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CHAPTER 1

Foundations

Basic Category Theory

Categories. We use the first order theory of Neumann-Bernays-Gödel (BNG) as described in [Men15, p. 231].

Definition 1.1 (Category). A *category* \mathcal{C} consists of

- A class $\text{ob}(\mathcal{C})$, called the *objects of* \mathcal{C} .
- A class $\text{mor}(\mathcal{C})$, called the *morphisms of* \mathcal{C} .
- Two functions $\text{dom} : \text{mor}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{C})$ and $\text{cod} : \text{mor}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{C})$, which assign to each morphism f in \mathcal{C} its **domain** and **codomain**, respectively.
- For each $X \in \text{ob}(\mathcal{C})$ a function $\text{ob}(\mathcal{C}) \rightarrow \text{mor}(\mathcal{C})$ which assigns a morphism id_X such that $\text{dom id}_X = \text{cod id}_X = X$.
- A function

$$\circ : \{(g, f) \in \text{mor}(\mathcal{C}) \times \text{mor}(\mathcal{C}) : \text{dom } g = \text{cod } f\} \rightarrow \text{mor}(\mathcal{C}) \quad (1)$$

mapping (g, f) to $g \circ f$, called **composition**, such that $\text{dom}(g \circ f) = \text{dom } f$ and $\text{cod}(g \circ f) = \text{cod } g$.

Subject to the following axioms:

- **(Associativity Axiom)** For all $f, g, h \in \text{mor}(\mathcal{C})$ with $\text{dom } h = \text{cod } g$ and $\text{dom } g = \text{cod } f$, we have that

$$(h \circ g) \circ f = h \circ (g \circ f). \quad (2)$$

- **(Unit Axiom)** For all $f \in \text{mor}(\mathcal{C})$ with $\text{dom } f = X$ and $\text{cod } f = Y$ we have that

$$f = f \circ \text{id}_X = \text{id}_Y \circ f. \quad (3)$$

Remark 1.1. Let \mathcal{C} be a category. For $X, Y \in \text{ob}(\mathcal{C})$ we will abbreviate

$$\mathcal{C}(X, Y) := \{f \in \text{mor}(\mathcal{C}) : \text{dom } f = X \text{ and } \text{cod } f = Y\}.$$

Moreover, $f \in \mathcal{C}(X, Y)$ is depicted as

$$f : X \rightarrow Y. \quad (4)$$

Example 1.1. Let $*$ be a single, not nearer specified object. Consider as morphisms the class of all cardinal numbers and as composition cardinal addition. By [Hal12, pp. 112–113], cardinal addition is associative and \emptyset serves for the identity id_* .

Definition 1.2 (Locally Small, Hom-Set). A category \mathcal{C} is said to be **locally small** if for all $X, Y \in \mathcal{C}$, $\mathcal{C}(X, Y)$ is a set. If \mathcal{C} is locally small, $\mathcal{C}(X, Y)$ is called a **hom-set** for all $X, Y \in \mathcal{C}$.

Functors.

Definition 1.3 (Functor). Let \mathcal{C} and \mathcal{D} be categories. A **functor** $F : \mathcal{C} \rightarrow \mathcal{D}$ is a pair of functions (F_1, F_2) , $F_1 : \text{ob}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{D})$, called the **object function** and $F_2 : \text{mor}(\mathcal{C}) \rightarrow \text{mor}(\mathcal{D})$, called the **morphism function**, such that for every morphism $f : X \rightarrow Y$ we have that $F_2(f) : F_1(X) \rightarrow F_1(Y)$ and (F_1, F_2) is subject to the following **compatibility conditions**:

- For all $X \in \text{ob}(\mathcal{C})$, $F_2(\text{id}_X) = \text{id}_{F_1(X)}$.
- For all $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ we have that $F_2(g \circ f) = F_2(g) \circ F_2(f)$.

Remark 1.2. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. It is convenient to denote the components F_1 and F_2 also with F .

Subcategories.

Definition 1.4 (Subcategory). Let \mathcal{C} be a category. A **subcategory** \mathcal{S} of \mathcal{C} consists of

- A subclass $\text{ob}(\mathcal{S}) \subseteq \text{ob}(\mathcal{C})$.
- A subclass $\text{mor}(\mathcal{S}) \subseteq \text{mor}(\mathcal{C})$.

Subject to the following conditions:

- For all $X \in \mathcal{S}$, $\text{id}_X \in \text{mor}(\mathcal{S})$.

Example 1.2 (Top_{*}). Define the objects of Top_* to be the class of all tuple (X, p) , where X is a topological space and $p \in X$. Moreover, given objects (X, p) and (Y, q) in Top_* , define $\text{Top}_*((X, p), (Y, q)) := \{f \in \text{Top}(X, Y) : f(p) = q\}$. It is easy to check that Top_* is a category, called the **category of pointed topological spaces**.

