Contents

CHAPTER 1

Foundations

Basic Category Theory

Categories. We use the first order theory of Neumann-Bernays-Gödel (BNG) as described in [Men15, p. 231].

Definition 1.1 (Category). *A **category** $\mathfrak{C}$ consists of*

- *A class* $\mathrm{ob}(\mathcal{C})$*, called the **objects of** $\mathfrak{C}$.*
- *A class* $\mathrm{mor}(\mathcal{C})$*, called the **morphisms of** $\mathfrak{C}$.*
- *Two functions* $\mathrm{dom} : \mathrm{mor}(\mathcal{C}) \to \mathrm{ob}(\mathcal{C})$ *and* $\mathrm{cod} : \mathrm{mor}(\mathcal{C}) \to \mathrm{ob}(\mathcal{C})$*, which assign to each morphism f in $\mathcal{C}$ its **domain** and **codomain**, respectively.*
- *For each $X \in \mathrm{ob}(\mathcal{C})$ a function* $\mathrm{ob}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C})$ *which assigns a morphism* id_X *such that* $\mathrm{dom}\,\mathrm{id}_X = \mathrm{cod}\,\mathrm{id}_X = X$.
- *A function*
$$\circ : \{(g, f) \in \mathrm{mor}(\mathcal{C}) \times \mathrm{mor}(\mathcal{C}) : \mathrm{dom}\, g = \mathrm{cod}\, f\} \to \mathrm{mor}(\mathcal{C}) \tag{1}$$
*mapping (g, f) to $g \circ f$, called **composition**, such that* $\mathrm{dom}(g \circ f) = \mathrm{dom}\, f$ *and* $\mathrm{cod}(g \circ f) = \mathrm{cod}\, g$.

Subject to the following axioms:

- ***(Associativity Axiom)** For all $f, g, h \in \mathrm{mor}(\mathcal{C})$ with $\mathrm{dom}\, h = \mathrm{cod}\, g$ and $\mathrm{dom}\, g = \mathrm{cod}\, f$, we have that*
$$(h \circ g) \circ f = h \circ (g \circ f). \tag{2}$$
- ***(Unit Axiom)** For all $f \in \mathrm{mor}(\mathcal{C})$ with $\mathrm{dom}\, f = X$ and $\mathrm{cod}\, f = Y$ we have that*
$$f = f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f. \tag{3}$$

Remark 1.1. Let $\mathcal{C}$ be a category. For $X, Y \in \mathrm{ob}(\mathcal{C})$ we will abreviate
$$\mathcal{C}(X, Y) := \{f \in \mathrm{mor}(\mathcal{C}) : \mathrm{dom}\, f = X \text{ and } \mathrm{cod}\, f = Y\}.$$
Moreover, $f \in \mathcal{C}(X, Y)$ is depicted as
$$f : X \to Y. \tag{4}$$

Example 1.1. Let $*$ be a single, not nearer specified object. Consider as morphisms the class of all cardinal numbers and as composition cardinal addition. By [Hal12, pp. 112–113], cardinal addition is associative and $\varnothing$ serves for the identity id_*.

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Definition 1.2 (Locally Small, Hom-Set). *A category $\mathcal{C}$ is said to be* ***locally small*** *if for all $X, Y \in \mathcal{C}$, $\mathcal{C}(X, Y)$ is a set. If $\mathcal{C}$ is locally small, $\mathcal{C}(X, Y)$ is called a* ***hom-set*** *for all $X, Y \in \mathcal{C}$.*

Functors.

Definition 1.3 (Functor). *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A* ***functor $F : \mathcal{C} \to \mathcal{D}$*** *is a pair of functions (F_1, F_2), $F_1 : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, called the* ***object function*** *and $F_2 : \mathrm{mor}(\mathcal{C}) \to \mathrm{mor}(\mathcal{D})$, called the* ***morphism function****, such that for every morphism $f : X \to Y$ we have that $F_2(f) : F_1(X) \to F_1(Y)$ and (F_1, F_2) is subject to the following* ***compatibility conditions****:*

- *For all $X \in \mathrm{ob}(\mathcal{C})$, $F_2(\mathrm{id}_X) = \mathrm{id}_{F_1(X)}$.*
- *For all $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ we have that $F_2(g \circ f) = F_2(g) \circ F_2(f)$.*

Remark 1.2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. It is convenient to denote the components F_1 and F_2 also with F.

Subcategories.

Definition 1.4 (Subcategory). *Let $\mathcal{C}$ be a category. A* ***subcategory $\mathcal{S}$ of $\mathcal{C}$*** *consists of*

- *A subclass $\mathrm{ob}(\mathcal{S}) \subseteq \mathrm{ob}(\mathcal{C})$.*
- *A subclass $\mathrm{mor}(\mathcal{S}) \subseteq \mathrm{mor}(\mathcal{C})$.*

Subject to the following conditions:

- *For all $X \in \mathcal{S}$, $\mathrm{id}_S \in \mathrm{mor}(\mathcal{S})$.*

Example 1.2 (Top_*). Define the objects of Top_* to be the class of all tuple (X, p), where X is a topological space and $p \in X$. Moreover, given objects (X, p) and (Y, q) in Top_*, define $\mathsf{Top}_*\big((X, p), (Y, q)\big) := \{f \in \mathsf{Top}(X, Y) : f(p) = q\}$. It is easy to check that Top_* is a category, called the ***category of pointed topological spaces***.

