SOLUTIONS SHEET 9

YANNIS BÄHNI

Exercise 1. We may assume that $f \neq 0$ since otherwise we would have convergence in norm. Thus the continuity of f implies $\|f\|_p \neq 0$ for all $1 \leq p < \infty$ since also $|f|^p$ is continuous.

Lemma 1.1. *Let $1 \leq p < \infty$. Then $g_n, h_n, k_n \in L^p(\mathbb{R})$ for all $n \in \mathbb{N}$.*

Proof. This immediately follows from the computations

$$\begin{aligned}
\|g_n\|_p^p &= \int_{\mathbb{R}} |f(x-n)|^p \, dx = \int_{\mathbb{R}} |f(y)| \, dy = \|f\|_p^p \,, \\
\|h_n\|_p^p &= n^{-1} \int_{\mathbb{R}} |f(x/n)|^p \, dx = \int_{\mathbb{R}} |f(y)|^p \, dy = \|f\|_p^p \,, \\
\|k_n\|_p^p &= \int_{\mathbb{R}} |f(x)e^{inx}|^p \, dx = \int_{\mathbb{R}} |f(x)|^p \, dx = \|f\|_p^p \,.
\end{aligned}$$

□

Lemma 1.2. *Let $1 < p < \infty$. Then $g_n, h_n, k_n \rightharpoonup 0$ in $L^p(\mathbb{R})$.*

Proof. We make use of lemma 6.2.1 and theorem 2.2.6, which provides an antilinear isometric isomorphism $\left(L^p(\mathbb{R})\right)^* \cong L^q(\mathbb{R})$ where q is the dual exponent of p. Since $f \in C_c^\infty(\mathbb{R})$, there exists some $M > 0$ such that $\operatorname{supp}(f) \subseteq [-M, M]$. It is easy to verify that $\operatorname{supp}(g_n) \subseteq [-M+n, M+n]$ for all $n \in \mathbb{N}$. Let $\varphi \in L^q(\mathbb{R})$. Then Hölder's inequality implies

$$\begin{aligned}
\left| \int_{\mathbb{R}} \overline{\varphi}(x) g_n(x) dx \right| &\leq \int_{\mathbb{R}} |\varphi(x) g_n(x)| \, dx \\
&= \int_{\mathbb{R}} |\varphi(x) g_n(x) \chi_{\operatorname{supp}(g_n)}(x)| \, dx \\
&\leq \int_{\mathbb{R}} |\varphi(x) g_n(x) \chi_{[-M+n, M+n]}(x)| \, dx \\
&= \|\varphi \chi_{[-M+n,M+n]} g_n\|_1 \\
&\leq \|\varphi \chi_{[-M+n,M+n]}\|_q \|g_n\|_p \\
&= \|\varphi \chi_{[-M+n,M+n]}\|_q \|f\|_p \to 0
\end{aligned}$$

since dominated convergence yields

$$\lim_{n\to\infty} \int_{\mathbb{R}} |\varphi(x)|^q \, \chi_{[-M+n,M+n]}(x) dx = \int_{\mathbb{R}} |\varphi(x)|^q \lim_{n\to\infty} \chi_{[-M+n,M+n]}(x) dx = 0$$

(Yannis Bähni) University of Zurich, Rämistrasse 71, 8006 Zurich
E-mail address: yannis.baehni@uzh.ch.

Yannis Bähni
yannis.baehni@uzh.ch

which application is justified by the fact that $|\varphi(x)|^q\,\chi_{[-M+n,M+n]}(x) \leq |\varphi(x)|^q \in L^1(\mathbb{R})$ and $\chi_{[-M+n,M+n]}(x) \to 0$ for all $x \in \mathbb{R}$ (take n just sufficiently large).
Let $\varepsilon > 0$. Since $1 < q < \infty$, by theorem 2.2.8 we find $\widetilde{\varphi} \in C_c^\infty(\mathbb{R})$ such that $\|\varphi - \widetilde{\varphi}\|_q \leq \varepsilon$. Since $f \in C_c^\infty(\mathbb{R})$, we find $M \geq 0$, such that $|f| \leq M$. Hence

