Classical Mechanics

Yannis Bähni*

Semester Paper under the Supervision of Prof. Dr. Ana Cannas Da Silva at ETH Zürich

*ETH Zürich, Rämistrasse 101, 8092 Zürich. *E-mail address*: baehniy@student.ethz.ch

Preface

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Yannis Bähni

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CHAPTER 1

Lagrangian Mechanics

Calculus of Variations

Classical mechanics deals with differential equations originating from extremals of *functionals*, i.e. functions defined on an infinite-dimensional function space. The study of such extremality properties of functionals is known as the *calculus of variations*. To illustrate this fundamental principle, let us consider the *variational formulation* of second order elliptic operators in divergence form based on [Str14, pp. 167–168].

Let $n \in \mathbb{N}$, $n \geq 1$, and $\Omega \subseteq\subseteq \mathbb{R}^n$ such that $\bar{\Omega}$ is a manifold with boundary. Moreover, let $H_0^1(\Omega)$ denote the Sobolev space $W_0^{1,2}(\Omega)$ with inner product

$$\langle u, v\rangle_{H_0^1(\Omega)} = \int_\Omega uv + \int_\Omega \nabla u \nabla v.$$

Suppose $a^{ij} \in C^\infty(\bar{\Omega})$ symmetric, $f \in C^\infty(\bar{\Omega})$ and consider the second order homogenous Dirichlet problem

$$\begin{cases} -\dfrac{\partial}{\partial x^j}\left(a^{ij}\dfrac{\partial u}{\partial x^i}\right) = f & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1}$$

Suppose $u \in C^\infty(\bar{\Omega})$ solves (1). Then integration by parts (see [Lee13, p. 436]) yields

$$\int_\Omega fv = -\int_\Omega \frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial u}{\partial x^i}\right)v = -\int_\Omega \operatorname{div}(X)v = \int_\Omega \langle X, \nabla v\rangle = \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^j}$$

for any $v \in C_c^\infty(\Omega)$, where $X := \left(a^{ij}\frac{\partial u}{\partial x^i}\right)_j$. Thus we say that $u \in H_0^1(\Omega)$ is a *weak solution* of (1) iff

$$\forall v \in C_c^\infty(\Omega) : \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^j} = \int_\Omega fv.$$

If $(a^{ij})_{ij}$ is *uniformly elliptic*, i.e. there exists $\lambda > 0$ such that

$$\forall x \in \Omega \forall \xi \in \mathbb{R}^n : a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2\,,$$

then (1) admits a unique weak solution $u \in H_0^1(\Omega)$ (in fact $u \in C^\infty(\Omega)$ using *regularity theory*, for more details see [Str14, p. 175]). Indeed, observe that

$$\langle\cdot,\cdot\rangle_a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

defined by

$$\langle u, v\rangle_a := \int_\Omega a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} \tag{2}$$

is an inner product on $H_0^1(\Omega)$ with induced norm equivalent to the standard one on $H_0^1(\Omega)$ due to Poincaré's inequality [Str14, p. 107]. Applying the Riesz Representation theorem [Str14, pp. 49–50] yields the result. Moreover, this solution can be characterized by a *variational principle*, i.e. if we define the *energy functional* $E : H_0^1(\Omega) \to \mathbb{R}$

$$E(v) := \frac{1}{2}\|v\|_a^2 - \int_\Omega f v,$$

for any $v \in H_0^1(\Omega)$, where $\|\cdot\|_a$ denotes the norm induced by the inner product (2), then $u \in H_0^1(\Omega)$ solves (1) if and only if

$$E(u) = \inf_{v \in H_0^1(\Omega)} E(v). \tag{3}$$

Indeed, suppose $u \in H_0^1(\Omega)$ is a solution of (1). Let $v \in H_0^1(\Omega)$. Then $u = v + w$ for $w := u - v \in H_0^1(\Omega)$ and we compute

$$E(v) = E(u+w) = \frac{1}{2}\|u\|_a^2 + \langle u, w\rangle_a + \frac{1}{2}\|w\|_a^2 - \int_\Omega f(u+w) = E(u) + \frac{1}{2}\|w\|_a^2 \geq E(u)$$

with equality if and only if $u = v$ a.e. Conversly, suppose the infimum is attained by some $u \in H_0^1(\Omega)$. Thus by elementary calculus

$$0 = \left.\frac{d}{dt}\right|_{t=0} E(u + tv) = \langle u, v\rangle_a - \int_\Omega f v \tag{4}$$

for all $v \in H_0^1(\Omega)$.

