# Contents

CHAPTER 1

# Foundations

## Basic Category Theory

**Categories.** We use the first order theory of Neumann-Bernays-Gödel (BNG) as described in [Men15, p. 231].

**Definition 1.1 (Category).** *A **category** $\mathcal{C}$ consists of*

- *A class* $\mathrm{ob}(\mathcal{C})$*, called the **objects of** $\mathcal{C}$.*
- *A class* $\mathrm{mor}(\mathcal{C})$*, called the **morphisms of** $\mathcal{C}$.*
- *Two functions* $\mathrm{dom} : \mathrm{mor}(\mathcal{C}) \to \mathrm{ob}(\mathcal{C})$ *and* $\mathrm{cod} : \mathrm{mor}(\mathcal{C}) \to \mathrm{ob}(\mathcal{C})$*, which assign to each morphism $f$ in $\mathcal{C}$ its **domain** and **codomain**, respectively.*
- *For each $X \in \mathrm{ob}(\mathcal{C})$ a function* $\mathrm{ob}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C})$ *which assigns a morphism* $\mathrm{id}_X$ *such that* $\mathrm{dom}\,\mathrm{id}_X = \mathrm{cod}\,\mathrm{id}_X = X$.
- *A function*
$$\circ : \{(g, f) \in \mathrm{mor}(\mathcal{C}) \times \mathrm{mor}(\mathcal{C}) : \mathrm{dom}\, g = \mathrm{cod}\, f\} \to \mathrm{mor}(\mathcal{C}) \tag{1}$$
*mapping $(g, f)$ to $g \circ f$, called **composition**, such that* $\mathrm{dom}(g \circ f) = \mathrm{dom}\, f$ *and* $\mathrm{cod}(g \circ f) = \mathrm{cod}\, g$.

*Subject to the following axioms:*

- ***(Associativity Axiom)** For all $f, g, h \in \mathrm{mor}(\mathcal{C})$ with* $\mathrm{dom}\, h = \mathrm{cod}\, g$ *and* $\mathrm{dom}\, g = \mathrm{cod}\, f$*, we have that*
$$(h \circ g) \circ f = h \circ (g \circ f). \tag{2}$$
- ***(Unit Axiom)** For all $f \in \mathrm{mor}(\mathcal{C})$ with* $\mathrm{dom}\, f = X$ *and* $\mathrm{cod}\, f = Y$ *we have that*
$$f = f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f. \tag{3}$$

**Remark 1.1.** Let $\mathcal{C}$ be a category. For $X, Y \in \mathrm{ob}(\mathcal{C})$ we will abreviate
$$\mathcal{C}(X, Y) := \{f \in \mathrm{mor}(\mathcal{C}) : \mathrm{dom}\, f = X \text{ and } \mathrm{cod}\, f = Y\}.$$
Moreover, $f \in \mathcal{C}(X, Y)$ is depicted as
$$f : X \to Y. \tag{4}$$

**Example 1.1.** Let $*$ be a single, not nearer specified object. Consider as morphisms the class of all cardinal numbers and as composition cardinal addition. By [Hal12, pp. 112–113], cardinal addition is associative and $\varnothing$ serves for the identity $\mathrm{id}_*$.

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**Definition 1.2 (Locally Small, Hom-Set).** *A category $\mathcal{C}$ is said to be* ***locally small*** *if for all $X, Y \in \mathcal{C}$, $\mathcal{C}(X, Y)$ is a set. If $\mathcal{C}$ is locally small, $\mathcal{C}(X, Y)$ is called a* ***hom-set*** *for all $X, Y \in \mathcal{C}$.*

**Functors.**

**Definition 1.3 (Functor).** *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A* ***functor*** $\boldsymbol{F : \mathcal{C} \to \mathcal{D}}$ *is a pair of functions $(F_1, F_2)$, $F_1 : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, called the* ***object function*** *and $F_2 : \mathrm{mor}(\mathcal{C}) \to \mathrm{mor}(\mathcal{D})$, called the* ***morphism function****, such that for every morphism $f : X \to Y$ we have that $F_2(f) : F_1(X) \to F_1(Y)$ and $(F_1, F_2)$ is subject to the following* ***compatibility conditions****:*

- *For all $X \in \mathrm{ob}(\mathcal{C})$, $F_2(\mathrm{id}_X) = \mathrm{id}_{F_1(X)}$.*
- *For all $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ we have that $F_2(g \circ f) = F_2(g) \circ F_2(f)$.*

**Remark 1.2.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. It is convenient to denote the components $F_1$ and $F_2$ also with $F$.

**Subcategories.**

**Definition 1.4 (Subcategory).** *Let $\mathcal{C}$ be a category. A* ***subcategory*** $\boldsymbol{\mathcal{S}}$ ***of*** $\boldsymbol{\mathcal{C}}$ *consists of*

- *A subclass $\mathrm{ob}(\mathcal{S}) \subseteq \mathrm{ob}(\mathcal{C})$.*
- *A subclass $\mathrm{mor}(\mathcal{S}) \subseteq \mathrm{mor}(\mathcal{C})$.*

*Subject to the following conditions:*

- *For all $X \in \mathcal{S}$, $\mathrm{id}_S \in \mathrm{mor}(\mathcal{S})$.*

**Example 1.2 ($\mathsf{Top}_*$).** Define the objects of $\mathsf{Top}_*$ to be the class of all tuple $(X, p)$, where $X$ is a topological space and $p \in X$. Moreover, given objects $(X, p)$ and $(Y, q)$ in $\mathsf{Top}_*$, define $\mathsf{Top}_*\big((X, p), (Y, q)\big) := \{f \in \mathsf{Top}(X, Y) : f(p) = q\}$. It is easy to check that $\mathsf{Top}_*$ is a category, called the ***category of pointed topological spaces***.

**Limits.**

**Definition 1.5 (Diagram).** *Let $\mathcal{C}$ be a category and $\mathsf{A}$ a small category. A functor $\mathsf{A} \to \mathcal{C}$ is called a* ***diagram in*** $\boldsymbol{\mathcal{C}}$ ***of shape*** $\boldsymbol{\mathsf{A}}$.

**Definition 1.6 (Cone and Limit).** *Let $\mathcal{C}$ be a category and $D : \mathsf{A} \to \mathcal{C}$ a diagram in $\mathcal{C}$ of shape $\mathsf{A}$. A* ***cone on*** $\boldsymbol{D}$ *is a tuple $\big(C, (f_\alpha)_{\alpha \in \mathsf{A}}\big)$, where $C \in \mathcal{C}$ is an object, called the* ***vertex*** *of the cone, and a family of arrows in $\mathcal{C}$*

$$\big(C \xrightarrow{f_\alpha} D(\alpha)\big)_{\alpha \in \mathsf{A}}. \tag{5}$$

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*such that for all morphisms $f \in \mathsf{A}$, $f : \alpha \to \beta$, the triangle*

$$\begin{tikzcd} & D(\alpha) \arrow[dd, "D(f)"] \\ C \arrow[ur, "f_\alpha"] \arrow[dr, "f_\beta"'] & \\ & D(\beta) \end{tikzcd}$$

*commutes. A **(small) limit of D** is a cone $\big(L, (\pi_\alpha)_{\alpha\in\mathsf{A}}\big)$ with the property that for any other cone $\big(C, (f_\alpha)_{\alpha\in\mathsf{A}}\big)$ there exists a unique morphism $\overline{f} : C \to L$ such that $\pi_\alpha \circ \overline{f} = f_\alpha$ holds for every $\alpha \in \mathsf{A}$.*

