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SOLUTIONS SHEET 9

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Exercise 1. We may assume that $f \neq 0$ since otherwise we would have convergence in norm. Thus the continuity of f implies $\|f\|_p \neq 0$ for all $1 \leq p < \infty$ since also $|f|^p$ is continuous.

Lemma 1.1. *Let $1 \leq p < \infty$. Then $g_n, h_n, k_n \in L^p(\mathbb{R})$ for all $n \in \mathbb{N}$.*

Proof. We have that

$$\begin{aligned}
\|g_n\|_p^p &= \int_{\mathbb{R}} |f(x-n)|^p \, dx = \int_{\mathbb{R}} |f(y)| \, dy = \|f\|_p^p \, , \\
\|h_n\|_p^p &= n^{-1} \int_{\mathbb{R}} |f(x/n)|^p \, dx = \int_{\mathbb{R}} |f(y)|^p \, dy = \|f\|_p^p \, , \\
\|k_n\|_p^p &= \int_{\mathbb{R}} |f(x)e^{inx}|^p \, dx = \int_{\mathbb{R}} |f(x)|^p \, dx = \|f\|_p^p
\end{aligned}$$

and since $f \in C_c^\infty(\mathbb{R})$ implies that $f \in L^p(\mathbb{R})$, the claim follows. □

Lemma 1.2. *Let $1 < p < \infty$. Then $g_n, h_n, k_n \rightharpoonup 0$ in $L^p(\mathbb{R})$.*

Proof. We make use of lemma 6.2.1 and theorem 2.2.6, which provides an antilinear isometric isomorphism $\left(L^p(\mathbb{R})\right)^* \cong L^q(\mathbb{R})$ where q is the dual exponent of p. Since $f \in C_c^\infty(\mathbb{R})$, there exists some $M > 0$ such that $\operatorname{supp}(f) \subseteq [-M, M]$. It is easy to verify that $\operatorname{supp}(g_n) \subseteq [-M+n, M+n]$ for all $n \in \mathbb{N}$. Let $\varphi \in L^q(\mathbb{R})$. Then Hölder's inequality implies

$$\begin{aligned}
\left| \int_{\mathbb{R}} \overline{\varphi}(x) g_n(x) dx \right| &\leq \int_{\mathbb{R}} |\varphi(x) g_n(x)| \, dx \\
&= \int_{\mathbb{R}} |\varphi(x) g_n(x) \chi_{\operatorname{supp}(g_n)}(x)| \, dx \\
&\leq \int_{\mathbb{R}} |\varphi(x) g_n(x) \chi_{[-M+n, M+n]}(x)| \, dx \\
&= \|\varphi \chi_{[-M+n, M+n]} g_n\|_1 \\
&\leq \|\varphi \chi_{[-M+n, M+n]}\|_q \|g_n\|_p \\
&= \|\varphi \chi_{[-M+n, M+n]}\|_q \|f\|_p \to 0
\end{aligned}$$

since dominated convergence yields

$$\lim_{n \to \infty} \int_{\mathbb{R}} |\varphi(x)|^q \, \chi_{[-M+n, M+n]}(x) dx = \int_{\mathbb{R}} |\varphi(x)|^q \lim_{n \to \infty} \chi_{[-M+n, M+n]}(x) dx = 0$$

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which application is justified by the fact that $|\varphi(x)|^q \chi_{[-M+n,M+n]}(x) \leq |\varphi(x)|^q \in L^1(\mathbb{R})$ and $\chi_{[-M+n,M+n]}(x) \to 0$ for all $x \in \mathbb{R}$ (take n just sufficiently large).
Observe that

$$\begin{aligned}\int_{\mathbb{R}} \overline{\varphi}(x)k_n(x)dx &= \sqrt{\frac{2\pi}{2\pi}}\int_{\mathbb{R}} \overline{\varphi}(x)f(x)e^{inx}dx \\ &= \sqrt{\frac{2\pi}{2\pi}}\int_{\mathbb{R}} \overline{\varphi}(x)f(x)e^{inx}dx \\ &= \sqrt{\frac{2\pi}{2\pi}}\int_{\mathbb{R}} \overline{\varphi}(x)f(x)e^{-i(-n)x}dx \\ &= \sqrt{2\pi}\widehat{\overline{\varphi}f}(-n) \to 0\end{aligned}$$

by the Riemann-Lebesgue lemma since by Hölders inequality, $\overline{\varphi}f \in L^1(\mathbb{R})$. □

Lemma 1.3. *Let X be a normed space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X such that $x_n \rightharpoonup x$. If $x_n \to y$ for some $y \in X$, then $x = y$.*

Proof. Suppose that $x_n \to y$. Then since $\mathcal{T}_W \subseteq \mathcal{T}_{\|\cdot\|}$, we have that $x_n \rightharpoonup y$. But $(X, \mathcal{T}_W)$ is Hausdorff and thus limits are unique. Hence $x = y$. □

Corollary 1.1. *Let $1 < p < \infty$. Then g_n, h_n and k_n do not converge in norm.*

Proof. Since all three sequences converge weakly to 0, we only have to show that they do not converge towards 0 in $L^p(\mathbb{R})$. However, this is immediate from the first lemma, since all sequences have constant norm $\|f\|_p \neq 0$ and hence the limit should have also nonzero norm. □

Let us now investigate the case $p = 1$.

