Yannis Bähni
yannis.baehni@uzh.ch

HOMEWORK 2: SYMPLECTIC FORMS VS. AREA AND VOLUME

YANNIS BÄHNI

Exercise 1.1. Let (M, ω) be a $2n$-dimensional symplectic manifold.

(a) ω^n is a volume form.
(b) Show that if M is compact, then $[\omega^n] \in H^{2n}_{\mathrm{dR}}(M)$ is nonzero.
(c) Conclude that $[\omega] \neq 0$.
(d) $\mathbb{S}^{2n}$ does not admit a symplectic structure for $n > 1$.

Solution 1.1. Part (a) immediately follows from the fact that for each $p \in M$ we have that $\omega^n_p \neq 0$. Thus ω^n is a nonvanishing form of top degree, hence a volume form.
For proving (b), assume that $[\omega^n] = 0$. Hence ω^n is exact. Thus there exists $\mu \in \Omega^{2n-1}(M)$ such that $\omega^n = d\mu$. But then Stoke's theorem [Lee13, p. 411] together with positivity [Lee13, p. 407] yields

$$0 < \int_M \omega^n = \int_M d\mu = \int_{\partial M} \mu = \int_{\varnothing} \mu = 0$$

since M is oriented by part (a) and ω^n is a positively oriented orientation form (see [Lee13, p. 381]).
For proving (c), observe that $[\omega^n] = [\omega] \cup \cdots \cup [\omega]$, where $\cup$ is the so-called cup product (see [Lee13, p. 464]). So if $[\omega] = 0$, we have by bilinearity also $[\omega^n] = 0$, which contradicts part (b).
For proving (d), by [Lee13, p. 450] we have that

$$H^p_{\mathrm{dR}}(\mathbb{S}^n) \cong \begin{cases} \mathbb{R} & p = 0 \text{ or } p = n, \\ 0 & 0 < p < n, \end{cases}$$

for $n \geq 1$. Let $n > 1$. Assume that $(\mathbb{S}^n, \omega)$ is a symplectic manifold. Since $\mathbb{S}^n$ is compact, part (c) implies that $[\omega] \neq 0$. But $[\omega] \in H^2_{\mathrm{dR}}(\mathbb{S}^{2n}) \cong 0$.

Example 1.1. Consider the symplectic manifold $(\mathbb{R}^{2n}, \omega_0)$, where ω_0 is the standard symplectic structure on $\mathbb{R}^{2n}$. Clearly, $\mathbb{R}^{2n}$ is not compact and ω_0 is exact since

$$d\left(\sum_{i=1}^n x^i dy^i\right) = \sum_{i=1}^n dx^i \wedge dy^i = \omega_0.$$

(Yannis Bähni) University of Zurich, Rämistrasse 71, 8006 Zurich
E-mail address: yannis.baehni@uzh.ch.

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Example 1.2. Let M be a smooth manifold. Then (T^*M, ω) is a symplectic manfiold, where ω is the canonical symplectic form on T^*M. It is an exact form, since $\omega = -d\alpha$, where α is the tautological 1-form. Moreover, T^*M is not compact by problem 10-19 [Lee13, p. 271].

Exercise 1.2. Let (M, ω) be a $2n$-dimensional symplectic manifold.

(a)

Solution 1.2. For proving (a), we have using [Lee13, p. 117]

$$T_p\mathbb{S}^n = \{v \in \mathbb{R}^{n+1} : \langle v, p\rangle = 0\}$$

for each $p \in \mathbb{S}^n$. Consider the ***Euler vector field*** V defined by

$$V := x^i \frac{\partial}{\partial x^i}.$$

Then V is a unit normal vector field along $\mathbb{S}^n$. Indeed, if $p \in \mathbb{S}^n$ and $v \in T_p\mathbb{S}^n$ we have that

$$\langle p, v\rangle_{\overline{g}} = \langle p, v\rangle = 0$$

and

$$|p|_{\overline{g}} = |p| = 1.$$

Hence by [Lee13, p. 390], the volume form $\omega_{\mathring{g}}$ on $(\mathbb{S}^n, \mathring{g})$ is given by

$$\omega_{\mathring{g}} = \iota^*_{\mathbb{S}^n}\big(i_V\omega_{\overline{g}}\big).$$

