ADDITIVE AND ABELIAN CATEGORIES

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Abstract. We define preadditive, additive and abelian categories, where the latter is the natural generalization of AbGrp to study the basic results of homological algebra in. Moreover, we state and comment on two foundational results in the theory of abelian categories, namely the *Mitchell Embedding Theorem* and the *Eilenberg-Watts Theorem*, which roughly speaking, connect the category of modules with abelian categories.

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1. Introduction

Consider the following diagram in AbGrp:

$$\begin{tikzcd}
& 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d, "\varphi"] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \\
0 & 0 & & 0.
\end{tikzcd}$$

Those who are familiar with algebraic topology recognise it as the setting of the *five lemma*. This basic result in *homological algebra* simply says that φ is an isomorphism.

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The basic proof technique used to establish this result is called *diagram chasing*. It turns out, that abelian categories are the right generalization for this type of proof (for more details, see [Lan78, pp. 202–208]).

2. Preadditive Catgeories

Let $G, H \in \mathrm{ob}(\mathsf{AbGrp})$ and $\varphi, \psi \in \mathsf{AbGrp}(G, H)$. Define $\varphi + \psi$ pointwise. Since H is abelian, it follows that $\varphi + \psi \in \mathsf{AbGrp}(G, H)$. Moreover, it is easy to check, that with this operation defined above, $\mathsf{AbGrp}(G, H)$ is an abelian group and

$$\circ : \mathsf{AbGrp}(H, K) \times \mathsf{AbGrp}(G, H) \to \mathsf{AbGrp}(G, K)$$

is bilinear for each $K \in \mathrm{ob}(\mathsf{AbGrp})$. This motivates the following definition.

Definition 2.1 (Preadditive Category). *A* ***preadditive category*** *is a locally small category $\mathscr{C}$ in which all hom-sets $\mathscr{C}(X, Y)$ can be equipped with the structure of an abelian group and composition is bilinear, i.e. for all mophisms $f, f' : X \to Y$ and $g, g' : Y \to Z$ in $\mathscr{C}$ we have that*

$$(g + g') \circ (f + f') = g \circ f + g \circ f' + g' \circ f + g' \circ f'. \tag{1}$$

3. Additive Categories

Let us again consider AbGrp. As in Grp, the trivial group 0 is both an initial and a terminal object. Unlike in Grp, we have that $G \coprod H \cong G \prod H$ for all $G, H \in \mathrm{ob}(\mathsf{AbGrp})$. In a somewhat weaker sense, we will generalize this. Define $\iota_1 : G \to G \prod H$ and $\iota_2 : H \to G \prod H$ by

$$\iota_1(g) := (g, 0) \qquad \text{and} \qquad \iota_2(h) := (0, h),$$

respectively. Then it is easy to verify that

$$\pi_1 \circ \iota_1 = \mathrm{id}_G, \quad \pi_2 \circ \iota_2 = \mathrm{id}_H \quad \text{and} \quad \iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = \mathrm{id}_{G \prod H}$$

holds. Those observations motivate the following definitions.

Definition 3.1 (Null Object). *Let $\mathscr{C}$ be a category. A* ***null object in $\mathscr{C}$*** *is a an object of $\mathscr{C}$ which is both initial and terminal.*

Definition 3.2 (Biproduct Diagram). *Let $\mathscr{C}$ be a preadditive category and $X, Y \in \mathrm{ob}(\mathscr{C})$. A* ***biproduct diagram for X and Y*** *is a diagram*

$$\begin{tikzcd} X \arrow[r, shift right, "\iota_1"'] & Z \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & Y \arrow[l, shift left, "\iota_2"] \end{tikzcd}$$

such that

$$\pi_1 \circ \iota_1 = \mathrm{id}_X, \quad \pi_2 \circ \iota_2 = \mathrm{id}_Y \quad \textit{and} \quad \iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = \mathrm{id}_Z$$

holds.

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Definition 3.3 (Additive Category). *An **additive category** is a preadditive category which has a null object and a biproduct for each pair of its objects.*

4. Abelian Categories

Definition 4.1 (Zero Arrow). *Let $\mathscr{C}$ be a category with a null object 0. For $X, Y \in \mathrm{ob}(\mathscr{C})$, the unique composition $X \to 0 \to Y$ is called the **zero arrow from X to Y**, denoted by $0_{X,Y}$.*

Definition 4.2 (Kernel and Cokernel). *Let $\mathscr{C}$ be a category with a null object 0. A **kernel of a morphism $f : X \to Y$** is defined to be an equalizer of*

$$X \underset{0_{X,Y}}{\overset{f}{\rightrightarrows}} Y.$$

*Dually, a **cokernel of a morphism $f : X \to Y$** is a coequalizer of the above diagram.*

Lemma 4.1. *In* AbGrp, *every monic is a kernel and every epic is a cokernel.*

Proof. Let $m : G \to H$ be a monic in AbGrp. Consider the fork

$$G \xrightarrow{m} H \underset{0}{\overset{\pi}{\rightrightarrows}} H/m(G).$$

Then one can check that this is in fact a universal fork. Similarly, one can check that

$$\ker e \underset{0}{\overset{\iota}{\rightrightarrows}} G \xrightarrow{e} H$$

is a universal cofork for any epic $e : G \to H$. □

Definition 4.3 (Abelian Category). *An **abelian category** is an additive category satisfying the following additional conditions:*

(*a*) *Every morphism admits a kernel.*
(*b*) *Every morphism admits a cokernel.*
(*c*) *Every monic is a kernel.*
(*d*) *Every epic is a cokernel.*

Examples 4.1. AbGrp, Vect_K, ${}_R\mathsf{Mod}$ and Mod_R.

5. The Mitchell Embedding Theorem and its Consequences

Definition 5.1 (Exact Functor). *A functor $F : \mathscr{C} \to \mathscr{D}$ between two abelian categories $\mathscr{C}$ and $\mathscr{D}$ is called **exact**, if F preserves all finite limits and all finite colimits.*

Theorem 5.1 (Mitchell Embedding). *For every small abelian category there is an exact, full and faithful functor into ${}_R\mathsf{Mod}$ for some ring R.*

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A usefull application of the Mitchell embedding is that one can do proofs of basic homological algebra results in a familiar environment like ${}_R\mathsf{Mod}$ by diagram chasing. However, as [Lan78, pp. 202–208] shows, this can also be done without using the Mitchell embedding.

6. The Eilenberg-Watts Theorem and its Consequences