SOLUTIONS SHEET 7

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Exercise 1.

Lemma 1.1. *$q : X \to \mathbb{R}$ is a sublinear functional and $f \leq q$ on Y.*

Proof. Let $\lambda \geq 0$. Moreover, let $\{A_1, \dots, A_n\} \subseteq G$ for some $n \in \mathbb{N}$. Then for any $x \in X$ we have that

$$\begin{aligned}\frac{1}{n} p\left(A_1(\lambda x) + \cdots + A_n(\lambda x)\right) &= \frac{1}{n} p\left(\lambda A_1(x) + \cdots + \lambda A_n(x)\right)\\ &= \frac{1}{n} p\left(\lambda\left(A_1(x) + \cdots + A_n(x)\right)\right)\\ &= \lambda \frac{1}{n} p(A_1(x) + \cdots + A_n(x))\end{aligned}$$

since each A_i is linear and p is a sublinear functional on X. Thus also $q(\lambda x) = \lambda q(x)$. Let $x, y \in X$. Furthermore, fix some $\varepsilon > 0$. By definition of the infimum, we find $A_1, \dots, A_n \in G$ and $B_1, \dots, B_m \in G$, such that

$$q(x) \leq \frac{1}{n} p\left(A_1(x) + \cdots + A_n(x)\right) \leq q(x) + \frac{\varepsilon}{2}$$

and

$$q(y) \leq \frac{1}{m} p\left(B_1(y) + \cdots + B_m(y)\right) \leq q(y) + \frac{\varepsilon}{2}.$$

We estimate

$$\begin{aligned}\frac{1}{n} p\left(A_1(x) + \cdots + A_n(x)\right) &= \frac{m}{mn} p\left(A_1(x) + \cdots + A_n(x)\right)\\ &= \frac{1}{mn} \sum_{k=1}^{m} p\left(A_1(x) + \cdots + A_n(x)\right)\\ &\geq \frac{1}{mn} \sum_{k=1}^{m} p\left(B_k\left(A_1(x) + \cdots + A_n(x)\right)\right)\\ &= \frac{1}{mn} \sum_{k=1}^{m} p\left(B_k A_1(x) + \cdots + B_k A_n(x)\right)\end{aligned}$$

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by the linearity of elements in G, the closedness of G under composition and the property that $p(Ax) \leq p(x)$ holds for all $x \in X$ and $A \in G$. Similarly we estimate

$$\begin{aligned}
\frac{1}{m} p\left(B_1(y) + \cdots + B_m(y)\right) &= \frac{n}{mn} p\left(B_1(y) + \cdots + B_m(y)\right) \\
&= \frac{1}{mn} \sum_{k=1}^{n} p\left(B_1(y) + \cdots + B_m(y)\right) \\
&\geq \frac{1}{mn} \sum_{k=1}^{n} p\left(A_k\left(B_1(y) + \cdots + B_m(y)\right)\right) \\
&= \frac{1}{mn} \sum_{k=1}^{n} p\left(A_k B_1(y) + \cdots + A_k B_m(y)\right).
\end{aligned}$$

