# INTRODUCTION TO CATEGORY THEORY AND ITS APPLICATIONS

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## 1. Representable Functors

### 1.1. The Yoneda Lemma.

**Proposition 1.1.** *Let $\mathscr{C}$ be a locally small category and $X \in \mathscr{C}$ an object. Define* $\mathrm{Hom}_{\mathscr{C}}(X,-) : \mathscr{C} \to \mathsf{Set}$ *on objects $Y \in \mathscr{C}$ by* $\mathrm{Hom}_{\mathscr{C}}(X,Y) := \mathscr{C}(X,Y)$ *and on morphisms $f : Y \to Z$ by post-composition with $f$, i.e.*

$$\mathrm{Hom}_{\mathscr{C}}(X,f) : \mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{C}}(X,Z)$$

*is defined by* $\mathrm{Hom}_{\mathscr{C}}(X,f)(g) := f \circ g$. *Then* $\mathrm{Hom}_{\mathscr{C}}(X,-)$ *is a functor.*

*Proof.* □

**Proposition 1.2.** *Let $\mathscr{C}$ be a locally small category and $f \in \mathscr{C}(X,X')$ a morphism. Define $\eta^f := \big(\eta^f_A\big)_{A\in\mathscr{C}}$ by letting $\eta^f_A$ :* $\mathrm{Hom}_{\mathscr{C}}(X',A) \to \mathrm{Hom}_{\mathscr{C}}(X,A)$ *be pre-composition with $f$, i.e. $\eta^f_A(g) := g \circ f$. Then $\eta^f$ :* $\mathrm{Hom}_{\mathscr{C}}(X',-) \Rightarrow \mathrm{Hom}_{\mathscr{C}}(X,-)$.

*Proof.* □

**Proposition 1.3.** *Let $\mathscr{C}$ be a locally small category. Define $H^\bullet : \mathscr{C}^{\mathrm{op}} \to [\mathscr{C},\mathsf{Set}]$ on objects $X \in \mathscr{C}$ by $H^\bullet(X) :=$* $\mathrm{Hom}_{\mathscr{C}}(X,-)$ *and on morphisms $f \in \mathscr{C}^{\mathrm{op}}$ by $H^\bullet(f) := \eta^f$. Then $H^\bullet$ is a functor.*

*Proof.* □


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**Definition 1.1 (Representable and Corepresentable Functor).** *Let $\mathscr{C}$ be a locally small category. A covariant functor $F$ is said to be* ***representable****, if there exists an object $X \in \mathscr{C}$, such that $F \cong \mathrm{Hom}_{\mathscr{C}}(X, -)$. A contravariant functor $F$ is said to be* ***corepresentable****, if there exists $X \in \mathscr{C}$ such that $F \cong \mathrm{Hom}_{\mathscr{C}}(-, X)$, where $\mathrm{Hom}_{\mathscr{C}}(-, X) := \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, -)$.*

The functor defined in the dualized statement of proposition 1.3 has its own name.

**Definition 1.2 (Yoneda embedding of $\mathscr{C}$).** *Let $\mathscr{C}$ be a locally small category. Then the functor $H_\bullet := H^\bullet : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathsf{Set}]$ is called the* ***Yoneda embedding of $\mathscr{C}$****.*

**Theorem 1.1 (Yoneda Lemma).** *Let $\mathscr{C}$ be a locally small category. For any functor $F : \mathscr{C} \to \mathsf{Set}$ and for every object $X \in \mathscr{C}$ there is a bijection*

$$[\mathscr{C}, \mathsf{Set}]\left(\mathrm{Hom}_{\mathscr{C}}(X, -), F\right) \cong F(X) \tag{1}$$

*that associates to each $\alpha : \mathrm{Hom}_{\mathscr{C}}(X, -) \Rightarrow F$ the element $\alpha_X(\mathrm{id}_X) \in F(X)$. Moreover, the correspondence is natural in both $X$ and $F$.*

## 2. Adjoints

### 2.1. Adjunctions.

**Definition 2.1.** *Let $\mathscr{C}$ and $\mathscr{D}$. An* ***adjunction from $\mathscr{C}$ to $\mathscr{D}$*** *is a triple $(F, G, \varphi)$ consisting of two functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ and a function $\varphi$, which assigns to each $X \in \mathscr{C}$ and $Y \in \mathscr{D}$ a bijection*

$$\varphi_{X,Y} : \mathscr{D}\left(F(X), Y\right) \cong \mathscr{C}\left(X, G(Y)\right) \tag{2}$$

*which is natural in both $X$ and $Y$.*