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HOMEWORK 1: SYMPLECTIC LINEAR ALGEBRA

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Exercise 1.1. Let (V, Ω) be a symplectic vector space and $Y, W \subseteq V$ be linear subspaces.

(a) $\dim Y + \dim Y^\Omega = \dim V$.
(b) $\left(Y^\Omega\right)^\Omega = Y$.
(c) $Y \subseteq W \Leftrightarrow W^\Omega \subseteq Y^\Omega$.
(d) $\Omega|_Y$ nondegenerate $\Leftrightarrow Y \cap Y^\Omega = \{0\} \Leftrightarrow V = Y \oplus Y^\Omega$.
(e) If $Y \subseteq Y^\Omega$, then $\dim Y \leq \frac{1}{2} \dim V$.
(f) If $\dim Y = 1$, then Y is isotropic.
(g) If Y is of codimension 1, then Y is coisotropic.
(h) Y lagrangian $\Leftrightarrow$ Y isotropic and coisotropic $\Leftrightarrow Y = Y^\Omega$.

Solution 1.1. For proving (a), consider the mapping $\Phi : V \to Y^*$ defined by $\Phi(v) := \Omega(v, \cdot)|_Y$. Clearly, $\ker \Phi = Y^\Omega$. Let $\varphi \in Y^*$. By exercise B.13 [Lee13, p. 623], there exists an extension $\widetilde{\varphi} \in V^*$ of φ, i.e. $\widetilde{\varphi}|_Y = \varphi$. Since $\widetilde{\Omega}$ is an isomorphism, there exists $v \in V$ such that $\widetilde{\varphi} = \Omega(v, \cdot)$. Which implies $\widetilde{\varphi}|_Y = \Omega(v, \cdot)|_Y$. Hence we get that Φ is surjective and thus the rank-nullity law [Lee13, p. 627] implies that

$$\dim V = \dim(\Phi(V)) + \dim(\ker \Phi) = \dim Y^* + \dim Y^\Omega = \dim Y + \dim Y^\Omega$$

since V is finite dimensional.
For proving (b), let $v \in Y$. Then for any $u \in Y^\Omega$ we have that $\Omega(v, u) = -\Omega(u, v) = 0$ and thus $Y \subseteq \left(Y^\Omega\right)^\Omega$. Hence Y is a linear subspace of $\left(Y^\Omega\right)^\Omega$. Furthermore part (a) yields

$$\dim Y = \dim V - \dim Y^\Omega = \dim \left(Y^\Omega\right)^\Omega.$$

Thus exercise B.4. (b) [Lee13, p. 620] implies that $\left(Y^\Omega\right)^\Omega = Y$.
For proving (c), suppose that $Y \subseteq W$ and let $v \in W^\Omega$. Then for any $u \in Y$ we have that $\Omega(v, u) = 0$ and thus $W^\Omega \subseteq Y^\Omega$. Conversly, suppose that $W^\Omega \subseteq Y^\Omega$. By part (b) we can also show that $\left(Y^\Omega\right)^\Omega \subseteq \left(W^\Omega\right)^\Omega$ holds. But this is easily seen.
For proving (d), we show the two equivalences separately. The first equivalence follows immediately from the observation that

$$\ker \widetilde{\Omega|_Y} = \{v \in Y : \forall u \in Y \ (\Omega|_Y(v, u) = 0)\} = Y \cap Y^\Omega.$$

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For showing the second equivalence, assume that $Y \cap Y^{\Omega} = \{0\}$. Part (a) and exercise B.8 [Lee13, p. 621] yield

$$\dim(Y + Y^{\Omega}) = \dim Y + \dim Y^{\Omega} - \dim(Y \cap Y^{\Omega}) = \dim Y + \dim Y^{\Omega} = \dim V.$$

