Yannis Bähni

An Introduction to General and Algebraic Topology

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CHAPTER 1

Constructions

1. Limits

Definition 1.1 (Diagram). *Let $\mathcal{C}$ be a category and A a small category. A functor $\mathsf{A} \to \mathcal{C}$ is called a **diagram in $\mathcal{C}$ of shape A**.*

Definition 1.2 (Cone and Limit). *Let $\mathcal{C}$ be a category and $D : \mathsf{A} \to \mathcal{C}$ a diagram in $\mathcal{C}$ of shape A. A **cone on D** is a tuple $\big(C, (f_\alpha)_{\alpha \in \mathsf{A}}\big)$, where $C \in \mathcal{C}$ is an object, called the **vertex** of the cone, and a family of arrows in $\mathcal{C}$*

$$\big(C \xrightarrow{f_\alpha} D(\alpha)\big)_{\alpha \in \mathsf{A}} . \tag{1}$$

such that for all morphisms $f \in \mathsf{A}$, $f : \alpha \to \beta$, the triangle

$$\begin{tikzcd}
 & D(\alpha) \arrow[dd, "D(f)"] \\
C \arrow[ur, "f_\alpha"] \arrow[dr, "f_\beta"'] & \\
 & D(\beta)
\end{tikzcd}$$

*commutes. A **(small) limit of D** is a cone $\big(L, (\pi_\alpha)_{\alpha \in \mathsf{A}}\big)$ with the property that for any other cone $\big(C, (f_\alpha)_{\alpha \in \mathsf{A}}\big)$ there exists a unique morphism $\overline{f} : A \to L$ such that $\pi_\alpha \circ \overline{f} = f_\alpha$ holds for every $\alpha \in \mathsf{A}$.*

Remark 1.1. In the setting of definition 1.2, if $\big(L, (\pi_\alpha)_{\alpha \in \mathsf{A}}\big)$ is a limit of D, we sometimes reffering to L only as the limit of D and we write

$$L = \lim_{\leftarrow \mathsf{A}} D. \tag{2}$$

Definition 1.3 (Product). *Let $\mathcal{C}$ be a category and A a set. Define A to be the discrete category with $\mathsf{ob}(\mathsf{A}) := A$. Moreover, let D be a diagram in $\mathcal{C}$ of shape A. A **(small) product of D** is a limit of D.*

Remark 1.2. In the setting of definition 1.3, D yields a family $(X_\alpha)_{\alpha \in A}$ in $\mathcal{C}$ and thus we we speak also of a product of the family $(X_\alpha)_{\alpha \in A}$. If a product exists in $\mathcal{C}$, we write

$$\prod_{\alpha \in A} X_\alpha := \lim_{\leftarrow \mathsf{A}} D. \tag{3}$$

CHAPTER 2

The Fundamental Group

1. Homotopies

Proposition 1.1. *Being homotopic is a congruence on* Top.

Proof. First we show that being homotopic induces an equivalence relation on

$$\bigcup_{(X,Y)\in \mathrm{ob}(\mathsf{Top})\times \mathrm{ob}(\mathsf{Top})} C(X,Y)$$

$\square$