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Introduction to Functional Analysis

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Contents

CHAPTER 1

Topological Spaces

1. Definitions and Basic Notions

1.1. Topologies.

Definition 1.1. *Let X be a set. A **topology** on X is a collection $\mathcal{T}$ of subsets of X satisfying the following properties:*

(i) $\varnothing, X \in \mathcal{T}$.
(ii) *If* $(U_\alpha)_{\alpha \in A}$ *is a family of elements of* $\mathcal{T}$*, then* $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.
(iii) *If* $U_1, \ldots, U_n \in \mathcal{T}$*, then* $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.

*A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space** and elements of $\mathcal{T}$ are called **open sets**.*

Example 1.1 (Topologies).

(a) Let $(X, \mathcal{T})$ be a topological space and let $S \subseteq X$. Then the collection $\mathcal{T}_S := \{S \cap U : U \in \mathcal{T}\}$ is a topology on S.

Definition 1.2. *Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The **closure of A in X**, denoted by $\overline{A}$, is defined by*

$$\overline{A} := \bigcap \{B \subseteq X : A \subseteq B, B^c \in \mathcal{T}\}. \tag{1}$$

*The **interior of A in X**, denoted by* $\operatorname{Int} A$*, is defined by*

$$\operatorname{Int} A := \bigcup \{C \subseteq X : C \subseteq A, C \in \mathcal{T}\}. \tag{2}$$

There is an eminent characterization of a point being in the closure of a subset $A \subseteq X$ of a topological space X.

Proposition 1.1. *Let X be a topological space and $A \subseteq X$. Then $x \in \overline{A}$ if and only if the following condition holds: Every neighbourhood U of x contains a point belonging to A.*

Proof. Assume that there exists a neighbourhood U of x such that $U \cap A = \varnothing$. Then U^c is closed and $A \subseteq U^c$. But $x \notin U^c$ and thus $x \notin \overline{A}$. Conversly, assume $x \notin \overline{A}$. Thus we find a closed set B such that $A \subseteq B$ and $x \notin B$. But then B^c is open and $B^c \cap A = \varnothing$. $\square$

1.2. Hausdorff Spaces.

Definition 1.3. *Let X be a topological space. X is called a **Hausdorff space** if given $p, p' \in X$ with $p \neq p'$ we find neighbourhoods U and U' of p and p', respectively, such that $U \cap U' = \varnothing$*

1.3. Bases and Countability.

Definition 1.4. *Let $(X, \mathcal{T})$ be a topological space. A collection $\mathcal{B}$ of subsets of X is called a* ***basis for the topology of X*** *if the following two conditions hold:*

(i) $\mathcal{B} \subseteq \mathcal{T}$.
(ii) *For any $U \in \mathcal{T}$ we have $U = \bigcup_{\alpha \in A} B_\alpha$ where $B_\alpha \in \mathcal{B}$ for any $\alpha \in A$.*

As we shall see later, a topology $\mathcal{T}$ on a set X may have several bases but topologies having the same basis, are equal.

Corollary 1.1. *If X is a set, $\mathcal{T}$ and $\mathcal{T}'$ are topologies on X and $\mathcal{B}$ is a basis for each of the topologies $\mathcal{T}$ and $\mathcal{T}'$, then $\mathcal{T} = \mathcal{T}'$.*

Proof. This is immediate by the definition of a basis for a topology 1.4. □

Proposition 1.2 (Basis Criterion). *Let X be a topological space and $\mathcal{B}$ be a basis for the topology on X. Then U is open in X if and only if for each $p \in U$ there exists $B \in \mathcal{B}$ such that $p \in B \subseteq U$.*

Proof. Assume U is open. Since $\mathcal{B}$ is a basis, we have $U = \bigcup_{\alpha \in A} B_\alpha$ where $B_\alpha \in \mathcal{B}$ for each $\alpha \in A$. Thus for each $p \in U$ we have $p \in \bigcup_{\alpha \in A} B_\alpha$ and so $p \in B_\alpha$ for some $\alpha \in A$. But since also $\bigcup_{\alpha \in A} B_\alpha \subseteq U$ we have $B_\alpha \subseteq U$. Conversly, we can write $U = \bigcup_{p \in U} B_p$ for some $U \subseteq X$ where for each $p \in U$ we have $B_p \in \mathcal{B}$. Since each basis element is open, U is open as a union of open sets. □

Definition 1.5. *Let X be a set and $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for some topology on X if and only if it satisfies the following two conditions:*

(i) $\bigcup_{B \in \mathcal{B}} B = X$.
(ii) *If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists an element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

Example 1.2. Let $(X_\alpha)_{\alpha \in A}$ be a family of topological spaces. The ***product topology*** on $\prod_{\alpha \in A} X_\alpha$ is defined to be the topology generated by the basis consisting of all subsets of $\prod_{\alpha \in A} X_\alpha$ of the form $\prod_{\alpha \in A} U_\alpha$ where U_α is open in X_α for any $\alpha \in A$ and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in A$. The reader may verify that this is indeed a basis for a topology.

Definition 1.6. *Let X be a topological space. X is called* ***second countable*** *if there exists a countable basis for the topology of X.*

1.4. Continuity and Convergence.

Definition 1.7. *Let X and Y be two topological spaces and $f : X \to Y$. The map f is said to be* ***continuous*** *if for any open set $U \subseteq Y$ we have that $f^{-1}(Y)$ is open in X.*

Proposition 1.3 (Characteristic Property of Infinite Product Spaces). *Let $(X_\alpha)_{\alpha \in A}$ be a family of topological spaces. For any topological space Y, a mapping $f : Y \to \prod_{\alpha \in A} X_\alpha$ is continuous if and only if each of its component functions $f_\alpha := \pi_\alpha \circ f$ is continuous, where $\pi_\alpha : \prod_{\alpha \in A} X_\alpha \to X_\alpha$ denotes the* ***canonical projection****.*

Proof. It is enough to verify the statements for basis sets only. Let $U_\alpha \subseteq X_\alpha$ be open. Then $\pi_\alpha^{-1}(U_\alpha) = \prod_{\beta \in A} U_\beta$ where $U_\beta = X_\beta$ whenever $\beta \neq \alpha$. But this set is open in $\prod_{\alpha \in A} X_\alpha$ and hence by the continuity of f

$$f_\alpha^{-1}(U_\alpha) = (\pi_\alpha \circ f)^{-1}(U_\alpha) = f^{-1}(\pi_\alpha^{-1}(U_\alpha)) \tag{3}$$

is open in Y. Conversly, assume that f_α is continuous for every $\alpha \in A$. Let B belong to the basis of the topology of $\prod_{\alpha \in A} X_\alpha$. Then $B = \bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_{\alpha_i})$ for some open subsets $U_{\alpha_i} \subseteq X_{\alpha_i}$. But then

$$f^{-1}(B) = \bigcap_{i=1}^n f^{-1}(\pi_{\alpha_i}^{-1}(U_{\alpha_i})) = \bigcap_{i=1}^n (\pi_{\alpha_i} \circ f)^{-1}(U_{\alpha_i}) \tag{4}$$

is open in Y. $\square$

Corollary 1.2. *Let $(X_\alpha)_{\alpha \in A}$ be a family of topological spaces. Each canonical projection $\pi_\alpha : \prod_{\alpha \in A} X_\alpha \to X_\alpha$ is continuous.*

Proof. Choose $Y = \prod_{\alpha \in A} X_\alpha$ equipped with the product topology and $f = \mathrm{id}$ in proposition 1.3. $\square$

Proposition 1.4 (Uniqueness of the Product Topology). *Let $(X_\alpha)_{\alpha \in A}$ be a family of topological spaces. The product topology on $\prod_{\alpha \in A} X_\alpha$ is the unique topology satisfying the characteristic property 1.3.*

