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CHAPTER 1

Foundations

Basic Category Theory

Categories. We use the first order theory of Neumann-Bernays-Gödel (BNG) as described in [Men15, p. 231].

Definition 1.1 (Category). A *category* \mathcal{C} consists of

- A class $\text{ob}(\mathcal{C})$, called the *objects of* \mathcal{C} .
- A class $\text{mor}(\mathcal{C})$, called the *morphisms of* \mathcal{C} .
- Two functions $\text{dom} : \text{mor}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{C})$ and $\text{cod} : \text{mor}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{C})$, which assign to each morphism f in \mathcal{C} its *domain* and *codomain*, respectively.
- For each $X \in \text{ob}(\mathcal{C})$ a function $\text{ob}(\mathcal{C}) \rightarrow \text{mor}(\mathcal{C})$ which assigns a morphism id_X such that $\text{dom id}_X = \text{cod id}_X = X$.
- A function

$$\circ : \{(g, f) \in \text{mor}(\mathcal{C}) \times \text{mor}(\mathcal{C}) : \text{dom } g = \text{cod } f\} \rightarrow \text{mor}(\mathcal{C}) \quad (1)$$

mapping (g, f) to $g \circ f$, called *composition*, such that $\text{dom}(g \circ f) = \text{dom } f$ and $\text{cod}(g \circ f) = \text{cod } g$.

Subject to the following axioms:

- **(Associativity Axiom)** For all $f, g, h \in \text{mor}(\mathcal{C})$ with $\text{dom } h = \text{cod } g$ and $\text{dom } g = \text{cod } f$, we have that

$$(h \circ g) \circ f = h \circ (g \circ f). \quad (2)$$

- **(Unit Axiom)** For all $f \in \text{mor}(\mathcal{C})$ with $\text{dom } f = X$ and $\text{cod } f = Y$ we have that

$$f = f \circ \text{id}_X = \text{id}_Y \circ f. \quad (3)$$

Remark 1.1. Let \mathcal{C} be a category. For $X, Y \in \text{ob}(\mathcal{C})$ we will abbreviate

$$\mathcal{C}(X, Y) := \{f \in \text{mor}(\mathcal{C}) : \text{dom } f = X \text{ and } \text{cod } f = Y\}.$$

Moreover, $f \in \mathcal{C}(X, Y)$ is depicted as

$$f : X \rightarrow Y. \quad (4)$$

Example 1.1. Let $*$ be a single, not nearer specified object. Consider as morphisms the class of all cardinal numbers and as composition cardinal addition. By [Hal12, pp. 112–113], cardinal addition is associative and \emptyset serves for the identity id_* .

Definition 1.2 (Locally Small, Hom-Set). A category \mathcal{C} is said to be **locally small** if for all $X, Y \in \mathcal{C}$, $\mathcal{C}(X, Y)$ is a set. If \mathcal{C} is locally small, $\mathcal{C}(X, Y)$ is called a **hom-set** for all $X, Y \in \mathcal{C}$.

Functors.

Definition 1.3 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A **functor** $F : \mathcal{C} \rightarrow \mathcal{D}$ is a pair of functions (F_1, F_2) , $F_1 : \text{ob}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{D})$, called the **object function** and $F_2 : \text{mor}(\mathcal{C}) \rightarrow \text{mor}(\mathcal{D})$, called the **morphism function**, such that for every morphism $f : X \rightarrow Y$ we have that $F_2(f) : F_1(X) \rightarrow F_1(Y)$ and (F_1, F_2) is subject to the following **compatibility conditions**:

- For all $X \in \text{ob}(\mathcal{C})$, $F_2(\text{id}_X) = \text{id}_{F_1(X)}$.
- For all $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ we have that $F_2(g \circ f) = F_2(g) \circ F_2(f)$.

Remark 1.2. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. It is convenient to denote the components F_1 and F_2 also with F .

Subcategories.

Definition 1.4 (Subcategory). Let \mathcal{C} be a category. A **subcategory** \mathcal{S} of \mathcal{C} consists of

- A subclass $\text{ob}(\mathcal{S}) \subseteq \text{ob}(\mathcal{C})$.
- A subclass $\text{mor}(\mathcal{S}) \subseteq \text{mor}(\mathcal{C})$.

Subject to the following conditions:

- For all $X \in \mathcal{S}$, $\text{id}_X \in \text{mor}(\mathcal{S})$.

Example 1.2 (Top_{*}). Define the objects of Top_* to be the class of all tuple (X, p) , where X is a topological space and $p \in X$. Moreover, given objects (X, p) and (Y, q) in Top_* , define $\text{Top}_*((X, p), (Y, q)) := \{f \in \text{Top}(X, Y) : f(p) = q\}$. It is easy to check that Top_* is a category, called the **category of pointed topological spaces**.

Limits.

Definition 1.5 (Diagram). Let \mathcal{C} be a category and \mathbf{A} a small category. A functor $\mathbf{A} \rightarrow \mathcal{C}$ is called a **diagram in \mathcal{C} of shape \mathbf{A}** .

Definition 1.6 (Cone and Limit). Let \mathcal{C} be a category and $D : \mathbf{A} \rightarrow \mathcal{C}$ a diagram in \mathcal{C} of shape \mathbf{A} . A **cone on D** is a tuple $(C, (f_\alpha)_{\alpha \in \mathbf{A}})$, where $C \in \mathcal{C}$ is an object, called the **vertex** of the cone, and a family of arrows in \mathcal{C}

$$(C \xrightarrow{f_\alpha} D(\alpha))_{\alpha \in \mathbf{A}}. \quad (5)$$

such that for all morphisms $f \in \mathbf{A}$, $f : \alpha \rightarrow \beta$, the triangle

$$\begin{array}{ccc} & D(\alpha) & \\ f_\alpha \nearrow & \downarrow D(f) & \\ C & & D(\beta) \\ f_\beta \searrow & & \end{array}$$

commutes. A **(small) limit of D** is a cone $(L, (\pi_\alpha)_{\alpha \in \mathbf{A}})$ with the property that for any other cone $(C, (f_\alpha)_{\alpha \in \mathbf{A}})$ there exists a unique morphism $\bar{f} : C \rightarrow L$ such that $\pi_\alpha \circ \bar{f} = f_\alpha$ holds for every $\alpha \in \mathbf{A}$.

Remark 1.3. In the setting of definition 1.6, if $(L, (\pi_\alpha)_{\alpha \in \mathbf{A}})$ is a limit of D , we sometimes referring to L only as the limit of D and we write

$$L = \lim_{\leftarrow \mathbf{A}} D. \quad (6)$$

Basic Algebra

The Isomorphism Theorems.

Basic Point-Set Topology

The Closed Map Lemma.

Lemma 1.1 (Closed Map Lemma). Let $X, Y \in \text{ob}(\text{Top})$ such that X is compact and Y is Hausdorff, and $f \in \text{Top}(X, Y)$. Then:

- (a) f is a closed map.
- (b) If f is injective, it is a topological embedding.
- (c) If f is surjective, it is a quotient map.
- (d) If f is bijective, it is a homeomorphism.

CHAPTER 2

The Fundamental Group

The Fundamental Grupoid

Construction of the fundamental Grupoid.

Lemma 2.1 (Gluing Lemma). *Let $X, Y \in \text{ob}(\text{Top})$, $(X_\alpha)_{\alpha \in A}$ a finite closed cover of X and $(f_\alpha)_{\alpha \in A}$ a finite family of maps $f_\alpha \in \text{Top}(X_\alpha, Y)$ such that $f_\alpha|_{X_\alpha \cap X_\beta} = f_\beta|_{X_\alpha \cap X_\beta}$ for all $\alpha, \beta \in A$. Then there exists a unique $f \in \text{Top}(X, Y)$ such that $f|_{X_\alpha} = f_\alpha$ for all $\alpha \in A$.*

Proof. Let $x \in X$. Since $(X_\alpha)_{\alpha \in A}$ is a cover of X , we find $\alpha \in A$ such that $x \in X_\alpha$. Define $f(x) := f_\alpha(x)$. This is well defined, since if $x \in X_\alpha \cap X_\beta$ for some $\beta \in A$, we have that $f(x) = f_\beta(x) = f_\alpha(x)$. Clearly $f|_{X_\alpha} = f_\alpha$ for all $\alpha \in A$ and f is unique. Let us show continuity. To this end, let $K \subseteq Y$ be closed. Then

$$\begin{aligned} f^{-1}(K) &= X \cap f^{-1}(K) \\ &= \bigcup_{\alpha \in A} X_\alpha \cap f^{-1}(K) \\ &= \bigcup_{\alpha \in A} (X_\alpha \cap f^{-1}(K)) \\ &= \bigcup_{\alpha \in A} (X_\alpha \cap f_\alpha^{-1}(K)). \end{aligned}$$

Since each f_α is continuous, $f_\alpha^{-1}(K)$ is closed in X_α for each $\alpha \in A$ and thus since X_α is closed, $f^{-1}(K)$ is closed as a finite union of closed sets. \square

Theorem 2.1. *There is a functor $\text{Top} \rightarrow \text{Grpd}$.*

Proof. The proof is divided into several steps. Let us denote $\Pi : \text{Top} \rightarrow \text{Grpd}$ for the claimed functor.

Step 1: Definition of Π on objects. Let $X, Y \in \text{ob}(\text{Top})$, $f, g \in \text{Top}(X, Y)$ and $A \subseteq X$. A map $F \in \text{Top}(X \times I, Y)$ is called a **homotopy from X to Y relative to A** , if

- $F(x, 0) = f(x)$, for all $x \in X$.
- $F(x, 1) = g(x)$, for all $x \in X$.
- $F(x, t) = f(x) = g(x)$, for all $x \in A$ and for all $t \in I$.

