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Differential Topology and Differential Geometry with Applications to General Relativity

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Preface

The process of writing this book started in autumn of 2018. The book is mainly based on the course notes of the course *Differential Geometry I* and *Differential Geometry II* held by *Prof. Dr. Will J. Merry* at *ETH Zurich* in the autumn semester 2018 and spring semester 2019, I visited. They can be found here:

https://www.merry.io/differential-geometry/

Moreover, the part concerning general relativity, is freely taken from the course *General Relativity* taught by *Prof. Dr. Renato Renner* at ETH Zurich in the autumn semester 2018.

Winterthur, Yannis Bähni
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Contents

CHAPTER 1

Differential Topology

The Category of Smooth Manifolds

Example 1.1 (*n*-Spheres). Let $n \in \omega$. If $n = 0$, we have that $\mathbb{S}^0 = \{\pm 1\}$. It is easily seen that $\mathbb{S}^0$ is a smooth manifold of dimension 0. Let $n \geq 1$. Define $N := e_{n+1}$ and $S := -e_{n+1}$, where e_{n+1} denotes the $n+1$-th standard basis vector of $\mathbb{R}^{n+1}$. Moreover, set

$$U_+ := \mathbb{S}^n \setminus S \qquad \text{and} \qquad U_- := \mathbb{S}^n \setminus N.$$

Then U_+ and U_- are open subsets of $\mathbb{S}^n$, the upper and lower hemisphere, respectively. Then the functions $\varphi_\pm : U_\pm \to \mathbb{R}^n$ defined by

$$\varphi_\pm(x) := \frac{1}{1 \pm x_{n+1}}(x_1, \ldots, x_n),$$

are homeomorphisms. Indeed, one can check that $\psi_\pm : \mathbb{R}^n \to U_\pm$ defined by

$$\psi_\pm(x) := \left(\frac{2x}{1 + |x|^2}, \frac{\pm(1 - |x|^2)}{1 + |x|^2} \right)$$

is a continuous inverse for φ_+ and φ_-, respectively. We claim that $\{(U_\pm, \varphi_\pm)\}$ is a smooth atlas for $\mathbb{S}^n$. Clearly, $\mathbb{S}^n$ is covered by the two charts. Next we have to calculate the transition functions $\varphi_\mp \circ \varphi_\pm^{-1} = \varphi_\mp \circ \psi_\pm : \varphi_\pm(U_+ \cap U_-) \to \varphi_\mp(U_+ \cap U_-)$. It is easy to see that $\varphi_\pm(U_+ \cap U_-) = \mathbb{R}^n \setminus \{0\}$ and that

$$\varphi_\mp \circ \psi_\pm = \frac{x}{|x|^2},$$

which is smooth. Since $\mathbb{S}^n$ is Hausdorff as a metric space and as a subspace of a second countable space, itself second countable, $\mathbb{S}^n$ equipped with the smooth structure induced by the smooth atlas constructed above, is a smooth manifold of dimension n.

The Tangent Bundle

Vector Fields.

The Cotangent Bundle

Tensor Bundles and Tensor Fields

Let $M \in \mathsf{Diff}$ and $(k,l) \in \omega \times \omega$. Consider

$$T^{(k,l)}M := \coprod_{p\in M} T^{(k,l)}(T_pM).$$

Define $\pi : T^{(k,l)}M \to M$ by setting $\pi(p,A) := p$ for $A \in T^{(k,l)}(T_pM)$. Clearly π is surjective. Thus let $p \in M$ be arbitrary. Then there exists a chart (U,φ) around p

