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SOLUTIONS SHEET 8

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Exercise 1.

Exercise 2.

Exercise 3.

a. Suppose that $M \setminus \overline{A}$ is dense in M. Towards a contradiction, assume that A is not nowhere dense. Hence $\mathring{\overline{A}} \neq \varnothing$. Since $\mathring{\overline{A}}$ is open by definition of the interior of a set, there exists $\varepsilon > 0$ and $x \in \mathring{\overline{A}}$ such that $B_\varepsilon(x) \subseteq \mathring{\overline{A}}$. Moreover, $\mathring{\overline{A}} \subseteq \overline{A}$ and thus $B_\varepsilon(x) \subseteq \overline{A}$. This implies that $B_\varepsilon(x)$ and $M \setminus \overline{A}$ are disjoint. But $M \setminus \overline{A}$ is dense in M, hence we find a sequence $(x_n)_{n\in\mathbb{N}}$ in $M \setminus \overline{A}$ such that $x_n \to x$. Hence there exists $N \in \mathbb{N}$ such that $x_n \in B_\varepsilon(x)$ for all $n \geq N$. This is not possible since $B_\varepsilon(x)$ does not contain any elements of $M \setminus \overline{A}$. Contradiction.

b.

Lemma 1.1. *Let $(x_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}$. For $n \in \mathbb{N}$ define*

$$E_n := \bigcup_{k\in\mathbb{N}} \left(x_k - \frac{1}{2^k n}, x_k + \frac{1}{2^k n} \right) \tag{1}$$

and

$$E := \bigcap_{n\in\mathbb{N}} E_n. \tag{2}$$

Set $A := E^c$. Then A is meager and $\lambda(A^c) = 0$, where λ denotes the ordinary Lebesgue measure on $\mathbb{R}$.

Proof. We show first that $\lambda(A^c) = 0$. First observe that $(E_n)_{n\in\mathbb{N}}$ is a decreasing sequence of λ-measurable sets. Moreover, for any $n \in \mathbb{N}$ we have that

$$\lambda(E_n) \leq \frac{1}{n} \sum_{k\in\mathbb{N}} \frac{1}{2^{k-1}} = \frac{1}{n} \sum_{k\in\mathbb{N}_0} \frac{1}{2^k} = \frac{2}{n} < \infty$$

by subadditivity of the measure. Elementary measure theory now tells us that

$$\lambda(A^c) = \lambda(E) = \lambda\left(\cap_{n\in\mathbb{N}} E_n\right) = \lim_{n\to\infty} \lambda(E_n) \leq \lim_{n\to\infty} \frac{2}{n} = 0.$$

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Let us show that A is meager. Since $A = E^c = \cup_{n\in\mathbb{N}} E_n^c$, we show that E_n^c is nowhere dense for all $n \in \mathbb{N}$. By part **a.** we can also show that $\mathbb{R} \setminus \bar{E}_n^c$ is dense in $\mathbb{R}$. For fixed $n \in \mathbb{N}$ we have that

$$E_n^c = \bigcap_{k\in\mathbb{N}} \left(\left(-\infty, x_k - \frac{1}{2^k n}\right] \cup \left[x_k + \frac{1}{2^k n}, \infty\right) \right).$$

Thus E_n^c is a closed set (finite unions and countable intersection of closed intervals) and so $\bar{E}_n^c = E_n^c$. So $\mathbb{R} \setminus \bar{E}_n^c = E_n$. But $\mathbb{Q} \subseteq E_n$ for all $n \in \mathbb{N}$ and thus E_n is dense in $\mathbb{R}$. $\square$

Exercise 4.

a. If $A = \varnothing$, we have that $\cap_{\alpha\in A}\mathcal{T}_\alpha = \mathcal{P}(X)$ since topologies on X are subsets of $\mathcal{P}(X)$. Hence the intersection of the empty family of topologies on X is the discrete topology.
Consider now $A \neq \varnothing$. Clearly, $\varnothing, X \in \cap_{\alpha\in A}\mathcal{T}_\alpha$ since $\varnothing, X \in \mathcal{T}_\alpha$ for all $\alpha \in A$. Let $U_1, \dots, U_n \in \cap_{\alpha\in A}\mathcal{T}_\alpha$. Hence $U_1, \dots, U_n \in \mathcal{T}_\alpha$ for all $\alpha \in A$ and so $U_1 \cap \dots \cap U_n \in \mathcal{T}_\alpha$ for all $\alpha \in A$. Hence $U_1 \cap \dots \cap U_n \in \cap_{\alpha\in A}\mathcal{T}_\alpha$. Finally, suppose that $(U_\beta)_{\beta\in B}$ is a family in $\cap_{\alpha\in A}\mathcal{T}_\alpha$. Hence for all $\alpha \in A$ we have that $U_\beta \in \mathcal{T}_\alpha$ for all $\beta \in B$. So $\cup_{\beta\in B} U_\beta \in \mathcal{T}_\alpha$ for all $\alpha \in A$ and therefore $\cup_{\beta\in B} U_\beta \in \cap_{\alpha\in A}\mathcal{T}_\alpha$.

b. Define

$$\mathcal{B} := \{U_1 \cap \dots \cap U_n : n \in \mathbb{N}, U_i \in \mathcal{S} \text{ for all } i = 1, \dots, n\}$$

and

$$\mathcal{T} := \{\cup_{\alpha\in A} B_\alpha : B_\alpha \in A \text{ for all } \alpha \in A\}.$$

Lemma 1.2. $\mathcal{T}_\mathcal{F} = \mathcal{T}$.

