SEMINAR: INTRODUCTION TO FUCHSIAN GROUPS

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Abstract. Aim of this talk is to present the first ten pages in the book *Fuchsian groups* by *Svetlana Katok*. It is a short introduction into the rudiments of *hyperbolic geometry* and introduces an action of the *projective special linear group* on the *upper half-plane*. We show that the *geodesics* of the hyperbolic plane are precisely the straight lines parallel to the imaginary axis and the semicircles with center on the real axis and conclude with a description of the *isometries* of the hyperbolic plane.

Contents

1. Hyperbolic Geometry

1.1. The Hyperbolic Metric. The main object of our study is the so-called *hyperbolic plane*. There are three common equivalent, i.e. in some sense, which we will explain later, compatible, models of this space, each of which is useful in certain contexts. We are only concerned with two of them and introduce now the first.

Definition 1.1 (Poincaré Half-Space Model). *The upper half-plane*

$$\mathcal{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\} \tag{1}$$

equipped with the metric

$$g := \frac{dx^2 + dy^2}{y^2} \tag{2}$$

is a model for the hyperbolic plane, the ***Poincaré half-space model****.*

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Remark 1.1. The tuple $(\mathcal{H}, g)$ defined in 1.1 is an example of a *Riemannian manifold* (if we equip $\mathcal{H}$ with the smooth structure generated by the atlas $\{(\mathcal{H}, \mathrm{id}_{\mathcal{H}})\}$). We will not use this level of abstraction. However it is important to know, that the metric g induces an inner product $g_p : T_p\mathcal{H} \times T_p\mathcal{H} \to \mathbb{R}$ on the tangent space $T_p\mathcal{H}$ at each $p \in \mathcal{H}$. Since $\mathcal{H} \subseteq \mathbb{R}^2$ is open, we have that $T_p\mathcal{H} \cong \mathbb{C}$ ([Lee13, p. 56] together with [Lee13, p. 53]). Explicitely, if $p := (x_0, y_0) \in \mathcal{H}$ and $\xi_1 + i\eta_1, \xi_2 + i\eta_2 \in \mathbb{C}$ we have that

$$g_p(\xi_1 + i\eta_1, \xi_2 + i\eta_2) = \frac{1}{y_0^2}(\xi_1\xi_2 + \eta_1\eta_2). \tag{3}$$

Because of this, we can define the notion of *lengths* and *angles* on more abstract objects. But it is important to observe, that those notions strongly depend on the choice of metric and maybe contradict our intuition.

Definition 1.2 (Hyperbolic Length). *Let $I := [0,1]$ and $\gamma := x(t) + iy(t) \in C^1_{\mathrm{pw}}(I, \mathcal{H})$. Then the* ***hyperbolic length of γ****, written $\mathrm{h}(\gamma)$, is defined to be*

$$\mathrm{h}(\gamma) := \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)} dt = \int_0^1 \frac{\left|\frac{d\gamma}{dt}\right|}{y(t)} dt. \tag{4}$$

Definition 1.3 (Hyperbolic Distance). *Let $z, w \in \mathcal{H}$. The* ***hyperbolic distance between z and w****, written $\rho(z, w)$, is defined to be*

$$\rho(z, w) := \inf_{\substack{\gamma \in C^1_{\mathrm{pw}}(I, \mathcal{H}) \\ \gamma(0) = z \\ \gamma(1) = w}} \mathrm{h}(\gamma) \tag{5}$$

Recall that $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$.

Definition 1.4. *Let $n \in \mathbb{Z}$, $n \geq 1$ and $\det : \mathrm{M}(n, \mathbb{R}) \to \mathbb{R}$ denote the determinant function. Then we define:*

$$\begin{array}{ll}
\mathrm{GL}(n, \mathbb{R}) := \det^{-1}(\mathbb{R}^\times) & \textit{(}\textbf{\textit{General Linear Group}}\textit{),} \\
\mathrm{GL}^+(n, \mathbb{R}) := \det^{-1}((0, \infty)) & \textit{(}\textbf{\textit{Positive General Linear Group}}\textit{),} \\
\mathrm{SL}(n, \mathbb{R}) := \det^{-1}(1) & \textit{(}\textbf{\textit{Special Linear Group}}\textit{),} \\
\mathrm{PSL}(n, \mathbb{R}) := \mathrm{SL}(n, \mathbb{R})/Z(\mathrm{SL}(n, \mathbb{R})) & \textit{(}\textbf{\textit{Projective Special Linear Group}}\textit{).}
\end{array}$$

Remark 1.2. We have that $Z(\mathrm{SL}(n, \mathbb{R})) = \{\pm I\}$. Thus $\mathrm{PSL}(n, \mathbb{R}) = \mathrm{SL}(n, \mathbb{R})/\{\pm I\}$.

