INTRODUCTION TO CATEGORY THEORY AND ITS APPLICATIONS

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Contents

1. Representable Functors

1.1. The Yoneda Lemma.

Proposition 1.1. *Let $\mathscr{C}$ be a locally small category and $X \in \mathscr{C}$ an object. Define $\operatorname{Hom}_{\mathscr{C}}(X, -) : \mathscr{C} \to \mathsf{Set}$ on objects $Y \in \mathscr{C}$ by $\operatorname{Hom}_{\mathscr{C}}(X, Y) := \mathscr{C}(X, Y)$ and on morphisms $f : Y \to Z$ by post-composition with f, i.e.*

$$\operatorname{Hom}_{\mathscr{C}}(X, f) : \operatorname{Hom}_{\mathscr{C}}(X, Y) \to \operatorname{Hom}_{\mathscr{C}}(X, Z)$$

is defined by $\operatorname{Hom}_{\mathscr{C}}(X, f)(g) := f \circ g$. Then $\operatorname{Hom}_{\mathscr{C}}(X, -)$ is a functor.

Proof. □

Proposition 1.2. *Let $\mathscr{C}$ be a locally small category and $f \in \mathscr{C}(X, X')$ a morphism. Define $\eta^f := (\eta^f_A)_{A \in \mathscr{C}}$ by letting $\eta^f_A : \operatorname{Hom}_{\mathscr{C}}(X', A) \to \operatorname{Hom}_{\mathscr{C}}(X, A)$ be pre-composition with f, i.e. $\eta^f_A(g) := g \circ f$. Then $\eta^f : \operatorname{Hom}_{\mathscr{C}}(X', -) \Rightarrow \operatorname{Hom}_{\mathscr{C}}(X, -)$.*

Proof. □

Proposition 1.3. *Let $\mathscr{C}$ be a locally small category. Define $H^\bullet : \mathscr{C}^{\mathrm{op}} \to [\mathscr{C}, \mathsf{Set}]$ on objects $X \in \mathscr{C}$ by $H^\bullet(X) := \operatorname{Hom}_{\mathscr{C}}(X, -)$ and on morphisms $f \in \mathscr{C}^{\mathrm{op}}$ by $H^\bullet(f) := \eta^f$. Then $H^\bullet$ is a functor.*

Proof. □

Definition 1.1 (Representable and Corepresentable Functor). *Let $\mathscr{C}$ be a locally small category. A covariant functor F is said to be **representable**, if there exists an object $X \in \mathscr{C}$, such that $F \cong \operatorname{Hom}_{\mathscr{C}}(X, -)$. A contravariant functor F is said to be **corepresentable**, if there exists $X \in \mathscr{C}$ such that $F \cong \operatorname{Hom}_{\mathscr{C}}(-, X)$, where $\operatorname{Hom}_{\mathscr{C}}(-, X) := \operatorname{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, -)$.*

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The functor defined in the dualized statement of proposition 1.3 has its own name.

Definition 1.2 (Yoneda embedding of $\mathscr{C}$). *Let $\mathscr{C}$ be a locally small category. Then the functor $H_\bullet := H^\bullet : \mathscr{C} \to [\mathscr{C}^{\mathrm{op}}, \mathsf{Set}]$ is called the* ***Yoneda embedding of $\mathscr{C}$****.*

Theorem 1.1 (Yoneda Lemma). *Let $\mathscr{C}$ be a locally small category. For any functor $F : \mathscr{C} \to \mathsf{Set}$ and for every object $X \in \mathscr{C}$ there is a bijection*

$$[\mathscr{C}, \mathsf{Set}]\big(\mathrm{Hom}_{\mathscr{C}}(X, -), F\big) \cong F(X) \tag{1}$$

that associates to each $\alpha : \mathrm{Hom}_{\mathscr{C}}(X, -) \Rightarrow F$ the element $\alpha_X(\mathrm{id}_X) \in F(X)$. Moreover, the correspondence is natural in both X and F.