If there exists a homotopy between f and g relative to A we say that f and g are **homotopic relative to A** and write $f \simeq_A g$. If we want to emphasize the homotopy relative to A , we write $F : f \simeq_A g$.

Lemma 2.2. *Let $X, Y \in \text{ob}(\text{Top})$ and $A \subseteq X$. Then being homotopic relative to A is an equivalence relation on $\text{Top}(X, Y)$.*

Proof. Define a binary relation $R_A \subseteq \text{Top}(X, Y) \times \text{Top}(X, Y)$ by

$$f R_A g \quad :\Leftrightarrow \quad f \simeq_A g.$$

Let $f \in \text{Top}(X, Y)$. Define $F \in \text{Top}(X \times I, Y)$ by

$$F(x, t) := f(x).$$

Then clearly $F : f \simeq_A f$. Hence R_A is reflexive.

Let $g \in \text{Top}(X, Y)$ and assume that $f R_A g$. Thus $G : f \simeq_A g$. Define $F \in \text{Top}(X \times I, Y)$ by

$$F(x, t) := G(x, 1 - t).$$

Then it is easy to check that $F : g \simeq_A f$ and so R_A is symmetric.

Finally, let $h \in \text{Top}(X, Y)$ and suppose that $f R_A g$ and $g R_A h$. Hence $F_1 : f \simeq_A g$ and $F_2 : g \simeq_A h$. Define $F \in \text{Top}(X \times I, Y)$ by

$$F(x, t) := \begin{cases} F_1(x, 2t) & 0 \leq t \leq \frac{1}{2}, \\ F_2(x, 2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Continuity of F follows by an application of the gluing lemma 2.1. Then it is easy to check that $F : f \simeq_A h$ and hence R_A is transitive. \square

Let $X \in \text{ob}(\text{Top})$ and u a path in X from p to q . Define the **path class $[u]$ of u** by $[u] := [u]_{R_{\partial I}}$. Define now

- $\text{ob}(\Pi(X)) := X$.
- $\Pi(X)(p, q) := \{[u] : u \text{ is a path from } p \text{ to } q\}$ for all $p, q \in X$.
- Let $p \in X$. Then define $\text{id}_p \in \Pi(X)(p, p)$ by $\text{id}_p := [c_p]$, where c_p is the constant path defined by $c_p(s) := p$ for all $s \in I$.
- And $\Pi(X)(q, r) \times \Pi(X)(p, q) \rightarrow \Pi(X)(p, r)$ by

$$([v], [u]) \mapsto [u * v]$$

Where $u * v \in \text{Top}(p, r)$ is the **concatenated path of u and v** , defined by

$$(u * v)(s) := \begin{cases} u(2s) & 0 \leq s \leq \frac{1}{2}, \\ v(2s - 1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Continuity follows again from the gluing lemma 2.1 whereas well definedness follows from the next lemma.

Lemma 2.3. Suppose that $[u_1], [u_2] \in \Pi(X)(p, q)$ and $[v_1], [v_2] \in \Pi(X)(q, r)$ such that $[u_1] = [u_2]$ and $[v_1] = [v_2]$. Then $[u_1 * v_1] = [u_2 * v_2]$.

Proof. By assumption we have $G : u_1 \simeq_{\partial I} u_2$ and $H : v_1 \simeq_{\partial I} v_2$. Define $F \in \text{Top}(I \times I, X)$ by

$$F(s, t) := \begin{cases} G(2s, t) & 0 \leq s \leq \frac{1}{2}, \\ H(2s - 1, t) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Again, continuity follows from the gluing lemma 2.1 and it is easy to check that $F : u_1 * v_1 \simeq_{\partial I} u_2 * v_2$. \square

Let us now check that $\Pi(X)$ is indeed a category. Let $[u] \in \Pi(X)(p, q)$. We want to show that $u \simeq_{\partial I} c_p * u$. To this end, we consider figure 1a and conclude that a suitable homotopy is given by $F \in \text{Top}(I \times I, X)$ defined by

$$F(s, t) := \begin{cases} p & 0 \leq 2s \leq t, \\ u\left(\frac{2s - t}{2 - t}\right) & t \leq 2s \leq 2. \end{cases}$$

Similarly, considering figure 1b leads to $F \in \text{Top}(I \times I, X)$ defined by

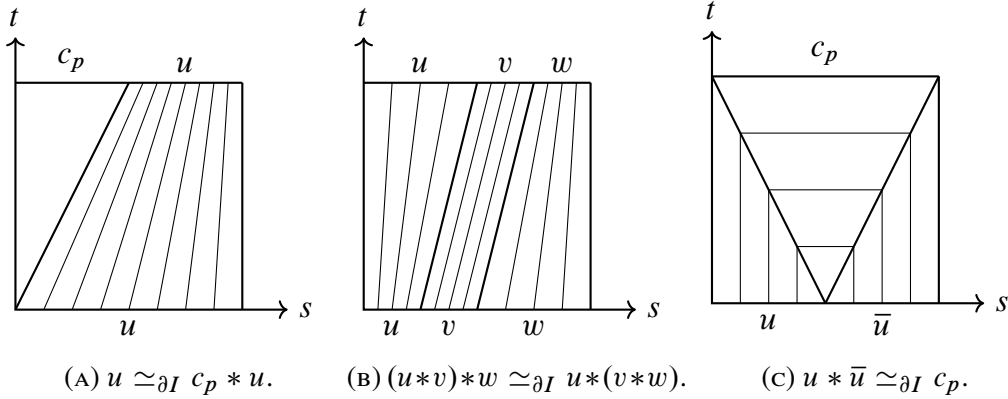


FIGURE 1. Visualization of the proof that $\Pi(X)$ is a grupoid object.

$$F(s, t) := \begin{cases} u\left(\frac{4s}{t+1}\right) & -1 \leq 4s - 1 \leq t, \\ v(4s - t - 1) & t \leq 4s - 1 \leq t + 1, \\ w\left(\frac{4s - t - 2}{4 - t - 2}\right) & t + 1 \leq 4s - 1 \leq 3. \end{cases}$$

Lastly, we check that $\Pi(X)$ is a grupoid. To this end, for a path u from p to q , define its **reverse path** \bar{u} by

$$\bar{u}(s) := u(1 - s).$$

We claim that $u * \bar{u} \simeq_{\partial I} c_p$. From figure 1c we deduce that $F \in \text{Top}(I \times I, X)$ is given by

$$F(s, t) := \begin{cases} u(2s) & 0 \leq 2s \leq 1 - t, \\ u(1 - t) & 1 - t \leq 2s \leq t + 1, \\ \bar{u}(2s - 1) & t + 1 \leq 2s \leq 2. \end{cases}$$

Step 2: Definition of Π on morphisms. Let $f \in \text{Top}(X, Y)$. Then $\Pi(f)$ is a functor from $\Pi(X)$ to $\Pi(Y)$. Define $\Pi(f)$ as follows:

- Let $p \in \text{ob}(\Pi(X))$. Then define $\Pi(f)(p) := f(p) \in \text{ob}(\Pi(Y))$.
- Let $[u] \in \Pi(X)(p, q)$. Then define $\Pi(f)[u] := [f \circ u] \in \Pi(Y)(f(p), f(q))$. We have to check that this definition is independent of the choice of the representative.

Lemma 2.4. *Let u and v be paths from p to q in X and suppose that $[u] = [v]$. Then for any $f \in \text{Top}(X, Y)$ we also have that $[f \circ u] = [f \circ v]$.*

Proof. Suppose that $H : u \simeq_{\partial I} v$. Define $F \in \text{Top}(I \times I, Y)$ by

$$F(s, t) := (f \circ H)(s, t).$$

Then $F : f \circ u \simeq_{\partial I} f \circ v$. □

Checking that Π satisfies the functorial properties is left as an exercise. □

Exercise 2.1. Check that $\Pi : \text{Top} \rightarrow \text{Grpd}$ is indeed a functor.

The Fundamental Group.

Lemma 2.5. *Let \mathcal{G} be a locally small grupoid. Then for every $X \in \text{ob}(\mathcal{G})$, $\mathcal{G}(X, X)$ can be equipped with a group structure.*

Proof. Since \mathcal{G} is locally small, $\mathcal{G}(X, X)$ is a set for every $X \in \text{ob}(\mathcal{G})$. Define a multiplication $\mathcal{G}(X, X) \times \mathcal{G}(X, X) \rightarrow \mathcal{G}(X, X)$ by $gh := h \circ g$. Clearly, this multiplication is associative. Moreover, the identity element is given by $\text{id}_X \in \mathcal{G}(X, X)$ and since every $g \in \mathcal{G}(X, X)$ is an isomorphism, the multiplicative inverse is given by the inverse in $\mathcal{G}(X, X)$. □

Proposition 2.1. *There is a functor $\text{Top}_* \rightarrow \text{Grp}$.*

Proof. Define $\pi_1 : \text{Top}_* \rightarrow \text{Grp}$ on objects $(X, p) \in \text{Top}_*$ by

$$\pi_1(X, p) := \Pi(X)(p, p).$$

By theorem 2.1 together with lemma 2.5, $\pi_1(X, p)$ is actually a group, called the **fundamental group of X with basepoint p** . On morphisms $f \in \text{Top}_*((X, p), (Y, q))$, define

$$\pi_1(f) := \Pi(f) : \Pi(X)(p, p) \rightarrow \Pi(Y)(q, q).$$

Let $[u], [v] \in \pi_1(X, p)$. Then

$$\begin{aligned} \pi_1([u] [v]) &= \Pi(f)([u] [v]) \\ &= \Pi(f) [u * v] \\ &= [f \circ (u * v)] \\ &= [(f \circ u) * (f \circ v)] \\ &= \Pi(f) [u] \Pi(f) [v] \\ &= \pi_1(f) [u] \pi_1(f) [v]. \end{aligned}$$