Suppose now that $u \in C^\infty(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$ solves the variational formulation (3). Then again integration by parts yields

$$\langle u, v\rangle_a - \int_\Omega f v = -\int_\Omega \operatorname{div}(X) v - \int_\Omega f v = \int_\Omega \left(-\frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial u}{\partial x^i}\right) - f\right) v$$

for all $v \in C_c^\infty(\Omega)$ and where $X := \left(a^{ij}\frac{\partial u}{\partial x^i}\right)_j$. Hence (4) implies

$$\forall v \in C_c^\infty(\Omega) : \int_\Omega \left(-\frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial u}{\partial x^i}\right) - f\right) v = 0.$$

We might expect that this implies

$$-\frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial u}{\partial x^i}\right) = f.$$

That this is indeed the case, is guaranteed by a foundational result in the *calculus of variations* (therefore the name).

Proposition 1.1 (Fundamental Lemma of Calculus of Variations). *Let $\Omega \subseteq \mathbb{R}^n$ open and $f \in L^1_{\text{loc}}(\Omega)$. If*

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega f\varphi = 0,$$

then $f = 0$ a.e.

Proof. See [Str14, p. 40]. □

Thus we recovered a second order partial differential equation from the variational formulation. In fact, this is exactly the boundary value problem (1) from the beginning of our exposition. This technique, and in particular the fundamental lemma of calculus of variations 1.1 will play an important role in our treatment of classical mechanics.

Exercise 1.2. [1] Let $\Omega \subseteq\subseteq \mathbb{R}^n, 2 \leq p < \infty$ and define $\mathcal{B} := \{v \in C^\infty(\bar{\Omega}) : v|_{\partial\Omega} = 0\}$. Moreover, define $E_p : \mathcal{B} \to \mathbb{R}$ by $E_p(v) := \int_\Omega |\nabla v|^p$. Derive the partial differential equation satisfied by minimizers $u \in \mathcal{B}$ of the variational problem $E(u) = \inf_{v\in\mathcal{B}} E(v)$.

Lagrangian Systems and the Principle of Least Action

Definition 1.3 (Lagrangian System). *A **Lagrangian system** is defined to be a tuple (M, L) consisting of an object $M \in \mathsf{Diff}$ and a morphism $L \in \mathsf{Diff}(TM \times \mathbb{R}, \mathbb{R})$, called a **Lagrangian function**.*

Definition 1.4 (Path Space). *Let $M \in \mathsf{Diff}$, $q_0, q_1 \in M$ and $t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$. Define the **path space of M connecting (q_0, t_0) and (q_1, t_1)** to be the set*

$$\mathcal{P}(M)_{q_1,t_1}^{q_0,t_0} := \{\gamma \in \mathsf{Diff}([t_0, t_1], M) : \gamma(t_0) = q_0 \text{ and } \gamma(t_1) = q_1\}. \tag{5}$$

Remark 1.5. For the sake of simplicity, we will just use the terminology *path space* for $\mathcal{P}(M)_{q_1,t_1}^{q_0,t_0}$ and simply write $\mathcal{P}(M)$. We implicitely assume the conditions of definition 1.4, however.

Definition 1.6 (Variation). *Let $\mathcal{P}(M)$ be a path space and $\gamma \in \mathcal{P}(M)$. A **variation of γ** is defined to be a morphism $\Gamma \in \mathsf{Diff}([t_0, t_1] \times [-\varepsilon_0, \varepsilon_0], M)$ for some $\varepsilon_0 > 0$ and such that*

- *$\Gamma(t, 0) = \gamma$ for all $t \in [t_1, t_0]$.*
- *$\Gamma(t_0, \varepsilon) = q_0$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.*
- *$\Gamma(t_1, \varepsilon) = q_1$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.*

Remark 1.7. If Γ is a variation of $\gamma \in \mathcal{P}(M)$, we write $\gamma_\varepsilon(-) := \Gamma(-, \varepsilon)$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

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[1] This is exercise 1.2.(b) from exercise sheet 1 of the course *Functional Analysis II* taught by *Prof. Dr. A. Carlotto* at ETHZ in the spring of 2018, which can be found here.

Example 1.8 (Perturbation of a Path along a Single Direction). Let $M \in \mathsf{Diff}$ of dimension n, (U, φ) a chart and suppose that γ is a path in U. With respect to this chart, we can write the coordinate representation of γ as

$$\gamma(t) = \left(\gamma^1(t), \ldots, \gamma^n(t)\right)$$

for any $t \in [t_0, t_1]$. Let $f \in C_c^\infty(t_0, t_1)$. Consider the family $\Gamma : [t_0, t_1] \times [-\varepsilon_0, \varepsilon_0] \to M$ defined by

$$\Gamma(t, \varepsilon) := (\iota \circ \varphi^{-1}) \left(\gamma^1(t), \ldots, \gamma^i(t) + \varepsilon f(t), \ldots, \gamma^n(t)\right)$$

where $\iota : U \hookrightarrow M$ denotes inclusion and $\varepsilon_0 > 0$ is to be determined. By exercise 1.9, there exists $\delta > 0$ such that

$$U_\delta := \{x \in \mathbb{R}^n : \operatorname{dist}(x, \gamma([t_0, t_1])) < \delta\} \subseteq \varphi(U).$$

