Contents

CHAPTER 1

Foundations

Basic Category Theory

Categories. We use the first order theory of Neumann-Bernays-Gödel (BNG) as described in [Men15, p. 231].

Definition 1.1 (Category). *A **category** $\mathfrak{C}$ consists of*

- *A class* $\operatorname{ob}(\mathcal{C})$*, called the **objects of** $\mathfrak{C}$.*
- *A class* $\operatorname{mor}(\mathcal{C})$*, called the **morphisms of** $\mathfrak{C}$.*
- *Two functions* $\operatorname{dom} : \operatorname{mor}(\mathcal{C}) \to \operatorname{ob}(\mathcal{C})$ *and* $\operatorname{cod} : \operatorname{mor}(\mathcal{C}) \to \operatorname{ob}(\mathcal{C})$*, which assign to each morphism f in $\mathcal{C}$ its **domain** and **codomain**, respectively.*
- *For each $X \in \operatorname{ob}(\mathcal{C})$ a function $\operatorname{ob}(\mathcal{C}) \to \operatorname{mor}(\mathcal{C})$ which assigns a morphism id_X such that* $\operatorname{dom} \mathrm{id}_X = \operatorname{cod} \mathrm{id}_X = X$.
- *A function*
$$\circ : \{(g, f) \in \operatorname{mor}(\mathcal{C}) \times \operatorname{mor}(\mathcal{C}) : \operatorname{dom} g = \operatorname{cod} f\} \to \operatorname{mor}(\mathcal{C}) \tag{1}$$
*mapping (g, f) to $g \circ f$, called **composition**, such that* $\operatorname{dom}(g \circ f) = \operatorname{dom} f$ *and* $\operatorname{cod}(g \circ f) = \operatorname{cod} g$.

Subject to the following axioms:

- ***(Associativity Axiom)** For all $f, g, h \in \operatorname{mor}(\mathcal{C})$ with $\operatorname{dom} h = \operatorname{cod} g$ and $\operatorname{dom} g = \operatorname{cod} f$, we have that*
$$(h \circ g) \circ f = h \circ (g \circ f). \tag{2}$$
- ***(Unit Axiom)** For all $f \in \operatorname{mor}(\mathcal{C})$ with $\operatorname{dom} f = X$ and $\operatorname{cod} f = Y$ we have that*
$$f = f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f. \tag{3}$$

Remark 1.1. Let $\mathcal{C}$ be a category. For $X, Y \in \operatorname{ob}(\mathcal{C})$ we will abreviate
$$\mathcal{C}(X, Y) := \{f \in \operatorname{mor}(\mathcal{C}) : \operatorname{dom} f = X \text{ and } \operatorname{cod} f = Y\}.$$
Moreover, $f \in \mathcal{C}(X, Y)$ is depicted as
$$f : X \to Y. \tag{4}$$

Example 1.1. Let $*$ be a single, not nearer specified object. Consider as morphisms the class of all cardinal numbers and as composition cardinal addition. By [Hal12, pp. 112–113], cardinal addition is associative and $\varnothing$ serves for the identity id_*.

Yannis Bähni
yannis.baehni@uzh.ch

Definition 1.2 (Locally Small, Hom-Set). *A category $\mathcal{C}$ is said to be* ***locally small*** *if for all $X, Y \in \mathcal{C}$, $\mathcal{C}(X, Y)$ is a set. If $\mathcal{C}$ is locally small, $\mathcal{C}(X, Y)$ is called a* ***hom-set*** *for all $X, Y \in \mathcal{C}$.*

Functors.

Definition 1.3 (Functor). *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A* ***functor*** $\boldsymbol{F : \mathcal{C} \to \mathcal{D}}$ *is a pair of functions (F_1, F_2), $F_1 : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, called the* ***object function*** *and $F_2 : \mathrm{mor}(\mathcal{C}) \to \mathrm{mor}(\mathcal{D})$, called the* ***morphism function****, such that for every morphism $f : X \to Y$ we have that $F_2(f) : F_1(X) \to F_1(Y)$ and (F_1, F_2) is subject to the following* ***compatibility conditions****:*

- *For all $X \in \mathrm{ob}(\mathcal{C})$, $F_2(\mathrm{id}_X) = \mathrm{id}_{F_1(X)}$.*
- *For all $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ we have that $F_2(g \circ f) = F_2(g) \circ F_2(f)$.*

Remark 1.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. It is convenient to denote the components F_1 and F_2 also with F.

Subcategories.

Definition 1.4 (Subcategory). *Let $\mathcal{C}$ be a category. A* ***subcategory*** $\boldsymbol{\mathcal{S}}$ ***of*** $\boldsymbol{\mathcal{C}}$ *consists of*

- *A subclass $\mathrm{ob}(\mathcal{S}) \subseteq \mathrm{ob}(\mathcal{C})$.*
- *A subclass $\mathrm{mor}(\mathcal{S}) \subseteq \mathrm{mor}(\mathcal{C})$.*

Subject to the following conditions:

- *For all $X \in \mathcal{S}$, $\mathrm{id}_S \in \mathrm{mor}(\mathcal{S})$.*

Example 1.2 (Top_*). Define the objects of Top_* to be the class of all tuple (X, p), where X is a topological space and $p \in X$. Moreover, given objects (X, p) and (Y, q) in Top_*, define $\mathsf{Top}_*\big((X, p), (Y, q)\big) := \{f \in \mathsf{Top}(X, Y) : f(p) = q\}$. It is easy to check that Top_* is a category, called the ***category of pointed topological spaces***.

Limits.

Definition 1.5 (Diagram). *Let $\mathcal{C}$ be a category and A a small category. A functor $\mathsf{A} \to \mathcal{C}$ is called a* ***diagram in*** $\boldsymbol{\mathcal{C}}$ ***of shape*** $\boldsymbol{\mathsf{A}}$.

Definition 1.6 (Cone and Limit). *Let $\mathcal{C}$ be a category and $D : \mathsf{A} \to \mathcal{C}$ a diagram in $\mathcal{C}$ of shape A. A* ***cone on*** $\boldsymbol{D}$ *is a tuple $\big(C, (f_\alpha)_{\alpha \in \mathsf{A}}\big)$, where $C \in \mathcal{C}$ is an object, called the* ***vertex*** *of the cone, and a family of arrows in $\mathcal{C}$*

$$\big(C \xrightarrow{f_\alpha} D(\alpha)\big)_{\alpha \in \mathsf{A}}. \tag{5}$$

Yannis Bähni
yannis.baehni@uzh.ch

such that for all morphisms $f \in \mathsf{A}$, $f : \alpha \to \beta$, the triangle

$$\begin{tikzcd} & D(\alpha) \arrow[dd, "D(f)"] \\ C \arrow[ur, "f_\alpha"] \arrow[dr, "f_\beta"'] & \\ & D(\beta) \end{tikzcd}$$

*commutes. A **(small) limit of D** is a cone $\big(L, (\pi_\alpha)_{\alpha\in\mathsf{A}}\big)$ with the property that for any other cone $\big(C, (f_\alpha)_{\alpha\in\mathsf{A}}\big)$ there exists a unique morphism $\overline{f} : C \to L$ such that $\pi_\alpha \circ \overline{f} = f_\alpha$ holds for every $\alpha \in \mathsf{A}$.*

Remark 1.3. In the setting of definition 1.6, if $\big(L, (\pi_\alpha)_{\alpha\in\mathsf{A}}\big)$ is a limit of D, we sometimes reffering to L only as the limit of D and we write

$$L = \lim_{\leftarrow \mathsf{A}} D. \tag{6}$$

CHAPTER 2

The Fundamental Group

The Fundamental Grupoid

Construction of the fundamental Grupoid.

