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CHAPTER 1

Differential Topology

Smooth Manifolds

Example 1.1 (*n*-Spheres). Let $n \in \omega$. If $n = 0$, we have that $\mathbb{S}^0 = \{\pm 1\}$. It is easily seen that $\mathbb{S}^0$ is a smooth manifold of dimension 0. Let $n \geq 1$. Define $N := e_{n+1}$ and $S := -e_{n+1}$, where e_{n+1} denotes the $n+1$-th standard basis vector of $\mathbb{R}^{n+1}$. Moreover, set

$$U_+ := \mathbb{S}^n \setminus S \qquad \text{and} \qquad U_- := \mathbb{S}^n \setminus N.$$

Then U_+ and U_- are open subsets of $\mathbb{S}^n$, the upper and lower hemisphere, respectively. Then the functions $\varphi_\pm : U_\pm \to \mathbb{R}^n$ defined by

$$\varphi_\pm(x) := \frac{1}{1 \pm x_{n+1}}(x_1, \ldots, x_n),$$

are homeomorphisms. Indeed, one can check that $\psi_\pm : \mathbb{R}^n \to U_\pm$ defined by

$$\psi_\pm(x) := \left(\frac{2x}{1 + |x|^2}, \frac{\pm(1 - |x|^2)}{1 + |x|^2} \right)$$

is a continuous inverse for φ_+ and φ_-, respectively. We claim that $\{(U_\pm, \varphi_\pm)\}$ is a smooth atlas for $\mathbb{S}^n$. Clearly, $\mathbb{S}^n$ is covered by the two charts. Next we have to calculate the transition functions $\varphi_\mp \circ \varphi_\pm^{-1} = \varphi_\mp \circ \psi_\pm : \varphi_\pm(U_+ \cap U_-) \to \varphi_\mp(U_+ \cap U_-)$. It is easy to see that $\varphi_\pm(U_+ \cap U_-) = \mathbb{R}^n \setminus \{0\}$ and that

$$\varphi_\mp \circ \psi_\pm = \frac{x}{|x|^2},$$

which is smooth. Since $\mathbb{S}^n$ is Hausdorff as a metric space and as a subspace of a second countable space, itself second countable, $\mathbb{S}^n$ equipped with the smooth structure induced by the smooth atlas constructed above, is a smooth manifold of dimension n.