Limits.

Definition 1.5 (Diagram). *Let $\mathcal{C}$ be a category and A a small category. A functor $\mathsf{A} \to \mathcal{C}$ is called a* ***diagram in $\mathcal{C}$ of shape A****.*

Definition 1.6 (Cone and Limit). *Let $\mathcal{C}$ be a category and $D : \mathsf{A} \to \mathcal{C}$ a diagram in $\mathcal{C}$ of shape A. A* ***cone on D*** *is a tuple $\big(C, (f_\alpha)_{\alpha \in \mathsf{A}}\big)$, where $C \in \mathcal{C}$ is an object, called the* ***vertex*** *of the cone, and a family of arrows in $\mathcal{C}$*

$$\big(C \xrightarrow{f_\alpha} D(\alpha)\big)_{\alpha \in \mathsf{A}}. \tag{5}$$

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such that for all morphisms $f \in \mathsf{A}$, $f : \alpha \to \beta$, *the triangle*

$$\begin{tikzcd} & D(\alpha) \arrow[dd, "D(f)"] \\ C \arrow[ur, "f_\alpha"] \arrow[dr, "f_\beta"'] & \\ & D(\beta) \end{tikzcd}$$

commutes. A ***(small) limit of D*** *is a cone* $\big(L, (\pi_\alpha)_{\alpha \in \mathsf{A}}\big)$ *with the property that for any other cone* $\big(C, (f_\alpha)_{\alpha \in \mathsf{A}}\big)$ *there exists a unique morphism* $\overline{f} : A \to L$ *such that* $\pi_\alpha \circ \overline{f} = f_\alpha$ *holds for every* $\alpha \in \mathsf{A}$.

Remark 1.3. In the setting of definition 1.6, if $\big(L, (\pi_\alpha)_{\alpha \in \mathsf{A}}\big)$ is a limit of D, we sometimes reffering to L only as the limit of D and we write

$$L = \lim_{\leftarrow \mathsf{A}} D. \tag{6}$$

CHAPTER 2

The Fundamental Group

The Fundamental Grupoid

Construction of the fundamental Grupoid.

Lemma 2.1 (Gluing Lemma). *Let $X, Y \in \mathrm{ob}(\mathsf{Top})$, $(X_\alpha)_{\alpha\in A}$ a finite closed cover of X and $(f_\alpha)_{\alpha\in A}$ a finite family of maps $f_\alpha \in \mathsf{Top}(X_\alpha, Y)$ such that $f_\alpha|_{X_\alpha\cap X_\beta} = f_\beta|_{X_\alpha\cap X_\beta}$ for all $\alpha, \beta \in A$. Then there exists a unique $f \in \mathsf{Top}(X, Y)$ such that $f|_{X_\alpha} = f_\alpha$ for all $\alpha \in A$.*

Proof. Let $x \in X$. Since $(X_\alpha)_{\alpha\in A}$ is a cover of X, we find $\alpha \in A$ such that $x \in X_\alpha$. Define $f(x) := f_\alpha(x)$. This is well defined, since if $x \in X_\alpha \cap X_\beta$ for some $\beta \in A$, we have that $f(x) = f_\beta(x) = f_\alpha(x)$. Clearly $f|_{X_\alpha} = f_\alpha$ for all $\alpha \in A$ and f is unique. Let us show continuity. To this end, let $K \subseteq Y$ be closed. Then

$$\begin{aligned} f^{-1}(K) &= X \cap f^{-1}(K) \\ &= \bigcup_{\alpha\in A} X_\alpha \cap f^{-1}(K) \\ &= \bigcup_{\alpha\in A} \left(X_\alpha \cap f^{-1}(K)\right) \\ &= \bigcup_{\alpha\in A} \left(X_\alpha \cap f_\alpha^{-1}(K)\right). \end{aligned}$$

Since each f_α is continuous, $f_\alpha^{-1}(K)$ is closed in X_α for each $\alpha \in A$ and thus since X_α is closed, $f^{-1}(K)$ is closed as a finite union of closed sets. □

Theorem 2.1. *There is a functor* $\mathsf{Top} \to \mathsf{Grpd}$.

Proof. The proof is divided into several steps. Let us denote $\Pi : \mathsf{Top} \to \mathsf{Grpd}$ for the claimed functor.

Step 1: Definition of Π *on objects.* Let $X, Y \in \mathrm{ob}(\mathsf{Top})$, $f, g \in \mathsf{Top}(X, Y)$ and $A \subseteq X$. A map $F \in \mathsf{Top}(X \times I, Y)$ is called a ***homotopy from X to Y relative to A***, if

- $F(x, 0) = f(x)$, for all $x \in X$.
- $F(x, 1) = g(x)$, for all $x \in X$.
- $F(x, t) = f(x) = g(x)$, for all $x \in A$ and for all $t \in I$.

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If there exists a homotopy between f and g relative to A we say that f and g are ***homotopic relative to A*** and write $f \simeq_A g$. If we want to emphasize the homotpoy relative to A, we write $F : f \simeq_A g$.

