MAT602 - FUNCTIONAL ANALYSIS

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Contents

1. Structures

1.1. Topological Spaces.

Theorem 1.1 (Urysohn's Lemma). *Suppose X is a normal topological space. Given disjoint closed subsets $A, B \subseteq X$, there exists a continuous function $f : X \to [0,1]$ such that $f|_A = 0$ and $f|_B = 1$.*

1.2. Metric Spaces.

1.3. Normed Spaces.

Proposition 1.1 (Sequence Spaces). *For $1 \leq p < \infty$ define*

$$\ell^p(\mathbb{K}) := \left\{x \in \mathbb{K}^{\mathbb{N}} : \textstyle\sum_{k\in\mathbb{N}} |x_k| < \infty\right\} \tag{1}$$

and for $p = \infty$

$$\ell^\infty(\mathbb{K}) := \left\{x \in \mathbb{K}^{\mathbb{N}} : \sup_{k\in\mathbb{N}} |x_k| < \infty\right\}. \tag{2}$$

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Moreover, for $x \in \ell^p(\mathbb{K})$ *set*

$$\|x\|_p := \left(\sum_{k\in\mathbb{N}} |x_k|^p\right)^{1/p} \tag{3}$$

for $1 \leq p < \infty$ *and*

$$\|x\|_\infty := \sup_{k\in\mathbb{N}} |x_k| . \tag{4}$$

Then $(\ell^p, \|\cdot\|_p)$ *is a Banach space for all* $1 \leq p \leq \infty$*.*

Theorem 1.2 (Completion of Normed Spaces). *Every normed space* X *has a completion which is unique up to isometric isomorphisms.*

2. Linear Operators

2.1. Continuous Operators.

Definition 2.1. *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be two normed spaces. An* ***operator*** *is a linear mapping* $T : X \to Y$*. Moreover, we say that an operator* $T : X \to Y$ *is* ***bounded*** *if there exists* $c > 0$ *such that*

$$\|T(x)\|_Y \leq c\|x\|_X \tag{5}$$

holds for all $x \in X$*.*

2.2. The Hahn-Banach Theorem.

Lemma 2.1. *Let* V *be a real vector space,* $S \subsetneq V$ *a linear subspace,* $p : V \to \mathbb{R}$ *a sublinear functional,* $f : S \to \mathbb{R}$ *linear and* $x_0 \in V \setminus S$*. Moreover, assume that* $f \leq p$ *on* S*. Then there exists* $F : S + \mathbb{R}x_0 \to \mathbb{R}$ *linear such that* $F \leq p$ *on* $S + \mathbb{R}x_0$ *and* $F|_S = f$*.*

Theorem 2.1 (Hahn-Banach, $\mathbb{R}$). *Let* V *be a vector space over* $\mathbb{R}$*,* $S \subseteq V$ *a linear subspace and* $f : S \to \mathbb{R}$ *linear. Moreover, let* $p : V \to \mathbb{R}$ *be a sublinear functional such that* $f \leq p$ *on* S*. Then there exists* $F : V \to \mathbb{R}$ *linear such that* $F \leq p$ *on* V *and* $F|_S = f$*.*

Theorem 2.2 (Hahn-Banach, $\mathbb{R}$ or $\mathbb{C}$). *Let* V *be a vector space over* $\mathbb{K}$*,* $q : V \to \mathbb{R}$ *a seminorm,* $S \subseteq V$ *a linear subspace and* $f : S \to \mathbb{K}$ *linear with* $|f| \leq q$ *on* S*. Then there exists* $F : V \to \mathbb{K}$ *linear with* $F|_S = f$ *and* $|F| \leq q$ *on* V*.*

Corollary 2.1 (Extension). *Let* $(X, \|\cdot\|)$ *be a normed space over* $\mathbb{K}$*,* $S \subseteq X$ *a linear subspace and* $f \in S^*$*. Then there exists* $F \in X^*$ *such that* $F|_S = f$ *and* $\|F\|_{X^*} = \|f\|_{S^*}$*.*

Corollary 2.2 (Separation). *Let* $(X, \|\cdot\|)$ *be a normed space over* $\mathbb{K}$ *and* $x_0 \in X \setminus \{0\}$*. Then there exists* $f \in X^*$ *with* $\|f\| = 1$ *and* $f(x_0) = \|x_0\|$*.*

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2.3. Reflexivity.

Proposition 2.1. *Let X be a normed vector space over $\mathbb{K}$. Then the mapping $\Phi : X \to X^{**}$ defined by $\Phi(x) := \varphi_x$, where $\varphi_x : X^* \to \mathbb{R}$ is defined by $\varphi_x(f) := f(x)$, is a linear isometry.*

Theorem 2.3. *Let X be a Banach space. Then X is reflexive if and only if X^* is reflexive.*

2.4. Hilbert Space Methods.

Theorem 2.4 (Riesz's Representation Theorem). *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$. The mapping $\Psi : H \to H^*$ defined by $\big(\Psi(x)\big)(y) := \langle x, y \rangle$ is an anti-linear isometric isomorphism.*

Corollary 2.3. *Every Hilbert space is reflexive.*

Theorem 2.5 (Lax-Milgram). *Let H be a Hilbert space over $\mathbb{K}$ and let $a : H \times H \to \mathbb{K}$ be a sesquilinear form. Moreover, suppose that there are constants $0 < c_0 \leq C_0 < \infty$ such that*

$$\begin{aligned} |a(x,y)| &\leq C_0 \|x\| \|y\| \quad (\textbf{\textit{Continuity}}), \\ \operatorname{Re} a(x,x) &\geq c_0 \|x\|^2 \quad (\textbf{\textit{Coercivity}}), \end{aligned}$$

for all $x, y \in H$. Then there exists a unique $A \in \mathcal{L}(H)$ such that

$$a(x,y) = \langle Ax, y \rangle \tag{6}$$

for all $x, y \in H$. Moreover, A is invertible with

$$\|A\| \leq C_0 \qquad \text{and} \qquad \|A^{-1}\| \leq \frac{1}{c_0}. \tag{7}$$

3. Baire Category Theorem

3.1. Baire Category Theorem and Banach-Steinhaus.

Theorem 3.1 (Baire Category Theorem). *Every complete metric space is a Baire space.*

Theorem 3.2 (Banach-Steinhaus). *Let X be a Banach space, Y a normed space and $\mathcal{F} \subseteq \mathcal{L}(X,Y)$. Assume that for all $x \in X$ there exists $c_x \geq 0$ such that*

$$\sup_{T \in \mathcal{F}} \|T(x)\| \leq c_x. \tag{8}$$

Then there exists $c \geq 0$ with

$$\sup_{T \in \mathcal{F}} \|T\| \leq c. \tag{9}$$

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3.2. The Open Mapping and Closed Graph Theorems.

Theorem 3.3 (Open Mapping Theorem). *Let X and Y be two Banach spaces and $T \in \mathcal{L}(X, Y)$ surjective. Then $T(U)$ is open for all $U \subseteq X$ open.*

Theorem 3.4 (Inverse Mapping Theorem). *Let X and Y be two Banach spaces and $T \in \mathcal{L}(X, Y)$ bijective. Then $T^{-1} \in \mathcal{L}(Y, X)$.*

Theorem 3.5 (Closed Graph Theorem). *Let X and Y be Banach spaces and $T : X \to Y$ linear. The following statements are equivalent:*

(i) $T \in \mathcal{L}(X, Y)$.

(ii) The graph of f, $\Gamma(f)$, is closed in $(X \times Y, \|\cdot\|_X + \|\cdot\|_Y)$.