More precisely, in the case $n = 2$ we have

$$\begin{aligned} i_V\omega_{\overline{g}} &= i_V(dx \wedge dy \wedge dz) \\ &= (i_V dx) \wedge dy \wedge dz - dx \wedge i_V(dy \wedge dz) \\ &= (i_V dx) \wedge dy \wedge dz - dx \wedge (i_V dy) \wedge dz + dx \wedge dy \wedge (i_V dz) \\ &= x dy \wedge dz + y dz \wedge dx + z dx \wedge dy. \end{aligned}$$

For $v, w \in T_p\mathbb{S}^2$, $p \in \mathbb{S}^2$, we have that

$$\omega_{\mathring{g}}|_p(v, w) = (i_V\omega_{\overline{g}})_{\iota(p)}\big(d\iota_p(v), d\iota_p(w)\big) = (i_V\omega_{\overline{g}})|_p(v, w)$$

under the usual identification of $T_p\mathbb{S}^n$ as a linear subspace of $T_p\mathbb{R}^{n+1}$. Finally

$$\begin{aligned} \omega_{\mathring{g}}(v, w) &= (x dy \wedge dz + y dz \wedge dx + z dx \wedge dy)(v, w) \\ &= x \det\begin{pmatrix} dy(v) & dz(v) \\ dy(w) & dz(w) \end{pmatrix} + y \det\begin{pmatrix} dz(v) & dx(v) \\ dz(w) & dx(w) \end{pmatrix} + z \det\begin{pmatrix} dx(v) & dy(v) \\ dx(w) & dy(w) \end{pmatrix} \\ &= x(v^2w^3 - w^2v^3) + y(v^3w^1 - w^3v^1) + z(v^1w^2 - w^1v^2) \\ &= \langle p, v \times w\rangle \end{aligned}$$

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for $p := (x, y, z) \in \mathbb{S}^2$ using [Lee13, p. 356].
For proving (b), consider cylindrical polar coordinates (θ, z) on $\mathbb{S}^2$ given by

$$(x, y, z) = (\sqrt{1-z^2}\cos\theta, \sqrt{1-z^2}\sin\theta, z).$$

Then we get

$$\begin{aligned}
i_V\omega_{\bar{g}} &= \mathrm{id}^*(i_V\omega_{\bar{g}})\\
&= \mathrm{id}^*(xdy \wedge dz + ydz \wedge dx + zdx \wedge dy)\\
&= \sqrt{1-z^2}\cos\theta d(\sqrt{1-z^2}\sin\theta) \wedge dz + \sqrt{1-z^2}\sin\theta dz \wedge d(\sqrt{1-z^2}\cos\theta)\\
&\quad + zd(\sqrt{1-z^2}\cos\theta) \wedge d(\sqrt{1-z^2}\sin\theta)\\
&= \sqrt{1-z^2}\cos\theta\left(\sqrt{1-z^2}\cos\theta d\theta - \frac{z}{\sqrt{1-z^2}}\sin\theta dz\right) \wedge dz\\
&\quad - \sqrt{1-z^2}\sin\theta dz \wedge \left(\sqrt{1-z^2}\sin\theta d\theta + \frac{z}{\sqrt{1-z^2}}\cos\theta dz\right)\\
&\quad - z\left(\sqrt{1-z^2}\sin\theta d\theta + \frac{z}{\sqrt{1-z^2}}\cos\theta dz\right)\\
&\quad \wedge \left(\sqrt{1-z^2}\cos\theta d\theta - \frac{z}{\sqrt{1-z^2}}\sin\theta dz\right)\\
&= (1-z^2)\cos^2\theta d\theta \wedge dz - (1-z^2)\sin^2\theta dz \wedge d\theta + z^2\sin^2\theta d\theta \wedge dz\\
&\quad - z^2\cos^2\theta dz \wedge d\theta\\
&= d\theta \wedge dz.
\end{aligned}$$

For proving (c), just observe that

$$\mathrm{Vol}(\mathbb{S}^2) = \int_{\mathbb{S}^2} \omega_{\mathring{g}} = \int_{(0,2\pi)\times(-1,1)} d\theta \wedge dz = 4\pi.$$

Exercise 1.3. Let (M, ω) be a $2n$-dimensional symplectic manifold.

(a)

Solution 1.3. For (a), observe that by exercise 1.1, the manifold must be orientable. Thus surfaces like the sphere, torus and connected sums of these (classification theorem) do possess a symplectic structure, whereas the non-orientable surfaces like the real projective plane and the Möbius strip do not possess any symplectic structures.