Hence the sublinearity of p together with the commutativity of G yields

$$\begin{aligned}
q(x) + q(y) + \varepsilon &\geq \frac{1}{n} p\left(A_1(x) + \cdots + A_n(x)\right) + \frac{1}{m} p\left(B_1(y) + \cdots + B_m(y)\right) \\
&\geq \frac{1}{mn} \sum_{k=1}^{m} p\left(B_k A_1(x) + \cdots + B_k A_n(x)\right) \\
&\quad + \frac{1}{mn} \sum_{k=1}^{n} p\left(A_k B_1(y) + \cdots + A_k B_m(y)\right) \\
&= \frac{1}{mn} \left(\sum_{k=1}^{m} p\left(\sum_{\ell=1}^{n} B_k A_\ell(x) \right) + \sum_{k=1}^{n} p\left(\sum_{\ell=1}^{m} A_k B_\ell(y) \right) \right) \\
&\geq \frac{1}{mn} \left(p\left(\sum_{k=1}^{m} \sum_{\ell=1}^{n} B_k A_\ell(x) \right) + p\left(\sum_{k=1}^{n} \sum_{\ell=1}^{m} A_k B_\ell(y) \right) \right) \\
&\geq \frac{1}{mn} p\left(\sum_{k=1}^{m} \sum_{\ell=1}^{n} B_k A_\ell(x) + \sum_{k=1}^{n} \sum_{\ell=1}^{m} A_k B_\ell(y) \right) \\
&= \frac{1}{mn} p\left(\sum_{k=1}^{m} \sum_{\ell=1}^{n} B_k A_\ell(x) + \sum_{k=1}^{n} \sum_{\ell=1}^{m} B_\ell A_k(y) \right) \\
&= \frac{1}{mn} p\left(\sum_{k=1}^{m} \sum_{\ell=1}^{n} B_k A_\ell(x) + \sum_{\ell=1}^{m} \sum_{k=1}^{n} B_\ell A_k(y) \right)
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{mn} p \left(\sum_{k=1}^{m} \sum_{\ell=1}^{n} B_k A_\ell(x) + \sum_{k=1}^{m} \sum_{\ell=1}^{n} B_k A_\ell(y) \right) \\
&= \frac{1}{mn} p \left(\sum_{k=1}^{m} \sum_{\ell=1}^{n} \big(B_k A_\ell(x) + B_k A_\ell(y) \big) \right) \\
&= \frac{1}{mn} p \left(\sum_{k=1}^{m} \sum_{\ell=1}^{n} B_k A_\ell(x+y) \right) \\
&\geq q(x+y).
\end{aligned}
$$

Since ε was arbitrary, we conclude that

$$q(x+y) \leq q(x) + q(y)$$

holds for all $x, y \in X$. Lastly, we show that $f \leq q$ on Y. Let $y \in Y$. Moreover, let $A_1, \dots, A_n \in G$. Then

$$
\begin{aligned}
f(y) &= \frac{1}{n} n f(y) \\
&= \frac{1}{n} \sum_{k=1}^{n} f(y) \\
&= \frac{1}{n} \sum_{k=1}^{n} f(A_k y) \\
&= \frac{1}{n} f(A_1 y + \cdots + A_n y) \\
&\leq \frac{1}{n} p(A_1 y + \cdots + A_n y)
\end{aligned}
$$

by the assumption that $f(Ay) = f(y)$, $Ay \in Y$, f is linear and $f \leq p$ on Y for all $y \in Y$ and $A \in G$. Taking the infimum over all finite sets of G finally yields the result. □

An application of *Hahn-Banach* now yields the existence of a linear mapping $F : X \to \mathbb{R}$ with $F|_Y = f$ and $F \leq q$ on X. Observe, that $q \leq p$ simply by choosing the finite set to be $\{\mathrm{id}_X\}$. Hence $F \leq p$ on X. Thus we have to show a final lemma.

Lemma 1.2. $\forall x \in X \, \forall A \in G \, \big(F(Ax) = F(x) \big)$.

Proof. Fix $x \in X$ and $A \in G$. Let $n \in \mathbb{N}$ and consider $\left\{ \mathrm{id}_X, A, A^2, \dots, A^{n-1} \right\} \subseteq G$. On one hand we have that

$$F(Ax) - F(x) = F(Ax - x)$$

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$$
\begin{aligned}
&\leq q(Ax - x) \\
&\leq \frac{1}{n} p\left(\sum_{k=0}^{n-1} A^k(Ax - x)\right) \\
&= \frac{1}{n} p\left(\sum_{k=0}^{n-1} \left(A^{k+1}x - A^k x\right)\right) \\
&= \frac{1}{n} p(A^n x - x) \\
&\leq \frac{1}{n} \big(p(A^n x) + p(-x)\big) \\
&\leq \frac{1}{n} \big(p(x) + p(-x)\big)
\end{aligned}
$$

