SOLUTIONS SHEET 1

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Exercise 1.

(a) Clearly, this holds for the discrete topology, i.e. $\mathcal{T}_a = 2^X$. Let $\mathcal{T}_a'$ be another topology on X satisfying the property. Consider the fucntions $f : (X, 2^X) \to (X, \mathcal{T}_a')$ and $g : (X, \mathcal{T}_a') \to (X, 2^X)$ defined by $f(x) := x$ and $g(x) := x$, respectively. Since both topologies on X have the property, we get that f and g are continuous. Moreover
$$g \circ f = \mathrm{id}_{(X,2^X)} \qquad \text{and} \qquad f \circ g = \mathrm{id}_{(X,\mathcal{T}_a')}.$$
Thus $(X, 2^X) \approx (X, \mathcal{T}_a')$.

Exercise 2.

Exercise 3.

Exercise 4. Let $g, \tilde{g} : Y \to X$ be inverses of f. Then we have
$$g = g \circ \mathrm{id}_Y = g \circ (f \circ \tilde{g}) = (g \circ f) \circ \tilde{g} = \mathrm{id}_X \circ \tilde{g} = \tilde{g}.$$
Thus we can unambiguously write $f^{-1} := g$.

Exercise 5. We have that
$$\begin{aligned}\big((g \circ f)^{-1} \circ g\big) \circ f &= (g \circ f)^{-1} \circ (g \circ f) = \mathrm{id}_X \\ \big((h \circ g)^{-1} \circ h\big) \circ g &= (h \circ g)^{-1} \circ (h \circ g) = \mathrm{id}_Y \\ h \circ \big(g \circ (h \circ g)^{-1}\big) &= (h \circ g) \circ (h \circ g)^{-1} = \mathrm{id}_W\end{aligned}$$

Exercise 6. Assume $f : X \to Y$ has the left cancellation property. Let $x, y \in X$ such that $f(x) = f(y)$. Now let $Z := \{x, y\}$. Define two functions $c_x, c_y : Z \to X$ by $c_x(z) := x$ and $c_y(z) := y$, respectively. Now
$$f \circ c_x = f(x) = f(y) = f \circ c_y$$
holds by assumption. Thus the left cancellation property implies that $c_x = c_y$, hence $x = y$ and f is injective. Conversly, assume that f is injective. Let $\alpha, \beta : Z \to X$ such that $f \circ \alpha = f \circ \beta$ and $z \in Z$. Then we have that $f(\alpha(z)) = f(\beta(z))$ and thus by injectivity, $\alpha(z) = \beta(z)$.

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