Lemma 1.2. *Let $n,k \in \mathbb{Z}$, $n,k \geq 1$. Let V be an n-dimensional real vector space. Then*

$$V \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{k} \cong L(\underbrace{V,\ldots,V}_{k};V) \tag{1}$$

canonically. If (e_ν) is a basis of V and (e_ν^*) the corresponding basis of V^*, then $f \in$ $\mathrm{End}(V)$ *corresponds to*

$$\sum_{\nu=1}^{n} f(e_\nu) \otimes e_\nu^*. \tag{2}$$

Proof. It is easily checked that

$$\Psi : \begin{cases} V \times V^* \times \cdots \times V^* \to L(V,\ldots,V;V) \\ (v, f_1,\ldots,f_k) \mapsto \big((v_1,\ldots,v_k) \mapsto f_1(v_1)\cdots f_k(v_k)v\big) \end{cases}$$

is multilinear. Thus by the universal property of the tensor product there exists a unique linear mapping $\widetilde{\Psi} \in \mathrm{Hom}_{\mathbb{R}}\left(T^{(1,k)}(V); L(V,\ldots,V;V)\right)$ such that $\Psi = \widetilde{\Psi} \circ \otimes$. It is also easily checked that $\widetilde{\Psi}$ is an isomorphism. Let $f \in \mathrm{End}(V)$. Then for any $v \in V$ we have

$$\begin{aligned}
\widetilde{\Psi}\left(\sum_{\nu=1}^{n} f(e_\nu)\otimes e_\nu^*\right)(v) &= \sum_{\nu=1}^{n} \widetilde{\Psi}\left(f(e_\nu)\otimes e_\nu^*\right)(v) \\
&= \sum_{\nu=1}^{n} e_\nu^*(v) f(e_\nu) \\
&= f\left(\sum_{\nu=1}^{n} e_\nu^*(v) e_\nu\right) \\
&= f(v).
\end{aligned}$$

□

Proposition 1.3. *The bundle of mixed tensors of type (k,l) has a unique natural structure as a smooth vector bundle of rank n^{k+l} over M.*

Proof. For each $p \in M$ let $E_p := T^{(k,l)}(T_pM)$. By [Lee13, p. 57] and [Lee13, p. 313] $\dim E_p = n^{k+l}$. Furthermore, let $E := T^{(k,l)}TM$ and $\pi : E \to M$ be defined by $\pi(p, A) := p$. Let $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ be an atlas for M. For each $\alpha \in A$ define

$$\Phi_\alpha : \begin{cases} \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{n^{k+l}} \\ (p, A) \mapsto \left(p, (A^{i_1\dots i_k}_{j_1\dots j_l})\right) \end{cases}$$

Clearly, the inverse is given by

$$\Phi_\alpha^{-1} : \begin{cases} U_\alpha \times \mathbb{R}^{n^{k+l}} \to \pi^{-1}(U_\alpha) \\ \left(p, (A^{i_1\dots i_k}_{j_1\dots j_l})\right) \mapsto (p, A) \end{cases} .$$

Hence each Φ_α is bijective. Now we have to check, that $\Phi_\alpha|_{E_p}$ is an isomorphism. So let $\lambda \in \mathbb{R}$ and $B \in E_p$. Then

$$\begin{aligned} \Phi_\alpha|_{E_p}(p, \lambda A + B) &= \left(p, (\lambda A + B)^{i_1\dots i_k}_{j_1\dots j_l})\right) \\ &= \left(p, \lambda(A^{i_1\dots i_k}_{j_1\dots j_l}) + (B^{i_1\dots i_k}_{j_1\dots j_l})\right) \\ &= \lambda\Phi_\alpha|_{E_p}(p, A) + \Phi_\alpha|_{E_p}(p, B). \end{aligned}$$

Now let $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$. We consider the mapping

$$\Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^{n^{k+l}} \to (U_\alpha \cap U_\beta) \times \mathbb{R}^{n^{k+l}}.$$

Define $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n^{k+l}, \mathbb{R})$ by

$$\tau_{\alpha\beta} := (\delta^i_j).$$

Then we have that

$$(\Phi_\alpha \circ \Phi_\beta^{-1})\left(p, (A^{i_1\dots i_k}_{j_1\dots j_l})\right) = \left(p, (A^{i_1\dots i_k}_{j_1\dots j_l})\right) = \left(p, \tau_{\alpha\beta}(p)(A^{i_1\dots i_k}_{j_1\dots j_l})\right).$$

Since $\tau_{\alpha\beta}$ is constant, it is smooth (see [Lee13, p. 36]). Hence we can apply the vector bundle chart lemma [Lee13, p. 253] and the result follows. ☐

Proposition 1.4 (Smoothness Criteria for Tensor Fields). *Let M be smooth manifold and let $A : M \to T^{(1,k)}TM$ be a rough section. Then the following are equivalent:*

(*a*) $A \in \Gamma(T^{(1,k)}TM)$.

