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HOMEWORK 3: EXACT SYMPLECTIC MANIFOLDS

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Exercise 1.1. Let M and N be smooth manifolds, $F : M \to N$ a diffeomorphism and $A \in \Gamma\left(T^{(0,k)}TN\right)$, $k \in \mathbb{Z}, k \geq 1$. Then

$$F^*A(X_1, \dots, X_k) = A(F_*X_1, \dots, F_*X_k) \circ F \tag{1}$$

holds for all $X_1, \dots, X_k \in \mathfrak{X}(M)$.

Solution 1.1. Let $p \in M$. Then

$$\begin{aligned} F^*A(X_1, \dots, X_k)(p) &= (F^*A)_p(X_1|_p, \dots, X_k|_p) \\ &= A_{F(p)}\left(dF_p(X_1|_p), \dots, dF_p(X_k|_p)\right) \\ &= A_{F(p)}\left((F_*X_1)_{F(p)}, \dots, (F_*X_k)_{F(p)}\right) \\ &= A\left(F_*X_1, \dots, F_*X_k\right)\left(F(p)\right). \end{aligned}$$

Exercise 1.2. (a)

Solution 1.2. For (a), consider the tangent-cotangent isomorphism $\widetilde{\omega} : TM \to T^*M$. Set $X := \widetilde{\omega}^{-1}(-\alpha)$. As a composition of smooth functions, $X : M \to TM$ is smooth. Moreover, $X_p = \widetilde{\omega}^{-1}(-\alpha_p) \in T_pM$. Thus $X \in \mathfrak{X}(M)$. Moreover

$$i_X\omega = \widetilde{\omega}\left(\widetilde{\omega}^{-1}(-\alpha)\right) = -\alpha.$$

Since $\widetilde{\omega}$ is an isomorphism, X is unique. Cartan's magic formula together with the assumption $\omega = -d\alpha$ yields

$$\mathcal{L}_X\omega = di_X\omega + i_Xd\omega = di_X\omega = -d\alpha = \omega.$$

For proving (b), assume that $L_X\omega = \omega$ for some $X \in \mathfrak{X}(M)$. Again, Cartan's magic formula yields

$$\mathcal{L}_X\omega = di_X\omega + i_Xd\omega = di_X\omega = \omega.$$

Now $i_X\omega \in \Omega^1(M)$ and thus ω is exact.
For proving (c), an application of the Fisherman's formula yields

$$\frac{d}{dt}(\exp tX)^*\omega = (\exp tX)^*\mathcal{L}_X\omega = (\exp tX)^*\omega$$

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and the property of the flow $\exp tX$ yields

$$(\exp tX|_0)^*\omega = \mathrm{id}_M^*\,\omega = \omega.$$

Also we have that $\frac{d}{dt}e^t\omega = e^t\omega$ and $e^0\omega = \omega$. Hence $(\exp tX)^*\omega$ and $e^t\omega$ solve the same locally uniquely solvable initial value problem and are therefore locally equal.
For proving (d), observe that for $Y \in \mathfrak{X}(M)$ we have that

$$\begin{aligned}
\omega(g_*X, Y) \circ g &= (g^*\omega)(X, g_*^{-1}Y)\\
&= \omega\left(X, g_*^{-1}Y\right)\\
&= i_X\omega\left(g_*^{-1}Y\right)\\
&= -\alpha\left(g_*^{-1}Y\right) \circ g^{-1}\\
&= -(g^*\alpha)\left(g_*^{-1}Y\right)\\
&= -\alpha(Y) \circ g\\
&= i_X\omega(Y) \circ g\\
&= \omega(X, Y) \circ g.
\end{aligned}$$

Thus $\widetilde{\omega}(g_*X) = \widetilde{\omega}(X)$ and since $\widetilde{\omega}$ is an isomorphism, we have that $g_*X = X$.
For proving (e), let us define $\rho_t := g \circ \exp tX \circ g^{-1}$. Then

$$\rho_0 = g \circ \exp tX|_{t=0} \circ g^{-1} = \mathrm{id}_M$$

and

$$\begin{aligned}
\frac{d}{dt}\rho_t(p) &= \frac{d}{dt} g\left(\exp tX\left(g^{-1}(p)\right)\right)\\
&= dg_{\exp tX(g^{-1}(p))}\frac{d}{dt}\exp tX\left(g^{-1}(p)\right)\\
&= dg_{\exp tX(g^{-1}(p))}X\left(\exp tX\left(g^{-1}(p)\right)\right)\\
&= (g_*X)_{(g\circ\exp tX\circ g^{-1})(p)}.
\end{aligned}$$

Thus ρ_t is the flow of the vector field g_*X. By part (d) g_*X coincides with X and thus ρ_t is also the flow of X. But flows are unique and thus

$$g \circ \exp tX \circ g^{-1} = \exp tX$$

from which the claim follows.
For (f), let us compute $\mathcal{L}_X\omega_0$. We have

$$\begin{aligned}
\mathcal{L}_X\omega_0 &= d\,i_X\omega_0\\
&= d\sum_{i=1}^n \left((i_X dx_i) \wedge dy_i - dx_i \wedge (i_X dy_i)\right)
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{2} d \sum_{i=1}^{n} (x_i dy_i - y_i dx_i) \\
&= \frac{1}{2} \sum_{i=1}^{n} (dx_i \wedge dy_i - dy_i \wedge dx_i) \\
&= \omega_0 .
\end{aligned}
$$

Exercise 1.3. (a)

Solution 1.3. For showing (a), let us consider $X := f^i \frac{\partial}{\partial x^i} + g^i \frac{\partial}{\partial \xi^i}$. Then

$$
\begin{aligned}
i_X \omega &= i_X \sum_{i=1}^{n} dx_i \wedge d\xi_i \\
&= \sum_{i=1}^{n} \big((i_X dx_i) \wedge d\xi_i - dx_i \wedge (i_X d\xi_i) \big) \\
&= \sum_{i=1}^{n} \left(f^i d\xi_i - g^i dx_i \right) .
\end{aligned}
$$

Comparing this with $-\alpha$ yields $f^i = 0$ and $g^i = \xi^i$ for all $i = 1, \dots, n$. Hence

$$
X = \xi^i \frac{\partial}{\partial \xi^i} .
$$

Appendix A. The Tubular Neighbourhood Theorem

Theorem A.1 (Generalization of the Inverse Function Theorem). *Let M and N be smooth manifolds and S a compact immersed submanifold of M. Moreover, let $f : M \to N$ be smooth such that $f|_S$ is injective. Suppose that df_p is an isomorphism for all $x \in S$.*