Choose $\varepsilon_0 > 0$ such that $0 < \varepsilon_0 < \delta / \|f\|_\infty$. Then in coordinates

$$\operatorname{dist}\left(\gamma_\varepsilon(t), \gamma([t_0, t_1])\right) \leq |\gamma_\varepsilon(t) - \gamma(t)| = |\varepsilon| \|f\|_\infty \leq \varepsilon_0 \|f\|_\infty < \delta$$

for all $t \in [t_0, t_1]$. Hence $\gamma_\varepsilon(t) \in U_\delta$ and thus $\gamma_\varepsilon(t) \in \varphi(U)$. Therefore, Γ is indeed well-defined. Moreover, it is easy to show that the properties of definition 1.6 holds, therefore, Γ is a variation of γ. In fact, this example shows, that any path γ contained in a single chart admits infinitely many variations. An example of such a variation is shown in figure 1.

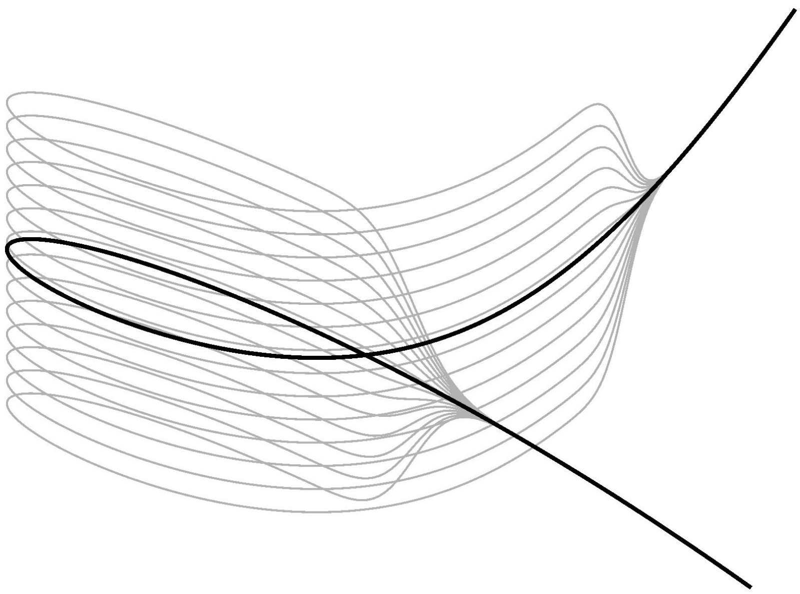

Figure 1. Example of a variation along the second coordinate using a smooth bump function as in [Lee13, p. 42].

Exercise 1.9. Let $U \subseteq \mathbb{R}^n$ open and $A \subseteq U$ closed. Then there exists $\delta > 0$ such that

$$U_\delta := \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) < \delta\} \subseteq U.$$

Definition 1.10 (Action Functional). *Let (M, L) be a Lagrangian system and $\mathcal{P}(M)$ be a path space. The morphism $S : \mathcal{P}(M) \to \mathbb{R}$ defined by*

$$S(\gamma) := \int_{t_0}^{t_1} L(\gamma(t), \dot{\gamma}(t), t)dt$$

*is called the **action functional**.*

Axiom 1 (Hamilton's Principle of Least Action). *Let (M, L) be a Lagrangian system and $\mathcal{P}(M)$ be a path space. A path $\gamma \in \mathsf{Diff}([t_0, t_1], M)$ describes a motion of (M, L) between (q_0, t_0) and (q_1, t_1) if and only if*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\gamma_\varepsilon) = 0 \tag{6}$$

for all variations γ_ε of γ.

Theorem 1.11 (Euler-Lagrange Equations). *Let (M, L) be a Lagrangian system. A path $\gamma \in \mathsf{Diff}([t_0, t_1], M)$ describes a motion of (M, L) between (q_0, t_0) and (q_1, t_1) if and only if with respect to any chart (U, q^i)*

$$\frac{\partial L}{\partial q}\big(\gamma(t), \dot{\gamma}(t), t\big) - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\big(\gamma(t), \dot{\gamma}(t), t\big) = 0 \tag{7}$$

*holds, where $(q, \dot{q})$ denotes the standard coordinates on TM. The system of equations (7) is referred to as the **Euler-Lagrange equations**.*

Proof. The proof is divided into three steps.

Step 1: *Suppose that the extremal γ is contained in a chart (U, q^i).*

□

CHAPTER 2

Hamiltonian Mechanics

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