Thus again exercise B.4. (b) [Lee13, p. 620] implies that $Y + Y^{\Omega} = V$. Therefore $V = Y \oplus Y^{\Omega}$. The other implication simply holds by the definition of the direct sum.
(e) directly follows from (a) and exercise B.4. [Lee13, p. 620] since

$$\dim V = \dim Y + \dim Y^{\Omega} \geq 2 \dim Y.$$

For proving (f), let $v \in Y \setminus \{0\}$. Then every element of Y can be written uniquely as λv for some $\lambda \in \mathbb{R}$. Thus by the alternating property of Ω we get that

$$\Omega(\lambda v, \mu v) = \lambda\mu\Omega(v, v) = 0$$

for all $\lambda, \mu \in \mathbb{R}$ and so $Y \subseteq Y^{\Omega}$.
For proving (g), we observe that part (a) yields $\dim Y^{\Omega} = 1$. Thus Y^{Ω} is isotropic by part (f) and therefore $Y^{\Omega} \subseteq (Y^{\Omega})^{\Omega} = Y$ by part (b).
For proving (h), we first observe that the second equivalence is trivially true. We show that Y is lagrangian if and only if $Y = Y^{\Omega}$. Assume that Y is lagrangian. From part (a) immediately follows that $\dim Y = \dim Y^{\Omega}$. Since $Y \subseteq Y^{\Omega}$ we get that $Y = Y^{\Omega}$. Conversly, assume that $Y = Y^{\Omega}$. Using again part (a) we get that $2 \dim Y = \dim V$.

Exercise 1.2. Let (V, Ω) be a symplectic vector space and E be a finite dimenisonal real vector space.

(a) $(E \oplus E^*, \Omega_0)$ is a symplectic vector space where

$$\Omega_0(u \oplus \alpha, v \oplus \beta) := \beta(u) - \alpha(v). \tag{1}$$

Moreover, if (e_i) is a symplectic basis for E, then $(e_i \oplus 0, 0 \oplus e_i^*)$ is a symplectic basis for $(E \oplus E^*, \Omega_0)$.
(b) If Y is a lagrangian subspace of V, then V is symplectomorphic to $(Y \oplus Y^*, \Omega_0)$.
(c) If Y is a lagrangian subspace of V, then any basis (e_i) of Y can be extended to a symplectic basis for V.

Remark 1.1. We intentionally switched the order of exercises which does feel in our view more natural. Clearly, the exericse was not mentioned to solve that way since we will use a more advanced concept from chapter 12. But we think that this solution adds some more generality to the theory.

Solution 1.2. For proving (a), we observe that bilinearity and skew-symmetry is immediate from the definition of Ω_0. Hence we have to show that Ω_0 is nondegenerate. Assume that $u \oplus \alpha \in \ker \Omega_0$. Hence we have that $\beta(u) = \alpha(v)$ for all $v \oplus \beta \in E \oplus E^*$. Assume that $u \neq 0$. Then we find $\beta \in E^*$ such that $\beta(u) \neq 0$. Setting $v = 0$ yields a contradiction and thus $u = 0$. Assume that $\alpha \neq 0$. Hence we find $v \in E$ such that $\alpha(v) \neq 0$. Hence setting $\beta = 0$ again yields a contradiction. Thus $\alpha = 0$ and so Ω_0 is

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nondegenerate. The symplectic form is canonical in the sense that its definition does not depend on a choice of a basis for $E \oplus E^*$. That $(e_i \oplus 0, 0 \oplus e_i^*)$ is a symplectic basis follows directly from

$$\Omega_0(e_i \oplus 0, e_j \oplus 0) = 0 = \Omega_0(0 \oplus e_i^*, 0 \oplus e_j^*)$$

and

$$\Omega_0(e_i \oplus 0, 0 \oplus e_j^*) = e_j^*(e_i) = \delta_{ij}.$$

To prove (b), let J be a compatible complex structure on (V, Ω) (the existence is assured by [Sil08, p. 84]). Thus lemma A.1 implies that $V = Y \oplus J(Y)$. Define a mapping $\varphi : Y \oplus J(Y) \to Y \oplus Y^*$ by

