MAT602 - FUNCTIONAL ANALYSIS

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Contents

1. Structures

1.1. Topological Spaces.

Theorem 1.1 (Urysohn's Lemma). *Suppose X is a normal topological space. Given disjoint closed subsets $A, B \subseteq X$, there exists a continuous function $f : X \to [0,1]$ such that $f|_A = 0$ and $f|_B = 1$.*

1.2. Metric Spaces.

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1.3. Normed Spaces.

Proposition 1.1 (Sequence Spaces). *For $1 \leq p < \infty$ define*

$$\ell^p(\mathbb{K}) := \left\{x \in \mathbb{K}^{\mathbb{N}} : \textstyle\sum_{k\in\mathbb{N}} |x_k| < \infty\right\} \tag{1}$$

and for $p = \infty$

$$\ell^\infty(\mathbb{K}) := \left\{x \in \mathbb{K}^{\mathbb{N}} : \sup_{k\in\mathbb{N}} |x_k| < \infty\right\}. \tag{2}$$

Moreover, for $x \in \ell^p(\mathbb{K})$ set

$$\|x\|_p := \left(\sum_{k\in\mathbb{N}} |x_k|^p\right)^{1/p} \tag{3}$$

for $1 \leq p < \infty$ and

$$\|x\|_\infty := \sup_{k\in\mathbb{N}} |x_k|. \tag{4}$$

Then $(\ell^p, \|\cdot\|_p)$ is a Banach space for all $1 \leq p \leq \infty$.

Theorem 1.2 (Completion of Normed Spaces). *Every normed space X has a completion which is unique up to isometric isomorphisms.*

1.4. Hilbert Spaces.

Lemma 1.1 (Cauchy-Schwarz Inequality). *Let $(H, \langle\cdot,\cdot\rangle)$ be a pre-Hilbert space. Then*

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle \tag{5}$$

for all $x, y \in H$.

Theorem 1.3. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subseteq H$ a closed convex set in H and $x_0 \in H$. Then there exists a unique $y \in K$ such that*

$$\|x_0 - y\| = \operatorname{dist}(x_0, K) = \inf_{x\in K} \|x_0 - x\|. \tag{6}$$

Theorem 1.4 (Orthogonal Complement). *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $M \subseteq H$ a closed linear subspace. Then $M^\perp \subseteq H$ is a closed linear subspace and $H = M \oplus M^\perp$.*

Theorem 1.5. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and $(x_\alpha)_{\alpha\in A}$ an orthonormal system. Then the following statements are equivalent:*

(i) *$(x_\alpha)_{\alpha\in A}$ is a Hilbert space basis.*

(ii) *For all $x \in H$ we have that $x = \sum_{\alpha\in A} \langle x_\alpha, x\rangle x_\alpha$.*

(iii) *For all $x \in H$ we have that $\|x\|^2 = \sum_{\alpha\in A} |\langle x_\alpha, x\rangle|^2$.*

(iv) *$\langle x_\alpha, x\rangle = 0$ for all $\alpha \in A$ implies $x = 0$.*

(v) *$(x_\alpha)_{\alpha\in A}$ is a maximal orthonormal system.*

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2. Function Spaces

2.1. Continuous Functions on Compact Spaces.

Theorem 2.1 (Stone-Weierstrass, $\mathbb{R}$). *Let A be a subalgebra of $C_{\mathbb{R}}(K)$ separating the points of K. Then we have either $\overline{A} = C_{\mathbb{R}}(K)$ or there exists a unique $x_0 \in K$ such that $\overline{A} = \{f \in C_{\mathbb{R}}(K) : f(x_0) = 0\}$.*

3. Linear Operators

3.1. Continuous Operators.

Definition 3.1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. An **operator** is a linear mapping $T : X \to Y$. Moreover, we say that an operator $T : X \to Y$ is **bounded** if there exists $c > 0$ such that*

$$\|T(x)\|_Y \leq c\|x\|_X \tag{7}$$

holds for all $x \in X$.