Limits.

Definition 1.5 (Diagram). Let \mathcal{C} be a category and \mathbf{A} a small category. A functor $\mathbf{A} \rightarrow \mathcal{C}$ is called a **diagram in \mathcal{C} of shape \mathbf{A}** .

Definition 1.6 (Cone and Limit). Let \mathcal{C} be a category and $D : \mathbf{A} \rightarrow \mathcal{C}$ a diagram in \mathcal{C} of shape \mathbf{A} . A **cone on D** is a tuple $(C, (f_\alpha)_{\alpha \in \mathbf{A}})$, where $C \in \mathcal{C}$ is an object, called the **vertex** of the cone, and a family of arrows in \mathcal{C}

$$(C \xrightarrow{f_\alpha} D(\alpha))_{\alpha \in \mathbf{A}}. \quad (5)$$

such that for all morphisms $f \in \mathbf{A}$, $f : \alpha \rightarrow \beta$, the triangle

$$\begin{array}{ccc}
 & D(\alpha) & \\
 f_\alpha \nearrow & \downarrow D(f) & \\
 C & & \\
 f_\beta \searrow & \downarrow & \\
 & D(\beta) &
 \end{array}$$

commutes. A **(small) limit of D** is a cone $(L, (\pi_\alpha)_{\alpha \in \mathbf{A}})$ with the property that for any other cone $(C, (f_\alpha)_{\alpha \in \mathbf{A}})$ there exists a unique morphism $\bar{f} : C \rightarrow L$ such that $\pi_\alpha \circ \bar{f} = f_\alpha$ holds for every $\alpha \in \mathbf{A}$.

Remark 1.3. In the setting of definition 1.6, if $(L, (\pi_\alpha)_{\alpha \in \mathbf{A}})$ is a limit of D , we sometimes referring to L only as the limit of D and we write

$$L = \lim_{\leftarrow \mathbf{A}} D. \quad (6)$$

CHAPTER 2

The Fundamental Group

The Fundamental Grupoid

Construction of the fundamental Grupoid.

Lemma 2.1 (Gluing Lemma). *Let $X, Y \in \text{ob}(\text{Top})$, $(X_\alpha)_{\alpha \in A}$ a finite closed cover of X and $(f_\alpha)_{\alpha \in A}$ a finite family of maps $f_\alpha \in \text{Top}(X_\alpha, Y)$ such that $f_\alpha|_{X_\alpha \cap X_\beta} = f_\beta|_{X_\alpha \cap X_\beta}$ for all $\alpha, \beta \in A$. Then there exists a unique $f \in \text{Top}(X, Y)$ such that $f|_{X_\alpha} = f_\alpha$ for all $\alpha \in A$.*

Proof. Let $x \in X$. Since $(X_\alpha)_{\alpha \in A}$ is a cover of X , we find $\alpha \in A$ such that $x \in X_\alpha$. Define $f(x) := f_\alpha(x)$. This is well defined, since if $x \in X_\alpha \cap X_\beta$ for some $\beta \in A$, we have that $f(x) = f_\beta(x) = f_\alpha(x)$. Clearly $f|_{X_\alpha} = f_\alpha$ for all $\alpha \in A$ and f is unique. Let us show continuity. To this end, let $K \subseteq Y$ be closed. Then

$$\begin{aligned} f^{-1}(K) &= X \cap f^{-1}(K) \\ &= \bigcup_{\alpha \in A} X_\alpha \cap f^{-1}(K) \\ &= \bigcup_{\alpha \in A} (X_\alpha \cap f^{-1}(K)) \\ &= \bigcup_{\alpha \in A} (X_\alpha \cap f_\alpha^{-1}(K)). \end{aligned}$$

Since each f_α is continuous, $f_\alpha^{-1}(K)$ is closed in X_α for each $\alpha \in A$ and thus since X_α is closed, $f^{-1}(K)$ is closed as a finite union of closed sets. \square

Theorem 2.1. *There is a functor $\text{Top} \rightarrow \text{Grpd}$.*

Proof. The proof is divided into several steps. Let us denote $\Pi : \text{Top} \rightarrow \text{Grpd}$ for the claimed functor.

Step 1: Definition of Π on objects. Let $X, Y \in \text{ob}(\text{Top})$, $f, g \in \text{Top}(X, Y)$ and $A \subseteq X$. A map $F \in \text{Top}(X \times I, Y)$ is called a **homotopy from X to Y relative to A** , if

- $F(x, 0) = f(x)$, for all $x \in X$.
- $F(x, 1) = g(x)$, for all $x \in X$.
- $F(x, t) = f(x) = g(x)$, for all $x \in A$ and for all $t \in I$.