Lemma 1.1. $\mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{GL}^+(2, \mathbb{R})/\{\lambda I : \lambda \in \mathbb{R}^\times\}$.

Proof. Let us consider the mapping $\Phi : \mathrm{GL}^+(2, \mathbb{R}) \to \mathrm{PSL}(2, \mathbb{R})$ defined by

$$\Phi(A) := \frac{1}{\sqrt{\det(A)}} A\{\pm I\}.$$

This mapping is well-defined since $\det(\Phi(A)) = 1$. Moreover, Φ is a homomorphism of groups, surjective since $\mathrm{SL}(2, \mathbb{R}) \subseteq \mathrm{GL}^+(2, \mathbb{R})$ and $\ker \Phi = \{\lambda I : \lambda \in \mathbb{R}^\times\}$. Thus the first isomorphism theorem yields the desired statement. □

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Recall that if G is a group and X a set, a *left action of G on X* is a mapping $G \times X \to X$, written $(g, x) \mapsto g \cdot x$, with the following properties:

(i) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $x \in X$ and $g_1, g_2 \in G$.
(ii) $1 \cdot x = x$ for all $x \in X$.

If X is a topological space and G acts on X, we say that the action is an *action by homeomorphisms* if for each $g \in G$ the mapping $x \mapsto g \cdot x$ is a homeomorphism (see [Lee11, pp. 78–79]). Moreover, an action of a group G on a set X induces a homomorphism of G into S_X (see [Gri07, p. 54]).

Proposition 1.1. *Let $z \in \mathcal{H}$. Define*

$$A \cdot z := \frac{az+b}{cz+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}^+(2, \mathbb{R}). \tag{6}$$

This defines an action by homeomorphisms of $\mathrm{GL}^+(2, \mathbb{R})$ on $\mathcal{H}$.

Proof. First we show that $A \cdot z$ is well-defined. However, this is clear since $z \neq -d/c$. Next we show that $A \cdot z \in \mathcal{H}$ for any $A \in \mathrm{GL}^+(2, \mathbb{R})$ and $z \in \mathcal{H}$. This immediately follows by

$$\operatorname{Im}\left(A \cdot z\right) = \frac{1}{2i}\left(A \cdot z - \overline{A \cdot z}\right) = \frac{1}{2i}\frac{(ad-bc)(z-\overline{z})}{|cz+d|^2} = \frac{\det(A)\operatorname{Im}(z)}{|cz+d|^2} > 0 \tag{7}$$

since $\det(A), \operatorname{Im}(z) > 0$. Furthermore it is easy to check that this defines indeed an action. It is an action by homeomorphisms since $z \mapsto A \cdot z$ is a bijection and clearly continuous as a well-defined rational function. □

Corollary 1.1. *The action of $\mathrm{GL}^+(2, \mathbb{R})$ on $\mathcal{H}$ defined in proposition 1.1 descends to an action by homeomorphisms of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathcal{H}$.*

Proof. Since $\{\lambda I : \lambda \in \mathbb{R}^\times\} \subseteq \ker(\mathrm{GL}^+(2, \mathbb{R}) \to S_{\mathcal{H}})$, the mapping $\mathrm{GL}^+(2, \mathbb{R}) \to S_{\mathcal{H}}$ factors uniquely through $\pi : \mathrm{GL}^+(2, \mathbb{R}) \to \mathrm{GL}^+(2, \mathbb{R}) / \{\lambda I : \lambda \in \mathbb{R}^\times\}$ by [Gri07, p. 23] which is isomorphic to $\mathrm{PSL}(2, \mathbb{R})$ by lemma 1.1. □

Definition 1.5 (Möbius Transformations). *Define the group of* ***Möbius transformations on $\mathcal{H}$*** *by*

$$\text{Möb}(\mathcal{H}) := \{z \mapsto A \cdot z : A \in \mathrm{PSL}(2, \mathbb{R})\}. \tag{8}$$

Examples 1.1 (Möbius Transformations).

