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SOLUTIONS SHEET 1

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Exercise 1.

a. The first part can be shown for an arbitrary set X. Clearly $\varnothing, X \in \mathcal{T}$ since $X^c = \varnothing$ is countable. Let $(U_\iota)_{\iota \in I}$ be a family of sets in $\mathcal{T}$. If $U_\iota = \varnothing$ for all $\iota \in I$ we have that $\bigcup_{\iota \in I} U_\iota = \varnothing \in \mathcal{T}$. So assume that $U_{\iota_0} \neq \varnothing$ for some $\iota_0 \in I$. But then $U_{\iota_0}^c$ is countable, and so is $(\bigcup_{\iota \in I} U_\iota)^c = \bigcap_{\iota \in I} U_\alpha^c \subseteq U_{\iota_0}^c$. Lastly, let $U_1, \dots, U_n \in \mathcal{T}$ for $n \in \mathbb{Z}$, $n \geq 1$. If $U_\iota = \varnothing$ for some ι, then $\bigcap_{\iota=1}^n U_\iota = \varnothing$ and thus $\bigcap_{\iota=1}^n U_\iota \in \mathcal{T}$. So assume that $U_\iota \neq \varnothing$ for $\iota = 1, \dots, n$. Then $(\bigcap_{\iota=1}^n U_\iota)^c = \bigcup_{\iota=1}^n U_\iota^c$ which is a finite union of countable sets, which is countable. Hence $\mathcal{T}$ is indeed a topology on X.
We claim that $(X, \mathcal{T})$ is not Hausdorff when X is uncountable. Towards a contradiction assume that $(X, \mathcal{T})$ is Hausdorff. Let $p, q \in X$ with $p \neq q$. Hence there exist (open) neighbourhoods U and V of p and q respectively such that $U \cap V = \varnothing$. Now $X = U \cup U^c$, where U^c is countable and clearly nonempty. But $U \cap V = \varnothing$ implies $U \subseteq V^c$ which therefore yields that U is also countable. Hence X is a union of two countable sets and thus countable. Contradiction.

b. We prove both times the contrapositive. Assume that there is a family $(A_\iota)_{\iota \in I}$ of closed subsets of X having the finite intersection property such that $\bigcap_{\iota \in I} A_\iota = \varnothing$. Then $\bigcup_{\iota \in I} A_\iota^c = (\bigcap_{\iota \in I} A_\iota)^c = X$. Since each A_ι is closed, A_ι^c is open for all $\iota \in I$ and thus $(A_\iota^c)_{\iota \in I}$ is an open cover for X. We claim that $(A_\iota^c)_{\iota \in I}$ does not admit any finite subcover. Towards a contradiction, assume that it does. Hence we find $\iota_1, \dots, \iota_n \in I$, $n \in \mathbb{Z}$, $n \geq 1$, such that $\bigcup_{k=1}^n A_{\iota_k}^c = X$. But then $\bigcap_{k=1}^n A_{\iota_k} = \varnothing$, contradicting the finite intersection property of the family $(A_\iota)_{\iota \in I}$.
Conversly, suppose that there exists an open cover $(A_\iota)_{\iota \in I}$ of X which does not admit a finite subcover. We claim that the closed family $(A_\iota^c)_{\iota \in I}$ has the finite intersection property and $\bigcap_{\iota \in I} A_\iota = \varnothing$. Let $\iota_1, \dots, \iota_n \in I$, $n \in \mathbb{Z}$, $n \geq 1$. Since $(A_{\iota_k})_{k=1}^n$ cannot cover X, otherwise it would be a finite subcover of $(A_\iota)_{\iota \in I}$, we have that $\bigcap_{k=1}^n A_{\iota_k}^c \neq \varnothing$. Thus $(A_\iota^c)_{\iota \in I}$ has the finite intersection property. Since $(A_\iota)_{\iota \in I}$ covers X we have that $\bigcap_{\iota \in I} A_\iota^c = \varnothing$.

Exercise 2.

a. Clearly, $\varnothing, X \in \mathcal{T}_d$. Let $(U_\iota)_{\iota \in I}$ be a family of elements in $\mathcal{T}_d$ and $x \in \bigcup_{\iota \in I} U_\iota$. Then there exists $\iota \in I$ such that $x \in U_\iota$. Furthermore, we find $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U_\iota$. Hence $B_\varepsilon(x) \subseteq \bigcup_{\iota \in I} U_\iota$. Let $U_1, \dots, U_n \in \mathcal{T}$ for $n \in \mathbb{Z}$, $n \geq 1$, and $x \in \bigcap_{\iota=1}^n U_\iota$. Hence

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there exist $\varepsilon_1, \ldots, \varepsilon_n > 0$ such that $B_{\varepsilon_\iota}(x) \subseteq U_\iota$ for $\iota = 1, \ldots, n$ and so $B_{\widetilde{\varepsilon}}(x) \subseteq \cap_{\iota=1}^n U_\iota$ for $\widetilde{\varepsilon} := \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. Thus $\mathcal{T}_d$ is a topology on X.

b. We will use the fact that two metrics induce the same topology if and only if they induce the same convergence. Let $\widetilde{M} := (0, \infty)$. Define $f : \widetilde{M} \to \widetilde{M}$ by $f(x) := 1/x$. Then clearly $d_2 = \widetilde{d}_2|_M$ and $d_1 = \widetilde{d}_1|_M$, where

$$\widetilde{d}_2 : \widetilde{M} \times \widetilde{M} \xrightarrow{f \times f} \widetilde{M} \times \widetilde{M} \xrightarrow{|\cdot,\cdot|} \mathbb{R}$$

and

$$\widetilde{d}_1 : \widetilde{M} \times \widetilde{M} \xrightarrow{f \times f} \widetilde{M} \times \widetilde{M} \xrightarrow{\widetilde{d}_2} \mathbb{R}.$$

It is easy to show that $\widetilde{d}_2$ is a metric. Let $x \in M$ and $(x_n)_{n \in \mathbb{N}}$ be a sequence in M. Assume that $x_n \xrightarrow{d_1} x$. Then

$$d_2(x_n, x) = \widetilde{d}_1\big(f(x_n), f(x)\big) \to 0$$

and

$$d_1(x_n, x) = \widetilde{d}_2\big(f(x_n), f(x)\big) \to 0$$

by the continuity of f on $\widetilde{M}$.
(M, d_1) is complete since M is a closed subset of the complete metric space $\mathbb{R}$. Consider the sequence $(n)_{n \in \mathbb{N}}$ in M. Clearly, it is a Cauchy sequence in (M, d_2) since $\frac{1}{n} \xrightarrow{|\cdot|} 0$. Assume that it converges also in (M, d_2). Since the induced topologies of d_1 and d_2 are the same, we would get that $(n)_{n \in \mathbb{N}}$ also converges in (M, d_1). But this is absurd. Hence (M, d_2) cannot be complete.