ADDITIVE AND ABELIAN CATEGORIES

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Abstract. We define preadditive, additive and abelian categories, where the latter is the natural generalization of AbGrp to study the basic results of homological algebra in. Moreover, we state and comment on two foundational results in the theory of abelian categories, namely the *Mitchell Embedding Theorem* and the *Eilenberg-Watts Theorem*, which roughly speaking, connect the category of modules with abelian categories.

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1. Introduction

Given a diagram

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in AbGrp, we say that the diagram is *exact at B*, if $\operatorname{im} f = \ker g$. Recall that the *cokernel of f* is defined to be $\operatorname{coker} f := B/\operatorname{im} f$. Consider the following commutative diagram with exact rows in AbGrp:

$$\begin{tikzcd}
 & \bullet \arrow[r] \arrow[d, "f"] & \bullet \arrow[r] \arrow[d, "g"] & \bullet \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet &
\end{tikzcd}$$

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Those who are familiar with algebraic topology recognise it as the setting of the *snake lemma*. This basic result in *homological algebra* yields the existence of a morphism of groups $\delta \in \mathsf{AbGrp}(\ker h, \operatorname{coker} f)$ such that the sequence

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h$$

is exact. The basic proof technique used to establish this result is called *diagram chasing*. It turns out, that abelian categories are the right generalization for this type of proof.

2. Preadditive Catgeories

Let $G, H \in \operatorname{ob}(\mathsf{AbGrp})$ and $\varphi, \psi \in \mathsf{AbGrp}(G, H)$. Define $\varphi + \psi$ pointwise. Since H is abelian, it follows that $\varphi + \psi \in \mathsf{AbGrp}(G, H)$. Moreover, it is easy to check, that with this operation defined above, $\mathsf{AbGrp}(G, H)$ is an abelian group and

$$\circ : \mathsf{AbGrp}(H, K) \times \mathsf{AbGrp}(G, H) \to \mathsf{AbGrp}(G, K)$$

is bilinear for each $K \in \operatorname{ob}(\mathsf{AbGrp})$. This motivates the following definition.

Definition 2.1 (Preadditive Category [Lan78]). *A **preadditive category** is a locally small category $\mathscr{C}$ in which all hom-sets $\mathscr{C}(X, Y)$ can be equipped with the structure of an abelian group and composition is bilinear, i.e. for all mophisms $f, f' : X \to Y$ and $g, g' : Y \to Z$ in $\mathscr{C}$ we have that*

$$(g + g') \circ (f + f') = g \circ f + g \circ f' + g' \circ f + g' \circ f'. \tag{1}$$

Remark 2.1. In a preadditive category $\mathscr{C}$, we have that $\mathscr{C}(X, Y) \neq \varnothing$ for all $X, Y \in \operatorname{ob}(\mathscr{C})$.

Lemma 2.1. *Let $\mathscr{C}$ be a preadditive category. Then compositions with the zero elements are again zero elements of the correpsonding abelian groups.*

Proof. This simply follows from $0 \circ f = (0 + 0) \circ f = 0 \circ f + 0 \circ f$. The other case is similar. □

3. Additive Categories

As in Grp, the trivial group in AbGrp is both an initial and a terminal object. Objects with this property have a special name.

Definition 3.1 (Null Object [Lan78]). *Let $\mathscr{C}$ be a category. A **null object in $\mathscr{C}$** is a an object of $\mathscr{C}$ which is both initial and terminal.*

Definition 3.2 (Zero Arrow [Lan78]). *Let $\mathscr{C}$ be a category with a null object 0. For $X, Y \in \operatorname{ob}(\mathscr{C})$, the unique composition $X \to 0 \to Y$ is called the **zero arrow from X to Y**, denoted by $0 : X \to Y$.*

Lemma 3.1. *Let $\mathscr{C}$ be a preadditive category with null object and $X, Y \in \operatorname{ob}(\mathscr{C})$. Then the zero arrow $0 : X \to Y$ is the zero element of the group $\mathscr{C}(X, Y)$.*

Proof. The zero arrow $0 : X \to Y$ is the unique composition

$$X \longrightarrow 0 \longrightarrow Y.$$

However, since 0 is a null object, we have that the two morphisms are the two zero objects in the corresponding abelian group structures. Hence lemma 2.1 yields the result. $\square$

Let $A, B \in \mathrm{ob}(\mathsf{AbGrp})$. Then we have seen that $A \coprod B \cong A \prod B$. This can be generalized to preadditive categories.