Proof. Assume there exists another topology on $\prod_{\alpha \in A} X_\alpha$ which satisfies the characteristic property 1.3. Then setting $Y = \prod_{\alpha \in A} X_\alpha$ equipped with this topolohy in proposition 1.3 and using that by corollary 1.2 the mappings $f_\alpha = \pi_\alpha \circ f$ are continuous by composition of continuous functions yields that id is continuous and so the product topology is contained in the other one. Exchanging the roles of Y and $\prod_{\alpha \in A} X_\alpha$ yields the desired equality. $\square$

Proposition 1.5 (Minimality of the Product Topology). *Let $(X_\alpha)_{\alpha \in A}$ be a family of topological spaces. Endow $\prod_{\alpha \in A} X_\alpha$ with a topology such that every canoncical projection $\pi_\alpha : \prod_{\alpha \in A} X_\alpha \to X_\alpha$ is continuous. Then this topology contains the product topology.*

Proof. Let B be a basis element of the basis of the product topology on $\prod_{\alpha \in A} X_\alpha$. Thus

$$B = \bigcap_{i=1}^n \pi_{\alpha_i}^{-1}(U_{\alpha_i}) \tag{5}$$

for some open subsets $U_{\alpha_i} \subseteq X_{\alpha_i}$. Since each canonical projection π_α is continuous, we have that B is contained in the topology. $\square$

Definition 1.8. *Let X be a topological space, $(x_n)_{n \in \mathbb{N}}$ be a sequence in X and $x \in X$. The sequence $(x_n)_{n \in \mathbb{N}}$ is said to **converge to x** if for every neighbourhood U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for any $n > N$.*

Corollary 1.3. *Let X be a topological space and $A \subseteq X$. If $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence in A, i.e. $x_n \in A$ for any $n \in \mathbb{N}$, then its limit belongs to $\overline{A}$.*

Proof. This is immediate by the characterization of proposition 1.1. $\square$

Proposition 1.6. *Let $(X_\alpha)_{\alpha\in A}$ be a family of topological spaces and $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\prod_{\alpha\in A} X_\alpha$. Then $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} \pi_\alpha(x_n) = \pi_\alpha(x)$ for any $\alpha \in A$.*

Proof. Assume $\lim_{n\to\infty} x_n = x \in \prod_{\alpha\in A} X_\alpha$. Fix some $\alpha \in A$ and consider some neighbourhood U of $\pi_\alpha(x)$. Then $\prod_{\beta\in A} U_\beta$ where $U_\beta = X_\beta$ for any $\beta \neq \alpha$ and $U_\beta = U$ for $\beta = \alpha$ is a neighbourhood of x in $\prod_{\alpha\in A} X_\alpha$. Thus there exists some $N \in \mathbb{N}$ such that $x_n \in \prod_{\beta\in A} U_\beta$ whenever $n > N$. But then $\pi_\alpha(x_n) \in U$ for any $n > N$ and thus $\lim_{n\to\infty} \pi_\alpha(x_n) = \pi_\alpha(x)$. Conversly, suppose $\lim_{n\to\infty} \pi_\alpha(x_n) = \pi_\alpha(x)$ for any $\alpha \in A$. Let U be some neighbourhood of x. Then by the basis criterion 1.2 we find some basis element $B = \prod_{\alpha\in A} B_\alpha$, where B_α is open in X_α and $B_\alpha = X_\alpha$ for all but finitely many $\alpha \in A$, such that $x \in B \subseteq U$. If $B_\alpha \neq X_\alpha$, define $N_\alpha \in \mathbb{N}$ to be the number such that $n > N_\alpha$ implies $\pi_\alpha(x_n) \in B_\alpha$, otherwise let $N_\alpha := 1$. Thus $N := \max\{N_\alpha : \alpha \in A\}$ is bounded above and is therefore well defined. Hence $x_n \in U$ whenever $n > N$ and thus we have convergence. □

1.5. Connectedness and Compactness.

Definition 1.9. *Let X be a topological space. If $X = U \cup V$ for some disjoint open sets $U, V \neq \varnothing$, X is called* ***disconnected****, otherwise X is said to be* ***connected****.*

Definition 1.10. *Let X be a topological space. An* ***open cover*** *of X is a family $(U_\alpha)_{\alpha\in A}$ of open subsets of X such that $X = \bigcup_{\alpha\in A} U_\alpha$. A* ***subcover*** *of $(U_\alpha)_{\alpha\in A}$ is a subfamily $(U_\alpha)_{\alpha\in A'}$, $A' \subseteq A$.*

Definition 1.11. *A topological space X is said to be* ***compact*** *if any open cover has a finite subcover.*

Theorem 1.1 (Main Theorem on Compactness). *Let X and Y be topological spaces and $f : X \to Y$ be continuous.If X is compact, then $f(X)$ is compact.*

Proposition 1.7. *Every closed, bounded interval in $\mathbb{R}$ is compact.*

Theorem 1.2 (Extreme Value Theorem). *If X is a compact space and $f : X \to \mathbb{R}$ is continuous, then f is bounded and attains its maximum and minimum values on X.*

Exercises

Exercise 1.1. Prove that a mapping $f : X \to Y$ between two topological spaces X and Y is continuous if and only if for all $A \subseteq Y$ closed, $f^{-1}(Y)$ is closed in X. *Hint:* Use that for $A, B \subseteq Y$ we have $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

2. Metric Spaces

2.1. Basic Definitions and Properties.

We follow [Lee11, pp. 396 – 398].

Definition 2.1. *Let M be a set. A* ***metric*** *on M is a function*

$$d : M \times M \to \mathbb{R} \tag{6}$$

having the following properties:

(*i*) $d(x, y) \geq 0$ *for all* $x, y \in M$.
(*ii*) $d(x, y) = 0$ *if and only if* $x = y$.
(*iii*) $d(x, y) = d(y, x)$ *for all* $x, y \in M$.
(*iv*) (Triangle Inequality) $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in M$.

*If on a set M a metric d has been specified, the tuple (M, d) is called a **metric space**.*

Definition 2.2. *Let (M, d) be a metric space. For any $x \in M$ and $r > 0$, the **open ball of radius r around x** is the set*

$$B_r(x) := \{y \in M : d(y, x) < r\}. \tag{7}$$

Proposition 2.1. *Let (M, d) be a metric space.*

(*a*) *The collection*

$$\mathcal{T}_d := \{A \subseteq M : \forall x \in A \exists r > 0 \text{ such that } B_r(x) \subseteq A\} \tag{8}$$

*is a topology on M, called the **metric topology induced by the metric d**.*

(*b*) *For each $x \in M$ and $r > 0$ we have $B_r(x) \in \mathcal{T}_d$.*
(*c*) *$A \subseteq M$ is in $\mathcal{T}_d$ if and only if A can be written as a union of some open balls.*