Lemma 2.1 (Gluing Lemma). *Let* $X, Y \in \mathrm{ob}(\mathsf{Top})$*,* $(X_\alpha)_{\alpha\in A}$ *a finite closed cover of* X *and* $(f_\alpha)_{\alpha\in A}$ *a finite family of maps* $f_\alpha \in \mathsf{Top}(X_\alpha, Y)$ *such that* $f_\alpha|_{X_\alpha\cap X_\beta} = f_\beta|_{X_\alpha\cap X_\beta}$ *for all* $\alpha, \beta \in A$*. Then there exists a unique* $f \in \mathsf{Top}(X, Y)$ *such that* $f|_{X_\alpha} = f_\alpha$ *for all* $\alpha \in A$*.*

Proof. Let $x \in X$. Since $(X_\alpha)_{\alpha\in A}$ is a cover of X, we find $\alpha \in A$ such that $x \in X_\alpha$. Define $f(x) := f_\alpha(x)$. This is well defined, since if $x \in X_\alpha \cap X_\beta$ for some $\beta \in A$, we have that $f(x) = f_\beta(x) = f_\alpha(x)$. Clearly $f|_{X_\alpha} = f_\alpha$ for all $\alpha \in A$ and f is unique. Let us show continuity. To this end, let $K \subseteq Y$ be closed. Then

$$\begin{aligned}
f^{-1}(K) &= X \cap f^{-1}(K) \\
&= \bigcup_{\alpha\in A} X_\alpha \cap f^{-1}(K) \\
&= \bigcup_{\alpha\in A} \left(X_\alpha \cap f^{-1}(K)\right) \\
&= \bigcup_{\alpha\in A} \left(X_\alpha \cap f_\alpha^{-1}(K)\right).
\end{aligned}$$

Since each f_α is continuous, $f_\alpha^{-1}(K)$ is closed in X_α for each $\alpha \in A$ and thus since X_α is closed, $f^{-1}(K)$ is closed as a finite union of closed sets. $\square$

Theorem 2.1. *There is a functor* $\mathsf{Top} \to \mathsf{Grpd}$.

Proof. The proof is divided into several steps. Let us denote $\Pi : \mathsf{Top} \to \mathsf{Grpd}$ for the claimed functor.

Step 1: Definition of Π *on objects.* Let $X, Y \in \mathrm{ob}(\mathsf{Top})$, $f, g \in \mathsf{Top}(X, Y)$ and $A \subseteq X$. A map $F \in \mathsf{Top}(X \times I, Y)$ is called a ***homotopy from*** $\boldsymbol{X}$ ***to*** $\boldsymbol{Y}$ ***relative to*** $\boldsymbol{A}$, if

- $F(x, 0) = f(x)$, for all $x \in X$.
- $F(x, 1) = g(x)$, for all $x \in X$.
- $F(x, t) = f(x) = g(x)$, for all $x \in A$ and for all $t \in I$.

Yannis Bähni
yannis.baehni@uzh.ch

If there exists a homotopy between f and g relative to A we say that f and g are ***homotopic relative to A*** and write $f \simeq_A g$. If we want to emphasize the homotpoy relative to A, we write $F : f \simeq_A g$.

Lemma 2.2. *Let $X, Y \in \operatorname{ob}(\mathsf{Top})$ and $A \subseteq X$. Then being homotopic relative to A is an equivalence relation on $\mathsf{Top}(X, Y)$.*

Proof. Define a binary relation $R_A \subseteq \mathsf{Top}(X, Y) \times \mathsf{Top}(X, Y)$ by

$$fR_Ag \quad :\Leftrightarrow \quad f \simeq_A g.$$

Let $f \in \mathsf{Top}(X, Y)$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x, t) := f(x).$$

Then clearly $F : f \simeq_A f$. Hence R_A is reflexive.
Let $g \in \mathsf{Top}(X, Y)$ and assume that fR_Ag. Thus $G : f \simeq_A g$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x, t) := G(x, 1 - t).$$

Then it is easy to check that $F : g \simeq_A f$ and so R_A is symmetric.
Finally, let $h \in \mathsf{Top}(X, Y)$ and suppose that fR_Ag and gR_Ah. Hence $F_1 : f \simeq_A g$ and $F_2 : g \simeq_A h$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x, t) := \begin{cases} F_1(x, 2t) & 0 \leq t \leq \frac{1}{2}, \\ F_2(x, 2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Continuity of F follows by an application of the gluing lemma 2.1. Then it is easy to check that $F : f \simeq_A h$ and hence R_A is transitive. □

Let $X \in \operatorname{ob}(\mathsf{Top})$ and u a path in X from p to q. Define the ***path class [u] of u*** by $[u] := [u]_{R_{\partial I}}$. Define now

- $\operatorname{ob}\big(\Pi(X)\big) := X$.
- $\Pi(X)(p, q) := \{[u] : u \text{ is a path from } p \text{ to } q\}$ for all $p, q \in X$.
- Let $p \in X$. Then define $\mathrm{id}_p \in \Pi(X)(p, p)$ by $\mathrm{id}_p := [c_p]$, where c_p is the constant path defined by $c_p(s) := p$ for all $s \in I$.
- And $\Pi(X)(q, r) \times \Pi(X)(p, q) \to \Pi(X)(p, r)$ by

  $$([v], [u]) \mapsto [u * v]$$

  Where $u * v \in \mathsf{Top}(p, r)$ is the ***concatenated path of u and v***, defined by

  $$(u * v)(s) := \begin{cases} u(2s) & 0 \leq t \leq \frac{1}{2}, \\ v(2s - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

  Continuity follows again from the gluing lemma 2.1 whereas well definedness follows from the next lemma.

Yannis Bähni
yannis.baehni@uzh.ch

Lemma 2.3. *Suppose that* $[u_1], [u_2] \in \Pi(X)(p,q)$ *and* $[v_1], [v_2] \in \Pi(X)(q,r)$ *such that* $[u_1] = [u_2]$ *and* $[v_1] = [v_2]$. *Then* $[u_1 * v_1] = [u_2 * v_2]$.

Proof. By assumption we have $G : u_1 \simeq_{\partial I} u_2$ and $H : v_1 \simeq_{\partial I} v_2$. Define $F \in \mathsf{Top}(I \times I, X)$ by

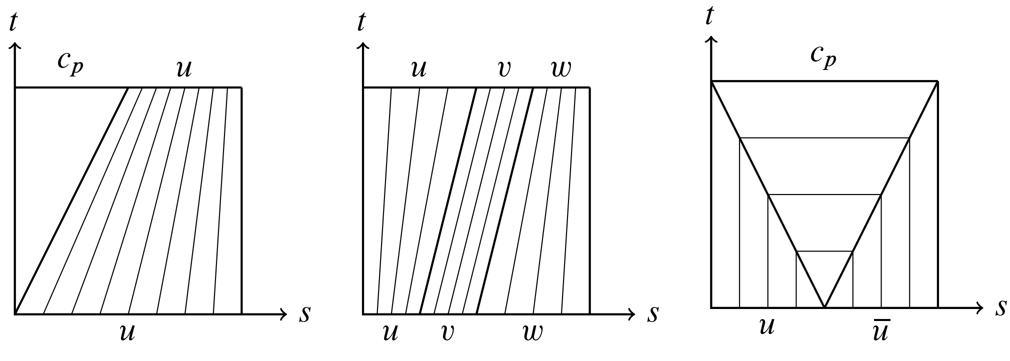
$$F(s,t) := \begin{cases} G(2s,t) & 0 \leq s \leq \frac{1}{2}, \\ H(2s-1,t) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Again, continuity follows from the gluing lemma 2.1 and it is easy to check that $F : u_1 * v_1 \simeq_{\partial I} u_2 * v_2$. □

Let us now check that $\Pi(X)$ is indeed a category. Let $[u] \in \Pi(X)(p,q)$. We want to show that $u \simeq_{\partial I} c_p * u$. To this end, we consider figure 1a and conclude that a suitable homotopy is given by $F \in \mathsf{Top}(I \times I, X)$ defined by

$$F(s,t) := \begin{cases} p & 0 \leq 2s \leq t, \\ u\left(\dfrac{2s-t}{2-t}\right) & t \leq 2s \leq 2. \end{cases}$$

Similarly, considering figure 1b leads to $F \in \mathsf{Top}(I \times I, X)$ defined by



(A) $u \simeq_{\partial I} c_p * u$. (B) $(u*v)*w \simeq_{\partial I} u*(v*w)$. (C) $u * \overline{u} \simeq_{\partial I} c_p$.