(a) ***Translations***: $z \mapsto z + b$, $b \in \mathbb{R}$.
(b) ***Homothety***: $z \mapsto az$, $a \in (0, \infty)$.
(c) ***Inversion***: $z \mapsto -1/z$.

Definition 1.6 (Isometries). *A transformation of $\mathcal{H}$ onto itself is called an* **isometry** *if it preserves the hyperbolic distance on $\mathcal{H}$. We shall denote the set of all isometries on $\mathcal{H}$ by* $\mathrm{Isom}(\mathcal{H})$.

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Proposition 1.2. $\text{Möb}(\mathscr{H}) \subseteq \text{Isom}(\mathscr{H})$.

Proof. Let $f \in \text{Möb}(\mathscr{H})$ and $\gamma := x(t) + iy(t) \in C^1_{\text{pw}}(I, \mathscr{H})$. Then there exists a partition $a_0 < \cdots < a_n$ of I such that γ restricted to each subinterval is continuously differentiable. By (7) we have that

$$\text{Im}(f \circ \gamma) = \frac{y}{|c\gamma + d|^2}$$

and [FL03, p. 25] yields

$$\frac{d(f \circ \gamma)}{dt} = \frac{\partial f}{\partial z}\frac{d\gamma}{dt} + \frac{\partial f}{\partial \overline{z}}\frac{d\overline{\gamma}}{dt} = \frac{\partial f}{\partial z}\frac{d\gamma}{dt} = \frac{1}{(c\gamma + d)^2}\frac{d\gamma}{dt}$$

for all $t \neq a_\nu$, $\nu = 0, \ldots, n$, by the holomorphicity of f. Thus

$$\text{h}(f \circ \gamma) = \sum_{\nu=1}^{n} \int_{a_{\nu-1}}^{a_\nu} \frac{\left|\frac{d(f\circ\gamma)}{dt}\right|}{\text{Im}(f(\gamma(t)))} dt = \sum_{\nu=1}^{n} \int_{a_{\nu-1}}^{a_\nu} \frac{\left|\frac{d\gamma}{dt}\right|}{y(t)} dt = \text{h}(\gamma).$$

□

1.2. Geodesics. The fifth postulate of Euclid's geometry says that given any line in the plane and a point which does not belong to it, then there is a unique line through this point never intersecting the other line. This no longer holds in the hyperbolic plane. Thus we speak of non-Euclidean geometry. In this section we find out, what curves in $\mathscr{H}$ correspond to straight lines in the Eudlidean plane.

Definition 1.7 (Geodesics). *A* ***geodesic*** *in* $(\mathscr{H}, g)$ *is the image of a curve* $\gamma \in C^1_{\text{pw}}(\mathbb{R}, \mathscr{H})$*, such that for all* $a, b \in \mathbb{R}$*,* $a < b$*, we have that* $\rho\big(\gamma(a), \gamma(b)\big) = \text{h}(\gamma|_{[a,b]})$*.*

The next proposition is exercise 1.1. [Kat92, p. 21].

Proposition 1.3. *Let* $A \subseteq \mathscr{H}$ *be a semicircle with center on the real axis or a line parallel to the imaginary axis. Then there exists* $f \in \text{Möb}(\mathscr{H})$ *such that* $f(A) = i\mathbb{R}_{>0}$*.*

Proof. First consider the case where A is a line parallel to the imaginary axis. A may be written as the image of the curve $\gamma : (0, \infty) \to \mathscr{H}$ defined by $\gamma(t) := \alpha + it$, where $\alpha \in \mathbb{R}$ is the thought point of intersection of A with the real axis. For $\beta \in \mathbb{R}$ define $f_\beta : \mathscr{H} \to \mathbb{C}$ by

$$f_\beta(z) := -\frac{1}{z - \alpha} + \beta. \tag{9}$$

It is immediate that $f_\beta \in \text{Möb}(\mathscr{H})$ since $f_\beta(z) = A \cdot z$ for

$$A := \begin{pmatrix} \beta & -(\alpha\beta + 1) \\ 1 & -\alpha \end{pmatrix}.$$

