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## CHAPTER 1

### Foundations

#### Basic Category Theory

**Categories.** We use the first order theory of Neumann-Bernays-Gödel (BNG) as described in [Men15, p. 231].

**Definition 1.1 (Category).** A *category*  $\mathcal{C}$  consists of

- A class  $\text{ob}(\mathcal{C})$ , called the *objects of*  $\mathcal{C}$ .
- A class  $\text{mor}(\mathcal{C})$ , called the *morphisms of*  $\mathcal{C}$ .
- Two functions  $\text{dom} : \text{mor}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{C})$  and  $\text{cod} : \text{mor}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{C})$ , which assign to each morphism  $f$  in  $\mathcal{C}$  its **domain** and **codomain**, respectively.
- For each  $X \in \text{ob}(\mathcal{C})$  a function  $\text{ob}(\mathcal{C}) \rightarrow \text{mor}(\mathcal{C})$  which assigns a morphism  $\text{id}_X$  such that  $\text{dom id}_X = \text{cod id}_X = X$ .
- A function

$$\circ : \{(g, f) \in \text{mor}(\mathcal{C}) \times \text{mor}(\mathcal{C}) : \text{dom } g = \text{cod } f\} \rightarrow \text{mor}(\mathcal{C}) \quad (1)$$

mapping  $(g, f)$  to  $g \circ f$ , called **composition**, such that  $\text{dom}(g \circ f) = \text{dom } f$  and  $\text{cod}(g \circ f) = \text{cod } g$ .

Subject to the following axioms:

- **(Associativity Axiom)** For all  $f, g, h \in \text{mor}(\mathcal{C})$  with  $\text{dom } h = \text{cod } g$  and  $\text{dom } g = \text{cod } f$ , we have that

$$(h \circ g) \circ f = h \circ (g \circ f). \quad (2)$$

- **(Unit Axiom)** For all  $f \in \text{mor}(\mathcal{C})$  with  $\text{dom } f = X$  and  $\text{cod } f = Y$  we have that

$$f = f \circ \text{id}_X = \text{id}_Y \circ f. \quad (3)$$

**Remark 1.1.** Let  $\mathcal{C}$  be a category. For  $X, Y \in \text{ob}(\mathcal{C})$  we will abbreviate

$$\mathcal{C}(X, Y) := \{f \in \text{mor}(\mathcal{C}) : \text{dom } f = X \text{ and } \text{cod } f = Y\}.$$

Moreover,  $f \in \mathcal{C}(X, Y)$  is depicted as

$$f : X \rightarrow Y. \quad (4)$$

**Example 1.1.** Let  $*$  be a single, not nearer specified object. Consider as morphisms the class of all cardinal numbers and as composition cardinal addition. By [Hal12, pp. 112–113], cardinal addition is associative and  $\emptyset$  serves for the identity  $\text{id}_*$ .

**Definition 1.2 (Locally Small, Hom-Set).** A category  $\mathcal{C}$  is said to be **locally small** if for all  $X, Y \in \mathcal{C}$ ,  $\mathcal{C}(X, Y)$  is a set. If  $\mathcal{C}$  is locally small,  $\mathcal{C}(X, Y)$  is called a **hom-set** for all  $X, Y \in \mathcal{C}$ .

### Functors.

**Definition 1.3 (Functor).** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. A **functor**  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a pair of functions  $(F_1, F_2)$ ,  $F_1 : \text{ob}(\mathcal{C}) \rightarrow \text{ob}(\mathcal{D})$ , called the **object function** and  $F_2 : \text{mor}(\mathcal{C}) \rightarrow \text{mor}(\mathcal{D})$ , called the **morphism function**, such that for every morphism  $f : X \rightarrow Y$  we have that  $F_2(f) : F_1(X) \rightarrow F_1(Y)$  and  $(F_1, F_2)$  is subject to the following **compatibility conditions**:

- For all  $X \in \text{ob}(\mathcal{C})$ ,  $F_2(\text{id}_X) = \text{id}_{F_1(X)}$ .
- For all  $f \in \mathcal{C}(X, Y)$  and  $g \in \mathcal{C}(Y, Z)$  we have that  $F_2(g \circ f) = F_2(g) \circ F_2(f)$ .

**Remark 1.2.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. It is convenient to denote the components  $F_1$  and  $F_2$  also with  $F$ .

### Subcategories.

**Definition 1.4 (Subcategory).** Let  $\mathcal{C}$  be a category. A **subcategory**  $\mathcal{S}$  of  $\mathcal{C}$  consists of

- A subclass  $\text{ob}(\mathcal{S}) \subseteq \text{ob}(\mathcal{C})$ .
- A subclass  $\text{mor}(\mathcal{S}) \subseteq \text{mor}(\mathcal{C})$ .

Subject to the following conditions:

- For all  $X \in \mathcal{S}$ ,  $\text{id}_X \in \text{mor}(\mathcal{S})$ .

**Example 1.2 (Top<sub>\*</sub>).** Define the objects of  $\text{Top}_*$  to be the class of all tuple  $(X, p)$ , where  $X$  is a topological space and  $p \in X$ . Moreover, given objects  $(X, p)$  and  $(Y, q)$  in  $\text{Top}_*$ , define  $\text{Top}_*((X, p), (Y, q)) := \{f \in \text{Top}(X, Y) : f(p) = q\}$ . It is easy to check that  $\text{Top}_*$  is a category, called the **category of pointed topological spaces**.

### Limits.

**Definition 1.5 (Diagram).** Let  $\mathcal{C}$  be a category and  $\mathbf{A}$  a small category. A functor  $\mathbf{A} \rightarrow \mathcal{C}$  is called a **diagram in  $\mathcal{C}$  of shape  $\mathbf{A}$** .