If there exists a homotopy between f and g relative to A we say that f and g are **homotopic relative to A** and write $f \simeq_A g$. If we want to emphasize the homotopy relative to A , we write $F : f \simeq_A g$.

Lemma 2.2. *Let $X, Y \in \text{ob}(\text{Top})$ and $A \subseteq X$. Then being homotopic relative to A is an equivalence relation on $\text{Top}(X, Y)$.*

Proof. Define a binary relation $R_A \subseteq \text{Top}(X, Y) \times \text{Top}(X, Y)$ by

$$f R_A g \quad :\Leftrightarrow \quad f \simeq_A g.$$

Let $f \in \text{Top}(X, Y)$. Define $F \in \text{Top}(X \times I, Y)$ by

$$F(x, t) := f(x).$$

Then clearly $F : f \simeq_A f$. Hence R_A is reflexive.

Let $g \in \text{Top}(X, Y)$ and assume that $f R_A g$. Thus $G : f \simeq_A g$. Define $F \in \text{Top}(X \times I, Y)$ by

$$F(x, t) := G(x, 1 - t).$$

Then it is easy to check that $F : g \simeq_A f$ and so R_A is symmetric.

Finally, let $h \in \text{Top}(X, Y)$ and suppose that $f R_A g$ and $g R_A h$. Hence $F_1 : f \simeq_A g$ and $F_2 : g \simeq_A h$. Define $F \in \text{Top}(X \times I, Y)$ by

$$F(x, t) := \begin{cases} F_1(x, 2t) & 0 \leq t \leq \frac{1}{2}, \\ F_2(x, 2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Continuity of F follows by an application of the gluing lemma 2.1. Then it is easy to check that $F : f \simeq_A h$ and hence R_A is transitive. \square

Let $X \in \text{ob}(\text{Top})$ and u a path in X from p to q . Define the **path class $[u]$ of u** by $[u] := [u]_{R_{\partial I}}$. Define now

- $\text{ob}(\Pi(X)) := X$.
- $\Pi(X)(p, q) := \{[u] : u \text{ is a path from } p \text{ to } q\}$ for all $p, q \in X$.
- Let $p \in X$. Then define $\text{id}_p \in \Pi(X)(p, p)$ by $\text{id}_p := [c_p]$, where c_p is the constant path defined by $c_p(s) := p$ for all $s \in I$.
- And $\Pi(X)(q, r) \times \Pi(X)(p, q) \rightarrow \Pi(X)(p, r)$ by

$$([v], [u]) \mapsto [u * v]$$

Where $u * v \in \text{Top}(p, r)$ is the **concatenated path of u and v** , defined by

$$(u * v)(s) := \begin{cases} u(2s) & 0 \leq s \leq \frac{1}{2}, \\ v(2s - 1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Continuity follows again from the gluing lemma 2.1 whereas well definedness follows from the next lemma.

Lemma 2.3. Suppose that $[u_1], [u_2] \in \Pi(X)(p, q)$ and $[v_1], [v_2] \in \Pi(X)(q, r)$ such that $[u_1] = [u_2]$ and $[v_1] = [v_2]$. Then $[u_1 * v_1] = [u_2 * v_2]$.

Proof. By assumption we have $G : u_1 \simeq_{\partial I} u_2$ and $H : v_1 \simeq_{\partial I} v_2$. Define $F \in \text{Top}(I \times I, X)$ by

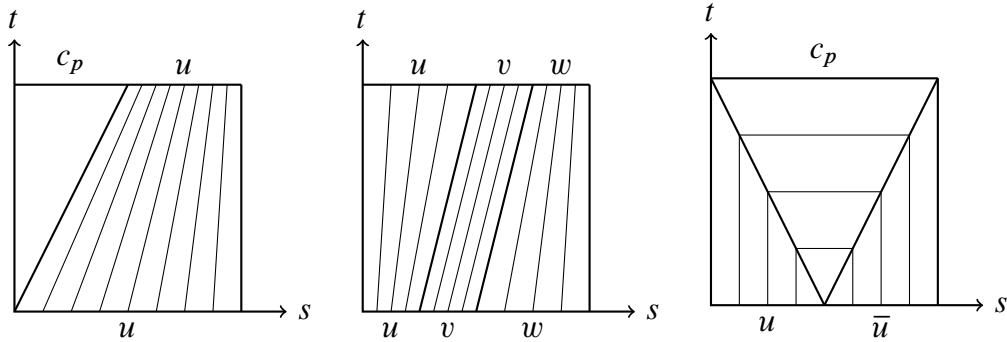
$$F(s, t) := \begin{cases} G(2s, t) & 0 \leq s \leq \frac{1}{2}, \\ H(2s - 1, t) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Again, continuity follows from the gluing lemma 2.1 and it is easy to check that $F : u_1 * v_1 \simeq_{\partial I} u_2 * v_2$. \square

