MAT602 - FUNCTIONAL ANALYSIS

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Contents

1. Linear Operators

1.1. Continuous Operators.

Definition 1.1. *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be two normed spaces. An* ***operator*** *is a linear mapping* $T : X \to Y$*. Moreover, we say that an operator* $T : X \to Y$ *is* ***bounded*** *if there exists* $c > 0$ *such that*

$$\|T(x)\|_Y \leq c\|x\|_X \tag{1}$$

holds for all $x \in X$*.*

1.2. The Hahn-Banach Theorem.

Lemma 1.1. *Let* V *be a real vector space,* $S \subsetneq V$ *a linear subspace,* $p : V \to \mathbb{R}$ *a sublinear functional,* $f : S \to \mathbb{R}$ *linear and* $x_0 \in V \setminus S$*. Moreover, assume that* $f \leq p$ *on* S*. Then there exists* $F : S + \mathbb{R}x_0 \to \mathbb{R}$ *linear such that* $F \leq p$ *on* $S + \mathbb{R}x_0$ *and* $F|_S = f$*.*

Theorem 1.1 (Hahn-Banach, $\mathbb{R}$). *Let* V *be a vector space over* $\mathbb{R}$*,* $S \subseteq V$ *a linear subspace and* $f : S \to \mathbb{R}$ *linear. Moreover, let* $p : V \to \mathbb{R}$ *be a sublinear functional such that* $f \leq p$ *on* S*. Then there exists* $F : V \to \mathbb{R}$ *linear such that* $F \leq p$ *on* V *and* $F|_S = f$*.*

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Theorem 1.2 (Hahn-Banach, $\mathbb{R}$ or $\mathbb{C}$). *Let V be a vector space over $\mathbb{K}$, $q : V \to \mathbb{R}$ a seminorm, $S \subseteq V$ a linear subspace and $f : S \to \mathbb{K}$ linear with $|f| \leq q$ on S. Then there exists $F : V \to \mathbb{K}$ linear with $F|_S = f$ and $|F| \leq q$ on V.*

Corollary 1.1 (Extension). *Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K}$, $S \subseteq X$ a linear subspace and $f \in S^*$. Then there exists $F \in X^*$ such that $F|_S = f$ and $\|F\|_{X^*} = \|f\|_{S^*}$.*

Corollary 1.2 (Separation). *Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{K}$ and $x_0 \in X \setminus \{0\}$. Then there exists $f \in X^*$ with $\|f\| = 1$ and $f(x_0) = \|x_0\|$.*

1.3. Reflexivity.

Proposition 1.1. *Let X be a normed vector space over $\mathbb{K}$. Then the mapping $\Phi : X \to X^{**}$ defined by $\Phi(x) := \varphi_x$, where $\varphi_x : X^* \to \mathbb{R}$ is defined by $\varphi_x(f) := f(x)$, is a linear isometry.*

Theorem 1.3. *Let X be a Banach space. Then X is reflexive if and only if X^* is reflexive.*

1.4. Hilbert Space Methods.

Theorem 1.4 (Riesz's Representation Theorem). *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$. The mapping $\Psi : H \to H^*$ defined by $\big(\Psi(x)\big)(y) := \langle x, y\rangle$ is an anti-linear isometric isomorphism.*

Corollary 1.3. *Every Hilbert space is reflexive.*

Theorem 1.5 (Lax-Milgram). *Let H be a Hilbert space over $\mathbb{K}$ and let $a : H \times H \to \mathbb{K}$ be a sesquilinear form. Moreover, suppose that there are constants $0 < c_0 \leq C_0 < \infty$ such that*

$$\begin{aligned} |a(x,y)| &\leq C_0 \|x\|\|y\| \quad (\textit{\textbf{Continuity}}), \\ \operatorname{Re} a(x,x) &\geq c_0 \|x\|^2 \quad (\textit{\textbf{Coercivity}}), \end{aligned}$$

for all $x, y \in H$. Then there exists a unique $A \in \mathcal{L}(H)$ such that

$$a(x,y) = \langle Ax, y\rangle \tag{2}$$

for all $x, y \in H$. Moreover, A is invertible with

$$\|A\| \leq C_0 \qquad \textit{and} \qquad \|A^{-1}\| \leq \frac{1}{c_0}. \tag{3}$$

2. Baire Category Theorem

2.1. Baire Category Theorem and Banach-Steinhaus.

Theorem 2.1 (Baire Category Theorem). *Every complete metric space is a Baire space.*

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Theorem 2.2 (Banach-Steinhaus). *Let X be a Banach space, Y a normed space and $\mathcal{F} \subseteq \mathcal{L}(X, Y)$. Assume that for all $x \in X$ there exists $c_x \geq 0$ such that*

$$\sup_{T \in \mathcal{F}} \|T(x)\| \leq c_x. \tag{4}$$

Then there exists $c \geq 0$ with

$$\sup_{T \in \mathcal{F}} \|T\| \leq c. \tag{5}$$