**Remark 1.3.** In the setting of definition 1.6, if $\big(L, (\pi_\alpha)_{\alpha\in\mathsf{A}}\big)$ is a limit of $D$, we sometimes reffering to $L$ only as the limit of $D$ and we write

$$L = \lim_{\leftarrow \mathsf{A}} D. \tag{6}$$

CHAPTER 2

# The Fundamental Group

## The Fundamental Grupoid

### Construction of the fundamental Grupoid.

**Lemma 2.1 (Gluing Lemma).** *Let* $X, Y \in \mathrm{ob}(\mathsf{Top})$*,* $(X_\alpha)_{\alpha\in A}$ *a finite closed cover of* $X$ *and* $(f_\alpha)_{\alpha\in A}$ *a finite family of maps* $f_\alpha \in \mathsf{Top}(X_\alpha, Y)$ *such that* $f_\alpha|_{X_\alpha\cap X_\beta} = f_\beta|_{X_\alpha\cap X_\beta}$ *for all* $\alpha, \beta \in A$*. Then there exists a unique* $f \in \mathsf{Top}(X, Y)$ *such that* $f|_{X_\alpha} = f_\alpha$ *for all* $\alpha \in A$*.*

*Proof.* Let $x \in X$. Since $(X_\alpha)_{\alpha\in A}$ is a cover of $X$, we find $\alpha \in A$ such that $x \in X_\alpha$. Define $f(x) := f_\alpha(x)$. This is well defined, since if $x \in X_\alpha \cap X_\beta$ for some $\beta \in A$, we have that $f(x) = f_\beta(x) = f_\alpha(x)$. Clearly $f|_{X_\alpha} = f_\alpha$ for all $\alpha \in A$ and $f$ is unique. Let us show continuity. To this end, let $K \subseteq Y$ be closed. Then

$$\begin{aligned} f^{-1}(K) &= X \cap f^{-1}(K) \\ &= \bigcup_{\alpha\in A} X_\alpha \cap f^{-1}(K) \\ &= \bigcup_{\alpha\in A} \left(X_\alpha \cap f^{-1}(K)\right) \\ &= \bigcup_{\alpha\in A} \left(X_\alpha \cap f_\alpha^{-1}(K)\right). \end{aligned}$$

Since each $f_\alpha$ is continuous, $f_\alpha^{-1}(K)$ is closed in $X_\alpha$ for each $\alpha \in A$ and thus since $X_\alpha$ is closed, $f^{-1}(K)$ is closed as a finite union of closed sets. $\square$

**Theorem 2.1.** *There is a functor* $\mathsf{Top} \to \mathsf{Grpd}$.

*Proof.* The proof is divided into several steps. Let us denote $\Pi : \mathsf{Top} \to \mathsf{Grpd}$ for the claimed functor.

*Step 1: Definition of* $\Pi$ *on objects.* Let $X, Y \in \mathrm{ob}(\mathsf{Top})$, $f, g \in \mathsf{Top}(X, Y)$ and $A \subseteq X$. A map $F \in \mathsf{Top}(X \times I, Y)$ is called a ***homotopy from*** $\boldsymbol{X}$ ***to*** $\boldsymbol{Y}$ ***relative to*** $\boldsymbol{A}$, if

- $F(x, 0) = f(x)$, for all $x \in X$.
- $F(x, 1) = g(x)$, for all $x \in X$.
- $F(x, t) = f(x) = g(x)$, for all $x \in A$ and for all $t \in I$.

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If there exists a homotopy between $f$ and $g$ relative to $A$ we say that $f$ and $g$ are ***homotopic relative to $A$*** and write $f \simeq_A g$. If we want to emphasize the homotpoy relative to $A$, we write $F : f \simeq_A g$.

**Lemma 2.2.** *Let $X, Y \in \mathrm{ob}(\mathsf{Top})$ and $A \subseteq X$. Then being homotopic relative to $A$ is an equivalence relation on $\mathsf{Top}(X, Y)$.*

*Proof.* Define a binary relation $R_A \subseteq \mathsf{Top}(X,Y) \times \mathsf{Top}(X,Y)$ by

$$fR_Ag \quad :\Leftrightarrow \quad f \simeq_A g.$$

Let $f \in \mathsf{Top}(X,Y)$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x,t) := f(x).$$

Then clearly $F : f \simeq_A f$. Hence $R_A$ is reflexive.
Let $g \in \mathsf{Top}(X,Y)$ and assume that $fR_Ag$. Thus $G : f \simeq_A g$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x,t) := G(x, 1-t).$$

Then it is easy to check that $F : g \simeq_A f$ and so $R_A$ is symmetric.
Finally, let $h \in \mathsf{Top}(X,Y)$ and suppose that $fR_Ag$ and $gR_Ah$. Hence $F_1 : f \simeq_A g$ and $F_2 : g \simeq_A h$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x,t) := \begin{cases} F_1(x, 2t) & 0 \leq t \leq \frac{1}{2}, \\ F_2(x, 2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Continuity of $F$ follows by an application of the gluing lemma 2.1. Then it is easy to check that $F : f \simeq_A h$ and hence $R_A$ is transitive. $\square$

Let $X \in \mathrm{ob}(\mathsf{Top})$ and $u$ a path in $X$ from $p$ to $q$. Define the ***path class $[u]$ of $u$*** by $[u] := [u]_{R_{\partial I}}$. Define now

- $\mathrm{ob}\big(\Pi(X)\big) := X$.
- $\Pi(X)(p,q) := \{[u] : u \text{ is a path from } p \text{ to } q\}$ for all $p, q \in X$.
- Let $p \in X$. Then define $\mathrm{id}_p \in \Pi(X)(p,p)$ by $\mathrm{id}_p := [c_p]$, where $c_p$ is the constant path defined by $c_p(s) := p$ for all $s \in I$.
- And $\Pi(X)(q,r) \times \Pi(X)(p,q) \to \Pi(X)(p,r)$ by

  $$([v], [u]) \mapsto [u * v]$$

  Where $u * v \in \mathsf{Top}(p, r)$ is the ***concatenated path of $u$ and $v$***, defined by

  $$(u * v)(s) := \begin{cases} u(2s) & 0 \leq t \leq \frac{1}{2}, \\ v(2s-1) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

  Continuity follows again from the gluing lemma 2.1 whereas well definedness follows from the next lemma.

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**Lemma 2.3.** *Suppose that* $[u_1], [u_2] \in \Pi(X)(p,q)$ *and* $[v_1], [v_2] \in \Pi(X)(q,r)$ *such that* $[u_1] = [u_2]$ *and* $[v_1] = [v_2]$*. Then* $[u_1 * v_1] = [u_2 * v_2]$*.*

*Proof.* By assumption we have $G : u_1 \simeq_{\partial I} u_2$ and $H : v_1 \simeq_{\partial I} v_2$. Define $F \in \mathsf{Top}(I \times I, X)$ by

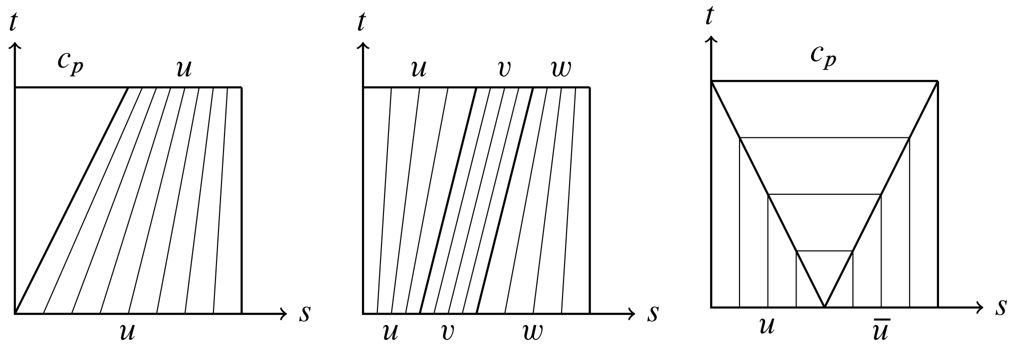
$$F(s,t) := \begin{cases} G(2s,t) & 0 \leq s \leq \frac{1}{2}, \\ H(2s-1,t) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Again, continuity follows from the gluing lemma 2.1 and it is easy to check that $F : u_1 * v_1 \simeq_{\partial I} u_2 * v_2$. □

Let us now check that $\Pi(X)$ is indeed a category. Let $[u] \in \Pi(X)(p,q)$. We want to show that $u \simeq_{\partial I} c_p * u$. To this end, we consider figure 1a and conclude that a suitable homotopy is given by $F \in \mathsf{Top}(I \times I, X)$ defined by

$$F(s,t) := \begin{cases} p & 0 \leq 2s \leq t, \\ u\left(\dfrac{2s-t}{2-t}\right) & t \leq 2s \leq 2. \end{cases}$$

Similarly, considering figure 1b leads to $F \in \mathsf{Top}(I \times I, X)$ defined by




(A) $u \simeq_{\partial I} c_p * u$. (B) $(u*v)*w \simeq_{\partial I} u*(v*w)$. (C) $u * \overline{u} \simeq_{\partial I} c_p$.