Let us now check that $\Pi(X)$ is indeed a category. Let $[u] \in \Pi(X)(p, q)$. We want to show that $u \simeq_{\partial I} c_p * u$. To this end, we consider figure 1a and conclude that a suitable homotopy is given by $F \in \text{Top}(I \times I, X)$ defined by

$$F(s, t) := \begin{cases} p & 0 \leq 2s \leq t, \\ u\left(\frac{2s - t}{2 - t}\right) & t \leq 2s \leq 2. \end{cases}$$

Similarly, considering figure 1b leads to $F \in \text{Top}(I \times I, X)$ defined by



(A) $u \simeq_{\partial I} c_p * u$.

(B) $(u * v) * w \simeq_{\partial I} u * (v * w)$.

(C) $u * \bar{u} \simeq_{\partial I} c_p$.

FIGURE 1. Visualization of the proof that $\Pi(X)$ is a groupoid object.

$$F(s, t) := \begin{cases} u\left(\frac{4s}{t+1}\right) & -1 \leq 4s - 1 \leq t, \\ v(4s - t - 1) & t \leq 4s - 1 \leq t + 1, \\ w\left(\frac{4s - t - 2}{4 - t - 2}\right) & t + 1 \leq 4s - 1 \leq 3. \end{cases}$$

Lastly, we check that $\Pi(X)$ is a grupoid. To this end, for a path u from p to q , define its **reverse path** \bar{u} by

$$\bar{u}(s) := u(1 - s).$$

We claim that $u * \bar{u} \simeq_{\partial I} c_p$. From figure 1c we deduce that $F \in \text{Top}(I \times I, X)$ is given by

$$F(s, t) := \begin{cases} u(2s) & 0 \leq 2s \leq 1 - t, \\ u(1 - t) & 1 - t \leq 2s \leq t + 1, \\ \bar{u}(2s - 1) & t + 1 \leq 2s \leq 2. \end{cases}$$

Step 2: Definition of Π on morphisms. Let $f \in \text{Top}(X, Y)$. Then $\Pi(f)$ is a functor from $\Pi(X)$ to $\Pi(Y)$. Define $\Pi(f)$ as follows:

- Let $p \in \text{ob}(\Pi(X))$. Then define $\Pi(f)(p) := f(p) \in \text{ob}(\Pi(Y))$.
- Let $[u] \in \Pi(X)(p, q)$. Then define $\Pi(f)[u] := [f \circ u] \in \Pi(Y)(f(p), f(q))$. We have to check that this definition is independent of the choice of the representative.

Lemma 2.4. *Let u and v be paths from p to q in X and suppose that $[u] = [v]$. Then for any $f \in \text{Top}(X, Y)$ we also have that $[f \circ u] = [f \circ v]$.*

Proof. Suppose that $H : u \simeq_{\partial I} v$. Define $F \in \text{Top}(I \times I, Y)$ by

$$F(s, t) := (f \circ H)(s, t).$$

Then $F : f \circ u \simeq_{\partial I} f \circ v$. □

Checking that Π satisfies the functorial properties is left as an exercise. □

Exercise 0.1. Check that $\Pi : \text{Top} \rightarrow \text{Grpd}$ is indeed a functor.

The Fundamental Group.

Lemma 2.5. *Let \mathcal{G} be a locally small grupoid. Then for every $X \in \text{ob}(\mathcal{G})$, $\mathcal{G}(X, X)$ can be equipped with a group structure.*

Proof. Since \mathcal{G} is locally small, $\mathcal{G}(X, X)$ is a set for every $X \in \text{ob}(\mathcal{G})$. Define a multiplication $\mathcal{G}(X, X) \times \mathcal{G}(X, X) \rightarrow \mathcal{G}(X, X)$ by $gh := h \circ g$. Clearly, this multiplication is associative. Moreover, the identity element is given by $\text{id}_X \in \mathcal{G}(X, X)$ and since every $g \in \mathcal{G}(X, X)$ is an isomorphism, the multiplicative inverse is given by the inverse in $\mathcal{G}(X, X)$. □