and on the other

$$
\begin{aligned}
F(Ax) - F(x) &= F(Ax - x) \\
&= -F(x - Ax) \\
&\geq -q(x - Ax) \\
&\geq -\frac{1}{n} p\left(\sum_{k=0}^{n-1} A^k(x - Ax)\right) \\
&= -\frac{1}{n} p\left(\sum_{k=0}^{n-1} \left(A^k x - A^{k+1}x\right)\right) \\
&= -\frac{1}{n} p(x - A^n x) \\
&\geq -\frac{1}{n} \big(p(x) + p(-A^n x)\big) \\
&= -\frac{1}{n} \big(p(x) + p(A^n(-x))\big) \\
&\geq -\frac{1}{n} \big(p(x) + p(-x)\big).
\end{aligned}
$$

Hence

$$
|F(Ax) - F(x)| \leq \frac{1}{n} \big(p(x) + p(-x)\big).
$$

Since $n \in \mathbb{N}$ was arbitrary, we conclude that

$$
|F(Ax) - F(x)| = 0
$$

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and thus
$$F(Ax) = F(x).$$
$\square$

Exercise 2. See separate sheet.

Exercise 3.

Lemma 1.3. *Let $y \in H$ and define a mapping $\varphi_y : H \to \mathbb{K}$ by $\varphi_y(x) := \langle A(y), x\rangle$. Then $\varphi_y \in \mathcal{L}(H, \mathbb{K})$.*

Proof. Clearly, φ_y is linear since $\langle \cdot, \cdot \rangle$ is linear in the second component. Moreover, φ_y is bounded. Indeed, using Cauchy-Schwarz yields
$$|\varphi_y(x)| = |\langle A(y), x\rangle| \leq \|A(y)\| \|x\|$$
for all $x \in H$. $\square$

Thus we may define a family
$$\mathcal{F} := \left\{\varphi_y : y \in \partial B_1(0)\right\} \subseteq \mathcal{L}(H, \mathbb{K}).$$
Let $x \in H$. Then for any $y \in \partial B_1(0)$ we have that
$$|\varphi_y(x)| = |\langle A(y), x\rangle| = |\langle y, A(x)\rangle| \leq \|y\| \|A(x)\| = \|A(x)\|$$
by symmetry and again Cauchy-Schwarz. Hence
$$\sup_{T \in \mathcal{F}} |T(x)| = \sup_{y \in \partial B_1(0)} |\varphi_y(x)| \leq \|A(x)\|$$
for all $x \in H$. Since any Hilbert space is a Banach space, an application of *Banach-Steinhaus* yields the existence of a constant $c > 0$ such that
$$\sup_{T \in \mathcal{F}} \|T\| = \sup_{y \in \partial B_1(0)} \|\varphi_y\| \leq c.$$
For $x \in H$ such that $A(x) \neq 0$ we have that
$$\begin{aligned}
\|A(x)\|^2 &= \langle A(x), A(x)\rangle \\
&= \|x\| \, \langle A(x/\|x\|), A(x)\rangle \\
&= \|x\| \, \varphi_{x/\|x\|}(A(x)) \\
&\leq \|x\| \, |\varphi_{x/\|x\|}(A(x))| \\
&\leq \|x\| \|A(x)\| \, \|\varphi_{x/\|x\|}\| \\
&\leq c\|x\| \|A(x)\|
\end{aligned}$$
and thus dividing both sides by $\|A(x)\|$ yields the boundedness of A.

Exercise 4.
Exercise 5.