Thus $\pi_1(f)$ is a morphism in Grp. Functoriality of π_1 immediately follows from the functoriality of Π . \square

Lemma 2.6. *Let $X \in \text{ob}(\text{Top})$, $p \in X$ and A be the path component of X containing p . Then $\pi_1(\iota)$, where $\iota : A \hookrightarrow X$ denotes the inclusion, is an isomorphism.*

Proof. Suppose $[u] \in \ker \pi_1(\iota)$. Then $[\iota \circ u] = [c_p]$ and Hence $F : \iota \circ u \simeq_{\partial I} c_p$. Since $I \times I$ is path connected and $p \in F(I \times I)$, it follows that $F(I \times I) \subseteq A$ and thus $F : u \simeq_{\partial I} c_p$ in A and hence $[u] = [c_p]$. To see that $\pi_1(\iota)$ is surjective, just observe that $u(I) \subseteq A$ for $[u] \in \pi_1(X, p)$ since $u(I)$ is path connected and $p \in u(I)$. \square

Lemma 2.7. *Let $X \in \text{ob}(\text{Top})$ be path connected and $p, q \in X$. Then*

$$\pi_1(X, p) \cong \pi_1(X, q).$$

Proof. Since X is path connected we find a path v from p to q in X . Define a mapping $\Phi_v : \pi_1(X, p) \rightarrow \pi_1(X, q)$

$$\Phi_v [u] := [\bar{v} * u * v].$$

Clearly, Φ_v is invertible with inverse $\Phi_{\bar{v}}$. Moreover, for $[u], [w] \in \pi_1(X, p)$ we have that

$$\begin{aligned} \Phi_v([u] [w]) &= \Phi_v [u * w] \\ &= [\bar{v} * u * w * v] \\ &= [\bar{v} * u * v * \bar{v} * w * v] \\ &= [\bar{v} * u * v] [\bar{v} * w * v] \\ &= \Phi_v [u] \Phi_v [w]. \end{aligned}$$

\square

$\pi_1(\mathbb{S}^1)$.

Definition 2.1 (Exponential Quotient Map and Fundamental Loop). *The mapping $\varepsilon : \mathbb{R} \rightarrow \mathbb{S}^1$ defined by*

$$\varepsilon(x) := e^{2\pi i x} \quad (7)$$

*is called the **exponential quotient map**. Moreover, the **fundamental loop** ω is defined to be the restriction $\omega := \varepsilon|_I$.*

Proposition 2.2 (Lifting Property of the Circle). *Let $n \in \mathbb{Z}$, $n \geq 0$, $X \subseteq \mathbb{R}^n$ compact and convex, $p \in X$, $f \in \text{Top}_*((X, p), (\mathbb{S}^1, 1))$ and $m \in \mathbb{Z}$. Then there exists a unique map $\tilde{f} \in \text{Top}_*((X, p), (\mathbb{R}, m))$, called the **lifting of f** , such that*

$$\begin{array}{ccc} & & (\mathbb{R}, m) \\ & \nearrow \tilde{f} & \downarrow \varepsilon \\ (X, p) & \xrightarrow{f} & (\mathbb{S}^1, 1) \end{array}$$

commutes.

Proof. We show first existence and then uniqueness.

Step 1: Existence. Since X is compact and f is continuous, f is uniformly continuous on X . Thus we find $\delta > 0$ such that $|f(x) - f(y)| < 2$, whenever $|x - y| < \delta$, i.e. $f(x)$ and $f(y)$ are not antipodal points. Moreover, since X is compact, X is bounded and hence we find $N \in \mathbb{N}$, such that $|x - y| < N\delta$ holds for all $x, y \in X$. Let $x \in X$. For $0 \leq k \leq N$, define $L_k : X \rightarrow X$ by

$$L_k(x) := \left(1 - \frac{k}{N}\right)p + \frac{k}{N}x.$$

Those are well defined functions since X is convex. Moreover, each L_k is continuous. Indeed, it is easy to check that L_k is Lipschitz. Also, for each $0 \leq k < N$, $f(L_k(x))$ and $f(L_{k+1}(x))$ are not antipodal for all $x \in X$. Indeed, it is easy to check that $|L_k(x) - L_{k+1}(x)| < \delta$ holds for all $x \in X$. For $0 \leq k < N$ define $g_k : X \rightarrow \mathbb{S}^1 \setminus \{-1\}$ by

$$g_k(x) := \frac{f(L_{k+1}(x))}{f(L_k(x))}.$$

Clearly g_k is well defined and continuous as a composition of continuous functions. Let $\text{Log} : \mathbb{S}^1 \setminus \{-1\} \rightarrow \mathbb{C}$ denote the principal branch of the logarithm. Define $\tilde{f} : X \rightarrow \mathbb{R}$ by

$$\tilde{f}(x) := m + \frac{1}{2\pi i} \sum_{k=0}^{N-1} \text{Log}(g_k(x)).$$

Clearly, \tilde{f} is continuous and moreover we have that $\tilde{f} = m$ since $g_k(p) = 1$ for all $0 \leq k < N$. Finally, for any $x \in X$ we have that

$$(\varepsilon \circ \tilde{f})(x) = \varepsilon(m) \prod_{k=0}^{N-1} g_k(x) = \frac{f(L_N(x))}{f(L_0(x))} = \frac{f(x)}{f(p)} = f(x).$$

Step 2: Uniqueness. Suppose $\tilde{g} \in \text{Top}_*((X, p), (\mathbb{R}, m))$ is another such function. Define $\varphi \in \text{Top}_*((X, p), (\mathbb{R}, 0))$ by

$$\varphi(x) := \tilde{f}(x) - \tilde{g}(x).$$

Then clearly $\varepsilon \circ \varphi = 1$ and thus $\varphi(X) \subseteq \mathbb{Z}$. Since X is convex, X is connected and so $\varphi = 0$. □

Corollary 2.1. *Let $u, v \in \Omega(\mathbb{S}^1, 1)$ such that $[u] = [v]$. If $\tilde{u}, \tilde{v} : (I, 0) \rightarrow (\mathbb{R}, 0)$ are the liftings of u and v , respectively, then $[\tilde{u}] = [\tilde{v}]$.*

Proof. Let $F : u \simeq_{\partial I} v$. By proposition 2.2, we find $\tilde{F} \in \text{Top}_*((I \times I, (0, 0)), (\mathbb{R}, 0))$, such that $\varepsilon \circ \tilde{F} = F$. We claim that $\tilde{F} : \tilde{u} \simeq_{\partial I} \tilde{v}$. For $s \in I$ define $\tilde{u}_0(s) := \tilde{F}(s, 0)$. Then $\tilde{u}_0(0) = \tilde{F}(0, 0) = 0$ and since \tilde{u}_0 is continuous we have that $\tilde{u}_0 \in \text{Top}_*((I, 0), (\mathbb{R}, 0))$. Moreover

$$(\varepsilon \circ \tilde{u}_0)(s) = \varepsilon(\tilde{F}(s, 0)) = F(s, 0) = u(s)$$

for all $s \in I$ and thus \tilde{u}_0 is a lifting of u . But by proposition 2.2, liftings are unique and thus $\tilde{u}_0 = \tilde{u}$. Next define $\tilde{w}_0(t) := \tilde{F}(0, t)$ for all $t \in I$. Then $\tilde{w}_0(0) = \tilde{F}(0, 0) = 0$ and so $\tilde{w}_0 \in \text{Top}_*((I, 0), (\mathbb{R}, 0))$. Moreover

$$(\varepsilon \circ \tilde{w}_0)(t) = \varepsilon(\tilde{F}(0, t)) = F(0, t) = u(0) = v(0) = 1.$$

for all $t \in I$. Thus

$$\begin{array}{ccc} & & (\mathbb{R}, 0) \\ & \nearrow \tilde{w}_0 & \downarrow \varepsilon \\ (I, 0) & \xrightarrow{c_1} & (\mathbb{S}^1, 1) \end{array}$$

commutes. But also c_0 makes the above diagram commute. By uniqueness, $\tilde{w}_0 = c_0$. Define $\tilde{v}_0(s) := \tilde{F}(s, 1)$ for all $s \in I$. Then $\tilde{v}_0(0) = \tilde{F}(0, 1) = \tilde{w}_0(1) = 0$ and it is easy to check that \tilde{v}_0 is a lift for v . Hence $\tilde{v}_0 = \tilde{v}$. Finally, define $\tilde{w}_1(t) := \tilde{F}(1, t)$ for all $t \in I$. Then $\tilde{w}_1(0) = \tilde{F}(1, 0) = \tilde{u}(1)$ and thus $\tilde{w}_1 \in \text{Top}_*((I, 0), (\mathbb{R}, \tilde{u}(0)))$. Moreover

$$(\varepsilon \circ \tilde{w}_1)(t) = \varepsilon(\tilde{F}(1, t)) = F(1, t) = v(1) = u(1) = 1$$

for all $t \in I$. By proposition 2.2, we have again that $\tilde{w}_1 = c_{\tilde{u}(1)}$. So $F : \tilde{u} \simeq_{\partial I} \tilde{v}$. □

Definition 2.2 (Degree). Let $u \in \Omega(\mathbb{S}^1, 1)$. The **degree of u** , written $\deg u$, is defined by $\deg u := \tilde{u}(1)$, where \tilde{u} is the unique lift of u such that $\tilde{u}(0) = 0$.