Proposition 3.1. *Let $\mathscr{C}$ be a preadditive category admitting all finite coproducts. Then any n-ary coproduct is also an n-ary product for all $n \in \omega$. In particular, $\mathscr{C}$ admits all finite products and a null object.*

Proof. Step 1: Zero-ary case [Lan78, p. 194]. Since $\mathscr{C}$ has the empty coproduct, $\mathscr{C}$ has an initial object $\varnothing$. Since $\varnothing$ is initial, there exists a unique map $\varnothing \to \varnothing$, namely $\mathrm{id}_\varnothing$. But $\mathscr{C}(\varnothing, \varnothing)$ is a group and thus $\mathrm{id}_\varnothing = 0$. Hence for any morphism $f : X \to \varnothing$, lemma 2.1 yields $f = \mathrm{id}_\varnothing \circ f = 0 \circ f = 0$.
Step 2: Binary case. By the zero-ary case we know that $\mathscr{C}$ admits a null object 0. Let $X, Y \in \mathrm{ob}(\mathscr{C})$. We want to show that $X \coprod Y$ is also a product of X and Y. By the universal property of the coproduct we have a commutative diagram

$$\begin{tikzcd}
 & X & \\
X \arrow[ur, "\mathrm{id}_X"] \arrow[r, "\iota_X"] \arrow[dr, "0"'] & X \coprod Y \arrow[u, "{(\mathrm{id}_X, 0)}"'] \arrow[d, "{(0, \mathrm{id}_Y)}"] & Y \arrow[l, "\iota_Y"'] \arrow[ul, "0"'] \arrow[dl, "\mathrm{id}_Y"] \\
 & Y. &
\end{tikzcd}$$

Suppose (Z, p_X, p_Y) is another product cone. Define $f : Z \to X \coprod Y$ by

$$f := \iota_X \circ p_X + \iota_Y \circ p_Y.$$

Using lemma 3.1 and 2.1, we compute

$$\begin{aligned}
(\mathrm{id}_X, 0) \circ f &= (\mathrm{id}_X, 0) \circ \iota_X \circ p_X + (\mathrm{id}_X, 0) \circ \iota_Y \circ p_Y \\
&= \mathrm{id}_X \circ p_X + 0 \circ p_Y \\
&= p_X + 0 \\
&= p_X,
\end{aligned}$$

and simiarly $(0, \mathrm{id}_Y) \circ f = p_Y$. Now we have to check uniqueness. This is the hardest part of the proof and involves the *Yoneda embedding* $\mathscr{Y} : \mathscr{C} \hookrightarrow [\mathscr{C}^{\mathrm{op}}, \mathsf{Set}]$. We want to show that

$$\iota_X \circ (\mathrm{id}_X, 0) + \iota_Y \circ (0, \mathrm{id}_Y) = \mathrm{id}_{X \coprod Y}.$$

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Applying the Yoneda embedding to the category $\mathscr{C}^{\mathrm{op}}$, we get that it is enough to show that

$$f \circ \iota_X \circ (\mathrm{id}_X, 0) + \iota_Y \circ (0, \mathrm{id}_Y) = f \circ \mathrm{id}_{X \coprod Y}$$

holds for all morphisms $f \in \mathscr{C}(X \coprod Y, C)$. Let $(\alpha, \beta) : X \coprod Y \to C$ be any morphism (where $(\alpha, \beta) \circ \iota_X) = \alpha$ and $(\alpha, \beta) \circ \iota_Y = \beta)$. Using the universal property of the corpoduct it is easy to show that

$$(\alpha, \beta) \circ \iota_X \circ (\mathrm{id}_X, 0) = (\alpha, 0) \quad \text{and} \quad (\alpha, \beta) \circ \iota_Y \circ (0, \mathrm{id}_Y) = (0, \beta),$$

and moreover one can show that for any other morphism $(\alpha', \beta') : X \coprod Y \to C$ we have

$$(\alpha, \beta) + (\alpha', \beta') = (\alpha + \alpha', \beta + \beta').$$

Thus

$$(\alpha, \beta) \circ \mathrm{id}_{X \coprod Y} = (\alpha, \beta) = (\alpha, 0) + (0, \beta) = (\alpha, \beta) \circ \left(\iota_X \circ (\mathrm{id}_X, 0) + \iota_Y \circ (0, \mathrm{id}_Y)\right).$$

Now if $f' : Z \to X \coprod Y$ is another morphism making the diagram commute, we have that

$$\begin{aligned} f - f' &= \mathrm{id}_{X \coprod Y} \circ (f - f') \\ &= \left(\iota_X \circ (\mathrm{id}_X, 0) + \iota_Y \circ (0, \mathrm{id}_Y)\right) \circ (f - f') \\ &= \iota_X \circ (p_X - p_X) + \iota_Y \circ (p_Y - p_Y) \\ &= \iota_X \circ 0 + \iota_Y \circ 0 \\ &= 0, \end{aligned}$$

by lemma 2.1.
Step 3: n-ary case. Induction over $n \in \omega$.

$\square$

Definition 3.3 (Additive Category). *An* ***additive category*** *is a preadditive category which admits all finite coproducts.*

Remark 3.1. Let $\mathscr{C}$ be an additive category. Then by proposition 3.1, $\mathscr{C}$ admits all finite products which coincide with the coproducts.