(*b*) *In every smooth coordinate chart, the component functions of A are smooth.*

(*c*) *For all $X_1, \dots, X_k \in \mathfrak{X}(M)$, the rough section $A(X_1, \dots, X_k) : M \to TM$ defined by*

$$A(X_1, \dots, X_k)(p) := A_p(X_1|_p, \dots, X_k|_p) \tag{3}$$

is a smooth vector field.

(*d*) *If $X_1, \dots, X_k$ are smooth vector fields on some open subset $U \subseteq M$, then also $A(X_1, \dots, X_k)$ is a smooth vector field on U.*

Proof. We prove (a)$\Leftrightarrow$(b) and (b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(b).
To prove (a)$\Leftrightarrow$(b), let $(U,(x^i))$ be a smooth chart. Actually, we can prove this for general tensor fields of type (k,l). Proposition 1.3 together with the proof of the vector bundle chart lemma [Lee13, pp. 253–254] implies, that the corresponding chart on $T^{(k,l)}TM$ is given by $\left(\pi^{-1}(U),\widetilde{\varphi}\right)$, where $\widetilde{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^{n^{k+l}}$ is defined by

$$\widetilde{\varphi} := (\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}}) \circ \Phi$$

where $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^{n^{k+l}}$ is given as in the proof of proposition 1.3. Now we consider the coordinate representation $\widehat{A}$ in the given charts (see [Lee13, p. 35]). Since A is a rough section, we have that

$$A^{-1}\left(\pi^{-1}(U)\right) = (\pi \circ A)^{-1}(U) = \mathrm{id}_M^{-1}(U) = U.$$

Hence $\varphi\left(U \cap A^{-1}(\pi^{-1}(U))\right) = \varphi(U)$, which is open, and $\widehat{A} : \varphi(U) \to \widetilde{\varphi}\left(\pi^{-1}(U)\right)$ is given by

$$\begin{aligned}\widehat{A}(x) &= (\widetilde{\varphi} \circ A \circ \varphi^{-1})(x)\\ &= (\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}})\left(\Phi(\varphi^{-1}(x), A_{\varphi^{-1}(x)})\right)\\ &= (\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}})\left(\varphi^{-1}(x), \left(A^{i_1\dots i_k}_{j_1\dots j_l}(\varphi^{-1}(x))\right)\right)\\ &= \left(x, \left(\widehat{A}^{i_1\dots i_k}_{j_1\dots j_l}(x)\right)\right).\end{aligned}$$

By [Lee13, p. 35] A is smooth if and only if in any chart $\widehat{A}$ is smooth. This is furthermore equivalent to that each $\widehat{A}^{i_1\dots i_k}_{j_1\dots j_l}$ is smooth and thus equivalent to that $A^{i_1\dots i_k}_{j_1\dots j_l}$ is smooth (see [Lee13, p. 33]).
To prove (b)$\Rightarrow$(c), let $(U,(x^i))$ be a smooth chart. Then write $X_1,\dots,X_k \in \mathfrak{X}(M)$ as

$$X_\nu = X_\nu^{\mu_\nu}\frac{\partial}{\partial x^{\mu_\nu}}.$$

for $\nu = 1,\dots,k$. For $p \in U$ lemma 1.2 implies

$$\begin{aligned}A(X_1,\dots,X_n)(p) &= A_p(X_1|_p,\dots,X_k|_p)\\ &= A_p\left(X_1^{\mu_1}(p)\left.\frac{\partial}{\partial x^{\mu_1}}\right|_p,\dots,X_1^{\mu_k}(p)\left.\frac{\partial}{\partial x^{\mu_k}}\right|_p\right)\\ &= X_1^{\mu_1}(p)\cdots X_k^{\mu_k}(p)A_p\left(\left.\frac{\partial}{\partial x^{\mu_1}}\right|_p,\dots,\left.\frac{\partial}{\partial x^{\mu_k}}\right|_p\right)\\ &= X_1^{\mu_1}(p)\cdots X_k^{\mu_k}(p)A^i_{\mu_1\dots\mu_k}(p)\left.\frac{\partial}{\partial x^i}\right|_p.\end{aligned}$$