FIGURE 1. Visualization of the proof that $\Pi(X)$ is a grupoid object.

$$F(s,t) := \begin{cases} u\left(\dfrac{4s}{t+1}\right) & -1 \leq 4s-1 \leq t, \\ v(4s-t-1) & t \leq 4s-1 \leq t+1, \\ w\left(\dfrac{4s-t-2}{4-t-2}\right) & t+1 \leq 4s-1 \leq 3. \end{cases}$$

Yannis Bähni
yannis.baehni@uzh.ch

Lastly, we check that $\Pi(X)$ is a grupoid. To this end, for a path u from p to q, define its ***reverse path*** $\bar{u}$ by

$$\bar{u}(s) := u(1-s).$$

We claim that $u * \bar{u} \simeq_{\partial I} c_p$. From figure 1c we deduce that $F \in \mathsf{Top}(I \times I, X)$ is given by

$$F(s,t) := \begin{cases} u(2s) & 0 \leq 2s \leq 1-t, \\ u(1-t) & 1-t \leq 2s \leq t+1, \\ \bar{u}(2s-1) & t+1 \leq 2s \leq 2. \end{cases}$$

Step 2: Definition of Π *on morphisms.* Let $f \in \mathsf{Top}(X,Y)$. Then $\Pi(f)$ is a functor from $\Pi(X)$ to $\Pi(Y)$. Define $\Pi(f)$ as follows:

- Let $p \in \mathrm{ob}\big(\Pi(X)\big)$. Then define $\Pi(f)(p) := f(p) \in \mathrm{ob}\big(\Pi(Y)\big)$.
- Let $[u] \in \Pi(X)(p,q)$. Then define $\Pi(f)\,[u] := [f \circ u] \in$. We have to check that this definition is independent of the choice of the representative.

Lemma 2.4. *Let u and v be paths from p to q in X and suppose that $[u] = [v]$. Then for any $f \in \mathsf{Top}(X,Y)$ we also have that $[f \circ u] = [f \circ v]$.*

Proof. Suppose that $H : u \simeq_{\partial I} v$. Define $F \in \mathsf{Top}(I \times I, Y)$ by

$$F(s,t) := (f \circ F)(s,t).$$

Then $F : f \circ u \simeq_{\partial I} f \circ v$. □

Checking that Π satisfies the functorial properties is left as an exercise. □

Exercise 2.1. Check that $\Pi : \mathsf{Top} \to \mathsf{Grpd}$ is indeed a functor.

The Fundamental Group.

Lemma 2.5. *Let $\mathcal{G}$ be a locally small grupoid. Then for every $X \in \mathrm{ob}(\mathcal{G})$, $\mathcal{G}(X,X)$ can be equipped with a group structure.*

Proof. Since $\mathcal{G}$ is locally small, $\mathcal{G}(X,X)$ is a set for every $X \in \mathrm{ob}(\mathcal{G})$. Define a multiplication $\mathcal{G}(X,X) \times \mathcal{G}(X,X) \to \mathcal{G}(X,X)$ by $gh := h \circ g$. Clearly, this multiplication is associative. Moreover, the identity element is given by $\mathrm{id}_X \in \mathcal{G}(X,X)$ and since every $g \in \mathcal{G}(X,X)$ is an isomorphism, the multiplicative inverse is given by the inverse in $\mathcal{G}(X,X)$. □

Proposition 2.1. *There is a functor* $\mathsf{Top}_* \to \mathsf{Grp}$.

Proof. Define $\pi_1 : \mathsf{Top}_* \to \mathsf{Grp}$ on objects $(X,p) \in \mathsf{Top}_*$ by

$$\pi_1(X,p) := \Pi(X)(p,p).$$

Yannis Bähni
yannis.baehni@uzh.ch

By theorem 2.1 together with lemma 2.5, $\pi_1(X,p)$ is actually a group, called the ***fundamental group of X with basepoint p***. On morphisms $f \in \mathsf{Top}_*\big((X,p),(Y,q)\big)$, define

$$\pi_1(f) := \Pi(f) : \Pi(X)(p,p) \to \Pi(Y)(q,q).$$

Let $[u], [v] \in \pi_1(X,p)$. Then

$$\begin{aligned}\pi_1([u]\,[v]) &= \Pi(f)([u]\,[v])\\ &= \Pi(f)\,[u * v]\\ &= [f \circ (u * v)]\\ &= [(f \circ u) * (f \circ v)]\\ &= \Pi(f)\,[u]\,\Pi(f)\,[v]\\ &= \pi_1(f)\,[u]\,\pi_1(f)\,[v]\,.\end{aligned}$$

Thus $\pi_1(f)$ is a morphism in Grp. Functoriality of π_1 immediately follows from the functoriality of Π. □

Lemma 2.6. *Let $X \in \mathrm{ob}(\mathsf{Top})$, $p \in X$ and A be the path component of X containing p. Then $\pi_1(\iota)$, where $\iota : A \hookrightarrow X$ denotes the inclusion, is an isomorphism.*

Proof. Suppose $[u] \in \ker \pi_1(\iota)$. Then $[\iota \circ u] = [c_p]$ and Hence $F : \iota \circ u \simeq_{\partial I} c_p$. Since $I \times I$ is path connected and $p \in F(I \times I)$, it follows that $F(I \times I) \subseteq A$ and thus $F : u \simeq_{\partial I} c_p$ in A and hence $[u] = [c_p]$. To see that $\pi_1(\iota)$ is surjective, just observe that $u(I) \subseteq A$ for $[u] \in \pi_1(X,p)$ since $u(I)$ is path connected and $p \in u(I)$. □

Lemma 2.7. *Let $X \in \mathrm{ob}(\mathsf{Top})$ be path connected and $p, q \in X$. Then*

$$\pi_1(X,p) \cong \pi_1(X,q).$$

Proof. Since X is path connected we find a path v from p to q in X. Define a mapping $\Phi_v : \pi_1(X,p) \to \pi_1(X,q)$

$$\Phi_v\,[u] := [\overline{v} * u * v]\,.$$

Clearly, Φ_v is invertible with inverse $\Phi_{\overline{v}}$. Moreover, for $[u], [w] \in \pi_1(X,p)$ we have that

$$\begin{aligned}\Phi_v([u]\,[w]) &= \Phi_v\,[u * w]\\ &= [\overline{v} * u * w * v]\\ &= [\overline{v} * u * v * \overline{v} * w * v]\\ &= [\overline{v} * u * v]\,[\overline{v} * w * v]\\ &= \Phi_v\,[u]\,\Phi_v\,[w]\,.\end{aligned}$$

□

Yannis Bähni
yannis.baehni@uzh.ch

$\pi_1(\mathbb{S}^1)$.

Definition 2.1 (Exponential Quotient Map). *The mapping $\varepsilon : \mathbb{R} \to \mathbb{S}^1$ defined by*

$$\varepsilon(x) := e^{2\pi i x} \tag{7}$$

*is called the **exponential quotient map**.*

Proposition 2.2 (Lifting Property of the Circle). *Let $n \in \mathbb{Z}$, $n \geq 0$, $X \subseteq \mathbb{R}^n$ compact and convex, $p \in X$, $f \in \mathsf{Top}_*\left((X,p),(\mathbb{S}^1,1)\right)$ and $m \in \mathbb{Z}$. Then there exists a unique map $\widetilde{f} \in \mathsf{Top}_*\left((X,p),(\mathbb{R},m)\right)$, called the **lifting of f**, such that*

$$\begin{tikzcd} & (\mathbb{R},m) \arrow[d, "\varepsilon"] \\ (X,p) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (\mathbb{S}^1,1) \end{tikzcd}$$

commutes.

Proof. We show first existence and then uniqueness.

