Contents

CHAPTER 1

The Fundamental Group

1. The Fundamental Grupoid

Lemma 1.1 (Gluing Lemma). *Let* $X, Y \in \mathrm{ob}(\mathsf{Top})$*,* $(X_\alpha)_{\alpha\in A}$ *a finite closed cover of* X *and* $(f_\alpha)_{\alpha\in A}$ *a finite family of maps* $f_\alpha \in \mathsf{Top}(X_\alpha, Y)$ *such that* $f_\alpha|_{X_\alpha\cap X_\beta} = f_\beta|_{X_\alpha\cap X_\beta}$ *for all* $\alpha, \beta \in A$*. Then there exists a unique* $f \in \mathsf{Top}(X, Y)$ *such that* $f|_{X_\alpha} = f_\alpha$ *for all* $\alpha \in A$*.*

Proof. Let $x \in X$. Since $(X_\alpha)_{\alpha\in A}$ is a cover of X, we find $\alpha \in A$ such that $x \in X_\alpha$. Define $f(x) := f_\alpha(x)$. This is well defined, since if $x \in X_\alpha \cap X_\beta$ for some $\beta \in A$, we have that $f(x) = f_\beta(x) = f_\alpha(x)$. Clearly $f|_{X_\alpha} = f_\alpha$ for all $\alpha \in A$ and f is unique. Let us show continuity. To this end, let $K \subseteq Y$ be closed. Then

$$\begin{aligned} f^{-1}(K) &= X \cap f^{-1}(K) \\ &= \bigcup_{\alpha\in A} X_\alpha \cap f^{-1}(K) \\ &= \bigcup_{\alpha\in A} \left(X_\alpha \cap f^{-1}(K)\right) \\ &= \bigcup_{\alpha\in A} \left(X_\alpha \cap f_\alpha^{-1}(K)\right). \end{aligned}$$

Since each f_α is continuous, $f_\alpha^{-1}(K)$ is closed in X_α for each $\alpha \in A$ and thus since X_α is closed, $f^{-1}(K)$ is closed as a finite union of closed sets. □

Theorem 1.1. *There is a functor* $\mathsf{Top} \to \mathsf{Grpd}$.

Proof. The proof is divided into several steps. Let us denote $\Pi : \mathsf{Top} \to \mathsf{Grpd}$ for the claimed functor.

Step 1: Definition of Π *on objects.* Let $X, Y \in \mathrm{ob}(\mathsf{Top})$, $f, g \in \mathsf{Top}(X, Y)$ and $A \subseteq X$. A map $F \in \mathsf{Top}(X \times I, Y)$ is called a ***homotopy from*** $\boldsymbol{X}$ ***to*** $\boldsymbol{Y}$ ***relative to*** $\boldsymbol{A}$, if

- $F(x, 0) = f(x)$, for all $x \in X$.
- $F(x, 1) = g(x)$, for all $x \in X$.
- $F(x, t) = f(x) = g(x)$, for all $x \in A$ and for all $t \in I$.

Yannis Bähni
yannis.baehni@uzh.ch

If there exists a homotopy between f and g relative to A we say that f and g are ***homotopic relative to A*** and write $f \simeq_A g$. If we want to emphasize the homotpoy relative to A, we write $F : f \simeq_A g$.

Lemma 1.2. *Let $X, Y \in \mathrm{ob}(\mathsf{Top})$ and $A \subseteq X$. Then being homotopic relative to A is an equivalence relation on $\mathsf{Top}(X, Y)$.*

Proof. Define a binary relation $R_A \subseteq \mathsf{Top}(X, Y) \times \mathsf{Top}(X, Y)$ by

$$fR_A g \quad :\Leftrightarrow \quad f \simeq_A g.$$

Let $f \in \mathsf{Top}(X, Y)$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x, t) := f(x).$$

Then clearly $F : f \simeq_A f$. Hence R_A is reflexive.
Let $g \in \mathsf{Top}(X, Y)$ and assume that $fR_A g$. Thus $G : f \simeq_A g$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x, t) := G(x, 1 - t).$$

Then it is easy to check that $F : g \simeq_A f$ and so R_A is symmetric.
Finally, let $h \in \mathsf{Top}(X, Y)$ and suppose that $fR_A g$ and $gR_A h$. Hence $F_1 : f \simeq_A g$ and $F_2 : g \simeq_A h$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x, t) := \begin{cases} F_1(x, 2t) & 0 \leq t \leq \frac{1}{2}, \\ F_2(x, 2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Continuity of F follows by an application of the gluing lemma 1.1. Then it is easy to check that $F : f \simeq_A h$ and hence R_A is transitive. □

