ADDITIVE AND ABELIAN CATEGORIES

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Abstract.

Contents

1. Preadditive Catgeories

Let $G, H \in \mathrm{ob}(\mathsf{AbGrp})$ and $\varphi, \psi \in \mathsf{AbGrp}(G, H)$. Define $\varphi + \psi$ pointwise. Since H is abelian, it follows that $\varphi + \psi \in \mathsf{AbGrp}(G, H)$. Moreover, it is easy to check, that with this operation defined above, $\mathsf{AbGrp}(G, H)$ is an abelian group and

$$\circ : \mathsf{AbGrp}(H, K) \times \mathsf{AbGrp}(G, H) \to \mathsf{AbGrp}(G, K)$$

is bilinear for each $K \in \mathrm{ob}(\mathsf{AbGrp})$. This motivates the following definition.

Definition 1.1 (Preadditive Category). *A **preadditive category** is a locally small category $\mathscr{C}$ in which all hom-sets $\mathscr{C}(X, Y)$ can be equipped with the structure of an abelian group and composition is bilinear, i.e. for all mophisms $f, f' : X \to Y$ and $g, g' : Y \to Z$ in $\mathscr{C}$ we have that*

$$(g + g') \circ (f + f') = g \circ f + g \circ f' + g' \circ f + g' \circ f'. \tag{1}$$

Examples 1.1. AbGrp, Vect_K, ${}_R\mathsf{Mod}$ and Mod_R.

2. Additive Categories

Let us again consider AbGrp. As in Grp, the trivial group 0 is both an initial and a terminal object. Unlike in Grp, we have that $G \coprod H \cong G \prod H$ for all $G, H \in \mathrm{ob}(\mathsf{AbGrp})$.

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In a somewhat weaker sense, we will generalize this. Define $\iota_1 : G \to G \prod H$ and $\iota_2 : H \to G \prod H$ by

$$\iota_1(g) := (g, 0) \qquad \text{and} \qquad \iota_2(h) := (0, h),$$

respectively. Then it is easy to verify that

$$\pi_1 \circ \iota_1 = \mathrm{id}_G, \quad \pi_2 \circ \iota_2 = \mathrm{id}_H \quad \text{and} \quad \iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = \mathrm{id}_{G \prod H}$$

holds. Those observations motivate the following definitions.

Definition 2.1 (Null Object). *Let $\mathscr{C}$ be a category. A **null object in $\mathscr{C}$** is a an object of $\mathscr{C}$ which is both initial and terminal.*

Definition 2.2 (Biproduct Diagram). *Let $\mathscr{C}$ be a preadditive category and $X, Y \in \mathrm{ob}(\mathscr{C})$. A **biproduct diagram for X and Y** is a diagram*

$$\begin{tikzcd} X \arrow[r, shift right, "\iota_1"'] & Z \arrow[l, shift right, "\pi_1"'] \arrow[r, shift left, "\pi_2"] & Y \arrow[l, shift left, "\iota_2"] \end{tikzcd}$$

such that

$$\pi_1 \circ \iota_1 = \mathrm{id}_X, \quad \pi_2 \circ \iota_2 = \mathrm{id}_Y \quad \textit{and} \quad \iota_1 \circ \pi_1 + \iota_2 \circ \pi_2 = \mathrm{id}_Z$$

holds.

Definition 2.3 (Additive Category). *An **additive category** is a preadditive category which has a null object and a biproduct for each pair of its objects.*

Examples 2.1. AbGrp, Vect_K, ${}_R\mathsf{Mod}$ and Mod_R.

3. Abelian Categories

Definition 3.1 (Zero Arrow). *Let $\mathscr{C}$ be a category with a null object 0. For $X, Y \in \mathrm{ob}(\mathscr{C})$, the unique composition $X \to 0 \to Y$ is called the **zero arrow from X to Y**, denoted by* $0_{X,Y}$.

Definition 3.2 (Kernel and Cokernel). *Let $\mathscr{C}$ be a category with a null object 0. A **kernel of a morphism $f : X \to Y$** is defined to be an equalizer of*

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "0_{X,Y}"'] & Y. \end{tikzcd}$$

*Dually, a **cokernel of a morphism $f : X \to Y$** is a coequalizer of the above diagram.*

Definition 3.3 (Abelian Category). *An **abelian category** is an additive category satisfying the following additional conditions:*

(*a*) *Every morphism admits a kernel.*
(*b*) *Every morphism admits a cokernel.*
(*c*) *Every monic is a kernel.*

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(*d*) *Every epic is a cokernel.*

Examples 3.1. AbGrp, ${}_R\mathsf{Mod}$ and Mod_R.