Step 1: Existence. Since X is compact and f is continuous, f is uniformly continuous on X. Thus we find $\delta > 0$ such that $|f(x) - f(y)| < 2$, whenever $|x - y| < \delta$, i.e. $f(x)$ and $f(y)$ are not antipodal points. Moreover, since X is compact, X is bounded and hence we find $N \in \mathbb{N}$, such that $|x - y| < N\delta$ holds for all $x, y \in X$. Let $x \in X$. For $0 \leq k \leq N$, define $L_k : X \to X$ by

$$L_k(x) := \left(1 - \frac{k}{N}\right)p + \frac{k}{N}x.$$

Those are well defined functions since X is convex. Moreover, each L_k is continuous. Indeed, it is easy to check that L_k is Lipschitz. Also, for each $0 \leq k < N$, $f(L_k(x))$ and $f(L_{k+1}(x))$ are not antipodal for all $x \in X$. Indeed, it is easy to check that $|L_k(x) - L_{k+1}(x)| < \delta$ holds for all $x \in X$. For $0 \leq k < N$ define $g_k : X \to \mathbb{S}^1 \setminus \{-1\}$ by

$$g_k(x) := \frac{f(L_{k+1}(x))}{f(L_k(x))}.$$

Clearly g_k is well defined and continuous as a composition of continuous functions. Let $\mathrm{Log} : \mathbb{S}^1 \setminus \{-1\} \to \mathbb{C}$ denote the principal branch of the logarithm. Define $\widetilde{f} : X \to \mathbb{R}$ by

$$\widetilde{f}(x) := m + \frac{1}{2\pi i}\sum_{k=0}^{N-1} \mathrm{Log}(g_k(x)).$$

Clearly, $\widetilde{f}$ is continuous and moreover we have that $\widetilde{f} = m$ since $g_k(p) = 1$ for all $0 \leq k < N$. Finally, for any $x \in X$ we have that

$$(\varepsilon \circ \widetilde{f})(x) = \varepsilon(m) \prod_{k=0}^{N-1} g_k(x) = \frac{f(L_N(x))}{f(L_0(x))} = \frac{f(x)}{f(p)} = f(x).$$

Step 2: Uniqueness. Suppose $\widetilde{g} \in \mathsf{Top}_*\big((X,p),(\mathbb{R},m)\big)$ is another such function. Define $\varphi \in \mathsf{Top}_*\big((X,p),(\mathbb{R},0)\big)$ by

$$\varphi(x) := \widetilde{f}(x) - \widetilde{g}(x).$$

Then clearly $\varepsilon \circ \varphi = 1$ and thus $\varphi(X) \subseteq \mathbb{Z}$. Since X is convex, X is connected and so $\varphi = 0$.

$\square$

Corollary 2.1. *Let $u, v \in \Omega(\mathbb{S}^1, 1)$ such that $[u] = [v]$. If $\widetilde{u}, \widetilde{v} : (I,0) \to (\mathbb{R},0)$ are the liftings of u and v, respectively, then $[\widetilde{u}] = [\widetilde{v}]$.*

Proof. Let $F : u \simeq_{\partial I} v$. By proposition 2.2, we find $\widetilde{F} \in \mathsf{Top}_*\big((I \times I, (0,0)), (\mathbb{R},0)\big)$, such that $\varepsilon \circ \widetilde{F} = F$. We claim that $\widetilde{F} : \widetilde{u} \simeq_{\partial I} \widetilde{v}$. For $s \in I$ define $\widetilde{u}_0(s) := \widetilde{F}(s,0)$. Then $\widetilde{u}_0(0) = \widetilde{F}(0,0) = 0$ and since $\widetilde{u}_0$ is continuous we have that $\widetilde{u}_0 \in \mathsf{Top}_*\big((I,0),(\mathbb{R},0)\big)$. Moreover

$$(\varepsilon \circ \widetilde{u}_0)(s) = \varepsilon\left(\widetilde{F}(s,0)\right) = F(s,0) = u(s)$$

for all $s \in I$ and thus $\widetilde{u}_0$ is a lifting of u. But by proposition 2.2, liftings are unique and thus $\widetilde{u}_0 = \widetilde{u}$. Next define $\widetilde{w}_0(t) := \widetilde{F}(0,t)$ for all $t \in I$. Then $\widetilde{w}_0(0) = \widetilde{F}(0,0) = 0$ and so $\widetilde{w}_0 \in \mathsf{Top}_*\big((I,0),(\mathbb{R},0)\big)$. Moreover

$$(\varepsilon \circ \widetilde{w}_0)(t) = \varepsilon\left(\widetilde{F}(0,t)\right) = F(0,t) = u(0) = v(0) = 1.$$

for all $t \in I$. Thus

$$\begin{tikzcd}
 & (\mathbb{R},0) \arrow[d, "\varepsilon"] \\
(I,0) \arrow[ur, "\widetilde{w}_0"] \arrow[r, "c_1"'] & (\mathbb{S}^1,1)
\end{tikzcd}$$

commutes. But also c_0 makes the above diagram commute. By uniqueness, $\widetilde{w}_0 = c_0$. Define $\widetilde{v}_0(s) := \widetilde{F}(s,1)$ for all $s \in I$. Then $\widetilde{v}_0(0) = \widetilde{F}(0,1) = \widetilde{w}_0(1) = 0$ and it is easy to check that $\widetilde{v}_0$ is a lift for v. Hence $\widetilde{v}_0 = \widetilde{v}$. Finally, define $\widetilde{w}_1(t) := \widetilde{F}(1,t)$ for all $t \in I$. Then $\widetilde{w}_1(0) = \widetilde{F}(1,0) = \widetilde{u}(1)$ and thus $\widetilde{w}_1 \in \mathsf{Top}_*\big((I,0),(\mathbb{R},\widetilde{u}(0))\big)$. Moreover

$$(\varepsilon \circ \widetilde{w}_1)(t) = \varepsilon\left(\widetilde{F}(1,t)\right) = F(1,t) = v(1) = u(1) = 1$$

for all $t \in I$. By proposition 2.2, we have again that $\widetilde{w}_1 = c_{\widetilde{u}(1)}$. So $F : \widetilde{u} \simeq_{\partial I} \widetilde{v}$. $\square$

Yannis Bähni
yannis.baehni@uzh.ch

Definition 2.2 (Degree). *Let* $u \in \Omega(\mathbb{S}^1, 1)$*. The* ***degree of u****, written* $\deg u$*, is defined by* $\deg u := \widetilde{u}(1)$*, where* $\widetilde{u}$ *is the unique lift of* u *such that* $\widetilde{u}(0) = 0$*.*

Theorem 2.2 (Fundamental Group of the Circle). $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.

Proof. Define $\deg : \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}$ by $\deg [u] := \deg u$. This is well defined by corollary 2.1, since if $[u] = [v]$, then $[\widetilde{u}] = [\widetilde{v}]$ and in particular $\widetilde{u}(1) = \widetilde{v}(1)$.
Step 1: $\deg \in \mathsf{Grp}\big(\pi_1(\mathbb{S}^1, 1), (\mathbb{Z}, +)\big)$. Let $[u], [v] \in \pi_1(\mathbb{S}^1, 1)$ and $m := \deg [u]$, $n := \deg [v]$. Moreover, let $\widetilde{u}$ and $\widetilde{v}$ denote the unique liftings of u and v, respectively, such that $\widetilde{u}(0) = 0$ and $\widetilde{v}(0) = 0$. Define

$$\widetilde{w}(s) := \begin{cases} \widetilde{u}(2s) & 0 \leq s \leq \frac{1}{2}, \\ m + \widetilde{v}(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Clearly $\widetilde{w}$ is continuous and $\widetilde{w}(0) = 0$. Hence $\widetilde{w} \in \mathsf{Top}_*\big((I, 0), (\mathbb{R}, 0)\big)$. Also we have that $\varepsilon \circ \widetilde{w} = u * v$ and thus $\widetilde{w}$ is the lift of $u * v$. But $\widetilde{w}(1) = m + n$ and so

$$\deg([u]\,[v]) = \deg [u * v] = \deg(u * v) = \widetilde{w}(1) = m + n = \deg [u] + \deg [v] .$$

Step 2: $\deg$ *is injective.* Suppose $\deg [u] = 0$. Then $\widetilde{u}(1) = 0$ and thus $\widetilde{u} \in \Omega(\mathbb{R}, 0)$. Since $\mathbb{R}$ is contractible, we have that $[\widetilde{u}] = [c_0]$ and thus

$$[u] = [\varepsilon \circ \widetilde{u}] = \pi_1(\varepsilon)\,[\widetilde{u}] = \pi_1(\varepsilon)\,[c_0] = [c_1] .$$

Thus $\ker(\deg)$ is trivial.

Step 3: $\deg$ *is surjective.* Let $m \in \mathbb{Z}$. Then

$$\deg [\varepsilon^m] = \deg \varepsilon^m = \widetilde{\varepsilon^m}(1) = m.$$

□

The Seifert-Van Kampen Theorem

Coproducts and Pushouts in Grp.