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For $t \in (0, \infty)$ we have that

$$(f_\beta \circ \gamma)(t) = \beta + i\frac{1}{t}.$$

Thus f_0 maps A to the positive imaginary axis. Next consider A to be a semicircle with radius $r > 0$ and center on the real axis. Let $\alpha \in \mathbb{R}$ denote the left thought point of intersection of A with the real axis. Then A is the image of the curve $\gamma : (0, \pi) \to \mathscr{H}$ defined by $\gamma(t) := re^{it} + \alpha + r$. For any $t \in (0, \pi)$ we have

$$(f_\beta \circ \gamma)(t) = -\frac{1}{r}\frac{1}{e^{it} + 1} + \beta = -\frac{1}{2r}\frac{e^{-it} + 1}{1 + \cos t} + \beta = -\frac{1}{2r} + \frac{i}{2r}\frac{\sin t}{1 + \cos t} + \beta.$$

Hence $f_{1/2r}$ maps A to the positive imaginary axis. $\square$

Theorem 1.1 (Geodesics of the Hyperbolic Plane). *The geodesics in $\mathscr{H}$ are precisely the semicircles with center on the real axis and the straight lines parallel to the imaginary axis. Furthermore, through any two points belonging to $\mathscr{H}$ there is exactly one geodesic segment connecting them.*

Proof. Let $a, b \in (0, \infty)$ such that $a < b$ and $\gamma := x(t) + iy(t) \in C^1_{\mathrm{pw}}(I, \mathscr{H})$ joining ia and ib. Then we have that

$$\mathrm{h}(\gamma) = \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)} dt \geq \int_0^1 \frac{\left|\frac{dy}{dt}\right|}{y(t)} dt \geq \int_0^1 \frac{\frac{dy}{dt}}{y(t)} dt = \int_a^b \frac{dy}{y} = \log\frac{b}{a} = \mathrm{h}(\gamma_0)$$

where $\gamma_0 := (1 - t)ia + tib$ is the straight line on the imaginary axis from ia to ib.
Let $z, w \in \mathscr{H}$. Then there exists a unique circle with center on the real axis or a unique line parallel to the imaginary axis which goes through those points, say A. By proposition 1.3 there exists $f \in \text{Möb}(\mathscr{H})$ such that $f(A)$ is a subset of the positive imaginary axis. The statement now follows from proposition 1.2 and the computation above. $\square$

Definition 1.8. *Let $z, w \in \mathscr{H}$. The unique geodesic segment joining them is denoted by $[z, w]$.*

Corollary 1.2. *Let $z, w \in \mathscr{H}$ with $z \neq w$. Then*

$$\rho(z, w) = \rho(z, \xi) + \rho(\xi, w) \tag{10}$$

holds if and only if $\xi \in [z, w]$. Moreover, any transformation in Möb$(\mathscr{H})$ *maps geodesics onto geodesics.*

Theorem 1.2 (Explicit Formulas for the Hyperbolic Distance). *For $z, w \in \mathscr{H}$ we have that*

(a)

$$\rho(z, w) = \log\frac{|z - \overline{w}| + |z - w|}{|z - \overline{w}| - |z - w|}; \tag{11}$$

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(*b*)

$$\cosh \rho(z,w) = 1 + \frac{|z-w|^2}{2\operatorname{Im}(z)\operatorname{Im}(w)}; \tag{12}$$

(*c*)

$$\sinh\left(\frac{1}{2}\rho(z,w)\right) = \frac{|z-w|}{2\sqrt{\operatorname{Im}(z)\operatorname{Im}(w)}}; \tag{13}$$

(*d*)

$$\cosh\left(\frac{1}{2}\rho(z,w)\right) = \frac{|z-\overline{w}|}{2\sqrt{\operatorname{Im}(z)\operatorname{Im}(w)}}; \tag{14}$$

(*e*)

$$\tanh\left(\frac{1}{2}\rho(z,w)\right) = \left|\frac{z-w}{z-\overline{w}}\right|. \tag{15}$$