**Definition 1.6 (Cone and Limit).** Let  $\mathcal{C}$  be a category and  $D : \mathbf{A} \rightarrow \mathcal{C}$  a diagram in  $\mathcal{C}$  of shape  $\mathbf{A}$ . A **cone on  $D$**  is a tuple  $(C, (f_\alpha)_{\alpha \in \mathbf{A}})$ , where  $C \in \mathcal{C}$  is an object, called the **vertex** of the cone, and a family of arrows in  $\mathcal{C}$

$$(C \xrightarrow{f_\alpha} D(\alpha))_{\alpha \in \mathbf{A}}. \quad (5)$$

such that for all morphisms  $f \in \mathbf{A}$ ,  $f : \alpha \rightarrow \beta$ , the triangle

$$\begin{array}{ccc}
 & D(\alpha) & \\
 f_\alpha \nearrow & \downarrow D(f) & \\
 C & & D(\beta) \\
 f_\beta \searrow & & 
 \end{array}$$

commutes. A **(small) limit of  $D$**  is a cone  $(L, (\pi_\alpha)_{\alpha \in \mathbf{A}})$  with the property that for any other cone  $(C, (f_\alpha)_{\alpha \in \mathbf{A}})$  there exists a unique morphism  $\bar{f} : C \rightarrow L$  such that  $\pi_\alpha \circ \bar{f} = f_\alpha$  holds for every  $\alpha \in \mathbf{A}$ .

**Remark 1.3.** In the setting of definition 1.6, if  $(L, (\pi_\alpha)_{\alpha \in \mathbf{A}})$  is a limit of  $D$ , we sometimes referring to  $L$  only as the limit of  $D$  and we write

$$L = \lim_{\leftarrow \mathbf{A}} D. \quad (6)$$

## CHAPTER 2

### The Fundamental Group

#### The Fundamental Grupoid

##### Construction of the fundamental Grupoid.

**Lemma 2.1 (Gluing Lemma).** *Let  $X, Y \in \text{ob}(\text{Top})$ ,  $(X_\alpha)_{\alpha \in A}$  a finite closed cover of  $X$  and  $(f_\alpha)_{\alpha \in A}$  a finite family of maps  $f_\alpha \in \text{Top}(X_\alpha, Y)$  such that  $f_\alpha|_{X_\alpha \cap X_\beta} = f_\beta|_{X_\alpha \cap X_\beta}$  for all  $\alpha, \beta \in A$ . Then there exists a unique  $f \in \text{Top}(X, Y)$  such that  $f|_{X_\alpha} = f_\alpha$  for all  $\alpha \in A$ .*

*Proof.* Let  $x \in X$ . Since  $(X_\alpha)_{\alpha \in A}$  is a cover of  $X$ , we find  $\alpha \in A$  such that  $x \in X_\alpha$ . Define  $f(x) := f_\alpha(x)$ . This is well defined, since if  $x \in X_\alpha \cap X_\beta$  for some  $\beta \in A$ , we have that  $f(x) = f_\beta(x) = f_\alpha(x)$ . Clearly  $f|_{X_\alpha} = f_\alpha$  for all  $\alpha \in A$  and  $f$  is unique. Let us show continuity. To this end, let  $K \subseteq Y$  be closed. Then

$$\begin{aligned} f^{-1}(K) &= X \cap f^{-1}(K) \\ &= \bigcup_{\alpha \in A} X_\alpha \cap f^{-1}(K) \\ &= \bigcup_{\alpha \in A} (X_\alpha \cap f^{-1}(K)) \\ &= \bigcup_{\alpha \in A} (X_\alpha \cap f_\alpha^{-1}(K)). \end{aligned}$$

Since each  $f_\alpha$  is continuous,  $f_\alpha^{-1}(K)$  is closed in  $X_\alpha$  for each  $\alpha \in A$  and thus since  $X_\alpha$  is closed,  $f^{-1}(K)$  is closed as a finite union of closed sets.  $\square$

**Theorem 2.1.** *There is a functor  $\text{Top} \rightarrow \text{Grpd}$ .*

*Proof.* The proof is divided into several steps. Let us denote  $\Pi : \text{Top} \rightarrow \text{Grpd}$  for the claimed functor.

*Step 1: Definition of  $\Pi$  on objects.* Let  $X, Y \in \text{ob}(\text{Top})$ ,  $f, g \in \text{Top}(X, Y)$  and  $A \subseteq X$ . A map  $F \in \text{Top}(X \times I, Y)$  is called a **homotopy from  $X$  to  $Y$  relative to  $A$** , if

- $F(x, 0) = f(x)$ , for all  $x \in X$ .
- $F(x, 1) = g(x)$ , for all  $x \in X$ .
- $F(x, t) = f(x) = g(x)$ , for all  $x \in A$  and for all  $t \in I$ .

If there exists a homotopy between  $f$  and  $g$  relative to  $A$  we say that  $f$  and  $g$  are **homotopic relative to  $A$**  and write  $f \simeq_A g$ . If we want to emphasize the homotopy relative to  $A$ , we write  $F : f \simeq_A g$ .

**Lemma 2.2.** *Let  $X, Y \in \text{ob}(\text{Top})$  and  $A \subseteq X$ . Then being homotopic relative to  $A$  is an equivalence relation on  $\text{Top}(X, Y)$ .*

*Proof.* Define a binary relation  $R_A \subseteq \text{Top}(X, Y) \times \text{Top}(X, Y)$  by

$$f R_A g \quad :\Leftrightarrow \quad f \simeq_A g.$$

Let  $f \in \text{Top}(X, Y)$ . Define  $F \in \text{Top}(X \times I, Y)$  by

$$F(x, t) := f(x).$$

Then clearly  $F : f \simeq_A f$ . Hence  $R_A$  is reflexive.

Let  $g \in \text{Top}(X, Y)$  and assume that  $f R_A g$ . Thus  $G : f \simeq_A g$ . Define  $F \in \text{Top}(X \times I, Y)$  by

$$F(x, t) := G(x, 1 - t).$$

Then it is easy to check that  $F : g \simeq_A f$  and so  $R_A$  is symmetric.

Finally, let  $h \in \text{Top}(X, Y)$  and suppose that  $f R_A g$  and  $g R_A h$ . Hence  $F_1 : f \simeq_A g$  and  $F_2 : g \simeq_A h$ . Define  $F \in \text{Top}(X \times I, Y)$  by

$$F(x, t) := \begin{cases} F_1(x, 2t) & 0 \leq t \leq \frac{1}{2}, \\ F_2(x, 2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Continuity of  $F$  follows by an application of the gluing lemma 2.1. Then it is easy to check that  $F : f \simeq_A h$  and hence  $R_A$  is transitive.  $\square$

Let  $X \in \text{ob}(\text{Top})$  and  $u$  a path in  $X$  from  $p$  to  $q$ . Define the **path class  $[u]$  of  $u$**  by  $[u] := [u]_{R_{\partial I}}$ . Define now

- $\text{ob}(\Pi(X)) := X$ .
- $\Pi(X)(p, q) := \{[u] : u \text{ is a path from } p \text{ to } q\}$  for all  $p, q \in X$ .
- Let  $p \in X$ . Then define  $\text{id}_p \in \Pi(X)(p, p)$  by  $\text{id}_p := [c_p]$ , where  $c_p$  is the constant path defined by  $c_p(s) := p$  for all  $s \in I$ .
- And  $\Pi(X)(q, r) \times \Pi(X)(p, q) \rightarrow \Pi(X)(p, r)$  by

$$([v], [u]) \mapsto [u * v]$$

Where  $u * v \in \text{Top}(p, r)$  is the **concatenated path of  $u$  and  $v$** , defined by

$$(u * v)(s) := \begin{cases} u(2s) & 0 \leq s \leq \frac{1}{2}, \\ v(2s - 1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Continuity follows again from the gluing lemma 2.1 whereas well definedness follows from the next lemma.

