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SOLUTIONS SHEET 7

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Exercise 1.

Exercise 2.

Exercise 3.

Lemma 1.1. *Let $y \in H$ and define a mapping $\varphi_y : H \to \mathbb{C}$ by $\varphi_y(x) := \langle A(y), x\rangle$. Then $\varphi_y \in \mathscr{L}(H, \mathbb{C})$.*

Proof. Clearly, φ_y is linear since $\langle\cdot,\cdot\rangle$ is linear in the second component. Moreover, φ_y is bounded. Indeed, using Cauchy-Schwarz yields

$$|\varphi_y(x)| = |\langle A(y), x\rangle| \leq \|A(y)\|\|x\|$$

for all $x \in H$. □

Thus we may define a family

$$\mathscr{F} := \left\{\varphi_y : y \in \partial B_1(0)\right\} \subseteq \mathscr{L}(H, \mathbb{C}).$$

Let $x \in H$. Then for any $y \in \partial B_1(0)$ we have that

$$|\varphi_y(x)| = |\langle A(y), x\rangle| = |\langle y, A(x)\rangle| \leq \|y\|\|A(x)\| = \|A(x)\|$$

by symmetry and again Cauchy-Schwarz. Hence

$$\sup_{T\in\mathscr{F}} |T(x)| = \sup_{y\in\partial B_1(0)} |\varphi_y(x)| \leq \|A(x)\|$$

for all $x \in H$. Since any Hilbert space is a Banach space, an application of *Banach-Steinhaus* yields the existence of a constant $c > 0$ such that

$$\sup_{T\in\mathscr{F}} \|T\| = \sup_{y\in\partial B_1(0)} \|\varphi_y\| \leq c.$$

For $x \in H$ such that $A(x) \neq 0$ we have that

$$\begin{aligned}
\|A(x)\|^2 &= \langle A(x), A(x)\rangle \\
&= \|x\| \langle A(x/\|x\|), A(x)\rangle \\
&= \|x\| \varphi_{x/\|x\|}(A(x))
\end{aligned}$$

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$$
\begin{aligned}
&\leq \|x\| \, |\varphi_{x/\|x\|}(A(x))| \\
&\leq \|x\| \|A(x)\| \, \|\varphi_{x/\|x\|}\| \\
&\leq c\|x\| \|A(x)\|
\end{aligned}
$$

and thus dividing both sides by $\|A(x)\|$ yields the boundedness of A.