FIGURE 1. Visualization of the proof that $\Pi(X)$ is a grupoid object.

$$F(s,t) := \begin{cases} u\left(\dfrac{4s}{t+1}\right) & -1 \leq 4s-1 \leq t, \\ v(4s-t-1) & t \leq 4s-1 \leq t+1, \\ w\left(\dfrac{4s-t-2}{4-t-2}\right) & t+1 \leq 4s-1 \leq 3. \end{cases}$$

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Lastly, we check that $\Pi(X)$ is a grupoid. To this end, for a path $u$ from $p$ to $q$, define its ***reverse path*** $\overline{u}$ by

$$\overline{u}(s) := u(1-s).$$

We claim that $u * \overline{u} \simeq_{\partial I} c_p$. From figure 1c we deduce that $F \in \mathsf{Top}(I \times I, X)$ is given by

$$F(s,t) := \begin{cases} u(2s) & 0 \leq 2s \leq 1-t, \\ u(1-t) & 1-t \leq 2s \leq t+1, \\ \overline{u}(2s-1) & t+1 \leq 2s \leq 2. \end{cases}$$

*Step 2: Definition of* $\Pi$ *on morphisms.* Let $f \in \mathsf{Top}(X,Y)$. Then $\Pi(f)$ is a functor from $\Pi(X)$ to $\Pi(Y)$. Define $\Pi(f)$ as follows:

- Let $p \in \mathrm{ob}\big(\Pi(X)\big)$. Then define $\Pi(f)(p) := f(p) \in \mathrm{ob}\big(\Pi(Y)\big)$.
- Let $[u] \in \Pi(X)(p,q)$. Then define $\Pi(f)\,[u] := [f \circ u] \in$. We have to check that this definition is independent of the choice of the representative.

**Lemma 2.4.** *Let $u$ and $v$ be paths from $p$ to $q$ in $X$ and suppose that $[u] = [v]$. Then for any $f \in \mathsf{Top}(X,Y)$ we also have that $[f \circ u] = [f \circ v]$.*

*Proof.* Suppose that $H : u \simeq_{\partial I} v$. Define $F \in \mathsf{Top}(I \times I, Y)$ by

$$F(s,t) := (f \circ F)(s,t).$$

Then $F : f \circ u \simeq_{\partial I} f \circ v$. □

Checking that $\Pi$ satisfies the functorial properties is left as an exercise. □

**Exercise 2.1.** Check that $\Pi : \mathsf{Top} \to \mathsf{Grpd}$ is indeed a functor.

**The Fundamental Group.**

**Lemma 2.5.** *Let $\mathcal{G}$ be a locally small grupoid. Then for every $X \in \mathrm{ob}(\mathcal{G})$, $\mathcal{G}(X,X)$ can be equipped with a group structure.*

*Proof.* Since $\mathcal{G}$ is locally small, $\mathcal{G}(X,X)$ is a set for every $X \in \mathrm{ob}(\mathcal{G})$. Define a multiplication $\mathcal{G}(X,X) \times \mathcal{G}(X,X) \to \mathcal{G}(X,X)$ by $gh := h \circ g$. Clearly, this multiplication is associative. Moreover, the identity element is given by $\mathrm{id}_X \in \mathcal{G}(X,X)$ and since every $g \in \mathcal{G}(X,X)$ is an isomorphism, the multiplicative inverse is given by the inverse in $\mathcal{G}(X,X)$. □

**Proposition 2.1.** *There is a functor* $\mathsf{Top}_* \to \mathsf{Grp}$.

*Proof.* Define $\pi_1 : \mathsf{Top}_* \to \mathsf{Grp}$ on objects $(X,p) \in \mathsf{Top}_*$ by

$$\pi_1(X,p) := \Pi(X)(p,p).$$

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By theorem 2.1 together with lemma 2.5, $\pi_1(X,p)$ is actually a group, called the ***fundamental group of $X$ with basepoint $p$***. On morphisms $f \in \mathsf{Top}_*\big((X,p),(Y,q)\big)$, define

$$\pi_1(f) := \Pi(f) : \Pi(X)(p,p) \to \Pi(Y)(q,q).$$

Let $[u], [v] \in \pi_1(X,p)$. Then

$$\begin{aligned}\pi_1([u][v]) &= \Pi(f)([u][v]) \\ &= \Pi(f)[u*v] \\ &= [f\circ(u*v)] \\ &= [(f\circ u)*(f\circ v)] \\ &= \Pi(f)[u]\,\Pi(f)[v] \\ &= \pi_1(f)[u]\,\pi_1(f)[v].\end{aligned}$$

Thus $\pi_1(f)$ is a morphism in $\mathsf{Grp}$. Functoriality of $\pi_1$ immediately follows from the functoriality of $\Pi$. ☐

**Lemma 2.6.** *Let $X \in \mathrm{ob}(\mathsf{Top})$, $p \in X$ and $A$ be the path component of $X$ containing $p$. Then $\pi_1(\iota)$, where $\iota : A \hookrightarrow X$ denotes the inclusion, is an isomorphism.*

*Proof.* Suppose $[u] \in \ker \pi_1(\iota)$. Then $[\iota\circ u] = [c_p]$ and Hence $F : \iota\circ u \simeq_{\partial I} c_p$. Since $I\times I$ is path connected and $p \in F(I\times I)$, it follows that $F(I\times I) \subseteq A$ and thus $F : u \simeq_{\partial I} c_p$ in $A$ and hence $[u] = [c_p]$. To see that $\pi_1(\iota)$ is surjective, just observe that $u(I) \subseteq A$ for $[u] \in \pi_1(X,p)$ since $u(I)$ is path connected and $p \in u(I)$. ☐

**Lemma 2.7.** *Let $X \in \mathrm{ob}(\mathsf{Top})$ be path connected and $p,q \in X$. Then*

$$\pi_1(X,p) \cong \pi_1(X,q).$$

*Proof.* Since $X$ is path connected we find a path $v$ from $p$ to $q$ in $X$. Define a mapping $\Phi_v : \pi_1(X,p) \to \pi_1(X,q)$

$$\Phi_v[u] := [\overline{v}*u*v].$$

Clearly, $\Phi_v$ is invertible with inverse $\Phi_{\overline{v}}$. Moreover, for $[u], [w] \in \pi_1(X,p)$ we have that

$$\begin{aligned}\Phi_v([u][w]) &= \Phi_v[u*w] \\ &= [\overline{v}*u*w*v] \\ &= [\overline{v}*u*v*\overline{v}*w*v] \\ &= [\overline{v}*u*v][\overline{v}*w*v] \\ &= \Phi_v[u]\,\Phi_v[w].\end{aligned}$$

☐

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**$\pi_1(\mathbb{S}^1)$.**

**Definition 2.1 (Exponential Quotient Map).** *The mapping $\varepsilon : \mathbb{R} \to \mathbb{S}^1$ defined by*

$$\varepsilon(x) := e^{2\pi i x} \tag{7}$$

*is called the* ***exponential quotient map****.*

**Proposition 2.2 (Lifting Property of the Circle).** *Let $n \in \mathbb{Z}$, $n \geq 0$, $X \subseteq \mathbb{R}^n$ compact and convex, $p \in X$, $f \in \mathsf{Top}_*\big((X,p),(\mathbb{S}^1,1)\big)$ and $m \in \mathbb{Z}$. Then there exists a unique map $\widetilde{f} \in \mathsf{Top}_*\big((X,p),(\mathbb{R},m)\big)$, called the* ***lifting of $f$****, such that*

$$\begin{tikzcd} & (\mathbb{R},m) \arrow[d, "\varepsilon"] \\ (X,p) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (\mathbb{S}^1,1) \end{tikzcd}$$

*commutes.*

*Proof.* We show first existence and then uniqueness.