**Lemma 2.3.** Suppose that  $[u_1], [u_2] \in \Pi(X)(p, q)$  and  $[v_1], [v_2] \in \Pi(X)(q, r)$  such that  $[u_1] = [u_2]$  and  $[v_1] = [v_2]$ . Then  $[u_1 * v_1] = [u_2 * v_2]$ .

*Proof.* By assumption we have  $G : u_1 \simeq_{\partial I} u_2$  and  $H : v_1 \simeq_{\partial I} v_2$ . Define  $F \in \text{Top}(I \times I, X)$  by

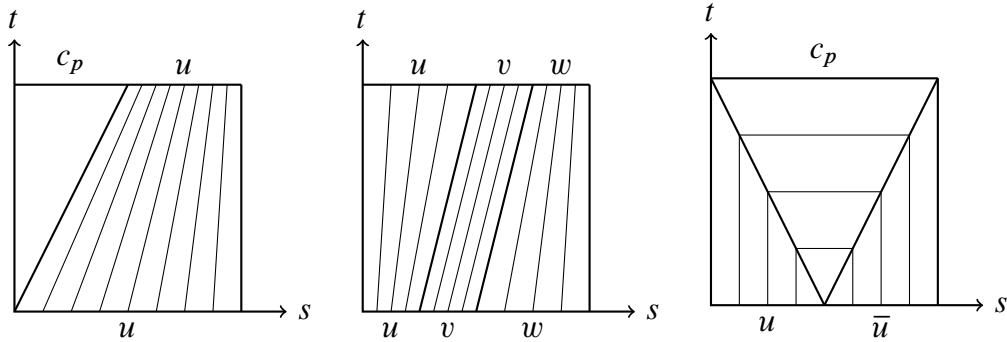
$$F(s, t) := \begin{cases} G(2s, t) & 0 \leq s \leq \frac{1}{2}, \\ H(2s - 1, t) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Again, continuity follows from the gluing lemma 2.1 and it is easy to check that  $F : u_1 * v_1 \simeq_{\partial I} u_2 * v_2$ .  $\square$

Let us now check that  $\Pi(X)$  is indeed a category. Let  $[u] \in \Pi(X)(p, q)$ . We want to show that  $u \simeq_{\partial I} c_p * u$ . To this end, we consider figure 1a and conclude that a suitable homotopy is given by  $F \in \text{Top}(I \times I, X)$  defined by

$$F(s, t) := \begin{cases} p & 0 \leq 2s \leq t, \\ u\left(\frac{2s - t}{2 - t}\right) & t \leq 2s \leq 2. \end{cases}$$

Similarly, considering figure 1b leads to  $F \in \text{Top}(I \times I, X)$  defined by



(A)  $u \simeq_{\partial I} c_p * u$ .

(B)  $(u * v) * w \simeq_{\partial I} u * (v * w)$ .

(C)  $u * \bar{u} \simeq_{\partial I} c_p$ .

FIGURE 1. Visualization of the proof that  $\Pi(X)$  is a groupoid object.

$$F(s, t) := \begin{cases} u\left(\frac{4s}{t+1}\right) & -1 \leq 4s - 1 \leq t, \\ v(4s - t - 1) & t \leq 4s - 1 \leq t + 1, \\ w\left(\frac{4s - t - 2}{4 - t - 2}\right) & t + 1 \leq 4s - 1 \leq 3. \end{cases}$$

Lastly, we check that  $\Pi(X)$  is a grupoid. To this end, for a path  $u$  from  $p$  to  $q$ , define its **reverse path**  $\bar{u}$  by

$$\bar{u}(s) := u(1 - s).$$

We claim that  $u * \bar{u} \simeq_{\partial I} c_p$ . From figure 1c we deduce that  $F \in \text{Top}(I \times I, X)$  is given by

$$F(s, t) := \begin{cases} u(2s) & 0 \leq 2s \leq 1 - t, \\ u(1 - t) & 1 - t \leq 2s \leq t + 1, \\ \bar{u}(2s - 1) & t + 1 \leq 2s \leq 2. \end{cases}$$

*Step 2: Definition of  $\Pi$  on morphisms.* Let  $f \in \text{Top}(X, Y)$ . Then  $\Pi(f)$  is a functor from  $\Pi(X)$  to  $\Pi(Y)$ . Define  $\Pi(f)$  as follows:

- Let  $p \in \text{ob}(\Pi(X))$ . Then define  $\Pi(f)(p) := f(p) \in \text{ob}(\Pi(Y))$ .
- Let  $[u] \in \Pi(X)(p, q)$ . Then define  $\Pi(f)[u] := [f \circ u] \in \Pi(Y)(f(p), f(q))$ . We have to check that this definition is independent of the choice of the representative.

