Contents

CHAPTER 1

Singular Homology

1. Free Abelian Groups

Proposition 1.1. *The forgetful functor $U : \mathsf{Ab} \to \mathsf{Set}$ admits a left adjoint.*

Proof. We have to construct a functor $F : \mathsf{Set} \to \mathsf{Ab}$. Let S be a set. Define

$$F(S) := \left\{ f \in \mathbb{Z}^S : \operatorname{supp} f \text{ is finite} \right\}.$$

Equipped with pointwise addition, $F(S)$ is an abelian group. There is a natural inclusion $\iota : S \hookrightarrow U\left(F(S)\right)$ sending $x \in S$ to the function taking the value one at x and zero else. Hence we may regard elements of $F(S)$ as formal linear combinations $\sum_{x \in S} m_x x$, where $m_x \in \mathbb{Z}$ for all $x \in S$. Let $G \in \operatorname{ob}(\mathsf{Ab})$ be an abelian group and $\varphi \in \mathsf{Ab}\left(F(S), G\right)$ a morphism of groups. Define $\overline{\varphi} \in \mathsf{Set}\left(S, U(G)\right)$ by $\overline{\varphi} := U(\varphi) \circ \iota$. Conversely, if we have $f \in \mathsf{Set}\left(S, U(G)\right)$, define $\overline{f} \in \mathsf{Ab}\left(F(S), G\right)$ by $\overline{f}\left(\sum_{x \in S} m_x x\right) := \sum_{x \in S} m_x f\left(\iota^{-1}(x)\right)$. This is well defined since all but finitely many m_x are zero. It is easy to check that $\overline{f}$ is indeed a morphism of groups. Let $\varphi \in \mathsf{Ab}\left(F(S), G\right)$. Then

$$\begin{aligned}
\overline{\overline{\varphi}}\left(\sum_{x \in S} m_x x\right) &= \sum_{x \in S} m_x \overline{\varphi}\left(\iota^{-1}(x)\right) \\
&= \sum_{x \in S} m_x \left(U(\varphi) \circ \iota\right)\left(\iota^{-1}(x)\right) \\
&= \sum_{x \in S} m_x U(\varphi)(x) \\
&= \sum_{x \in S} m_x \varphi(x) \\
&= \varphi\left(\sum_{x \in S} m_x x\right).
\end{aligned}$$

And for $f \in \mathsf{Set}\left(S, U(G)\right)$ we have that

$$\overline{\overline{f}}(x) = \left(U(\overline{f}) \circ \iota\right)(x) = \overline{f}\left(\iota(x)\right) = f(x).$$

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Hence $\overline{\overline{\varphi}} = \varphi$ and $\overline{\overline{f}} = f$ and so we have a bijection

$$\mathsf{Ab}\left(F(S), G\right) \cong \mathsf{Set}\left(S, U(G)\right).$$

The mapping $f \mapsto \overline{f}$ will be referred to as ***extending by linearity***. To check naturality in S and G is left as an exercise. ☐

Exercise 1.1. Check the naturality of the bijection in proposition 1.1. Also check that $F : \mathsf{Set} \to \mathsf{Ab}$ is indeed a functor. F is called the ***free functor from* Set *to* Ab**.

Definition 1.1 (Free Abelian Group). *Let $F : \mathsf{Set} \to \mathsf{Ab}$ be the free functor. For any set S, we call $F(S)$ the **free group generated by S**.*