\phi(p) = q, \qquad d\phi(p) = \Phi.$$

Proof. That (i) implies (ii) follows immediately from Theorem 6.4.8 and the global C-A-H Theorem 6.1.7. Conversely assume (ii). Then, for every pair of points $p, q \in M$ and every orthogonal linear isomorphism $\Phi : T_pM \to T_qM$, it follows from Theorem 5.3.1 (Theorema Egregium) that $\Phi_* R_p = R_q$ and so $K(p, E) = K(q, \Phi E)$ for every 2-dimensional linear subspace $E \subset T_pM$. Since, for every pair of points $p, q \in M$ and of 2-dimensional linear subspaces $E \subset T_pM$ and $F \subset T_qM$, we can find an orthogonal linear isomorphism $\Phi : T_pM \to T_qM$ such that $\Phi E = F$, this implies (i). □

Corollary 6.4.13 asserts that a connected, simply connected, complete Riemannian m-manifold M has constant sectional curvature if and only if the isometry group $\mathcal{I}(M)$ acts transitively on its orthonormal frame bundle $\mathcal{O}(M)$. Note that, by Lemma 5.1.10, this group action is also free.

6.4.3 Examples and Exercises

Example 6.4.14. Any flat Riemannian manifold has constant sectional curvature $k = 0$.

Example 6.4.15. The manifold $M = \mathbb{R}^m$ with its standard metric is, up to isometry, the unique connected, simply connected, complete Riemannian m-manifold with constant sectional curvature

$$k = 0.$$

Example 6.4.16. For $m \geq 2$ the unit sphere $M = S^m$ with its standard metric is, up to isometry, the unique connected, simply connected, complete Riemannian m-manifold with constant sectional curvature

$$k = 1.$$

Hence, by Corollary 6.4.12, every connected simply connected, complete Riemannian manifold with positive sectional curvature $k = 1$ is compact. Moreover, by Corollary 6.4.13, the isometry group $\mathcal{I}(S^m)$ is isomorphic to the group $\mathrm{O}(m+1)$ of orthogonal linear transformations of $\mathbb{R}^{m+1}$. Thus, by Corollary 6.4.13, the orthonormal frame bundle $\mathcal{O}(S^m)$ is diffeomorphic to $\mathrm{O}(m+1)$. This follows also from the fact that, if

$$v_1, \ldots, v_m$$

is an orthonormal basis of $T_pS^m = p^\perp$ then

$$p, v_1, \ldots, v_m$$

is an orthonormal basis of $\mathbb{R}^{m+1}$.

Example 6.4.17. A product of spheres is *not* a space of constant sectional curvature, but it *is* a symmetric space. **Exercise:** Prove this.

Example 6.4.18. For $n \geq 4$ the orthogonal group $\mathrm{O}(n)$ is not a space of constant sectional curvature, but it is a symmetric space and has nonnegative sectional curvature (see Example 6.4.4).

6.4.4 Hyperbolic Space

The **hyperbolic space** $\mathbb{H}^m$ is, up to isometry, the unique connected, simply connected, complete Riemannian m-manifold with constant sectional curvature $k = -1$. A model for $\mathbb{H}^m$ can be constructed as follows. A point in $\mathbb{R}^{m+1}$ will be denoted by

$$p = (x_0, x), \qquad x_0 \in \mathbb{R}, \qquad x = (x_1, \dots, x_m) \in \mathbb{R}^m.$$

Let $Q : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}$ denote the symmetric bilinear form given by

$$Q(p, q) := -x_0 y_0 + x_1 y_1 + \cdots + x_m y_m$$

for $p = (x_0, x), q = (y_0, y) \in \mathbb{R}^{m+1}$. Since Q is nondegenerate the space

$$\mathbb{H}^m := \left\{ p = (x_0, x) \in \mathbb{R}^{m+1} \,|\, Q(p, p) = -1,\, x_0 > 0 \right\}$$

is a smooth m-dimensional submanifold of $\mathbb{R}^{m+1}$ and the tangent space of $\mathbb{H}^m$ at p is given by

$$T_p\mathbb{H}^m = \left\{ v \in \mathbb{R}^{m+1} \,|\, Q(p, v) = 0 \right\}.$$

For $p = (x_0, x) \in \mathbb{R}^{m+1}$ and $v = (\xi_0, \xi) \in \mathbb{R}^{m+1}$ we have

$$\begin{aligned} p \in \mathbb{H}^m \quad &\iff \quad x_0 = \sqrt{1 + |x|^2}, \\ v \in T_p\mathbb{H}^m \quad &\iff \quad \xi_0 = \frac{\langle \xi, x \rangle}{\sqrt{1 + |x|^2}}. \end{aligned}$$

Now let us define a Riemannian metric on $\mathbb{H}^m$ by

$$g_p(v, w) := Q(v, w) = \langle \xi, \eta \rangle - \xi_0 \eta_0 = \langle \xi, \eta \rangle - \frac{\langle \xi, x \rangle \langle \eta, x \rangle}{1 + |x|^2} \tag{6.4.8}$$

for $v = (\xi_0, \xi) \in T_p\mathbb{H}^m$ and $w = (\eta_0, \eta) \in T_p\mathbb{H}^m$.

Theorem 6.4.19. *$\mathbb{H}^m$ is a connected, simply connected, complete Riemannian m-manifold with constant sectional curvature $k = -1$.*

Proof. See page 273. □

We remark that the manifold $\mathbb{H}^m$ does not quite fit into the extrinsic framework of most of this book as it is not exhibited as a submanifold of Euclidean space but rather of "pseudo-Euclidean space": the positive definite inner product $\langle v, w \rangle$ of the ambient space $\mathbb{R}^{m+1}$ is replaced by a nondegenerate symmetric bilinear form $Q(v,w)$. However, all the theory developed thus far goes through (reading $Q(v,w)$ for $\langle v,w\rangle$) provided we make the additional hypothesis (true in the example $M=\mathbb{H}^m$) that the first fundamental form $g_p = Q|_{T_pM}$ is positive definite. For then $Q|_{T_pM}$ is nondegenerate and we may define the orthogonal projection $\Pi(p)$ onto T_pM as before. The next lemma summarizes the basic observations; the proof is an exercise in linear algebra.

Lemma 6.4.20. *Let Q be a symmetric bilinear form on a vector space V and for each subspace E of V define its orthogonal complement by*

$$E^{\perp_Q} := \{u \in V \,|\, Q(u,v) = 0 \;\forall v \in E\}.$$

Assume Q is nondegenerate, i.e. $V^{\perp_Q} = \{0\}$. Then, for every linear subspace $E \subset V$, we have

$$V = E \oplus E^{\perp_Q} \qquad \iff \qquad E \cap E^{\perp_Q} = \{0\},$$

i.e. $E^{\perp_Q}$ is a vector space complement of E if and only if the restriction of Q to E is nondegenerate.

Proof of Theorem 6.4.19. The proofs of the various properties of $\mathbb{H}^m$ are entirely analogous to the corresponding proofs for S^m. Thus the unit normal field to $\mathbb{H}^m$ is given by $\nu(p) = p$ for $p \in \mathbb{H}^m$ although the "square of its length" is $Q(p,p) = -1$.

For $p \in \mathbb{H}^m$ we introduce the Q-orthogonal projection $\Pi(p)$ of $\mathbb{R}^{m+1}$ onto $T_p\mathbb{H}^m$. It is characterized by the conditions

$$\Pi(p)^2 = \Pi(p), \qquad \ker \Pi(p) \perp_Q \operatorname{im}\Pi(p), \qquad \operatorname{im}\Pi(p) = T_p\mathbb{H}^m,$$

and is given by the explicit formula

$$\Pi(p)v = v + Q(v,p)p$$

for $v \in \mathbb{R}^{m+1}$. The covariant derivative of a vector field $X \in \operatorname{Vect}(\gamma)$ along a smooth curve $\gamma : \mathbb{R} \to \mathbb{H}^m$ is given by

$$\begin{aligned}\nabla X(t) &= \Pi(\gamma(t))\dot{X}(t) \\ &= \dot{X}(t) + Q(\dot{X}(t), \gamma(t))\gamma(t) \\ &= \dot{X}(t) - Q(X(t), \dot{\gamma}(t))\gamma(t).\end{aligned}$$

The last identity follows by differentiating the equation $Q(X,\gamma) \equiv 0$. This can be interpreted as the hyperbolic Gauß–Weingarten formula as follows.