Theorem 2.2 (Fundamental Group of the Circle). $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.

Proof. Define $\deg : \pi_1(\mathbb{S}^1, 1) \rightarrow \mathbb{Z}$ by $\deg [u] := \deg u$. This is well defined by corollary 2.1, since if $[u] = [v]$, then $[\tilde{u}] = [\tilde{v}]$ and in particular $\tilde{u}(1) = \tilde{v}(1)$.

Step 1: $\deg \in \text{Grp}(\pi_1(\mathbb{S}^1, 1), (\mathbb{Z}, +))$. Let $[u], [v] \in \pi_1(\mathbb{S}^1, 1)$ and $m := \deg [u]$, $n := \deg [v]$. Moreover, let \tilde{u} and \tilde{v} denote the unique liftings of u and v , respectively, such that $\tilde{u}(0) = 0$ and $\tilde{v}(0) = 0$. Define

$$\tilde{w}(s) := \begin{cases} \tilde{u}(2s) & 0 \leq s \leq \frac{1}{2}, \\ m + \tilde{v}(2s - 1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Clearly \tilde{w} is continuous and $\tilde{w}(0) = 0$. Hence $\tilde{w} \in \text{Top}_*((I, 0), (\mathbb{R}, 0))$. Also we have that $\varepsilon \circ \tilde{w} = u * v$ and thus \tilde{w} is the lift of $u * v$. But $\tilde{w}(1) = m + n$ and so

$$\deg([u][v]) = \deg[u * v] = \deg(u * v) = \tilde{w}(1) = m + n = \deg [u] + \deg [v].$$

Step 2: \deg is injective. Suppose $\deg [u] = 0$. Then $\tilde{u}(1) = 0$ and thus $\tilde{u} \in \Omega(\mathbb{R}, 0)$. Since \mathbb{R} is contractible, we have that $[\tilde{u}] = [c_0]$ and thus

$$[u] = [\varepsilon \circ \tilde{u}] = \pi_1(\varepsilon) [\tilde{u}] = \pi_1(\varepsilon) [c_0] = [c_1].$$

Thus $\ker(\deg)$ is trivial.

Step 3: \deg is surjective. Let $m \in \mathbb{Z}$. Then

$$\deg [\varepsilon^m] = \deg \varepsilon^m = \tilde{\varepsilon}^m(1) = m.$$

□

The Seifert-Van Kampen Theorem

Coproducts and Pushouts in Grp.

Proposition 2.3 (Coproducts in Grp). Grp has all small coproducts.

Proof. Let $A \in \text{ob}(\text{Set})$ and \mathbf{A} be the small category defined as the discrete category with $\text{ob}(\mathbf{A}) := A$, i.e.

$$\bullet \quad \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \quad \bullet$$

Let $D : \mathbf{A} \rightarrow \text{Grp}$ be a functor. Hence we get a family $(G_\alpha)_{\alpha \in A}$ in Grp, where $G_\alpha := D(\alpha)$ for all $\alpha \in A$. A **word** in $(G_\alpha)_{\alpha \in A}$ is a finite sequence in $\coprod_{\alpha \in A} G_\alpha$. A word in $(G_\alpha)_{\alpha \in A}$ will simply be written as (g_1, \dots, g_n) , where $g_k \in G_\alpha$ for some $\alpha \in A$. The **empty word** is denoted by $()$. Let \mathcal{W} denote the set of all words in $(G_\alpha)_{\alpha \in A}$. On \mathcal{W} define a multiplication by **concatenation**

$$(g_1, \dots, g_n)(h_1, \dots, h_m) := (g_1, \dots, g_n, h_1, \dots, h_m).$$

An **elementary reduction** is an operation of one of the following forms:

- $(g_1, \dots, g_k, g_{k+1}, \dots, g_n) \mapsto (g_1, \dots, g_k g_{k+1}, \dots, g_n)$, where $g_k, g_{k+1} \in G_\alpha$ for some $\alpha \in A$.
- $(g_1, \dots, g_{k-1}, 1_\alpha, g_{k+1}, \dots, g_n) \mapsto (g_1, \dots, g_{k-1}, g_{k+1}, \dots, g_n)$.

Let \sim denote the equivalence relation on \mathcal{W} generated by elementary reductions.

Lemma 2.8. \mathcal{W}/\sim together with concatenation of representatives is an element of Grp.

Proof. Define

$$[(g_1, \dots, g_n)] [(h_1, \dots, h_m)] := [(g_1, \dots, g_n, h_1, \dots, h_m)].$$

It is left to the reader to show that this is well defined and that \mathcal{W}/\sim is indeed a group. \square

The group defined in lemma 2.8 will be denoted by $\bigstar_{\alpha \in A} G_\alpha$ and called the **free product of $(G_\alpha)_{\alpha \in A}$** . Let us define a cocone on D . For this consider the inclusions $\iota_\alpha : G_\alpha \rightarrow \bigstar_{\alpha \in A} G_\alpha$ defined by

$$\iota_\alpha(g) := [(g)]$$

for all $\alpha \in A$. It is immediate from

$$\iota_\alpha(gh) = [(gh)] = [(g, h)] = [(g)] [(h)] = \iota_\alpha(g) \iota_\alpha(h)$$

for $g, h \in G_\alpha$, that ι_α is a morphism of groups. Since there are only the identity morphisms in A , $(\bigstar_{\alpha \in A} G_\alpha, (\iota_\alpha)_{\alpha \in A})$ is a cocone on D . Let us show that this is in fact a universal cocone. To this end, suppose that $(C, (\varphi_\alpha)_{\alpha \in A})$ is another cocone on D . Define a mapping $\bar{f} : \bigstar_{\alpha \in A} G_\alpha \rightarrow C$ by

$$\bar{f} [(g_1, \dots, g_n)] := \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$$

where $g_k \in G_{\alpha_k}$. Then \bar{f} is easily seen to be well defined since each φ_α is a morphism of groups. Moreover, if $g \in G_\alpha$, then

$$(\bar{f} \circ \iota_\alpha)(g) = \bar{f} [(g)] = \varphi_\alpha(g)$$

for all $\alpha \in A$. Suppose that $f : \bigstar_{\alpha \in A} G_\alpha \rightarrow C$ is another homomorphism of groups such that $f \circ \iota_\alpha = \varphi_\alpha$ for all $\alpha \in A$. Then for $[(g_1, \dots, g_n)] \in \bigstar_{\alpha \in A} G_\alpha$ we have

$$\begin{aligned} f [(g_1, \dots, g_n)] &= f([(g_1)] \cdots [(g_n)]) \\ &= f [(g_1)] \cdots f [(g_n)] \\ &= f (\iota_{\alpha_1}(g_1)) \cdots f (\iota_{\alpha_n}(g_n)) \\ &= \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n) \\ &= \bar{f} [(g_1, \dots, g_n)]. \end{aligned}$$

\square

Exercise 2.2. Check that \mathcal{W}/\sim is indeed a group with the declared group structure and that \bar{f} is indeed well defined.

Proposition 2.4 (Pushouts in Grp). *Grp has all pushouts.*

Proof. Consider the diagram $D : \mathbf{A} \rightarrow \mathbf{Grp}$

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array} \quad \xrightarrow{D} \quad \begin{array}{ccc} G & \xrightarrow{\varphi_1} & H_1 \\ \varphi_2 \downarrow & & \downarrow \\ H_2 & & \end{array}$$

and define N to be the normal subgroup of $H_1 * H_2$ generated by elements of the form $[(\varphi_1(g^{-1}), \varphi_2(g))]$ for $g \in G$. Let $K := (H_1 * H_2)/N$. Then

$$\begin{array}{ccc} G & \xrightarrow{\varphi_1} & H_1 \\ \varphi_2 \downarrow & & \downarrow \pi \circ \iota_1 \\ H_2 & \xrightarrow{\pi \circ \iota_2} & K \end{array}$$

commutes. Indeed, if $g \in G$, we have that $(\pi \circ \iota_1 \circ \varphi_1)(g) = [(\varphi_1(g))]$ N and similarly $(\pi \circ \iota_2 \circ \varphi_2)(g) = [(\varphi_2(g))]$ N . Then

$$[(\varphi_1(g))^{-1}] [(\varphi_2(g))] = [(\varphi_1(g^{-1}))] [(\varphi_2(g))] = [(\varphi_1(g^{-1}))] [(\varphi_2(g))] \in N.$$

Suppose that we have another cocone on D :

$$\begin{array}{ccc} G & \xrightarrow{\varphi_1} & H_1 \\ \varphi_2 \downarrow & & \downarrow \psi_1 \\ H_2 & \xrightarrow{\psi_2} & C \end{array}$$

By proposition 2.3, there exists a unique morphism of groups $f : H_1 * H_2 \rightarrow C$ and we thus get the following diagram:

$$\begin{array}{ccccc} G & \xrightarrow{\varphi_1} & H_1 & & \\ \varphi_2 \downarrow & & \downarrow \iota_1 & & \searrow \psi_1 \\ H_2 & \xrightarrow{\iota_2} & H_1 * H_2 & \xrightarrow{\pi} & K \\ & & \searrow f & \dashrightarrow & \bar{f} \\ & & & & C \end{array}$$

ψ_2 (curved arrow from H_2 to C)

To show that $N \subseteq \ker f$ is left as an exercise. Hence by the factorization theorem (see [Gri07, p. 23]), f factors uniquely through π , i.e. there exists a unique morphism of groups $\bar{f} : K \rightarrow C$ such that $\bar{f} \circ \pi = f$. \square

Exercise 2.3. In the previous proposition, verify that $N \subseteq \ker f$.