Lemma 2.2. *Let $X, Y \in \mathrm{ob}(\mathsf{Top})$ and $A \subseteq X$. Then being homotopic relative to A is an equivalence relation on $\mathsf{Top}(X, Y)$.*

Proof. Define a binary relation $R_A \subseteq \mathsf{Top}(X, Y) \times \mathsf{Top}(X, Y)$ by

$$fR_Ag \quad :\Leftrightarrow \quad f \simeq_A g.$$

Let $f \in \mathsf{Top}(X, Y)$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x, t) := f(x).$$

Then clearly $F : f \simeq_A f$. Hence R_A is reflexive.
Let $g \in \mathsf{Top}(X, Y)$ and assume that fR_Ag. Thus $G : f \simeq_A g$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x, t) := G(x, 1 - t).$$

Then it is easy to check that $F : g \simeq_A f$ and so R_A is symmetric.
Finally, let $h \in \mathsf{Top}(X, Y)$ and suppose that fR_Ag and gR_Ah. Hence $F_1 : f \simeq_A g$ and $F_2 : g \simeq_A h$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x, t) := \begin{cases} F_1(x, 2t) & 0 \leq t \leq \frac{1}{2}, \\ F_2(x, 2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Continuity of F follows by an application of the gluing lemma 2.1. Then it is easy to check that $F : f \simeq_A h$ and hence R_A is transitive. □

Let $X \in \mathrm{ob}(\mathsf{Top})$ and u a path in X from p to q. Define the ***path class*** $[u]$ ***of*** u by $[u] := [u]_{R_{\partial I}}$. Define now

- $\mathrm{ob}\big(\Pi(X)\big) := X$.
- $\Pi(X)(p, q) := \{[u] : u \text{ is a path from } p \text{ to } q\}$ for all $p, q \in X$.
- Let $p \in X$. Then define $\mathrm{id}_p \in \Pi(X)(p, p)$ by $\mathrm{id}_p := [c_p]$, where c_p is the constant path defined by $c_p(s) := p$ for all $s \in I$.
- And $\Pi(X)(q, r) \times \Pi(X)(p, q) \to \Pi(X)(p, r)$ by

  $$([v], [u]) \mapsto [u * v]$$

  Where $u * v \in \mathsf{Top}(p, r)$ is the ***concatenated path of u and v***, defined by

  $$(u * v)(s) := \begin{cases} u(2s) & 0 \leq t \leq \frac{1}{2}, \\ v(2s - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

  Continuity follows again from the gluing lemma 2.1 whereas well definedness follows from the next lemma.

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Lemma 2.3. *Suppose that* $[u_1], [u_2] \in \Pi(X)(p,q)$ *and* $[v_1], [v_2] \in \Pi(X)(q,r)$ *such that* $[u_1] = [u_2]$ *and* $[v_1] = [v_2]$*. Then* $[u_1 * v_1] = [u_2 * v_2]$*.*

Proof. By assumption we have $G : u_1 \simeq_{\partial I} u_2$ and $H : v_1 \simeq_{\partial I} v_2$. Define $F \in \mathsf{Top}(I \times I, X)$ by

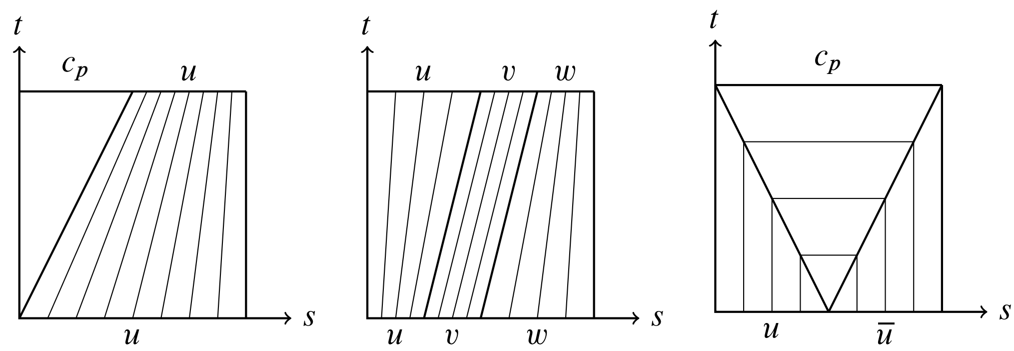
$$F(s,t) := \begin{cases} G(2s,t) & 0 \leq s \leq \frac{1}{2}, \\ H(2s-1,t) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Again, continuity follows from the gluing lemma 2.1 and it is easy to check that $F : u_1 * v_1 \simeq_{\partial I} u_2 * v_2$. □

Let us now check that $\Pi(X)$ is indeed a category. Let $[u] \in \Pi(X)(p,q)$. We want to show that $u \simeq_{\partial I} c_p * u$. To this end, we consider figure 1a and conclude that a suitable homotopy is given by $F \in \mathsf{Top}(I \times I, X)$ defined by

$$F(s,t) := \begin{cases} p & 0 \leq 2s \leq t, \\ u\left(\dfrac{2s-t}{2-t}\right) & t \leq 2s \leq 2. \end{cases}$$

Similarly, considering figure 1b leads to $F \in \mathsf{Top}(I \times I, X)$ defined by



(A) $u \simeq_{\partial I} c_p * u$. (B) $(u*v)*w \simeq_{\partial I} u*(v*w)$. (C) $u * \overline{u} \simeq_{\partial I} c_p$.