*Step 1: Existence.* Since $X$ is compact and $f$ is continuous, $f$ is uniformly continuous on $X$. Thus we find $\delta > 0$ such that $|f(x) - f(y)| < 2$, whenever $|x - y| < \delta$, i.e. $f(x)$ and $f(y)$ are not antipodal points. Moreover, since $X$ is compact, $X$ is bounded and hence we find $N \in \mathbb{N}$, such that $|x - y| < N\delta$ holds for all $x, y \in X$. Let $x \in X$. For $0 \leq k \leq N$, define $L_k : X \to X$ by

$$L_k(x) := \left(1 - \frac{k}{N}\right) p + \frac{k}{N} x.$$

Those are well defined functions since $X$ is convex. Moreover, each $L_k$ is continuous. Indeed, it is easy to check that $L_k$ is Lipschitz. Also, for each $0 \leq k < N$, $f(L_k(x))$ and $f(L_{k+1}(x))$ are not antipodal for all $x \in X$. Indeed, it is easy to check that $|L_k(x) - L_{k+1}(x)| < \delta$ holds for all $x \in X$. For $0 \leq k < N$ define $g_k : X \to \mathbb{S}^1 \setminus \{-1\}$ by

$$g_k(x) := \frac{f(L_{k+1}(x))}{f(L_k(x))}.$$

Clearly $g_k$ is well defined and continuous as a composition of continuous functions. Let $\operatorname{Log} : \mathbb{S}^1 \setminus \{-1\} \to \mathbb{C}$ denote the principal branch of the logarithm. Define $\widetilde{f} : X \to \mathbb{R}$ by

$$\widetilde{f}(x) := m + \frac{1}{2\pi i} \sum_{k=0}^{N-1} \operatorname{Log}(g_k(x)).$$

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Clearly, $\widetilde{f}$ is continuous and moreover we have that $\widetilde{f} = m$ since $g_k(p) = 1$ for all $0 \leq k < N$. Finally, for any $x \in X$ we have that

$$(\varepsilon \circ \widetilde{f})(x) = \varepsilon(m) \prod_{k=0}^{N-1} g_k(x) = \frac{f(L_N(x))}{f(L_0(x))} = \frac{f(x)}{f(p)} = f(x).$$

*Step 2: Uniqueness.* Suppose $\widetilde{g} \in \mathsf{Top}_* \big((X,p),(\mathbb{R},m)\big)$ is another such function. Define $\varphi \in \mathsf{Top}_* \big((X,p),(\mathbb{R},0)\big)$ by

$$\varphi(x) := \widetilde{f}(x) - \widetilde{g}(x).$$

Then clearly $\varepsilon \circ \varphi = 1$ and thus $\varphi(X) \subseteq \mathbb{Z}$. Since $X$ is convex, $X$ is connected and so $\varphi = 0$.

□

**Corollary 2.1.** *Let $u, v \in \Omega(\mathbb{S}^1, 1)$ such that $[u] = [v]$. If $\widetilde{u}, \widetilde{v} : (I,0) \to (\mathbb{R},0)$ are the liftings of $u$ and $v$, respectively, then $[\widetilde{u}] = [\widetilde{v}]$.*

*Proof.* Let $F : u \simeq_{\partial I} v$. By proposition 2.2, we find $\widetilde{F} \in \mathsf{Top}_* \big((I \times I, (0,0)), (\mathbb{R},0)\big)$, such that $\varepsilon \circ \widetilde{F} = F$. We claim that $\widetilde{F} : \widetilde{u} \simeq_{\partial I} \widetilde{v}$. For $s \in I$ define $\widetilde{u}_0(s) := \widetilde{F}(s,0)$. Then $\widetilde{u}_0(0) = \widetilde{F}(0,0) = 0$ and since $\widetilde{u}_0$ is continuous we have that $\widetilde{u}_0 \in \mathsf{Top}_* \big((I,0),(\mathbb{R},0)\big)$. Moreover

$$(\varepsilon \circ \widetilde{u}_0)(s) = \varepsilon\left(\widetilde{F}(s,0)\right) = F(s,0) = u(s)$$

for all $s \in I$ and thus $\widetilde{u}_0$ is a lifting of $u$. But by proposition 2.2, liftings are unique and thus $\widetilde{u}_0 = \widetilde{u}$. Next define $\widetilde{w}_0(t) := \widetilde{F}(0,t)$ for all $t \in I$. Then $\widetilde{w}_0(0) = \widetilde{F}(0,0) = 0$ and so $\widetilde{w}_0 \in \mathsf{Top}_* \big((I,0),(\mathbb{R},0)\big)$. Moreover

$$(\varepsilon \circ \widetilde{w}_0)(t) = \varepsilon\left(\widetilde{F}(0,t)\right) = F(0,t) = u(0) = v(0) = 1.$$

for all $t \in I$. Thus

$$\begin{tikzcd} & (\mathbb{R},0) \arrow[d, "\varepsilon"] \\ (I,0) \arrow[ur, "\widetilde{w}_0"] \arrow[r, "c_1"'] & (\mathbb{S}^1,1) \end{tikzcd}$$

commutes. But also $c_0$ makes the above diagram commute. By uniqueness, $\widetilde{w}_0 = c_0$. Define $\widetilde{v}_0(s) := \widetilde{F}(s,1)$ for all $s \in I$. Then $\widetilde{v}_0(0) = \widetilde{F}(0,1) = \widetilde{w}_0(1) = 0$ and it is easy to check that $\widetilde{v}_0$ is a lift for $v$. Hence $\widetilde{v}_0 = \widetilde{v}$. Finally, define $\widetilde{w}_1(t) := \widetilde{F}(1,t)$ for all $t \in I$. Then $\widetilde{w}_1(0) = \widetilde{F}(1,0) = \widetilde{u}(1)$ and thus $\widetilde{w}_1 \in \mathsf{Top}_* \big((I,0),(\mathbb{R},\widetilde{u}(0))\big)$. Moreover

$$(\varepsilon \circ \widetilde{w}_1)(t) = \varepsilon\left(\widetilde{F}(1,t)\right) = F(1,t) = v(1) = u(1) = 1$$

for all $t \in I$. By proposition 2.2, we have again that $\widetilde{w}_1 = c_{\widetilde{u}(1)}$. So $F : \widetilde{u} \simeq_{\partial I} \widetilde{v}$. □

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**Definition 2.2 (Degree).** *Let $u \in \Omega(\mathbb{S}^1, 1)$. The **degree of $u$**, written $\deg u$, is defined by $\deg u := \widetilde{u}(1)$, where $\widetilde{u}$ is the unique lift of $u$ such that $\widetilde{u}(0) = 0$.*