Definition 2.3 (Amalgamated Free Product). *The pushout of a diagram*

$$\begin{array}{ccc} G & \xrightarrow{\varphi_1} & H_1 \\ \varphi_2 \downarrow & & \\ & & H_2 \end{array}$$

in \mathbf{Grp} is called the *amalgamated free product of H_1 and H_2 along $(G, \varphi_1, \varphi_2)$* , written $H_1 *_{(G, \varphi_1, \varphi_2)} H_2$.

The Seifert-Van Kampen Theorem and its Consequences.

Theorem 2.3 (Seifert-Van Kampen). *Let $X \in \mathbf{ob}(\mathbf{Top})$, (U, V) an open cover for X , such that U , V and $U \cap V$ are path connected. Moreover, let $p \in U \cap V$. Then*

$$\pi_1(X, p) \cong \pi_1(U, p) *_{(\pi_1(U \cap V, p), \pi_1(\iota_U), \pi_1(\iota_V))} \pi_1(V, p), \quad (8)$$

where $\iota_U : U \cap V \hookrightarrow U$ and $\iota_V : U \cap V \hookrightarrow V$ denote inclusion.

CHAPTER 3

Singular Homology

Construction of the Singular Homology Functor

Aim of this section is to construct for each $n \in \omega$ a functor $H_n : \mathbf{Top} \rightarrow \mathbf{AbGrp}$, called the n -th singular homology functor.

Free Abelian Groups.

Proposition 3.1. *The forgetful functor $U : \mathbf{AbGrp} \rightarrow \mathbf{Set}$ admits a left adjoint.*

Proof. We have to construct a functor $F : \mathbf{Set} \rightarrow \mathbf{AbGrp}$. Let S be a set. Define

$$F(S) := \{f \in \mathbb{Z}^S : \text{supp } f \text{ is finite}\}.$$

Equipped with pointwise addition, $F(S)$ is an abelian group. There is a natural inclusion $\iota : S \hookrightarrow U(F(S))$ sending $x \in S$ to the function taking the value one at x and zero else. Hence we may regard elements of $F(S)$ as formal linear combinations $\sum_{x \in S} m_x x$, where $m_x \in \mathbb{Z}$ for all $x \in S$. On morphisms $f : S \rightarrow T$ in \mathbf{Set} , define $F(f) : F(S) \rightarrow F(T)$ simply by setting $F(f)(\sum_{x \in S} m_x x) := \sum_{x \in S} m_x f(x)$.

Let $G \in \text{ob}(\mathbf{AbGrp})$ be an abelian group and $\varphi \in \mathbf{AbGrp}(F(S), G)$ a morphism of groups. Define $\bar{\varphi} \in \mathbf{Set}(S, U(G))$ by $\bar{\varphi} := U(\varphi)$. Conversely, if we have $f \in \mathbf{Set}(S, U(G))$, define $\bar{f} \in \mathbf{AbGrp}(F(S), G)$ by $\bar{f}(\sum_{x \in S} m_x x) := \sum_{x \in S} m_x f(x)$. This is well defined since all but finitely many m_x are zero and G is abelian. It is easy to check that \bar{f} is indeed a morphism of groups. Let $\varphi \in \mathbf{AbGrp}(F(S), G)$. Then

$$\begin{aligned} \bar{\varphi}\left(\sum_{x \in S} m_x x\right) &= \sum_{x \in S} m_x \bar{\varphi}(x) \\ &= \sum_{x \in S} m_x U(\varphi)(x) \\ &= \sum_{x \in S} m_x \varphi(x) \\ &= \varphi\left(\sum_{x \in S} m_x x\right). \end{aligned}$$

And for $f \in \text{Set}(S, U(G))$ we have that

$$\bar{\bar{f}}(x) = U(\bar{f})(x) = \bar{f}(x) = f(x).$$

Hence $\bar{\bar{\varphi}} = \varphi$ and $\bar{\bar{f}} = f$ and so we have a bijection

$$\text{AbGrp}(F(S), G) \cong \text{Set}(S, U(G)).$$

The mapping $f \mapsto \bar{f}$ will be referred to as **extending by linearity**. To check naturality in S and G is left as an exercise. \square

Exercise 3.1. In proposition 3.1, check that $F : \text{Set} \rightarrow \text{AbGrp}$ is indeed a functor, called the **free functor from Set to AbGrp**, and the naturality of the bijection in both arguments.

Definition 3.1 (Free Abelian Group). Let $F : \text{Set} \rightarrow \text{AbGrp}$ be the free functor. For any set S , we call $F(S)$ the **free group generated by S** .

Chain Complexes.

Definition 3.2 (Chain Complex). A **chain complex** is a tuple $(C_\bullet, \partial_\bullet)$ consisting of a sequence $(C_n)_{n \in \mathbb{Z}}$ in $\text{ob}(\text{AbGrp})$ and a sequence $(\partial_n)_{n \in \mathbb{Z}}$ in $\text{mor}(\text{AbGrp})$, called **boundary operators**, such that we have $\partial_n \in \text{AbGrp}(C_n, C_{n-1})$ and $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$.

Definition 3.3 (Chain Maps). Let $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ be two chain complexes. A **chain map** $f_\bullet : C_\bullet \rightarrow C'_\bullet$ is a sequence $(f_n)_{n \in \mathbb{Z}}$ in $\text{mor}(\text{AbGrp})$ such that $f_n \in \text{AbGrp}(C_n, C'_n)$ and the diagram

$$\begin{array}{ccc} C_n & \xrightarrow{\partial_n} & C_{n-1} \\ f_n \downarrow & & \downarrow f_{n-1} \\ C'_n & \xrightarrow{\partial'_n} & C'_{n-1} \end{array}$$

commutes for all $n \in \mathbb{Z}$.

Proposition 3.2. There is a category with objects chain complexes and morphisms chain maps.

Proof. Let $f_\bullet : C_\bullet \rightarrow C'_\bullet$ and $g_\bullet : C'_\bullet \rightarrow C''_\bullet$ be chain maps. Define a map $g_\bullet \circ f_\bullet$ by $g_n \circ f_n$ for each $n \in \mathbb{Z}$. This defines a chain map. Moreover, for each chain complex C_\bullet define id_{C_\bullet} by id_{C_n} for all $n \in \mathbb{Z}$. It is easy to check, that then \circ is associative and the identity laws hold. \square

Definition 3.4 (Comp). The category in 3.2 is called the **category of chain complexes** and we refer to it as **Comp**.

Theorem 3.1. There is a functor $\text{Top} \rightarrow \text{Comp}$.

Proof. The proof is divided into several steps. Let us denote $C_\bullet : \text{Top} \rightarrow \text{Comp}$ for the claimed functor.

Step 1: Construction of a sequence of abelian groups. Let $v_0, \dots, v_k \in \mathbb{R}^n$ for some $n, k \in \omega$. We say that (v_0, \dots, v_k) is **affinely independent** if $(v_1 - v_0, \dots, v_k - v_0)$ is linearly independent. We define the **k -simplex spanned by (v_0, \dots, v_k)** , written $[v_0, \dots, v_k]$, to be

$$[v_0, \dots, v_k] := \left\{ \sum_{i=0}^k s_i v_i : s_i \geq 0 \text{ for all } i = 0, \dots, k \text{ and } \sum_{i=0}^k s_i = 1 \right\}. \quad (9)$$

equipped with the subspace topology. Moreover, we define the **standard n -simplex Δ^n** to be the n -simplex spanned by (e_0, \dots, e_n) where $e_0 := 0 \in \mathbb{R}^n$ and (e_1, \dots, e_n) is the standard ordered basis of \mathbb{R}^n . Let $X \in \text{ob}(\text{Top})$. Define a **singular n -simplex in X** to be a morphism $\sigma \in \text{Top}(\Delta^n, X)$. Let $n \in \mathbb{Z}$. Define

$$C_n(X) := \begin{cases} F(\text{Top}(\Delta^n, X)) & n \geq 0, \\ 0 & n < 0. \end{cases} \quad (10)$$

We will call elements of $C_n(X)$ **singular n -chains**.

Step 2: Construction of boundary operators. Let $X \in \text{ob}(\text{Top})$ and σ a singular n -simplex in X for $n \geq 1$. We define $\varphi_k^n : \Delta^{n-1} \rightarrow \Delta^n$, called the **k -th face map**, to be the unique affine map determined by the vertex map

$$\begin{array}{ccc} & \varphi_k^n & \\ e_0 & \mapsto & e_0 \\ \vdots & & \vdots \\ e_{k-1} & \mapsto & e_{k-1} \\ e_k & \mapsto & e_{k+1} \\ \vdots & & \vdots \\ e_{n-1} & \mapsto & e_n. \end{array}$$

Explicitly, given $\sum_{i=0}^{n-1} s_i e_i \in \Delta^{n-1}$, we have that (see [Lee11, p. 152])

$$\varphi_k^n \left(\sum_{i=0}^{n-1} s_i e_i \right) = \sum_{i=0}^{n-1} s_i \varphi_k^n(e_i).$$

Define now

$$\partial \sigma := \sum_{k=0}^n (-1)^k \sigma \circ \varphi_k^n \in U(C_{n-1}(X)) \quad (11)$$

to be the **boundary of σ** . Moreover, the **singular boundary operator** is defined to be $\bar{\partial}_n$ and $\bar{\partial}_n := 0$ for $n \leq 0$.

