Yannis Bähni
yannis.baehni@uzh.ch

THE TUBULAR NEIGHBOURHOOD THEOREM

YANNIS BÄHNI

1. Prerequisites

Definition 1.1. *Let (X, d) be a metric space and $A \subseteq X$. For $x \in X$, define the* ***distance from x to A****, written* $\operatorname{dist}(x, A)$*, by*

$$\operatorname{dist}(x, A) := \inf_{a \in A} d(x, a).$$

Lemma 1.1. *Let (X, d) be a metric space and $A \subseteq X$ nonempty. Then* $\operatorname{dist}(\cdot, A) : X \to \mathbb{R}$ *is a continuous function.*

Proof. We show that $\operatorname{dist}(\cdot, A)$ is in fact Lipschitz continuous. Let $x, y \in X$. Then for any $a \in A$ we have that

$$\operatorname{dist}(x, A) \leq d(x, a) \leq d(x, y) + d(y, a).$$

Hence $\operatorname{dist}(x, A) - d(x, y)$ is a lower bound for $d(y, a)$ for any $a \in A$. But this means

$$\operatorname{dist}(x, A) - d(x, y) \leq \operatorname{dist}(y, A).$$

Reversing the roles of x and y in the previous argument and applying the symmetry of the metric, we get that

$$|\operatorname{dist}(x, A) - \operatorname{dist}(y, A)| \leq d(x, y).$$

□

Lemma 1.2. *Let (X, d) be a metric space and $K \subseteq X$ be compact and nonempty. If* $\operatorname{dist}(x, K) = 0$ *for some $x \in X$, then $x \in K$.*

Proof. For any $\varepsilon > 0$, we find $y \in K$ such that

$$\operatorname{dist}(x, K) \leq d(x, y) < \operatorname{dist}(x, K) + \varepsilon.$$

Thus we find a sequence $(y_n)_{n \in \mathbb{N}}$, such that $\operatorname{dist}(x, y_n) \to 0$. Since K is compact, there exists a subsequence y_{n_k} in K such that $y_{n_k} \to y$, where $y \in K$. But then

$$d(x, y) = \lim_{k \to \infty} d(x, y_{n_k}) = 0$$

which implies $x = y$ and so $x \in K$. □

(Yannis Bähni) University of Zurich, Rämistrasse 71, 8006 Zurich
E-mail address: yannis.baehni@uzh.ch.

Yannis Bähni
yannis.baehni@uzh.ch

Theorem 1.1 (Inverse Function Theorem Generalization, Compact Case). *Let M and N be smooth manifolds, K a compact subspace of M and $F : M \to N$ a smooth mapping, such that $F|_K$ is injective and dF_p is nonsingular for any $p \in K$. Then there exists a neighbourhood U of K in M and a neighbourhood V of $F(K)$ in N such that $F|_U : U \to V$ is a diffeomorphism.*

Proof. By corollary 13.30 [Lee13, p. 341], every smooth manifold is metrizable. Hence we can equip M with a metric d. Moreover, the metric topology on M induced by d is the same as the original manifold topology. By proposition 1.12 [Lee13, p. 9], every topological manifold is locally compact, hence by proposition 4.63 [Lee11, pp. 104–105], each point of M has a precompact neighbourhood. Since $K \subseteq M$, we find for any $p \in K$ a precompact neighbourhood V_p of p. Thus $(V_p)_{p\in K}$ is an open cover of K and the compactness of K implies that there exists a finite subcover $V_{p_1}, \dots, V_{p_n}$ of K.
For any $\varepsilon > 0$, define

$$U_\varepsilon := \{p \in M : \operatorname{dist}(p, K) < \varepsilon\}.$$

By lemma 1.1, U_ε is open since $U_\varepsilon = \operatorname{dist}(\cdot, A)^{-1}((-\infty, \varepsilon))$. Thus

$$W_\varepsilon := \bigcup_{i=1}^{n} (V_{p_i} \cap U_\varepsilon)$$

is open and clearly $K \subseteq W_\varepsilon$ for any $\varepsilon > 0$. Hence W_ε is a neighbourhood of K.
Assume now that F is not injective on any neighbourhood of K. For any $n \in \mathbb{N}$ we thus find $x_n, y_n \in W_{1/n}$ such that $x_n \neq y_n$ but $F(x_n) = F(y_n)$. Hence we have constructed two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in W_1. Now by