FIGURE 1. Visualization of the proof that $\Pi(X)$ is a grupoid object.

$$F(s,t) := \begin{cases} u\left(\dfrac{4s}{t+1}\right) & -1 \leq 4s-1 \leq t, \\ v(4s-t-1) & t \leq 4s-1 \leq t+1, \\ w\left(\dfrac{4s-t-2}{4-t-2}\right) & t+1 \leq 4s-1 \leq 3. \end{cases}$$

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Lastly, we check that $\Pi(X)$ is a grupoid. To this end, for a path u from p to q, define its ***reverse path*** $\bar{u}$ by

$$\bar{u}(s) := u(1-s).$$

We claim that $u * \bar{u} \simeq_{\partial I} c_p$. From figure 1c we deduce that $F \in \mathsf{Top}(I \times I, X)$ is given by

$$F(s,t) := \begin{cases} u(2s) & 0 \leq 2s \leq 1-t, \\ u(1-t) & 1-t \leq 2s \leq t+1, \\ \bar{u}(2s-1) & t+1 \leq 2s \leq 2. \end{cases}$$

Step 2: Definition of Π *on morphisms.* Let $f \in \mathsf{Top}(X,Y)$. Then $\Pi(f)$ is a functor from $\Pi(X)$ to $\Pi(Y)$. Define $\Pi(f)$ as follows:

- Let $p \in \mathrm{ob}\big(\Pi(X)\big)$. Then define $\Pi(f)(p) := f(p) \in \mathrm{ob}\big(\Pi(Y)\big)$.
- Let $[u] \in \Pi(X)(p,q)$. Then define $\Pi(f)[u] := [f \circ u] \in$. We have to check that this definition is independent of the choice of the representative.

Lemma 2.4. *Let u and v be paths from p to q in X and suppose that $[u] = [v]$. Then for any $f \in \mathsf{Top}(X,Y)$ we also have that $[f \circ u] = [f \circ v]$.*

Proof. Suppose that $H : u \simeq_{\partial I} v$. Define $F \in \mathsf{Top}(I \times I, Y)$ by

$$F(s,t) := (f \circ F)(s,t).$$

Then $F : f \circ u \simeq_{\partial I} f \circ v$. □

Checking that Π satisfies the functorial properties is left as an exercise. □

Exercise 0.1. Check that $\Pi : \mathsf{Top} \to \mathsf{Grpd}$ is indeed a functor.

The Fundamental Group.

Lemma 2.5. *Let $\mathcal{G}$ be a locally small grupoid. Then for every $X \in \mathrm{ob}(\mathcal{G})$, $\mathcal{G}(X,X)$ can be equipped with a group structure.*

Proof. Since $\mathcal{G}$ is locally small, $\mathcal{G}(X,X)$ is a set for every $X \in \mathrm{ob}(\mathcal{G})$. Define a multiplication $\mathcal{G}(X,X) \times \mathcal{G}(X,X) \to \mathcal{G}(X,X)$ by $gh := h \circ g$. Clearly, this multiplication is associative. Moreover, the identity element is given by $\mathrm{id}_X \in \mathcal{G}(X,X)$ and since every $g \in \mathcal{G}(X,X)$ is an isomorphism, the multiplicative inverse is given by the inverse in $\mathcal{G}(X,X)$. □

Proposition 2.1. *There is a functor* $\mathsf{Top}_* \to \mathsf{Grp}$.

Proof. Define $\pi_1 : \mathsf{Top}_* \to \mathsf{Grp}$ on objects $(X,p) \in \mathsf{Top}_*$ by

$$\pi_1(X,p) := \Pi(X)(p,p).$$

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By theorem 2.1 together with lemma 2.5, $\pi_1(X, p)$ is actually a group, called the ***fundamental group of X with basepoint p***. On morphisms $f \in \mathsf{Top}_*\big((X,p),(Y,q)\big)$, define

$$\pi_1(f) := \Pi(f) : \Pi(X)(p,p) \to \Pi(Y)(q,q).$$

Let $[u], [v] \in \pi_1(X,p)$. Then

$$\begin{aligned}
\pi_1([u][v]) &= \Pi(f)([u][v]) \\
&= \Pi(f)[u * v] \\
&= [f \circ (u * v)] \\
&= [(f \circ u) * (f \circ v)] \\
&= \Pi(f)[u]\,\Pi(f)[v] \\
&= \pi_1(f)[u]\,\pi_1(f)[v].
\end{aligned}$$

Thus $\pi_1(f)$ is a morphism in Grp. Functoriality of π_1 immediately follows from the functoriality of Π. □

Lemma 2.6. *Let $X \in \mathrm{ob}(\mathsf{Top})$, $p \in X$ and A be the path component of X containing p. Then $\pi_1(\iota)$, where $\iota : A \hookrightarrow X$ denotes the inclusion, is an isomorphism.*

Proof. Suppose $[u] \in \ker \pi_1(\iota)$. Then $[\iota \circ u] = [c_p]$ and Hence $F : \iota \circ u \simeq_{\partial I} c_p$. Since $I \times I$ is path connected and $p \in F(I \times I)$, it follows that $F(I \times I) \subseteq A$ and thus $F : u \simeq_{\partial I} c_p$ in A and hence $[u] = [c_p]$. To see that $\pi_1(\iota)$ is surjective, just observe that $u(I) \subseteq A$ for $[u] \in \pi_1(X,p)$ since $u(I)$ is path connected and $p \in u(I)$. □

$\pi_1(\mathbb{S}^1)$.