**Theorem 2.2 (Fundamental Group of the Circle).** $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.

*Proof.* Define $\deg : \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}$ by $\deg[u] := \deg u$. This is well defined by corollary 2.1, since if $[u] = [v]$, then $[\widetilde{u}] = [\widetilde{v}]$ and in particular $\widetilde{u}(1) = \widetilde{v}(1)$.
*Step 1:* $\deg \in \mathsf{Grp}\big(\pi_1(\mathbb{S}^1, 1), (\mathbb{Z}, +)\big)$. Let $[u], [v] \in \pi_1(\mathbb{S}^1, 1)$ and $m := \deg[u]$, $n := \deg[v]$. Moreover, let $\widetilde{u}$ and $\widetilde{v}$ denote the unique liftings of $u$ and $v$, respectively, such that $\widetilde{u}(0) = 0$ and $\widetilde{v}(0) = 0$. Define

$$\widetilde{w}(s) := \begin{cases} \widetilde{u}(2s) & 0 \leq s \leq \frac{1}{2}, \\ m + \widetilde{v}(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Clearly $\widetilde{w}$ is continuous and $\widetilde{w}(0) = 0$. Hence $\widetilde{w} \in \mathsf{Top}_*\big((I, 0), (\mathbb{R}, 0)\big)$. Also we have that $\varepsilon \circ \widetilde{w} = u * v$ and thus $\widetilde{w}$ is the lift of $u * v$. But $\widetilde{w}(1) = m + n$ and so

$$\deg([u]\,[v]) = \deg[u * v] = \deg(u * v) = \widetilde{w}(1) = m + n = \deg[u] + \deg[v]\,.$$

*Step 2:* $\deg$ *is injective.* Suppose $\deg[u] = 0$. Then $\widetilde{u}(1) = 0$ and thus $\widetilde{u} \in \Omega(\mathbb{R}, 0)$. Since $\mathbb{R}$ is contractible, we have that $[\widetilde{u}] = [c_0]$ and thus

$$[u] = [\varepsilon \circ \widetilde{u}] = \pi_1(\varepsilon)\,[\widetilde{u}] = \pi_1(\varepsilon)\,[c_0] = [c_1]\,.$$

Thus $\ker(\deg)$ is trivial.

*Step 3:* $\deg$ *is surjective.* Let $m \in \mathbb{Z}$. Then

$$\deg[\varepsilon^m] = \deg \varepsilon^m = \widetilde{\varepsilon^m}(1) = m.$$

$\square$

## The Seifert-Van Kampen Theorem

### Coproducts and Pushouts in Grp.

**Proposition 2.3 (Coproducts in Grp).** $\mathsf{Grp}$ *has all small coproducts.*

*Proof.* Let $A \in \operatorname{ob}(\mathsf{Set})$ and $\mathsf{A}$ be the small category defined as the discrete category with $\operatorname{ob}(\mathsf{A}) := A$, i.e.

$$\bullet \qquad \bullet \qquad \bullet \qquad \cdots \qquad \bullet \qquad \bullet \qquad \bullet$$

Let $D : \mathsf{A} \to \mathsf{Grp}$ be a functor. Hence we get a family $(G_\alpha)_{\alpha \in A}$ in $\mathsf{Grp}$, where $G_\alpha := D(\alpha)$ for all $\alpha \in A$. A ***word*** in $(G_\alpha)_{\alpha \in A}$ is a finite sequence in $\coprod_{\alpha \in A} G_\alpha$. A word in $(G_\alpha)_{\alpha \in A}$ will simply be written as $(g_1, \ldots, g_n)$, where $g_k \in G_\alpha$ for some $\alpha \in A$. The ***empty word*** is denoted by $()$. Let $\mathcal{W}$ denote the set of all words in $(G_\alpha)_{\alpha \in A}$. On $\mathcal{W}$ define a multiplication by ***concatenation***

$$(g_1, \ldots, g_n)(h_1, \ldots, h_m) := (g_1, \ldots, g_n, h_1, \ldots, h_m).$$

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An ***elementary reduction*** is an operation of one of the following forms:

- $(g_1, \dots, g_k, g_{k+1}, \dots, g_n) \mapsto (g_1, \dots, g_k g_{k+1}, \dots, g_n,$ where $g_k, g_{k+1} \in G_\alpha$ for some $\alpha \in A$.
- $(g_1, \dots, g_{k-1}, 1_\alpha, g_{k+1}, \dots, g_n) \mapsto (g_1, \dots, g_{k-1}, g_{k+1}, \dots, g_n)$.

Let $\sim$ denote the equivalence relation on $\mathcal{W}$ generated by elementary reductions.

**Lemma 2.8.** *$\mathcal{W}/\sim$ together with concatenation of representatives is an element of* Grp.

*Proof.* Define

$$[(g_1, \dots, g_n)]\,[(h_1, \dots, h_m)] := [(g_1, \dots, g_n, h_1, \dots, h_m)]\,.$$

It is left to the reader to show that this is well defined and that $\mathcal{W}/\sim$ is indeed a group. $\square$

The group defined in lemma 2.8 will be denoted by $\mathop{\ast}_{\alpha \in A} G_\alpha$ and called the ***free product of $(G_\alpha)_{\alpha \in A}$***. Let us define a cocone on $D$. For this consider the inclusions $\iota_\alpha : G_\alpha \to \mathop{\ast}_{\alpha \in A} G_\alpha$ defined by

$$\iota_\alpha(g) := [(g)]$$

for all $\alpha \in A$. It is immediate from

$$\iota_\alpha(gh) = [(gh)] = [(g,h)] = [(g)]\,[(h)] = \iota_\alpha(g)\iota_\alpha(h)$$

for $g, h \in G_\alpha$, that $\iota_\alpha$ is a morphism of groups. Since there are only the identity morphisms in A, $\left(\mathop{\ast}_{\alpha \in A} G_\alpha, (\iota_\alpha)_{\alpha \in A}\right)$ is a cocone on $D$. Let us show that this is in fact a universal cocone. To this end, suppose that $\left(C, (\varphi_\alpha)_{\alpha \in A}\right)$ is another cocone on $D$. Define a mapping $\bar{f} : \mathop{\ast}_{\alpha \in A} G_\alpha \to C$ by

$$\bar{f}\,[(g_1, \dots, g_n)] := \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$$

where $g_k \in G_{\alpha_k}$. Then $\bar{f}$ is easily seen to be well defined since each $\varphi_\alpha$ is a morphism of groups. Moreover, if $g \in G_\alpha$, then

$$(\bar{f} \circ \iota_\alpha)(g) = \bar{f}\,[(g)] = \varphi_\alpha(g)$$

for all $\alpha \in A$. Suppose that $f : \mathop{\ast}_{\alpha \in A} G_\alpha \to C$ is another homomorphism of groups such that $f \circ \iota_\alpha = \varphi_\alpha$ for all $\alpha \in A$. Then for $[(g_1, \dots, g_n)] \in \mathop{\ast}_{\alpha \in A} G_\alpha$ we have

$$\begin{aligned} f\,[(g_1, \dots, g_n)] &= f([(g_1)] \cdots [(g_n)]) \\ &= f\,[(g_1)] \cdots f\,[(g_n)] \\ &= f\,(\iota_{\alpha_1}(g_1)) \cdots f\,(\iota_{\alpha_n}(g_n)) \\ &= \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n) \\ &= \bar{f}\,[(g_1, \dots, g_n)]\,. \end{aligned}$$

$\square$

**Exercise 2.2.** Check that $\mathcal{W}/\sim$ is indeed a group with the declared group structure and that $\bar{f}$ is indeed well defined.