Proposition 2.3 (Coproducts in Grp). Grp *has all small coproducts.*

Proof. Let $A \in \mathrm{ob}(\mathsf{Set})$ and A be the small category defined as the discrete category with $\mathrm{ob}(\mathsf{A}) := A$, i.e.

$$\bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet \quad \bullet \quad \bullet$$

Let $D : \mathsf{A} \to \mathsf{Grp}$ be a functor. Hence we get a family $(G_\alpha)_{\alpha \in A}$ in Grp, where $G_\alpha := D(\alpha)$ for all $\alpha \in A$. A ***word*** in $(G_\alpha)_{\alpha \in A}$ is a finite sequence in $\coprod_{\alpha \in A} G_\alpha$. A word in $(G_\alpha)_{\alpha \in A}$ will simply be written as $(g_1, \dots, g_n)$, where $g_k \in G_\alpha$ for some $\alpha \in A$. The ***empty word*** is denoted by $()$. Let $\mathcal{W}$ denote the set of all words in $(G_\alpha)_{\alpha \in A}$. On $\mathcal{W}$ define a multiplication by ***concatenation***

$$(g_1, \dots, g_n)(h_1, \dots, h_m) := (g_1, \dots, g_n, h_1, \dots, h_m).$$

Yannis Bähni
yannis.baehni@uzh.ch

An ***elementary reduction*** is an operation of one of the following forms:

- $(g_1, \dots, g_k, g_{k+1}, \dots, g_n) \mapsto (g_1, \dots, g_k g_{k+1}, \dots, g_n$, where $g_k, g_{k+1} \in G_\alpha$ for some $\alpha \in A$.
- $(g_1, \dots, g_{k-1}, 1_\alpha, g_{k+1}, \dots, g_n) \mapsto (g_1, \dots, g_{k-1}, g_{k+1}, \dots, g_n)$.

Let $\sim$ denote the equivalence relation on $\mathcal{W}$ generated by elementary reductions.

Lemma 2.8. *$\mathcal{W}/\sim$ together with concatenation of representatives is an element of* Grp.

Proof. Define
$$[(g_1, \dots, g_n)]\,[(h_1, \dots, h_m)] := [(g_1, \dots, g_n, h_1, \dots, h_m)]\,.$$
It is left to the reader to show that this is well defined and that $\mathcal{W}/\sim$ is indeed a group. $\square$

The group defined in lemma 2.8 will be denoted by $\mathop{\Large\ast}_{\alpha \in A} G_\alpha$ and called the ***free product of $(G_\alpha)_{\alpha \in A}$***. Let us define a cocone on D. For this consider the inclusions $\iota_\alpha : G_\alpha \to \mathop{\Large\ast}_{\alpha \in A} G_\alpha$ defined by
$$\iota_\alpha(g) := [(g)]$$
for all $\alpha \in A$. It is immediate from
$$\iota_\alpha(gh) = [(gh)] = [(g, h)] = [(g)]\,[(h)] = \iota_\alpha(g)\iota_\alpha(h)$$
for $g, h \in G_\alpha$, that ι_α is a morphism of groups. Since there are only the identity morphisms in A, $\left(\mathop{\Large\ast}_{\alpha \in A} G_\alpha, (\iota_\alpha)_{\alpha \in A}\right)$ is a cocone on D. Let us show that this is in fact a universal cocone. To this end, suppose that $\left(C, (\varphi_\alpha)_{\alpha \in A}\right)$ is another cocone on D. Define a mapping $\overline{f} : \mathop{\Large\ast}_{\alpha \in A} G_\alpha \to C$ by
$$\overline{f}\,[(g_1, \dots, g_n)] := \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$$
where $g_k \in G_{\alpha_k}$. Then $\overline{f}$ is easily seen to be well defined since each φ_α is a morphism of groups. Moreover, if $g \in G_\alpha$, then
$$(\overline{f} \circ \iota_\alpha)(g) = \overline{f}\,[(g)] = \varphi_\alpha(g)$$
for all $\alpha \in A$. Suppose that $f : \mathop{\Large\ast}_{\alpha \in A} G_\alpha \to C$ is another homomorphism of groups such that $f \circ \iota_\alpha = \varphi_\alpha$ for all $\alpha \in A$. Then for $[(g_1, \dots, g_n)] \in \mathop{\Large\ast}_{\alpha \in A} G_\alpha$ we have
$$\begin{aligned}
f\,[(g_1, \dots, g_n)] &= f([(g_1)] \cdots [(g_n)]) \\
&= f\,[(g_1)] \cdots f\,[(g_n)] \\
&= f\,(\iota_{\alpha_1}(g_1)) \cdots f\,(\iota_{\alpha_n}(g_n)) \\
&= \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n) \\
&= \overline{f}\,[(g_1, \dots, g_n)]\,.
\end{aligned}$$
$\square$

Exercise 2.2. Check that $\mathcal{W}/\sim$ is indeed a group with the declared group structure and that $\overline{f}$ is indeed well defined.

Yannis Bähni
yannis.baehni@uzh.ch

Proposition 2.4 (Pushouts in Grp). Grp *has all pushouts.*

Proof. Consider the diagram $D : \mathsf{A} \to \mathsf{Grp}$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \quad \xrightarrow{D} \quad \begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \\ H_2 & \end{tikzcd}$$

and define N to be the normal subgroup of $H_1 * H_2$ generated by elements of the form $[(\varphi_1(g^{-1}), \varphi_2(g))]$ for $g \in G$. Let $K := (H_1 * H_2)/N$. Then

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \arrow[d, "\pi \circ \iota_1"] \\ H_2 \arrow[r, "\pi \circ \iota_2"'] & K \end{tikzcd}$$

commutes. Indeed, if $g \in G$, we have that $(\pi \circ \iota_1 \circ \varphi_1)(g) = [(\varphi_1(g))]\,N$ and similarly $(\pi \circ \iota_2 \circ \varphi_2)(g) = [(\varphi_2(g))]\,N$. Then

$$[(\varphi_1(g))]^{-1}\,[(\varphi_2(g))] = [(\varphi_1(g)^{-1})]\,[(\varphi_2(g))] = [(\varphi_1(g^{-1}))]\,[(\varphi_2(g))] \in N.$$

Suppose that we have another cocone on D:

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \arrow[d, "\psi_1"] \\ H_2 \arrow[r, "\psi_2"'] & C \end{tikzcd}$$

By proposition 2.3, there exists a unique morphism of groups $f : H_1 * H_2 \to C$ and we thus get the following diagram:

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \arrow[d, "\iota_1"] \arrow[ddrr, bend left, "\psi_1"] & & \\ H_2 \arrow[r, "\iota_2"'] \arrow[drrr, bend right, "\psi_2"'] & H_1 * H_2 \arrow[dr, "\pi"] \arrow[drr, bend right, "f"'] & & \\ & & K \arrow[r, dashed, "\bar{f}"] & C \end{tikzcd}$$

To show that $N \subseteq \ker f$ is left as an exercise. Hence by the factorization theorem (see [Gri07, p. 23]), f factors uniquely through π, i.e. there exists a unique morphism of groups $\bar{f} : K \to C$ such that $\bar{f} \circ \pi = f$. $\square$

Yannis Bähni
yannis.baehni@uzh.ch

Exercise 2.3. In the previous proposition, verify that $N \subseteq \ker f$.

Definition 2.3 (Amalgamated Free Product). *The pushout of a diagram*

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \\ H_2 & \end{tikzcd}$$

in Grp *is called the* ***amalgamated free product of H_1 and H_2 along $(G, \varphi_1, \varphi_2)$****, written* $H_1 *_{(G,\varphi_1,\varphi_2)} H_2$.

The Seifert-Van Kampen Theorem and its Consequences.

Theorem 2.3 (Seifert-Van Kampen). *Let $X \in \mathrm{ob}(\mathsf{Top})$, (U, V) an open cover for X, such that U, V and $U \cap V$ are path connected. Moreover, let $p \in U \cap V$. Then*

$$\pi_1(X, p) \cong \pi_1(U, p) *_{(\pi_1(U\cap V,p),\pi_1(\iota_U),\pi_1(\iota_V))} \pi_1(V, p), \tag{8}$$

where $\iota_U : U \cap V \hookrightarrow U$ and $\iota_V : U \cap V \hookrightarrow V$ denote inclusion.

CHAPTER 3

Singular Homology

Construction of the Singular Homology Functor

Free Abelian Groups.