$$\varphi\left(x + J(y)\right) := x \oplus -\Phi\left(J(y)\right)$$

where $\Phi : J(y) \to Y^*$ is the isomorphism constructed in lemma A.1. Φ is easily seen to be an isomorphism. Moreover

$$\begin{aligned}(\varphi^*\Omega_0)\left(x + J(y), x' + J(y')\right) &= \Omega_0\left(\varphi(x + J(y)), \varphi(x' + J(y'))\right)\\ &= \Omega_0\left(x \oplus -\Omega\left(J(y), \cdot\right)\big|_Y, x' \oplus -\Omega\left(J(y'), \cdot\right)\big|_Y\right)\\ &= -\Omega\left(J(y'), x\right) + \Omega\left(J(y), x'\right)\\ &= \Omega\left(x + J(y), x' + J(y')\right).\end{aligned}$$

Hence φ is a symplectomorphism.
To prove (c), we observe that by part (a) and (b) and lemma A.2, $\left(\varphi^{-1}(e_i \oplus 0), \varphi^{-1}(0 \oplus e_i^*)\right)$ is a symplectic basis for V, but $\varphi^{-1}\left(u \oplus \alpha\right) = u - \Phi^{-1}(\alpha)$ and thus (e_i) is part of the symplectic basis.

Exercise 1.3. Let V be a finite dimensional real vector space.

(a) Any $\Omega \in \Lambda^2(V^*)$ is of the form

$$\Omega = \sum_{i=1}^{n} e_i^* \wedge f_i^* \tag{2}$$

where (u_i, e_i, f_i) is a basis for V provided by the standard form theorem [Sil08, p. 3].

(b) Assume $\dim V = 2n$ and $\Omega \in \Lambda^2(V^*)$. Then Ω is symplectic if and only if $\Omega^n \neq 0$.

Solution 1.3. For proving (a), we have that if (v_i) is any basis for V, then

$$\Omega = \sum_{i<j} \Omega(v_i, v_j) v_i^* \wedge v_j^*$$

by [Lee13, p. 353]. Thus the statement directly follows from the standard form theorem.
For proving (b), let (e_i, f_i) be a symplectic basis for V. By part (a) we have that

$$\Omega = \sum_{i=1}^{n} e_i^* \wedge f_i^*.$$

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We claim that

$$\left(\sum_{i=1}^{n} e_i^* \wedge f_i^*\right)^n = n!\left(e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^*\right). \tag{3}$$

We do a proof by induction over n. The formula trivially holds for $n = 1$. So assume that it holds for some $n \geq 1$. The binomial theorem yields

$$\begin{aligned}
\left(\sum_{i=1}^{n+1} e_i^* \wedge f_i^*\right)^{n+1} &= \left(\sum_{i=1}^{n+1} e_i^* \wedge f_i^*\right)^{n} \wedge \left(\sum_{i=1}^{n+1} e_i^* \wedge f_i^*\right) \\
&= \left(\sum_{i=1}^{n} e_i^* \wedge f_i^* + e_{n+1}^* \wedge f_{n+1}^*\right)^{n} \wedge \left(\sum_{i=1}^{n+1} e_i^* \wedge f_i^*\right) \\
&= \left(\sum_{k=0}^{n} \binom{n}{k} \left(\sum_{i=1}^{n} e_i^* \wedge f_i^*\right)^{k} \wedge \left(e_{n+1}^* \wedge f_{n+1}^*\right)^{n-k}\right) \wedge \left(\sum_{i=1}^{n+1} e_i^* \wedge f_i^*\right) \\
&= \left(\sum_{i=1}^{n} e_i^* \wedge f_i^*\right)^{n} \wedge \left(\sum_{i=1}^{n+1} e_i^* \wedge f_i^*\right) \\
&\quad + n\left(\sum_{i=1}^{n} e_i^* \wedge f_i^*\right)^{n-1} \wedge \left(e_{n+1}^* \wedge f_{n+1}^*\right) \wedge \left(\sum_{i=1}^{n+1} e_i^* \wedge f_i^*\right) \\
&= n!\left(e_1^* \wedge f_1^* \wedge \cdots \wedge e_{n+1}^* \wedge f_{n+1}^*\right) \\
&\quad + n\left(\sum_{i=1}^{n} e_i^* \wedge f_i^*\right)^{n} \wedge \left(e_{n+1}^* \wedge f_{n+1}^*\right) \\
&= (n+1)!\left(e_1^* \wedge f_1^* \wedge \cdots \wedge e_{n+1}^* \wedge f_{n+1}^*\right).
\end{aligned}$$