Let $X \in \mathrm{ob}(\mathsf{Top})$ and u a path in X from p to q. Define the ***path class $[u]$ of u*** by $[u] := [u]_{R_{\partial I}}$. Define now

- $\mathrm{ob}\big(\Pi(X)\big) := X$.
- $\Pi(X)(p, q) := \{[u] : u \text{ is a path from } p \text{ to } q\}$ for all $p, q \in X$.
- And $\Pi(X)(q, r) \times \Pi(X)(p, q) \to \Pi(X)(p, r)$ by

  $$([v], [u]) \mapsto [u * v]$$

  Where $u * v \in \mathsf{Top}(p, r)$ is the ***concatenated path of u and v***, defined by

  $$(u * v)(s) := \begin{cases} u(2s) & 0 \leq t \leq \frac{1}{2}, \\ v(2s - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

  Continuity follows again from the gluing lemma 1.1 whereas well definedness follows from the next lemma.

Lemma 1.3. *Suppose that $[u_1], [u_2] \in \Pi(X)(p, q)$ and $[v_1], [v_2] \in \Pi(X)(q, r)$ such that $[u_1] = [u_2]$ and $[v_1] = [v_2]$. Then $[u_1 * v_1] = [u_2 * v_2]$.*

Yannis Bähni
yannis.baehni@uzh.ch

Proof. By assumption we have $G : u_1 \simeq_{\partial I} u_2$ and $H : v_1 \simeq_{\partial I} v_2$. Define $F \in \mathsf{Top}(I \times I, X)$ by

$$F(s,t) := \begin{cases} G(2s,t) & 0 \leq s \leq \frac{1}{2}, \\ H(2s-1,t) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Again, continuity follows from the gluing lemma 1.1 and it is easy to check that $F : u_1 * v_1 \simeq_{\partial I} u_2 * v_2$. ☐

- Let $p \in X$. Then define $\mathrm{id}_p \in \Pi(X)(p,p)$ by $\mathrm{id}_p := [c_p]$, where c_p is the constant path defined by $c_p(s) := p$ for all $s \in I$.

Let us now check that $\Pi(X)$ is indeed a category.

Lemma 1.4.

☐

2. The Fundamental Group

Lemma 1.5. *Let $\mathscr{G}$ be a locally small grupoid. Then for every $X \in \mathrm{ob}(\mathscr{G})$, $\mathscr{G}(X,X)$ can be equipped with a group structure.*

Proof. Since $\mathscr{G}$ is locally small, $\mathscr{G}(X,X)$ is a set for every $X \in \mathrm{ob}(\mathscr{G})$. Define a multiplication $\mathscr{G} \times \mathscr{G} \to \mathscr{G}$ by $gh := h \circ g$. ☐

CHAPTER 2

Singular Homology

Free Abelian Groups

Proposition 2.1. *The forgetful functor $U : \mathsf{Ab} \to \mathsf{Set}$ admits a left adjoint.*

Proof. We have to construct a functor $F : \mathsf{Set} \to \mathsf{Ab}$. Let S be a set. Define

$$F(S) := \left\{ f \in \mathbb{Z}^S : \operatorname{supp} f \text{ is finite} \right\}.$$

Equipped with pointwise addition, $F(S)$ is an abelian group. There is a natural inclusion $\iota : S \hookrightarrow U\left(F(S)\right)$ sending $x \in S$ to the function taking the value one at x and zero else. Hence we may regard elements of $F(S)$ as formal linear combinations $\sum_{x\in S} m_x x$, where $m_x \in \mathbb{Z}$ for all $x \in S$. Let $G \in \operatorname{ob}(\mathsf{Ab})$ be an abelian group and $\varphi \in \mathsf{Ab}\left(F(S), G\right)$ a morphism of groups. Define $\overline{\varphi} \in \mathsf{Set}\left(S, U(G)\right)$ by $\overline{\varphi} := U(\varphi)$. Conversly, if we have $f \in \mathsf{Set}\left(S, U(G)\right)$, define $\overline{f} \in \mathsf{Ab}\left(F(S), G\right)$ by $\overline{f}\left(\sum_{x\in S} m_x x\right) := \sum_{x\in S} m_x f(x)$. This is well defined since all but finitely many m_x are zero and G is abelian. It is easy to check that $\overline{f}$ is indeed a morphism of groups. Let $\varphi \in \mathsf{Ab}\left(F(S), G\right)$. Then

$$\begin{aligned}
\overline{\overline{\varphi}}\left(\sum_{x\in S} m_x x\right) &= \sum_{x\in S} m_x \overline{\varphi}(x) \\
&= \sum_{x\in S} m_x U(\varphi)(x) \\
&= \sum_{x\in S} m_x \varphi(x) \\
&= \varphi\left(\sum_{x\in S} m_x x\right).
\end{aligned}$$