$$\begin{aligned}
\left|\int_{\mathbb{R}} \overline{\varphi}(x) h_n(x) dx\right| &\leq \int_{\mathbb{R}} |\varphi(x)||h_n(x)|\,dx \\
&= \int_{\mathbb{R}} |\varphi(x) - \widetilde{\varphi}(x) + \widetilde{\varphi}(x)||h_n(x)|\,dx \\
&\leq \int_{\mathbb{R}} |\varphi(x) - \widetilde{\varphi}(x)||h_n(x)|\,dx + \int_{\mathbb{R}} |\widetilde{\varphi}(x)||h_n(x)|\,dx \\
&\leq \|\varphi - \widetilde{\varphi}\|_q \|h_n\|_p + \int_{\mathbb{R}} |\widetilde{\varphi}(x)||h_n(x)|\,dx \\
&= \|\varphi - \widetilde{\varphi}\|_q \|f\|_p + \int_{\mathbb{R}} |\widetilde{\varphi}(x)||h_n(x)|\,dx \\
&\leq \varepsilon\|f\|_p + \int_{\mathbb{R}} |\widetilde{\varphi}(x)||h_n(x)|\,dx \\
&= \varepsilon\|f\|_p + n^{-1/p}\int_{\mathbb{R}} |\widetilde{\varphi}(x)||f(x/n)|\,dx \\
&\leq \varepsilon\|f\|_p + n^{-1/p} M \int_{\mathbb{R}} |\widetilde{\varphi}(x)|\,dx \\
&= \varepsilon\|f\|_p + n^{-1/p} M\, \|\widetilde{\varphi}\|_1
\end{aligned}$$

and thus

$$\left|\int_{\mathbb{R}} \overline{\varphi}(x) h_n(x) dx\right| \xrightarrow{n\to\infty} \varepsilon\|f\|_p\,.$$

Since ε was arbitrary, we conclude that

$$\int_{\mathbb{R}} \overline{\varphi}(x) h_n(x) dx \xrightarrow{n\to\infty} 0$$

and since $\varphi \in L^q(\mathbb{R})$ was arbitrary, we conclude that

$$h_n \rightharpoonup 0$$

in $L^p(\mathbb{R})$.
Observe that

$$\begin{aligned}
\int_{\mathbb{R}} \overline{\varphi}(x) k_n(x) dx &= \sqrt{\frac{2\pi}{2\pi}} \int_{\mathbb{R}} \overline{\varphi}(x) f(x) e^{inx} dx \\
&= \sqrt{\frac{2\pi}{2\pi}} \int_{\mathbb{R}} \overline{\varphi}(x) f(x) e^{inx} dx \\
&= \sqrt{\frac{2\pi}{2\pi}} \int_{\mathbb{R}} \overline{\varphi}(x) f(x) e^{-i(-n)x} dx
\end{aligned}$$

Yannis Bähni
yannis.baehni@uzh.ch

$$= \sqrt{2\pi}\widehat{\overline{\varphi} f}(-n) \to 0$$

by the Riemann-Lebesgue lemma since by Hölders inequality, $\overline{\varphi} f \in L^1(\mathbb{R})$. □

Lemma 1.3. *Let X be a normed space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X such that $x_n \rightharpoonup x$. If $x_n \to y$ for some $y \in X$, then $x = y$.*

Proof. Suppose that $x_n \to y$. Then since $\mathcal{T}_W \subseteq \mathcal{T}_{\|\cdot\|}$, we have that $x_n \rightharpoonup y$. But $(X, \mathcal{T}_W)$ is Hausdorff and thus limits are unique. Hence $x = y$. □

Corollary 1.1. *Let $1 < p < \infty$. Then g_n, h_n and k_n do not converge in norm.*

Proof. Since all three sequences converge weakly to 0, we only have to show that they do not converge towards 0 in $L^p(\mathbb{R})$. However, this is immediate from the first lemma, since all sequences have constant norm $\|f\|_p \neq 0$ and hence the limit should have also nonzero norm. □

Exercise 2.