**Lemma 2.4.** *Let  $u$  and  $v$  be paths from  $p$  to  $q$  in  $X$  and suppose that  $[u] = [v]$ . Then for any  $f \in \text{Top}(X, Y)$  we also have that  $[f \circ u] = [f \circ v]$ .*

*Proof.* Suppose that  $H : u \simeq_{\partial I} v$ . Define  $F \in \text{Top}(I \times I, Y)$  by

$$F(s, t) := (f \circ H)(s, t).$$

Then  $F : f \circ u \simeq_{\partial I} f \circ v$ . □

Checking that  $\Pi$  satisfies the functorial properties is left as an exercise. □

**Exercise 2.1.** Check that  $\Pi : \text{Top} \rightarrow \text{Grpd}$  is indeed a functor.

### The Fundamental Group.

**Lemma 2.5.** *Let  $\mathcal{G}$  be a locally small grupoid. Then for every  $X \in \text{ob}(\mathcal{G})$ ,  $\mathcal{G}(X, X)$  can be equipped with a group structure.*

*Proof.* Since  $\mathcal{G}$  is locally small,  $\mathcal{G}(X, X)$  is a set for every  $X \in \text{ob}(\mathcal{G})$ . Define a multiplication  $\mathcal{G}(X, X) \times \mathcal{G}(X, X) \rightarrow \mathcal{G}(X, X)$  by  $gh := h \circ g$ . Clearly, this multiplication is associative. Moreover, the identity element is given by  $\text{id}_X \in \mathcal{G}(X, X)$  and since every  $g \in \mathcal{G}(X, X)$  is an isomorphism, the multiplicative inverse is given by the inverse in  $\mathcal{G}(X, X)$ . □

**Proposition 2.1.** *There is a functor  $\text{Top}_* \rightarrow \text{Grp}$ .*

*Proof.* Define  $\pi_1 : \text{Top}_* \rightarrow \text{Grp}$  on objects  $(X, p) \in \text{Top}_*$  by

$$\pi_1(X, p) := \Pi(X)(p, p).$$



By theorem 2.1 together with lemma 2.5,  $\pi_1(X, p)$  is actually a group, called the **fundamental group of  $X$  with basepoint  $p$** . On morphisms  $f \in \text{Top}_*((X, p), (Y, q))$ , define

$$\pi_1(f) := \Pi(f) : \Pi(X)(p, p) \rightarrow \Pi(Y)(q, q).$$

Let  $[u], [v] \in \pi_1(X, p)$ . Then

$$\begin{aligned} \pi_1([u] [v]) &= \Pi(f)([u] [v]) \\ &= \Pi(f) [u * v] \\ &= [f \circ (u * v)] \\ &= [(f \circ u) * (f \circ v)] \\ &= \Pi(f) [u] \Pi(f) [v] \\ &= \pi_1(f) [u] \pi_1(f) [v]. \end{aligned}$$

Thus  $\pi_1(f)$  is a morphism in Grp. Functoriality of  $\pi_1$  immediately follows from the functoriality of  $\Pi$ .  $\square$

**Lemma 2.6.** *Let  $X \in \text{ob}(\text{Top})$ ,  $p \in X$  and  $A$  be the path component of  $X$  containing  $p$ . Then  $\pi_1(\iota)$ , where  $\iota : A \hookrightarrow X$  denotes the inclusion, is an isomorphism.*

*Proof.* Suppose  $[u] \in \ker \pi_1(\iota)$ . Then  $[\iota \circ u] = [c_p]$  and Hence  $F : \iota \circ u \simeq_{\partial I} c_p$ . Since  $I \times I$  is path connected and  $p \in F(I \times I)$ , it follows that  $F(I \times I) \subseteq A$  and thus  $F : u \simeq_{\partial I} c_p$  in  $A$  and hence  $[u] = [c_p]$ . To see that  $\pi_1(\iota)$  is surjective, just observe that  $u(I) \subseteq A$  for  $[u] \in \pi_1(X, p)$  since  $u(I)$  is path connected and  $p \in u(I)$ .  $\square$

**Lemma 2.7.** *Let  $X \in \text{ob}(\text{Top})$  be path connected and  $p, q \in X$ . Then*

$$\pi_1(X, p) \cong \pi_1(X, q).$$

*Proof.* Since  $X$  is path connected we find a path  $v$  from  $p$  to  $q$  in  $X$ . Define a mapping  $\Phi_v : \pi_1(X, p) \rightarrow \pi_1(X, q)$

$$\Phi_v [u] := [\bar{v} * u * v].$$

Clearly,  $\Phi_v$  is invertible with inverse  $\Phi_{\bar{v}}$ . Moreover, for  $[u], [w] \in \pi_1(X, p)$  we have that

$$\begin{aligned} \Phi_v([u] [w]) &= \Phi_v [u * w] \\ &= [\bar{v} * u * w * v] \\ &= [\bar{v} * u * v * \bar{v} * w * v] \\ &= [\bar{v} * u * v] [\bar{v} * w * v] \\ &= \Phi_v [u] \Phi_v [w]. \end{aligned}$$

$\square$

$\pi_1(\mathbb{S}^1)$ .

**Definition 2.1 (Exponential Quotient Map).** The mapping  $\varepsilon : \mathbb{R} \rightarrow \mathbb{S}^1$  defined by

$$\varepsilon(x) := e^{2\pi i x} \quad (7)$$

is called the *exponential quotient map*.

**Proposition 2.2 (Lifting Property of the Circle).** Let  $n \in \mathbb{Z}$ ,  $n \geq 0$ ,  $X \subseteq \mathbb{R}^n$  compact and convex,  $p \in X$ ,  $f \in \text{Top}_*((X, p), (\mathbb{S}^1, 1))$  and  $m \in \mathbb{Z}$ . Then there exists a unique map  $\tilde{f} \in \text{Top}_*((X, p), (\mathbb{R}, m))$ , called the *lifting of  $f$* , such that

$$\begin{array}{ccc} & (\mathbb{R}, m) & \\ \tilde{f} \nearrow & \downarrow \varepsilon & \\ (X, p) & \xrightarrow{f} & (\mathbb{S}^1, 1) \end{array}$$

commutes.

*Proof.* We show first existence and then uniqueness.

*Step 1: Existence.* Since  $X$  is compact and  $f$  is continuous,  $f$  is uniformly continuous on  $X$ . Thus we find  $\delta > 0$  such that  $|f(x) - f(y)| < 2$ , whenever  $|x - y| < \delta$ , i.e.  $f(x)$  and  $f(y)$  are not antipodal points. Moreover, since  $X$  is compact,  $X$  is bounded and hence we find  $N \in \mathbb{N}$ , such that  $|x - y| < N\delta$  holds for all  $x, y \in X$ . Let  $x \in X$ . For  $0 \leq k \leq N$ , define  $L_k : X \rightarrow X$  by

$$L_k(x) := \left(1 - \frac{k}{N}\right)p + \frac{k}{N}x.$$

Those are well defined functions since  $X$  is convex. Moreover, each  $L_k$  is continuous. Indeed, it is easy to check that  $L_k$  is Lipschitz. Also, for each  $0 \leq k < N$ ,  $f(L_k(x))$  and  $f(L_{k+1}(x))$  are not antipodal for all  $x \in X$ . Indeed, it is easy to check that  $|L_k(x) - L_{k+1}(x)| < \delta$  holds for all  $x \in X$ . For  $0 \leq k < N$  define  $g_k : X \rightarrow \mathbb{S}^1 \setminus \{-1\}$  by