Proof. First we prove (a). Obviously, $\varnothing, M \in \mathcal{T}_d$. Consider a family $(U_\alpha)_{\alpha \in A} \in \mathcal{T}_d$ and let $x \in \bigcup_{\alpha \in A} U_\alpha$. Thus $x \in U_\alpha$ for some $\alpha \in A$. Since $U_\alpha \in \mathcal{T}_d$, we find $r > 0$ such that $B_r(x) \subseteq U_\alpha$. Hence $B_r(x) \subseteq \bigcup_{\alpha \in A} U_\alpha$ and so $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_d$. Now assume $U_1, \ldots, U_n \in \mathcal{T}_d$ and let $x \in U_1 \cap \cdots \cap U_n$. Since each $U_i \in \mathcal{T}_d$, we find $r_i > 0$ such that $B_{r_i}(x) \subseteq U_i$ for each $i = 1, \ldots, n$. Setting $r := \min\{r_1, \ldots, r_n\}$ we have that $B_r(x) \subseteq U_1 \cap \cdots \cap U_n$ and thus $U_1 \cap \cdots \cap U_n \in \mathcal{T}_d$. To prove (b), let $y \in B_r(x)$. Then for $z \in B_{r - d(x,y)}(y)$ we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r - d(x, y) = r \tag{9}$$

and hence $z \in B_r(x)$. To prove (c), let $A \in \mathcal{T}_d$. Then for any $x \in A$ we find r_x such that $B_{r_x}(x) \subseteq A$. But then $\bigcup_{x \in A} B_{r_x}(x) = A$. Conversely, assume $A = \bigcup_{\alpha \in A} B_{r_\alpha}(x_\alpha)$. By (b) we have that $B_{r_\alpha}(x_\alpha) \in \mathcal{T}_d$ for each $\alpha \in A$. Thus by (a) we have that $\bigcup_{\alpha \in A} B_{r_\alpha}(x_\alpha) \in \mathcal{T}_d$. □

Proposition 2.2. *Let (M, d) and (M', d') be metric spaces and $f : M \to M'$. The mapping f is continuous if and only if the following condition holds: For any $x \in M$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$ for every $y \in M$.*

Proof. Assume f is continuous. Let $x \in M$ and $\varepsilon > 0$. Then $f^{-1}(B_\varepsilon(f(x)))$ is open in M since f is continuous and thus we find $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$. Conversely, let $U \subseteq M'$ be open. For any □

Proposition 2.3 (Sequence Criterion for Continuity). *Let (M, d) and (M', d') be metric spaces. A mapping $f : M \to M'$ is continuous if and only if the following criterion holds: If $(x_n)_{n \in \mathbb{N}}$ is a sequence in M which converges to some $x \in M$, then $\lim_{n \to \infty} f(x_n) = f(x)$.*

Definition 2.3. *Two metrics d and d' on a set M are said to be **equivalent** if $\mathcal{T}_d = \mathcal{T}_{d'}$.*

A useful criterion to determine wether two metrics d and d' are equivalent or not is stated in the following proposition.

Proposition 2.4. *Let d and d' be two metrics on a set M. Then d and d' are equivalent if and only if the following condition is satisfied: for every $x \in M$ and every $r > 0$ there exist $r_1, r_2 > 0$ such that $B_{r_1}^{(d')}(x) \subseteq B_r^{(d)}(x)$ and $B_{r_2}^{(d)}(x) \subseteq B_r^{(d')}(x)$.*

Proof. Assume $\mathcal{T}_d = \mathcal{T}_{d'}$. Then it is obvious that the condition is satisfied with $r_1 = r_2 = r$. Conversly, assume $U \in \mathcal{T}_d$. □

Definition 2.4. *Two metrics d and d' on a set M are said to be **strongly equivalent** if there are $c, c' > 0$ such that for all $x, y \in M$*

$$d(x,y) \leq c'd'(x,y) \qquad \text{and} \qquad d'(x,y) \leq cd(x,y). \tag{10}$$

Corollary 2.1. *Strongly equivalent metrics are equivalent.*

Proof. This follows immediately from proposition 2.4 by setting $r_1 := r/c'$ and $r_2 := r/c$. □

The following theorem is taken from [Eng89, p. 259].

Theorem 2.1. *Let $(M_i)_{i\in\mathbb{N}}$ be a sequence of metric spaces with respective metrics $(d_i)_{i\in\mathbb{N}}$ where $d_i \leq 1$ for all $i \in \mathbb{N}$. Then*

$$d(x,y) := \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x_i, y_i) \tag{11}$$

is a metric on $\prod_{i=1}^{\infty} X_i$. Moreover, the topology induced by the above metric coincides with the product topology on $\prod_{i=1}^{\infty} X_i$.

Proof. That d defines a metric is clear. The well-definedness follows from the fact that the series is majorized by the convergent geometric series $\sum_{i=1}^{\infty} \frac{1}{2^i}$. Let $\prod_{i=1}^{\infty} M_i$ be equipped with the metric topology. Let $\varepsilon > 0$ and $k \in \mathbb{N}$. Then if $d(x,y) < \varepsilon/2^k$ we have

$$d_k(x_k, y_k) \leq 2^k \sum_{i=1}^{\infty} \frac{1}{2^i} d_i(x_i, y_i) < \varepsilon. \tag{12}$$

Hence π_k is continuous and by the minimality of the product topology 1.5 we get that the product topology is contained in the metric topology induced by d. Conversly, let $U \subseteq \prod_{i=1}^{\infty} X_i$ open with respect to the topology induced by d. Therefore there exists $r > 0$ such that $B_r(x) \subseteq U$. Let $k \in \mathbb{N}$ so such that

$$\sum_{i=k+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^k} < \frac{1}{2} r \tag{13}$$

and for $i = 1, \dots, k$ let

$$U_i := B_{r/2}(x_i) = \{z \in X_i : d_i(x_i, z) < r/2\}. \tag{14}$$

Now for $y \in \bigcap_{i=1}^{k} \pi_i^{-1}(U_i)$ we have

$$d(x,y) = \sum_{i=1}^{k} \frac{1}{2^i} d_i(x_i, y_i) + \sum_{i=k+1}^{\infty} \frac{1}{2^i} d_i(x_i, y_i) < r \tag{15}$$

Therefore

$$\bigcap_{i=1}^{k} \pi_i^{-1}(U_i) \subseteq B_r(x) \subseteq U \tag{16}$$

and clearly $\bigcap_{i=1}^{k} \pi_i^{-1}(U_i)$ belongs to the basis of the product topology. Hence U is open in the product topology. □

It is obvious, that if we have a finite number of metric spaces $M_1, \ldots, M_n$, theorem 2.1 can be also applied to non-bounded metrics d_i. This yields a usefull corollary.

Corollary 2.2. *The product topology on $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$, where $(\mathbb{R}, \mathcal{T}_{|\cdot|})$, is the same as the one induced by the metric $|\cdot|$.*

Proof. □

Exercises

Exercise 2.1. Let (M, d) be a metric space. Show that M is a Hausdorff space.

Exercise 2.2. In this exercise we show that $(\mathbb{R}^n, \mathcal{T}_{|\cdot|})$ is second countable.

(a) For $a < b$ show that $(a, b) \subseteq R$ contains a rational number.
(b) Show that $\mathbb{Q}$ is dense in $\mathbb{R}$. *Hint:* Prove that for any real point there is a rational sequence converging to it and use corollary 1.3.
(c) Show that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. *Hint:* Use proposition 1.6 and 2.2.
(d) Show that the collection consisting of all open balls $B_p(q) \subseteq \mathbb{R}^n$ where $p \in \mathbb{Q}$ and $q \in \mathbb{Q}^n$ is a countable basis of $(\mathbb{R}^n, \mathcal{T}_{|\cdot|})$.

Exercise 2.3. Let (M, d) be a metric space , $(x_n)_{n\in\mathbb{N}}$ a sequence in M and $x \in M$. The sequence $(x_n)_{n\in\mathbb{N}}$ converges to x if and only if the following condition is satisfied: For any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ whenever $n > N$.

Exercise 2.4. Let (M, d) be a metric space and $(x_n)_{n\in\mathbb{N}}$. Then $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} d(x_n, x) = 0$, i.e. the sequence $(d(x_n, x))_{n\in\mathbb{N}}$ converges to zero in $(\mathbb{R}, |\cdot|)$.

Exercise 2.5. Prove proposition 2.3.