For $p \in \mathbb{H}^m$ and $u \in T_p\mathbb{H}^m$ we introduce, as before, the second fundamental form $h_p(u) : T_p\mathbb{H}^m \to (T_p\mathbb{H}^m)^{\perp_Q}$ via

$$h_p(u)v := \big(d\Pi(p)u\big)v$$

and denote its Q-adjoint by

$$h_p(u)^* : (T_p\mathbb{H}^m)^{\perp_Q} \to T_p\mathbb{H}^m.$$

For every $p \in \mathbb{R}^{m+1}$ we have

$$\Big(d\Pi(p)u\Big)v = \frac{d}{dt}\bigg|_{t=0} \big(v + Q(v, p + tu)(p + tu)\big) = Q(v,p)u + Q(v,u)p,$$

where the first summand on the right is tangent to $\mathbb{H}^m$ and the second summand is Q-orthogonal to $T_p\mathbb{H}^m$. Hence

$$h_p(u)v = Q(v,u)p, \qquad h_p(u)^*w = Q(w,p)u \tag{6.4.9}$$

for $v \in T_p\mathbb{H}^m$ and $w \in (T_p\mathbb{H}^m)^{\perp_Q}$.

With this understood, the Gauß-Weingarten formula

$$\dot{X} = \nabla X + h_\gamma(\dot\gamma)X$$

extends to the present setting. The reader may verify that the operators $\nabla : \mathrm{Vect}(\gamma) \to \mathrm{Vect}(\gamma)$ thus defined satisfy the axioms of Theorem 3.7.8 and hence define the Levi-Civita connection on $\mathbb{H}^m$.

Now a smooth curve $\gamma : I \to \mathbb{H}^m$ is a geodesic if and only if it satisfies the equivalent conditions

$$\nabla\dot\gamma \equiv 0 \quad\iff\quad \ddot\gamma(t) \perp_Q T_{\gamma(t)}\mathbb{H}^m \;\; \forall\, t \in I \quad\iff\quad \ddot\gamma = Q(\dot\gamma,\dot\gamma)\gamma.$$

A geodesic must satisfy the equation

$$\frac{d}{dt}Q(\dot\gamma,\dot\gamma) = 2Q(\ddot\gamma,\dot\gamma) = 0$$

because $\ddot\gamma$ is a scalar multiple of γ, and so $Q(\dot\gamma,\dot\gamma)$ is constant. Let $p \in \mathbb{H}^m$ and $v \in T_p\mathbb{H}^m$ be given with $Q(v,v) = 1$. Then the geodesic $\gamma : \mathbb{R} \to \mathbb{H}^m$ with $\gamma(0) = p$ and $\dot\gamma(0) = v$ is given by

$$\gamma(t) = \cosh(t)p + \sinh(t)v, \tag{6.4.10}$$

where

$$\cosh(t) := \frac{e^t + e^{-t}}{2}, \qquad \sinh(t) := \frac{e^t - e^{-t}}{2}.$$

In fact we have $\ddot\gamma(t) = \gamma(t) \perp_Q T_{\gamma(t)}\mathbb{H}^m$. It follows that the geodesics exist for all time and hence $\mathbb{H}^m$ is geodesically complete. Moreover, being diffeomorphic to Euclidean space, $\mathbb{H}^m$ is connected and simply connected.

It remains to prove that $\mathbb{H}^m$ has constant sectional curvature $k = -1$. To see this we use the Gauß–Codazzi formula in the hyperbolic setting, i.e.

$$R_p(u,v) = h_p(u)^* h_p(v) - h_p(v)^* h_p(u). \tag{6.4.11}$$

By equation (6.4.9), this gives

$$\begin{aligned}\langle R_p(u,v)v, u\rangle &= Q(h_p(u)u, h_p(v)v) - Q(h_p(v)u, h_p(u)v) \\ &= Q(Q(u,u)p, Q(v,v)p) - Q(Q(u,v)p, Q(u,v)p) \\ &= -Q(u,u)Q(v,v) + Q(u,v)^2 \\ &= -g_p(u,u)g_p(v,v) + g_p(u,v)^2\end{aligned}$$

for all $u, v \in T_p\mathbb{H}^m$. Hence, for every $p \in M$ and every 2-dimensional linear subspace $E \subset T_pM$ with a basis $u, v \in E$ we have

$$K(p,E) = \frac{\langle R_p(u,v)v,u\rangle}{g_p(u,u)g_p(v,v) - g_p(u,v)^2} = -1.$$

This proves Theorem 6.4.19. □

Exercise 6.4.21. Prove that the pullback of the metric on $\mathbb{H}^m$ under the diffeomorphism

$$\mathbb{R}^m \to \mathbb{H}^m : x \mapsto \left(\sqrt{1+|x|^2}, x\right)$$

is given by

$$|\xi|_x = \sqrt{|\xi|^2 - \frac{\langle x,\xi\rangle^2}{1+|x|^2}}$$

or, equivalently, by the metric tensor,

$$g_{ij}(x) = \delta_{ij} - \frac{x_i x_j}{1+|x|^2} \tag{6.4.12}$$

for $x = (x_1, \dots, x_m) \in \mathbb{R}^m$.

Exercise 6.4.22. The **Poincaré model** of hyperbolic space is the open unit disc $\mathbb{D}^m \subset \mathbb{R}^m$ equipped with the **Poincaré metric**

$$|\eta|_y = \frac{2\,|\eta|}{1-|y|^2}$$

for $y \in \mathbb{D}^m$ and $\eta \in \mathbb{R}^m = T_y\mathbb{D}^m$. Thus the metric tensor is given by

$$g_{ij}(y) = \frac{4\delta_{ij}}{\left(1-|y|^2\right)^2}, \qquad y \in \mathbb{D}^m. \tag{6.4.13}$$

Prove that the diffeomorphism

$$\mathbb{D}^m \to \mathbb{H}^m : y \mapsto \left(\frac{1+|y|^2}{1-|y|^2}, \frac{2y}{1-|y|^2} \right)$$

is an isometry with inverse

$$\mathbb{H}^m \to \mathbb{D}^m : (x_0, x) \mapsto \frac{x}{1+x_0}.$$

Interpret this map as a stereographic projection from the *south pole* $(-1, 0)$.

Exercise 6.4.23. The composition of the isometries in Exercise 6.4.21 and Exercise 6.4.22 is the diffeomorphism $\mathbb{R}^m \to \mathbb{D}^m : x \mapsto y$ given by

$$y = \frac{x}{\sqrt{1+|x|^2}+1}, \qquad x = \frac{2y}{1-|y|^2}, \qquad \sqrt{1+|x|^2} = \frac{1+|y|^2}{1-|y|^2}.$$

Prove that this is an isometry intertwining the Riemannian metrics (6.4.12) and (6.4.13). Find a formula for the geodesics in the Poincaré disc $\mathbb{D}^m$. **Hint:** Use Exercise 6.4.25 below.

Exercise 6.4.24. Prove that the isometry group of $\mathbb{H}^m$ is the pseudo-orthogonal group

$$\mathcal{I}(\mathbb{H}^m) = \mathrm{O}(m, 1) := \left\{ g \in \mathrm{GL}(m+1) \;\middle|\; \begin{array}{l} Q(gv, gw) = Q(v, w) \\ \text{for all } v, w \in \mathbb{R}^{m+1} \end{array} \right\}.$$

Thus, by Corollary 6.4.13, the orthonormal frame bundle $\mathcal{O}(\mathbb{H}^m)$ is diffeomorphic to $\mathrm{O}(m, 1)$.

Exercise 6.4.25. Prove that the exponential map

$$\exp_p : T_p\mathbb{H}^m \to \mathbb{H}^m$$

is given by

$$\exp_p(v) = \cosh\left(\sqrt{Q(v,v)}\right) p + \frac{\sinh\left(\sqrt{Q(v,v)}\right)}{\sqrt{Q(v,v)}} v \tag{6.4.14}$$

for $v \in T_p\mathbb{H}^m = p^{\perp_Q}$. Prove that this map is a diffeomorphism for every $p \in \mathbb{H}^m$. Thus any two points in $\mathbb{H}^m$ are connected by a unique geodesic. Prove that the intrinsic distance function on hyperbolic space is given by

$$d(p, q) = \cosh^{-1}\left(Q(p, q)\right) \tag{6.4.15}$$

for $p, q \in \mathbb{H}^m$. Compare this with Example 4.3.11.

6.5 Nonpositive Sectional Curvature

In the previous section we have seen that any two points in a connected, simply connected, complete manifold M of constant negative curvature can be connected by a unique geodesic (see Exercise 6.4.25). Thus the entire manifold M is geodesically convex and its injectivity radius is infinity. This continues to hold in much greater generality for manifolds with nonpositive sectional curvature. It is convenient, at this point, to extend the discussion to Riemannian manifolds in the intrinsic setting. In particular, at some point in the proof of the main theorem of this section and in our main example, we shall work with a Riemannian metric that does not arise (in any obvious way) from an embedding.

