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SOLUTIONS SHEET 8

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Exercise 1.

Exercise 2.

Exercise 3.

Exercise 4.

a. If $A = \varnothing$, we have that $\cap_{\alpha\in A}\mathcal{T}_\alpha = \mathcal{P}(X)$ since topologies on X are subsets of $\mathcal{P}(X)$. Hence the intersection of the empty family of topologies on X is the discrete topology. Consider now $A \neq \varnothing$. Clearly, $\varnothing, X \in \cap_{\alpha\in A}\mathcal{T}_\alpha$ since $\varnothing, X \in \mathcal{T}_\alpha$ for all $\alpha \in A$. Let $U_1, \dots, U_n \in \cap_{\alpha\in A}\mathcal{T}_\alpha$. Hence $U_1, \dots, U_n \in \mathcal{T}_\alpha$ for all $\alpha \in A$ and so $U_1 \cap \dots \cap U_n \in \mathcal{T}_\alpha$ for all $\alpha \in A$. Hence $U_1 \cap \dots \cap U_n \in \cap_{\alpha\in A}\mathcal{T}_\alpha$. Finally, suppose that $(U_\beta)_{\beta\in B}$ is a family in $\cap_{\alpha\in A}\mathcal{T}_\alpha$. Hence for all $\alpha \in A$ we have that $U_\beta \in \mathcal{T}_\alpha$ for all $\beta \in B$. So $\cup_{\beta\in B}U_\beta \in \mathcal{T}_\alpha$ for all $\alpha \in A$ and therefore $\cup_{\beta\in B}U_\beta \in \cap_{\alpha\in A}\mathcal{T}_\alpha$.

b. Define

$$\mathcal{B} := \{U_1 \cap \dots \cap U_n : n \in \mathbb{N}, U_i \in \mathcal{S} \text{ for all } i = 1, \dots, n\}$$

and

$$\mathcal{T} := \{\cup_{\alpha\in A}B_\alpha : B_\alpha \in A \text{ for all } \alpha \in A\}.$$

Lemma 1.1. $\mathcal{T}_\mathcal{F} = \mathcal{T}$.

Proof. By part **a.**, $\mathcal{T}_\mathcal{F}$ is a topology. We show that also $\mathcal{T}$ is a topology. By [Lee11, p. 34] it is enough to show that $\mathcal{B}$ satisfies the following two conditions:

(i) $\cup_{B\in\mathcal{B}}B = X$.

(ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists an element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Then $\mathcal{T}$ is the unique topology on X generated by $\mathcal{B}$, i.e. the collection of arbitrary unions of elements of $\mathcal{B}$. Since $\mathcal{F}$ is nonempty, there exists $f \in \mathcal{F}$. Clearly $X = f^{-1}(Y_f)$ and Y_f is open in Y_f. Hence $f^{-1}(Y_f) \in \mathcal{S}$ and thus $X \in \cup_{B\in\mathcal{B}}B$. Suppose that $B_1, B_2 \in \mathcal{B}$ such that $B_1 \cap B_2 \neq \varnothing$. Hence we find $U_1, \dots, U_n, V_1, \dots, V_m \in \mathcal{S}$ such that $B_1 = U_1 \cap \dots \cap U_n$ and $B_2 = V_1 \cap \dots \cap V_m$. Suppose $x \in B_1 \cap B_2$. Then also $x \in U_1 \cap \dots \cap U_n \cap V_1 \cap \dots \cap V_m$. But

$$U_1 \cap \dots \cap U_n \cap V_1 \cap \dots \cap V_m \in \mathcal{B}$$

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as a finite intersection of elements of $\mathcal{S}$. Hence $\mathcal{T}$ is a topology.
Clearly, $\mathcal{S} \subseteq \mathcal{T}$, since already $\mathcal{S} \subseteq \mathcal{B}$. Since $\mathcal{T}_{\mathcal{F}}$ is the smallest topology containing $\mathcal{S}$, we get that $\mathcal{T}_{\mathcal{F}} \subseteq \mathcal{T}$.
Let $U \in \mathcal{T}$. Then $U = \cup_{\alpha \in A} B_\alpha$ for some index set A and $B_\alpha \in \mathcal{B}$ for all $\alpha \in A$. But each B_α is a finite intersection of elements of $\mathcal{S}$ and thus since $\mathcal{T}_{\mathcal{F}}$ is a topology containing $\mathcal{S}$, we have that $B_\alpha \in \mathcal{T}_{\mathcal{F}}$ for all $\alpha \in A$. But then also $U \in \mathcal{T}_{\mathcal{F}}$ as a union of sets in $\mathcal{T}_{\mathcal{F}}$. Hence $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{F}}$. $\square$

Exercise 5.

References

[Lee11] John M. Lee. *Introduction to Topological Manifolds*. Second Edition. Springer Science+Business Media, 2011.