The use of the binomial theorem is justified since elements of $\Lambda^2(V^*)$ commute under the wedge product by [Lee13, p. 356]. Hence

$$\Omega^n = n!\left(e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^*\right) \neq 0.$$

Conversly, assume that $\Omega^n \neq 0$. Assume that Ω is degenerate. Hence there exists a basis

$$u_1, \ldots, u_k, e_1, \ldots, e_m, f_1, \ldots, f_m$$

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of f such that $m < n$. But then part (a) together with (3) yield

$$\Omega^n = m!(e_1^* \wedge f_1^* \wedge \cdots \wedge e_m^* \wedge f_m^*) \wedge \left(\sum_{i=1}^m e_i^* \wedge f_i^*\right)^{n-m} = 0.$$

Appendix A. Lagrangian Subspaces, Compatible Structures and Symplectic Bases

The next lemma is adapted from exercise 3. (a), homework 8 [Sil08, p. 88].

Lemma A.1. *Let (V, Ω) be a symplectic vector space, J be a compatible complex structure on V and Y a lagrangian subspace of V. Then $J(Y)$ is a lagrangian subspace, $V = Y \oplus J(Y)$ and $J(Y) \cong Y^*$.*

Proof. Clearly $\dim J(Y) = \dim Y = \frac{1}{2}\dim V$ since J is invertible and $J(Y) \subseteq J(Y)^\Omega$ since $\Omega(J\cdot, J\cdot) = \Omega(\cdot,\cdot)$ and $Y = Y^\Omega$. Hence $J(Y)$ is lagrangian and thus

$$V = Y \oplus Y^\perp = Y \oplus J(Y)^\Omega = Y \oplus J(Y)$$

by exercise B.45. [Lee13, p. 637]. Consider $\Phi : J(Y) \to Y^*$ defined by

$$\Phi\left(J(y)\right) := \Omega\left(J(y), \cdot\right)\big|_Y.$$

Let $J(y) \in \ker \Phi$. Then $\Omega\left(J(y), w\right) = 0$ for all $w \in Y$. Especially $\Omega\left(J(y), y\right) = 0$. But this is only possible if $y = 0$. Thus Φ is injective and due to dimensional reasons surjective, hence an isomorphism. □

Lemma A.2. *Let (V, Ω) and (V', Ω') be symplectic vector spaces and $\varphi : V \to V'$ a symplectomorphism. If (e_i, f_i) is a symplectic basis for V, then $\left(\varphi(e_i), \varphi(f_i)\right)$ is a symplectic basis for V'.*

Proof. We have that

$$\begin{aligned}
\Omega'\left(\varphi(e_i), \varphi(e_j)\right) &= (\varphi^*\Omega')(e_i, e_j) = \Omega(e_i, e_j) = 0\\
\Omega'\left(\varphi(f_i), \varphi(f_j)\right) &= (\varphi^*\Omega')(f_i, f_j) = \Omega(f_i, f_j) = 0\\
\Omega'\left(\varphi(e_i), \varphi(f_j)\right) &= (\varphi^*\Omega')(e_i, f_j) = \Omega(e_i, f_j) = \delta_{ij}.
\end{aligned}$$

□

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