Proof. Using the definitions of the hyperbolic functions and the identities

$$\begin{aligned}
\sinh^2\frac{x}{2} &= \frac{\cosh x - 1}{2};\\
\cosh x &= \sqrt{1+\sinh^2 x};\\
\sinh x &= \operatorname{sgn} x\sqrt{\cosh^2 x - 1};\\
\operatorname{arctanh} x &= \frac{1}{2}\log\frac{1+x}{1-x}
\end{aligned}$$

one shows that (a)⇒(b)⇒(c)⇒(d)⇒(e)⇒(a). We shall prove (c). By proposition 1.2 we have that the left-hand side is invariant under $f \in \text{Möb}(\mathscr{H})$. We show that also the right-hand side is invariant under f. Equation (7) and some algebraic manipulations immediately yield

$$\frac{|f(z)-f(w)|}{2\sqrt{\operatorname{Im}(f(z))\operatorname{Im}(f(w))}} = \frac{|z-w|}{2\sqrt{\operatorname{Im}(z)\operatorname{Im}(w)}}$$

Hence by an application of proposition 1.3 we may assume that $z = ia$ and $w = ib$ for $a, b \in (0,\infty)$, $a < b$. Theorem 1.1 yields $\rho(z,w) = \log\frac{b}{a}$ and it is easy to see that the equality in (c) holds. □

Recall that $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ denotes the *Riemann sphere* and $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

Definition 1.9 (Cross-Ratio). *The* ***cross-ratio*** *of distinct points $z_1, z_2, z_3, z_4 \in \widehat{\mathbb{C}}$ is defined to be*

$$(z_1, z_2; z_3, z_4) := \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}. \tag{16}$$

The next theorem uses results about fractional linear transformations and thus we only state but do not prove it.

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Theorem 1.3. *Let $z, w \in \mathcal{H}$ be distinct and let the geodesic joining z and w have endpoints $z^*, w^* \in \hat{\mathbb{R}}$, choosen in such a way that z lies between z^* and w. Then*

$$\rho(z, w) = \log(w, z^*; z, w^*). \tag{17}$$

Definition 1.10 (Poincaré Ball Model). *The unit disc*

$$\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\} \tag{18}$$

equipped with the metric

$$\tilde{g} := 4\frac{|dz|^2}{\left(1 - |z|^2\right)^2} \tag{19}$$

is a model for the hyperbolic plane, the ***Poincaré ball model****.*

Recall, that if (M, g) and $(\tilde{M}, \tilde{g})$ are two Riemannian manifolds, a diffeomorphism $F : M \to \tilde{M}$ is said to be a *Riemannian isometry* if $F^*\tilde{g} = g$. If there exists a Riemannian isometry for two Riemannian manifolds, they are said to be *isometric*.

Proposition 1.4. *$(\mathcal{H}, g)$ and $(\mathcal{U}, \tilde{g})$ are isometric.*

Proof. Consider the mapping $f : \mathcal{H} \to \mathbb{C}$ defined by

$$f(z) := \frac{iz + 1}{z + i}. \tag{20}$$

Since f is a fractional linear transformation, it is clearly a diffeomorphism. Moreover

$$f'(z) = -\frac{2}{(z + i)^2} \qquad \text{and} \qquad 1 - |f(z)|^2 = 4\frac{\operatorname{Im}(z)}{|z + i|^2}.$$

For $z \in \mathcal{H}$, the second formula yields $|f(z)| < 1$ and thus actually $f : \mathcal{H} \to \mathcal{U}$. Also

$$f^*\tilde{g} = 4\frac{f^*(dzd\bar{z})}{\left(1 - |f(z)|^2\right)^2} = 4\frac{|f'(z)|^2\, dzd\bar{z}}{\left(1 - |f(z)|^2\right)^2} = \frac{dzd\bar{z}}{(\operatorname{Im}(z))^2} = g$$

by [Lee97, p. 41]. □

Remark 1.3. Riemannian isometries preserve distance, i.e. if (M, g) and $(\tilde{M}, \tilde{g})$ are connected Riemannian manifolds and $F : M \to \tilde{M}$ is a Riemannian isometry, then $d_{\tilde{g}}\big(F(p), F(q)\big) = d_g(p, q)$ where $p, q \in M$ and $d_g, d_{\tilde{g}}$ are the associated Riemannian distances, respectively (see [Lee13, p. 338]).