Exercise 2.6. Let M be a set. Show that equivalence of metrics on M is an equivalence relation on the set of all metrics on M.

Exercise 2.7. Let M be a set. Show that strong equivalence of metrics on M is an equivalence relation on the set of all metrics on M.

Exercise 2.8. Let X be a topological space. A subset of X is called an ***F_σ-set*** if it is a countable union of closed sets and a ***G_δ-set*** if it is a countable intersection of open sets (see [OB98, p. 61]). Assume we are given a metric d on X. For a nonempty subset A of X we define the real valued ***distance function ρ_A*** by $\rho_A(x) := \inf \{d(x, a) : a \in A\}$ for any $x \in X$.

(a) Show that ρ_A is continuous. *Hint:* Show that ρ_A is in fact Lipschitz continuous.
(b) Show that $\rho_A^{-1}(\{0\}) = \overline{A}$. *Hint:* Use corollary 1.3.
(c) Show that any closed subset of X is a G_δ-set.
(d) Show that any open subset of X is an F_σ-set.

3. Normed Spaces

Definition 3.1. *Let X be a real or complex vector space. A mapping*

$$\|\cdot\| : X \to \mathbb{R} \tag{17}$$

*is called **norm**, if*

(*i*) $\|\lambda x\| = |\lambda| \, \|x\|$ *for all* $\lambda \in \mathbb{C}$ *and* $x \in X$.
(*ii*) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$.
(*iii*) $\|x\| = 0$ *implies* $x = 0$.

If only (*i*) *and* (*ii*) *hold,* $\|\cdot\|$ *is called a **seminorm**. If a norm has been specified on a vector space* X*, the tuple* $(X, \|\cdot\|)$ *is called a **normed space**.*

Definition 3.2. *Two norms* $\|\cdot\|$ *and* $|||x|||$ *are said to be **equivalent** if the induced metrics are strongly equivalent.*

There is a usefull result in the finite dimenisonal case (see [Wer11, p. 26]).

Theorem 3.1. *Let X be a finite dimensional real or complex vector space. Then any two norms are equivalent.*

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis of X, $x := \sum_{k=1}^{n} x_k e_k \in X$ and $\|\cdot\|$ be a norm on X. We show that $\|\cdot\|$ is equivalent to the euclidean norm. Hölder's inequality for series (see [Els11, p. 224]) yields

$$\|x\| \leq \left(\sum_{k=1}^{n} |x_k|^2 \right)^{1/2} \left(\sum_{k=1}^{n} \|e_k\|^2 \right)^{1/2} \leq \sqrt{n} \max \{\|e_1\|, \ldots, \|e_n\|\} \|x\|_2 \tag{18}$$

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $(X, \|\cdot\|_2)$ which converges to $x \in X$. By estimate (18) and the reverse triangle inequality (see exercise 3.3) we get

$$0 \leq |\|x_n\| - \|x_n\|| \leq \|x_n - x\| \leq \sqrt{n} \max \{\|e_1\|, \ldots, \|e_n\|\} \|x_n - x\|_2 \tag{19}$$

which implies by double application of exercise 2.4

$$\lim_{n \to \infty} \|x_n\| = \|x\| . \tag{20}$$

Hence $\|\cdot\|$ is a continuous function on $(X, \|\cdot\|_2)$. Furthermore, $\mathbb{S}^{n-1} = \|\cdot\|_2^{-1}(\{1\})$ and $\mathbb{S}^{n-1}$ is clearly bounded since $\mathbb{S}^{n-1} \subseteq \overline{B}_1(0)$. Therefore $\mathbb{S}^{n-1}$ is compact by Heine-Borel and by the extreme value theorem 1.2 we get that $\|\cdot\|$ attains its minimum value $\|x_0\|$ on $\mathbb{S}^{n-1}$. Since $\|\cdot\|$ is a norm, we must have $\|x_0\| > 0$ and since $x / \|x\|_2 \in \mathbb{S}^{n-1}$ for $x \neq 0$ we get

$$\|x_0\| \|x\|_2 \leq \|x\| . \tag{21}$$

The case $x = 0$ holds trivially. □

Theorem 3.1 yields a particularly important result together with the following proposition.

Exercises

Exercise 3.1. Show that if $\|\cdot\| : X \to \mathbb{R}$ is a norm, then $\|x\| \geq 0$ for any $x \in X$.

Exercise 3.2. Show that any norm $\|\cdot\|$ induces a metric $d_{\|\cdot\|}$ on X by setting $d_{\|\cdot\|}(x, y) := \|x - y\|$ for $x, y \in X$.

Exercise 3.3. Show that if $(X, \|\cdot\|)$ is a normed space, then the ***reverse triangle inequality*** $|\|x\| - \|y\|| \leq \|x - y\|$ holds for any $x, y \in X$. Deduce that $\|\cdot\|$ is a Lipschitz continuous function on $(X, \|\cdot\|)$.

CHAPTER 2

Measure Spaces

1. Definitions and Basic Notions

Proposition 1.1. *Let X and Y be topological spaces. Then $\mathcal{B}(X) \otimes \mathcal{B}(Y) \subseteq \mathcal{B}(X \times Y)$. If additionally X and Y are both second countable, then $\mathcal{B}(X \times Y) = \mathcal{B}(X) \otimes \mathcal{B}(Y)$.*

Proposition 1.2. *Let $(X, \mathcal{A}, \mu)$ be a measure space, $(X, \mathcal{A}_\mu, \overline{\mu})$ be its completion and $f : X \to [-\infty, \infty]$ $\mathcal{A}_\mu$-measurable. Then there exist $\mathcal{A}$-measurable functions $g, h : X \to [-\infty, \infty]$ with $g \leq f \leq h$ and $g = h$ μ-a.e.*

Proof. First assume $f \in \Sigma^+$. Then

$$f = \sum_{i=1}^{n} a_i \chi_{A_i} \tag{22}$$

where $a_i \leq 0$ and $A_i \in \mathcal{A}_\mu$ for $i = 1, \ldots, n$. Since $\mathcal{A}_\mu$ is the completion of $\mathcal{A}$, there exist $E_i, F_i \in \mathcal{A}$ such that

$$E_i \subseteq A_i \subseteq F_i \qquad \text{and} \qquad \mu(F_i \setminus E_i) \tag{23}$$

for $i = 1, \ldots, n$. Then

$$g := \sum_{i=1}^{n} a_i \chi_{E_i} \qquad \text{and} \qquad h := \sum_{i=1}^{n} a_i \chi_{F_i} \tag{24}$$

have the desired properties.Now let f be a $\mathcal{A}_\mu$-measurable nonnegative function. Then we find a sequence $(\varphi_n)_{n \in \mathbb{N}}$ of $\mathcal{A}_\mu$-measurable simple functions $0 \leq \varphi_n$ such that $\varphi_n \nearrow f$. By the first part we find sequences $(g_n)_{n \in \mathbb{N}}$ and $(h_n)_{n \in \mathbb{N}}$ of $\mathcal{A}$-measurable functions such that

$$g_n \leq \varphi_n \leq h_n \qquad \text{and} \qquad g_n = h_n \ \mu\text{-a.e.} \tag{25}$$

for any $n \in \mathbb{N}$. □

Exercises

Exercise 1.1. Let X be a set. Determine $\sigma(\{\{x\} : x \in X\})$.

Exercise 1.2. Let X, Y be sets, $A \subseteq \mathcal{P}(Y)$ and $f : X \to Y$ be surjective. Then

$$m\left(\{f^{-1}(S) : S \in A\}\right) = \{f^{-1}(S) : S \in m(A)\}. \tag{26}$$

Hint: For the inclusion $\supseteq$ consider the set

$$\mathcal{M} := \{B \subseteq Y : B \in m(A) \text{ and } f^{-1}(B) \in m\left(\{f^{-1}(S) : S \in A\}\right)\}. \tag{27}$$

Exercise 1.3. Show that there is no countable σ-algebra.