By the smoothness criterion for vector fields [Lee13, p. 175] we have that each component function $X_\nu^{\mu_n}$ is smooth. Thus if A is smooth, we have by that each $A^i_{j_1\dots j_k}$ is smooth and since $C^\infty(M)$ is an $\mathbb{R}$-algebra (see [Lee13, p. 33]), we have that

$$X_1^{\mu_1} \cdots X_k^{\mu_k} A^i_{\mu_1 \dots \mu_k}$$

is smooth for $i = 1, \dots, n$. Thus again by the smoothness criterion together with the localness of smoothness [Lee13, p. 35] we get that $A(X_1, \dots, X_k) \in \mathfrak{X}(M)$.
To prove (c)⇒(d), we use that smoothness is a local property (see [Lee13, p. 35]). Let $p \in U$. Then by [Cat17, p. 14] we find a smooth bump function ψ supported in U and identically equal to 1 on some neighbourhood V of p. Set

$$\widetilde{X}_\nu|_p := \begin{cases} \psi(p) X_\nu|_p & p \in \operatorname{supp} \psi \\ 0 & p \in M \setminus \operatorname{supp} \psi \end{cases}.$$

Then the gluing lemma for smooth maps [Lee13, p. 35] implies $\widetilde{X}_1, \dots, \widetilde{X}_k \in \mathfrak{X}(M)$. Hence by (c) we get that $A(\widetilde{X}_1, \dots, \widetilde{X}_k) \in$
$mathfrakX(M)$ and so the restriction $A(\widetilde{X}_1, \dots, \widetilde{X}_k)|_V$ is smooth. But $A(\widetilde{X}_1, \dots, \widetilde{X}_k)|_V =$ $A(X_1, \dots, X_k)$ and so we are done.
Lasty to prove (d)⇒(b), each vector field locally defined by

$$X_{j_\nu} = \delta^{\mu_\nu}_{j_\nu} \frac{\partial}{\partial x^{\mu_\nu}}.$$

is smooth. Thus by

$$A(X_1, \dots, X_n)(p) = \delta^{\mu_1}_{j_1} \cdots \delta^{\mu_k}_{j_k} A^i_{\mu_1 \dots \mu_k}(p) \left.\frac{\partial}{\partial x^i}\right|_p = A^i_{j_1 \dots j_k}(p) \left.\frac{\partial}{\partial x^i}\right|_p$$

we get that $A^i_{j_1 \dots j_k}$ is smooth and hence by (b) also A. □

Theorem 1.5 (Tensor Characterization Lemma). *A mapping*

$$\underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k} \to C^\infty(M) \qquad \textit{or} \qquad \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k} \to \mathfrak{X}(M)$$

is induced by an element of $\Gamma(T^{(0,k)}TM)$ *or* $\Gamma(T^{(1,k)}TM)$*, respectively, if and only if they are multilinear over* $C^\infty(M)$*.*

Proof. We are proving only the second statement. Any element in $\Gamma(T^{(1,k)}TM)$ induces a mapping $\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ by part (c) of the smoothness criteria for tensor fields 1.4. Thus we have to show that $\mathcal{A}$ is multilinear over $C^\infty(M)$. Let $f \in C^\infty(M)$ and $X_\nu, \widetilde{X}_\nu \in \mathfrak{X}(M)$, $\nu = 1, \dots, k$. Then for any $p \in M$ we have that