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a. We define

$$\mathcal{F} := \{B(\cdot, y) : y \in \partial B_1(0)\}.$$

Lemma 1.4. *We have that $\mathcal{F} \subseteq \mathcal{L}(X, \mathbb{K})$ and for all $x \in X$, there exists $c_x \geq 0$ such that $\sup_{T \in \mathcal{F}} |T(x)| \leq c_x$.*

Proof. Let $y \in \partial B_1(0)$. Then $B(\cdot, y)$ is linear by definition of a bilinear functional. Moreover, for any $x \in X$ we have that

$$|B(x, y)| \leq c_y \|x\|$$

for some $c_y \geq 0$ by continuity of B in the first argument. Hence $\mathcal{F} \subseteq \mathcal{L}(X, \mathbb{K})$. Let $x \in X$. Then

$$|B(x, y)| \leq c_x \|y\| = c_x$$

for some $c_x \geq 0$ by continuity of B in the second argument. Thus

$$\sup_{T \in \mathcal{F}} |T(x)| = \sup_{y \in \partial B_1(0)} |B(x, y)| \leq c_x$$

for all $x \in X$. $\square$

An application of *Banach-Steinhaus* on the family $\mathcal{F}$ yields the existence of a constant $c \geq 0$ such that

$$\sup_{T \in \mathcal{F}} \|T\| \leq c.$$

Let $x, y \in X$. Then

$$\begin{aligned} |B(x, y)| &= \|x\| \|y\| \, |B(x/\|x\|, y/\|y\|)| \\ &\leq \|x\| \|y\| \sup_{\|\xi\|=1} |B(\xi, y/\|y\|)| \\ &\leq \|x\| \|y\| \sup_{\|\zeta\|=1} \sup_{\|\xi\|=1} |B(\xi, \zeta)| \\ &= \|x\| \|y\| \sup_{\|\zeta\|=1} \|B(\cdot, \zeta)\| \\ &\leq c \|x\| \|y\|. \end{aligned}$$

Lemma 1.5. *Equip $X \times X$ with the norm $\|(x, y)\| := \|x\| + \|y\|$. Then B is continuous.*

Proof. Let $(x, y) \in X \times X$ and $(x_n, y_n)_{n \in \mathbb{N}}$ be a sequence in $X \times X$ converging to (x, y). We claim that $x_n \to x$ and $y_n \to y$ in X. Indeed

$$\|x_n - x\| \leq \|x_n - x\| + \|y_n - y\| = \|(x_n, y_n) - (x, y)\| \to 0$$

as $n \to \infty$ and similarly

$$\|y_n - y\| \leq \|x_n - x\| + \|y_n - y\| = \|(x_n, y_n) - (x, y)\| \to 0.$$

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Moreover, since $y_n \to y$, y_n is bounded, i.e. there exists some $M \geq 0$ such that $\|y_n\| \leq M$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned}
|B(x_n, y_n) - B(x, y)| &= |B(x_n, y_n) - B(x, y_n) + B(x, y_n) - B(x, y)| \\
&= |B(x_n - x, y_n) + B(x, y_n - y)| \\
&\leq |B(x_n - x, y_n)| + |B(x, y_n - y)| \\
&\leq c\|x_n - x\|\|y_n\| + c\|x\|\|y_n - y\| \\
&\leq cM\|x_n - x\| + c\|x\|\|y_n - y\| \to 0
\end{aligned}$$

as $n \to \infty$. □

b.

Lemma 1.6. *B is a bilinear functional on $\mathcal{P}$ which is continuous in each argument separately.*

Proof. The bilinearity of B directly follows from the linearity of the integral. Fix $q \in \mathcal{P}$. Then for any $p \in \mathcal{P}$ we have that

$$|B(p,q)| = \left|\int_0^1 p(t)q(t)dt\right| \leq \int_0^1 |p(t)||q(t)|\,dt \leq \sup_{t\in[0,1]} |q(t)| \int_0^1 |p(t)|\,dt = c_q\|p\|$$

since q is continuous. Similarly, for each fixed $p \in \mathcal{P}$ we get that $|B(p,q)| \leq c_p\|q\|$ for all $q \in \mathcal{P}$. □