Definition 2.1 (Exponential Quotient Map). *The mapping $\varepsilon : \mathbb{R} \to \mathbb{S}^1$ defined by*

$$\varepsilon(x) := e^{2\pi i x} \tag{7}$$

*is called the **exponential quotient map**.*

Proposition 2.2 (Lifting Property of the Circle). *Let $n \in \mathbb{Z}$, $n \geq 0$, $X \subseteq \mathbb{R}^n$ compact and convex, $p \in X$, $f \in \mathsf{Top}_*\big((X,p),(\mathbb{S}^1,1)\big)$ and $m \in \mathbb{Z}$. Then there exists a unique map $\widetilde{f} \in \mathsf{Top}_*\big((X,p),(\mathbb{R},m)\big)$, called the **lifting of f**, such that*

$$\begin{tikzcd}
 & (\mathbb{R}, m) \arrow[d, "\varepsilon"] \\
(X,p) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (\mathbb{S}^1, 1)
\end{tikzcd}$$

commutes.

Proof. We show first existence and then uniqueness.

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Step 1: Existence. Since X is compact and f is continuous, f is uniformly continuous on X. Thus we find $\delta > 0$ such that $|f(x) - f(y)| < 2$, whenever $|x - y| < \delta$, i.e. $f(x)$ and $f(y)$ are not antipodal points. Moreover, since X is compact, X is bounded and hence we find $N \in \mathbb{N}$, such that $|x - y| < N\delta$ holds for all $x, y \in X$. Let $x \in X$. For $0 \leq k \leq N$, define $L_k : X \to X$ by

$$L_k(x) := \left(1 - \frac{k}{N}\right) p + \frac{k}{N} x.$$

Those are well defined functions since X is convex. Moreover, each L_k is continuous. Indeed, it is easy to check that L_k is Lipschitz. Also, for each $0 \leq k < N$, $f(L_k(x))$ and $f(L_{k+1}(x))$ are not antipodal for all $x \in X$. Indeed, it is easy to check that $|L_k(x) - L_{k+1}(x)| < \delta$ holds for all $x \in X$. For $0 \leq k < N$ define $g_k : X \to \mathbb{S}^1 \setminus \{-1\}$ by

$$g_k(x) := \frac{f(L_{k+1}(x))}{f(L_k(x))}.$$

Clearly g_k is well defined and continuous as a composition of continuous functions. Let $\operatorname{Log} : \mathbb{S}^1 \setminus \{-1\} \to \mathbb{C}$ denote the principal branch of the logarithm. Define $\widetilde{f} : X \to \mathbb{R}$ by

$$\widetilde{f}(x) := m + \frac{1}{2\pi i} \sum_{k=0}^{N-1} \operatorname{Log}(g_k(x)).$$

Clearly, $\widetilde{f}$ is continuous and moreover we have that $\widetilde{f} = m$ since $g_k(p) = 1$ for all $0 \leq k < N$. Finally, for any $x \in X$ we have that

$$(\varepsilon \circ \widetilde{f})(x) = \varepsilon(m) \prod_{k=0}^{N-1} g_k(x) = \frac{f(L_N(x))}{f(L_0(x))} = \frac{f(x)}{f(p)} = f(x).$$

Step 2: Uniqueness. Suppose $\widetilde{g} \in \mathsf{Top}_*\big((X, p), (\mathbb{R}, m)\big)$ is another such function. Define $\varphi \in \mathsf{Top}_*\big((X, p), (\mathbb{R}, 0)\big)$ by

$$\varphi(x) := \widetilde{f}(x) - \widetilde{g}(x).$$

Then clearly $\varepsilon \circ \varphi = 1$ and thus $\varphi(X) \subseteq \mathbb{Z}$. Since X is convex, X is connected and so $\varphi = 0$.

$\square$

Corollary 2.1. *Let $u, v \in \Omega(\mathbb{S}^1, 1)$ such that $[u] = [v]$. If $\widetilde{u}, \widetilde{v} : (I, 0) \to (\mathbb{R}, 0)$ are the liftings of u and v, respectively, then $[\widetilde{u}] = [\widetilde{v}]$.*

Proof. Let $F : u \simeq_{\partial I} v$. By proposition 2.2, we find $\widetilde{F} \in \mathsf{Top}_*\big((I \times I, (0, 0)), (\mathbb{R}, 0)\big)$, such that $\varepsilon \circ \widetilde{F} = F$. We claim that $\widetilde{F} : \widetilde{u} \simeq_{\partial I} \widetilde{v}$. For $s \in I$ define $\widetilde{u}_0(s) := \widetilde{F}(s, 0)$. Then

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$\widetilde{u}_0(0) = \widetilde{F}(0,0) = 0$ and since $\widetilde{u}_0$ is continuous we have that $\widetilde{u}_0 \in \mathsf{Top}_*\left((I,0),(\mathbb{R},0)\right)$. Moreover

$$(\varepsilon \circ \widetilde{u}_0)(s) = \varepsilon\left(\widetilde{F}(s,0)\right) = F(s,0) = u(s)$$

for all $s \in I$ and thus $\widetilde{u}_0$ is a lifting of u. But by proposition 2.2, liftings are unique and thus $\widetilde{u}_0 = \widetilde{u}$. Next define $\widetilde{w}_0(t) := \widetilde{F}(0,t)$ for all $t \in I$. Then $\widetilde{w}_0(0) = \widetilde{F}(0,0) = 0$ and so $\widetilde{w}_0 \in \mathsf{Top}_*\left((I,0),(\mathbb{R},0)\right)$. Moreover

