

SOLUTIONS SHEET 8

YANNIS BÄHNI

Exercise 1.

Exercise 2.

Exercise 3.

Exercise 4.

a.

b. Define

$$\mathcal{B} := \{U_1 \cap \cdots \cap U_n : n \in \mathbb{N}, U_i \in \mathcal{S} \text{ for all } i = 1, \dots, n\}$$

and

$$\mathcal{T} := \{\cup_{\alpha \in A} B_\alpha : B_\alpha \in \mathcal{B} \text{ for all } \alpha \in A\}.$$

Lemma 1.1. $\mathcal{T}_{\mathcal{F}} = \mathcal{T}$.

Proof. By part **a.**, $\mathcal{T}_{\mathcal{F}}$ is a topology. We show that also \mathcal{T} is a topology. By [Lee11, p. 34] it is enough to show that \mathcal{B} satisfies the following two conditions:

(i) $\cup_{B \in \mathcal{B}} B = X$.

(ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists an element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Then \mathcal{T} is the unique topology on X generated by \mathcal{B} , i.e. the collection of arbitrary unions of elements of \mathcal{B} . Since \mathcal{F} is nonempty, there exists $f \in \mathcal{F}$. Clearly $X = f^{-1}(Y_f)$ and Y_f is open in Y_f . Hence $f^{-1}(Y_f) \in \mathcal{S}$ and thus $X \in \cup_{B \in \mathcal{B}} B$. Suppose that $B_1, B_2 \in \mathcal{B}$ such that $B_1 \cap B_2 \neq \emptyset$. Hence we find $U_1, \dots, U_n, V_1, \dots, V_m \in \mathcal{S}$ such that $B_1 = U_1 \cap \cdots \cap U_n$ and $B_2 = V_1 \cap \cdots \cap V_m$. Suppose $x \in B_1 \cap B_2$. Then also $x \in U_1 \cap \cdots \cap U_n \cap V_1 \cap \cdots \cap V_m$. But

$$U_1 \cap \cdots \cap U_n \cap V_1 \cap \cdots \cap V_m \in \mathcal{B}$$

as a finite intersection of elements of \mathcal{S} . Hence \mathcal{T} is a topology.

Clearly, $\mathcal{S} \subseteq \mathcal{T}$, since already $\mathcal{S} \subseteq \mathcal{B}$. Since $\mathcal{T}_{\mathcal{F}}$ is the smallest topology containing \mathcal{S} , we get that $\mathcal{T}_{\mathcal{F}} \subseteq \mathcal{T}$.

Let $U \in \mathcal{T}$. Then $U = \cup_{\alpha \in A} B_\alpha$ for some index set A and $B_\alpha \in \mathcal{B}$ for all $\alpha \in A$. But each B_α is a finite intersection of elements of \mathcal{S} and thus since $\mathcal{T}_{\mathcal{F}}$ is a topology containing

\mathcal{S} , we have that $B_\alpha \in \mathcal{T}_{\mathcal{F}}$ for all $\alpha \in A$. But then also $U \in \mathcal{T}_{\mathcal{F}}$ as a union of sets in $\mathcal{T}_{\mathcal{F}}$. Hence $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{F}}$. \square

Exercise 5.

References

[Lee11] John M. Lee. *Introduction to Topological Manifolds*. Second Edition. Springer Science+Business Media, 2011.