As an application of the ball model we give the following proposition.

Proposition 1.5. *$(\mathcal{H}, \rho)$ is a metric space whose metric topology is the same as the standard topology on $\mathcal{H}$ (i.e. the metric topology induced by the Euclidean metric).*

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Proof. Let $z, w \in \mathscr{H}$. Then clearly $\rho(z, w) \geq 0$ and by parametrization invariance we get that $\rho(z, w) = \rho(w, z)$. Also the triangle inequality holds. Also if $z = w$, we have that $\rho(z, w) = 0$. Assume that $z \neq w$. Then there exists a unique geodesic segment $[z, w]$ joining them. But $\mathrm{h}([z, w]) > 0$ as one can easily see. Thus ρ is indeed a metric.
Left to show is that the metric topology induced by the hyperbolic distance function coincides with the usual Euclidean topology. However, since the proof is quite long if one to do it formally, we only state a reference. By [And05, p. 127] any hyperbolic circle in $\mathscr{U}$ is a Euclidean one and vice versa (with different centers in general). Then exercise 4.7 [And05, p. 129] shows that if $x + iy \in \mathscr{H}$ is a Euclidean center of a Euclidean circle with radius r, then the hyperbolic center is given by $x + i\sqrt{y^2 - r^2}$ and the hyperbolic radius R satisfies $r = y \tanh(R)$. From this it follows that Euclidean discs and hyperbolic disks coincide (with possibly different radius and center). However, since the set of all disks in a metric space is a basis for the topology induced by the metric, we have that the Euclidean and the hyperbolic topologies are the same. □

Proposition 1.6. *Let $f \in \mathrm{Isom}(\mathscr{H})$. Then f is continuous.*

Proof. By proposition 1.5 it is enough to show that f is continuous with respect to the hyperbolic metric. But this is immediate since $\rho\big(f(z), f(w)\big) = \rho(z, w)$ for all $z, w \in \mathscr{H}$ implies that f is Lipschitz continuous. □

Definition 1.11 (Euclidean Boundaries and Closures). *Define*

$$\begin{aligned}
\partial\mathscr{U} &:= \Sigma := \{z \in \mathbb{C} : |z| = 1\} && \textit{(Principal Circle, Boundary of } \mathscr{U}\textit{)},\\
\partial\mathscr{H} &:= \widehat{\mathbb{R}} && \textit{(Boundary of } \mathscr{H} \textit{ in } \widehat{\mathbb{C}}\textit{)},\\
\bar{\mathscr{U}} &:= \mathscr{U} \cup \Sigma && \textit{(Closure of } \mathscr{U}\textit{)},\\
\bar{\mathscr{H}} &:= \mathscr{H} \cup \widehat{\mathbb{R}} && \textit{(Closure of } \mathscr{H}\textit{)}.
\end{aligned}$$

1.3. Isometries. We have already seen that Möbius transformations are isometries on $\mathscr{H}$. Is is also true that $\mathrm{Isom}(\mathscr{H}) \subseteq \mathrm{Möb}(\mathscr{H})$? Unfortunately, the answer is no. However, $\mathrm{Isom}(\mathscr{H}) \setminus \mathrm{Möb}(\mathscr{H})$ is not as complicated as one might think. Roughly speaking, the isometries on $\mathscr{H}$ are the union of so-called orientation-preserving, i.e. Möbius transformations and orientation-reversing transformations.

Proposition 1.7. $(\mathrm{Isom}(\mathscr{H}), \circ)$ *is a group.*

Proof. We show $\mathrm{Isom}(\mathscr{H}) \leq S_{\mathscr{H}}$. Let $f \in \mathrm{Isom}(\mathscr{H})$. By definition, f is surjective. Let $z, w \in \mathscr{H}$ such that $f(z) = f(w)$. Then

$$\rho(z, w) = \rho\big(f(z), f(w)\big) = 0$$

which implies that $z = w$ since ρ is a metric by proposition 1.5. Thus f is bijective. Clearly $\mathrm{id}_{\mathscr{H}} \in \mathrm{Isom}(\mathscr{H})$ and for $g \in \mathrm{Isom}(\mathscr{H})$ we have that

$$\rho\big((g \circ f)(z), (g \circ f)(w)\big) = \rho\big(f(z), f(w)\big) = \rho(z, w)$$