And for $f \in \mathsf{Set}\left(S, U(G)\right)$ we have that

$$\overline{\overline{f}}(x) = U(\overline{f})(x) = \overline{f}(x) = f(x).$$

Hence $\overline{\overline{\varphi}} = \varphi$ and $\overline{\overline{f}} = f$ and so we have a bijection

$$\mathsf{Ab}\left(F(S), G\right) \cong \mathsf{Set}\left(S, U(G)\right).$$

Yannis Bähni
yannis.baehni@uzh.ch

The mapping $f \mapsto \overline{f}$ will be referred to as ***extending by linearity***. To check naturality in S and G is left as an exercise. $\square$

Exercise 0.1. Check the naturality of the bijection in proposition 2.1. Also check that $F : \mathsf{Set} \to \mathsf{Ab}$ is indeed a functor. F is called the ***free functor from* Set *to* Ab**.

Definition 2.1 (Free Abelian Group). *Let $F : \mathsf{Set} \to \mathsf{Ab}$ be the free functor. For any set S, we call $F(S)$ the* ***free group generated by S***.

Chain Complexes

Definition 2.2 (Chain Complex). *A* ***chain complex*** *is a tuple $(C_\bullet, \partial_\bullet)$ consisting of a sequence $(C_n)_{n\in\mathbb{Z}}$ in $\mathrm{ob}(\mathsf{Ab})$ and a sequence $(\partial_n)_{n\in\mathbb{Z}}$ in $\mathrm{mor}(\mathsf{Ab})$, called* ***boundary operators****, such that we have $\partial_n \in \mathsf{Ab}(C_n, C_{n-1})$ and $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$.*

Definition 2.3 (Chain Maps). *Let $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ be two chain complexes. A* ***chain map $f_\bullet : C_\bullet \to C'_\bullet$*** *is a sequence $(f_n)_{n\in\mathbb{Z}}$ in $\mathrm{mor}(\mathsf{Ab})$ such that $f_n \in \mathsf{Ab}(C_n, C'_n)$ and the diagram*

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \end{tikzcd}$$

commutes for all $n \in \mathbb{Z}$.

Proposition 2.2. *There is a category with objects chain complexes and morphisms chain maps.*

Proof. Let $f_\bullet : C_\bullet \to C'_\bullet$ and $g_\bullet : C'_\bullet \to C''_\bullet$ be chain maps. Define a map $g_\bullet \circ f_\bullet$ by $g_n \circ f_n$ for each $n \in \mathbb{Z}$. This defines a chain map. Moreover, for each chain complex $C_\bullet$ define $\mathrm{id}_{C_\bullet}$ by id_{C_n} for all $n \in \mathbb{Z}$. It is easy to check, that then $\circ$ is associative and the identity laws hold. $\square$

Definition 2.4 (Comp). *The category in 2.2 is called the* ***category of chain complexes*** *and we refer to it as* Comp.

Theorem 2.1. *There is a functor* $\mathsf{Top} \to \mathsf{Comp}$.

Proof. The proof is divided into several steps. Let us denote $C_\bullet : \mathsf{Top} \to \mathsf{Comp}$ for the claimed functor.

Step 1: Construction of a sequence of abelian groups. Let $v_0, \dots, v_k \in \mathbb{R}^n$ for some $n, k \in \mathbb{N}$. We say that $(v_0, \dots, v_k)$ is ***affinely independent*** if $(v_1 - v_0, \dots, v_k - v_0)$

Yannis Bähni
yannis.baehni@uzh.ch

is linearly independent. We define the ***k*-simplex spanned by $(v_0, \dots, v_k)$**, written $[v_0, \dots, v_k]$, to be

$$[v_0, \dots, v_k] := \left\{\sum_{i=0}^k s_i v_i : s_i \geq 0 \text{ for all } i = 0, \dots, k \text{ and } \sum_{i=0}^k s_i = 1\right\}. \tag{1}$$

equipped with the subspace topology. Moreover, we define the ***standard n*-simplex Δ^n** to be the n-simplex spanned by $(e_0, \dots, e_n)$ where $(e_{i+1})_i$ is the standard basis of $\mathbb{R}^{n+1}$. Let $X \in \operatorname{ob}(\mathsf{Top})$. Define a ***singular n*-simplex in X** to be a map $\sigma \in \mathsf{Top}(\Delta^n, X)$. Let $n \in \mathbb{Z}$. Define

$$C_n(X) := \begin{cases} F\left(\mathsf{Top}(\Delta^n, X)\right) & n \geq 0, \\ 0 & n < 0. \end{cases} \tag{2}$$

We will call elements of $C_n(X)$ ***singular n*-chains**.