Proposition 3.1. *The forgetful functor $U : \mathsf{AbGrp} \to \mathsf{Set}$ admits a left adjoint.*

Proof. We have to construct a functor $F : \mathsf{Set} \to \mathsf{AbGrp}$. Let S be a set. Define

$$F(S) := \left\{ f \in \mathbb{Z}^S : \operatorname{supp} f \text{ is finite} \right\}.$$

Equipped with pointwise addition, $F(S)$ is an abelian group. There is a natural inclusion $\iota : S \hookrightarrow U\left(F(S)\right)$ sending $x \in S$ to the function taking the value one at x and zero else. Hence we may regard elements of $F(S)$ as formal linear combinations $\sum_{x \in S} m_x x$, where $m_x \in \mathbb{Z}$ for all $x \in S$. On morphisms $f : S \to T$ in Set, define $F(f) : F(S) \to F(T)$ simply by setting $F(f)\left(\sum_{x \in S} m_x x\right) := \sum_{x \in S} m_x f(x)$.
Let $G \in \operatorname{ob}(\mathsf{AbGrp})$ be an abelian group and $\varphi \in \mathsf{AbGrp}\left(F(S), G\right)$ a morphism of groups. Define $\overline{\varphi} \in \mathsf{Set}\left(S, U(G)\right)$ by $\overline{\varphi} := U(\varphi)$. Conversly, if we have $f \in \mathsf{Set}\left(S, U(G)\right)$, define $\overline{f} \in \mathsf{AbGrp}\left(F(S), G\right)$ by $\overline{f}\left(\sum_{x \in S} m_x x\right) := \sum_{x \in S} m_x f(x)$. This is well defined since all but finitely many m_x are zero and G is abelian. It is easy to check that $\overline{f}$ is indeed a morphism of groups. Let $\varphi \in \mathsf{AbGrp}\left(F(S), G\right)$. Then

$$\begin{aligned}
\overline{\overline{\varphi}}\left(\sum_{x \in S} m_x x\right) &= \sum_{x \in S} m_x \overline{\varphi}(x) \\
&= \sum_{x \in S} m_x U(\varphi)(x) \\
&= \sum_{x \in S} m_x \varphi(x) \\
&= \varphi\left(\sum_{x \in S} m_x x\right).
\end{aligned}$$

And for $f \in \mathsf{Set}\left(S, U(G)\right)$ we have that

$$\overline{\overline{f}}(x) = U(\overline{f})(x) = \overline{f}(x) = f(x).$$

Yannis Bähni
yannis.baehni@uzh.ch

Hence $\overline{\overline{\varphi}} = \varphi$ and $\overline{\overline{f}} = f$ and so we have a bijection

$$\mathsf{AbGrp}\left(F(S), G\right) \cong \mathsf{Set}\left(S, U(G)\right).$$

The mapping $f \mapsto \overline{f}$ will be referred to as ***extending by linearity***. To check naturality in S and G is left as an exercise. □

Exercise 3.1. In proposition 3.1, check that $F : \mathsf{Set} \to \mathsf{AbGrp}$ is indeed a functor, called the ***free functor from*** **Set** ***to*** **AbGrp**, and the naturality of the bijection in both arguments.

Definition 3.1 (Free Abelian Group). *Let $F : \mathsf{Set} \to \mathsf{AbGrp}$ be the free functor. For any set S, we call $F(S)$ the* ***free group generated by S****.*

Chain Complexes.

Definition 3.2 (Chain Complex). *A* ***chain complex*** *is a tuple $(C_\bullet, \partial_\bullet)$ consisting of a sequence $(C_n)_{n\in\mathbb{Z}}$ in $\mathrm{ob}(\mathsf{AbGrp})$ and a sequence $(\partial_n)_{n\in\mathbb{Z}}$ in $\mathrm{mor}(\mathsf{AbGrp})$, called* ***boundary operators****, such that we have $\partial_n \in \mathsf{AbGrp}(C_n, C_{n-1})$ and $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$.*

Definition 3.3 (Chain Maps). *Let $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ be two chain complexes. A* ***chain map $f_\bullet : C_\bullet \to C'_\bullet$*** *is a sequence $(f_n)_{n\in\mathbb{Z}}$ in $\mathrm{mor}(\mathsf{AbGrp})$ such that $f_n \in \mathsf{AbGrp}(C_n, C'_n)$ and the diagram*

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \end{tikzcd}$$

commutes for all $n \in \mathbb{Z}$.

Proposition 3.2. *There is a category with objects chain complexes and morphisms chain maps.*

Proof. Let $f_\bullet : C_\bullet \to C'_\bullet$ and $g_\bullet : C'_\bullet \to C''_\bullet$ be chain maps. Define a map $g_\bullet \circ f_\bullet$ by $g_n \circ f_n$ for each $n \in \mathbb{Z}$. This defines a chain map. Moreover, for each chain complex $C_\bullet$ define $\mathrm{id}_{C_\bullet}$ by id_{C_n} for all $n \in \mathbb{Z}$. It is easy to check, that then $\circ$ is associative and the identity laws hold. □

Definition 3.4 (Comp). *The category in 3.2 is called the* ***category of chain complexes*** *and we refer to it as* Comp.

Theorem 3.1. *There is a functor* $\mathsf{Top} \to \mathsf{Comp}$.

Proof. The proof is divided into several steps. Let us denote $C_\bullet : \mathsf{Top} \to \mathsf{Comp}$ for the claimed functor.

Yannis Bähni
yannis.baehni@uzh.ch

Step 1: Construction of a sequence of abelian groups. Let $v_0, \dots, v_k \in \mathbb{R}^n$ for some $n, k \in \omega$. We say that $(v_0, \dots, v_k)$ is ***affinely independent*** if $(v_1 - v_0, \dots, v_k - v_0)$ is linearly independent. We define the ***k-simplex spanned by*** $\boldsymbol{(v_0, \dots, v_k)}$, written $[v_0, \dots, v_k]$, to be

$$[v_0, \dots, v_k] := \left\{\sum_{i=0}^k s_i v_i : s_i \geq 0 \text{ for all } i = 0, \dots, k \text{ and } \sum_{i=0}^k s_i = 1\right\}. \tag{9}$$

equipped with the subspace topology. Moreover, we define the ***standard n-simplex*** $\boldsymbol{\Delta^n}$ to be the n-simplex spanned by $(e_0, \dots, e_n)$ where $e_0 := 0 \in \mathbb{R}^n$ and $(e_1, \dots, e_n)$ is the standard ordered basis of $\mathbb{R}^n$. Let $X \in \operatorname{ob}(\mathsf{Top})$. Define a ***singular n-simplex in X*** to be a morphism $\sigma \in \mathsf{Top}(\Delta^n, X)$. Let $n \in \mathbb{Z}$. Define

$$C_n(X) := \begin{cases} F\left(\mathsf{Top}(\Delta^n, X)\right) & n \geq 0, \\ 0 & n < 0. \end{cases} \tag{10}$$

We will call elements of $C_n(X)$ ***singular n-chains***.

Step 2: Construction of boundary operators. Let $X \in \operatorname{ob}(\mathsf{Top})$ and σ a singular n-simplex in X for $n \geq 1$. We define $\varphi_k^n : \Delta^{n-1} \to \Delta^n$, called the ***k-th face map***, to be the unique affine map determined by the vertex map

$$\begin{array}{ccc} & \varphi_k^n & \\ e_0 & \mapsto & e_0 \\ \vdots & & \vdots \\ e_{k-1} & \mapsto & e_{k-1} \\ e_k & \mapsto & e_{k+1} \\ \vdots & & \vdots \\ e_{n-1} & \mapsto & e_n. \end{array}$$

Explicitely, given $\sum_{i=0}^{n-1} s_i e_i \in \Delta^{n-1}$, we have that (see [Lee11, p. 152])

$$\varphi_k^n\left(\sum_{i=0}^{n-1} s_i e_i\right) = \sum_{i=0}^{n-1} s_i \varphi_k^n(e_i).$$

Define now

$$\partial\sigma := \sum_{k=0}^{n} (-1)^k \sigma \circ \varphi_k^n \in U\left(C_{n-1}(X)\right) \tag{11}$$

to be the ***boundary of*** $\boldsymbol{\sigma}$. Moreover, the ***singular boundary operator*** is defined to be $\overline{\partial}_n$ and $\partial_n := 0$ for $n \leq 0$.