Exercise 2.

a.

b. Suppose that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. By lemma 6.2.1. we have that $f(x_n) \to f(x)$ for all $f \in H^*$. Using the *Riesz representation theorem* this is equivalent to $\langle y, x_n\rangle \to \langle y, x\rangle$ for all $y \in H$. But then

$$\|x - x_n\|^2 = \langle x - x_n, x - x_n\rangle = \|x\|^2 - 2\,\mathrm{Re}\,\langle x, x_n\rangle + \|x_n\|^2 \to 0$$

since Re is a continuous function and $\langle x, x_n\rangle \to \|x\|^2$.

Exercise 3.

a.

Lemma 1.4. *Let $0 < \varepsilon < 1$ and define*

$$I_\varepsilon(f) := \varepsilon^{-1}\int_0^\varepsilon f(x)dx$$

for $f \in L^\infty(0,1)$. Then $I_\varepsilon \in \left(L^\infty(0,1)\right)^*$ and $\|I_\varepsilon\| = 1$ for all $0 < \varepsilon < 1$.

Proof. Let $f \in L^\infty(0,1)$. Then we have that $|f| \leq \|f\|_\infty$ λ-a.e. Hence

$$|I_\varepsilon(f)| = \varepsilon^{-1}\left|\int_0^\varepsilon f(x)dx\right| \leq \varepsilon^{-1}\int_0^\varepsilon |f(x)|\,dx \leq \|f\|_\infty \tag{1}$$

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and thus I_ε is bounded and thus continuous. Clearly I_ε is $\mathbb{C}$-linear by the $\mathbb{C}$-linearity of the integral. Moreover, using (1) we get that

$$\|I_\varepsilon\| = \sup_{\|f\|_\infty = 1} |I_\varepsilon(f)| \leq \sup_{\|f\|=1} \|f\|_\infty = 1.$$

Conversly, setting $f := \chi_{(0,1)} \in L^\infty(0,1)$, we get that $|I_\varepsilon(f)| = 1$ and hence by $\|f\| = 1$

$$\|I_\varepsilon\| = \sup_{\|g\|_\infty = 1} |I_\varepsilon(g)| \geq |I_\varepsilon(f)| = 1.$$

□

Exercise 4.

a. First we show that $\|\cdot\|_\sigma$ is well defined. Let $x^* \in X^*$, then we have

$$\|x^*\|_\sigma = \sum_{k=1}^\infty 2^{-k} |x^*(x_k)| \leq \sum_{k=1}^\infty 2^{-k} \|x^*\|\|x_k\| = \|x^*\| \sum_{k=1}^\infty 2^{-k} = \|x^*\| < \infty$$

since $x_k \in S_X$ and $\sum_{k=1}^\infty 2^{-k} = 1$. Hence $\|x^*\|_\sigma \leq \|x^*\|$ holds. Let $\lambda \in \mathbb{K}$. Then we have that

$$\|\lambda x^*\|_\sigma = \sum_{k=1}^\infty 2^{-k} |\lambda x^*(x_k)| = \sum_{k=1}^\infty 2^{-k}|\lambda| |x^*(x_k)| = |\lambda| \sum_{k=1}^\infty 2^{-k} |x^*(x_k)| = |\lambda| \|x^*\|_\sigma .$$

Let $y^* \in X^*$. Then the triangle inequality follows from

$$\begin{aligned}\|x^* + y^*\|_\sigma &= \sum_{k=1}^\infty 2^{-k} |x^*(x_k) + y^*(x_k)| \\ &\leq \sum_{k=1}^\infty 2^{-k} |x^*(x_k)| + \sum_{k=1}^\infty 2^{-k} |y^*(x_k)| \\ &= \|x^*\|_\sigma + \|y^*\|_\sigma .\end{aligned}$$

Lastly, clearly $\|x^*\| = 0$ if $x^* = 0$. Conversly, suppose that $\|x^*\| = 0$. Hence $x^*(x_k) = 0$ for all $k \in \mathbb{N}$. Let $Y \in X$. Since $\overline{\operatorname{span}\{x_k : k \in \mathbb{N}\}} = X$, we find a sequence $(y_n)_{n\in\mathbb{N}}$ in $\operatorname{span}\{x_k : k \in \mathbb{N}\}$, such that $y_n \to y$. Moreover, for each $n \in \mathbb{N}$ we have that $y_n = \sum_{k=1}^\infty \lambda_k^{(n)} x_k$ for $\lambda_k^{(n)} \in \mathbb{K}$ and $\lambda_k^{(n)} = 0$ for all but finitely many $k \in \mathbb{N}$. Hence

$$x^*(y) = \lim_{n\to\infty} x^*(y_n) = \lim_{n\to\infty} \sum_{k=1}^\infty \lambda_k^{(n)} x^*(x_k) = 0$$

by the continuity of x^* and so $x^* = 0$.