Yannis Bähni
yannis.baehni@uzh.ch

HOMEWORK 2: SYMPLECTIC FORMS VS. AREA AND VOLUME

YANNIS BÄHNI

Exercise 1.1. Let (M, ω) be a $2n$-dimensional symplectic manifold.

(a) ω^n is a volume form.
(b) Show that if M is compact, then $[\omega^n] \in H^{2n}_{\mathrm{dR}}(M)$ is nonzero.
(c) Conclude that $[\omega] \neq 0$.
(d) $\mathbb{S}^{2n}$ does not admit a symplectic structure for $n > 1$.

Solution 1.1. Part (a) immediately follows from the fact that for each $p \in M$ we have that $\omega^n_p \neq 0$. Thus ω^n is a nonvanishing form of top degree, hence a volume form.
For proving (b), assume that $[\omega^n] = 0$. Hence ω^n is exact. Thus there exists $\mu \in \Omega^{2n-1}(M)$ such that $\omega^n = d\mu$. But then Stoke's theorem [Lee13, p. 411] together with positivity [Lee13, p. 407] yields

$$0 < \int_M \omega^n = \int_M d\mu = \int_{\partial M} \mu = \int_{\varnothing} \mu = 0$$

since M is oriented by part (a) and ω^n is a positively oriented orientation form (see [Lee13, p. 381]).
For proving (c), observe that $[\omega^n] = [\omega] \cup \cdots \cup [\omega]$, where $\cup$ is the so-clalled cup product (see [Lee13, p. 464]). So if $[\omega] = 0$, we have by bilinearity also $[\omega^n] = 0$, which contradicts part (b).
For proving (d), by [Lee13, p. 450] we have that

$$H^p_{\mathrm{dR}}(\mathbb{S}^n) \cong \begin{cases} \mathbb{R} & p = 0 \text{ or } p = n, \\ 0 & 0 < p < n, \end{cases}$$

for $n \geq 1$. Let $n > 1$. Assume that $(\mathbb{S}^n, \omega)$ is a symplectic manifold. Since $\mathbb{S}^n$ is compact, part (c) implies that $[\omega] \neq 0$. But $[\omega] \in H^2_{\mathrm{dR}}(\mathbb{S}^{2n}) \cong 0$.

Example 1.1. Consider the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$, where ω_0 is the standard symplectic structure on $\mathbb{R}^{2n}$. Clearly, $\mathbb{R}^{2n}$ is not compact and ω_0 is exact since

$$d\left(\sum_{i=1}^n x^i dy^i\right) = \sum_{i=1}^n dx^i \wedge dy^i = \omega_0.$$

(Yannis Bähni) University of Zurich, Rämistrasse 71, 8006 Zurich
E-mail address: yannis.baehni@uzh.ch.

Yannis Bähni
yannis.baehni@uzh.ch

Example 1.2. Let M be a smooth manifold. Then (T^*M, ω) is a symplectic manfiold, where ω is the canonical symplectic form on T^*M. It is an exact form, since $\omega = -d\alpha$, where α is the tautological 1-form. Moreover, T^*M is not compact by problem 10-19 [Lee13, p. 271].

Exercise 1.2. Let (M, ω) be a $2n$-dimensional symplectic manifold.

(a)

Solution 1.2. For proving (a), we have using [Lee13, p. 117]

$$T_p\mathbb{S}^n = \{v \in \mathbb{R}^{n+1} : \langle v, p\rangle = 0\}$$

for each $p \in \mathbb{S}^n$. Consider the ***Euler vector field*** V defined by

$$V := x^i \frac{\partial}{\partial x^i}.$$

Then V is a unit normal vector field along $\mathbb{S}^n$. Indeed, if $p \in \mathbb{S}^n$ and $v \in T_p\mathbb{S}^n$ we have that

$$\langle p, v\rangle_{\overline{g}} = \langle p, v\rangle = 0$$

and

$$|p|_{\overline{g}} = |p| = 1.$$

Hence by [Lee13, p. 390], the volume form $\omega_{\mathring{g}}$ on $(\mathbb{S}^n, \mathring{g})$ is given by

$$\omega_{\mathring{g}} = \iota_{\mathbb{S}^n}^*\big(i_V \omega_{\overline{g}}\big).$$