$$\begin{aligned} \mathcal{A}(X_1, \dots, fX_\nu + \widetilde{X}_\nu, \dots, X_k)_p =& A_p(X_1|_p, \dots, (fX_\nu + \widetilde{X}_\nu)_p, \dots, X_k|_p) \\ =& A_p(X_1|_p, \dots, f(p)X_\nu|_p + \widetilde{X}_\nu|_p, \dots, X_k|_p) \\ =& f(p) A_p(X_1|_p, \dots, X_\nu|_p, \dots, X_k|_p) \\ &+ A_p(X_1|_p, \dots, \widetilde{X}_\nu|_p, \dots, X_k|_p) \\ =& f(p)\mathcal{A}(X_1, \dots, X_\nu, \dots, X_k)_p \end{aligned}$$

$$\begin{aligned}&+\mathcal{A}(X_1,\dots,\widetilde{X}_\nu,\dots,X_k)_p\\ &=\big(f\mathcal{A}(X_1,\dots,X_\nu,\dots,X_k)\big)_p\\ &\quad+\mathcal{A}(X_1,\dots,\widetilde{X}_\nu,\dots,X_k)_p.\end{aligned}$$

Conversly, suppose that $\mathcal{A}:\mathfrak{X}(M)\times\cdots\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ is multilinear over $C^\infty(M)$. Let $p\in M$. First we show that $\mathcal{A}$ acts locally, i.e. if $X_\nu=\widetilde{X}_\nu$ in some neighbourhood U of p implies that also

$$\mathcal{A}(X_1,\dots,X_\nu,\dots,X_k)=\mathcal{A}(X_1,\dots,\widetilde{X}_\nu,\dots,X_k)$$

on U. By the multilinearity of $\mathcal{A}$ it is enough to show that if X_ν vanishes on U then so does $\mathcal{A}$. There exists a smooth bump function ψ for $\{p\}$ supported in U (see [Lee13, p. 44]). Hence $\psi X_\nu=0$ on M and $\psi(p)=1$. Thus

$$0=\mathcal{A}(X_1,\dots,\psi X_\nu,\dots,X_k)_p=\psi(p)\mathcal{A}(X_1,\dots,X_\nu,\dots,X_k)_p.$$

and since $\psi(p)=1$ we have that

$$\mathcal{A}(X_1,\dots,X_\nu,\dots,X_k)_p=0$$

for any $p\in U$.

Next we show that $\mathcal{A}$ actually acts pointwise, i.e. if $X_\nu|_p$ vanishes so does $\mathcal{A}$. Let $(U,(x^i))$ be a chart containing p and $X_\nu=X_\nu^i\frac{\partial}{\partial x^i}$ on U. The same construction as used showing the implication (c)$\Rightarrow$(d) in the proof of proposition 1.4 yields the existence of $f^1,\dots,f^n\in C^\infty(M)$ and $\widetilde{X}_1,\dots,\widetilde{X}_n\in\mathfrak{X}(M)$ such that $f^i=X_\nu^i$ and $\widetilde{X}_i=\frac{\partial}{\partial x^i}$ on a neighbourhood $V\subseteq U$ of p. Thus by the previous localization, we get that

$$\mathcal{A}(X_1,\dots,X_\nu,\dots,X_k)=\mathcal{A}(X_1,\dots,f^i\widetilde{X}_i,\dots,X_k)=f^i\mathcal{A}(X_1,\dots,\widetilde{X}_i,\dots,X_k)$$

in U. Since $0=X_\nu^i(p)=f^i(p)$, $\mathcal{A}$ vanishes at p. Hence $\mathcal{A}$ depends only on the value of X_ν at p. Thus define a rough section $A:M\to T^{(1,k)}TM$ by

$$A_p(v_1,\dots,v_k):=\mathcal{A}(V_1,\dots,V_k)(p)$$

where $V_1,\dots,V_k\in\mathfrak{X}(M)$ are any extensions of $v_1,\dots,v_k\in T_pM$ (see [Lee13, p. 177]). By the above, the choice of the extensions does not matter and the resulting rough section is smooth by proposition 1.4 part (c), hence $A\in\Gamma(T^{(1,k)}TM)$. $\square$

CHAPTER 2

Differential Geometry

CHAPTER 3

General Relativity

Classical Mechanics

We follow closely the exposition provided in [Tak08, pp. 4–61].

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