

## HOMEWORK 3: EXACT SYMPLECTIC MANIFOLDS

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**Exercise 1.1.** Let  $M$  and  $N$  be smooth manifolds,  $F : M \rightarrow N$  a diffeomorphism and  $A \in \Gamma(T^{(0,k)}TN)$ ,  $k \in \mathbb{Z}$ ,  $k \geq 1$ . Then

$$F^*A(X_1, \dots, X_k) = A(F_*X_1, \dots, F_*X_k) \circ F \quad (1)$$

holds for all  $X_1, \dots, X_k \in \mathfrak{X}(M)$ .

**Solution 1.1.** Let  $p \in M$ . Then

$$\begin{aligned} F^*A(X_1, \dots, X_k)(p) &= (F^*A)_p(X_1|_p, \dots, X_k|_p) \\ &= A_{F(p)}(dF_p(X_1|_p), \dots, dF_p(X_k|_p)) \\ &= A_{F(p)}((F_*X_1)_{F(p)}, \dots, (F_*X_k)_{F(p)}) \\ &= A(F_*X_1, \dots, F_*X_k)(F(p)). \end{aligned}$$

**Exercise 1.2.** (a)

**Solution 1.2.** For (a), consider the tangent-cotangent isomorphism  $\tilde{\omega} : TM \rightarrow T^*M$ . Set  $X := \tilde{\omega}^{-1}(-\alpha)$ . As a composition of smooth functions,  $X : M \rightarrow TM$  is smooth. Moreover,  $X_p = \tilde{\omega}^{-1}(-\alpha_p) \in T_pM$ . Thus  $X \in \mathfrak{X}(M)$ . Moreover

$$i_X\omega = \tilde{\omega}(\tilde{\omega}^{-1}(-\alpha)) = -\alpha.$$

Since  $\tilde{\omega}$  is an isomorphism,  $X$  is unique. Cartan's magic formula together with the assumption  $\omega = -d\alpha$  yields

$$\mathcal{L}_X\omega = di_X\omega + i_Xd\omega = di_X\omega = -d\alpha = \omega.$$

For proving (b), assume that  $L_X\omega = \omega$  for some  $X \in \mathfrak{X}(M)$ . Again, Cartan's magic formula yields

$$\mathcal{L}_X\omega = di_X\omega + i_Xd\omega = di_X\omega = \omega.$$

Now  $i_X\omega \in \Omega^1(M)$  and thus  $\omega$  is exact.

For proving (c), an application of the Fisherman's formula yields

$$\frac{d}{dt}(\exp tX)^*\omega = (\exp tX)^*\mathcal{L}_X\omega = (\exp tX)^*\omega$$

and the property of the flow  $\exp tX$  yields

$$(\exp tX|_0)^*\omega = \text{id}_M^*\omega = \omega.$$

Also we have that  $\frac{d}{dt}e^t\omega = e^t\omega$  and  $e^0\omega = \omega$ . Hence  $(\exp tX)^*\omega$  and  $e^t\omega$  solve the same locally uniquely solvable initial value problem and are therefore locally equal.

For proving (d), observe that for  $Y \in \mathfrak{X}(M)$  we have that

$$\begin{aligned}\omega(g_*X, Y) \circ g &= (g^*\omega)(X, g_*^{-1}Y) \\ &= \omega(X, g_*^{-1}Y) \\ &= i_X\omega(g_*^{-1}Y) \\ &= -\alpha(g_*^{-1}Y) \circ g^{-1} \\ &= -(g^*\alpha)(g_*^{-1}Y) \\ &= -\alpha(Y) \circ g \\ &= i_X\omega(Y) \circ g \\ &= \omega(X, Y) \circ g.\end{aligned}$$

Thus  $\tilde{\omega}(g_*X) = \tilde{\omega}(X)$  and since  $\tilde{\omega}$  is an isomorphism, we have that  $g_*X = X$ .

For proving (e), let us define  $\rho_t := g \circ \exp tX \circ g^{-1}$ . Then

$$\rho_0 = g \circ \exp tX|_{t=0} \circ g^{-1} = \text{id}_M$$

and

$$\begin{aligned}\frac{d}{dt}\rho_t(p) &= \frac{d}{dt}g(\exp tX(g^{-1}(p))) \\ &= dg_{\exp tX(g^{-1}(p))}\frac{d}{dt}\exp tX(g^{-1}(p)) \\ &= dg_{\exp tX(g^{-1}(p))}X(\exp tX(g^{-1}(p))) \\ &= (g_*X)_{(g \circ \exp tX \circ g^{-1})(p)}.\end{aligned}$$

Thus  $\rho_t$  is the flow of the vector field  $g_*X$ . By part (d)  $g_*X$  coincides with  $X$  and thus  $\rho_t$  is also the flow of  $X$ . But flows are unique and thus

$$g \circ \exp tX \circ g^{-1} = \exp tX$$

from which the claim follows.

For (f), let us compute  $\mathcal{L}_X\omega_0$ . We have

$$\begin{aligned}\mathcal{L}_X\omega_0 &= di_X\omega_0 \\ &= d\sum_{i=1}^n((i_X dx_i) \wedge dy_i - dx_i \wedge (i_X dy_i))\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} d \sum_{i=1}^n (x_i dy_i - y_i dx_i) \\
 &= \frac{1}{2} \sum_{i=1}^n (dx_i \wedge dy_i - dy_i \wedge dx_i) \\
 &= \omega_0.
 \end{aligned}$$

**Exercise 1.3.** (a)

**Solution 1.3.** For showing (a), let us consider  $X := f^i \frac{\partial}{\partial x^i} + g^i \frac{\partial}{\partial \xi^i}$ . Then

$$\begin{aligned}
 i_X \omega &= i_X \sum_{i=1}^n dx_i \wedge d\xi_i \\
 &= \sum_{i=1}^n ((i_X dx_i) \wedge d\xi_i - dx_i \wedge (i_X d\xi_i)) \\
 &= \sum_{i=1}^n (f^i d\xi_i - g^i dx_i).
 \end{aligned}$$

Comparing this with  $-\alpha$  yields  $f^i = 0$  and  $g^i = \xi^i$  for all  $i = 1, \dots, n$ . Hence

$$X = \xi^i \frac{\partial}{\partial \xi^i}.$$