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and so $g \circ f \in \mathrm{Isom}(\mathscr{H})$. Moreover f^{-1} is distance preserving by

$$\rho\big(f^{-1}(z), f^{-1}(w)\big) = \rho\big((f \circ f^{-1})(z), (f \circ f^{-1})(w)\big) = \rho(z, w).$$

$\square$

Definition 1.12. *Let* $\det : \mathrm{M}(2, \mathbb{R}) \to \mathbb{R}$ *denote the determinant function. Define*

$$\mathrm{PS^*L}(2, \mathbb{R}) := \mathrm{S^*L}(2, \mathbb{R}) / \{\pm I\}. \tag{21}$$

where $\mathrm{S^*L}(2, \mathbb{R}) := \det^{-1}(\pm 1)$.

Proposition 1.8. $\mathrm{PSL}(2, \mathbb{R}) \leq \mathrm{PS^*L}(2, \mathbb{R})$ *and* $[\mathrm{PS^*L}(2, \mathbb{R}) : \mathrm{PSL}(2, \mathbb{R})] = 2$.

Proof. Clearly $\mathrm{PSL}(2, \mathbb{R}) \leq \mathrm{PS^*L}(2, \mathbb{R})$. We claim that

$$\{A \cdot \mathrm{PSL}(2, \mathbb{R}) : A \in \mathrm{PS^*L}(2, \mathbb{R})\} = \{\mathrm{PSL}(2, \mathbb{R}), \pi(E) \cdot \mathrm{PSL}(2, \mathbb{R})\}$$

for

$$E := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The inclusion $\supseteq$ trivially holds. So assume that $A \cdot \mathrm{PSL}(2, \mathbb{R})$ belongs to the right-hand side. Since $A \in \mathrm{PS^*L}(2, \mathbb{R})$, we have that either $A = A_1 \{\pm I\}$ where $\det(A_1) = 1$ or $A = A_2 \{\pm I\}$ where $\det(A_2) = -1$. In the first case simply $A \in \mathrm{PSL}(2, \mathbb{R})$ and in the second we write $A = E(EA_2)\{\pm I\}$. Then $\det(EA_2) = \det(E)\det(A_2) = 1$ and thus $A \in E \cdot \mathrm{PSL}(2, \mathbb{R})$. $\square$

Theorem 1.4 (Isometries of the Hyperbolic Plane). *We have that*

$$\mathrm{Isom}(\mathscr{H}) = \mathrm{M\ddot{o}b}(\mathscr{H}) \cup \{z \mapsto A \cdot \overline{z} : A \in \det^{-1}(-1)\}. \tag{22}$$

Furthermore

$$\mathrm{Isom}(\mathscr{H}) \cong \mathrm{PS^*L}(2, \mathbb{R}) \tag{23}$$

and $[\mathrm{Isom}(\mathscr{H}) : \mathrm{M\ddot{o}b}(\mathscr{H})] = 2$.

Proof. Let $f \in \mathrm{Isom}(\mathscr{H})$ and I denote the positive imaginary axis. Then $f(I)$ is a geodesic. By proposition 1.3 there exists $g \in \mathrm{M\ddot{o}b}(\mathscr{H})$ such that $g(f(I)) = I$. Assume that $(g \circ f)(i) \neq i$. Thus $(g \circ f)(i) = ia$, for some $a \in \mathbb{R}_{>0}$. Hence applying some dilation $z \mapsto \frac{z}{a}$ fixes i. Also we can force $g \circ f$ to map (i, ∞) and $(0, i)$ by composing with $z \mapsto -\frac{1}{z}$. By proposition 1.2 those two maps are isometries. Hence we may assume that $g \circ f$ fixes I. Let $z := x + iy \in \mathscr{H}$ and $u + iv := (g \circ f)(z)$. Furthermore let $t > 0$. Then

$$\rho(z, it) = \rho\big((g \circ f)(z), (g \circ f)(it)\big) = \rho(u + iv, it).$$