$$g_k(x) := \frac{f(L_{k+1}(x))}{f(L_k(x))}.$$

Clearly  $g_k$  is well defined and continuous as a composition of continuous functions. Let  $\text{Log} : \mathbb{S}^1 \setminus \{-1\} \rightarrow \mathbb{C}$  denote the principal branch of the logarithm. Define  $\tilde{f} : X \rightarrow \mathbb{R}$  by

$$\tilde{f}(x) := m + \frac{1}{2\pi i} \sum_{k=0}^{N-1} \text{Log}(g_k(x)).$$

Clearly,  $\tilde{f}$  is continuous and moreover we have that  $\tilde{f} = m$  since  $g_k(p) = 1$  for all  $0 \leq k < N$ . Finally, for any  $x \in X$  we have that

$$(\varepsilon \circ \tilde{f})(x) = \varepsilon(m) \prod_{k=0}^{N-1} g_k(x) = \frac{f(L_N(x))}{f(L_0(x))} = \frac{f(x)}{f(p)} = f(x).$$

*Step 2: Uniqueness.* Suppose  $\tilde{g} \in \text{Top}_*((X, p), (\mathbb{R}, m))$  is another such function. Define  $\varphi \in \text{Top}_*((X, p), (\mathbb{R}, 0))$  by

$$\varphi(x) := \tilde{f}(x) - \tilde{g}(x).$$

Then clearly  $\varepsilon \circ \varphi = 1$  and thus  $\varphi(X) \subseteq \mathbb{Z}$ . Since  $X$  is convex,  $X$  is connected and so  $\varphi = 0$ . □

**Corollary 2.1.** *Let  $u, v \in \Omega(\mathbb{S}^1, 1)$  such that  $[u] = [v]$ . If  $\tilde{u}, \tilde{v} : (I, 0) \rightarrow (\mathbb{R}, 0)$  are the liftings of  $u$  and  $v$ , respectively, then  $[\tilde{u}] = [\tilde{v}]$ .*

*Proof.* Let  $F : u \simeq_{\partial I} v$ . By proposition 2.2, we find  $\tilde{F} \in \text{Top}_*((I \times I, (0, 0)), (\mathbb{R}, 0))$ , such that  $\varepsilon \circ \tilde{F} = F$ . We claim that  $\tilde{F} : \tilde{u} \simeq_{\partial I} \tilde{v}$ . For  $s \in I$  define  $\tilde{u}_0(s) := \tilde{F}(s, 0)$ . Then  $\tilde{u}_0(0) = \tilde{F}(0, 0) = 0$  and since  $\tilde{u}_0$  is continuous we have that  $\tilde{u}_0 \in \text{Top}_*((I, 0), (\mathbb{R}, 0))$ . Moreover

$$(\varepsilon \circ \tilde{u}_0)(s) = \varepsilon(\tilde{F}(s, 0)) = F(s, 0) = u(s)$$

for all  $s \in I$  and thus  $\tilde{u}_0$  is a lifting of  $u$ . But by proposition 2.2, liftings are unique and thus  $\tilde{u}_0 = \tilde{u}$ . Next define  $\tilde{w}_0(t) := \tilde{F}(0, t)$  for all  $t \in I$ . Then  $\tilde{w}_0(0) = \tilde{F}(0, 0) = 0$  and so  $\tilde{w}_0 \in \text{Top}_*((I, 0), (\mathbb{R}, 0))$ . Moreover

$$(\varepsilon \circ \tilde{w}_0)(t) = \varepsilon(\tilde{F}(0, t)) = F(0, t) = u(0) = v(0) = 1.$$

for all  $t \in I$ . Thus

$$\begin{array}{ccc} & & (\mathbb{R}, 0) \\ & \nearrow \tilde{w}_0 & \downarrow \varepsilon \\ (I, 0) & \xrightarrow{c_1} & (\mathbb{S}^1, 1) \end{array}$$

commutes. But also  $c_0$  makes the above diagram commute. By uniqueness,  $\tilde{w}_0 = c_0$ . Define  $\tilde{v}_0(s) := \tilde{F}(s, 1)$  for all  $s \in I$ . Then  $\tilde{v}_0(0) = \tilde{F}(0, 1) = \tilde{w}_0(1) = 0$  and it is easy to check that  $\tilde{v}_0$  is a lift for  $v$ . Hence  $\tilde{v}_0 = \tilde{v}$ . Finally, define  $\tilde{w}_1(t) := \tilde{F}(1, t)$  for all  $t \in I$ . Then  $\tilde{w}_1(0) = \tilde{F}(1, 0) = \tilde{u}(1)$  and thus  $\tilde{w}_1 \in \text{Top}_*((I, 0), (\mathbb{R}, \tilde{u}(0)))$ . Moreover

$$(\varepsilon \circ \tilde{w}_1)(t) = \varepsilon(\tilde{F}(1, t)) = F(1, t) = v(1) = u(1) = 1$$

for all  $t \in I$ . By proposition 2.2, we have again that  $\tilde{w}_1 = c_{\tilde{u}(1)}$ . So  $F : \tilde{u} \simeq_{\partial I} \tilde{v}$ . □

**Definition 2.2 (Degree).** Let  $u \in \Omega(\mathbb{S}^1, 1)$ . The **degree of  $u$** , written  $\deg u$ , is defined by  $\deg u := \tilde{u}(1)$ , where  $\tilde{u}$  is the unique lift of  $u$  such that  $\tilde{u}(0) = 0$ .

**Theorem 2.2 (Fundamental Group of the Circle).**  $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ .

*Proof.* Define  $\deg : \pi_1(\mathbb{S}^1, 1) \rightarrow \mathbb{Z}$  by  $\deg[u] := \deg u$ . This is well defined by corollary 2.1, since if  $[u] = [v]$ , then  $[\tilde{u}] = [\tilde{v}]$  and in particular  $\tilde{u}(1) = \tilde{v}(1)$ .