Step 3: $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. It is enough to consider $n \geq 1$, since $\partial_n \circ \partial_{n+1} = 0$ holds trivially in the other cases. Let $X \in \text{ob}(\text{Top})$ and $\sigma \in \text{Top}(\Delta^{n+1}, X)$. Then we have

$$\begin{aligned}
 (\partial_n \circ \partial_{n+1})(\sigma) &= \partial_n \left(\sum_{k=0}^{n+1} (-1)^k \sigma \circ \varphi_k^{n+1} \right) \\
 &= \sum_{k=0}^{n+1} (-1)^k \partial_n (\sigma \circ \varphi_k^{n+1}) \\
 &= \sum_{k=0}^{n+1} \sum_{j=0}^n (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\
 &= \sum_{0 \leq k \leq j \leq n} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n + \sum_{0 \leq j < k \leq n+1} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\
 &= \sum_{0 \leq j \leq k \leq n} (-1)^{k+j} \sigma \circ \varphi_j^{n+1} \circ \varphi_k^n + \sum_{0 \leq j < k \leq n+1} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\
 &= \sum_{0 \leq j < k \leq n+1} ((-1)^{k+j-1} \sigma \circ \varphi_j^{n+1} \circ \varphi_{k-1}^n + (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n)
 \end{aligned}$$

Since $\varphi_j^{n+1} \circ \varphi_{k-1}^n = \varphi_k^{n+1} \circ \varphi_j^n$, it follows that

$$\partial_n \circ \partial_{n+1} = 0.$$

Indeed, consider the following chart of vertex maps:

	φ_{k-1}^n		φ_j^{n+1}		φ_j^n		φ_k^{n+1}	
e_0	\mapsto	e_0	\mapsto	e_0	e_0	\mapsto	e_0	\mapsto
\vdots		\vdots		\vdots	\vdots		\vdots	
e_{j-1}	\mapsto	e_{j-1}	\mapsto	e_{j-1}	e_{j-1}	\mapsto	e_{j-1}	\mapsto
e_j	\mapsto	e_j	\mapsto	e_{j+1}	e_j	\mapsto	e_{j+1}	\mapsto
\vdots		\vdots		\vdots	\vdots		\vdots	
e_{k-1}	\mapsto	e_{k-1}	\mapsto	e_{k+1}	e_{k-1}	\mapsto	e_k	\mapsto
e_k	\mapsto	e_{k+1}	\mapsto	e_{k+2}	e_k	\mapsto	e_{k+1}	\mapsto
\vdots		\vdots		\vdots	\vdots		\vdots	
e_{n-1}	\mapsto	e_n	\mapsto	e_{n+1}	e_{n-1}	\mapsto	e_n	\mapsto

Step 4: Construction of chain maps. Let $X, Y \in \text{ob}(\text{Top})$ and $f \in \text{Top}(X, Y)$. For $n \geq 0$, define $f_n^\# : \text{Top}(\Delta^n, X) \rightarrow U(C_n(Y))$ by $f_n^\# := f \circ \sigma$. Extending this map by linearity yields a homomorphism $f_n^\# : C_n(X) \rightarrow C_n(Y)$. Moreover, set $f_n^\# := 0$ for $n < 0$. Let

$n \geq 1$ and $\sigma \in \text{Top}(\Delta^n, X)$. Then on one hand we have

$$(f_{n-1}^\# \circ \partial_n)(\sigma) = f_{n-1}^\# \left(\sum_{k=0}^n (-1)^k \sigma \circ \varphi_k^n \right) = \sum_{k=0}^n (-1)^k f \circ \sigma \circ \varphi_k^n$$

and on the other

$$(\partial_n \circ f_n^\#)(\sigma) = \partial_n(f \circ \sigma) = \sum_{k=0}^n (-1)^k f \circ \sigma \circ \varphi_k^n.$$

Checking, that C_\bullet is indeed a functor is left as an exercise. \square

Exercise 3.2. Show that $C_\bullet : \text{Top} \rightarrow \text{Comp}$ is a functor.

The Homology Functor.

Proposition 3.3. For each $n \in \mathbb{Z}$ there exists a functor $\text{Comp} \rightarrow \text{AbGrp}$.

Proof. Let $(C_\bullet, \partial_\bullet)$ be a chain complex. Let $x \in \text{im } \partial_{n+1}$. Hence there exists $y \in C_{n+1}$ such that $x = \partial_{n+1}y$. But then $\partial_n x = (\partial_n \circ \partial_{n+1})(y) = 0$ and thus $\text{im } \partial_{n+1} \subseteq \ker \partial_n$. Define

$$H_n(C_\bullet, \partial_\bullet) := \frac{\ker \partial_n}{\text{im } \partial_{n+1}} \in \text{ob}(\text{AbGrp}).$$

Let $(C'_\bullet, \partial'_\bullet)$ be a chain complex and $f_\bullet : C_\bullet \rightarrow C'_\bullet$ a chain map. Then $f_n(\ker \partial_n) \subseteq \ker \partial'_n$. Indeed, if $y \in f_n(\ker \partial_n)$, there exists $x \in \ker \partial_n$, such that $y = f_n(x)$. Since f_\bullet is a chain map, we thus have $\partial'_n y = (\partial'_n \circ f_n)(x) = (f_{n-1} \circ \partial_n)(x) = 0$. Moreover, we have that $\text{im } \partial_{n+1} \subseteq \ker \pi'_n \circ f_n$, where $\pi'_n : \ker \partial'_n \rightarrow H_n(C'_\bullet, \partial'_\bullet)$ is the usual projection. Indeed, if $y \in \text{im } \partial_{n+1}$, we find $x \in C_{n+1}$, such that $y = \partial_{n+1}x$. Since again f_\bullet is a chain map, we have that $f_n y = (f_n \circ \partial_{n+1})(x) = (\partial'_{n+1} \circ f_{n+1})(x) \in \text{im } \partial'_{n+1} = \ker \pi'_n$. Hence $\pi'_n \circ f_n$ factors uniquely through $\pi_n : \ker \partial_n \rightarrow H_n(C_\bullet, \partial_\bullet)$. Define $H_n(f_\bullet)$ to be this map. \square

Remark 3.1. Let $(C_\bullet, \partial_\bullet)$ be a chain complex and $n \in \mathbb{Z}$. Then we will write $\langle x \rangle$ for an element in $H_n(C_\bullet, \partial_\bullet)$, the so-called *homology class*. Hence if $(C'_\bullet, \partial'_\bullet)$ is another chain complex and $f_\bullet : C_\bullet \rightarrow C'_\bullet$ a chain map, then $H_n(f_\bullet)\langle c \rangle = \langle f_n c \rangle$.

Definition 3.5 (Cycles and Boundaries). Let $(C_\bullet, \partial_\bullet)$ be a chain complex and $n \in \mathbb{Z}$. Then elements of $\ker \partial_n$ are called ***n-cycles*** and elements of $\text{im } \partial_{n+1}$ are called ***n-boundaries***.

Definition 3.6 (Homology Functor). Let $n \in \mathbb{Z}$ and $H_n : \text{Comp} \rightarrow \text{AbGrp}$ be the functor defined in proposition 3.3. We call H_n the ***n-th homology functor***.

Definition 3.7 (Singular Homology Functor). Let $n \in \mathbb{Z}$. The composition

$$H_n \circ C_\bullet : \text{Top} \rightarrow \text{AbGrp} \tag{12}$$

of the singular chain complex functor C_\bullet in theorem 3.1 and the n -th homology functor of proposition 3.3 is called the ***singular homology functor***, written H_n^{sing} .

Remark 3.2. For notational purposes we will often refer to the functor H_n^{sing} simply as H_n .

First Properties of Singular Homology.

Proposition 3.4 (Zeroth Singular Homology Group). *Let $X \in \text{ob}(\text{Top})$ be non empty and path connected. Then $H_0(X) \cong \mathbb{Z}$.*

Proof. Since $\partial_0 : C_0(X) \rightarrow 0$, $\ker \partial_0 = C_0(X)$. Moreover, a map in $\text{Top}(\Delta^0, X)$ can be identified with a point in X and hence an element of $C_0(X)$ can be written as $\sum_{x \in X} m_x x$. Define a mapping $\Phi : C_0(X) \rightarrow \mathbb{Z}$ by $\Phi(\sum_{x \in X} m_x x) := \sum_{x \in X} m_x$. This mapping is well defined since all but finitely many m_x are zero. It is also easy to check, that Φ is a morphism of groups and that Φ is surjective. We claim that $\ker \Phi = \text{im } \partial_1$. Indeed, if $\sum_{x \in X} m_x x \in \ker \Phi$, then $\sum_{x \in X} m_x = 0$. Let $p \in X$. Since X is path connected, we find for each $x \in X$ a path σ_x from p to x . Consider the singular 1-chain $\sum_{x \in X} m_x \sigma_x$. Then we have

$$\partial_1 \left(\sum_{x \in X} m_x \sigma_x \right) = \sum_{x \in X} m_x (\sigma_x(1) - \sigma_x(0)) = \sum_{x \in X} m_x (x - p) = \sum_{x \in X} m_x x.$$

Hence $\sum_{x \in X} m_x x \in \text{im } \partial_1$. Conversely, it is enough to show the claim on basis elements $\sigma \in \text{Top}(\Delta^1, X)$. We have

$$\Phi(\partial_1 \sigma) = \Phi(\sigma(1) - \sigma(0)) = 1 - 1 = 0.$$

Hence the first isomorphism theorem [Gri07, p. 23] implies that $H_0(X) \cong \mathbb{Z}$. \square

Proposition 3.5 (The Dimension Axiom). *Let $*$ $\in \text{ob}(\text{Top})$ be a one point space. Then $H_n(*) = 0$ for all $n \in \mathbb{Z}$, $n > 0$.*

The Homotopy Axiom

Theorem 3.2 (The Homotopy Axiom). *Let $f, g \in \text{Top}(X, Y)$ be freely homotopic. Then $H_n(f) = H_n(g)$ for all $n \in \mathbb{Z}$.*

The Hurewicz Theorem

Abelianizations.