More precisely, in the case $n = 2$ we have

$$\begin{aligned}
i_V\omega_{\overline{g}} &= i_V(dx \wedge dy \wedge dz)\\
&= (i_V dx) \wedge dy \wedge dz - dx \wedge i_V(dy \wedge dz)\\
&= (i_V dx) \wedge dy \wedge dz - dx \wedge (i_V dy) \wedge dz + dx \wedge dy \wedge (i_V dz)\\
&= x dy \wedge dz + y dz \wedge dx + z dx \wedge dy.
\end{aligned}$$

For $v, w \in T_p\mathbb{S}^2$, $p \in \mathbb{S}^2$, we have that

$$\omega_{\mathring{g}}|_p(v, w) = (i_V\omega_{\overline{g}})_{\iota(p)}\big(d\iota_p(v), d\iota_p(w)\big) = (i_V\omega_{\overline{g}})|_p(v, w)$$

under the usual identification of $T_p\mathbb{S}^n$ as a linear subspace of $T_p\mathbb{R}^{n+1}$. Finally

$$\begin{aligned}
\omega_{\mathring{g}}(v, w) &= (x dy \wedge dz + y dz \wedge dx + z dx \wedge dy)(v, w)\\
&= x \det\begin{pmatrix} dy(v) & dz(v)\\ dy(w) & dz(w)\end{pmatrix} + y \det\begin{pmatrix} dz(v) & dx(v)\\ dz(w) & dx(w)\end{pmatrix} + z \det\begin{pmatrix} dx(v) & dy(v)\\ dx(w) & dy(w)\end{pmatrix}\\
&= x(v^2w^3 - w^2v^3) + y(v^3w^1 - w^3v^1) + z(v^1w^2 - w^1v^2)\\
&= \langle p, v \times w\rangle
\end{aligned}$$

Yannis Bähni
yannis.baehni@uzh.ch

for $p := (x,y,z) \in \mathbb{S}^2$ using [Lee13, p. 356].
For proving (b), consider cylindrical polar coordinates (θ, z) on $\mathbb{S}^2$ given by

$$(x,y,z) = (\sqrt{1-z^2}\cos\theta, \sqrt{1-z^2}\sin\theta, z).$$

Then we get

$$\begin{aligned}
i_V\omega_{\overline{g}} &= \mathrm{id}^*(i_V\omega_{\overline{g}}) \\
&= \mathrm{id}^*(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy) \\
&= \sqrt{1-z^2}\cos\theta d(\sqrt{1-z^2}\sin\theta) \wedge dz + \sqrt{1-z^2}\sin\theta dz \wedge d(\sqrt{1-z^2}\cos\theta) \\
&\quad + zd(\sqrt{1-z^2}\cos\theta) \wedge d(\sqrt{1-z^2}\sin\theta) \\
&= \sqrt{1-z^2}\cos\theta\left(\sqrt{1-z^2}\cos\theta d\theta - \frac{z}{\sqrt{1-z^2}}\sin\theta dz\right) \wedge dz \\
&\quad - \sqrt{1-z^2}\sin\theta dz \wedge \left(\sqrt{1-z^2}\sin\theta d\theta + \frac{z}{\sqrt{1-z^2}}\cos\theta dz\right) \\
&\quad - z\left(\sqrt{1-z^2}\sin\theta d\theta + \frac{z}{\sqrt{1-z^2}}\cos\theta dz\right) \\
&\quad \wedge \left(\sqrt{1-z^2}\cos\theta d\theta - \frac{z}{\sqrt{1-z^2}}\sin\theta dz\right) \\
&= (1-z^2)\cos^2\theta d\theta \wedge dz - (1-z^2)\sin^2\theta dz \wedge d\theta + z^2\sin^2\theta d\theta \wedge dz \\
&\quad - z^2\cos^2\theta dz \wedge d\theta \\
&= d\theta \wedge dz.
\end{aligned}$$

For proving (c), just observe that

$$\mathrm{Vol}(\mathbb{S}^2) = \int_{\mathbb{S}^2} \omega_{\mathring{g}} = \int_{(0,2\pi)\times(-1,1)} d\theta \wedge dz = 4\pi.$$