*Step 1:*  $\deg \in \text{Grp}(\pi_1(\mathbb{S}^1, 1), (\mathbb{Z}, +))$ . Let  $[u], [v] \in \pi_1(\mathbb{S}^1, 1)$  and  $m := \deg[u]$ ,  $n := \deg[v]$ . Moreover, let  $\tilde{u}$  and  $\tilde{v}$  denote the unique liftings of  $u$  and  $v$ , respectively, such that  $\tilde{u}(0) = 0$  and  $\tilde{v}(0) = 0$ . Define

$$\tilde{w}(s) := \begin{cases} \tilde{u}(2s) & 0 \leq s \leq \frac{1}{2}, \\ m + \tilde{v}(2s - 1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Clearly  $\tilde{w}$  is continuous and  $\tilde{w}(0) = 0$ . Hence  $\tilde{w} \in \text{Top}_*((I, 0), (\mathbb{R}, 0))$ . Also we have that  $\varepsilon \circ \tilde{w} = u * v$  and thus  $\tilde{w}$  is the lift of  $u * v$ . But  $\tilde{w}(1) = m + n$  and so

$$\deg([u][v]) = \deg[u * v] = \deg(u * v) = \tilde{w}(1) = m + n = \deg[u] + \deg[v].$$

*Step 2:  $\deg$  is injective.* Suppose  $\deg[u] = 0$ . Then  $\tilde{u}(1) = 0$  and thus  $\tilde{u} \in \Omega(\mathbb{R}, 0)$ . Since  $\mathbb{R}$  is contractible, we have that  $[\tilde{u}] = [c_0]$  and thus

$$[u] = [\varepsilon \circ \tilde{u}] = \pi_1(\varepsilon)[\tilde{u}] = \pi_1(\varepsilon)[c_0] = [c_1].$$

Thus  $\ker(\deg)$  is trivial.

*Step 3:  $\deg$  is surjective.* Let  $m \in \mathbb{Z}$ . Then

$$\deg[\varepsilon^m] = \deg \varepsilon^m = \tilde{\varepsilon}^m(1) = m.$$

□

## The Seifert-Van Kampen Theorem

### Coproducts and Pushouts in Grp.

**Proposition 2.3 (Coproducts in Grp).** Grp has all small coproducts.

*Proof.* Let  $A \in \text{ob}(\text{Set})$  and  $\mathbf{A}$  be the small category defined as the discrete category with  $\text{ob}(\mathbf{A}) := A$ , i.e.

$$\bullet \quad \bullet \quad \bullet \quad \dots \quad \bullet \quad \bullet \quad \bullet$$

Let  $D : \mathbf{A} \rightarrow \text{Grp}$  be a functor. Hence we get a family  $(G_\alpha)_{\alpha \in A}$  in Grp, where  $G_\alpha := D(\alpha)$  for all  $\alpha \in A$ . A **word** in  $(G_\alpha)_{\alpha \in A}$  is a finite sequence in  $\coprod_{\alpha \in A} G_\alpha$ . A word in  $(G_\alpha)_{\alpha \in A}$  will simply be written as  $(g_1, \dots, g_n)$ , where  $g_k \in G_\alpha$  for some  $\alpha \in A$ . The **empty word** is denoted by  $()$ . Let  $\mathcal{W}$  denote the set of all words in  $(G_\alpha)_{\alpha \in A}$ . On  $\mathcal{W}$  define a multiplication by **concatenation**

$$(g_1, \dots, g_n)(h_1, \dots, h_m) := (g_1, \dots, g_n, h_1, \dots, h_m).$$

An **elementary reduction** is an operation of one of the following forms:

- $(g_1, \dots, g_k, g_{k+1}, \dots, g_n) \mapsto (g_1, \dots, g_k g_{k+1}, \dots, g_n)$ , where  $g_k, g_{k+1} \in G_\alpha$  for some  $\alpha \in A$ .
- $(g_1, \dots, g_{k-1}, 1_\alpha, g_{k+1}, \dots, g_n) \mapsto (g_1, \dots, g_{k-1}, g_{k+1}, \dots, g_n)$ .

Let  $\sim$  denote the equivalence relation on  $\mathcal{W}$  generated by elementary reductions.

**Lemma 2.8.**  $\mathcal{W}/\sim$  together with concatenation of representatives is an element of Grp.

*Proof.* Define

$$[(g_1, \dots, g_n)] [(h_1, \dots, h_m)] := [(g_1, \dots, g_n, h_1, \dots, h_m)].$$

It is left to the reader to show that this is well defined and that  $\mathcal{W}/\sim$  is indeed a group.  $\square$

The group defined in lemma 2.8 will be denoted by  $\bigstar_{\alpha \in A} G_\alpha$  and called the **free product of  $(G_\alpha)_{\alpha \in A}$** . Let us define a cocone on  $D$ . For this consider the inclusions  $\iota_\alpha : G_\alpha \rightarrow \bigstar_{\alpha \in A} G_\alpha$  defined by

$$\iota_\alpha(g) := [(g)]$$

for all  $\alpha \in A$ . It is immediate from

$$\iota_\alpha(gh) = [(gh)] = [(g, h)] = [(g)] [(h)] = \iota_\alpha(g) \iota_\alpha(h)$$

for  $g, h \in G_\alpha$ , that  $\iota_\alpha$  is a morphism of groups. Since there are only the identity morphisms in  $A$ ,  $(\bigstar_{\alpha \in A} G_\alpha, (\iota_\alpha)_{\alpha \in A})$  is a cocone on  $D$ . Let us show that this is in fact a universal cocone. To this end, suppose that  $(C, (\varphi_\alpha)_{\alpha \in A})$  is another cocone on  $D$ . Define a mapping  $\bar{f} : \bigstar_{\alpha \in A} G_\alpha \rightarrow C$  by

$$\bar{f}[(g_1, \dots, g_n)] := \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$$

where  $g_k \in G_{\alpha_k}$ . Then  $\bar{f}$  is easily seen to be well defined since each  $\varphi_\alpha$  is a morphism of groups. Moreover, if  $g \in G_\alpha$ , then

$$(\bar{f} \circ \iota_\alpha)(g) = \bar{f}[(g)] = \varphi_\alpha(g)$$

for all  $\alpha \in A$ . Suppose that  $f : \bigstar_{\alpha \in A} G_\alpha \rightarrow C$  is another homomorphism of groups such that  $f \circ \iota_\alpha = \varphi_\alpha$  for all  $\alpha \in A$ . Then for  $[(g_1, \dots, g_n)] \in \bigstar_{\alpha \in A} G_\alpha$  we have

$$\begin{aligned} f[(g_1, \dots, g_n)] &= f([(g_1)] \cdots [(g_n)]) \\ &= f[(g_1)] \cdots f[(g_n)] \\ &= f(\iota_{\alpha_1}(g_1)) \cdots f(\iota_{\alpha_n}(g_n)) \\ &= \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n) \\ &= \bar{f}[(g_1, \dots, g_n)]. \end{aligned}$$

$\square$

**Exercise 2.2.** Check that  $\mathcal{W}/\sim$  is indeed a group with the declared group structure and that  $\bar{f}$  is indeed well defined.