Exercise 1.4.

2. Interpolation of L^p Spaces

2.1. The Lemma of I. I. Hirschman and Hadamard's Three Lines Lemma.

Lemma 2.1 (I. I. Hirschman). *Let F be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \mathrm{Re} z < 1\}$ and continuous on $\overline{S}$, such that for some $0 < A < \infty$ and $0 \le \tau_0 < \pi$ we have $\log|F(z)| \le Ae^{\tau_0 |\mathrm{Im}\, z|}$ for every $z \in \overline{S}$. Then*

$$|F(z)| \le \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log|F(it+iy)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log|F(1+it+iy)|}{\cosh(\pi t)+\cos(\pi x)}\right]\mathrm{d}t\right)$$

whenever $z := x + iy \in S$.

Proof. We will first prove the case $\underline{y = 0}$. Assume F to be not identically zero (the case where F is identically zero is trivial). Let h be as in lemma (**??**) and let $\zeta := \rho e^{i\theta}$, $0 \le \rho < 1$. Since $\zeta \in D$, we have $0 < \mathrm{Re}\, h(\zeta) < 1$ and thus the hypothesis on F and lemma (**??**) yields

$$\log|F(h(\zeta))| \le Ae^{\frac{\tau_0}{\pi}|\log|(1+\zeta)/(1-\zeta)||} \le Ae^{\tau_0/\pi}\frac{1}{|\cos(\theta/2)|^{\tau_0/\pi}}\frac{1}{|\sin(\theta/2)|^{\tau_0/\pi}} \tag{28}$$

for $1/(2e-1) \le \rho$. Since $0 < \tau_0 < \pi$, inequality (27) asserts, that $\log|F(h(\zeta))|$ is bounded from above by an integrable function of θ, independently of $\rho \ge 1/(2e-1)$. Furthermore we have

$$M := \sup\left\{\log|F(h(\zeta))| : \zeta \in \overline{B}_{1/(2e-1)}\right\} < \infty \tag{29}$$

since a upper semicontinuous function on a compact space attains its supremum (see lemma **??**). Hence

$$\log|F(h(\rho e^{i\theta}))| \le \max\left\{M, Ae^{\tau_0/\pi}\frac{1}{|\cos(\theta/2)|^{\tau_0/\pi}}\frac{1}{|\sin(\theta/2)|^{\tau_0/\pi}}\right\} =: g(\theta) \tag{30}$$

for any $0 \le \rho < 1$ where $g \in L^1[-\pi, \pi]$. Let $0 \le \rho < R < 1$ and $a_1, \ldots, a_n$ denote the zeros of $F(h(\zeta))$ for $|\zeta| < R$ (since $F \circ h$ is holomorphic for $|\zeta| < 1$ there are indeed only finitely many ones) multiple zeros being repeated. Then for $F(h(\zeta)) \neq 0$ we have by the *Poisson-Jensen formula* (see [Ahl79, p. 208])

$$\log|F(h(\zeta))| = -\sum_{k=1}^{n}\log\left|\frac{R^2 - \overline{a}_k\zeta}{R(\zeta - a_k)}\right| + \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{Re}\left[\frac{Re^{it}+\zeta}{Re^{it}-\zeta}\right]\log|F(h(Re^{it}))|\,\mathrm{d}t \tag{31}$$

Therefore by

$$\begin{aligned}\mathrm{Re}\left[\frac{Re^{it}+\zeta}{Re^{it}-\zeta}\right] &= \mathrm{Re}\left[\frac{R^2 - 2i\,\mathrm{Im}\,[\zeta Re^{-it}] - |\zeta|^2}{R^2 - 2\,\mathrm{Re}\,[\zeta Re^{-it}] + |\zeta|^2}\right]\\ &= \mathrm{Re}\left[\frac{R^2 - 2iR\rho\sin(\theta - t) - \rho^2}{R^2 - 2R\rho\cos(\theta - t) + \rho^2}\right]\\ &= \frac{R^2 - \rho^2}{R^2 - 2R\rho\cos(\theta - t) + \rho^2}\end{aligned}$$

and since $(R^2 - |a_k|^2)(R^2 - \rho^2) \geq 0$ for all $k = 1, \ldots, n$ implies $|R^2 - \overline{a}_k \zeta| \geq |R(\zeta - a_k)|$ the estimate

$$\log|F(h(\zeta))| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - \rho^2}{R^2 - 2R\rho\cos(\theta - t) + \rho^2} \log|F(h(Re^{it}))| \,\mathrm{d}t \quad (32)$$

is valid for every $|\zeta| < R$. By [Rud87, p. 236] we have

$$\frac{R-\rho}{R+\rho} \leq \frac{R^2 - \rho^2}{R^2 - 2R\rho\cos(\theta - \varphi) + \rho^2} \leq \frac{R+\rho}{R-\rho} \quad (33)$$

for $0 \leq \rho < R$. Combining (29) and (32) yields

$$\frac{R^2 - \rho^2}{R^2 - 2R\rho\cos(\theta - t) + \rho^2} \log|F(h(\zeta))| \leq \frac{R+\rho}{R-\rho} g(\theta) =: G(\theta) \quad (34)$$

where $G \in L^1[-\pi, \pi]$. For $0 < R < 1$ let

$$f_R(\varphi) := \frac{R^2 - \rho^2}{R^2 - 2R\rho\cos(\theta - \varphi) + \rho^2} \log|F(h(Re^{i\varphi}))|$$

and for $\varphi \notin \{0, \pi\}$

$$f(\varphi) := \frac{1 - \rho^2}{1 - 2\rho\cos(\theta - \varphi) + \rho^2} \log|F(h(e^{i\varphi}))|$$

Since $\log|F(h(\zeta))|$ is upper semicontinuous on $\overline{D} \setminus \{\pm 1\}$ by lemma **??** we get

$$\begin{aligned}
\limsup_{R \nearrow 1} f_R(\varphi) &= \limsup_{R \nearrow 1} \left[\frac{R^2 - \rho^2}{R^2 - 2R\rho\cos(\theta - \varphi) + \rho^2} \log|F(h(Re^{i\varphi}))| \right] \\
&= \lim_{R \nearrow 1} \frac{R^2 - \rho^2}{R^2 - 2R\rho\cos(\theta - \varphi) + \rho^2} \limsup_{R \nearrow 1} \log|F(h(Re^{i\varphi}))| \\
&= \frac{1 - \rho^2}{1 - 2\rho\cos(\theta - \varphi) + \rho^2} \log|F(h(e^{i\varphi}))| = f(\varphi)
\end{aligned}$$

using [Bou95, p. 363] and proposition **??**. The functions $G - f_R$ being non-negative, an application of Fatou's lemma yields

$$\int_{-\pi}^{\pi} \liminf_{R \nearrow 1} [G(\varphi) - f_R(\varphi)] \,\mathrm{d}\varphi \leq \liminf_{R \nearrow 1} \int_{-\pi}^{\pi} [G(\varphi) - f_R(\varphi)] \,\mathrm{d}\varphi$$