Step 2: Construction of boundary operators. Let $X \in \operatorname{ob}(\mathsf{Top})$ and σ a singular n-simplex in X for $n \geq 1$. We define $\varphi_k^n : \Delta^{n-1} \to \Delta^n$, called the ***k*-th face map**, by

$$\varphi_k^n(s_0, \dots, s_{n-1}) := \begin{cases} (0, s_0, \dots, s_{n-1}) & k = 0, \\ (s_0, \dots, s_{k-1}, 0, s_k, \dots, s_{n-1}) & 1 \leq k \leq n-1. \end{cases} \tag{3}$$

Define now

$$\partial\sigma := \sum_{k=0}^n (-1)^k \sigma \circ \varphi_k^n \in U\left(C_{n-1}(X)\right) \tag{4}$$

to be the ***boundary of σ***. Moreover, the ***singular boundary operator*** is defined to be $\overline{\partial_n}$ and $\partial_n := 0$ for $n \leq 0$.

Step 3: $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. It is enough to consider $n \geq 1$, since $\partial_n \circ \partial_{n+1} = 0$ holds trivially in the other cases. Let $X \in \operatorname{ob}(\mathsf{Top})$ and $\sigma \in \mathsf{Top}(\Delta^{n+1}, X)$. Then we have

$$\begin{aligned}
(\partial_n \circ \partial_{n+1})(\sigma) &= \partial_n\left(\sum_{k=0}^{n+1}(-1)^k \sigma \circ \varphi_k^{n+1}\right) \\
&= \sum_{k=0}^{n+1}(-1)^k \partial_n\left(\sigma \circ \varphi_k^{n+1}\right) \\
&= \sum_{k=0}^{n+1}\sum_{j=0}^{n}(-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\
&= \sum_{0 \leq k \leq j \leq n}(-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n + \sum_{0 \leq j < k \leq n+1}(-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\
&= \sum_{0 \leq j \leq k \leq n}(-1)^{k+j} \sigma \circ \varphi_j^{n+1} \circ \varphi_k^n + \sum_{0 \leq j < k \leq n+1}(-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n
\end{aligned}$$

Yannis Bähni
yannis.baehni@uzh.ch

$$= \sum_{0 \leq j < k \leq n+1} \left((-1)^{k+j-1} \sigma \circ \varphi_j^{n+1} \circ \varphi_{k-1}^n + (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \right)$$

Step 4: Construction of chain maps. Let $X, Y \in \mathrm{ob}(\mathsf{Top})$ and $f \in \mathsf{Top}(X, Y)$. For $n \geq 0$, define $f_n^\# : \mathsf{Top}(\Delta^n, X) \to U\left(C_n(Y)\right)$ by $f^\# := f \circ \sigma$. Extending this map by linearity yields a homomorphism $f_n^\# : C_n(X) \to C_n(Y)$. Moreover, set $f_n^\# = 0$ for $n < 0$. Let $n \geq 1$ and $\sigma \in \mathsf{Top}(\Delta^n, X)$. Then on one hand we have

$$(f_{n-1}^\# \circ \partial_n)(\sigma) = f_{n-1}^\# \left(\sum_{k=0}^n (-1)^k \sigma \circ \varphi_k^n \right) = \sum_{k=0}^n (-1)^k f \circ \sigma \circ \varphi_k^n$$

and on the other

$$(\partial_n \circ f_n^\#)(\sigma) = \partial_n(f \circ \sigma) = \sum_{k=0}^n (-1)^k f \circ \sigma \circ \varphi_k^n.$$

Step 5: Checking functorial properties. We are ready to define the functor $C_\bullet : \mathsf{Top} \to \mathsf{Comp}$. Let $C_\bullet(X)$ be the chain complex consisting of $\left(C_n(X)\right)_{n \in \mathbb{Z}}$ and $(\partial_n)_{n \in \mathbb{Z}}$.

$\square$

APPENDIX A

Set Theory

1. Basic Concepts

Problem 1.1. Let $n \in \mathbb{N}$ and $a_{kj} \in \mathbb{C}$ for $k = 0, \ldots, n+1$, $j = 0, \ldots, n$. Show that

$$\sum_{k=0}^{n+1} \sum_{j=0}^{n} a_{kj} = \sum_{0 \le k \le j \le n} a_{kj} + \sum_{0 \le j < k \le n+1} a_{kj}.$$