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SOLUTIONS SHEET 7

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Exercise 1.

Exercise 2.

Exercise 3.

Lemma 1.1. *Let $y \in H$ and define a mapping $\varphi_y : H \to \mathbb{C}$ by $\varphi_y(x) := \langle A(y), x\rangle$. Then $\varphi_y \in \mathscr{L}(H, \mathbb{C})$.*

Proof. Clearly, φ_y is linear since $\langle\cdot,\cdot\rangle$ is linear in the second component. Moreover, φ_y is bounded. Indeed, using Cauchy-Schwarz yields

$$|\varphi_y(x)| = |\langle A(y), x\rangle| \leq \|A(y)\|\|x\|$$

for all $x \in H$. $\square$

Thus we may define a family

$$\mathscr{F} := \{\varphi_y : y \in \partial B_1(0)\} \subseteq \mathscr{L}(H, \mathbb{C}).$$

Let $x \in H$. Then for any $y \in \partial B_1(0)$ we have that

$$|\varphi_y(x)| = |\langle A(y), x\rangle| = |\langle y, A(x)\rangle| \leq \|y\|\|A(x)\| = \|A(x)\|$$

by symmetry and again Cauchy-Schwarz. Hence

$$\sup_{T\in\mathscr{F}} |T(x)| = \sup_{y\in\partial B_1(0)} |\varphi_y(x)| \leq \|A(x)\|$$

for all $x \in H$. Since any Hilbert space is a Banach space, an application of *Banach-Steinhaus* yields the existence of a constant $c > 0$ such that

$$\sup_{T\in\mathscr{F}} \|T\| = \sup_{y\in\partial B_1(0)} \|\varphi_y\| \leq c.$$

For $x \in H$ such that $A(x) \neq 0$ we have that

$$\begin{aligned}\|A(x)\|^2 &= \langle A(x), A(x)\rangle \\ &= \|x\| \langle A(x/\|x\|), A(x)\rangle \\ &= \|x\| \varphi_{x/\|x\|}(A(x))\end{aligned}$$

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$$
\begin{aligned}
&\le \|x\| \, |\varphi_{x/\|x\|}(A(x))| \\
&\le \|x\| \|A(x)\| \, \|\varphi_{x/\|x\|}\| \\
&\le c\|x\| \|A(x)\|
\end{aligned}
$$

and thus dividing both sides by $\|A(x)\|$ yields the boundedness of A.

Exercise 4.
Exercise 5.
a. We define

$$\mathscr{F} := \{B(\cdot, y) : y \in \partial B_1(0)\}.$$

Lemma 1.2. *We have that $\mathscr{F} \subseteq \mathscr{L}(X, \mathbb{K})$ and for all $x \in X$, there exists $c_x \ge 0$ such that* $\sup_{T\in\mathscr{F}} |T(x)| \le c_x$.

Proof. Let $y \in \partial B_1(0)$. Then $B(\cdot, y)$ is linear by definition of a bilinear functional. Moreover, for any $x \in X$ we have that

$$|B(x, y)| \le c_y \|x\|$$

for some $c_y \ge 0$ by continuity of B in the first argument. Hence $\mathscr{F} \subseteq \mathscr{L}(X, \mathbb{K})$. Let $x \in X$. Then

$$|B(x, y)| \le c_x \|y\| = c_x$$

for some $c_x \ge 0$ by continuity of B in the second argument. Thus

$$\sup_{T\in\mathscr{F}} |T(x)| = \sup_{y\in\partial B_1(0)} |B(x, y)| \le c_x$$

for all $x \in X$. □

An application of *Banach-Steinhaus* on the family $\mathscr{F}$ yields the existence of a constant $c \ge 0$ such that

$$\sup_{T\in\mathscr{F}} \|T\| \le c.$$

Let $x, y \in X$. Then

$$
\begin{aligned}
|B(x, y)| &= \|x\| \|y\| \, |B(x/\|x\|, y/\|y\|)| \\
&\le \|x\| \|y\| \sup_{\|\xi\|=1} |B(\xi, y/\|y\|)| \\
&\le \|x\| \|y\| \sup_{\|\zeta\|=1} \sup_{\|\xi\|=1} |B(\xi, \zeta)| \\
&= \|x\| \|y\| \sup_{\|\zeta\|=1} \|B(\cdot, \zeta)\| \\
&\le c\|x\| \|y\|.
\end{aligned}
$$

Lemma 1.3. *Equip $X \times X$ with the norm $\|(x, y)\| := \|x\| + \|y\|$. Then B is continuous.*

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Proof. Let $(x, y) \in X \times X$ and $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence in $X \times X$ converging to (x, y). We claim that $x_n \to x$ and $y_n \to y$ in X. Indeed

$$\|x_n - x\| \leq \|x_n - x\| + \|y_n - y\| = \|(x_n, y_n) - (x, y)\| \to 0$$

as $n \to \infty$ and similarly

$$\|y_n - y\| \leq \|x_n - x\| + \|y_n - y\| = \|(x_n, y_n) - (x, y)\| \to 0.$$

Moreover, since $y_n \to y$, y_n is bounded, i.e. there exists some $M \geq 0$ such that $\|y_n\| \leq M$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned}|B(x_n, y_n) - B(x, y)| &= |B(x_n, y_n) - B(x, y_n) + B(x, y_n) - B(x, y)| \\ &= |B(x_n - x, y_n) + B(x, y_n - y)| \\ &\leq |B(x_n - x, y_n)| + |B(x, y_n - y)| \\ &\leq c\|x_n - x\|\|y_n\| + c\|x\|\|y_n - y\| \\ &\leq cM\|x_n - x\| + c\|x\|\|y_n - y\| \to 0\end{aligned}$$

as $n \to \infty$. □

b.

Lemma 1.4. *B is a bilinear functional on $\mathcal{P}$ which is continuous in each argument separately.*

Proof. The bilinearity of B directly follows from the linearity of the integral. Fix $q \in \mathcal{P}$. Then for any $p \in \mathcal{P}$ we have that

$$|B(p,q)| = \left|\int_0^1 p(t)q(t)dt\right| \leq \int_0^1 |p(t)||q(t)|\,dt \leq \sup_{t\in[0,1]} |q(t)| \int_0^1 |p(t)|\,dt = c_q\|p\|$$

since q is continuous. Similarly, for each fixed $p \in \mathcal{P}$ we get that $|B(p,q)| \leq c_p\|q\|$ for all $q \in \mathcal{P}$. □