$$(\varepsilon \circ \widetilde{w}_0)(t) = \varepsilon\left(\widetilde{F}(0,t)\right) = F(0,t) = u(0) = v(0) = 1.$$

for all $t \in I$. Thus

$$\begin{tikzcd} & (\mathbb{R},0) \arrow[d, "\varepsilon"] \\ (I,0) \arrow[ur, "\widetilde{w}_0"] \arrow[r, "c_1"'] & (\mathbb{S}^1,1) \end{tikzcd}$$

commutes. But also c_0 makes the above diagram commute. By uniqueness, $\widetilde{w}_0 = c_0$. Define $\widetilde{v}_0(s) := \widetilde{F}(s,1)$ for all $s \in I$. Then $\widetilde{v}_0(0) = \widetilde{F}(0,1) = \widetilde{w}_0(1) = 0$ and it is easy to check that $\widetilde{v}_0$ is a lift for v. Hence $\widetilde{v}_0 = \widetilde{v}$. Finally, define $\widetilde{w}_1(t) := \widetilde{F}(1,t)$ for all $t \in I$. Then $\widetilde{w}_1(0) = \widetilde{F}(1,0) = \widetilde{u}(1)$ and thus $\widetilde{w}_1 \in \mathsf{Top}_*\left((I,0),(\mathbb{R},\widetilde{u}(0))\right)$. Moreover

$$(\varepsilon \circ \widetilde{w}_1)(t) = \varepsilon\left(\widetilde{F}(1,t)\right) = F(1,t) = v(1) = u(1) = 1$$

for all $t \in I$. By proposition 2.2, we have again that $\widetilde{w}_1 = c_{\widetilde{u}(1)}$. So $F : \widetilde{u} \simeq_{\partial I} \widetilde{v}$. ☐

Definition 2.2 (Degree). *Let $u \in \Omega(\mathbb{S}^1,1)$. The **degree of u**, written $\deg u$, is defined by $\deg u := \widetilde{u}(1)$, where $\widetilde{u}$ is the unique lift of u such that $\widetilde{u}(0) = 0$.*

Theorem 2.2 (Fundamental Group of the Circle). $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.

Proof. Define $\deg : \pi_1(\mathbb{S}^1,1) \to \mathbb{Z}$ by $\deg[u] := \deg u$. This is well defined by corollary 2.1, since if $[u] = [v]$, then $[\widetilde{u}] = [\widetilde{v}]$ and in particular $\widetilde{u}(1) = \widetilde{v}(1)$.

Step 1: $\deg \in \mathsf{Grp}\left(\pi_1(\mathbb{S}^1,1),(\mathbb{Z},+)\right)$. Let $[u],[v] \in \pi_1(\mathbb{S}^1,1)$ and $m := \deg[u]$, $n := \deg[v]$. Moreover, let $\widetilde{u}$ and $\widetilde{v}$ denote the unique liftings of u and v, respectively, such that $\widetilde{u}(0) = 0$ and $\widetilde{v}(0) = 0$. Define

$$\widetilde{w}(s) := \begin{cases} \widetilde{u}(2s) & 0 \leq s \leq \frac{1}{2}, \\ m + \widetilde{v}(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Clearly $\widetilde{w}$ is continuous and $\widetilde{w}(0) = 0$. Hence $\widetilde{w} \in \mathsf{Top}_*\left((I,0),(\mathbb{R},0)\right)$. Also we have that $\varepsilon \circ \widetilde{w} = u * v$ and thus $\widetilde{w}$ is the lift of $u * v$. But $\widetilde{w}(1) = m + n$ and so

$$\deg([u]\,[v]) = \deg[u * v] = \deg(u * v) = \widetilde{w}(1) = m + n = \deg[u] + \deg[v].$$

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Step 2: $\deg$ is injective. Suppose $\deg[u] = 0$. Then $\widetilde{u}(1) = 0$ and thus $\widetilde{u} \in \Omega(\mathbb{R}, 0)$. Since $\mathbb{R}$ is contractible, we have that $[\widetilde{u}] = [c_0]$ and thus

$$[u] = [\varepsilon \circ \widetilde{u}] = \pi_1(\varepsilon)\,[\widetilde{u}] = \pi_1(\varepsilon)\,[c_0] = [c_1]\,.$$

Thus $\ker(\deg)$ is trivial.

Step 3: $\deg$ is surjective. Let $m \in \mathbb{Z}$. Then

$$\deg[\varepsilon^m] = \deg\varepsilon^m = \widetilde{\varepsilon^m}(1) = m\,.$$

□

CHAPTER 3

Singular Homology

Free Abelian Groups

Proposition 3.1. *The forgetful functor* $U : \mathsf{Ab} \to \mathsf{Set}$ *admits a left adjoint.*

Proof. We have to construct a functor $F : \mathsf{Set} \to \mathsf{Ab}$. Let S be a set. Define

$$F(S) := \left\{ f \in \mathbb{Z}^S : \operatorname{supp} f \text{ is finite} \right\}.$$