Definition 6.5.1. *A Riemannian manifold M is said to have* **nonpositive sectional curvature** *if $K(p,E) \le 0$ for every $p \in M$ and every 2-dimensional linear subspace $E \subset T_pM$ or, equivalently, $\langle R_p(u,v)v,u\rangle \le 0$ for all $p \in M$ and all $u,v \in T_pM$.*

6.5.1 The Theorem of Hadamard and Cartan

The next theorem shows that every connected, simply connected, complete Riemannian manifold with nonpositive sectional curvature is diffeomorphic to Euclidean space and has infinite injectivity radius. This is in sharp contrast to positive curvature manifolds as the example $M = S^m$ shows.

Theorem 6.5.2 (Cartan–Hadamard). *Let M be a connected, simply connected, complete Riemannian manifold. The following are equivalent.*

(i) *M has nonpositive sectional curvature.*

(ii) *The derivative of each exponential map is length increasing, i.e.*

$$\left|d\exp_p(v)\hat{v}\right| \ge |\hat{v}|$$

for all $p \in M$ and all $v, \hat{v} \in T_pM$.

(iii) *Each exponential map is* **distance increasing***, i.e.*

$$d(\exp_p(v_0), \exp_p(v_1)) \ge |v_0 - v_1|$$

for all $p \in M$ and all $v_0, v_1 \in T_pM$.

Moreover, if these equivalent conditions are satisfied then the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism for every $p \in M$. Thus any two points in M can be connected by a unique geodesic.

Proof. See page 279. □

Lemma 6.5.3. *Let M and M' be connected, simply connected, complete Riemannian manifolds and $\phi : M \to M'$ be a local isometry. Then ϕ is bijective and hence is an isometry.*

Proof. This follows by combining the weak and strong versions of the global C-A-H Theorem 6.1.7. Fix a point $p_0 \in M$ and define

$$p_0' := \phi(p_0), \qquad \Phi_0 := d\phi(p_0).$$

Then the tuple M, M', p_0, p_0', Φ_0 satisfies condition (i) of the weak version of Theorem 6.1.7. Hence this tuple also satisfies condition (iv) of Theorem 6.1.7. Since M and M' are connected, simply connected, and complete we may apply the strong version of Theorem 6.1.7 to obtain an isometry

$$\psi : M \to M'$$

satisfying

$$\psi(p_0) = p_0', \qquad d\psi(p_0) = \Phi_0.$$

Since every isometry is also a local isometry and M is connected it follows from Lemma 5.1.10 that $\phi(p) = \psi(p)$ for all $p \in M$. Hence ϕ is an isometry, as required. □

Remark 6.5.4. Refining the argument in the proof of Lemma 6.5.3 one can show that a local isometry $\phi : M \to M'$ must be surjective whenever M is complete and M' is connected. None of these assumptions can be removed. (Take an isometric embedding of a disc in the plane or an embedding of a complete space M into a space with two components, one of which is isometric to M.)

Likewise, one can show that a local isometry $\phi : M \to M'$ must be injective whenever M is complete and connected and M' is simply connected. Again none of these asumptions can be removed. (Take a covering $\mathbb{R} \to S^1$, or a covering of a disjoint union of two isometric complete simply connected spaces onto one copy of this space, or some noninjective immersion of a disc into the plane and choose the pullback metric on the disc.)

Exercise 6.5.5. Let $\xi : [0, \infty) \to \mathbb{R}^n$ be a smooth function such that

$$\xi(0) = 0, \qquad \dot{\xi}(0) \neq 0, \qquad \xi(t) \neq 0 \quad \forall\, t > 0.$$

Prove that the function $f : [0, \infty) \to \mathbb{R}$ given by $f(t) := |\xi(t)|$ is smooth. **Hint:** The function $\eta : [0, \infty) \to \mathbb{R}^n$ defined by

$$\eta(t) := \begin{cases} t^{-1}\xi(t), & \text{for } t > 0, \\ \dot{\xi}(0), & \text{for } t = 0, \end{cases}$$

is smooth. Show that f is differentiable and $\dot{f} = |\eta|^{-1} \langle \eta, \dot{\xi} \rangle$.

Exercise 6.5.6. Let $\xi : \mathbb{R} \to \mathbb{R}^n$ be a smooth function such that

$$\xi(0) = 0, \qquad \ddot{\xi}(0) = 0.$$

Prove that there are constant $\varepsilon > 0$ and $c > 0$ such that, for all $t \in \mathbb{R}$:

$$|t| < \varepsilon \qquad \Longrightarrow \qquad |\xi(t)|^2 \, |\dot{\xi}(t)|^2 - \langle \xi(t), \dot{\xi}(t) \rangle^2 \le c \, |t|^6 .$$

Hint: Write $\xi(t) = tv + \eta(t)$ and $\dot{\xi}(t) = v + \dot{\eta}(t)$ with $\eta(t) = O(t^3)$ and $\dot{\eta}(t) = O(t^2)$. Show that the terms of order 2 and 4 cancel in the Taylor expansion at $t = 0$.

Proof of Theorem 6.5.2. We prove that (i) implies (ii). Let $p \in M$ and $v, \hat{v} \in T_pM$ be given. Assume without loss of generality that $\hat{v} \neq 0$ and define $\gamma : \mathbb{R} \to M$ and $X \in \mathrm{Vect}(\gamma)$ by

$$\gamma(t) := \exp_p(tv), \qquad X(t) := \left.\frac{\partial}{\partial \lambda}\right|_{\lambda=0} \exp_p(t(v + \lambda \hat{v})) \in T_{\gamma(t)}M \tag{6.5.1}$$

for $t \in \mathbb{R}$. Then

$$X(0) = 0, \qquad \nabla X(0) = \hat{v} \neq 0, \qquad X(t) = d\exp_p(tv)t\hat{v}, \tag{6.5.2}$$

and, by Lemma 6.1.15, X is a Jacobi field along γ:

$$\nabla\nabla X = R(\dot{\gamma}, X)\dot{\gamma}. \tag{6.5.3}$$

It follows from Exercise 6.5.5 with $\xi(t) := \Phi_\gamma(0,t)X(t)$ that the function $[0, \infty) \to \mathbb{R} : t \mapsto |X(t)|$ is smooth and

$$\left.\frac{d}{dt}\right|_{t=0} |X(t)| = |\nabla X(0)| = |\hat{v}| .$$

Moreover, for $t > 0$, we have

$$\begin{aligned}
\frac{d^2}{dt^2} |X| &= \frac{d}{dt} \frac{\langle X, \nabla X \rangle}{|X|} \\
&= \frac{|\nabla X|^2 + \langle X, \nabla\nabla X \rangle}{|X|} - \frac{\langle X, \nabla X \rangle^2}{|X|^3} \\
&= \frac{|X|^2 |\nabla X|^2 - \langle X, \nabla X \rangle^2}{|X|^3} + \frac{\langle X, R(\dot{\gamma}, X)\dot{\gamma} \rangle}{|X|} \\
&\ge 0.
\end{aligned} \tag{6.5.4}$$

Here the third equation follows from the fact that X is a Jacobi field along γ, and the last inequality follows from the nonpositive sectional curvature condition in (i) and from the Cauchy–Schwarz inequality. Thus the second derivative of the function $[0,\infty) \to \mathbb{R} : t \mapsto |X(t)| - t\,|\hat{v}|$ is nonnegative; so its first derivative is nondecreasing and it vanishes at $t=0$; thus

$$|X(t)| - t\,|\hat{v}| \geq 0$$

for every $t \geq 0$. In particular, for $t=1$ we obtain

$$\left|d\exp_p(v)\hat{v}\right| = |X(1)| \geq |\hat{v}|\,.$$

as claimed. Thus we have proved that (i) implies (ii).

We prove that (ii) implies (i). Assume, by contradiction, that (ii) holds but there is a point $p \in M$ and a pair of vectors $v, \hat{v} \in T_pM$ such that

$$\langle R_p(v,\hat{v})v,\hat{v}\rangle < 0. \tag{6.5.5}$$

Define $\gamma : \mathbb{R} \to M$ and $X \in \mathrm{Vect}(\gamma)$ by (6.5.1) so that (6.5.2) and (6.5.3) are satisfied. Thus X is a Jacobi field with

$$X(0) = 0, \qquad \nabla X(0) = \hat{v} \neq 0.$$

Hence it follows from Exercise 6.5.6 with $\xi(t) := \Phi_\gamma(0,t)X(t)$ that there is a constant $c > 0$ such that, for $t > 0$ sufficiently small, we have the inequality

$$|X(t)|^2\,|\nabla X(t)|^2 - \langle X(t), \nabla X(t)\rangle^2 \leq ct^6.$$

Moreover,

$$|X(t)| \geq \delta t, \qquad \langle X(t), R(\dot{\gamma}(t), X(t))\dot{\gamma}(t)\rangle \leq -\varepsilon t^2,$$

for t sufficiently small, where the second inequality follows from (6.5.5). Hence, by (6.5.4), we have

$$\frac{d^2}{dt^2}\,|X| = \frac{|X|^2|\nabla X|^2 - \langle X, \nabla X\rangle^2}{|X|^3} + \frac{\langle X, R(\dot{\gamma}, X)\dot{\gamma}\rangle}{|X|} \leq \frac{ct^3}{\delta^3} - \frac{\varepsilon t}{\delta}.$$

Integrating this inequality over an interval $[0,t]$ with $ct^2 < \varepsilon\delta^2$ we get

$$\frac{d}{dt}\,|X(t)| < \left.\frac{d}{dt}\right|_{t=0} |X(t)| = |\nabla X(0)|$$

Integrating this inequality again gives $|X(t)| < t\,|\nabla X(0)|$ for small t and so

$$\left|d\exp_p(tv)t\hat{v}\right| = |X(t)| < t\,|\nabla X(0)| = t\,|\hat{v}|\,.$$

This contradicts (ii).