3.2. The Hahn-Banach Theorem.

Lemma 3.1. *Let V be a real vector space, $S \subsetneq V$ a linear subspace, $p : V \to \mathbb{R}$ a sublinear functional, $f : S \to \mathbb{R}$ linear and $x_0 \in V \setminus S$. Moreover, assume that $f \leq p$ on S. Then there exists $F : S + \mathbb{R}x_0 \to \mathbb{R}$ linear such that $F \leq p$ on $S + \mathbb{R}x_0$ and $F|_S = f$.*

Theorem 3.1 (Hahn-Banach, $\mathbb{R}$). *Let V be a vector space over $\mathbb{R}$, $S \subseteq V$ a linear subspace and $f : S \to \mathbb{R}$ linear. Moreover, let $p : V \to \mathbb{R}$ be a sublinear functional such that $f \leq p$ on S. Then there exists $F : V \to \mathbb{R}$ linear such that $F \leq p$ on V and $F|_S = f$.*

Theorem 3.2 (Hahn-Banach, $\mathbb{R}$ or $\mathbb{C}$). *Let V be a vector space over $\mathbb{K}$, $q : V \to \mathbb{R}$ a seminorm, $S \subseteq V$ a linear subspace and $f : S \to \mathbb{K}$ linear with $|f| \leq q$ on S. Then there exists $F : V \to \mathbb{K}$ linear with $F|_S = f$ and $|F| \leq q$ on V.*

Corollary 3.1 (Extension). *Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K}$, $S \subseteq X$ a linear subspace and $f \in S^*$. Then there exists $F \in X^*$ such that $F|_S = f$ and $\|F\|_{X^*} = \|f\|_{S^*}$.*

Corollary 3.2 (Separation). *Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K}$ and $x_0 \in X \setminus \{0\}$. Then there exists $f \in X^*$ with $\|f\| = 1$ and $f(x_0) = \|x_0\|$.*

3.3. Reflexivity.

Proposition 3.1. *Let X be a normed vector space over $\mathbb{K}$. Then the mapping $\Phi : X \to X^{**}$ defined by $\Phi(x) := \varphi_x$, where $\varphi_x : X^* \to \mathbb{R}$ is defined by $\varphi_x(f) := f(x)$, is a linear isometry.*

Theorem 3.3. *Let X be a Banach space. Then X is reflexive if and only if X^* is reflexive.*

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3.4. Hilbert Space Methods.

Theorem 3.4 (Riesz's Representation Theorem). *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$. The mapping $\Psi : H \to H^*$ defined by $\big(\Psi(x)\big)(y) := \langle x, y\rangle$ is an anti-linear isometric isomorphism.*

Corollary 3.3. *Every Hilbert space is reflexive.*

Theorem 3.5 (Lax-Milgram). *Let H be a Hilbert space over $\mathbb{K}$ and let $a : H \times H \to \mathbb{K}$ be a sesquilinear form. Moreover, suppose that there are constants $0 < c_0 \le C_0 < \infty$ such that*

$$\begin{aligned} |a(x,y)| &\le C_0\|x\|\|y\| \quad \textit{(}\textbf{\textit{Continuity}}\textit{)}, \\ \operatorname{Re} a(x,x) &\ge c_0\|x\|^2 \quad \textit{(}\textbf{\textit{Coercivity}}\textit{)}, \end{aligned}$$

for all $x, y \in H$. Then there exists a unique $A \in \mathcal{L}(H)$ such that

$$a(x,y) = \langle Ax, y\rangle \tag{8}$$

for all $x, y \in H$. Moreover, A is invertible with

$$\|A\| \le C_0 \qquad \textit{and} \qquad \|A^{-1}\| \le \frac{1}{c_0}. \tag{9}$$

4. Baire Category Theorem

4.1. Baire Category Theorem and Banach-Steinhaus.

Theorem 4.1 (Baire Category Theorem). *Every complete metric space is a Baire space.*

Theorem 4.2 (Banach-Steinhaus). *Let X be a Banach space, Y a normed space and $\mathcal{F} \subseteq \mathcal{L}(X,Y)$. Assume that for all $x \in X$ there exists $c_x \ge 0$ such that*

$$\sup_{T\in\mathcal{F}} \|T(x)\| \le c_x. \tag{10}$$

Then there exists $c \ge 0$ with

$$\sup_{T\in\mathcal{F}} \|T\| \le c. \tag{11}$$

4.2. The Open Mapping and Closed Graph Theorems.

Theorem 4.3 (Open Mapping Theorem). *Let X and Y be two Banach spaces and $T \in \mathcal{L}(X,Y)$ surjective. Then $T(U)$ is open for all $U \subseteq X$ open.*

Theorem 4.4 (Inverse Mapping Theorem). *Let X and Y be two Banach spaces and $T \in \mathcal{L}(X,Y)$ bijective. Then $T^{-1} \in \mathcal{L}(Y,X)$.*

Theorem 4.5 (Closed Graph Theorem). *Let X and Y be Banach spaces and $T : X \to Y$ linear. The following statements are equivalent:*

(i) $T \in \mathcal{L}(X,Y)$.
(ii) The graph of f, $\Gamma(f)$, is closed in $(X \times Y, \|\cdot\|_X + \|\cdot\|_Y)$.