**Proposition 2.4 (Pushouts in Grp).** Grp has all pushouts.

*Proof.* Consider the diagram  $D : \mathbf{A} \rightarrow \mathbf{Grp}$

$$\begin{array}{ccc} \bullet & \xrightarrow{\quad} & \bullet \\ \downarrow & & \\ \bullet & & \end{array} \xrightarrow{D} \begin{array}{ccc} G & \xrightarrow{\varphi_1} & H_1 \\ \varphi_2 \downarrow & & \\ H_2 & & \end{array}$$

□

## CHAPTER 3

### Singular Homology

#### Free Abelian Groups

**Proposition 3.1.** *The forgetful functor  $U : \mathbf{Ab} \rightarrow \mathbf{Set}$  admits a left adjoint.*

*Proof.* We have to construct a functor  $F : \mathbf{Set} \rightarrow \mathbf{Ab}$ . Let  $S$  be a set. Define

$$F(S) := \{f \in \mathbb{Z}^S : \text{supp } f \text{ is finite}\}.$$

Equipped with pointwise addition,  $F(S)$  is an abelian group. There is a natural inclusion  $\iota : S \hookrightarrow U(F(S))$  sending  $x \in S$  to the function taking the value one at  $x$  and zero else. Hence we may regard elements of  $F(S)$  as formal linear combinations  $\sum_{x \in S} m_x x$ , where  $m_x \in \mathbb{Z}$  for all  $x \in S$ . Let  $G \in \mathbf{ob}(\mathbf{Ab})$  be an abelian group and  $\varphi \in \mathbf{Ab}(F(S), G)$  a morphism of groups. Define  $\bar{\varphi} \in \mathbf{Set}(S, U(G))$  by  $\bar{\varphi} := U(\varphi)$ . Conversely, if we have  $f \in \mathbf{Set}(S, U(G))$ , define  $\bar{f} \in \mathbf{Ab}(F(S), G)$  by  $\bar{f}(\sum_{x \in S} m_x x) := \sum_{x \in S} m_x f(x)$ . This is well defined since all but finitely many  $m_x$  are zero and  $G$  is abelian. It is easy to check that  $\bar{f}$  is indeed a morphism of groups. Let  $\varphi \in \mathbf{Ab}(F(S), G)$ . Then

$$\begin{aligned} \bar{\bar{\varphi}}\left(\sum_{x \in S} m_x x\right) &= \sum_{x \in S} m_x \bar{\varphi}(x) \\ &= \sum_{x \in S} m_x U(\varphi)(x) \\ &= \sum_{x \in S} m_x \varphi(x) \\ &= \varphi\left(\sum_{x \in S} m_x x\right). \end{aligned}$$

And for  $f \in \mathbf{Set}(S, U(G))$  we have that

$$\bar{\bar{f}}(x) = U(\bar{f})(x) = \bar{f}(x) = f(x).$$

Hence  $\bar{\bar{\varphi}} = \varphi$  and  $\bar{\bar{f}} = f$  and so we have a bijection

$$\mathbf{Ab}(F(S), G) \cong \mathbf{Set}(S, U(G)).$$

The mapping  $f \mapsto \bar{f}$  will be referred to as **extending by linearity**. To check naturality in  $S$  and  $G$  is left as an exercise.  $\square$

**Exercise 3.1.** Check the naturality of the bijection in proposition 3.1. Also check that  $F : \text{Set} \rightarrow \text{Ab}$  is indeed a functor.  $F$  is called the **free functor from Set to Ab**.

**Definition 3.1 (Free Abelian Group).** Let  $F : \text{Set} \rightarrow \text{Ab}$  be the free functor. For any set  $S$ , we call  $F(S)$  the **free group generated by  $S$** .

### Chain Complexes

**Definition 3.2 (Chain Complex).** A **chain complex** is a tuple  $(C_\bullet, \partial_\bullet)$  consisting of a sequence  $(C_n)_{n \in \mathbb{Z}}$  in  $\text{ob}(\text{Ab})$  and a sequence  $(\partial_n)_{n \in \mathbb{Z}}$  in  $\text{mor}(\text{Ab})$ , called **boundary operators**, such that we have  $\partial_n \in \text{Ab}(C_n, C_{n-1})$  and  $\partial_n \circ \partial_{n+1} = 0$  for all  $n \in \mathbb{Z}$ .

**Definition 3.3 (Chain Maps).** Let  $(C_\bullet, \partial_\bullet)$  and  $(C'_\bullet, \partial'_\bullet)$  be two chain complexes. A **chain map**  $f_\bullet : C_\bullet \rightarrow C'_\bullet$  is a sequence  $(f_n)_{n \in \mathbb{Z}}$  in  $\text{mor}(\text{Ab})$  such that  $f_n \in \text{Ab}(C_n, C'_n)$  and the diagram

$$\begin{array}{ccc} C_n & \xrightarrow{\partial_n} & C_{n-1} \\ f_n \downarrow & & \downarrow f_{n-1} \\ C'_n & \xrightarrow{\partial'_n} & C'_{n-1} \end{array}$$

commutes for all  $n \in \mathbb{Z}$ .

**Proposition 3.2.** There is a category with objects chain complexes and morphisms chain maps.

*Proof.* Let  $f_\bullet : C_\bullet \rightarrow C'_\bullet$  and  $g_\bullet : C'_\bullet \rightarrow C''_\bullet$  be chain maps. Define a map  $g_\bullet \circ f_\bullet$  by  $g_n \circ f_n$  for each  $n \in \mathbb{Z}$ . This defines a chain map. Moreover, for each chain complex  $C_\bullet$  define  $\text{id}_{C_\bullet}$  by  $\text{id}_{C_n}$  for all  $n \in \mathbb{Z}$ . It is easy to check, that then  $\circ$  is associative and the identity laws hold.  $\square$

**Definition 3.4 (Comp).** The category in 3.2 is called the **category of chain complexes** and we refer to it as **Comp**.