We prove that (ii) implies that the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism for every $p \in M$. By (ii) $\exp_p$ is a *local diffeomorphism*, i.e. its derivative $d\exp_p(v) : T_pM \to T_{\exp_p(v)}M$ is bijective for every $v \in T_pM$. Hence we can define a metric on $M' := T_pM$ by pulling back the metric on M under the exponential map. To make this more explicit we choose a basis $e_1, \dots, e_m$ of T_pM and define the map $\psi : \mathbb{R}^m \to M$ by

$$\psi(x) := \exp_p\left(\sum_{i=1}^m x^i e_i\right)$$

for $x = (x^1, \dots, x^m) \in \mathbb{R}^m$. Define the metric tensor by

$$g_{ij}(x) := \left\langle \frac{\partial\psi}{\partial x^i}(x), \frac{\partial\psi}{\partial x^j}(x)\right\rangle, \qquad i,j = 1,\dots,m.$$

Then $(\mathbb{R}^m, g)$ is a Riemannian manifold (covered by a single coordinate chart) and $\psi : (\mathbb{R}^m, g) \to M$ is a local isometry, by definition of g. The manifold $(\mathbb{R}^m, g)$ is clearly connected and simply connected. Moreover, for every tangent vector $\xi = (\xi^1, \dots, \xi^n) \in \mathbb{R}^m = T_0\mathbb{R}^m$, the curve $\mathbb{R} \to \mathbb{R}^m : t \mapsto t\xi$ is a geodesic with respect to g (because ψ is a local isometry and the image of the curve under ψ is a geodesic in M). Hence it follows from Theorem 4.6.5 that $(\mathbb{R}^m, g)$ is complete. Since both $(\mathbb{R}^m, g)$ and M are connected, simply connected, and complete, the local isometry ψ is bijective, by Lemma 6.5.3. Thus the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism as claimed. It follows that any two points in M are connected by a unique geodesic.

We prove that (ii) implies (iii). Fix a point $p \in M$ and two tangent vectors $v_0, v_1 \in T_pM$. Let $\gamma : [0,1] \to M$ be the geodesic with endpoints $\gamma(0) = \exp_p(v_0)$ and $\gamma(1) = \exp_p(v_1)$ and let $v : [0,1] \to T_pM$ be the unique curve satisfying $\exp_p(v(t)) = \gamma(t)$ for all t. Then $v(0) = v_0$, $v(1) = v_1$, and

$$\begin{aligned}
d(\exp_p(v_0), \exp_p(v_1)) &= L(\gamma) \\
&= \int_0^1 \left|d\exp_p(v(t))\dot v(t)\right| \, dt \\
&\geq \int_0^1 |\dot v(t)| \, dt \\
&\geq \left|\int_0^1 \dot v(t)\, dt\right| \\
&= |v_1 - v_0|.
\end{aligned}$$

Here the third inequality follows from (ii). This shows that (ii) implies (iii).

We prove that (iii) implies (ii). Fix a point $p \in M$ and a tangent vector $v \in T_pM$ and denote

$$q := \exp_p(v).$$

By (iii) the exponential map $\exp_q : T_qM \to M$ is injective and, since M is complete, it is bijective (see Theorem 4.6.6). Hence there is a unique geodesic from q to any other point in M and therefore, by Theorem 4.5.4, we have

$$|w| = d(q, \exp_q(w)) \tag{6.5.6}$$

for every $w \in T_qM$. Now define

$$\phi := \exp_q^{-1} \circ \exp_p : T_pM \to T_qM.$$

This map satisfies

$$\phi(v) = 0.$$

Moreover, it is differentiable in a neighborhood of v and, by the chain rule, we have

$$d\phi(v) = d\exp_p(v) : T_pM \to T_qM.$$

Now choose $w := \phi(v + \hat{v})$ in (6.5.6) with $\hat{v} \in T_pM$. Then

$$\exp_q(w) = \exp_q(\phi(v + \hat{v})) = \exp_p(v + \hat{v})$$

and hence

$$|\phi(v + \hat{v}))| = d(\exp_p(v), \exp_p(v + \hat{v})) \geq |\hat{v}|,$$

where the last inequality follows from (iii). This gives

$$\begin{aligned} \left|d\exp_p(v)\hat{v}\right| &= |d\phi(v)\hat{v}| \\ &= \lim_{t \to 0} \frac{|\phi(v + t\hat{v})|}{t} \\ &\geq \lim_{t \to 0} \frac{|t\hat{v}|}{t} \\ &= |\hat{v}|. \end{aligned}$$

Thus we have proved that (iii) implies (ii). This completes the proof of Theorem 6.5.2. □

6.5.2 Cartan's Fixed Point Theorem

Theorem 6.5.7 (Cartan). *Let M be a complete, connected, simply connected Riemannian manifold with nonpositive sectional curvature. Let G be a compact topological group that acts on M by isometries. Then there exists a point $p \in M$ such that $gp = p$ for every $g \in \mathrm{G}$.*

Proof. The proof has three steps and follows the argument given by Bill Casselmann in [3]. The second step is Serre's uniqueness result for the *circumcentre* of a bounded set in a *semi-hyperbolic space*.

Step 1. *Let $m \in M$ and $v \in T_mM$ and define*

$$p_0 := \exp_m(-v), \qquad p_1 := \exp_m(v).$$

Then

$$2d(m,q)^2 + \frac{d(p_0,p_1)^2}{2} \le d(p_0,q)^2 + d(p_1,q)^2$$

for every $q \in M$.

By Theorem 6.5.2 the exponential map $\exp_m : T_mM \to M$ is a diffeomorphism. Hence $d(p_0,p_1) = 2|v|$. Now let $q \in M$. Then there is a unique tangent vector $w \in T_mM$ such that

$$q = \exp_m(w), \qquad d(m,q) = |w|.$$

Since the exponential map is expanding, by Theorem 6.5.2, we have

$$d(p_0,q) \ge |w+v|, \qquad d(p_1,q) \ge |w-v|.$$

Hence

$$\begin{aligned} d(m,q)^2 &= |w|^2 \\ &= \frac{|w+v|^2 + |w-v|^2}{2} - |v|^2 \\ &\le \frac{d(p_0,q)^2 + d(p_1,q)^2}{2} - \frac{d(p_0,p_1)^2}{4}. \end{aligned}$$

This proves Step 1.

Step 2. *For $p \in M$ and $r \ge 0$ denote by $B(p,r) \subset M$ the closed ball of radius r centered at p. Let $\Omega \subset M$ be a nonempty bounded set and define*

$$r_\Omega := \inf\{r > 0 \,|\, \textit{there exists a } p \in M \textit{ such that } \Omega \subset B(p,r)\}$$

Then there exists a unique point $p_\Omega \in M$ such that $\Omega \subset B(p_\Omega, r_\Omega)$.

We prove existence. Choose a sequence $r_i > r_\Omega$ and a sequence $p_i \in M$ such that

$$\Omega \subset B(p_i, r_i), \qquad \lim_{i\to\infty} r_i = r_\Omega.$$

Choose $q \in \Omega$. Then $d(q, p_i) \le r_i$ for every i. Since the sequence r_i is bounded and M is complete, it follows that p_i has a convergent subsequence, still denoted by p_i. Its limit $p_\Omega := \lim_{i\to\infty} p_i$ satisfies $\Omega \subset B(p_\Omega, r_\Omega)$.

We prove uniqueness. Let $p_0, p_1 \in M$ such that

$$\Omega \subset B(p_0, r_\Omega) \cap B(p_1, r_\Omega).$$

Since the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism, by Theorem 6.5.2, there exists a unique vector $v_0 \in T_{p_0}M$ such that $p_1 = \exp_{p_0}(v_0)$. Denote the midpoint between p_0 and p_1 by

$$m := \exp_{p_0}\left(\tfrac{1}{2}v_0\right).$$

Then it follows from Step 1 that

$$\begin{aligned} d(m,q)^2 &\le \frac{d(p_0,q)^2 + d(p_1,q)^2}{2} - \frac{d(p_0,p_1)^2}{4} \\ &\le r_\Omega^2 - \frac{d(p_0,p_1)^2}{4} \end{aligned}$$

for every $q \in \Omega$. Since $\sup_{q\in\Omega} d(m,q) \ge r_\Omega$, by definition of r_Ω, it follows that $d(p_0, p_1) = 0$ and hence $p_0 = p_1$. This proves Step 2.

