SOLUTIONS SHEET 10

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Exercise 1.
a.

Lemma 1.1. *Let $[x] \in X/M$. Then*

$$\|[x]\|_{X/M} = \inf_{m \in M} \|x - m\|.$$

Proof. This immediately follows from

$$\{\|y\| : y \in [x]\} = \{\|x - m\| : m \in M\}.$$

Indeed, if $y \in [x]$, by definition $x - y \in M$ and thus there exists some $m \in M$ such that $x - y = m$ or equivalently $y = x - m$. Conversly, $x - m \in [x]$. ☐

There are four things to check.

- ***(Well definedness)*** Let $[x], [y] \in X/M$ such that $[x] = [y]$. Hence $x \sim y$ and thus we find $m_0 \in M$ such that $x - y = m_0$. Thus

$$\|[x]\|_{X/M} = \inf_{m \in M} \|x - m\| = \inf_{m \in M} \|y - (m - m_0)\| = \inf_{\widetilde{m} \in M} \|y - \widetilde{m}\| = \|[y]\|_{X/M}$$

since M is a linear subspace.

- ***(Positivity)*** Let $[x] \in X/M$. If $[x] = 0$ we have that $x \in M$. But then

$$\|[x]\|_{X/M} = \inf_{m \in M} \|x - m\| = 0.$$

Conversly, assume that $\|[x]\|_{X/M} = 0$. By the definition of the infimum, we can construct a sequence $(m_n)_{n \in \mathbb{N}}$ in M such that $\|x - m_n\| \to 0$. But then $m_n \to x$ and since M is closed we have that $x \in M$. Hence $[x] = 0$.

- ***(Homogeneity)*** Let $[x] \in X/M$ and $\lambda \in \mathbb{K}$. The case $\lambda = 0$ is clear. So assume $\lambda \neq 0$. Then

$$\begin{aligned}
\|\lambda [x]\|_{X/M} &= \|[\lambda x]\|_{X/M} \\
&= \inf_{m \in M} \|\lambda x - m\| \\
&= \inf_{m \in M} |\lambda| \|x - m/\lambda\| \\
&= |\lambda| \inf_{m \in M} \|x - m/\lambda\| \\
&= |\lambda| \inf_{\widetilde{m} \in M} \|x - \widetilde{m}\|
\end{aligned}$$

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$$= |\lambda| \|[x]\|_{X/M}$$

since M is a linear subspace.

- ***(Triangle inequality)*** Let $[x], [y] \in X/M$. Then

$$\begin{aligned}
\|[x]+[y]\|_{X/M} &= \|[x+y]\|_{X/M} \\
&= \inf_{m\in M} \|x+y-m\| \\
&= \inf_{m\in M} \|x+y-2m+m\| \\
&\leq \inf_{m\in M} \|x-m\| + \inf_{m\in M} \|y-m\| + \inf_{m\in M} \|m\| \\
&= \inf_{m\in M} \|x-m\| + \inf_{m\in M} \|y-m\| \\
&= \|[x]\|_{X/M} + \|[y]\|_{X/M}
\end{aligned}$$

since M is a linear subspace and thus $0 \in M$.

b. Let $x \in X$. By part **a.** we have that

$$\|\pi(x)\|_{X/M} = \|[x]\|_{X/M} = \inf_{m\in M} \|x-m\| \leq \inf_{m\in M} \|x\| + \inf_{m\in M} \|m\| = \|x\|.$$

c. Let $([x_n])_{n\in\mathbb{N}}$ be a Cauchy sequence in X/M. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Indeed, for any $m \in M$ we have that

$$\|x_n - x_k\| \leq \|x_n - x_k - m\| + \|m\|$$

And thus

$$\|x_n - x_k\| \leq \inf_{m\in M} \|x_n - x_k - m\| + \inf_{m\in M} \|m\| = \|[x_n - x_k]\|_{X/M} = \|[x_n] - [x_k]\|_{X/M} \xrightarrow{n,k\to\infty} 0.$$

Since X is a Banach space, there exists $x \in X$ such that $x_n \to x$. Then $[x_n] \to [x]$. Indeed, by part **b.** we have

$$\lim_{n\to\infty} [x_n] = \lim_{n\to\infty} \pi(x_n) = \pi(x) = [x].$$

d. Define $\widetilde{T} : X/\ker T \to T(X)$ by

$$\widetilde{T}([x]) := T(x).$$

This mapping is well defined. Indeed, if $[x] = [y] \in X/\ker T$, we have that $x - y \in \ker T$ and thus

$$\widetilde{T}([x]) = T(x) = T(x-y+y) = T(x-y) + T(y) = T(y) = \widetilde{T}([y])$$

by the linearity of T. Also $\widetilde{T}$ is linear. Let $\lambda \in \mathbb{K}$. Then we have

$$\widetilde{T}([x] + \lambda[y]) = \widetilde{T}([x+\lambda y]) = T(x+\lambda y) = T(x) + \lambda T(y) = \widetilde{T}([x]) + \lambda\widetilde{T}([y]).$$

Clearly, $\widetilde{T}$ is surjective. Also $\widetilde{T}$ is injective since if $[x] \in \ker \widetilde{T}$, we have that

$$0 = \widetilde{T}([x]) = T(x)$$

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and thus $x \in \ker T$ which implies $[x] = 0$. Next we verify the commutativity of the diagram. Let $x \in X$. Then

$$(\iota \circ \widetilde{T} \circ \pi)(x) = \iota\left(\widetilde{T}([x])\right) = \iota\left(T(x)\right) = T(x).$$

Lastly we show that $\|\widetilde{T}\| = \|T\|$ which in particular implies $\widetilde{T} \in \mathscr{L}\left(X/\ker T, T(X)\right)$. Indeed, by part **b.** we have that $\|\pi(x)\|_{X/M} \leq \|x\|$ for all $x \in X$ and thus

$$\|\widetilde{T}([x])\| \leq \|\widetilde{T}\|\|[x]\|_{X/M} = \|\widetilde{T}\|\|\pi(x)\|_{X/M} \leq \|\widetilde{T}\|\|x\| = \|T\|\|x\|$$

for all $[x] \in X/M$.

- ***($\|T\| \leq \|\widetilde{T}\|$)*** Observe that

$$\{x \in X : \|x\| \leq 1\} \subseteq \{x \in X : \|[x]\|_{X/M} \leq 1\}$$

by the continuity of π. Thus

$$\|T\| = \sup_{\|x\| \leq 1} \|T(x)\| \leq \sup_{\|[x]\|_{X/M} \leq 1} \|T(x)\| = \sup_{\|[x]\|_{X/M} \leq 1} \|T([x])\| = \|\widetilde{T}\|.$$