Applying theorem 1.2 (c) yields

$$v\big(x^2 + (y - t)^2\big) = y\big(u^2 + (v - t)^2\big)$$

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Thus dividing both sides by t^2 and letting $t \to \infty$ leads to $v = y$. But if $v = y$ the above equation gives that $x^2 = u^2$. Hence $u = \pm x$ and so we get that

$$(g \circ f)(z) = z \qquad \text{or} \qquad (g \circ f)(z) = -\overline{z}.$$

By proposition 1.6, any isometry is continuous. Consider the sets

$$A := \{g \circ f = \mathrm{id}_{\mathcal{H}}\} \qquad \text{and} \qquad B := \{g \circ f = -\overline{\mathrm{id}}_{\mathcal{H}}\}.$$

Let $(z_n)_{n \in \mathbb{N}}$ be a sequence in A such that $z_n \to z \in \mathcal{H}$. We get

$$(g \circ f)(z) = \lim_{n \to \infty} (g \circ f)(z_n) = \lim_{n \to \infty} z_n = z$$

and similarly for B since complex conjugation is also continuous. Hence A and B are closed subsets of $\mathcal{H}$. Also, they are open. Indeed, let $z_0 \in A$. Since $g \circ f$ is continuous at z_0 we find for any $\varepsilon > 0$ small enough $\delta > 0$, such that $|(g \circ f)(z) - (g \circ f)(z_0)| < \varepsilon$ for all $z \in B_\delta(z_0)$. But this cannot be possible if $(g \circ f)(z) = -\overline{z}$ and similarly for B. Thus A and B are open and closed, nonempty since $I \subseteq A$, $I \subseteq B$ and since $\mathcal{H}$ is connected, we get that $A = \mathcal{H}$ and $B = \mathcal{H}$ by [Lee11, p. 86]. Thus we have that either $g \circ f = \mathrm{id}_{\mathcal{H}}$ or $g \circ f = -\overline{\mathrm{id}}_{\mathcal{H}}$. If $g \circ f = \mathrm{id}_{\mathcal{H}}$, we have that $f = g^{-1} \in \text{Möb}(\mathcal{H})$. In the other case we have that $f = g^{-1} \circ -\overline{\mathrm{id}}_{\mathcal{H}}$.
Consider a mapping $\Phi : \mathrm{S^*L}(2, \mathbb{R}) \to \mathrm{Isom}(\mathcal{H})$ defined by

$$\Phi(A) := \begin{cases} z \mapsto \pi(A) \cdot z & \det(A) = 1, \\ z \mapsto A \cdot \overline{z} & \det(A) = -1. \end{cases}$$

This clearly defines a group homomorphism. We claim $\ker \Phi = \{\pm I\}$. Let $A \in \ker \Phi$. If $\det(A) = 1$, we have that

$$\frac{az + b}{cz + d} = z$$

for all $z \in \mathcal{H}$ or equivalently

$$cz^2 + (d - a)z - b = 0$$

for all $z \in \mathcal{H}$. The fundamental theorem of algebra now implies that $c = b = 0$ and $a = d$ which yields $a = d = \pm 1$. Moreover Φ is clearly surjective and thus the first isomorphism theorem [Gri07, p. 23] yields $\mathrm{Isom}(\mathcal{H}) \cong \mathrm{PS^*L}(2, \mathbb{R})$. Finally, by proposition 1.8 we have that $[\mathrm{PS^*L}(2, \mathbb{R}) : \mathrm{PSL}(2, \mathbb{R})] = 2$ and therefore $\text{Möb}(\mathcal{H}) \cong \mathrm{PSL}(2, \mathbb{R})$ yields $[\mathrm{Isom}(\mathcal{H}) : \text{Möb}(\mathcal{H})] = 2$. $\square$

Definition 1.13. *Let* $f \in \mathrm{Isom}(\mathcal{H})$ *and let* $A \in \mathrm{PS^*L}(2, \mathbb{R})$ *be the corresponding matrix. Then* $\det(A)$ *is called the* ***orientation of the isometry*** f. *If* $\det(A) = 1$, f *is said to be a* ***orientation preserving isometry****, while if* $\det(A) = -1$, f *is said to be a* ***orientation reversing isometry****.*

Remark 1.4. With this terminology, $\text{Möb}(\mathcal{H})$ is precisely the group of orientation-preserving isometries on $\mathcal{H}$.

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