Step 3. *We prove Theorem 6.5.7.*

Let $q \in M$ and consider the group orbit $\Omega := \{gq \,|\, g \in \mathrm{G}\}$. Since G is compact, this set is bounded. Let $r_\Omega \ge 0$ and $p_\Omega \in M$ be as in Step 2. Then

$$\Omega \subset B(p_\Omega, r_\Omega).$$

Since G acts on M by isometries, this implies

$$\Omega = g\Omega \subset B(gp_\Omega, r_\Omega)$$

for every $g \in \mathrm{G}$. Hence it follows from the uniqueness statement in Step 2 that $gp_\Omega = p_\Omega$ for every $g \in \mathrm{G}$. This proves Step 3 and Theorem 6.5.7. □

6.5.3 Positive Definite Symmetric Matrices

We close this section with an example of a nonpositive sectional curvature manifold which plays a key role in Donaldson's beautiful paper on Lie algebra theory [4]. Let n be a positive integer and consider the space

$$\mathscr{P} := \left\{ P \in \mathbb{R}^{n\times n} \,|\, P^{\mathsf{T}} = P > 0 \right\}$$

of positive definite symmetric $n\times n$-matrices. (The notation "$P > 0$" means $\langle x, Px\rangle > 0$ for every nonzero vector $x \in \mathbb{R}^n$.) Thus $\mathscr{P}$ is an open subset of the vector space

$$\mathscr{S} := \left\{ S \in \mathbb{R}^{n\times n} \,|\, S^{\mathsf{T}} = S \right\}$$

of symmetric matrices and hence the tangent space of $\mathscr{P}$ is $T_P\mathscr{P} = \mathscr{S}$ for every $P \in \mathscr{P}$. However, we do not use the metric inherited from the inclusion into $\mathscr{S}$ but define a Riemannian metric by

$$\langle S_1, S_2\rangle_P := \operatorname{trace}\left(S_1P^{-1}S_2P^{-1}\right) \tag{6.5.7}$$

for $P \in \mathscr{P}$ and $S_1, S_2 \in \mathscr{S} = T_P\mathscr{P}$.

Theorem 6.5.8. *The space $\mathscr{P}$ with the Riemannian metric* (6.5.7) *is a connected, simply connected, complete Riemannian manifold with nonpositive sectional curvature. Moreover, $\mathscr{P}$ is a symmetric space and the group* $\mathrm{GL}(n,\mathbb{R})$ *of nonsingular $n \times n$-matrices acts on $\mathscr{P}$ by isometries via*

$$g_*P := gPg^{\mathsf{T}} \tag{6.5.8}$$

for $g \in \mathrm{GL}(n,\mathbb{R})$ and $P \in \mathscr{P}$.

Proof. See page 286 □

Remark 6.5.9. The paper [4] by Donaldson contains an elementary direct proof that the manifold $\mathscr{P}$ with the metric (6.5.7) satisfies the assertions of Theorem 6.5.2.

Remark 6.5.10. The submanifold

$$\mathscr{P}_0 := \{P \in \mathscr{P} \,|\, \det(P) = 1\}$$

of positive definite symmetric matrices with determinant one is totally geodesic (see Remark 6.5.11 below). Hence all the assertions of Theorem 6.5.8 (with $\mathrm{GL}(n,\mathbb{R})$ replaced by $\mathrm{SL}(n,\mathbb{R})$) remain valid for $\mathscr{P}_0$.

Remark 6.5.11. Let M be a Riemannian manifold and $L \subset M$ be a submanifold. Then the following are equivalent.
(i) If $\gamma : I \to M$ is a geodesic on an open interval I such that $0 \in I$ and

$$\gamma(0) \in L, \qquad \dot\gamma(0) \in T_{\gamma(0)}L,$$

then there is a constant $\varepsilon > 0$ such that $\gamma(t) \in L$ for $|t| < \varepsilon$.
(ii) If $\gamma : I \to L$ is a smooth curve on an open interval I and Φ_γ denotes parallel transport along γ in M then

$$\Phi_\gamma(t,s)T_{\gamma(s)}L = T_{\gamma(t)}L \qquad \forall\, s,t \in I.$$

(iii) If $\gamma : I \to L$ is a smooth curve on an open interval I and $X \in \mathrm{Vect}(\gamma)$ is a vector field along γ (with values in TM) then

$$X(t) \in T_{\gamma(t)}L \quad \forall\, t \in I \qquad \Longrightarrow \qquad \nabla X(t) \in T_{\gamma(t)}L \quad \forall\, t \in I.$$

A submanifold that satisfies these equivalent conditions is called **totally geodesic**

Exercise 6.5.12. Prove the equivalence of (i), (ii), (iii) in Remark 6.5.11. **Hint:** Choose suitable coordinates and translate each of the three assertions into conditions on the Christoffel symbols.

Exercise 6.5.13. Prove that $\mathscr{P}_0$ is a totally geodesic submanifold of $\mathscr{P}$. Prove that, in the case $n = 2$, $\mathscr{P}_0$ is isometric to the hyperbolic space $\mathbb{H}^2$.

Proof of Theorem 6.5.8. The manifold $\mathscr{P}$ is obviously connected and simply connected as it is a convex open subset of a finite dimensional vector space. The remaining assertions will be proved in five steps.

Step 1. *Let $I \to \mathscr{P} : t \mapsto P(t)$ be a smoth path in $\mathscr{P}$ and $I \to \mathscr{S} : t \mapsto S(t)$ be a vector field along P. Then the covariant derivative of S is given by*

$$\nabla S = \dot S - \frac{1}{2}SP^{-1}\dot P - \frac{1}{2}\dot P P^{-1}S. \tag{6.5.9}$$

The formula (6.5.9) determines a family of linear operators on the spaces of vector fields along paths that satisfy the torsion-free condition

$$\nabla_s \partial_t P = \nabla_t \partial_s P$$

for every smooth map $\mathbb{R}^2 \to \mathscr{P} : (s,t) \mapsto P(s,t)$ and the Leibniz rule

$$\nabla \langle S_1, S_2\rangle_P = \langle \nabla S_1, S_2\rangle_P + \langle S_1, \nabla S_2\rangle_P$$

for any two vector fields S_1 and S_2 along P. These two conditions determine the covariant derivative uniquely (see Lemma 3.6.5 and Theorem 3.7.8).

Step 2. *The geodesics in $\mathscr{P}$ are given by*

$$\begin{aligned}\gamma(t) &= P\exp(tP^{-1}S)\\ &= \exp(tSP^{-1})P\\ &= P^{1/2}\exp(tP^{-1/2}SP^{-1/2})P^{1/2}\end{aligned}\tag{6.5.10}$$

for $P \in \mathscr{P}$, $S \in \mathscr{S} = T_P\mathscr{P}$, and $t \in \mathbb{R}$. In particular $\mathscr{P}$ is complete.

The curve $\gamma : \mathbb{R} \to \mathscr{P}$ defined by (6.5.10) satisfies

$$\dot\gamma(t) = S\exp(tP^{-1}S) = SP^{-1}\gamma(t).$$

Hence it follows from Step 1 that

$$\nabla\dot\gamma(t) = \ddot\gamma(t) - \dot\gamma(t)\gamma(t)^{-1}\dot\gamma(t) = \ddot\gamma(t) - SP^{-1}\dot\gamma(t) = 0$$

for every $t \in \mathbb{R}$. Hence γ is a geodesic. Note also that the curve $\gamma : \mathbb{R} \to \mathscr{P}$ in (6.5.10) satisfies $\gamma(0) = P$ and $\dot\gamma(0) = S$.

Step 3. *The curvature tensor on $\mathscr{P}$ is given by*

$$\begin{aligned}R_P(S,T)A = &-\frac{1}{4}SP^{-1}TP^{-1}A - \frac{1}{4}AP^{-1}TP^{-1}S\\ &+ \frac{1}{4}TP^{-1}SP^{-1}A + \frac{1}{4}AP^{-1}SP^{-1}T\end{aligned}\tag{6.5.11}$$

for $P \in \mathscr{P}$ and $S, T, A \in \mathscr{S}$.

