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SOLUTIONS SHEET 8

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Exercise 1.

Exercise 2. We will show that there exists a unique solution to the integral equation $f \in L^2(X)$.

Lemma 1.1. *Let $L \in L^2(X \times X)$ and $f \in L^2(X)$. For $x \in X$ define*

$$g_f(x) := \int_X L(x, y) f(y) dy.$$

Then $g_f \in L^2(X)$.

Proof. We have that

$$\begin{aligned}
\|g_f\|^2_{L^2(X)} &= \int_X |g_f(x)|^2 \, dx \\
&= \int_X \left| \int_X L(x, y) f(y) dy \right|^2 dx \\
&\leq \int_X \left(\int_X |L(x, y) f(y)| \, dy \right)^2 dx \\
&= \int_X \|L(x, \cdot) f\|^2_{L^1(X)} \, dx \\
&\leq \int_X \|L(x, \cdot)\|^2_{L^2(X)} \|f\|^2_{L^2(X)} \, dx \\
&= \|f\|^2_{L^2(X)} \int_X \int_X |L(x, y)|^2 \, dy dx \\
&= \|f\|^2_{L^2(X)} \|L\|^2_{L^2(X \times X)}
\end{aligned}$$

by Hölder and Fubini. □

Now we have to solve the equation

$$cf + u = g_f.$$

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Define $a : L^2(X) \times L^2(X) \to \mathbb{C}$ by

$$a(f, \varphi) := \langle cf - g_f, \varphi \rangle_{L^2(X)}.$$

If a satisfies the assumptions of *Lax-Milgram*, we find $A \in \mathscr{L}\left(L^2(X)\right)$, such that

$$a(f, \varphi) = \langle A(f), \varphi \rangle_{L^2(X)}$$

holds for all $f, \varphi \in L^2(X)$. Hence

$$\langle cf - g_f, \varphi \rangle_{L^2(X)} = \langle A(f), \varphi \rangle_{L^2(X)}$$

holds for all $f, \varphi \in L^2(X)$. Moreover, since A is invertible, we find a unique $f_0 \in L^2(X)$, such that $A(f_0) = -u$. Therefore

$$R_{L^2(X)}(cf_0 - g_{f_0})(\varphi) = \langle cf_0 - g_{f_0}, \varphi \rangle_{L^2(X)} = \langle -u, \varphi \rangle_{L^2(X)} = R_{L^2(X)}(-u)(\varphi)$$

holds for all $\varphi \in L^2(X)$. Thus $R_{L^2(X)}(cf_0 - g_{f_0}) = R_{L^2(X)}(-u)$ and so the *Riesz representation theorem* yields that

$$cf_0 - g_{f_0} = -u.$$

Lemma 1.2. *$a : L^2(X) \times L^2(X) \to \mathbb{C}$ is a continuous coercive sesquilinear form.*

Proof. Clearly a is sesquilinear, i.e. antilinear in the first and linear in the second argument, by the corresponding properties of the $L^2(X)$ inner product and the simple observation that

$$g_{f+\lambda h} = g_f + \lambda g_h$$

for $\lambda \in \mathbb{C}$ and $f, h \in L^2(X)$. Let $\varphi \in L^2(X)$. Then Cauchy-Schwarz together with lemma 1.1 yields

$$\begin{aligned}
|a(f, \varphi)| &= |\langle cf - g_f, \varphi \rangle_{L^2(X)}| \\
&\leq \|cf - g_f\|_{L^2(X)} \|\varphi\|_{L^2(X)} \\
&\leq \left(c \|f\|_{L^2(X)} + \|g_f\|_{L^2(X)}\right) \|\varphi\|_{L^2(X)} \\
&\leq \left(c\|f\|_{L^2(X)} + \|f\|_{L^2(X)} \|L\|_{L^2(X \times X)}\right) \|\varphi\|_{L^2(X)} \\
&\leq 2c\|f\|_{L^2(X)} \|\varphi\|_{L^2(X)}
\end{aligned}$$