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**Proposition 2.4 (Pushouts in Grp).** Grp *has all pushouts.*

*Proof.* Consider the diagram $D : \mathsf{A} \to \mathsf{Grp}$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \quad \xrightarrow{D} \quad \begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \\ H_2 & \end{tikzcd}$$

and define $N$ to be the normal subgroup of $H_1 * H_2$ generated by elements of the form $[(\varphi_1(g^{-1}), \varphi_2(g))]$ for $g \in G$. Let $K := (H_1 * H_2)/N$. Then

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \arrow[d, "\pi \circ \iota_1"] \\ H_2 \arrow[r, "\pi \circ \iota_2"'] & K \end{tikzcd}$$

commutes. Indeed, if $g \in G$, we have that $(\pi \circ \iota_1 \circ \varphi_1)(g) = [(\varphi_1(g))]\, N$ and similarly $(\pi \circ \iota_2 \circ \varphi_2)(g) = [(\varphi_2(g))]\, N$. Then

$$[(\varphi_1(g))]^{-1}\, [(\varphi_2(g))] = [(\varphi_1(g)^{-1})]\, [(\varphi_2(g))] = [(\varphi_1(g^{-1}))]\, [(\varphi_2(g))] \in N.$$

Suppose that we have another cocone on $D$:

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \arrow[d, "\psi_1"] \\ H_2 \arrow[r, "\psi_2"'] & C \end{tikzcd}$$

By proposition 2.3, there exists a unique morphism of groups $f : H_1 * H_2 \to C$ and we thus get the following diagram:

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \arrow[d, "\iota_1"] \arrow[ddrr, bend left, "\psi_1"] & & \\ H_2 \arrow[r, "\iota_2"'] \arrow[drrr, bend right, "\psi_2"'] & H_1 * H_2 \arrow[dr, "\pi"] \arrow[drr, bend right, "f"'] & & \\ & & K \arrow[r, dashed, "\bar{f}"] & C \end{tikzcd}$$

To show that $N \subseteq \ker f$ is left as an exercise. Hence by the factorization theorem (see [Gri07, p. 23]), $f$ factors uniquely through $\pi$, i.e. there exists a unique morphism of groups $\bar{f} : K \to C$ such that $\bar{f} \circ \pi = f$. □

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**Exercise 2.3.** In the previous proposition, verify that $N \subseteq \ker f$.

**Definition 2.3 (Amalgamated Free Product).** *The pushout of a diagram*

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \\ H_2 & \end{tikzcd}$$

*in* $\mathsf{Grp}$ *is called the **amalgamated free product of $H_1$ and $H_2$ along $(G, \varphi_1, \varphi_2)$**, written* $H_1 *_{(G,\varphi_1,\varphi_2)} H_2$.

**The Seifert-Van Kampen Theorem and its Consequences.**

**Theorem 2.3 (Seifert-Van Kampen).** *Let $X \in \mathrm{ob}(\mathsf{Top})$, $(U, V)$ an open cover for $X$, such that $U$, $V$ and $U \cap V$ are path connected. Moreover, let $p \in U \cap V$. Then*

$$\pi_1(X, p) \cong \pi_1(U, p) *_{(\pi_1(U \cap V, p), \pi_1(\iota_U), \pi_1(\iota_V))} \pi_1(V, p), \tag{8}$$

*where $\iota_U : U \cap V \hookrightarrow U$ and $\iota_V : U \cap V \hookrightarrow V$ denote inclusion.*

CHAPTER 3

# Singular Homology

## Construction of the Singular Homology Functor

### Free Abelian Groups.

**Proposition 3.1.** *The forgetful functor* $U : \mathsf{AbGrp} \to \mathsf{Set}$ *admits a left adjoint.*

*Proof.* We have to construct a functor $F : \mathsf{Set} \to \mathsf{AbGrp}$. Let $S$ be a set. Define

$$F(S) := \left\{ f \in \mathbb{Z}^S : \operatorname{supp} f \text{ is finite} \right\}.$$

Equipped with pointwise addition, $F(S)$ is an abelian group. There is a natural inclusion $\iota : S \hookrightarrow U\left(F(S)\right)$ sending $x \in S$ to the function taking the value one at $x$ and zero else. Hence we may regard elements of $F(S)$ as formal linear combinations $\sum_{x \in S} m_x x$, where $m_x \in \mathbb{Z}$ for all $x \in S$. Let $G \in \operatorname{ob}(\mathsf{AbGrp})$ be an abelian group and $\varphi \in \mathsf{AbGrp}\left(F(S), G\right)$ a morphism of groups. Define $\overline{\varphi} \in \mathsf{Set}\left(S, U(G)\right)$ by $\overline{\varphi} := U(\varphi)$. Conversely, if we have $f \in \mathsf{Set}\left(S, U(G)\right)$, define $\overline{f} \in \mathsf{AbGrp}\left(F(S), G\right)$ by $\overline{f}\left(\sum_{x \in S} m_x x\right) := \sum_{x \in S} m_x f(x)$. This is well defined since all but finitely many $m_x$ are zero and $G$ is abelian. It is easy to check that $\overline{f}$ is indeed a morphism of groups. Let $\varphi \in \mathsf{AbGrp}\left(F(S), G\right)$. Then

$$\begin{aligned}
\overline{\overline{\varphi}}\left(\sum_{x \in S} m_x x\right) &= \sum_{x \in S} m_x \overline{\varphi}(x) \\
&= \sum_{x \in S} m_x U(\varphi)(x) \\
&= \sum_{x \in S} m_x \varphi(x) \\
&= \varphi\left(\sum_{x \in S} m_x x\right).
\end{aligned}$$

And for $f \in \mathsf{Set}\left(S, U(G)\right)$ we have that

$$\overline{\overline{f}}(x) = U(\overline{f})(x) = \overline{f}(x) = f(x).$$

Hence $\overline{\overline{\varphi}} = \varphi$ and $\overline{\overline{f}} = f$ and so we have a bijection

$$\mathsf{AbGrp}\left(F(S), G\right) \cong \mathsf{Set}\left(S, U(G)\right).$$

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The mapping $f \mapsto \bar{f}$ will be referred to as ***extending by linearity***. To check naturality in $S$ and $G$ is left as an exercise. □

**Exercise 3.1.** Check the naturality of the bijection in proposition 3.1. Also check that $F : \mathsf{Set} \to \mathsf{AbGrp}$ is indeed a functor. $F$ is called the ***free functor from* Set *to* AbGrp**.

**Definition 3.1 (Free Abelian Group).** *Let $F : \mathsf{Set} \to \mathsf{AbGrp}$ be the free functor. For any set $S$, we call $F(S)$ the **free group generated by $S$**.*

**Chain Complexes.**

**Definition 3.2 (Chain Complex).** *A **chain complex** is a tuple $(C_\bullet, \partial_\bullet)$ consisting of a sequence $(C_n)_{n\in\mathbb{Z}}$ in $\mathrm{ob}(\mathsf{AbGrp})$ and a sequence $(\partial_n)_{n\in\mathbb{Z}}$ in $\mathrm{mor}(\mathsf{AbGrp})$, called **boundary operators**, such that we have $\partial_n \in \mathsf{AbGrp}(C_n, C_{n-1})$ and $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$.*

**Definition 3.3 (Chain Maps).** *Let $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ be two chain complexes. A **chain map $f_\bullet : C_\bullet \to C'_\bullet$** is a sequence $(f_n)_{n\in\mathbb{Z}}$ in $\mathrm{mor}(\mathsf{AbGrp})$ such that $f_n \in \mathsf{AbGrp}(C_n, C'_n)$ and the diagram*

$$\begin{tikzcd}
C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & C_{n-1} \arrow[d, "f_{n-1}"] \\
C'_n \arrow[r, "\partial'_n"'] & C'_{n-1}
\end{tikzcd}$$

*commutes for all $n \in \mathbb{Z}$.*

**Proposition 3.2.** *There is a category with objects chain complexes and morphisms chain maps.*

*Proof.* Let $f_\bullet : C_\bullet \to C'_\bullet$ and $g_\bullet : C'_\bullet \to C''_\bullet$ be chain maps. Define a map $g_\bullet \circ f_\bullet$ by $g_n \circ f_n$ for each $n \in \mathbb{Z}$. This defines a chain map. Moreover, for each chain complex $C_\bullet$ define $\mathrm{id}_{C_\bullet}$ by $\mathrm{id}_{C_n}$ for all $n \in \mathbb{Z}$. It is easy to check, that then $\circ$ is associative and the identity laws hold. □

**Definition 3.4 (Comp).** *The category in 3.2 is called the **category of chain complexes** and we refer to it as* Comp.

