SOLUTIONS SHEET 6

YANNIS BÄHNI

Exercise 1.

Exercise 2.

Exercise 3.

Exercise 4. Let $f \in X$. Define

$$f^+ := \max(f, 0) \qquad \text{and} \qquad f^- := \max(-f, 0).$$

Clearly $f^+, f^- \geq 0$, $f = f^+ - f^-$ and $|f| = f^+ + f^-$. Moreover, since by assumption also $|f| \in X$, by $f^+, f^- \leq |f|$ we have also that $f^+, f^- \in B(E, \mathbb{R})$. Hence we can define $p : B(E, \mathbb{R}) \to \mathbb{R}$ by

$$p(f) := \sup_{x \in E} f^+(x).$$

Lemma 1.1. *$p : B(E, \mathbb{R}) \to \mathbb{R}$ is a sublinear functional such that $T(f) \leq p(f)$ for all $f \in X$.*

Proof. Let $f, g \in B(E, \mathbb{R})$ and $\lambda \geq 0$. Then we have

$$\begin{aligned}
p(\lambda f) &= \sup_{x \in E} (\lambda f)^+(x) \\
&= \sup_{x \in E} \max(\lambda f(x), 0) \\
&= \sup_{x \in E} \frac{1}{2}\big(\lambda f(x) + |\lambda f(x)|\big) \\
&= \sup_{x \in E} \frac{1}{2}\big(\lambda f(x) + \lambda |f(x)|\big) \\
&= \lambda \sup_{x \in E} f^+(x) \\
&= \lambda p(f)
\end{aligned}$$

and

$$p(f + g) = \sup_{x \in E} (f + g)^+(x)$$

(Yannis Bähni) University of Zurich, Rämistrasse 71, 8006 Zurich
E-mail address: yannis.baehni@uzh.ch.

Yannis Bähni
yannis.baehni@uzh.ch

$$
\begin{aligned}
&= \sup_{x \in E} \frac{1}{2} \big(f(x) + g(x) + |f(x) + g(x)| \big) \\
&\leq \sup_{x \in E} \frac{1}{2} \big(f(x) + g(x) + |f(x)| + |g(x)| \big) \\
&= \sup_{x \in E} \frac{1}{2} \big(f(x) + |f(x)| \big) + \sup_{x \in E} \frac{1}{2} \big(g(x) + |g(x)| \big) \\
&= p(f) + p(g).
\end{aligned}
$$

Let $f \in X$. Then we have

$$
\begin{aligned}
T(f) &= T(f^+ - f^-) \\
&= T(f^+) - T(f^-) \\
&\leq T(f^+) \\
&\leq |T(f^+)| \\
&\leq \|f^+\| \\
&= \sup_{x \in E} |f^+(x)| \\
&= \sup_{x \in E} f^+(x) \\
&= p(f)
\end{aligned}
$$

since $T(f^-) \geq 0$. □

Lemma 1.1 together with the real version of Hahn-Banach yields the existence of a linear functional $\overline{T} : B(E, \mathbb{R}) \to \mathbb{R}$ such that $\overline{T}|_X = T$ and $\overline{T}(f) \leq p(f)$ for all $f \in B(E, \mathbb{R})$.

Lemma 1.2. *For all $f \in B(E, \mathbb{R})$ we have that $|\overline{T}(f)| \leq \|f\|$ and if $f \geq 0$ then also $\overline{T}(f) \geq 0$.*

Proof. We have that

$$
\overline{T}(f) \leq p(f) = \sup_{x \in E} f^+(x) \leq \sup_{x \in E} |f(x)| = \|f\|
$$

and

$$
\begin{aligned}
\overline{T}(f) &= -\overline{T}(-f) \\
&\geq -p(-f) \\
&= -\sup_{x \in E} (-f)^+(x) \\
&= -\sup_{x \in E} \max(-f(x), 0) \\
&= -\sup_{x \in E} f^-(x)
\end{aligned}
$$

Yannis Bähni
yannis.baehni@uzh.ch

$$\begin{aligned} &\geq -\sup_{x \in E} |f(x)| \\ &= -\|f\|. \end{aligned}$$

Hence $|\overline{T}(f)| \leq \|f\|$. Assume that $f \geq 0$. Then $f = f^+$ and so

$$\begin{aligned} -\overline{T}(f) &= \overline{T}(-f) \\ &\leq p(-f) \\ &= \sup_{x \in E} (-f)^+(x) \\ &= \sup_{x \in E} \max(-f, 0) \\ &= \sup_{x \in E} f^-(x) \\ &= 0 \end{aligned}$$

yields $\overline{T}(f) \geq 0$ $\square$