Proposition 3.6. *The forgetful functor $U : \text{AbGrp} \rightarrow \text{Grp}$ admits a left adjoint.*

Proof. Let $G \in \text{ob}(\text{Grp})$. For $g, h \in G$, define the **commutator of g and h** , written $[g, h]$, by $[g, h] := ghg^{-1}h^{-1}$. Moreover, set

$$X_G := \{[g, h] : g, h \in G\}$$

and define the **commutator subgroup of G** , written $[G, G]$, by $[G, G] := \langle X_G \rangle$.

Lemma 3.1. *For all $G \in \text{ob}(\text{Grp})$, $[G, G] \trianglelefteq G$.*

Proof. We follow [Lee11, p. 265]. Clearly, $[G, G] \leq G$. By [KM13, p. 31] we have that

$$\langle X \rangle = \{x_1 \cdots x_n : n \in \omega \setminus \{0\}, x_1, \dots, x_n \in X_G \cup X_G^{-1}\}.$$

It is easy to check that $X_G = X_G^{-1}$ and thus

$$\langle X \rangle = \{x_1 \cdots x_n : n \in \omega \setminus \{0\}, x_1, \dots, x_n \in X_G\}.$$

Let $k \in G$ and $x_1 \cdots x_n \in [G, G]$. Since

$$kx_1 \cdots x_n k^{-1} = kx_1 k^{-1} kx_2 k^{-1} k \cdots kx_n k^{-1}$$

it is enough to show that $k[g, h]k^{-1} \in [G, G]$ for all $g, h \in G$. But this immediately follows from

$$k[g, h]k^{-1} = kghg^{-1}h^{-1}k^{-1} = [kgk^{-1}, khk^{-1}].$$

Thus $[G, G] \trianglelefteq G$. □

Lemma 3.2. $G \in \text{ob}(\text{AbGrp})$ if and only if $[G, G] = \{1\}$.

Proof. Let $G \in \text{ob}(\text{AbGrp})$. Then $[g, h] = 1$ for all $g, h \in G$, which implies $X_G = \{1\}$ and thus $\langle X_G \rangle = \{1\}$. Conversely, since $X_G \subseteq [G, G] = \{1\}$, we have that $[g, h] = 1$ for all $g, h \in G$ which is equivalent to $gh = hg$ for all $g, h \in G$. □

Corollary 3.1. The quotient group $G/[G, G]$ is abelian.

Proof. By lemma 3.2 it is enough to show that $[G/[G, G], G/[G, G]]$ is trivial. We actually show that $X_{G/[G, G]} = \{1\}$. This immediately follows from

$$[g[G, G], h[G, G]] = ghg^{-1}h^{-1}[G, G] = [G, G]$$

for $g[G, G], h[G, G] \in G/[G, G]$. □

Hence define $\text{Ab} : \text{Grp} \rightarrow \text{AbGrp}$ on objects by

$$\text{Ab}(G) := G/[G, G].$$

The abelian group $\text{Ab}(G)$ is called the **abelianization of G** . On morphisms $\varphi : G \rightarrow H$ in Grp define $\text{Ab}(\varphi) : \text{Ab}(G) \rightarrow \text{Ab}(H)$ by setting $\text{Ab}(\varphi)(g[G, G]) := \varphi(g)[H, H]$. It is easy to check that this is a well defined morphism of abelian groups.

Let $H \in \text{ob}(\text{AbGrp})$ and $\psi \in \text{AbGrp}(\text{Ab}(G), H)$. Define $\bar{\psi} \in \text{Grp}(G, U(H))$ by setting $\bar{\psi}(g) := \psi(g[G, G])$. If $\varphi \in \text{Grp}(G, U(H))$, define $\bar{\varphi} \in \text{AbGrp}(\text{Ab}(G), H)$ by $\bar{\varphi}(g[G, G]) := \varphi(g)$. It is easy to check that this mapping is actually well defined and that $\bar{\bar{\psi}} = \psi$ and $\bar{\bar{\varphi}} = \varphi$ holds. □

Exercise 3.3. In proposition 3.6, check that $\text{Ab} : \text{Grp} \rightarrow \text{AbGrp}$ is indeed a functor and the naturality of the bijection in both arguments.

The Hurewicz Morphism. Since elements of $H_1(X)$ are homology classes of loops, one might suspect that there is a connection between the fundamental group $\pi_1(X, p)$ of a path connected space X at p and the first singular homology group $H_1(X)$. However, since $H_1(X)$ is always abelian and $\pi_1(X, p)$ is not necessarily abelian, they cannot be equal. In this section we use a little trick which makes matters simpler: if c is any singular n -chain, not necessarily an n -cycle, we can also take its equivalence class modulo n -boundaries. We shall denote this class also with $\langle c \rangle$. Clearly, if c is an n -cycle, then $\langle c \rangle$ is the usual homology class.

Theorem 3.3 (Hurewicz Theorem). *Let $X \in \text{ob}(\text{Top})$ be path connected and $p \in X$. Then $\text{Ab}(\pi_1(X, p)) \cong H_1(X)$.*

Proof. We show the result in a sequence of lemmata.

Lemma 3.3. *The mapping $h : \pi_1(X, p) \rightarrow H_1(X)$ defined by $h([u]) := \langle u \rangle$ is well defined.*

Proof. First of all, since $u \in \Omega(X, p)$, we have that $u \in C_1(X)$. Moreover, $\partial u = u(1) - u(0) = p - p = 0$. Thus u has a homology class $\langle u \rangle$. Let us check that h is well defined. Suppose that $[u] = [v]$. Hence $F : u \simeq_{\partial I} v$. Consider the fundamental loop $\omega \in \Omega(S^1, 1)$. By [Lee11, p. 70], ω is a quotient map. Since $u, v \in \Omega(X, p)$, there exist $\tilde{u}, \tilde{v} \in \text{Top}(S^1, X)$, such that $\tilde{u} \circ \omega = u$ and $\tilde{v} \circ \omega = v$ (see [Lee11, p. 72]). Since I is a locally compact Hausdorff space [Lee11, p. 107] implies that $\omega \times \text{id}_I$ is a quotient map. Thus F passes to the quotient and yields a map $\tilde{F} \in \text{Top}(S^1 \times I, X)$. Now it is easy to check that $\tilde{F} : \tilde{u} \simeq_{\{1\}} \tilde{v}$. Thus an application of the homotopy axiom yields

$$\langle u \rangle = \langle \tilde{u} \circ \omega \rangle = H_1(\tilde{u})\langle \omega \rangle = H_1(\tilde{v})\langle \omega \rangle = \langle \tilde{v} \circ \omega \rangle = \langle v \rangle.$$

□

Lemma 3.4. *Let u be a path in X from p to q . Then $\langle \bar{u} \rangle = -\langle u \rangle$.*

Proof. From figure 2a, we deduce that an appropriate definition of a singular 2-simplex σ would be

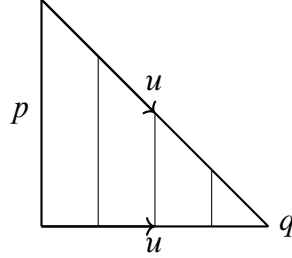
$$\sigma(x, y) := u(x).$$

Indeed

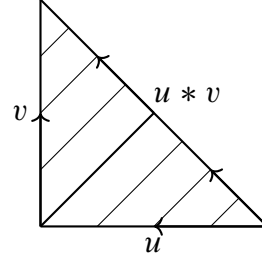
$$\partial \sigma = \bar{u} - c_p + u$$

and since c_p is the boundary of $\sigma_p \in \text{Top}(\Delta^2, X)$ defined by $\sigma_p(x, y) := p$, we have that $\bar{u} + u$ is a boundary. □

Lemma 3.5. *Let u and v be paths in X from p to q and from q to r , respectively. Then $\langle u * v \rangle = \langle u \rangle + \langle v \rangle$.*



(A) $\langle \bar{u} \rangle = -\langle u \rangle$.



(B) $\langle u * v \rangle = \langle u \rangle + \langle v \rangle$.

Proof. Consider figure 2b. The thin lines correspond to where $y - x$ is constant. Hence define $\sigma : \Delta^2 \rightarrow X$ by

$$\sigma(x, y) := \begin{cases} u(y - x + 1) & 0 \leq y \leq x \leq 1, \\ v(y - x) & 0 \leq x \leq y \leq 1. \end{cases}$$

An application of the gluing lemma shows that σ is actually a singular 2-simplex. Moreover

$$\partial\sigma = u * v - v + \bar{u}.$$

Hence lemma 3.4 yield

$$0 = \langle u * v - v + \bar{u} \rangle = \langle u * v \rangle - \langle v \rangle - \langle u \rangle.$$

□

Corollary 3.2. h is a morphism of groups.

Corollary 3.3. Let u, v, w be composable paths in X . Then $\langle (u * v) * w \rangle = \langle u * (v * w) \rangle$.

Lemma 3.6. h is surjective.

