SOLUTIONS SHEET 1

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Exercise 1.

(a) The pair $(D(X), i)$ has the universal property

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "\forall \text{ functions } f"'] & D(X) \arrow[d, "\exists! \text{ continuous } \overline{f}"] \\ & \forall(Y, \mathcal{T}_Y). \end{tikzcd}$$

Assume, that there is another pair $(D'(X), i')$ with this property. Thus we get the two commuting diagrams

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "i'"'] & D(X) \arrow[d, "\overline{i'}"] \\ & D'(X), \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "i'"] \arrow[dr, "i"'] & D'(X) \arrow[d, "\overline{i}"] \\ & D(X). \end{tikzcd}$$

Putting them together yields

$$\begin{tikzcd} & D(X) \arrow[d, "\overline{i'}"] \\ X \arrow[ur, "i"] \arrow[r, "i'"] \arrow[dr, "i"'] & D'(X) \arrow[d, "\overline{i}"] \\ & D(X), \end{tikzcd} \qquad \begin{tikzcd} & D'(X) \arrow[d, "\overline{i}"] \\ X \arrow[ur, "i'"] \arrow[r, "i"] \arrow[dr, "i'"'] & D(X) \arrow[d, "\overline{i'}"] \\ & D'(X). \end{tikzcd}$$

Hence

$$(\overline{i} \circ \overline{i'}) \circ i = i \qquad \text{and} \qquad (\overline{i'} \circ \overline{i}) \circ i' = i'.$$

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Since also $\mathrm{id}_{D(X)} \circ i = i$ and $\mathrm{id}_{D'(X)} \circ i' = i'$, uniqueness implies that

$$\overline{i} \circ \overline{i'} = \mathrm{id}_{D(X)} \qquad \text{and} \qquad \overline{i'} \circ \overline{i} = \mathrm{id}_{D'(X)}\,.$$

Thus $D(X) \cong D'(X)$ uniquely.

(b) Let $I(X) := (X, \{\varnothing, X\})$ be the ***indiscrete topological space***. Define a mapping $\pi : I(X) \to X$ by $\pi(x) := x$. Then the tuple $(I(X), \pi)$ has the following universal property:

$$\begin{tikzcd}
X & I(X) \arrow[l, "\pi"'] \\
 & \forall(Y, \mathcal{T}_Y). \arrow[ul, "\forall \text{ functions } f"] \arrow[u, "\exists! \text{ continuous } \overline{f}"']
\end{tikzcd}$$

Now the argumentation is the same as in part (a).

Exercise 2.

(a) First we show that $(\mathbb{Z}[X], X)$ has the claimed property. Let R be a unital ring with $r \in R$. Then there exists a unique homomorphism of rings $\varphi : \mathbb{Z} \to R$. Define $f : \mathbb{Z}[X] \to R$ by

$$f\left(\sum_{\iota=0}^{n} a_\iota X^\iota\right) := \sum_{\iota=0}^{n} \varphi(a_\iota) r^\iota.$$

Clearly, $f(X) = r$. Also it is easy to check that f is a homomorphism of rings. Assume that $g : \mathbb{Z}[X] \to R$ is a homomorphism of rings such that $g(X) = r$. Then

$$g\left(\sum_{\iota=0}^{n} a_\iota X^\iota\right) = \sum_{\iota=0}^{n} g(a_\iota) g(X)^\iota = \sum_{\iota=0}^{n} g(a_\iota) r^\iota = \sum_{\iota=0}^{n} \varphi(a_\iota) r^\iota = f\left(\sum_{\iota=0}^{n} a_\iota X^\iota\right)$$

by the uniqueness of φ (g induces a homomorphism of rings $\mathbb{Z} \to R$). Consider the following diagram:

$$(\mathbb{Z}[X], X) \xrightarrow{\exists! f, f(X)=a} (A, a) \xrightarrow{\exists! g, g(a)=X} (\mathbb{Z}[X], X) \xrightarrow{\exists! f, f(X)=a} (A, a)\,.$$

Now $\mathrm{id}_{(\mathbb{Z}[X], X)}(X) = X$ and $g(f(X)) = X$, thus by uniqueness $g \circ f = \mathrm{id}_{(\mathbb{Z}[X], X)}$ and similarly $f \circ g = \mathrm{id}_{(A,a)}$.

(b)

Exercise 3. Existence was shown in the lecture, the so-called ***free group***. Uniqueness is shown exactly as in **Exercise 1.**.

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Exercise 4. Let $g, \tilde{g} : Y \to X$ be inverses of f. Then we have

$$g = g \circ \mathrm{id}_Y = g \circ (f \circ \tilde{g}) = (g \circ f) \circ \tilde{g} = \mathrm{id}_X \circ \tilde{g} = \tilde{g}.$$

Thus we can unambiguously write $f^{-1} := g$.

Exercise 5. That $h \circ g \circ f$ is an isomorphism immediately follows by

$$\big((g \circ f)^{-1} \circ g \circ (h \circ g)^{-1}\big) \circ (h \circ g \circ f) = \mathrm{id}_X$$
$$(h \circ g \circ f) \circ \big((g \circ f)^{-1} \circ g \circ (h \circ g)^{-1}\big) = \mathrm{id}_W .$$

Moreover

$$\big((h \circ g)^{-1} \circ h\big) \circ g = (h \circ g)^{-1} \circ (h \circ g) = \mathrm{id}_Y$$
$$g \circ \big(f \circ (g \circ f)^{-1}\big) = (g \circ f) \circ (g \circ f)^{-1} = \mathrm{id}_Z .$$

Lemma 0.1. *Let C be a category and $f : X \to Y$. Assume that there exist $g, \tilde{g} : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ \tilde{g} = \mathrm{id}_Y$. Then f is an isomorphism with $f^{-1} = g = \tilde{g}$.*

Proof. We have that

$$g = g \circ \mathrm{id}_Y = g \circ (f \circ \tilde{g}) = (g \circ f) \circ \tilde{g} = \mathrm{id}_X \circ \tilde{g} = \tilde{g}.$$

□

Thus g is invertible.

Lemma 0.2. *Let C be a category and $f : X \to Y$, $g : Y \to Z$ isomorphisms. Then also $g \circ f$ is an isomorphism with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Proof. We have that

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = \mathrm{id}_X \qquad \text{and} \qquad (g \circ f) \circ (f^{-1} \circ g^{-1}) = \mathrm{id}_Z .$$

Hence the statement follows by the uniqueness of the inverse. □

Therefore also

$$f = (h \circ g)^{-1} \circ (h \circ g \circ f) \qquad \text{and} \qquad h = (h \circ g \circ f) \circ (g \circ f)^{-1}$$

are isomorphisms.

Exercise 6. Assume $f : X \to Y$ has the left cancellation property. Let $x, y \in X$ such that $f(x) = f(y)$. Now let $Z := \{x, y\}$. Define two functions $c_x, c_y : Z \to X$ by $c_x(z) := x$ and $c_y(z) := y$, respectively. Now

$$f \circ c_x = f(x) = f(y) = f \circ c_y$$

holds by assumption. Thus the left cancellation property implies that $c_x = c_y$, hence $x = y$ and f is injective. Conversly, assume that f is injective. Let $\alpha, \beta : Z \to X$ such that $f \circ \alpha = f \circ \beta$ and $z \in Z$. Then we have that $f(\alpha(z)) = f(\beta(z))$ and thus by injectivity, $\alpha(z) = \beta(z)$.