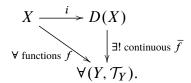
SOLUTIONS SHEET 1

YANNIS BÄHNI

Exercise 1.

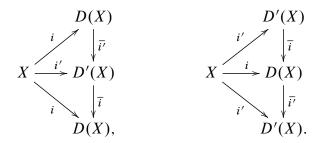
(a) The pair (D(X), i) has the universal property



Assume, that there is another pair (D'(X), i') with this property. Thus we get the two commuting diagrams



Putting them together yields



Hence

$$(\overline{i} \circ \overline{i'}) \circ i = i$$
 and $(\overline{i'} \circ \overline{i}) \circ i' = i'$.

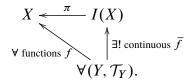
(Yannis Bähni) University of Zurich, Rämistrasse 71, 8006 Zurich *E-mail address*: yannis.baehni@uzh.ch.

Since also $\mathrm{id}_{D(X)} \circ i = i$ and $\mathrm{id}_{D'(X)} \circ i' = i'$, uniqueness implies that

$$\overline{i} \circ \overline{i'} = \mathrm{id}_{D(X)}$$
 and $\overline{i'} \circ \overline{i} = \mathrm{id}_{D'(X)}$.

Thus $D(X) \cong D'(X)$ uniquely.

(b) Let $I(X) := (X, \{\emptyset, X\})$ be the *indiscrete topological space*. Define a mapping $\pi : I(X) \to X$ by $\pi(x) := x$. Then the tuple $(I(X), \pi)$ has the following universal property:



Now the argumentation is the same as in part (a).

Exercise 2.

(b)

(a) First we show that $(\mathbb{Z}[X], X)$ has the claimed property. Let R be a unital ring with $r \in R$. Then there exists a unique homomorphism of rings $\varphi : \mathbb{Z} \to R$. Define $f : \mathbb{Z}[X] \to R$ by

$$f\left(\sum_{i=0}^{n} a_i X^i\right) := \sum_{i=0}^{n} \varphi(a_i) r^i.$$

Clearly, f(X) = r. Also it is easy to check that f is a homomorphism of rings. Assume that $g : \mathbb{Z}[X] \to R$ is a homomorphism of rings such that g(X) = r. Then

$$g\left(\sum_{i=0}^{n} a_{i} X^{i}\right) = \sum_{i=0}^{n} g(a_{i})g(X)^{i} = \sum_{i=0}^{n} g(a_{i})r^{i} = \sum_{i=0}^{n} \varphi(a_{i})r^{i} = f\left(\sum_{i=0}^{n} a_{i} X^{i}\right)$$

by the uniqueness of φ (g induces a homomorphism of rings $\mathbb{Z} \to R$). Consider the following diagram:

$$(\mathbb{Z}[X], X) \xrightarrow{\exists! f, f(X) = a} (A, a) \xrightarrow{\exists! g, g(a) = X} (\mathbb{Z}[X], X) \xrightarrow{\exists! f, f(X) = a} (A, a) .$$

Now $\mathrm{id}_{(\mathbb{Z}[X],X)}(X) = X$ and g(f(X)) = X, thus by uniqueness $g \circ f = \mathrm{id}_{(\mathbb{Z}[X],X)}$ and similarly $f \circ g = \mathrm{id}_{(A,a)}$.

Exercise 3. Existence was shown in the lecture, the so-called *free group*. Uniqueness is shown exactly as in **Exercise 1.**.

Exercise 4. Let $g, \tilde{g}: Y \to X$ be inverses of f. Then we have

$$g = g \circ id_Y = g \circ (f \circ \widetilde{g}) = (g \circ f) \circ \widetilde{g} = id_X \circ \widetilde{g} = \widetilde{g}.$$

Thus we can unambiguously write $f^{-1} := g$.

Exercise 5. That $h \circ g \circ f$ is an isomorphism immediately follows by

$$((g \circ f)^{-1} \circ g \circ (h \circ g)^{-1}) \circ (h \circ g \circ f) = \mathrm{id}_X$$

$$(h \circ g \circ f) \circ ((g \circ f)^{-1} \circ g \circ (h \circ g)^{-1}) = \mathrm{id}_W.$$

Moreover

$$((h \circ g)^{-1} \circ h) \circ g = (h \circ g)^{-1} \circ (h \circ g) = \mathrm{id}_Y$$

$$g \circ (f \circ (g \circ f)^{-1}) = (g \circ f) \circ (g \circ f)^{-1} = \mathrm{id}_Z.$$

Lemma 0.1. Let C be a category and $f: X \to Y$. Assume that there exsist $g, \tilde{g}: Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ \tilde{g} = \mathrm{id}_Y$. Then f is an isomorphism with $f^{-1} = g = \tilde{g}$. *Proof.* We have that

$$g = g \circ id_Y = g \circ (f \circ \widetilde{g}) = (g \circ f) \circ \widetilde{g} = id_X \circ \widetilde{g} = \widetilde{g}.$$

Thus *g* is invertible.

Lemma 0.2. Let C be a category and $f: X \to Y$, $g: Y \to Z$ isomorphisms. Then also $g \circ f$ is an isomorphism with $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof. We have that

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = \mathrm{id}_X$$
 and $(g \circ f) \circ (f^{-1} \circ g^{-1}) = \mathrm{id}_Z$.

Hence the statement follows by the uniqueness of the inverse.

Therefore also

$$f = (h \circ g)^{-1} \circ (h \circ g \circ f)$$
 and $h = (h \circ g \circ f) \circ (g \circ f)^{-1}$

are isomorphisms.

Exercise 6. Assume $f: X \to Y$ has the left cancellation property. Let $x, y \in X$ such that f(x) = f(y). Now let $Z := \{x, y\}$. Define two functions $c_x, c_y : Z \to X$ by $c_x(z) := x$ and $c_y(z) := y$, respectively. Now

$$f \circ c_x = f(x) = f(y) = f \circ c_y$$

holds by assumption. Thus the left cancellation property implies that $c_x = c_y$, hence x = y and f is injective. Conversly, assume that f is injective. Let $\alpha, \beta : Z \to X$ such that $f \circ \alpha = f \circ \beta$ and $z \in Z$. Then we have that $f(\alpha(z)) = f(\beta(z))$ and thus by injectivity, $\alpha(z) = \beta(z)$.