Proof. Let $x \in X$. If $x = p$, define $\gamma_p := c_p$. If $x \neq p$, by the path connectedness of X we can choose a path γ_x from p to x . Hence we get a map $\gamma : X \rightarrow \text{Top}(\Delta^1, X)$. Extending by linearity yields a mapping $\gamma : C_0(X) \rightarrow C_1(X)$. Let $c := \sum_{k=1}^n m_k \sigma_k$ be a 1-cycle in X . Consider

$$[u] := [\gamma_{\sigma_1(0)} * \sigma_1 * \overline{\gamma_{\sigma_1(1)}}]^{m_1} \cdots [\gamma_{\sigma_n(0)} * \sigma_n * \overline{\gamma_{\sigma_n(1)}}]^{m_n} \in \pi_1(X, p).$$

Now lemma 3.4 and 3.5, corollary 3.2 and 3.3 yields

$$\begin{aligned} h([u]) &= \sum_{k=1}^n m_k \langle \gamma_{\sigma_k(0)} * \sigma_k * \overline{\gamma_{\sigma_k(1)}} \rangle \\ &= \sum_{k=1}^n m_k (\langle \gamma_{\sigma_k(0)} \rangle + \langle \sigma_k \rangle + \langle \overline{\gamma_{\sigma_k(1)}} \rangle) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^n m_k (\langle \gamma_{\sigma_k(0)} \rangle + \langle \sigma_k \rangle - \langle \gamma_{\sigma_k(1)} \rangle) \\
&= \langle c \rangle - \sum_{k=1}^n m_k \langle \gamma_{\sigma_k(1) - \sigma_k(0)} \rangle \\
&= \langle c \rangle - \sum_{k=1}^n m_k \langle \gamma_{\partial \sigma_k} \rangle \\
&= \langle c \rangle - \langle \gamma_{\partial c} \rangle \\
&= \langle c \rangle.
\end{aligned}$$

□

Lastly, we want to show that $\ker h = [\pi_1(X, p), \pi_1(X, p)]$. Since then the first isomorphism theorem implies $\text{Ab}(\pi_1(X, p)) \cong H_1(X)$. Since $H_1(X)$ is abelian, clearly $[\pi_1(X, p), \pi_1(X, p)] \subseteq \ker h$ and thus h factors uniquely $\tilde{h} : \text{Ab}(\pi_1(X, p)) \rightarrow H_1(X)$. The next lemma will be useful.

Lemma 3.7. *Let $\sigma : \Delta^2 \rightarrow X$ be a singular 2-simplex. Define $\sigma^{(k)} := \sigma \circ \varphi_k^2$ for $k = 0, 1, 2$. Then $[\sigma^{(0)} * \overline{\sigma^{(1)}} * \sigma^{(2)}] = [c_{\sigma(e_1)}]$.*

Proof. Let $u := \sigma^{(0)} * \overline{\sigma^{(1)}} * \sigma^{(2)}$. Since $\mathbb{B}^2 \approx \Delta^2$, we can consider $\sigma : \mathbb{B}^2 \rightarrow X$. One can check that the circle representative \tilde{u} of u is the reparametrized restriction $\sigma|_{\mathbb{S}^1}$. Since reparametrizations are invariant under homotopies, we have that u is a nullhomotopic loop. □

Let $\sigma \in \text{Top}(\Delta^1, X)$. Define $g(\sigma) := [\gamma_{\sigma(0)} * \sigma * \overline{\gamma_{\sigma(1)}}]_{\text{Ab}}$, where $[u]_{\text{Ab}}$ denotes the equivalence class of $[u]$ in $\text{Ab}(\pi_1(X, p))$. Since $\text{Ab}(\pi_1(X, p))$ is abelian, extension by linearity yields a map $g : C_1(X) \rightarrow \text{Ab}(\pi_1(X, p))$.

Lemma 3.8. *g vanishes on $\text{im } \partial_2$.*

Proof. Let $\sigma \in \text{Top}(\Delta^2, X)$. Then lemma 3.7 yields

$$\begin{aligned}
g(\partial \sigma) &= g(\sigma^{(0)}) g(\sigma^{(1)})^{-1} g(\sigma^{(2)}) \\
&= [\gamma_{\sigma(e_1)} * \sigma^{(0)} * \overline{\gamma_{\sigma(e_2)}} * \gamma_{\sigma(e_2)} * \overline{\sigma^{(1)}} * \overline{\gamma_{\sigma(e_0)}} * \gamma_{\sigma(e_0)} * \sigma^{(2)} * \overline{\gamma_{\sigma(e_1)}}]_{\text{Ab}} \\
&= [\gamma_{\sigma(e_1)} * \sigma^{(0)} * \overline{\sigma^{(1)}} * \sigma^{(2)} * \overline{\gamma_{\sigma(e_1)}}]_{\text{Ab}} \\
&= [\gamma_{\sigma(e_1)} * c_{\sigma(e_1)} * \overline{\gamma_{\sigma(e_1)}}]_{\text{Ab}} \\
&= [c_p]_{\text{Ab}}.
\end{aligned}$$

□

By lemma 3.8, g passes to the quotient and yields a map $\tilde{g} : H_1(X) \rightarrow \text{Ab}(\pi_1(X, p))$.
Moreover

$$(\tilde{g} \circ \tilde{h}) [u]_{\text{Ab}} = \tilde{g} (h [u]) = \tilde{g} \langle u \rangle = g(u) = [c_p * u * \overline{c_p}]_{\text{Ab}} = [u]_{\text{Ab}}$$

and thus \tilde{h} admits a retraction in AbGrp which implies that \tilde{h} is injective. Hence $\ker \tilde{h}$ is trivial and thus if we write $\pi : \pi_1(X, p) \rightarrow \text{Ab}(\pi_1(X, p))$ for the canonical projection

$$\ker h = \ker(\tilde{h} \circ \pi) = (\tilde{h} \circ \pi)^{-1}(0) = \pi^{-1}(\tilde{h}^{-1}(0)) = \pi^{-1}(0) = [\pi_1(X, p), \pi_1(X, p)].$$

□

Definition 3.8 (Hurewicz Homomorphism). Let $X \in \text{ob}(\text{Top})$ and $p \in X$. The homomorphism $h : \pi_1(X, p) \rightarrow H_1(X)$ defined in theorem 3.3 is called the **Hurewicz homomorphism**.

Applications

The Brouwer Fixed Point Theorem.

Definition 3.9 (Retract). Let $X \in \text{ob}(\text{Top})$ and $S \subseteq X$ a subspace. We say that S is a **retract of X** , if the inclusion $\iota : S \hookrightarrow X$ admits a retraction in Top .

Lemma 3.9. Let $n \in \mathbb{Z}$, $n \geq 1$. Then S^n is not a retract of \mathbb{B}^{n+1} .

Proof.

□

Theorem 3.4 (Brouwer Fixed Point Theorem). Let $n \in \mathbb{Z}$, $n \geq 1$. Then every mapping $f \in \text{Top}(\mathbb{B}^n, \mathbb{B}^n)$ has a fixed point.

Proof.

□

CHAPTER 4

Homological Algebra

Basic Definitions

Diagram Lemmas

The Snake Lemma.

Proposition 4.1 (Snake Lemma). *Suppose we are given a commutative diagram of abelian groups with exact rows:*

$$\begin{array}{ccccccc}
 A & \xrightarrow{i} & B & \xrightarrow{j} & C & \longrightarrow & 0 \\
 \downarrow f & & \downarrow g & & \downarrow h & & \\
 0 & \longrightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C'
 \end{array}$$

Then there exists $\delta \in \text{AbGrp}(\ker h, \text{coker } f)$ such that the sequence

$$\ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \text{coker } f \longrightarrow \text{coker } g \longrightarrow \text{coker } h \quad (13)$$

is exact.

Proof. Consider the augmented diagram in figure 3, where the morphisms k, l, p and q are induced by i, j, i' and j' , respectively.

Step 1: Exactness at $\ker g$. Let $x \in \ker f$. Then $l(k(x)) = j(i(x)) = 0$ by exactness at B and thus $\text{im } k \subseteq \ker l$. Conversely, let $y \in \ker l$. Then $j(y) = 0$ and by exactness at B , there exists $x \in A$ such that $i(x) = y$. Moreover $0 = g(y) = g(i(x)) = i'(f(x))$ since $y \in \ker g$ and thus $f(x) = 0$ by injectivity of i' . Hence $\ker j \subseteq \text{im } k$.

Step 2: Exactness at $\text{coker } g$. Let $x + \text{im } f \in \text{coker } f$. Then

$$q(p(x + \text{im } f)) = j'(i'(x)) + \text{im } h = \text{im } h$$

by exactness at B' implies $\text{im } p \subseteq \ker q$. Conversely,

□

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \ker f & \xrightarrow{k} & \ker g & \xrightarrow{l} & \ker h & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & A & \xrightarrow{i} & B & \xrightarrow{j} & C & \longrightarrow 0 \\
 & \downarrow f & & \downarrow g & & \downarrow h & \\
 0 & \longrightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \operatorname{coker} f & \xrightarrow{p} & \operatorname{coker} g & \xrightarrow{q} & \operatorname{coker} h & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

FIGURE 3. Proof of the snake lemma.

$$\begin{array}{ccccc}
 \ker f & \xrightarrow{k} & \ker g & \xrightarrow{l} & \ker h \\
 & & & & \downarrow \\
 & & & & B \longleftarrow C \\
 & & & & \downarrow g \\
 & & & & A' \longleftarrow B' \\
 & & & & \downarrow \\
 \operatorname{coker} f & \xrightarrow{p} & \operatorname{coker} g & \xrightarrow{q} & \operatorname{coker} h
 \end{array}$$

FIGURE 4. Proof of the snake lemma.

APPENDIX A

Set Theory

Basic Concepts

Problem A.1. Let $n \in \mathbb{N}$ and $a_{kj} \in \mathbb{C}$ for $k = 0, \dots, n+1$, $j = 0, \dots, n$. Show that

$$\sum_{k=0}^{n+1} \sum_{j=0}^n a_{kj} = \sum_{0 \leq k \leq j \leq n} a_{kj} + \sum_{0 \leq j < k \leq n+1} a_{kj}.$$

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