Choose smooth maps $P : \mathbb{R}^2 \to \mathscr{P}$ and $A : \mathbb{R}^2 \to \mathscr{S}$ (understood as a vector field along P) and denote $S := \partial_s P$ and $T := \partial_t P$. Then

$$R(S,T)A = \nabla_s\nabla_t A - \nabla_t\nabla_s A$$

and $\partial_s T = \partial_t S$. By Step 1 we have

$$\begin{aligned}\nabla_s A &= \partial_s A - \frac{1}{2}AP^{-1}S - \frac{1}{2}SP^{-1}A,\\ \nabla_t A &= \partial_t A - \frac{1}{2}AP^{-1}T - \frac{1}{2}TP^{-1}A,\end{aligned}$$

and hence

$$\begin{aligned}R(S,T)A \;\;=\;\; &\partial_s\nabla_t A - \frac{1}{2}(\nabla_t A)P^{-1}S - \frac{1}{2}SP^{-1}(\nabla_t A)\\ &- \partial_t\nabla_s A + \frac{1}{2}(\nabla_s A)P^{-1}T + \frac{1}{2}TP^{-1}(\nabla_s A).\end{aligned}$$

Now Step 3 follows by a direct calculation which we leave to the reader.

Step 4. *The manifold $\mathscr{P}$ has nonpositive sectional curvature.*

By Step 3 with $A = T$ and equation (6.5.7) we have

$$\begin{aligned}
\langle S, R_P(S,T)T\rangle_P &= \operatorname{trace}\left(SP^{-1}R_P(S,T)TP^{-1}\right) \\
&= -\frac{1}{4}\operatorname{trace}\left(SP^{-1}SP^{-1}TP^{-1}TP^{-1}\right) \\
&\quad -\frac{1}{4}\operatorname{trace}\left(SP^{-1}TP^{-1}TP^{-1}SP^{-1}\right) \\
&\quad +\frac{1}{4}\operatorname{trace}\left(SP^{-1}TP^{-1}SP^{-1}TP^{-1}\right) \\
&\quad +\frac{1}{4}\operatorname{trace}\left(SP^{-1}TP^{-1}SP^{-1}TP^{-1}\right) \\
&= -\frac{1}{2}\operatorname{trace}\left(SP^{-1}TP^{-1}TP^{-1}SP^{-1}\right) \\
&\quad +\frac{1}{2}\operatorname{trace}\left(SP^{-1}TP^{-1}SP^{-1}TP^{-1}\right) \\
&= -\frac{1}{2}\operatorname{trace}\left(X^\mathsf{T}X\right) + \frac{1}{2}\operatorname{trace}\left(X^2\right),
\end{aligned}$$

where $X := SP^{-1}TP^{-1}$. Write $X =: (x_{ij})_{i,j=1,\dots,n}$. Then, by the Cauchy–Schwarz inequality, we have

$$\operatorname{trace}(X^2) = \sum_{i,j} x_{ij}x_{ji} \le \sum_{i,j} x_{ij}^2 = \operatorname{trace}(X^\mathsf{T}X)$$

for every matrix $X \in \mathbb{R}^{n\times n}$. Hence $\langle S, R_P(S,T)T\rangle_P \le 0$ for all $P \in \mathscr{P}$ and all $S,T \in \mathscr{S}$. This proves Step 4.

Step 5. *$\mathscr{P}$ is a symmetric space.*

Given $A \in \mathscr{P}$ define the map $\phi : \mathscr{P} \to \mathscr{P}$ by

$$\phi(P) := AP^{-1}A.$$

This map is a diffeomorphism, fixed the matrix $A = \phi(A)$, and satisfies

$$d\phi(P)S = -AP^{-1}SP^{-1}A$$

for $P \in \mathscr{P}$ and $S \in \mathscr{S}$. Hence $d\phi(A) = -\mathrm{id}$ and, for all $P \in \mathscr{P}$ and all $S \in \mathscr{S}$, we have

$$(d\phi(P)S)\phi(P)^{-1} = -AP^{-1}SA^{-1}$$

and therefore

$$|d\phi(P)S|^2_{\phi(P)} = \operatorname{trace}\left(\left(AP^{-1}SA^{-1}\right)^2\right) = \operatorname{trace}\left(\left(P^{-1}S\right)^2\right) = |S|^2_P.$$

Hence ϕ is an isometry and this proves Theorem 6.5.8. $\square$

Remark 6.5.14. The space $\mathscr{P}$ can be identified with the quotient space $\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n)$ via polar decomposition.

Remark 6.5.15. Theorem 6.5.8 carries over verbatim to the complex setting. Just replace $\mathscr{P}$ by the space $\mathscr{H}$ of positive definite Hermitian matrices

$$H = H^* > 0,$$

where H^* denotes the conjugate transposed matrix of $H \in \mathbb{C}^{n\times n}$. The inner product is then defined by the same formula as in the real case, namely

$$\left\langle \hat{H}_1, \hat{H}_2 \right\rangle_H := \mathrm{trace}\left(\hat{H}_1 H^{-1} \hat{H}_2 H^{-1}\right)$$

for $H \in \mathscr{H}$ and two Hermitian matrices $\hat{H}_1, \hat{H}_2 \in T_H\mathscr{H}$. The assertions of Theorem 6.5.8 remain valid with $\mathrm{GL}(n,\mathbb{R})$ replaced by $\mathrm{GL}(n,\mathbb{C})$. This space $\mathscr{H}$ can be identified with the quotient $\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$ and, likewise, the subspace $\mathscr{H}_0$ of positive definite Hermitian matrices with determinant one can be identified with the quotient $\mathrm{SL}(n,\mathbb{C})/\mathrm{SU}(n)$. This quotient (with nonpositive sectional curvature) can be viewed as a kind of dual of the Lie group $\mathrm{SU}(n)$ (with nonnegative sectional curvature). **Exercise:** Prove this! Show that, in the case $n = 2$, the space $\mathscr{H}_0$ is isometric to hyperbolic 3-space.

6.6 Positive Ricci Curvature

In this section we prove that every complete connected manifold $M \subset \mathbb{R}^n$ whose Ricci curvature satisfies a uniform positive lower bound is necessarily compact. If the sectional curvature is constant and positive, this follows from Corollary 6.4.12 as was noted in Example 6.4.16.

Definition 6.6.1. *Let $M \subset \mathbb{R}^n$ be an m-dimensional submanifold and fix an element $p \in M$. The* **Ricci tensor** *of M at p is the symmetric bilinear form*

$$\mathrm{Ric}_p : T_pM \times T_pM \to \mathbb{R}$$

defined by

$$\mathrm{Ric}_p(u,v) := \sum_{i=1}^m \langle R_p(e_i,u)v, e_i\rangle,$$

where $e_1, \dots, e_m$ is an orthonormal basis of T_pM. The Ricci tensor is independent of the choice of this orthonormal frame and is symmetric by equations (5.2.16) *and* (5.2.18) *in Theorem 5.2.13.*

Theorem 6.6.2 (Bonnet–Myers). *Let $M \subset \mathbb{R}^n$ be a complete manifold and suppose that there exists a constant $\delta > 0$ such that*

$$\mathrm{Ric}_p(v,v) \geq (m-1)\delta\,|v|^2 \tag{6.6.1}$$

for every $p \in M$ and every $v \in T_pM$. Then $d(p,q) \leq \pi/\sqrt{\delta}$ for all $p,q \in M$ and hence M is compact.

Proof. See page 291. □

Lemma 6.6.3. *Let $\mathbb{R} \times [0,1] \to M : (s,t) \mapsto \gamma_s(t)$ be a smooth map such that $\gamma := \gamma_0 : [0,1] \to M$ is a geodesic and $\gamma_s(0) = \gamma_0(0)$ and $\gamma_s(1) = \gamma_0(1)$ for all $s \in \mathbb{R}$. Define $X \in \mathrm{Vect}(\gamma)$ by*

$$X(t) := \left.\frac{\partial}{\partial s}\right|_{s=0} \gamma_s(t)$$

for $0 \leq t \leq 1$. Then

$$\left.\frac{d^2}{ds^2}\right|_{s=0} E(\gamma_s) = \int_0^1 \Big(|\nabla_t X|^2 - \langle R(X,\dot\gamma)\dot\gamma, X\rangle \Big)\,dt. \tag{6.6.2}$$

Proof. In the proof of Theorem 4.1.4 we have seen that

$$\frac{d}{ds} E(\gamma_s) = -\int_0^1 \langle \nabla_t \partial_t \gamma_s(t), \partial_s \gamma_s(t)\rangle\,dt$$

for all $s \in \mathbb{R}$ (see equation (4.1.10)). Differentiate this equation again with respect to s and use the identity $\nabla_s \partial_t = \nabla_t \partial_s$ to obtain

$$\begin{aligned}
\frac{d^2}{ds^2} E(\gamma_s) &= -\left.\frac{d}{ds}\right|_{s=0} \int_0^1 \langle \nabla_t \partial_t \gamma_s, \partial_s \gamma_s\rangle\,dt \\
&= -\int_0^1 \langle \nabla_s \nabla_t \partial_t \gamma_s, \partial_s \gamma_s\rangle\,dt - \int_0^1 \langle \nabla_t \partial_t \gamma_s, \nabla_s \partial_s \gamma_s\rangle\,dt \\
&= -\int_0^1 \langle R(\partial_s \gamma_s, \partial_t \gamma_s)\partial_t \gamma_s, \partial_s \gamma_s\rangle\,dt \\
&\quad - \int_0^1 \langle \nabla_t \nabla_t \partial_s \gamma_s, \partial_s \gamma_s\rangle\,dt - \int_0^1 \langle \nabla_t \partial_t \gamma_s, \nabla_s \partial_s \gamma_s\rangle\,dt.
\end{aligned}$$