a. Fix $1 < p < \infty$ and assume that $x^{(n)} \rightharpoonup x$ in $\ell^p(\mathbb{K})$. By lemma 6.2.1 this implies that $f\left(x^{(n)}\right) \to f(x)$ for all $f \in \left(\ell^p(\mathbb{K})\right)^*$. By [Wer11, p. 59], we have that $\left(\ell^p(\mathbb{K})\right)^* \cong \ell^q(\mathbb{K})$ isometrically where q is the dual exponent to p. Explicitly, $\varphi(y) \in \left(\ell^p(\mathbb{K})\right)^*$ is given by

$$\varphi(y)(x) \mapsto \sum_{k\in\mathbb{N}} y_k x_k$$

for $y \in \ell^q(\mathbb{K})$. Let $i \in \mathbb{N}$. Then $(\delta_{ik})_{k\in\mathbb{N}} \in \ell^q(\mathbb{K})$ and thus we have that

$$x_i^{(n)} = \sum_{k\in\mathbb{N}} \delta_{ik} x_k^{(n)} = \varphi\left((\delta_{ik})_{k\in\mathbb{N}}\right)\left(x^{(n)}\right) \to \varphi\left((\delta_{ik})_{k\in\mathbb{N}}\right)(x) = \sum_{k\in\mathbb{N}} \delta_{ik} x_k = x_i.$$

Moreover, $x^{(n)}$ is bounded by proposition 6.2.2. Conversly, consider the following lemma.

Lemma 1.4. *Let $1 \leq p < \infty$. Define*

$$A_p := \left\{x \in \ell^p(\mathbb{K}) : \operatorname{supp} x \text{ is finite}\right\}.$$

Then A_p is dense in $\ell^p(\mathbb{K})$.

Proof. Let $x \in \ell^p(\mathbb{K})$. Consider $\left(x^{(n)}\right)_{n\in\mathbb{N}}$ defined by

$$x_k^{(n)} := \begin{cases} x_k & 1 \leq k \leq n, \\ 0 & k > n \end{cases}.$$

By

$$\|x^{(n)}\|_p^p = \sum_{k\in\mathbb{N}} |x_k^{(n)}|^p = \sum_{k\leq n} |x_k^{(n)}|^p < \infty$$

immediately follows that $x^{(n)} \in \ell^p(\mathbb{K})$ for all $n \in \mathbb{N}$ and hence $x^{(n)} \in A_p$. Moreover

$$\|x - x^{(n)}\|_p^p = \sum_{k\in\mathbb{N}} |x_k - x_k^{(n)}|^p = \sum_{k\geq n} |x_k|^p \to 0$$

implies that A_p is dense in $\ell^p(\mathbb{K})$. □

Yannis Bähni
yannis.baehni@uzh.ch

We make use of exercise 5 on sheet 8. Since $\left(x^{(n)}\right)_{n\in\mathbb{N}}$ is bounded, we have $\sup_{n\in\mathbb{N}}\|x^{(n)}\|_p < \infty$. By the previous lemma, A_q is dense in $\ell^q(\mathbb{K})$ and thus also in $\left(\ell^p(\mathbb{K})\right)^*$ via the explicit isometric isomorphism (to be precise, the image of A_q under this mapping). Let $y \in A_q$. Then we find $N \in \mathbb{N}$, such that $y_k = 0$ for all $k > N$. Hence

$$\varphi(y)\left(x^{(n)}\right) = \sum_{k\in\mathbb{N}} y_k x_k^{(n)} = \sum_{k\leq N} y_k x_k^{(n)} \to \sum_{k\leq N} y_k x_k = \sum_{k\in\mathbb{N}} y_k x_k = \varphi(y)(x)$$

since by assumption $x_i^{(n)} \to x_i$ for all $i \in \mathbb{N}$ and the sum is finite. Thus by exercise 5 on sheet 8 we conclude that $x^{(n)} \rightharpoonup x$ in $\ell^p(\mathbb{K})$.

b. Suppose that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$. By lemma 6.2.1. we have that $f(x_n) \to f(x)$ for all $f \in H^*$. Using the *Riesz representation theorem* this is equivalent to $\langle y, x_n\rangle \to \langle y, x\rangle$ for all $y \in H$. But then

$$\|x - x_n\|^2 = \langle x - x_n, x - x_n\rangle = \|x\|^2 - 2\operatorname{Re}\langle x, x_n\rangle + \|x_n\|^2 \to 0$$

since Re is a continuous function and $\langle x, x_n\rangle \to \|x\|^2$.

Exercise 3.

a.