since $\|L\|_{L^2(X \times X)} < c$. Lastly, since $\|L\|_{L^2(X \times X)} < c$, we find c_0 such that we have $\|L\|_{L^2(X \times X)} < c_0 < c$ by trichotomy. Thus again lemma 1.1 together with Cauchy-Schwarz implies that

$$\begin{aligned}
\operatorname{Re} a(f, f) &= \operatorname{Re}\langle cf - g_f, f \rangle_{L^2(X)} \\
&= \operatorname{Re} c\langle f, f \rangle_{L^2(X)} - \operatorname{Re}\langle g_f, f \rangle_{L^2(X)} \\
&= c\|f\|^2_{L^2(X)} - \operatorname{Re}\langle g_f, f \rangle_{L^2(X)} \\
&\geq c\|f\|^2_{L^2(X)} - |\langle g_f, f \rangle_{L^2(X)}|
\end{aligned}$$

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$$\begin{aligned}
&\geq c\|f\|_{L^2(X)}^2 - \|g_f\|_{L^2(X)}\|f\|_{L^2(X)} \\
&\geq c\|f\|_{L^2(X)}^2 - \|f\|_{L^2(X)}^2\|L\|_{L^2(X\times X)} \\
&\geq (c - c_0)\|f\|_{L^2(X)}^2 .
\end{aligned}$$

Now $2c > 0$ since $c > \|L\|_{L^2(X\times X)}$, $c - c_0 > 0$ and $c - c_0 \leq c \leq 2c$ since $c_0 > 0$. □

Exercise 3.

a. Suppose that $M \setminus \overline{A}$ is dense in M. Towards a contradiction, assume that A is not nowhere dense. Hence $\overset{\circ}{\overline{A}} \neq \varnothing$. Since $\overset{\circ}{\overline{A}}$ is open by definition of the interior of a set, there exists $\varepsilon > 0$ and $x \in \overset{\circ}{\overline{A}}$ such that $B_\varepsilon(x) \subseteq \overset{\circ}{\overline{A}}$. Moreover, $\overset{\circ}{\overline{A}} \subseteq \overline{A}$ and thus $B_\varepsilon(x) \subseteq \overline{A}$. This implies that $B_\varepsilon(x)$ and $M \setminus \overline{A}$ are disjoint. But $M \setminus \overline{A}$ is dense in M, hence we find a sequence $(x_n)_{n\in\mathbb{N}}$ in $M \setminus \overline{A}$ such that $x_n \to x$. Hence there exists $N \in \mathbb{N}$ such that $x_n \in B_\varepsilon(x)$ for all $n \geq N$. This is not possible since $B_\varepsilon(x)$ does not contain any elements of $M \setminus \overline{A}$. Contradiction.

b.

Lemma 1.3. *Let $(x_k)_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}$. For $n \in \mathbb{N}$ define*

$$E_n := \bigcup_{k\in\mathbb{N}} \left(x_k - \frac{1}{2^k n}, x_k + \frac{1}{2^k n} \right) \tag{1}$$

and

$$E := \bigcap_{n\in\mathbb{N}} E_n. \tag{2}$$

Set $A := E^c$. Then A is meager and $\lambda(A^c) = 0$, where λ denotes the ordinary Lebesgue measure on $\mathbb{R}$.