Proof. By part **a.**, $\mathcal{T}_\mathcal{F}$ is a topology. We show that also $\mathcal{T}$ is a topology. By [Lee11, p. 34] it is enough to show that $\mathcal{B}$ satisfies the following two conditions:

(i) $\cup_{B\in\mathcal{B}} B = X$.
(ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists an element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Then $\mathcal{T}$ is the unique topology on X generated by $\mathcal{B}$, i.e. the collection of arbitrary unions of elements of $\mathcal{B}$. Since $\mathcal{F}$ is nonempty, there exists $f \in \mathcal{F}$. Clearly $X = f^{-1}(Y_f)$ and Y_f is open in Y_f. Hence $f^{-1}(Y_f) \in \mathcal{S}$ and thus $X \in \cup_{B\in\mathcal{B}} B$. Suppose that $B_1, B_2 \in \mathcal{B}$ such that $B_1 \cap B_2 \neq \varnothing$. Hence we find $U_1, \dots, U_n, V_1, \dots, V_m \in \mathcal{S}$ such that $B_1 = U_1 \cap \dots \cap U_n$ and $B_2 = V_1 \cap \dots \cap V_m$. Suppose $x \in B_1 \cap B_2$. Then also $x \in U_1 \cap \dots \cap U_n \cap V_1 \cap \dots \cap V_m$. But

$$U_1 \cap \dots \cap U_n \cap V_1 \cap \dots \cap V_m \in \mathcal{B}$$

as a finite intersection of elements of $\mathcal{S}$. Hence $\mathcal{T}$ is a topology.
Clearly, $\mathcal{S} \subseteq \mathcal{T}$, since already $\mathcal{S} \subseteq \mathcal{B}$. Since $\mathcal{T}_\mathcal{F}$ is the smallest topology containing $\mathcal{S}$, we get that $\mathcal{T}_\mathcal{F} \subseteq \mathcal{T}$.
Let $U \in \mathcal{T}$. Then $U = \cup_{\alpha\in A} B_\alpha$ for some index set A and $B_\alpha \in \mathcal{B}$ for all $\alpha \in A$. But each B_α is a finite intersection of elements of $\mathcal{S}$ and thus since $\mathcal{T}_\mathcal{F}$ is a topology containing

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$\mathcal{S}$, we have that $B_\alpha \in \mathcal{T}_\mathcal{F}$ for all $\alpha \in A$. But then also $U \in \mathcal{T}_\mathcal{F}$ as a union of sets in $\mathcal{T}_\mathcal{F}$. Hence $\mathcal{T} \subseteq \mathcal{T}_\mathcal{F}$. □

Exercise 5. Suppose that $x_n \rightharpoonup x$. Proposition 6.2.2 implies that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded, in particular $\sup_{n\in\mathbb{N}}\|x_n\| < \infty$. Moreover, lemma 6.2.1 yields $f(x_n) \to f(x)$ for all $f \in X^*$. Since $Y \subseteq X^*$ we also have $f(x_n) \to f(x)$ for all $f \in Y$.
Conversly, suppose $\|x_n\| \leq M$ for some $M \geq 0$ and $f(x_n) \to f(x)$ for all $f \in Y$. Let $f \in X^*$. Since Y is dense in X^*, we find a sequence $(f_k)_{k\in\mathbb{N}}$ in Y such that $\|f_k - f\| \to 0$. Hence

$$\begin{aligned} |f(x_n) - f(x)| &= |f(x_n) - f(x) + f_k(x_n) - f_k(x_n) + f_k(x) - f_k(x)| \\ &\leq |f(x_n) - f_k(x_n)| + |f_k(x_n) - f_k(x)| + |f_k(x) - f(x)| \\ &\leq \|f - f_k\| (\|x_n\| + \|x\|) + |f_k(x_n) - f_k(x)| \\ &\leq \|f - f_k\| (M + \|x\|) + |f_k(x_n) - f_k(x)| \end{aligned}$$

and so

$$\lim_{n\to\infty} |f(x_n) - f(x)| \leq \|f - f_k\| (M + \|x\|) \xrightarrow{k\to\infty} 0.$$

References

[Lee11] John M. Lee. *Introduction to Topological Manifolds*. Second Edition. Springer Science+Business Media, 2011.