**Theorem 3.1.** There is a functor  $\text{Top} \rightarrow \text{Comp}$ .

*Proof.* The proof is divided into several steps. Let us denote  $C_\bullet : \text{Top} \rightarrow \text{Comp}$  for the claimed functor.

*Step 1: Construction of a sequence of abelian groups.* Let  $v_0, \dots, v_k \in \mathbb{R}^n$  for some  $n, k \in \mathbb{N}$ . We say that  $(v_0, \dots, v_k)$  is **affinely independent** if  $(v_1 - v_0, \dots, v_k - v_0)$



is linearly independent. We define the ***k-simplex spanned by  $(v_0, \dots, v_k)$*** , written  $[v_0, \dots, v_k]$ , to be

$$[v_0, \dots, v_k] := \left\{ \sum_{i=0}^k s_i v_i : s_i \geq 0 \text{ for all } i = 0, \dots, k \text{ and } \sum_{i=0}^k s_i = 1 \right\}. \quad (8)$$

equipped with the subspace topology. Moreover, we define the ***standard  $n$ -simplex  $\Delta^n$***  to be the  $n$ -simplex spanned by  $(e_0, \dots, e_n)$  where  $(e_{i+1})_i$  is the standard basis of  $\mathbb{R}^{n+1}$ . Let  $X \in \text{ob}(\text{Top})$ . Define a ***singular  $n$ -simplex in  $X$***  to be a map  $\sigma \in \text{Top}(\Delta^n, X)$ . Let  $n \in \mathbb{Z}$ . Define

$$C_n(X) := \begin{cases} F(\text{Top}(\Delta^n, X)) & n \geq 0, \\ 0 & n < 0. \end{cases} \quad (9)$$

We will call elements of  $C_n(X)$  ***singular  $n$ -chains***.

*Step 2: Construction of boundary operators.* Let  $X \in \text{ob}(\text{Top})$  and  $\sigma$  a singular  $n$ -simplex in  $X$  for  $n \geq 1$ . We define  $\varphi_k^n : \Delta^{n-1} \rightarrow \Delta^n$ , called the  ***$k$ -th face map***, by

$$\varphi_k^n(s_0, \dots, s_{n-1}) := \begin{cases} (0, s_0, \dots, s_{n-1}) & k = 0, \\ (s_0, \dots, s_{k-1}, 0, s_k, \dots, s_{n-1}) & 1 \leq k \leq n-1. \end{cases} \quad (10)$$

Define now

$$\partial \sigma := \sum_{k=0}^n (-1)^k \sigma \circ \varphi_k^n \in U(C_{n-1}(X)) \quad (11)$$

to be the ***boundary of  $\sigma$*** . Moreover, the ***singular boundary operator*** is defined to be  $\bar{\partial}_n$  and  $\bar{\partial}_n := 0$  for  $n \leq 0$ .

*Step 3:  $\partial_n \circ \partial_{n+1} = 0$  for all  $n \in \mathbb{Z}$ .* It is enough to consider  $n \geq 1$ , since  $\partial_n \circ \partial_{n+1} = 0$  holds trivially in the other cases. Let  $X \in \text{ob}(\text{Top})$  and  $\sigma \in \text{Top}(\Delta^{n+1}, X)$ . Then we have

$$\begin{aligned} (\partial_n \circ \partial_{n+1})(\sigma) &= \partial_n \left( \sum_{k=0}^{n+1} (-1)^k \sigma \circ \varphi_k^{n+1} \right) \\ &= \sum_{k=0}^{n+1} (-1)^k \partial_n (\sigma \circ \varphi_k^{n+1}) \\ &= \sum_{k=0}^{n+1} \sum_{j=0}^n (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\ &= \sum_{0 \leq k \leq j \leq n} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n + \sum_{0 \leq j < k \leq n+1} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\ &= \sum_{0 \leq j \leq k \leq n} (-1)^{k+j} \sigma \circ \varphi_j^{n+1} \circ \varphi_k^n + \sum_{0 \leq j < k \leq n+1} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \end{aligned}$$

$$= \sum_{0 \leq j < k \leq n+1} ((-1)^{k+j-1} \sigma \circ \varphi_j^{n+1} \circ \varphi_{k-1}^n + (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n)$$

*Step 4: Construction of chain maps.* Let  $X, Y \in \text{ob}(\text{Top})$  and  $f \in \text{Top}(X, Y)$ . For  $n \geq 0$ , define  $f_n^\# : \text{Top}(\Delta^n, X) \rightarrow U(C_n(Y))$  by  $f_n^\# := f \circ \sigma$ . Extending this map by linearity yields a homomorphism  $f_n^\# : C_n(X) \rightarrow C_n(Y)$ . Moreover, set  $f_n^\# = 0$  for  $n < 0$ . Let  $n \geq 1$  and  $\sigma \in \text{Top}(\Delta^n, X)$ . Then on one hand we have

$$(f_{n-1}^\# \circ \partial_n)(\sigma) = f_{n-1}^\# \left( \sum_{k=0}^n (-1)^k \sigma \circ \varphi_k^n \right) = \sum_{k=0}^n (-1)^k f \circ \sigma \circ \varphi_k^n$$

and on the other

$$(\partial_n \circ f_n^\#)(\sigma) = \partial_n(f \circ \sigma) = \sum_{k=0}^n (-1)^k f \circ \sigma \circ \varphi_k^n.$$

*Step 5: Checking functorial properties.* We are ready to define the functor  $C_\bullet : \text{Top} \rightarrow \text{Comp}$ . Let  $C_\bullet(X)$  be the chain complex consisting of  $(C_n(X))_{n \in \mathbb{Z}}$  and  $(\partial_n)_{n \in \mathbb{Z}}$ . □

## APPENDIX A

### Set Theory

#### 1. Basic Concepts

**Problem A.1.** Let  $n \in \mathbb{N}$  and  $a_{kj} \in \mathbb{C}$  for  $k = 0, \dots, n+1$ ,  $j = 0, \dots, n$ . Show that

$$\sum_{k=0}^{n+1} \sum_{j=0}^n a_{kj} = \sum_{0 \leq k \leq j \leq n} a_{kj} + \sum_{0 \leq j < k \leq n+1} a_{kj}.$$

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