Proof. We show first that $\lambda(A^c) = 0$. First observe that $(E_n)_{n\in\mathbb{N}}$ is a decreasing sequence of λ-measurable sets. Moreover, for any $n \in \mathbb{N}$ we have that

$$\lambda(E_n) \leq \frac{1}{n}\sum_{k\in\mathbb{N}} \frac{1}{2^{k-1}} = \frac{1}{n}\sum_{k\in\mathbb{N}_0}\frac{1}{2^k} = \frac{2}{n} < \infty$$

by subadditivity of the measure. Elementary measure theory now tells us that

$$\lambda(A^c) = \lambda(E) = \lambda\left(\cap_{n\in\mathbb{N}} E_n\right) = \lim_{n\to\infty}\lambda(E_n) \leq \lim_{n\to\infty}\frac{2}{n} = 0.$$

Let us show that A is meager. Since $A = E^c = \cup_{n\in\mathbb{N}} E_n^c$, we show that E_n^c is nowhere dense for all $n \in \mathbb{N}$. By part **a.** we can also show that $\mathbb{R} \setminus \overline{E_n^c}$ is dense in $\mathbb{R}$. For fixed $n \in \mathbb{N}$ we have that

$$E_n^c = \bigcap_{k\in\mathbb{N}} \left(\left(-\infty, x_k - \frac{1}{2^k n}\right] \cup \left[x_k + \frac{1}{2^k n}, \infty\right) \right).$$

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Thus E_n^c is a closed set (finite unions and countable intersection of closed intervals) and so $\bar{E}_n^c = E_n^c$. So $\mathbb{R} \setminus \bar{E}_n^c = E_n$. But $\mathbb{Q} \subseteq E_n$ for all $n \in \mathbb{N}$ and thus E_n is dense in $\mathbb{R}$. $\square$

Exercise 4.

a. If $A = \emptyset$, we have that $\cap_{\alpha \in A} \mathcal{T}_\alpha = \mathcal{P}(X)$ since topologies on X are subsets of $\mathcal{P}(X)$. Hence the intersection of the empty family of topologies on X is the discrete topology. Consider now $A \neq \emptyset$. Clearly, $\emptyset, X \in \cap_{\alpha \in A} \mathcal{T}_\alpha$ since $\emptyset, X \in \mathcal{T}_\alpha$ for all $\alpha \in A$. Let $U_1, \ldots, U_n \in \cap_{\alpha \in A} \mathcal{T}_\alpha$. Hence $U_1, \ldots, U_n \in \mathcal{T}_\alpha$ for all $\alpha \in A$ and so $U_1 \cap \cdots \cap U_n \in \mathcal{T}_\alpha$ for all $\alpha \in A$. Hence $U_1 \cap \cdots \cap U_n \in \cap_{\alpha \in A} \mathcal{T}_\alpha$. Finally, suppose that $(U_\beta)_{\beta \in B}$ is a family in $\cap_{\alpha \in A} \mathcal{T}_\alpha$. Hence for all $\alpha \in A$ we have that $U_\beta \in \mathcal{T}_\alpha$ for all $\beta \in B$. So $\cup_{\beta \in B} U_\beta \in \mathcal{T}_\alpha$ for all $\alpha \in A$ and therefore $\cup_{\beta \in B} U_\beta \in \cap_{\alpha \in A} \mathcal{T}_\alpha$.

b. Define

$$\mathcal{B} := \{U_1 \cap \cdots \cap U_n : n \in \mathbb{N}, U_i \in \mathcal{S} \text{ for all } i = 1, \ldots, n\}$$

and

$$\mathcal{T} := \{\cup_{\alpha \in A} B_\alpha : B_\alpha \in A \text{ for all } \alpha \in A\}.$$

Lemma 1.4. $\mathcal{T}_\mathcal{F} = \mathcal{T}$.

Proof. By part **a.**, $\mathcal{T}_\mathcal{F}$ is a topology. We show that also $\mathcal{T}$ is a topology. By [Lee11, p. 34] it is enough to show that $\mathcal{B}$ satisfies the following two conditions:

(i) $\cup_{B \in \mathcal{B}} B = X$.

(ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists an element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Then $\mathcal{T}$ is the unique topology on X generated by $\mathcal{B}$, i.e. the collection of arbitrary unions of elements of $\mathcal{B}$. Since $\mathcal{F}$ is nonempty, there exists $f \in \mathcal{F}$. Clearly $X = f^{-1}(Y_f)$ and Y_f is open in Y_f. Hence $f^{-1}(Y_f) \in \mathcal{S}$ and thus $X \in \cup_{B \in \mathcal{B}} B$. Suppose that $B_1, B_2 \in \mathcal{B}$ such that $B_1 \cap B_2 \neq \emptyset$. Hence we find $U_1, \ldots, U_n, V_1, \ldots, V_m \in \mathcal{S}$ such that $B_1 = U_1 \cap \cdots \cap U_n$ and $B_2 = V_1 \cap \cdots \cap V_m$. Suppose $x \in B_1 \cap B_2$. Then also $x \in U_1 \cap \cdots \cap U_n \cap V_1 \cap \cdots \cap V_m$. But

$$U_1 \cap \cdots \cap U_n \cap V_1 \cap \cdots \cap V_m \in \mathcal{B}$$

as a finite intersection of elements of $\mathcal{S}$. Hence $\mathcal{T}$ is a topology.

Clearly, $\mathcal{S} \subseteq \mathcal{T}$, since already $\mathcal{S} \subseteq \mathcal{B}$. Since $\mathcal{T}_\mathcal{F}$ is the smallest topology containing $\mathcal{S}$, we get that $\mathcal{T}_\mathcal{F} \subseteq \mathcal{T}$.

Let $U \in \mathcal{T}$. Then $U = \cup_{\alpha \in A} B_\alpha$ for some index set A and $B_\alpha \in \mathcal{B}$ for all $\alpha \in A$. But each B_α is a finite intersection of elements of $\mathcal{S}$ and thus since $\mathcal{T}_\mathcal{F}$ is a topology containing $\mathcal{S}$, we have that $B_\alpha \in \mathcal{T}_\mathcal{F}$ for all $\alpha \in A$. But then also $U \in \mathcal{T}_\mathcal{F}$ as a union of sets in $\mathcal{T}_\mathcal{F}$. Hence $\mathcal{T} \subseteq \mathcal{T}_\mathcal{F}$. $\square$

Exercise 5. Suppose that $x_n \rightharpoonup x$. Proposition 6.2.2 implies that the sequence $(x_n)_{n \in \mathbb{N}}$ is bounded, in particular $\sup_{n \in \mathbb{N}} \|x_n\| < \infty$. Moreover, lemma 6.2.1 yields $f(x_n) \to f(x)$ for all $f \in X^*$. Since $Y \subseteq X^*$ we also have $f(x_n) \to f(x)$ for all $f \in Y$.

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Conversely, suppose $\|x_n\| \leq M$ for some $M \geq 0$ and $f(x_n) \to f(x)$ for all $f \in Y$. Let $f \in X^*$. Since Y is dense in X^*, we find a sequence $(f_k)_{k\in\mathbb{N}}$ in Y such that $\|f_k - f\| \to 0$. Hence

$$\begin{aligned}|f(x_n) - f(x)| &= |f(x_n) - f(x) + f_k(x_n) - f_k(x_n) + f_k(x) - f_k(x)| \\ &\leq |f(x_n) - f_k(x_n)| + |f_k(x_n) - f_k(x)| + |f_k(x) - f(x)| \\ &\leq \|f - f_k\| \left(\|x_n\| + \|x\|\right) + |f_k(x_n) - f_k(x)| \\ &\leq \|f - f_k\| \left(M + \|x\|\right) + |f_k(x_n) - f_k(x)|\end{aligned}$$

and so

$$\lim_{n\to\infty} |f(x_n) - f(x)| \leq \|f - f_k\| \left(M + \|x\|\right) \xrightarrow{k\to\infty} 0.$$

Thus $f(x_n) \to f(x)$ for all $f \in X^*$ and so lemma 6.2.1 implies $x_n \rightharpoonup x$.

References

[Lee11] John M. Lee. *Introduction to Topological Manifolds*. Second Edition. Springer Science+Business Media, 2011.