Yannis Bähni
yannis.baehni@uzh.ch

SOLUTIONS SHEET 1

YANNIS BÄHNI

Exercise 1. For reference, the topology in part a) is the so called ***countable complement topology*** (see [Lee11, p. 45]) and part b) can be found in [Mun00, p. 169] (of course, I did not take the proof from there).
a) Let X be an arbitrary set. Clearly $\varnothing, X \in \mathcal{T}$ since $X^c = \varnothing$ is countable. Let $(U_\iota)_{\iota \in I}$ be a family of sets in $\mathcal{T}$. If $U_\iota = \varnothing$, then $\cup_{\iota \in I} U_\alpha = \varnothing \in \mathcal{T}$. So assume that $U_{\iota_0} \neq \varnothing$ for some $\iota_0 \in A$. But then $U_{\iota_0}^c$ is countable, and so is $(\cup_{\iota \in I} U_\iota)^c = \cap_{\iota \in I} U_\alpha^c \subseteq U_{\iota_0}^c$. Lastly, let $U_1, \dots, U_n \in \mathcal{T}$ for $n \in \mathbb{Z}$, $n \geq 1$. If $U_\iota = \varnothing$ for some ι, then $\cap_{\iota=1}^n U_\iota = \varnothing$ and thus $\cap_{\iota=1}^n U_\iota \in \mathcal{T}$. So assume that $U_\iota \neq \varnothing$ for $\iota = 1, \dots, n$. Then $(\cap_{\iota=1}^n U_\iota)^c = \cup_{\iota=1}^n U_\iota^c$ which is a finite union of countable sets, which is countable. Hence $\mathcal{T}$ is indeed a topology on X.

b) Assume that there is a family $(A_\iota)_{\iota \in I}$ of closed subsets of X having the finite intersection property such that $\cap_{\iota \in I} A_\iota = \varnothing$. Then since each A_ι is closed and $\cup_{\iota \in I} A_\iota^c = (\cap_{\iota \in I} A_\iota)^c = X$ we have that $(A_\iota^c)_{\iota \in I}$ is a cover for X. Now for any $J \subseteq I$ finite, the intersection $\cap_{\iota \in J} Aa_\iota$ is nonempty. This implies, that $\cup_{\iota \in J} A_\iota^c \neq X$ and thus the cover $(A_\iota^c)_{\iota \in I}$ of X does not possess a finite subcover, hence X is not compact.
Conversly, suppose that there exists a cover $(A_\iota)_{\iota \in I}$ which does not posses a finite subcover. Thus we have for any $J \subseteq I$ finite, that $\cup_{\iota \in J} A_\iota \neq X$ or equivalently, $\cap_{\iota \in J} A_\iota^c \neq \varnothing$. Thus the family $(A_\iota^c)_{\iota \in I}$ has the finite intersection property and each A_ι^c is closed since A_ι is open. But since $(A_\iota)_{\iota \in I}$ is a cover for X, we have that $\cap_{\iota \in I} A_\iota^c = \varnothing$.
Exercise 2.
a) Clearly, $\varnothing, X \in \mathcal{T}_d$. Let $(U_\iota)_{\iota \in I}$ be a family of elements in $\mathcal{T}_d$ and let $x \in \cup_{\iota \in I} U_\iota$. Then there exists $\iota \in I$ such that $x \in U_\iota$. Thus there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U_\iota$. Hence $B_\varepsilon(x) \subseteq \cup_{\iota \in I} U_\iota$. Lastly, let $U_1, \dots, U_n \in \mathcal{T}$ for $n \in \mathbb{Z}$, $n \geq 1$, and $x \in \cap_{\iota=1}^n U_\iota$. Hence there exist $\varepsilon_1, \dots, \varepsilon_n > 0$ such that $B_{\varepsilon_\iota}(x) \subseteq U_\iota$ and so $B_{\min_\iota \varepsilon_\iota}(x) \subseteq \cap_{\iota=1}^n U_\iota$. Thus $\mathcal{T}_d$ is a topology on X.

b) Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) := 1/x$. Then clearly $d_2 = \widetilde{d_2}|_M$, where

$$\widetilde{d_2} : (0, \infty) \times (0, \infty) \xrightarrow{f \times f} f\big((0,\infty)\big) \times f\big((0,\infty)\big) \xrightarrow{|\cdot,\cdot|} \mathbb{R}$$

(Yannis Bähni) University of Zurich, Rämistrasse 71, 8006 Zurich
E-mail address: yannis.baehni@uzh.ch.

Yannis Bähni
yannis.baehni@uzh.ch

and

$$d_1 : M \times M \xrightarrow{f \times f} f(M) \times f(M) \xrightarrow{\widetilde{d_2}} \mathbb{R}.$$

By [Lee11, p. 62] $f \times f$ is continuous and by [Eng89, p. 260] $|\cdot, \cdot|$ and $\widetilde{d_2}$ are continuous. Since two metrics induce the same topology if and only if they induce the same convergence (see [Eng89, p. 250]), we let $x \in M$ and $(x_n)_{n \in \mathbb{N}}$ be a sequence in M. Assume that $x_n \xrightarrow{d_1} x$. Then

$$d_2(x, x_n) = d_1\left(f(x), f(x_n)\right) \to d_1\left(f(x), f(x)\right) = 0$$

and

$$d_1(x, x_n) = d_2\left(f(x), f(x_n)\right) \to d_2\left(f(x), f(x)\right) = 0$$

by [Eng89, p. 260].
Now consider the sequence $(n)_{n \in \mathbb{N}}$ in M. Clearly, it is a Cauchy sequence since $(1/n)_{n \in \mathbb{N}}$ is a Cauchy sequence regarding the standard Euclidean metric and it cannot converge, since then it would also converge with respect to d_1 which is not the case. The completeness of (M, d_1) directly follows from the fact that a closed subspace of a complete metric space is complete.

References

[Eng89] Ryszard Engelking. *General Topology*. Revised and completed edition. Heldermann Verlag, 1989.

[Lee11] John M. Lee. *Introduction to Topological Manifolds*. Second Edition. Springer Science+Business Media, 2011.

[Mun00] James R. Munkres. *Topology*. Second edition. Prentice Hall, 2000.