Proposition 2.1. *There is a functor $\text{Top}_* \rightarrow \text{Grp}$.*

Proof. Define $\pi_1 : \text{Top}_* \rightarrow \text{Grp}$ on objects $(X, p) \in \text{Top}_*$ by

$$\pi_1(X, p) := \Pi(X)(p, p).$$

By theorem 2.1 together with lemma 2.5, $\pi_1(X, p)$ is actually a group, called the **fundamental group of X with basepoint p** . On morphisms $f \in \text{Top}_*((X, p), (Y, q))$, define

$$\pi_1(f) := \Pi(f) : \Pi(X)(p, p) \rightarrow \Pi(Y)(q, q).$$

Let $[u], [v] \in \pi_1(X, p)$. Then

$$\begin{aligned} \pi_1([u][v]) &= \Pi(f)([u][v]) \\ &= \Pi(f)[u * v] \\ &= [f \circ (u * v)] \\ &= [(f \circ u) * (f \circ v)] \\ &= \Pi(f)[u] \Pi(f)[v] \\ &= \pi_1(f)[u] \pi_1(f)[v]. \end{aligned}$$

Thus $\pi_1(f)$ is a morphism in Grp. Functoriality of π_1 immediately follows from the functoriality of Π . \square

$\pi_1(\mathbb{S}^1)$.

Definition 2.1 (Exponential Quotient Map). The mapping $\varepsilon : \mathbb{R} \rightarrow \mathbb{S}^1$ defined by

$$\varepsilon(x) := e^{2\pi i x} \tag{7}$$

is called the **exponential quotient map**.

Proposition 2.2 (Lifting Property of the Circle). Let $n \in \mathbb{Z}$, $n \geq 0$, $X \subseteq \mathbb{R}^n$ compact and convex, $p \in X$, $f \in \text{Top}_*((X, p), (\mathbb{S}^1, 1))$ and $m \in \mathbb{Z}$. Then there exists a unique map $\tilde{f} \in \text{Top}_*((X, p), (\mathbb{R}, m))$, called the **lifting of f** , such that

$$\begin{array}{ccc} & (\mathbb{R}, m) & \\ \tilde{f} \nearrow & \downarrow \varepsilon & \\ (X, p) & \xrightarrow{f} & (\mathbb{S}^1, 1) \end{array}$$

commutes.

Proof. We show first existence and then uniqueness.

Step 1: Existence. Since X is compact and f is continuous, f is uniformly continuous on X . Thus we find $\delta > 0$ such that $|f(x) - f(y)| < 2$, whenever $|x - y| < \delta$, i.e. $f(x)$ and $f(y)$ are not antipodal points. Moreover, since X is compact, X is bounded and hence we find $N \in \mathbb{N}$, such that $|x - y| < N\delta$ holds for all $x, y \in X$. Let $x \in X$. For $0 \leq k \leq N$, define $L_k : X \rightarrow X$ by

$$L_k(x) := \left(1 - \frac{k}{N}\right)p + \frac{k}{N}x.$$

Those are well defined functions since X is convex. Moreover, each L_k is continuous. Indeed, it is easy to check that L_k is Lipschitz. Also, for each $0 \leq k < N$, $f(L_k(x))$ and $f(L_{k+1}(x))$ are not antipodal for all $x \in X$. Indeed, it is easy to check that $|L_k(x) - L_{k+1}(x)| < \delta$ holds for all $x \in X$. For $0 \leq k < N$ define $g_k : X \rightarrow \mathbb{S}^1 \setminus \{-1\}$ by

$$g_k(x) := \frac{f(L_{k+1}(x))}{f(L_k(x))}.$$

Clearly g_k is well defined and continuous as a composition of continuous functions. Let $\text{Log} : \mathbb{S}^1 \setminus \{-1\} \rightarrow \mathbb{C}$ denote the principal branch of the logarithm. Define $\tilde{f} : X \rightarrow \mathbb{R}$ by

$$\tilde{f}(x) := m + \frac{1}{2\pi i} \sum_{k=0}^{N-1} \text{Log}(g_k(x)).$$

Clearly, \tilde{f} is continuous and moreover we have that $\tilde{f} = m$ since $g_k(p) = 1$ for all $0 \leq k < N$. Finally, for any $x \in X$ we have that

$$(\varepsilon \circ \tilde{f})(x) = \varepsilon(m) \prod_{k=0}^{N-1} g_k(x) = \frac{f(L_N(x))}{f(L_0(x))} = \frac{f(x)}{f(p)} = f(x).$$

Step 2: Uniqueness. Suppose $\tilde{g} \in \text{Top}_*((X, p), (\mathbb{R}, m))$ is another such function. Define $\varphi \in \text{Top}_*((X, p), (\mathbb{R}, 0))$ by

$$\varphi(x) := \tilde{f}(x) - \tilde{g}(x).$$

Then clearly $\varepsilon \circ \varphi = 1$ and thus $\varphi(X) \subseteq \mathbb{Z}$. Since X is convex, X is connected and so $\varphi = 0$. □

Corollary 2.1. *Let $u, v \in \Omega(\mathbb{S}^1, 1)$ such that $[u] = [v]$. If $\tilde{u}, \tilde{v} : (I, 0) \rightarrow (\mathbb{R}, 0)$ are the liftings of u and v , respectively, then $[\tilde{u}] = [\tilde{v}]$.*