By [Bou95, p. 354], we get

$$\limsup_{R \nearrow 1} \int_{-\pi}^{\pi} [f_R(\varphi) - G(\varphi)] \,\mathrm{d}\varphi \leq \int_{-\pi}^{\pi} \limsup_{R \nearrow 1} [f_R(\varphi) - G(\varphi)] \,\mathrm{d}\varphi$$

and thus

$$\begin{aligned}\limsup_{R\nearrow 1}\int_{-\pi}^{\pi} f_R(\varphi)\,\mathrm{d}\varphi - \int_{-\pi}^{\pi} G(\varphi)\,\mathrm{d}\varphi &= \limsup_{R\nearrow 1}\int_{-\pi}^{\pi} f_R(\varphi)\,\mathrm{d}\varphi - \lim_{R\nearrow 1}\int_{-\pi}^{\pi} G(\varphi)\,\mathrm{d}\varphi \\ &= \limsup_{R\nearrow 1}\int_{-\pi}^{\pi} [f_R(\varphi) - G(\varphi)]\,\mathrm{d}\varphi \\ &\leq \int_{-\pi}^{\pi} \limsup_{R\nearrow 1} [f_R(\varphi) - G(\varphi)]\,\mathrm{d}\varphi \\ &\leq \int_{-\pi}^{\pi} \limsup_{R\nearrow 1} f_R(\varphi)\,\mathrm{d}\varphi - \int_{-\pi}^{\pi} \lim_{R\nearrow 1} G(\varphi)\,\mathrm{d}\varphi \\ &= \int_{-\pi}^{\pi} \limsup_{R\nearrow 1} f_R(\varphi)\,\mathrm{d}\varphi - \int_{-\pi}^{\pi} G(\varphi)\,\mathrm{d}\varphi\end{aligned}$$

by [Bou95, p. 358]. Hence

$$\limsup_{R\nearrow 1}\int_{-\pi}^{\pi} f_R(\varphi)\,\mathrm{d}\varphi \leq \int_{-\pi}^{\pi} \limsup_{R\nearrow 1} f_R(\varphi)\,\mathrm{d}\varphi$$

and so

$$\log|F(h(\zeta))| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-\rho^2}{1-2\rho\cos(\theta-\varphi)+\rho^2}\log|F(h(e^{i\varphi}))|\,\mathrm{d}\varphi \tag{35}$$

The lemma now follows from (34) by a change of variables. By stipulating $x := h(\zeta)$ we obtain

$$\begin{aligned}\zeta = h^{-1}(x) = \frac{e^{\pi i x} - i}{e^{\pi i x} + i} &= \frac{\cos(\pi x) + i\sin(\pi x) - i}{\cos(\pi x) + i\sin(\pi x) + i} \\ &= \frac{\cos(\pi x) + i\sin(\pi x) - i}{\cos(\pi x) + i\sin(\pi x) + i}\,\frac{\cos(\pi x) - i\sin(\pi x) - i}{\cos(\pi x) - i\sin(\pi x) - i} \\ &= -i\frac{\cos(\pi x)}{1+\sin(\pi x)} = \left(\frac{\cos(\pi x)}{1+\sin(\pi x)}\right) e^{-i\pi/2}\end{aligned} \tag{36}$$

by

$$\begin{aligned}&(\cos(\pi x) + i\sin(\pi x) - i)\,(\cos(\pi x) - i\sin(\pi x) - i) \\ &\quad = \cos^2(\pi x) - i\sin(\pi x)\cos(\pi x) - i\cos(\pi x) + i\sin(\pi x)\cos(\pi x) \\ &\qquad + \sin^2(\pi x) + \sin(\pi x) - i\cos(\pi x) - \sin(\pi x) - 1 = -2i\cos(\pi x)\end{aligned}$$

and

$$\begin{aligned}&(\cos(\pi x) + i\sin(\pi x) + i)\,(\cos(\pi x) - i\sin(\pi x) - i) \\ &\quad = \cos^2(\pi x) - i\sin(\pi x)\cos(\pi x) - i\cos(\pi x) + i\sin(\pi x)\cos(\pi x) \\ &\qquad + \sin^2(\pi x) + \sin(\pi x) + i\cos(\pi x) + \sin(\pi x) + 1 = 2 + 2\sin(\pi x)\end{aligned}$$

From equality (35) we deduce $\rho = \frac{\cos(\pi x)}{1+\sin(\pi x)}$, $\theta = -\frac{\pi}{2}$ if $0 < x \leq \frac{1}{2}$ and $\rho = -\frac{\cos(\pi x)}{1+\sin(\pi x)}$, $\theta = \frac{\pi}{2}$ if $\frac{1}{2} \leq x < 1$. Let $0 < x \leq \frac{1}{2}$. Then we have

$$\begin{aligned}
&\frac{1-\rho^2}{1-2\rho\cos(\theta-\varphi)+\rho^2} \\
&= \frac{1+2\sin(\pi x)+\sin^2(\pi x)-\cos^2(\pi x)}{1+2\sin(\pi x)+\sin^2(\pi x)+2\cos(\pi x)\sin(\varphi)(1+\sin(\pi x))+\cos^2(\pi x)} \\
&\quad= \frac{\sin(\pi x)+\sin^2(\pi x)}{1+\sin(\pi x)+\cos(\pi x)\sin(\varphi)(1+\sin(\pi x))} = \frac{\sin(\pi x)}{1+\cos(\pi x)\sin(\varphi)}
\end{aligned}$$

and also for $\frac{1}{2} \leq x < 1$. Let Φ and Ψ be defined as in lemma (**??**). We have

$$\begin{aligned}
e^{i\Phi(t)} &= h^{-1}(it) = \frac{e^{-\pi t}-i}{e^{-\pi t}+i}\frac{e^{-\pi t}-i}{e^{-\pi t}-i} = \frac{e^{-2\pi t}-2ie^{-\pi t}-1}{e^{-2\pi t}+1} = \frac{e^{-2\pi t}-1}{e^{-2\pi t}+1} - \frac{2ie^{-\pi t}}{e^{-2\pi t}+1} \\
&= \frac{e^{-2\pi t}-1}{e^{-2\pi t}+1} - \frac{2i}{e^{-\pi t}+e^{\pi t}} = \frac{1-e^{2\pi t}}{1+e^{2\pi t}} - \frac{2i}{e^{-\pi t}+e^{\pi t}} = -\tanh(\pi t) - i\operatorname{sech}(\pi t)
\end{aligned}$$

and thus

$$\begin{aligned}
\sin(\Phi(t))\cosh(\pi t) &= \sin(-i\log(-\tanh(\pi t)-i\operatorname{sech}(\pi t)))\cosh(\pi t) \\
&= \frac{1}{2i}\left[-\tanh(\pi t)-i\operatorname{sech}(\pi t)+\frac{1}{\tanh(\pi t)+i\operatorname{sech}(\pi t)}\right]\cosh(\pi t) \\
&= \frac{1}{2i}\left[\frac{\cosh(\pi t)-\tanh(\pi t)\sinh(\pi t)-2i\tanh(\pi t)+\operatorname{sech}(\pi t)}{\tanh(\pi t)+i\operatorname{sech}(\pi t)}\right] \\
&= \frac{1}{2i}\left[\frac{\cosh^2(\pi t)-\sinh^2(\pi t)-2i\sinh(\pi t)+1}{\sinh(\pi t)+i}\right] \\
&= \frac{1-i\sinh(\pi t)}{i\sinh(\pi t)-1} = -1
\end{aligned}$$