Now take $s = 0$ to obtain

$$\begin{aligned}
\left.\frac{d^2}{ds^2}\right|_{s=0} E(\gamma_s) &= -\int_0^1 \langle R(X,\dot\gamma)\dot\gamma, X\rangle\,dt - \int_0^1 \langle \nabla_t \nabla_t X, X\rangle\,dt \\
&= \int_0^1 \Big(|\nabla_t X|^2 - \langle R(X,\dot\gamma)\dot\gamma, X\rangle \Big)\,dt.
\end{aligned}$$

This proves Lemma 6.6.3. □

Proof of Theorem 6.6.2. Let $p, q \in M$. By the Hopf–Rinow Theorem 4.6.6 there exists a geodesic $\gamma : [0,1] \to M$ such that

$$\gamma(0) = p, \qquad \gamma(1) = q, \qquad L(\gamma) = d(p,q).$$

Let $X \in \mathrm{Vect}(\gamma)$ such that

$$X(0) = 0, \qquad X(1) = 0$$

and define $\gamma_s(t) := \exp_{\gamma(t)}(sX(t))$ for $s \in \mathbb{R}$ and $0 \le t \le 1$. Then $s = 0$ is the absolute minimum of the function $\mathbb{R} \to \mathbb{R} : s \mapsto E(\gamma_s)$ by Lemma 4.5.1. Hence $\frac{d^2}{ds^2}|_{s=0} E(\gamma_s) \ge 0$ and by Lemma 6.6.3 this implies

$$\int_0^1 \langle R(X, \dot\gamma)\dot\gamma, X\rangle \, dt \le \int_0^1 |\nabla_t X(t)|^2 \, dt. \tag{6.6.3}$$

Now choose an orthonormal frame $E_1, \dots, E_m$ along γ such that $E_1 = \dot\gamma/|\dot\gamma|$ and $\nabla_t E_i \equiv 0$ for $i = 1, \dots m$. Define

$$X_i(t) := \sin(\pi t) E_i(t)$$

for $i = 1, \dots m$ and $0 \le t \le 1$. Then $|\nabla_t X_i(t)| = \pi \cos(\pi t)$ for all i and t and

$$\delta(m-1) |\dot\gamma(t)|^2 \le \mathrm{Ric}_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t)) = \sum_{i=2}^m \langle R(E_i(t), \dot\gamma(t))\dot\gamma(t), E_i(t)\rangle$$

for $0 \le t \le 1$. Multiply this inequality by $\sin^2(\pi t)$, integrate over the unit interval, and use the identity $|\dot\gamma(t)| = d(p,q)$ to obtain

$$\begin{aligned}
\frac{\delta(m-1)}{2} d(p,q)^2 &= \int_0^1 \delta(m-1)\sin^2(\pi t) |\dot\gamma(t)|^2 \, dt \\
&\le \sum_{i=2}^m \int_0^1 \langle R(X_i(t), \dot\gamma(t))\dot\gamma(t), X_i(t)\rangle \, dt \\
&\le \sum_{i=2}^m \int_0^1 |\nabla_t X_i(t)|^2 \, dt \\
&= (m-1) \int_0^1 \pi^2 \cos^2(\pi t) \, dt \\
&= \frac{\pi^2(m-1)}{2}.
\end{aligned}$$

Here the second step follows from (6.6.3). Hence $d(p,q)^2 \le \pi^2/\delta$ and this proves Theorem 6.6.2. $\square$

Corollary 6.6.4. *Let $M \subset \mathbb{R}^n$ be a complete manifold and suppose that there exists a constant $\delta > 0$ such that*

$$K(p, E) \geq \delta \tag{6.6.4}$$

for every $p \in M$ and every 2-dimensional linear subspace $E \subset T_pM$. Then

$$d(p, q) \leq \frac{\pi}{\sqrt{\delta}}$$

for all $p, q \in M$ and hence M is compact.

Proof. The estimate (6.6.4) implies (6.6.1) and hence the assertion follows from Theorem 6.6.2. This proves Corollary 6.6.4. □

The example of the m-sphere shows that the estimate in Corollary 6.6.4 is sharp. Namely, $M := S^m$ satisfies (6.6.4) with $\delta = 1$ and has diameter π.

The Ricci Tensor in Local Coordinates

Let $\phi : U \to \Omega$ be a local coordinate chart on M with values in an open set $\Omega \subset \mathbb{R}^m$, denote its inverse by $\psi := \phi^{-1} : \Omega \to U$, and let

$$E_i(x) := \frac{\partial \psi}{\partial x^i}(x) \in T_{\psi(x)}M, \qquad x \in \Omega, \qquad i = 1, \ldots, m,$$

be the local frame of the tangent bundle determined by this coordinate chart. Let $\Gamma_{ij}^k : \Omega \to \mathbb{R}$ denote the Christoffel symbols and $R_{ijk}^\ell : \Omega \to \mathbb{R}$ the coefficients of the Riemann curvature tensor so that

$$\nabla_i E_j = \sum_k \Gamma_{ij}^k E_k, \qquad R(E_i, E_j)E_k = \sum_\ell R_{ijk}^\ell E_\ell.$$

Then

$$\mathrm{Ric}_{ij} := \mathrm{Ric}(E_i, E_j) = \sum_{\nu=1}^m R_{\nu ij}^\nu. \tag{6.6.5}$$

(**Exercise:** Verify this identity.)

Appendix A

Notes

A.1 Maps and Functions

The notation

$$f : X \to Y$$

means that f is a function which assigns to every point x in the set X a point $f(x)$ in the set Y. When $Y = \mathbb{R}$ we express this by saying that f is a *real valued* function defined on the set X and if Y is a vector space we may say that f is a *vector valued* function. However in general it is better to say that f is a **map** from X to Y and call the set X the **source** of the map and the set Y its **target**. The **graph** of f is the set

$$\mathrm{graph}(f) := \{(x, y) \in X \times Y \mid y = f(x)\}.$$

We always distinguish two maps with the same graph when their targets are different.

A map $f : X \to Y$ is said to be

$$\left\{\begin{array}{l} \textbf{injective} \\ \textbf{surjective} \\ \textbf{bijective} \end{array}\right\} \quad \text{iff} \quad \left\{\begin{array}{l} f(x_1) = f(x_2) \implies x_1 = x_2 \\ \forall y \in Y \exists x \in X \text{ s.t. } y = f(x) \\ \text{it is both injective and surjective.} \end{array}\right\}$$

Then

(a) f is injective $\iff$ it has a left inverse $g : Y \to X$ (i.e. $g \circ f = \mathrm{id}_X$);

(b) f is surjective $\iff$ it has a right inverse $g : Y \to X$ (i.e. $f \circ g = \mathrm{id}_Y$);

(c) f is bijective $\iff$ it has a two sided inverse $f^{-1} : Y \to X$.

(Item (b) is the *Axiom of Choice*.)

The analogous principle holds for linear maps: if $A \in \mathbb{R}^{m\times n}$ then the linear map $\mathbb{R}^n \to \mathbb{R}^m : x \mapsto Ax$ is

(a) injective $\iff$ $BA = \mathbb{1}_n$ for some $B \in \mathbb{R}^{n\times m}$;

(b) surjective $\iff$ $AB = \mathbb{1}_m$ for some $B \in \mathbb{R}^{n\times m}$;

(c) bijective $\iff$ A is invertible (i.e. $m = n$ and $\det(A) \neq 0$).

(Here $\mathbb{1}_k$ is the $k \times k$ identity matrix.) However, this principle fails completely for continuous maps: the map $f : [0, 2\pi) \to S^1$ defined by $f(\theta) = (\cos\theta, \sin\theta)$ is continuous and bijective but its inverse is not continuous. (Here $S^1 \subset \mathbb{R}^2$ is the unit circle $x^2 + y^2 = 1$.)

A.2 Normal Forms

The *Fundamental Idea of Differential Calculus* is that near a point $x_0 \in U$ a smooth map $f : U \to V$ behaves like its linear approximation, i.e.

$$f(x) \approx f(x_0) + df(x_0)(x - x_0).$$

The *Normal Form Theorem from Linear Algebra* says that if $A \in \mathbb{R}^{m\times n}$ has rank r then there are invertible matrices $P \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$ such that

$$P^{-1}AQ = \begin{pmatrix} \mathbb{1}_r & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{pmatrix}.$$

By the Fundamental Idea we can expect an analogous theorem for smooth maps.