Lemma 1.5. *Let* $0 < \varepsilon < 1$ *and define*

$$I_\varepsilon(f) := \varepsilon^{-1}\int_0^\varepsilon f(x)dx$$

for $f \in L^\infty(0,1)$. *Then* $I_\varepsilon \in \left(L^\infty(0,1)\right)^*$ *and* $\|I_\varepsilon\| = 1$ *for all* $0 < \varepsilon < 1$.

Proof. Let $f \in L^\infty(0,1)$. Then we have that $|f| \leq \|f\|_\infty$ λ-a.e. Hence

$$|I_\varepsilon(f)| = \varepsilon^{-1}\left|\int_0^\varepsilon f(x)dx\right| \leq \varepsilon^{-1}\int_0^\varepsilon |f(x)|\,dx \leq \|f\|_\infty \tag{1}$$

and thus I_ε is bounded and thus continuous. Clearly I_ε is $\mathbb{C}$-linear by the $\mathbb{C}$-linearity of the integral. Moreover, using (1) we get that

$$\|I_\varepsilon\| = \sup_{\|f\|_\infty = 1} |I_\varepsilon(f)| \leq \sup_{\|f\|=1} \|f\|_\infty = 1.$$

Conversly, setting $f := \chi_{(0,1)} \in L^\infty(0,1)$, we get that $|I_\varepsilon(f)| = 1$ and hence by $\|f\| = 1$

$$\|I_\varepsilon\| = \sup_{\|g\|_\infty = 1} |I_\varepsilon(g)| \geq |I_\varepsilon(f)| = 1.$$

□

Exercise 4.

a. First we show that $\|\cdot\|_\sigma$ is well defined. Let $x^* \in X^*$, then we have

$$\|x^*\|_\sigma = \sum_{k=1}^\infty 2^{-k}\,|x^*(x_k)| \leq \sum_{k=1}^\infty 2^{-k}\,\|x^*\|\|x_k\| = \|x^*\|\sum_{k=1}^\infty 2^{-k} = \|x^*\| < \infty$$

Yannis Bähni
yannis.baehni@uzh.ch

since $x_k \in S_X$ and $\sum_{k=1}^{\infty} 2^{-k} = 1$. Hence $\|x^*\|_\sigma \leq \|x^*\|$ holds. Let $\lambda \in \mathbb{K}$. Then we have that

$$\|\lambda x^*\|_\sigma = \sum_{k=1}^{\infty} 2^{-k} |\lambda x^*(x_k)| = \sum_{k=1}^{\infty} 2^{-k} |\lambda| |x^*(x_k)| = |\lambda| \sum_{k=1}^{\infty} 2^{-k} |x^*(x_k)| = |\lambda| \|x^*\|_\sigma .$$

Let $y^* \in X^*$. Then the triangle inequality follows from

$$\begin{aligned} \|x^* + y^*\|_\sigma &= \sum_{k=1}^{\infty} 2^{-k} |x^*(x_k) + y^*(x_k)| \\ &\leq \sum_{k=1}^{\infty} 2^{-k} |x^*(x_k)| + \sum_{k=1}^{\infty} 2^{-k} |y^*(x_k)| \\ &= \|x^*\|_\sigma + \|y^*\|_\sigma . \end{aligned}$$

Lastly, clearly $\|x^*\| = 0$ if $x^* = 0$. Conversely, suppose that $\|x^*\| = 0$. Hence $x^*(x_k) = 0$ for all $k \in \mathbb{N}$. Let $Y \in X$. Since $\overline{\operatorname{span}\{x_k : k \in \mathbb{N}\}} = X$, we find a sequence $(y_n)_{n\in\mathbb{N}}$ in $\operatorname{span}\{x_k : k \in \mathbb{N}\}$, such that $y_n \to y$. Moreover, for each $n \in \mathbb{N}$ we have that $y_n = \sum_{k=1}^{\infty} \lambda_k^{(n)} x_k$ for $\lambda_k^{(n)} \in \mathbb{K}$ and $\lambda_k^{(n)} = 0$ for all but finitely many $k \in \mathbb{N}$. Hence

$$x^*(y) = \lim_{n\to\infty} x^*(y_n) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \lambda_k^{(n)} x^*(x_k) = 0$$

by the continuity of x^* and so $x^* = 0$.

References

[Wer11] Dirk Werner. *Funktionalanalysis*. 7., korrigierte und erweiterte Auflage. Springer, 2011.