Problem 1.1. Let $n \in \omega$ and $a_{jk} \in \mathbb{C}$ for $j = 0, \ldots, n+1$, $k = 0, \ldots, n$. Show that

$$\sum_{j=0}^{n+1}\sum_{k=0}^{n} a_{jk} = \sum_{0\le j\le k\le n} a_{jk} + \sum_{0\le k<j\le n+1} a_{jk}.$$

Solution 1.1. Let $n = 0$. Then we have

$$\sum_{j=0}^{n+1}\sum_{k=0}^{n} a_{jk} = a_{00} + a_{10} = \sum_{0\le j\le k\le n} a_{jk} + \sum_{0\le k<j\le n+1} a_{jk}.$$

Now assume that the formula holds true for some $n \in \omega$. Then

$$\begin{aligned}
\sum_{j=0}^{n+2}\sum_{k=0}^{n+1} a_{jk} &= \sum_{j=0}^{n+2}\left(\sum_{k=0}^{n} a_{jk} + a_{j,n+1}\right)\\
&= \sum_{j=0}^{n+1}\left(\sum_{k=0}^{n} a_{jk} + a_{j,n+1}\right) + \left(\sum_{k=0}^{n} a_{n+2,k} + a_{n+2,n+1}\right)\\
&= \sum_{j=0}^{n+1}\sum_{k=0}^{n} a_{jk} + \sum_{j=0}^{n+1} a_{j,n+1} + \sum_{k=0}^{n+1} a_{n+2,k}\\
&= \sum_{0\le j\le k\le n} a_{jk} + \sum_{0\le k<j\le n+1} a_{jk} + \sum_{j=0}^{n+1} a_{j,n+1} + \sum_{k=0}^{n+1} a_{n+2,k}\\
&= \sum_{0\le j\le k\le n+1} a_{jk} + \sum_{0\le k<j\le n+2} a_{jk}.
\end{aligned}$$

Problem 1.2. Let $0 < x < 1$. Then

$$\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty} \frac{dt}{\cosh(\pi t) + \cos(\pi x)} = x.$$

Proof. We compute

$$\begin{aligned}
\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty} \frac{dt}{\cosh(\pi t) + \cos(\pi x)} &= \frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty} \frac{dt}{\frac{1}{2}(e^{\pi t} + e^{-\pi t}) + \cos(\pi x)}\\
&= \frac{\sin(\pi x)}{\pi}\int_{0}^{\infty} \frac{ds}{s^2 + 2\cos(\pi x)s + 1}\\
&= \frac{\sin(\pi x)}{\pi}\int_{0}^{\infty} \frac{ds}{(s + \cos(\pi x))^2 + \sin^2(\pi x)}\\
&= \frac{1}{\pi \sin(\pi x)}\int_{0}^{\infty} \frac{ds}{\left(\frac{s+\cos(\pi x)}{\sin(\pi x)}\right)^2 + 1}
\end{aligned}$$

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\begin{aligned}
&= \frac{1}{\pi} \int_{\cot(\pi x)}^{\infty} \frac{du}{u^2 + 1} \\
&= \frac{1}{\pi} \left(\frac{\pi}{2} - \arctan(\cot(\pi x)) \right) \\
&= x.
\end{aligned}
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$\square$