Yannis Bähni
yannis.baehni@uzh.ch

Step 3: $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. It is enough to consider $n \geq 1$, since $\partial_n \circ \partial_{n+1} = 0$ holds trivially in the other cases. Let $X \in \mathrm{ob}(\mathsf{Top})$ and $\sigma \in \mathsf{Top}(\Delta^{n+1}, X)$. Then we have

$$\begin{aligned}
(\partial_n \circ \partial_{n+1})(\sigma) &= \partial_n \left(\sum_{k=0}^{n+1}(-1)^k \sigma \circ \varphi_k^{n+1}\right) \\
&= \sum_{k=0}^{n+1}(-1)^k \partial_n \left(\sigma \circ \varphi_k^{n+1}\right) \\
&= \sum_{k=0}^{n+1}\sum_{j=0}^{n}(-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\
&= \sum_{0 \leq k \leq j \leq n} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n + \sum_{0 \leq j < k \leq n+1} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\
&= \sum_{0 \leq j \leq k \leq n} (-1)^{k+j} \sigma \circ \varphi_j^{n+1} \circ \varphi_k^n + \sum_{0 \leq j < k \leq n+1} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\
&= \sum_{0 \leq j < k \leq n+1} \left((-1)^{k+j-1} \sigma \circ \varphi_j^{n+1} \circ \varphi_{k-1}^n + (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n\right)
\end{aligned}$$

Since $\varphi_j^{n+1} \circ \varphi_{k-1}^n = \varphi_k^{n+1} \circ \varphi_j^n$, it follows that

$$\partial_n \circ \partial_{n+1} = 0.$$

Indeed, consider the following chart of vertex maps:

$$\begin{array}{ccccc}
 & \varphi_{k-1}^n & & \varphi_j^{n+1} & \\
e_0 & \mapsto & e_0 & \mapsto & e_0 \\
\vdots & & \vdots & & \vdots \\
e_{j-1} & \mapsto & e_{j-1} & \mapsto & e_{j-1} \\
e_j & \mapsto & e_j & \mapsto & e_{j+1} \\
\vdots & & \vdots & & \\
e_{k-1} & \mapsto & e_{k-1} & \mapsto & e_{k+1} \\
e_k & \mapsto & e_{k+1} & \mapsto & e_{k+2} \\
\vdots & & \vdots & & \\
e_{n-1} & \mapsto & e_n & \mapsto & e_{n+1}
\end{array}
\qquad
\begin{array}{ccccc}
 & \varphi_j^n & & \varphi_k^{n+1} & \\
e_0 & \mapsto & e_0 & \mapsto & e_0 \\
\vdots & & \vdots & & \vdots \\
e_{j-1} & \mapsto & e_{j-1} & \mapsto & e_{j-1} \\
e_j & \mapsto & e_{j+1} & \mapsto & e_{j+1} \\
\vdots & & \vdots & & \\
e_{k-1} & \mapsto & e_k & \mapsto & e_{k+1} \\
e_k & \mapsto & e_{k+1} & \mapsto & e_{k+2} \\
\vdots & & \vdots & & \\
e_{n-1} & \mapsto & e_n & \mapsto & e_{n+1}
\end{array}.$$

Step 4: Construction of chain maps. Let $X, Y \in \mathrm{ob}(\mathsf{Top})$ and $f \in \mathsf{Top}(X, Y)$. For $n \geq 0$, define $f_n^\# : \mathsf{Top}(\Delta^n, X) \to U\left(C_n(Y)\right)$ by $f^\# := f \circ \sigma$. Extending this map by linearity yields a homomorphism $f_n^\# : C_n(X) \to C_n(Y)$. Moreover, set $f_n^\# := 0$ for $n < 0$. Let

Yannis Bähni
yannis.baehni@uzh.ch

$n \geq 1$ and $\sigma \in \mathsf{Top}(\Delta^n, X)$. Then on one hand we have

$$(f_{n-1}^{\#} \circ \partial_n)(\sigma) = f_{n-1}^{\#}\left(\sum_{k=0}^{n}(-1)^k \sigma \circ \varphi_k^n\right) = \sum_{k=0}^{n}(-1)^k f \circ \sigma \circ \varphi_k^n$$

and on the other

$$(\partial_n \circ f_n^{\#})(\sigma) = \partial_n(f \circ \sigma) = \sum_{k=0}^{n}(-1)^k f \circ \sigma \circ \varphi_k^n.$$

Checking, that $C_\bullet$ is indeed a functor is left as an exercise. □

Exercise 3.2. Show that $C_\bullet : \mathsf{Top} \to \mathsf{Comp}$ is a functor.

The Homology Functor.

Proposition 3.3. *For each $n \in \mathbb{Z}$ there exists a functor* $\mathsf{Comp} \to \mathsf{AbGrp}$.

Proof. Let $(C_\bullet, \partial_\bullet)$ be a chain complex. Let $x \in \operatorname{im} \partial_{n+1}$. Hence there exists $y \in C_{n+1}$ such that $x = \partial_{n+1} y$. But then $\partial_n x = (\partial_n \circ \partial_{n+1})(y) = 0$ and thus $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$. Define

$$H_n(C_\bullet, \partial_\bullet) := \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} \in \operatorname{ob}(\mathsf{AbGrp}).$$

Let $(C'_\bullet, \partial'_\bullet)$ be a chain complex and $f_\bullet : C_\bullet \to C'_\bullet$ a chain map. Then $f_n(\ker \partial_n) \subseteq \ker \partial'_n$. Indeed, if $y \in f_n(\ker \partial_n)$, there exists $x \in \ker \partial_n$, such that $y = f_n(x)$. Since $f_\bullet$ is a chain map, we thus have $\partial'_n y = (\partial'_n \circ f_n)(x) = (f_{n-1} \circ \partial_n)(x) = 0$. Moreover, we have that $\operatorname{im} \partial_{n+1} \subseteq \ker \pi'_n \circ f_n$, where $\pi'_n : \ker \partial'_n \to H_n(C'_\bullet, \partial'_\bullet)$ is the usual projection. Indeed, if $y \in \operatorname{im} \partial_{n+1}$, we find $x \in C_{n+1}$, such that $y = \partial_{n+1} x$. Since again $f_\bullet$ is a chain map, we have that $f_n y = (f_n \circ \partial_{n+1})(x) = (\partial'_{n+1} \circ f_{n+1})(x) \in \operatorname{im} \partial'_{n+1} = \ker \pi'_n$. Hence $\pi'_n \circ f_n$ factors uniquely through $\pi_n : \ker \partial_n \to H_n(C_\bullet, \partial_\bullet)$. Define $H_n(f_\bullet)$ to be this map. □

Remark 3.1. Let $(C_\bullet, \partial_\bullet)$ be a chain complex and $n \in \mathbb{Z}$. Then we will write $\langle x \rangle$ for an element in $H_n(C_\bullet, \partial_\bullet)$, the so-called *homology class*. Hence if $(C'_\bullet, \partial'_\bullet)$ is another chain complex and $f_\bullet : C_\bullet \to C'_\bullet$ a chain map, then $H_n(f)\langle x \rangle = \langle f_n x \rangle$.

Definition 3.5 (Cycles and Boundaries). *Let $(C_\bullet, \partial_\bullet)$ be a chain complex and $n \in \mathbb{Z}$. Then elements of* $\ker \partial_n$ *are called **n-cycles** and elements of* $\operatorname{im} \partial_{n+1}$ *are called **n-boundaries**.*

Definition 3.6 (Homology Functor). *Let $n \in \mathbb{Z}$ and $H_n : \mathsf{Comp} \to \mathsf{AbGrp}$ be the functor defined in proposition 3.3. We call H_n the **n-th homology functor**.*

Definition 3.7 (Singular Homology Functor). *Let $n \in \mathbb{Z}$. The composition*

$$H_n \circ C_\bullet : \mathsf{Top} \to \mathsf{AbGrp} \tag{12}$$

*of the singular chain complex functor $C_\bullet$ in theorem 3.1 and the n-th homology functor of proposition 3.3 is called the **singular homology functor**, written H_n^{sing}.*

Yannis Bähni
yannis.baehni@uzh.ch

Remark 3.2. For notational purposes we will often refer to the functor H_n^{sing} simply as H_n.