Equipped with pointwise addition, $F(S)$ is an abelian group. There is a natural inclusion $\iota : S \hookrightarrow U\left(F(S)\right)$ sending $x \in S$ to the function taking the value one at x and zero else. Hence we may regard elements of $F(S)$ as formal linear combinations $\sum_{x \in S} m_x x$, where $m_x \in \mathbb{Z}$ for all $x \in S$. Let $G \in \operatorname{ob}(\mathsf{Ab})$ be an abelian group and $\varphi \in \mathsf{Ab}\left(F(S), G\right)$ a morphism of groups. Define $\overline{\varphi} \in \mathsf{Set}\left(S, U(G)\right)$ by $\overline{\varphi} := U(\varphi)$. Conversly, if we have $f \in \mathsf{Set}\left(S, U(G)\right)$, define $\overline{f} \in \mathsf{Ab}\left(F(S), G\right)$ by $\overline{f}\left(\sum_{x \in S} m_x x\right) := \sum_{x \in S} m_x f(x)$. This is well defined since all but finitely many m_x are zero and G is abelian. It is easy to check that $\overline{f}$ is indeed a morphism of groups. Let $\varphi \in \mathsf{Ab}\left(F(S), G\right)$. Then

$$\begin{aligned}
\overline{\overline{\varphi}}\left(\sum_{x \in S} m_x x\right) &= \sum_{x \in S} m_x \overline{\varphi}(x) \\
&= \sum_{x \in S} m_x U(\varphi)(x) \\
&= \sum_{x \in S} m_x \varphi(x) \\
&= \varphi\left(\sum_{x \in S} m_x x\right).
\end{aligned}$$

And for $f \in \mathsf{Set}\left(S, U(G)\right)$ we have that

$$\overline{\overline{f}}(x) = U(\overline{f})(x) = \overline{f}(x) = f(x).$$

Hence $\overline{\overline{\varphi}} = \varphi$ and $\overline{\overline{f}} = f$ and so we have a bijection

$$\mathsf{Ab}\left(F(S), G\right) \cong \mathsf{Set}\left(S, U(G)\right).$$

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The mapping $f \mapsto \overline{f}$ will be referred to as ***extending by linearity***. To check naturality in S and G is left as an exercise. $\square$

Exercise 0.1. Check the naturality of the bijection in proposition 3.1. Also check that $F : \mathsf{Set} \to \mathsf{Ab}$ is indeed a functor. F is called the ***free functor from* Set *to* Ab**.

Definition 3.1 (Free Abelian Group). *Let $F : \mathsf{Set} \to \mathsf{Ab}$ be the free functor. For any set S, we call $F(S)$ the **free group generated by S**.*

Chain Complexes

Definition 3.2 (Chain Complex). *A **chain complex** is a tuple $(C_\bullet, \partial_\bullet)$ consisting of a sequence $(C_n)_{n\in\mathbb{Z}}$ in $\operatorname{ob}(\mathsf{Ab})$ and a sequence $(\partial_n)_{n\in\mathbb{Z}}$ in $\operatorname{mor}(\mathsf{Ab})$, called **boundary operators**, such that we have $\partial_n \in \mathsf{Ab}(C_n, C_{n-1})$ and $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$.*

Definition 3.3 (Chain Maps). *Let $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ be two chain complexes. A **chain map $f_\bullet : C_\bullet \to C'_\bullet$** is a sequence $(f_n)_{n\in\mathbb{Z}}$ in $\operatorname{mor}(\mathsf{Ab})$ such that $f_n \in \mathsf{Ab}(C_n, C'_n)$ and the diagram*

$$\begin{tikzcd} C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\ C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \end{tikzcd}$$

commutes for all $n \in \mathbb{Z}$.

Proposition 3.2. *There is a category with objects chain complexes and morphisms chain maps.*

Proof. Let $f_\bullet : C_\bullet \to C'_\bullet$ and $g_\bullet : C'_\bullet \to C''_\bullet$ be chain maps. Define a map $g_\bullet \circ f_\bullet$ by $g_n \circ f_n$ for each $n \in \mathbb{Z}$. This defines a chain map. Moreover, for each chain complex $C_\bullet$ define $\mathrm{id}_{C_\bullet}$ by id_{C_n} for all $n \in \mathbb{Z}$. It is easy to check, that then $\circ$ is associative and the identity laws hold. $\square$

Definition 3.4 (Comp). *The category in 3.2 is called the **category of chain complexes** and we refer to it as* Comp.

Theorem 3.1. *There is a functor* $\mathsf{Top} \to \mathsf{Comp}$.

Proof. The proof is divided into several steps. Let us denote $C_\bullet : \mathsf{Top} \to \mathsf{Comp}$ for the claimed functor.

Step 1: Construction of a sequence of abelian groups. Let $v_0, \dots, v_k \in \mathbb{R}^n$ for some $n, k \in \mathbb{N}$. We say that $(v_0, \dots, v_k)$ is ***affinely independent*** if $(v_1 - v_0, \dots, v_k - v_0)$

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is linearly independent. We define the ***k-simplex spanned by*** $(\boldsymbol{v_0,\dots,v_k})$, written $[v_0,\dots,v_k]$, to be

$$[v_0,\dots,v_k] := \left\{\sum_{i=0}^k s_i v_i : s_i \geq 0 \text{ for all } i = 0,\dots,k \text{ and } \sum_{i=0}^k s_i = 1\right\}. \tag{8}$$

equipped with the subspace topology. Moreover, we define the ***standard n-simplex*** $\boldsymbol{\Delta^n}$ to be the n-simplex spanned by $(e_0,\dots,e_n)$ where $(e_{i+1})_i$ is the standard basis of $\mathbb{R}^{n+1}$. Let $X \in \mathrm{ob}(\mathsf{Top})$. Define a ***singular n-simplex in X*** to be a map $\sigma \in \mathsf{Top}(\Delta^n, X)$. Let $n \in \mathbb{Z}$. Define

$$C_n(X) := \begin{cases} F\left(\mathsf{Top}(\Delta^n, X)\right) & n \geq 0, \\ 0 & n < 0. \end{cases} \tag{9}$$

We will call elements of $C_n(X)$ ***singular n-chains***.