Proof. Let $F : u \simeq_{\partial I} v$. By proposition 2.2, we find $\tilde{F} \in \text{Top}_*((I \times I, (0, 0)), (\mathbb{R}, 0))$, such that $\varepsilon \circ \tilde{F} = F$. We claim that $\tilde{F} : \tilde{u} \simeq_{\partial I} \tilde{v}$. For $s \in I$ define $\tilde{u}_0(s) := \tilde{F}(s, 0)$. Then $\tilde{u}_0(0) = \tilde{F}(0, 0) = 0$ and since \tilde{u}_0 is continuous we have that $\tilde{u}_0 \in \text{Top}_*((I, 0), (\mathbb{R}, 0))$. Moreover

$$(\varepsilon \circ \tilde{u}_0)(s) = \varepsilon(\tilde{F}(s, 0)) = F(s, 0) = u(s)$$

for all $s \in I$ and thus \tilde{u}_0 is a lifting of u . But by proposition 2.2, liftings are unique and thus $\tilde{u}_0 = \tilde{u}$. Next define $\tilde{w}_0(t) := \tilde{F}(0, t)$ for all $t \in I$. Then $\tilde{w}_0(0) = \tilde{F}(0, 0) = 0$ and so $\tilde{w}_0 \in \text{Top}_*((I, 0), (\mathbb{R}, 0))$. Moreover

$$(\varepsilon \circ \tilde{w}_0)(t) = \varepsilon(\tilde{F}(0, t)) = F(0, t) = u(0) = v(0) = 1.$$

for all $t \in I$. Thus

$$\begin{array}{ccc} & & (\mathbb{R}, 0) \\ & \nearrow \tilde{w}_0 & \downarrow \varepsilon \\ (I, 0) & \xrightarrow{c_1} & (\mathbb{S}^1, 1) \end{array}$$

commutes. But also c_0 makes the above diagram commute. By uniqueness, $\tilde{w}_0 = c_0$. Define $\tilde{v}_0(s) := \tilde{F}(s, 1)$ for all $s \in I$. Then $\tilde{v}_0(0) = \tilde{F}(0, 1) = \tilde{w}_0(1) = 0$ and it is easy to check that \tilde{v}_0 is a lift for v . Hence $\tilde{v}_0 = \tilde{v}$. Finally, define $\tilde{w}_1(t) := \tilde{F}(1, t)$ for all $t \in I$. Then $\tilde{w}_1(0) = \tilde{F}(1, 0) = \tilde{u}(1)$ and thus $\tilde{w}_1 \in \text{Top}_*((I, 0), (\mathbb{R}, \tilde{u}(0)))$. Moreover

$$(\varepsilon \circ \tilde{w}_1)(t) = \varepsilon(\tilde{F}(1, t)) = F(1, t) = v(1) = u(1) = 1$$

for all $t \in I$. By proposition 2.2, we have again that $\tilde{w}_1 = c_{\tilde{u}(1)}$. So $F : \tilde{u} \simeq_{\partial I} \tilde{v}$. \square

Definition 2.2 (Degree). Let $u \in \Omega(\mathbb{S}^1, 1)$. The **degree of u** , written $\deg u$, is defined by $\deg u := \tilde{u}(1)$, where \tilde{u} is the unique lift of u such that $\tilde{u}(0) = 0$.

Theorem 2.2 (Fundamental Group of the Circle). $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.

Proof.

$\square<++>$

CHAPTER 3

Singular Homology

Free Abelian Groups

Proposition 3.1. *The forgetful functor $U : \mathbf{Ab} \rightarrow \mathbf{Set}$ admits a left adjoint.*

Proof. We have to construct a functor $F : \mathbf{Set} \rightarrow \mathbf{Ab}$. Let S be a set. Define

$$F(S) := \{f \in \mathbb{Z}^S : \text{supp } f \text{ is finite}\}.$$

Equipped with pointwise addition, $F(S)$ is an abelian group. There is a natural inclusion $\iota : S \hookrightarrow U(F(S))$ sending $x \in S$ to the function taking the value one at x and zero else. Hence we may regard elements of $F(S)$ as formal linear combinations $\sum_{x \in S} m_x x$, where $m_x \in \mathbb{Z}$ for all $x \in S$. Let $G \in \mathbf{ob}(\mathbf{Ab})$ be an abelian group and $\varphi \in \mathbf{Ab}(F(S), G)$ a morphism of groups. Define $\bar{\varphi} \in \mathbf{Set}(S, U(G))$ by $\bar{\varphi} := U(\varphi)$. Conversely, if we have $f \in \mathbf{Set}(S, U(G))$, define $\bar{f} \in \mathbf{Ab}(F(S), G)$ by $\bar{f}(\sum_{x \in S} m_x x) := \sum_{x \in S} m_x f(x)$. This is well defined since all but finitely many m_x are zero and G is abelian. It is easy to check that \bar{f} is indeed a morphism of groups. Let $\varphi \in \mathbf{Ab}(F(S), G)$. Then