First Properties of Singular Homolgy.

Proposition 3.4 (Zeroth Singular Homology Group). *Let $X \in \text{ob}(\mathsf{Top})$ be non empty and path connected. Then $H_0(X) \cong \mathbb{Z}$.*

Proof. Since $\partial_0 : C_0 \to 0$, $\ker \partial_0 = C_0$. Moreover, a map in $\mathsf{Top}(\Delta^0, X)$ can be identified with a point in X and hence an element of C_0 can be written as $\sum_{x \in X} m_x x$. Define a mapping $\Phi : C_0 \to \mathbb{Z}$ by $\Phi\left(\sum_{x \in X} m_x x\right) := \sum_{x \in X} m_x$. This mapping is well defined since all but finitely many m_x are zero. It is also easy to check, that Φ is a morphism of groups and that Φ is surjective. We claim that $\ker \Phi = \operatorname{im} \partial_1$. Indeed, if $\sum_{x \in X} m_x x \in \ker \Phi$, then $\sum_{x \in X} m_x = 0$. Let $p \in X$. Since X is path connected, we find for each $x \in X$ a path σ_x from p to x. Consider the singular 1-chain $\sum_{x \in X} m_x \sigma_x$. Then we have

$$\partial_1 \left(\sum_{x \in X} m_x \sigma_x \right) = \sum_{x \in X} m_x \left(\sigma_x(1) - \sigma_x(0)\right) = \sum_{x \in X} m_x (x - p) = \sum_{x \in X} m_x x.$$

Hence $\sum_{x \in X} m_x x \in \operatorname{im} \partial_1$. Conversely, it is enough to show the claim on basis elements $\sigma \in \mathsf{Top}(\Delta^1, X)$. We have

$$\Phi(\partial_1 \sigma) = \Phi\left(\sigma(1) - \sigma(0)\right) = 1 - 1 = 0.$$

Hence the first isomorphism theorem [Gri07, p. 23] implies that $H_0(X) \cong \mathbb{Z}$. □

Proposition 3.5 (The Dimension Axiom). *Let $* \in \text{ob}(\mathsf{Top})$ be a one point space. Then $H_n(*) = 0$ for all $n \in \mathbb{Z}$, $n > 0$.*

The Hurewicz Theorem

Abelianizations.

Proposition 3.6. *The forgetful functor $U : \mathsf{AbGrp} \to \mathsf{Grp}$ admits a left adjoint.*

Proof. Let $G \in \text{ob}(\mathsf{Grp})$. For $g, h \in G$, define the ***commutator of g and h***, written $[g, h]$, by $[g, h] := ghg^{-1}h^{-1}$. Moreover, set

$$X_G := \{[g, h] : g, h \in G\}$$

and define the ***commutator subgroup of G***, written $[G, G]$, by $[G, G] := \langle X_G \rangle$.

Lemma 3.1. *For all $G \in \text{ob}(\mathsf{Grp})$, $[G, G] \trianglelefteq G$.*

Proof. We follow [Lee11, p. 265]. Clearly, $[G, G] \leq G$. By [KM13, p. 31] we have that

$$\langle X \rangle = \{x_1 \cdots x_n : n \in \omega \setminus \{0\}, x_1, \dots, x_n \in X_G \cup X_G^{-1}\}.$$

Yannis Bähni
yannis.baehni@uzh.ch

It is easy to check that $X_G = X_G^{-1}$ and thus

$$\langle X \rangle = \{x_1 \cdots x_n : n \in \omega \setminus \{0\}, x_1, \dots, x_n \in X_G\}.$$

Let $k \in G$ and $x_1 \cdots x_n \in [G, G]$. Since

$$kx_1 \cdots x_n k^{-1} = kx_1k^{-1}kx_2k^{-1}k \cdots kx_nk^{-1}$$

it is enough to show that $k[g,h]k^{-1} \in [G,G]$ for all $g, h \in G$. But this immediately follows from

$$k[g,h]k^{-1} = kghg^{-1}h^{-1}k^{-1} = [kgk^{-1}, khk^{-1}].$$

Thus $[G,G] \trianglelefteq G$. $\square$

Lemma 3.2. $G \in \operatorname{ob}(\mathsf{AbGrp})$ *if and only if* $[G,G] = \{1\}$.

Proof. Let $G \in \operatorname{ob}(\mathsf{AbGrp})$. Then $[g,h] = 1$ for all $g, h \in G$, which implies $X_G = \{1\}$ and thus $\langle X_G \rangle = \{1\}$. Conversly, since $X_G \subseteq [G,G] = \{1\}$, we have that $[g,h] = 1$ for all $g, h \in G$ which is equivalent to $gh = hg$ for all $g, h \in G$. $\square$

Corollary 3.1. *The quotient group* $G/[G,G]$ *is abelian.*

Proof. By lemma 3.2 it is enough to show that $[G/[G,G], G/[G,G]]$ is trivial. We actually show that $X_{G/[G,G]} = \{1\}$. This immediately follows from

$$[g[G,G], h[G,G]] = ghg^{-1}h^{-1}[G,G] = [G,G]$$

for $g[G,G], h[G,G] \in G/[G,G]$. $\square$

Hence define $\mathrm{Ab} : \mathsf{Grp} \to \mathsf{AbGrp}$ on objects by

$$\mathrm{Ab}(G) := G/[G,G].$$

The abelian group $\mathrm{Ab}(G)$ is called the ***abelianization of G***. On morphisms $\varphi : G \to H$ in Grp define $\mathrm{Ab}(\varphi) : \mathrm{Ab}(G) \to \mathrm{Ab}(H)$ by setting $\mathrm{Ab}(\varphi)(g[G,G]) := \varphi(g)[H,H]$. It is easy to check that this is a well defined morphism of abelian groups.
Let $H \in \operatorname{ob}(\mathsf{AbGrp})$ and $\psi \in \mathsf{AbGrp}(\mathrm{Ab}(G), H)$. Define $\overline{\psi} \in \mathsf{Grp}(G, U(H))$ by setting $\overline{\psi}(g) := \psi(g[G,G])$. If $\varphi \in \mathsf{Grp}(G, U(H))$, define $\overline{\varphi} \in \mathsf{AbGrp}(\mathrm{Ab}(G), H)$ by $\overline{\varphi}(g[G,G]) := \varphi(g)$. It is easy to check that this mapping is actually well defined and that $\overline{\overline{\psi}} = \psi$ and $\overline{\overline{\varphi}} = \varphi$ holds. $\square$

Exercise 3.3. In proposition 3.6, check that $\mathrm{Ab} : \mathsf{Grp} \to \mathsf{AbGrp}$ is indeed a functor and the naturality of the bijection in both arguments.

The Homotopy Axiom

Theorem 3.2 (The Homotopy Axiom). *Let* $f, g \in \mathsf{Top}(X, Y)$ *be freely homotopic. Then* $H_n(f) = H_n(g)$ *for all* $n \in \mathbb{Z}$.

Yannis Bähni
yannis.baehni@uzh.ch

Applications

The Brouwer Fixed Point Theorem.

Definition 3.8 (Retract). *Let $X \in \mathrm{ob}(\mathsf{Top})$ and $S \subseteq X$ a subspace. We say that* ***S is a retract of X****, if the inclusion $\iota : S \hookrightarrow X$ admits a retraction in* Top.

Lemma 3.3. *Let $n \in \mathbb{Z}$, $n \geq 1$. Then $\mathbb{S}^n$ is not a retract of $\mathbb{B}^{n+1}$.*

Proof. □

Theorem 3.3 (Brouwer Fixed Point Theorem). *Let $n \in \mathbb{Z}$, $n \geq 1$. Then every mapping $f \in \mathsf{Top}(\mathbb{B}^n, \mathbb{B}^n)$ has a fixed point.*

Proof. □

APPENDIX A

Set Theory

Basic Concepts

Problem A.1. Let $n \in \mathbb{N}$ and $a_{kj} \in \mathbb{C}$ for $k = 0, \ldots, n+1$, $j = 0, \ldots, n$. Show that

$$\sum_{k=0}^{n+1} \sum_{j=0}^{n} a_{kj} = \sum_{0 \le k \le j \le n} a_{kj} + \sum_{0 \le j < k \le n+1} a_{kj}.$$

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