**Theorem 3.1.** *There is a functor* $\mathsf{Top} \to \mathsf{Comp}$.

*Proof.* The proof is divided into several steps. Let us denote $C_\bullet : \mathsf{Top} \to \mathsf{Comp}$ for the claimed functor.

*Step 1: Construction of a sequence of abelian groups.* Let $v_0, \dots, v_k \in \mathbb{R}^n$ for some $n, k \in \mathbb{N}$. We say that $(v_0, \dots, v_k)$ is ***affinely independent*** if $(v_1 - v_0, \dots, v_k - v_0)$

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is linearly independent. We define the ***k-simplex spanned by*** $(\boldsymbol{v_0, \dots, v_k})$, written $[v_0, \dots, v_k]$, to be

$$[v_0, \dots, v_k] := \left\{\textstyle\sum_{i=0}^k s_i v_i : s_i \geq 0 \text{ for all } i = 0, \dots, k \text{ and } \sum_{i=0}^k s_i = 1\right\}. \tag{9}$$

equipped with the subspace topology. Moreover, we define the ***standard n-simplex*** $\boldsymbol{\Delta^n}$ to be the $n$-simplex spanned by $(e_0, \dots, e_n)$ where $(e_{i+1})_i$ is the standard basis of $\mathbb{R}^{n+1}$. Let $X \in \mathrm{ob}(\mathsf{Top})$. Define a ***singular n-simplex in X*** to be a map $\sigma \in \mathsf{Top}(\Delta^n, X)$. Let $n \in \mathbb{Z}$. Define

$$C_n(X) := \begin{cases} F\left(\mathsf{Top}(\Delta^n, X)\right) & n \geq 0, \\ 0 & n < 0. \end{cases} \tag{10}$$

We will call elements of $C_n(X)$ ***singular n-chains***.

*Step 2: Construction of boundary operators.* Let $X \in \mathrm{ob}(\mathsf{Top})$ and $\sigma$ a singular $n$-simplex in $X$ for $n \geq 1$. We define $\varphi_k^n : \Delta^{n-1} \to \Delta^n$, called the ***k-th face map***, to be the unique affine map determined by the vertex map

$$\begin{array}{ccc} & \varphi_k^n & \\ e_0 & \mapsto & e_0 \\ \vdots & & \vdots \\ e_{k-1} & \mapsto & e_{k-1} \\ e_k & \mapsto & e_{k+1} \\ \vdots & & \vdots \\ e_{n-1} & \mapsto & e_n. \end{array}$$

Explicitely, given $\sum_{i=0}^{n-1} s_i e_i \in \Delta^{n-1}$, we have that (see [Lee11, p. 152])

$$\varphi_k^n\left(\sum_{i=0}^{n-1} s_i e_i\right) = \sum_{i=0}^{n-1} s_i \varphi_k^n(e_i).$$

Define now

$$\partial\sigma := \sum_{k=0}^{n} (-1)^k \sigma \circ \varphi_k^n \in U\left(C_{n-1}(X)\right) \tag{11}$$

to be the ***boundary of*** $\boldsymbol{\sigma}$. Moreover, the ***singular boundary operator*** is defined to be $\overline{\partial}_n$ and $\partial_n := 0$ for $n \leq 0$.

*Step 3:* $\partial_n \circ \partial_{n+1} = 0$ *for all* $n \in \mathbb{Z}$. It is enough to consider $n \geq 1$, since $\partial_n \circ \partial_{n+1} = 0$ holds trivially in the other cases. Let $X \in \mathrm{ob}(\mathsf{Top})$ and $\sigma \in \mathsf{Top}(\Delta^{n+1}, X)$. Then we have

$$(\partial_n \circ \partial_{n+1})(\sigma) = \partial_n\left(\sum_{k=0}^{n+1} (-1)^k \sigma \circ \varphi_k^{n+1}\right)$$

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$$
\begin{aligned}
&= \sum_{k=0}^{n+1} (-1)^k \partial_n \left(\sigma \circ \varphi_k^{n+1}\right) \\
&= \sum_{k=0}^{n+1} \sum_{j=0}^{n} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\
&= \sum_{0 \le k \le j \le n} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n + \sum_{0 \le j < k \le n+1} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\
&= \sum_{0 \le j \le k \le n} (-1)^{k+j} \sigma \circ \varphi_j^{n+1} \circ \varphi_k^n + \sum_{0 \le j < k \le n+1} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\
&= \sum_{0 \le j < k \le n+1} \left((-1)^{k+j-1} \sigma \circ \varphi_j^{n+1} \circ \varphi_{k-1}^n + (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n\right)
\end{aligned}
$$

Since $\varphi_j^{n+1} \circ \varphi_{k-1}^n = \varphi_k^{n+1} \circ \varphi_j^n$, it follows that

$$
\partial_n \circ \partial_{n+1} = 0.
$$

Indeed, consider the following chart of vertex maps:

$$
\begin{array}{ccccc}
 & \varphi_{k-1}^n & & \varphi_j^{n+1} & \\
e_0 & \mapsto & e_0 & \mapsto & e_0 \\
\vdots & & \vdots & & \vdots \\
e_{j-1} & \mapsto & e_{j-1} & \mapsto & e_{j-1} \\
e_j & \mapsto & e_j & \mapsto & e_{j+1} \\
\vdots & & \vdots & & \\
e_{k-1} & \mapsto & e_{k-1} & \mapsto & e_{k+1} \\
e_k & \mapsto & e_{k+1} & \mapsto & e_{k+2} \\
\vdots & & \vdots & & \\
e_{n-1} & \mapsto & e_n & \mapsto & e_{n+1}
\end{array}
\qquad
\begin{array}{ccccc}
 & \varphi_j^n & & \varphi_k^{n+1} & \\
e_0 & \mapsto & e_0 & \mapsto & e_0 \\
\vdots & & \vdots & & \vdots \\
e_{j-1} & \mapsto & e_{j-1} & \mapsto & e_{j-1} \\
e_j & \mapsto & e_{j+1} & \mapsto & e_{j+1} \\
\vdots & & \vdots & & \\
e_{k-1} & \mapsto & e_k & \mapsto & e_{k+1} \\
e_k & \mapsto & e_{k+1} & \mapsto & e_{k+2} \\
\vdots & & \vdots & & \\
e_{n-1} & \mapsto & e_n & \mapsto & e_{n+1}
\end{array}.
$$

*Step 4: Construction of chain maps.* Let $X, Y \in \mathrm{ob}(\mathsf{Top})$ and $f \in \mathsf{Top}(X, Y)$. For $n \ge 0$, define $f_n^\# : \mathsf{Top}(\Delta^n, X) \to U\left(C_n(Y)\right)$ by $f^\# := f \circ \sigma$. Extending this map by linearity yields a homomorphism $f_n^\# : C_n(X) \to C_n(Y)$. Moreover, set $f_n^\# := 0$ for $n < 0$. Let $n \ge 1$ and $\sigma \in \mathsf{Top}(\Delta^n, X)$. Then on one hand we have

$$
(f_{n-1}^\# \circ \partial_n)(\sigma) = f_{n-1}^\# \left(\sum_{k=0}^{n} (-1)^k \sigma \circ \varphi_k^n\right) = \sum_{k=0}^{n} (-1)^k f \circ \sigma \circ \varphi_k^n
$$

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and on the other

$$(\partial_n \circ f_n^\#)(\sigma) = \partial_n(f \circ \sigma) = \sum_{k=0}^{n} (-1)^k f \circ \sigma \circ \varphi_k^n.$$