$$\begin{aligned} \bar{\bar{\varphi}}\left(\sum_{x \in S} m_x x\right) &= \sum_{x \in S} m_x \bar{\varphi}(x) \\ &= \sum_{x \in S} m_x U(\varphi)(x) \\ &= \sum_{x \in S} m_x \varphi(x) \\ &= \varphi\left(\sum_{x \in S} m_x x\right). \end{aligned}$$

And for $f \in \mathbf{Set}(S, U(G))$ we have that

$$\bar{\bar{f}}(x) = U(\bar{f})(x) = \bar{f}(x) = f(x).$$

Hence $\bar{\bar{\varphi}} = \varphi$ and $\bar{\bar{f}} = f$ and so we have a bijection

$$\mathbf{Ab}(F(S), G) \cong \mathbf{Set}(S, U(G)).$$

The mapping $f \mapsto \bar{f}$ will be referred to as **extending by linearity**. To check naturality in S and G is left as an exercise. \square

Exercise 0.1. Check the naturality of the bijection in proposition 3.1. Also check that $F : \text{Set} \rightarrow \text{Ab}$ is indeed a functor. F is called the **free functor from Set to Ab**.

Definition 3.1 (Free Abelian Group). Let $F : \text{Set} \rightarrow \text{Ab}$ be the free functor. For any set S , we call $F(S)$ the **free group generated by S** .

Chain Complexes

Definition 3.2 (Chain Complex). A **chain complex** is a tuple $(C_\bullet, \partial_\bullet)$ consisting of a sequence $(C_n)_{n \in \mathbb{Z}}$ in $\text{ob}(\text{Ab})$ and a sequence $(\partial_n)_{n \in \mathbb{Z}}$ in $\text{mor}(\text{Ab})$, called **boundary operators**, such that we have $\partial_n \in \text{Ab}(C_n, C_{n-1})$ and $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$.

Definition 3.3 (Chain Maps). Let $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ be two chain complexes. A **chain map** $f_\bullet : C_\bullet \rightarrow C'_\bullet$ is a sequence $(f_n)_{n \in \mathbb{Z}}$ in $\text{mor}(\text{Ab})$ such that $f_n \in \text{Ab}(C_n, C'_n)$ and the diagram

$$\begin{array}{ccc} C_n & \xrightarrow{\partial_n} & C_{n-1} \\ f_n \downarrow & & \downarrow f_{n-1} \\ C'_n & \xrightarrow{\partial'_n} & C'_{n-1} \end{array}$$

commutes for all $n \in \mathbb{Z}$.

Proposition 3.2. There is a category with objects chain complexes and morphisms chain maps.

Proof. Let $f_\bullet : C_\bullet \rightarrow C'_\bullet$ and $g_\bullet : C'_\bullet \rightarrow C''_\bullet$ be chain maps. Define a map $g_\bullet \circ f_\bullet$ by $g_n \circ f_n$ for each $n \in \mathbb{Z}$. This defines a chain map. Moreover, for each chain complex C_\bullet define id_{C_\bullet} by id_{C_n} for all $n \in \mathbb{Z}$. It is easy to check, that then \circ is associative and the identity laws hold. \square

Definition 3.4 (Comp). The category in 3.2 is called the **category of chain complexes** and we refer to it as **Comp**.

Theorem 3.1. There is a functor $\text{Top} \rightarrow \text{Comp}$.

Proof. The proof is divided into several steps. Let us denote $C_\bullet : \text{Top} \rightarrow \text{Comp}$ for the claimed functor.

Step 1: Construction of a sequence of abelian groups. Let $v_0, \dots, v_k \in \mathbb{R}^n$ for some $n, k \in \mathbb{N}$. We say that (v_0, \dots, v_k) is **affinely independent** if $(v_1 - v_0, \dots, v_k - v_0)$

is linearly independent. We define the ***k-simplex spanned by (v_0, \dots, v_k)*** , written $[v_0, \dots, v_k]$, to be

$$[v_0, \dots, v_k] := \left\{ \sum_{i=0}^k s_i v_i : s_i \geq 0 \text{ for all } i = 0, \dots, k \text{ and } \sum_{i=0}^k s_i = 1 \right\}. \quad (8)$$

equipped with the subspace topology. Moreover, we define the ***standard n -simplex Δ^n*** to be the n -simplex spanned by (e_0, \dots, e_n) where $(e_{i+1})_i$ is the standard basis of \mathbb{R}^{n+1} . Let $X \in \text{ob}(\text{Top})$. Define a ***singular n -simplex in X*** to be a map $\sigma \in \text{Top}(\Delta^n, X)$. Let $n \in \mathbb{Z}$. Define

$$C_n(X) := \begin{cases} F(\text{Top}(\Delta^n, X)) & n \geq 0, \\ 0 & n < 0. \end{cases} \quad (9)$$

We will call elements of $C_n(X)$ ***singular n -chains***.