$$W_\varepsilon = \bigcup_{i=1}^{n} (V_{p_i} \cap U_\varepsilon) \subseteq \bigcup_{i=1}^{n} V_{p_i} \subseteq \bigcup_{i=1}^{n} \overline{V}_{p_i}$$

we get that W_ε is contained in a compact set. Thus we find $p_1, p_2 \in \cup_{i=1}^{n} \overline{V}_{p_i}$ such that $x_{n_k} \to p$ and $y_{n_k} \to q$. But

$$\operatorname{dist}(p, K) = \lim_{k\to\infty} \operatorname{dist}(x_{n_k}, A) \leq \lim_{k\to\infty} \frac{1}{n_k} = 0.$$

by the continuity of the distance function and so $\operatorname{dist}(p, K) = \operatorname{dist}(q, K) = 0$. But this implies $p, q \in K$ by lemma 1.2. Moreover, since F is continuous by [Lee13, p. 34] we have that

$$F(p) = \lim_{k\to\infty} F(x_{n_k}) = \lim_{k\to\infty} F(y_{n_k}) = F(q)$$

and so by injectivity of $F|_K$ we get that $p = q$.
Finally, since dF_p is nonsingular, the inverse function theorem for manifolds [Lee13, p. 79] guarantees the existence of neighbourhoods U_0 of p and V_0 of $F(p)$ such that $F|_{U_0} : U_0 \to V_0$ is a diffeomorphism. Since x_{n_k} and y_{n_k} both converge to p and $x_{n_k} \neq y_{n_k}$ for all $k \in \mathbb{N}$ but $F(x_{n_k}) = F(y_{n_k})$, we get that F cannot be injective, hence

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no diffeomorphism, which is a contradiction. Hence there exists a neighbourhood W of K such that $F|_W$ is injective.
Since dF_p is nonsingular for any $p \in K$, there exist neighbourhoods $U_{0,p}$ of p and $V_{0,p}$ of $F(p)$ such that $F|_{U_{0,p}} : U_{0,p} \to V_{0,p}$ is a diffeomorphism by the inverse function theorem. Moreover, for any $p \in K$ there exists r_p such that $B_{r_p}(p) \subseteq U_{0,p}$ and by shrinking r_p, if necessary, we may assume that $B_{r_p}(p) \subseteq W$. Set

$$U := \bigcup_{p \in K} B_{r_p}(p) \qquad \text{and} \qquad V := F(U) = \bigcup_{p \in K} F(B_{r_p}(p)).$$

Then $K \subseteq U$, U is open and $F(K) \subseteq F(U) = V$. Also $F(B_{r_p}(p))$ is open in N. Indeed, $F|_{U_{0,p}} : U_{0,p} \to V_{0,p}$ is a diffeomorphism and thus an open map. Since $B_{r_p}(p) = U_{0,p} \cap B_{r_p}(p)$, $B_{r_p}(p)$ is also open in $U_{0,p}$. So $F(B_{r_p}(p))$ is open in $V_{0,p}$. But this means that there exists an open set B in N such that $F(B_{r_p}(p)) = V_{0,p} \cap B$, the right hand side is open in N and so is $F(B_{r_p}(p))$. Hence V is open in N as a union of open sets. Moreover, F is bijective and a local diffeomorphism. Thus by proposition 4.6 (f) [Lee13, p. 80] $F|_U : U \to V$ is a diffeomorphism. $\square$

Definition 1.2 (Normal Bundle). *Let (M, g) be a Riemannian manifold and $S \subseteq M$ a Riemannian submanifold. Let $p \in S$. The **normal space to S at p** is defined by*

$$N_pS := \{v \in T_pM : \forall w \in T_pS(\langle v, w\rangle_g = 0)\} \tag{1}$$

*and the **normal bundle of S** as*

$$NS := \coprod_{p \in S} N_pS. \tag{2}$$