Checking, that $C_\bullet$ is indeed a functor is left as an exercise. □

**Exercise 3.2.** Show that $C_\bullet : \mathsf{Top} \to \mathsf{Comp}$ is a functor.

**The Homology Functor.**

**Proposition 3.3.** *For each $n \in \mathbb{Z}$ there exists a functor* $\mathsf{Comp} \to \mathsf{AbGrp}$.

*Proof.* Let $(C_\bullet, \partial_\bullet)$ be a chain complex. Let $x \in \operatorname{im} \partial_{n+1}$. Hence there exists $y \in C_{n+1}$ such that $x = \partial_{n+1} y$. But then $\partial_n x = (\partial_n \circ \partial_{n+1})(y) = 0$ and thus $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$. Define

$$H_n(C_\bullet, \partial_\bullet) := \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}} \in \operatorname{ob}(\mathsf{AbGrp}).$$

Let $(C'_\bullet, \partial'_\bullet)$ be a chain complex and $f_\bullet : C_\bullet \to C'_\bullet$ a chain map. Then $f_n(\ker \partial_n) \subseteq \ker \partial'_n$. Indeed, if $y \in f_n(\ker \partial_n)$, there exists $x \in \ker \partial_n$, such that $y = f_n(x)$. Since $f_\bullet$ is a chain map, we thus have $\partial'_n y = (\partial'_n \circ f_n)(x) = (f_{n-1} \circ \partial_n)(x) = 0$. Moreover, we have that $\operatorname{im} \partial_{n+1} \subseteq \ker \pi'_n \circ f_n$, where $\pi'_n : \ker \partial'_n \to H_n(C'_\bullet, \partial'_\bullet)$ is the usual projection. Indeed, if $y \in \operatorname{im} \partial_{n+1}$, we find $x \in C_{n+1}$, such that $y = \partial_{n+1} x$. Since again $f_\bullet$ is a chain map, we have that $f_n y = (f_n \circ \partial_{n+1})(x) = (\partial'_{n+1} \circ f_{n+1})(x) \in \operatorname{im} \partial'_{n+1} = \ker \pi'_n$. Hence $\pi'_n \circ f_n$ factors uniquely through $\pi_n : \ker \partial_n \to H_n(C_\bullet, \partial_\bullet)$. Define $H_n(f_\bullet)$ to be this map. □

**Remark 3.1.** Let $(C_\bullet, \partial_\bullet)$ be a chain complex and $n \in \mathbb{Z}$. Then we will write $\langle x \rangle$ for an element in $H_n(C_\bullet, \partial_\bullet)$, the so-called *homology class*. Hence if $(C'_\bullet, \partial'_\bullet)$ is another chain complex and $f_\bullet : C_\bullet \to C'_\bullet$ a chain map, then $H_n(f)\langle x \rangle = \langle f_n x \rangle$.

**Definition 3.5 (Cycles and Boundaries).** *Let $(C_\bullet, \partial_\bullet)$ be a chain complex and $n \in \mathbb{Z}$. Then elements of* $\ker \partial_n$ *are called **n-cycles** and elements of* $\operatorname{im} \partial_{n+1}$ *are called **n-boundaries**.*

**Definition 3.6 (Homology Functor).** *Let $n \in \mathbb{Z}$ and $H_n : \mathsf{Comp} \to \mathsf{AbGrp}$ be the functor defined in proposition 3.3. We call $H_n$ the **n-th homology functor**.*

**Definition 3.7 (Singular Homology Functor).** *Let $n \in \mathbb{Z}$. The composition*

$$H_n \circ C_\bullet : \mathsf{Top} \to \mathsf{AbGrp} \tag{12}$$

*of the singular chain complex functor $C_\bullet$ in theorem 3.1 and the $n$-th homology functor of proposition 3.3 is called the **singular homology functor**, written $H_n^{\text{sing}}$.*

**Remark 3.2.** For notational purposes we will often refer to the functor $H_n^{\text{sing}}$ simply as $H_n$.

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**First Properties of Singular Homolgy.**

**Proposition 3.4 (Zeroth Singular Homology Group).** *Let $X \in \mathrm{ob}(\mathsf{Top})$ be non empty and path connected. Then $H_0(X) \cong \mathbb{Z}$.*

*Proof.* Since $\partial_0 : C_0 \to 0$, $\ker \partial_0 = C_0$. Moreover, a map in $\mathsf{Top}(\Delta^0, X)$ can be identified with a point in $X$ and hence an element of $C_0$ can be written as $\sum_{x \in X} m_x x$. Define a mapping $\Phi : C_0 \to \mathbb{Z}$ by $\Phi\left(\sum_{x\in X} m_x x\right) := \sum_{x \in X} m_x$. This mapping is well defined since all but finitely many $m_x$ are zero. It is also easy to check, that $\Phi$ is a morphism of groups and that $\Phi$ is surjective. We claim that $\ker \Phi = \operatorname{im} \partial_1$. Indeed, if $\sum_{x\in X} m_x x \in \ker \Phi$, then $\sum_{x \in X} m_x = 0$. Let $p \in X$. Since $X$ is path connected, we find for each $x \in X$ a path $u_x$ from $p$ to $x$. Consider the singular 1-chain $\sum_{x\in X} m_x u_x'$. Then we have

$$\partial_1\left(\sum_{x\in X} m_x u_x'\right) = \sum_{x\in X} m_x(u_x' \circ \varphi_0^1 - u_x' \circ \varphi_1^1) = \sum_{x \in X} m_x(x-p) = \sum_{x\in X} m_x x.$$

Hence $\sum_{x\in X} m_x x \in \operatorname{im}\partial_1$. Conversely, it is enough to show the claim on basis elements $\sigma \in \mathsf{Top}(\Delta^1, X)$. We have

$$\Phi(\partial_1 \sigma) = \Phi(\sigma \circ \varphi_0^1 - \sigma \circ \varphi_1^1) = \Phi\left(\sigma(e_1) - \sigma(e_0)\right) = 1 - 1 = 0.$$

Hence the first isomorphism theorem [Gri07, p. 23] implies that $H_0(X) \cong \mathbb{Z}$. □

**Proposition 3.5 (The Dimension Axiom).** *Let $* \in \mathrm{ob}(\mathsf{Top})$ be a one point space. Then $H_n(*) = 0$ for all $n \in \mathbb{Z}$, $n > 0$.*

## The Homotopy Axiom

**Theorem 3.2 (The Homotopy Axiom).** *Let $f, g \in \mathsf{Top}(X, Y)$ be freely homotopic. Then $H_n(f) = H_n(g)$ for all $n \in \mathbb{Z}$.*

## Applications

**The Brouwer Fixed Point Theorem.**

**Definition 3.8 (Retract).** *Let $X \in \mathrm{ob}(\mathsf{Top})$ and $S \subseteq X$ a subspace. We say that* ***$S$ is a retract of $X$****, if the inclusion $\iota : S \hookrightarrow X$ admits a retraction in* $\mathsf{Top}$.

**Lemma 3.1.** *Let $n \in \mathbb{Z}$, $n \geq 1$. Then $\mathbb{S}^n$ is not a retract of $\mathbb{B}^{n+1}$.*

*Proof.*

□

**Theorem 3.3 (Brouwer Fixed Point Theorem).** *Let $n \in \mathbb{Z}$, $n \geq 1$. Then every mapping $f \in \mathsf{Top}(\mathbb{B}^n, \mathbb{B}^n)$ has a fixed point.*

*Proof.*

□

APPENDIX A

# Set Theory

## 1. Basic Concepts

**Problem A.1.** Let $n \in \mathbb{N}$ and $a_{kj} \in \mathbb{C}$ for $k = 0, \dots, n+1$, $j = 0, \dots, n$. Show that

$$\sum_{k=0}^{n+1} \sum_{j=0}^{n} a_{kj} = \sum_{0 \le k \le j \le n} a_{kj} + \sum_{0 \le j < k \le n+1} a_{kj}.$$

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