Step 2: Construction of boundary operators. Let $X \in \text{ob}(\text{Top})$ and σ a singular n -simplex in X for $n \geq 1$. We define $\varphi_k^n : \Delta^{n-1} \rightarrow \Delta^n$, called the ***k -th face map***, by

$$\varphi_k^n(s_0, \dots, s_{n-1}) := \begin{cases} (0, s_0, \dots, s_{n-1}) & k = 0, \\ (s_0, \dots, s_{k-1}, 0, s_k, \dots, s_{n-1}) & 1 \leq k \leq n-1. \end{cases} \quad (10)$$

Define now

$$\partial \sigma := \sum_{k=0}^n (-1)^k \sigma \circ \varphi_k^n \in U(C_{n-1}(X)) \quad (11)$$

to be the ***boundary of σ*** . Moreover, the ***singular boundary operator*** is defined to be $\bar{\partial}_n$ and $\bar{\partial}_n := 0$ for $n \leq 0$.

Step 3: $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. It is enough to consider $n \geq 1$, since $\partial_n \circ \partial_{n+1} = 0$ holds trivially in the other cases. Let $X \in \text{ob}(\text{Top})$ and $\sigma \in \text{Top}(\Delta^{n+1}, X)$. Then we have

$$\begin{aligned} (\partial_n \circ \partial_{n+1})(\sigma) &= \partial_n \left(\sum_{k=0}^{n+1} (-1)^k \sigma \circ \varphi_k^{n+1} \right) \\ &= \sum_{k=0}^{n+1} (-1)^k \partial_n (\sigma \circ \varphi_k^{n+1}) \\ &= \sum_{k=0}^{n+1} \sum_{j=0}^n (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\ &= \sum_{0 \leq k \leq j \leq n} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n + \sum_{0 \leq j < k \leq n+1} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\ &= \sum_{0 \leq j \leq k \leq n} (-1)^{k+j} \sigma \circ \varphi_j^{n+1} \circ \varphi_k^n + \sum_{0 \leq j < k \leq n+1} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \end{aligned}$$

$$= \sum_{0 \leq j < k \leq n+1} ((-1)^{k+j-1} \sigma \circ \varphi_j^{n+1} \circ \varphi_{k-1}^n + (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n)$$

Step 4: Construction of chain maps. Let $X, Y \in \text{ob}(\text{Top})$ and $f \in \text{Top}(X, Y)$. For $n \geq 0$, define $f_n^\# : \text{Top}(\Delta^n, X) \rightarrow U(C_n(Y))$ by $f_n^\# := f \circ \sigma$. Extending this map by linearity yields a homomorphism $f_n^\# : C_n(X) \rightarrow C_n(Y)$. Moreover, set $f_n^\# = 0$ for $n < 0$. Let $n \geq 1$ and $\sigma \in \text{Top}(\Delta^n, X)$. Then on one hand we have

$$(f_{n-1}^\# \circ \partial_n)(\sigma) = f_{n-1}^\# \left(\sum_{k=0}^n (-1)^k \sigma \circ \varphi_k^n \right) = \sum_{k=0}^n (-1)^k f \circ \sigma \circ \varphi_k^n$$

and on the other

$$(\partial_n \circ f_n^\#)(\sigma) = \partial_n(f \circ \sigma) = \sum_{k=0}^n (-1)^k f \circ \sigma \circ \varphi_k^n.$$

Step 5: Checking functorial properties. We are ready to define the functor $C_\bullet : \text{Top} \rightarrow \text{Comp}$. Let $C_\bullet(X)$ be the chain complex consisting of $(C_n(X))_{n \in \mathbb{Z}}$ and $(\partial_n)_{n \in \mathbb{Z}}$. □

APPENDIX A

Set Theory

1. Basic Concepts

Problem 1.1. Let $n \in \mathbb{N}$ and $a_{kj} \in \mathbb{C}$ for $k = 0, \dots, n+1$, $j = 0, \dots, n$. Show that

$$\sum_{k=0}^{n+1} \sum_{j=0}^n a_{kj} = \sum_{0 \leq k \leq j \leq n} a_{kj} + \sum_{0 \leq j < k \leq n+1} a_{kj}.$$

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