Therefore the transformation formula yields

$$\frac{1}{2\pi}\int_{-\pi}^{0}\frac{\sin(\pi x)}{1+\cos(\pi x)\sin(\varphi)}\log|F(h(e^{i\varphi}))|\,\mathrm{d}\varphi = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin(\pi x)}{\cosh(\pi t)-\cos(\pi x)}\log|F(it)|\,\mathrm{d}t$$

and in a similar manner

$$\frac{1}{2\pi}\int_{0}^{\pi}\frac{\sin(\pi x)}{1+\cos(\pi x)\sin(\varphi)}\log|F(h(e^{i\varphi}))|\,\mathrm{d}\varphi = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin(\pi x)}{\cosh(\pi t)+\cos(\pi x)}\log|F(1+it)|\,\mathrm{d}t$$

holds since

$$\begin{aligned}
\sin(\Psi(t))\cosh(\pi t) &= \sin(-i\log(-\tanh(\pi t)+i\operatorname{sech}(\pi t)))\cosh(\pi t) \\
&= \frac{1}{2i}\left[-\tanh(\pi t)+i\operatorname{sech}(\pi t)-\frac{1}{-\tanh(\pi t)+i\operatorname{sech}(\pi t)}\right]\cosh(\pi t) \\
&= \frac{1}{2i}\left[\frac{-\cosh(\pi t)+\tanh(\pi t)\sinh(\pi t)-2i\tanh(\pi t)-\operatorname{sech}(\pi t)}{-\tanh(\pi t)+i\operatorname{sech}(\pi t)}\right] \\
&= \frac{1}{2i}\left[\frac{-\cosh^2(\pi t)+\sinh^2(\pi t)-2i\sinh(\pi t)-1}{i-\sinh(\pi t)}\right] \\
&= \frac{1+i\sinh(\pi t)}{1+i\sinh(\pi t)} = 1
\end{aligned}$$

Thus the case $y = 0$ is prooven.
The case $\underline{y \neq 0.}$ follows easily from the previous one. Fix $y \neq 0$ and define $G(z) := F(z + iy)$ for $z \in \overline{S}$. Then G is a holomorphic function in S and continuous on $\overline{S}$ as a composition of continuous and holomorphic functions. Moreover, the hypothesis on F yields

$$\log|G(z)| = \log|F(z + iy)| \leq Ae^{\tau_0|\operatorname{Im} z+y|} \leq Ae^{\tau_0|\operatorname{Im} z|}e^{\tau_0|y|} \tag{37}$$

for all $z \in \overline{S}$. The previous case yields for G with A replaced by $Ae^{\tau_0|y|}$

$$|G(x)| \leq \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log|G(it)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log|G(1 + it)|}{\cosh(\pi t) + \cos(\pi x)}\right] \mathrm{d}t\right) \tag{38}$$

Now, observing $G(x) = F(x + iy)$, $G(it) = F(it + iy)$ and $G(1 + it) = F(1 + it + iy)$ yields the desired result. □

An almost immediate consequence of the Lemma of I. I. Hirschman 2.1 is the so called Hadamard's Three Lines Lemma.

Lemma 2.2 (Hadamard's Three Lines Lemma). *Let F be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, continuous and bounded on $\overline{S}$, such that $|F(z)| \leq B_0$ when $\operatorname{Re} z = 0$ and $|F(z)| \leq B_1$ when $\operatorname{Re} z = 1$, for some $0 < B_0, B_1 < \infty$. Then $|F(z)| \leq B_0^{1-x}B_1^x$ when $\operatorname{Re} z = x$, for any $0 < x < 1$.*

Proof. Let $0 < x < 1$. Then

$$\begin{aligned}
\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\frac{1}{\cosh(\pi t) + \cos(\pi x)}\,\mathrm{d}t &= \frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\frac{1}{\frac{1}{2}(e^{\pi t} + e^{-\pi t}) + \cos(\pi x)}\,\mathrm{d}t \\
&= \frac{\sin(\pi x)}{\pi}\int_0^{\infty}\frac{1}{s^2 + 2\cos(\pi x)s + 1}\,\mathrm{d}s \\
&= \frac{\sin(\pi x)}{\pi}\int_0^{\infty}\frac{1}{(s + \cos(\pi x))^2 + \sin^2(\pi x)}\,\mathrm{d}s \\
&= \frac{1}{\pi\sin(\pi x)}\int_0^{\infty}\frac{1}{\left(\frac{s+\cos(\pi x)}{\sin(\pi x)}\right)^2 + 1}\,\mathrm{d}s \\
&= \frac{1}{\pi}\int_{\cot(\pi x)}^{\infty}\frac{1}{u^2 + 1}\,\mathrm{d}u \\
&= \frac{1}{\pi}\left[\frac{\pi}{2} - \arctan(\cot(\pi x))\right] \\
&= x
\end{aligned}$$

and in the same manner

$$\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\frac{1}{\cosh(\pi t) - \cos(\pi x)}\,\mathrm{d}t = 1 - x$$

Assume that F is holomorphic in S, continuous and bounded on $\overline{S}$ with $|F(z)| \leq B_0$ when $\operatorname{Re} z = 0$ and $|F(z)| \leq B_1$ when $\operatorname{Re} z = 1$ for some $0 < B_0, B_1 < \infty$. If $|F(z)| \leq M$ for $0 < M < \infty$, F satisfies the hypothesis with $A := \log(M)$ and $\tau_0 = 0$. Therefore

$$\begin{aligned}
|F(z)| &\leq \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log|F(it+iy)|}{\cosh(\pi t)-\cos(\pi x)}+\frac{\log|F(1+it+iy)|}{\cosh(\pi t)+\cos(\pi x)}\right]\mathrm{d}t\right)\\
&\leq \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log B_0}{\cosh(\pi t)-\cos(\pi x)}+\frac{\log B_1}{\cosh(\pi t)+\cos(\pi x)}\right]\mathrm{d}t\right)\\
&= \exp(x\log B_0+(1-x)\log B_1)\\
&= B_0^x B_1^{1-x}
\end{aligned}$$

whenever $z := x+iy \in S$. □

2.2. The Stein-Weiss Interpolation Theorem for Analytic Families of Operators and the Riesz-Thorin Interpolation Theorem.

Definition 2.1 (Analytic family, admissible growth). *Let (X,μ), (Y,ν) be two σ-finite measure spaces and $(T_z)_{z\in\overline{S}}$, where T_z is defined Σ_X and taking values in the space of all measurable functions on Y such that*

$$\int_Y |T_z(\chi_A)\chi_B|\,\mathrm{d}\nu \tag{39}$$

whenever $\mu(A),\nu(B)<\infty$. The family $(T_z)_{z\in\overline{S}}$ is said to be analytic *if for all $f\in\Sigma_X$, $g\in\Sigma_Y$ we have that*

$$z\mapsto \int_Y T_z(f)g\,\mathrm{d}\nu \tag{40}$$

is analytic on S and continuous on $\overline{S}$. Further, an analytic family $(T_z)_{z\in\overline{S}}$ is called of admissible growth, *if there is a constant $\tau_0\in[0,\pi)$, such that for all $f\in\Sigma_X$, $g\in\Sigma_Y$ a constant $C(f,g)$ exists with*

$$\log\left|\int_Y T_z(f)g\,\mathrm{d}\nu\right| \leq C(f,g)e^{\tau_0|\operatorname{Im} z|} \tag{41}$$

for all $z\in\overline{S}$.