Theorem A.2.1 (Local Normal Form for Smooth Maps). *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open, $x_0 \in U$, and $f : U \to V$ be smooth. Assume that the derivative $df(x_0) \in \mathbb{R}^{m\times n}$ has rank r. Then there is an open neighborhood U_0 of x_0 in U, an open neighborhood V_0 of $f(x_0)$ in V, a diffeomorphism $\phi : U_1 \times U_2 \subset \mathbb{R}^r \times \mathbb{R}^{n-r}$, a diffeomorphism $\psi : V_0 \to U_1 \times V_2 \subset \mathbb{R}^r \times \mathbb{R}^{m-r}$, such that $\phi(x_0) = (0, 0)$, $\psi(f(x_0)) = (0, 0)$, and*

$$\psi^{-1} \circ f \circ \phi(x, y) = (x, g(x, y)) \qquad \text{and} \qquad dg(0, 0) = 0$$

for $(x, y) \in U_1 \times U_2$.

The Local Normal Form Theorem is an easy consequence of the Inverse Function Theorem.

Theorem A.2.2 (Inverse Function Theorem). *Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$, $x_0 \in U$ and $f : U \to V$ be a smooth map. If $df(x_0)$ is invertible, then ($m = n$ and) there are neighborhoods U_0 of x_0 in U and V_0 of $f(x_0)$ in V so that the restriction $f_{|U_0} : U_0 \to V_0$ is a diffeomorphism.*

Here follow some other consequences of the Inverse Function Theorem.[1]

Corollary A.2.3 (Submersion Theorem). *When $r = m$ the diffeomorphisms ϕ and ψ in Theorem A.2.1 may be chosen so that the local normal form is*

$$\psi^{-1} \circ f \circ \phi(x, y) = x.$$

Corollary A.2.4 (Immersion Theorem). *When $r = n$ the diffeomorphisms ϕ and ψ in Theorem A.2.1 may be chosen so that the local normal form is*

$$\psi^{-1} \circ f \circ \phi(x) = (x, 0).$$

Corollary A.2.5 (Rank Theorem). *If the rank of $df(x) = r$ for all $x \in U$ then for every $x_0 \in U$ the diffeomorphisms ϕ and ψ in Theorem A.2.1 may be chosen so that the local normal form is*

$$\psi^{-1} \circ f \circ \phi(x) = (x, 0).$$

Corollary A.2.6 (Implicit Function Theorem). *Let $U \subset \mathbb{R}^m \times \mathbb{R}^n$ be an open set, let $F : U \to \mathbb{R}^n$ be smooth, and let $(x_0, y_0) \in U$ with $x_0 \in \mathbb{R}^m$ and $y_0 \in \mathbb{R}^n$. Define the partial derivative $d_2F(x_0, y_0) \in \mathbb{R}^{n\times n}$ by*

$$d_2F(x_0, y_0)v := \left.\frac{d}{dt}\right|_{t=0} F(x_0, y_0 + tv)$$

for $v \in \mathbb{R}^n$. Assume that $F(x_0, y_0) = 0$ and that $d_2F(x_0, y_0)$ is invertible. Then there exist neighborhoods U_0 of x_0 in $\mathbb{R}^m$ and V_0 of y_0 in $\mathbb{R}^n$ and a smooth map $g : U_0 \to V_0$ such that

$$U_0 \times V_0 \subset U, \qquad g(x_0) = y_0$$

and

$$F(x, y) = 0 \qquad \Longleftrightarrow \qquad y = g(x)$$

for $x \in U_0$ and $y \in V_0$.

[1] The terms *submersion* and *immersion* are defined in §2.6.1 and Definition 2.3.2 of §2.3.

A.3 Euclidean Spaces

This is the arena of Euclidean geometry; i.e. every figure which is studied in Euclidean geometry is a subset of Euclidean space. To define it one could proceed axiomatically as Euclid did; one would then verify that the axioms characterized Euclidean space by constructing "Cartesian Co-ordinate Systems" which identify the n-dimensional Euclidean space E^n with the n-dimensional numerical space $\mathbb{R}^n$. This program was carried out rigorously by Hilbert. We shall adopt the mathematically simpler but philosophically less satisfying course of taking the characterization as the definition.

Definition A.3.1. *An* **affine subspace** *A in a vector space V is a translate of a vector subspace W of V.*

Thus $A = o + W$ for $o \in A$ and each choice of $o \in A$ gives a bijection $W \to A$. Whereas V contains the "preferred" point namely to origin $0 \in V$ has no preferred point; Such spaces E^n and $\mathbf{E}^n$ would arise in linear algebra by taking E^n to be the space of solutions of $k-n$ independent inhomogeneous linear equations in k unknowns while $\mathbf{E}^n$ is the space of solutions of the corresponding homogeneous equations. The correspondence between E^n and $\mathbf{E}^n$ illustrates the mantra

> *The general solution of an inhomogeneous system of linear equations is a particular solution plus the general solution of the corresponding homogeneous linear system.*

We shall use three closely related spaces: n-dimensional Euclidean affine space E^n, n-dimensional Euclidean vector space $\mathbf{E}^n$, and the space $\mathbb{R}^n$ of all n-tuples of real numbers. The distinction among them is a bit pedantic, especially if one views as the purpose of geometry the interpretation of calculations on $\mathbb{R}^n$. The purpose for distinguishing these three spaces is the same as in elementary vector calculus; it aids geometric intuition. Here is the precise definition.

Definition A.3.2. An n-dimensional **Euclidean vector space** is a real n-dimensional vector space $\mathbf{E}^n$ equipped with a (real valued symmetric positive definite) inner product $\mathbf{E}^n \times \mathbf{E}^n \to \mathbb{R} : (v, w) \mapsto \langle v, w\rangle$. An n-dimensional *Euclidean affine space* consists of a set E^n and an n-dimensional Euclidean vector space $\mathbf{E}^n$ and maps

$$E^n \times E^n \to \mathbf{E}^n : (p, q) \mapsto p - q, \qquad E^n \times \mathbf{E}^n \to E^n : (p, v) \mapsto p + v$$

satisfying $p + 0 = p$, $p + (v + w) = (p + v) + w$, and $q + (p - q) = p$ for $p, q \in E^n$ and $v, w \in \mathbf{E}^n$. The vector $p - q \in \mathbf{E}^n$ is called the **vector** from q to p and the point $p + v$ is called the **translate** of p by v. It follows easily that each choice of a point $o \in E^n$ determines a bijection $v \mapsto o + v$ from $\mathbf{E}^n$ onto E^n. The inner product on $\mathbf{E}^n$ equips the space E^n with a metric

$$\|p - q\| = \sqrt{\langle p - q, p - q \rangle}.$$

The **standard Euclidean space** of dimension n is $E^n = \mathbf{E}^n = \mathbb{R}^n$ with the usual matrix algebra operations $(x \pm y)^i = x^i \pm y^i$, $\langle x, y \rangle = \sum_i x^i y^i$.

Lemma A.3.3. *Any choice of an origin $o \in E^n$ and an orthonormal basis $e_1, \ldots, e_n$ for $\mathbf{E}^n$ determines an isometric bijection:*

$$\mathbb{R}^n \to E^n : (x^1, \ldots, x^n) \mapsto o + \sum_{i=1}^n x^i e_i$$

(the inverse of which is) called a **Cartesian co-ordinate system** *on E^n.*

Lemma A.3.4. *If $E^n \to \mathbb{R}^n : p \mapsto (x^1, \ldots, x^n)$, $(y^1, \ldots, y^n)$ are two Cartesian co-ordinate systems the* **change of co-ordinates map** *has form*

$$y^j(p) = \sum_{j=1}^n a_i^j x^i(p) + v^i$$

where the matrix $a = (a_i^j) \in \mathbb{R}^{n \times n}$ is an orthogonal matrix and $v \in \mathbb{R}^n$.

Example A.3.5. Any n-dimensional affine subspace of some numerical space $\mathbb{R}^k$ $(k > n)$ is an example of a Euclidean space. The corresponding vector space $\mathbf{E}^n$ is the unique vector subspace of $\mathbb{R}^k$ for which:

$$E^n = o + \mathbf{E}^n$$

for $o \in E^n$. This subspace is independent of the choice of $o \in E^n$. Note that $\mathbf{E}^n$ contains the "preferred" point 0 while E^n has no preferred point, Such spaces E^n and $\mathbf{E}^n$ would arise in linear algebra by taking E^n to be the space of solutions of $k - n$ independent inhomogeneous linear equations in k unknowns while $\mathbf{E}^n$ is the space of solutions of the corresponding homogeneous equations. The correspondence between E^n and $\mathbf{E}^n$ illustrates the mantra

> *The general solution of an inhomogeneous system of linear equations is a particular solution plus the general solution of the corresponding homogeneous linear system.*

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