4. Abelian Categories

Definition 4.1 (Kernel and Cokernel [Lan78]). *Let $\mathscr{C}$ be a category with a null object 0. A* ***kernel of a morphism $f : X \to Y$*** *is defined to be an equalizer of*

$$X \overset{f}{\underset{0}{\rightrightarrows}} Y.$$

Dually, a ***cokernel of a morphism $f : X \to Y$*** *is a coequalizer of the above diagram.*

Lemma 4.1. *In* Grp*, every monic is a kernel and every epic is a cokernel.*

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Proof. Let $m : G \to H$ be a monic in Grp. Consider the fork

$$G \xrightarrow{m} H \underset{0}{\overset{\pi}{\rightrightarrows}} \operatorname{coker} m.$$

Then one can check that this is in fact a universal fork. Similarly, one can check that

$$\ker e \underset{0}{\overset{\iota}{\rightrightarrows}} G \xrightarrow{e} H$$

is a universal cofork for any epic $e : G \to H$ in Grp. $\square$

Definition 4.2 (Abelian Category [Lan78]). *An **abelian category** is an additive category satisfying the following additional conditions:*

(a) *Every morphism admits a kernel.*
(b) *Every morphism admits a cokernel.*
(c) *Every monic is a kernel.*
(d) *Every epic is a cokernel.*

Examples 4.1. AbGrp, Vect_K, ${}_R\mathsf{Mod}$ and Mod_R.

5. Exact Sequences

We follow [Fre64, p. 44].

Lemma 5.1. *Given a diagram*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in AbGrp*, we have that above sequence is exact at B if and only if $g \circ f = 0$ and*

$$\left(\ker g \xhookrightarrow{\iota} B \xrightarrow{\pi} \operatorname{coker} f \right) = 0.$$

Proof. Trivial. $\square$

In lemma 5.1, the second condition involves statements about the kernel and the cokernel in the categorical sense. Indeed, in Grp we have that

$$\ker g \xhookrightarrow{\iota} B \underset{0}{\overset{g}{\rightrightarrows}} C \qquad \text{and} \qquad A \underset{0}{\overset{f}{\rightrightarrows}} B \xrightarrow{\pi} \operatorname{coker} f$$

are an equalizer and a coequalizer, respectively. Hence

$$\ker g = \left(\ker g \xhookrightarrow{\iota} B \right) \qquad \text{and} \qquad \operatorname{coker} f = \left(B \xrightarrow{\pi} \operatorname{coker} f \right).$$

Definition 5.1 (Exactness). *Let $\mathscr{C}$ be an abelian category. A sequence*

$$X \longrightarrow Y \longrightarrow Z$$

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of objects in $\mathcal{C}$ is said to be ***exact at Y****, if*

$$X \longrightarrow Y \longrightarrow Z \; = 0 \qquad \text{and} \qquad K \longrightarrow Y \longrightarrow C \; = 0,$$

where

$$K \to Y = \ker(Y \to Z) \qquad \text{and} \qquad Y \to C = \operatorname{coker}(X \to Y).$$

Definition 5.2 (Left and Right Exactness). *Let $\mathcal{C}$ be an abelian category. A* ***left exact sequence*** *is an exact sequence of the form*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z.$$

Similarly, a ***right exact sequence*** *is an exact sequence*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow 0.$$

Definition 5.3 (Exact Functor). *Let $\mathcal{C}$, $\mathcal{D}$ be abelian categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a* ***left exact functor****, if it preserves left exact sequences. Similarly, F is called a* ***right exact functor****, if it preserves right exact sequences. Finally, F is called an* ***exact functor****, if it preserves exact sequences.*

Remark 5.1. A functor between abelian categories is left exact if and only if it preserves kernels. Similarly, it is right exact if and only if it preserves cokernels. A functor between abelian categories is exact, if and only if it is both left and right exact.

6. The Mitchell Embedding Theorem

Theorem 6.1 (Mitchell Embedding [Fre64]). *For every small abelian category there is an exact, full and faithful functor into ${}_R\mathsf{Mod}$ for some ring R.*

A usefull application of the Mitchell embedding is that one can do proofs of basic homological algebra results in a familiar environment like ${}_R\mathsf{Mod}$ by diagram chasing. However, as [Lan78, pp. 202–208] shows, this can also be done without using the Mitchell embedding.

7. The Eilenberg-Watts Theorem

Definition 7.1 (Bimodule). *Let R and S be two unital rings. An* ***R-S-bimodule*** *is an abelian group B such that:*

(a) B is a left R-module and a right S-module.
(b) We have that $(rb)s = r(bs)$ for all $r \in R$, $s \in S$ and $b \in B$.

Theorem 7.1 (Eilenberg-Watts). *Let R and S be unital rings and B a R-S-bimodule. Then the tensor product functor*

$$(-) \otimes_R B : \mathsf{Mod}_R \to \mathsf{Mod}_S$$

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is right exact and preserves small coproducts. Conversely, if $F : \mathsf{Mod}_R \to \mathsf{Mod}_S$ is right-exact and preserves small coproducts, then it is naturally isomorphic to tensoring with a bimodule.

References

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[Lan78] Saunders Mac Lane. *Categories for the Working Mathematician*. Second Edition. Graduate Texts in Mathematics 5. Springer Science+Business Media New York, 1978.