Step 2: Construction of boundary operators. Let $X \in \mathrm{ob}(\mathsf{Top})$ and σ a singular n-simplex in X for $n \geq 1$. We define $\varphi_k^n : \Delta^{n-1} \to \Delta^n$, called the ***k-th face map***, by

$$\varphi_k^n(s_0,\dots,s_{n-1}) := \begin{cases} (0, s_0,\dots,s_{n-1}) & k = 0, \\ (s_0,\dots,s_{k-1},0,s_k,\dots,s_{n-1}) & 1 \leq k \leq n-1. \end{cases} \tag{10}$$

Define now

$$\partial\sigma := \sum_{k=0}^n (-1)^k \sigma \circ \varphi_k^n \in U\left(C_{n-1}(X)\right) \tag{11}$$

to be the ***boundary of*** $\boldsymbol{\sigma}$. Moreover, the ***singular boundary operator*** is defined to be $\overline{\partial_n}$ and $\partial_n := 0$ for $n \leq 0$.

Step 3: $\partial_n \circ \partial_{n+1} = 0$ *for all* $n \in \mathbb{Z}$. It is enough to consider $n \geq 1$, since $\partial_n \circ \partial_{n+1} = 0$ holds trivially in the other cases. Let $X \in \mathrm{ob}(\mathsf{Top})$ and $\sigma \in \mathsf{Top}(\Delta^{n+1}, X)$. Then we have

$$\begin{aligned}
(\partial_n \circ \partial_{n+1})(\sigma) &= \partial_n\left(\sum_{k=0}^{n+1}(-1)^k \sigma \circ \varphi_k^{n+1}\right) \\
&= \sum_{k=0}^{n+1}(-1)^k \partial_n\left(\sigma \circ \varphi_k^{n+1}\right) \\
&= \sum_{k=0}^{n+1}\sum_{j=0}^{n}(-1)^{k+j}\sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\
&= \sum_{0\leq k\leq j\leq n}(-1)^{k+j}\sigma \circ \varphi_k^{n+1} \circ \varphi_j^n + \sum_{0\leq j<k\leq n+1}(-1)^{k+j}\sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\
&= \sum_{0\leq j\leq k\leq n}(-1)^{k+j}\sigma \circ \varphi_j^{n+1} \circ \varphi_k^n + \sum_{0\leq j<k\leq n+1}(-1)^{k+j}\sigma \circ \varphi_k^{n+1} \circ \varphi_j^n
\end{aligned}$$

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$$= \sum_{0 \leq j < k \leq n+1} \left((-1)^{k+j-1}\sigma \circ \varphi_j^{n+1} \circ \varphi_{k-1}^n + (-1)^{k+j}\sigma \circ \varphi_k^{n+1} \circ \varphi_j^n\right)$$

Step 4: Construction of chain maps. Let $X, Y \in \mathrm{ob}(\mathsf{Top})$ and $f \in \mathsf{Top}(X, Y)$. For $n \geq 0$, define $f_n^\# : \mathsf{Top}(\Delta^n, X) \to U\left(C_n(Y)\right)$ by $f^\# := f \circ \sigma$. Extending this map by linearity yields a homomorphism $f_n^\# : C_n(X) \to C_n(Y)$. Moreover, set $f_n^\# = 0$ for $n < 0$. Let $n \geq 1$ and $\sigma \in \mathsf{Top}(\Delta^n, X)$. Then on one hand we have

$$(f_{n-1}^\# \circ \partial_n)(\sigma) = f_{n-1}^\# \left(\sum_{k=0}^n (-1)^k \sigma \circ \varphi_k^n\right) = \sum_{k=0}^n (-1)^k f \circ \sigma \circ \varphi_k^n$$

and on the other

$$(\partial_n \circ f_n^\#)(\sigma) = \partial_n(f \circ \sigma) = \sum_{k=0}^n (-1)^k f \circ \sigma \circ \varphi_k^n.$$

Step 5: Checking functorial properties. We are ready to define the functor $C_\bullet : \mathsf{Top} \to \mathsf{Comp}$. Let $C_\bullet(X)$ be the chain complex consisting of $\left(C_n(X)\right)_{n\in\mathbb{Z}}$ and $(\partial_n)_{n\in\mathbb{Z}}$.

□

APPENDIX A

Set Theory

1. Basic Concepts

Problem 1.1. Let $n \in \mathbb{N}$ and $a_{kj} \in \mathbb{C}$ for $k = 0, \ldots, n+1$, $j = 0, \ldots, n$. Show that

$$\sum_{k=0}^{n+1} \sum_{j=0}^{n} a_{kj} = \sum_{0 \le k \le j \le n} a_{kj} + \sum_{0 \le j < k \le n+1} a_{kj}.$$

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