Theorem 2.1 (Stein-Weiss Interpolation Theorem of Analytic Families of Operators). *Let $(T_z)_{z\in\overline{S}}$ be an analytic family of admissible growth, $1\leq p_0,p_1,q_0,q_1\leq\infty$ and suppose that M_0, M_1 are positive functions on the real line such that for some $\tau_1\in[0,\pi)$*

$$\sup_{-\infty<y<\infty} e^{-\tau_1|y|}\log M_0(y)<\infty \qquad \textit{and} \qquad \sup_{-\infty<y<\infty} e^{-\tau_1|y|}\log M_1(y)<\infty. \tag{42}$$

Fix $0<\theta<1$ and define

$$\frac{1}{p} := \frac{1-\theta}{p_0}+\frac{\theta}{p_1} \qquad \textit{and} \qquad \frac{1}{q} := \frac{1-\theta}{q_0}+\frac{\theta}{q_1}. \tag{43}$$

Further suppose that for all $f\in\Sigma_X$ and $y\in\mathbb{R}$ we have

$$\|T_{iy}(f)\|_{L^{q_0}} \leq M_0(y)\|f\|_{L^{p_0}} \qquad \textit{and} \qquad \|T_{1+iy}(f)\|_{L^{q_1}} \leq M_1(y)\|f\|_{L^{p_1}}. \tag{44}$$

Then for all $f\in\Sigma_X$ we have

$$\|T_\theta(f)\|_{L^q} \leq M(\theta)\,\|f\|_{L^p}$$

where for $0 < x < 1$

$$M(x) = \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log M_0(t)}{\cosh(\pi t)-\cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t)+\cos(\pi x)}\right]\mathrm{d}t\right).$$

Proof. Fix $0 < \theta < 1$ and $f \in \Sigma_X$, $g \in \Sigma_Y$ with $\|f\|_{L^p} = \|g\|_{L^{q'}} = 1$. Define f_z, g_z as in (**??**) and for $z \in \overline{S}$

$$F(z) := \int_Y T_z(f_z)g_z\,\mathrm{d}\nu$$

Since the family $(T_z)_{z\in\overline{S}}$ is of admissible growth we have that there exist constants $c(\chi_{A_j},\chi_{B_k})$ for any $j = 1,\dots,n$ and $k = 1,\dots,m$ such that

$$\log\left|\int_{B_k} T_z\left(\chi_{A_j}\right)\mathrm{d}\nu\right| \leq c\left(\chi_{A_j},\chi_{B_k}\right)e^{\tau_0|\operatorname{Im} z|}$$

For shortness we will denote these constants simply by $c(A_j, B_k)$ and get

$$\begin{aligned}
\log|F(z)| &= \log\left|\sum_{j=1}^{n}\sum_{k=1}^{m} a_j^{P(z)} b_j^{Q(z)} e^{i\alpha_j}e^{i\beta_k}\int_Y T_z(\chi_{A_j})(y)\chi_{B_k}(y)\,\mathrm{d}\nu(y)\right| \\
&\leq \log\left[\sum_{j=1}^{n}\sum_{k=1}^{m}\max\{1, a_j^{p/p_0+p/p_1}\}\max\{1, b_k^{q'/q_0'+q'/q_1'}\}\left|\int_{B_k} T_z(\chi_{A_j})\,\mathrm{d}\nu\right|\right] \\
&\leq \log\left[\sum_{j=1}^{n}\sum_{k=1}^{m}(1+a_j)^{p/p_0+p/p_1}\,(1+b_k)^{q'/q_0'+q'/q_1'}\,e^{c(A_j,B_k)e^{\tau_0|\operatorname{Im} z|}}\right] \\
&\leq \log\left[\sum_{j=1}^{n}\sum_{k=1}^{m} e^{\log((1+a_j)^{p/p_0+p/p_1}(1+b_k)^{q'/q_0'+q'/q_1'})+c(A_j,B_k)e^{\tau_0|\operatorname{Im} z|}}\right] \\
&\leq \log\left(mne^{\sum_{j=1}^{n}\sum_{k=1}^{m}\log((1+a_j)^{p/p_0+p/p_1}(1+b_k)^{q'/q_0'+q'/q_1'})+c(A_j,B_k)e^{\tau_0|\operatorname{Im} z|}}\right) \\
&= \log(mn) + \sum_{j=1}^{n}\sum_{k=1}^{m}\log\left((1+a_j)^{p/p_0+p/p_1}\,(1+b_k)^{q'/q_0'+q'/q_1'}\right) + c(A_j,B_k)e^{\tau_0|\operatorname{Im} z|}
\end{aligned}$$

since $\tau_0 \in [0,\pi)$ and thus $e^{\tau_0|\operatorname{Im} z|} \geq 1$, F satisfies the hypotheses of the extension of Hadamard's three lines lemma **??** with

$$A = \log(mn) + \sum_{j=1}^{n}\sum_{k=1}^{m}\Big(\frac{p}{p_0}+\frac{p}{p_1}\Big)\log(1+a_j) + \Big(\frac{q'}{q_0'}+\frac{q'}{q_1'}\Big)\log(1+b_k) + c(A_j,B_k)$$

The same calculations as in the proof of the Riesz-Thorin interpolation theorem **??** yield for $y \in \mathbb{R}$

$$\|f_{iy}\|_{L^{p_0}} = \|f\|_{L^p}^{p/p_0} = 1 = \|g\|_{L^{q'}}^{q'/q_0'} = \|g_{iy}\|_{L^{q_0'}}$$

and

$$\|f_{1+iy}\|_{L^{p_1}} = \|f\|_{L^p}^{p/p_1} = 1 = \|g\|_{L^{q'}}^{q'/q_1'} = \|g_{1+iy}\|_{L^{q_1'}}$$

Further

$$\|F(iy)\| \leq \|T_{iy}(f_{iy})\|_{L^{q_0}} \|g_{iy}\|_{L^{q_0'}} \leq M_0(y) \|f_{iy}\|_{L^{p_0}} \|g_{iy}\|_{L^{q_0'}} = M_0(y)$$

and

$$\|F(1+iy)\| \leq \|T_{1+iy}(f_{1+iy})\|_{L^{q_1}} \|g_{1+iy}\|_{L^{q_1'}} \leq M_1(y) \|f_{1+iy}\|_{L^{p_1}} \|g_{1+iy}\|_{L^{q_1'}} = M_1(y)$$

by Hölder's inequality and the hypotheses on the analytic family $(T_z)_{z\in\overline{S}}$. Therefore the extension of Hadamard's three lines lemma **??** yields

$$|F(x)| \leq \exp\left(\frac{\sin(\pi x)}{2}\int_{-\infty}^{\infty}\left[\frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi x)}\right]\mathrm{d}t\right) = M(x)$$

for every $0 < x < 1$. Furthermore observe that

$$F(\theta) = \int_Y T_\theta(f) g \,\mathrm{d}\nu$$

and thus by [Fol99, p. 189]

$$\begin{aligned} M_q(T_\theta(f)) &= \sup\left\{\left|\int_Y T_\theta(f) g \,\mathrm{d}\nu\right| : g \in \Sigma_Y, \|g\|_{L^{q'}}\right\} \\ &= \sup\{|F(\theta)| : g \in \Sigma_Y, \|g\|_{L^{q'}}\} \\ &\leq M(\theta) \end{aligned}$$

Since $M(\theta)$ is an absolutely convergent integral (this is immediate by the growth conditions (41)) for any $0 < \theta < 1$, $M_q(T_\theta(f)) < \infty$ and thus $M_q(T_\theta(f)) = \|T_\theta(f)\|_{L^q}$. The general statement follows by replacing f with $f/\|f\|_{L^p}$ when $\|f\|_{L^p} \neq 0$. The theorem is trivially true when $\|f\|_{L^p} = 0$. $\square$

Theorem 2.2 (Riesz-Thorin interpolation theorem). *Suppose that (X, μ), (Y, ν) are measure spaces and $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. If $q_0 = q_1 = \infty$, suppose also that ν is semifinite. Let T be a linear operator defined on Σ_X and taking values in the set of measurable functions on Y, such that for some $0 < M_0, M_1 < \infty$ the estimates*

$$\|T(f)\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}} \quad \text{and} \quad \|T(f)\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}} \tag{45}$$

hold for all $f \in \Sigma_X$. Then for all $0 \leq \theta \leq 1$ we have

$$\|T(f)\|_{L^q} \leq M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p} \tag{46}$$

for all $f \in \Sigma_X$, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

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