

YANNIS BÄHNI

INTRODUCTION TO FUNCTIONAL ANALYSIS

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CHAPTER 1

Topological Spaces

1. Definitions and Basic Notions

1.1. Topologies.

Definition 1.1. Let X be a set. A **topology** on X is a collection \mathcal{T} of subsets of X satisfying the following properties:

- (i) $\emptyset, X \in \mathcal{T}$.
- (ii) If $(U_\alpha)_{\alpha \in A}$ is a family of elements of \mathcal{T} , then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.
- (iii) If $U_1, \dots, U_n \in \mathcal{T}$, then $U_1 \cap \dots \cap U_n \in \mathcal{T}$.

A set X for which a topology \mathcal{T} has been specified is called a **topological space** and elements of \mathcal{T} are called **open sets**.

Example 1.1 (Topologies).

- (a) Let (X, \mathcal{T}) be a topological space and let $S \subseteq X$. Then the collection $\mathcal{T}_S := \{S \cap U : U \in \mathcal{T}\}$ is a topology on S .

Definition 1.2. Let (X, \mathcal{T}) be a topological space and $A \subseteq X$. The **closure of A in X** , denoted by \overline{A} , is defined by

$$\overline{A} := \bigcap \{B \subseteq X : A \subseteq B, B^c \in \mathcal{T}\}. \quad (1)$$

The **interior of A in X** , denoted by $\text{Int } A$, is defined by

$$\text{Int } A := \bigcup \{C \subseteq X : C \subseteq A, C \in \mathcal{T}\}. \quad (2)$$

There is an eminent characterization of a point being in the closure of a subset $A \subseteq X$ of a topological space X .

Proposition 1.1. Let X be a topological space and $A \subseteq X$. Then $x \in \overline{A}$ if and only if the following condition holds: Every neighbourhood U of x contains a point belonging to A .

Proof. Assume that there exists a neighbourhood U of x such that $U \cap A = \emptyset$. Then U^c is closed and $A \subseteq U^c$. But $x \notin U^c$ and thus $x \notin \overline{A}$. Conversely, assume $x \notin \overline{A}$. Thus we find a closed set B such that $A \subseteq B$ and $x \notin B$. But then B^c is open and $B^c \cap A = \emptyset$. \square

1.2. Hausdorff Spaces.

Definition 1.3. Let X be a topological space. X is called a **Hausdorff space** if given $p, p' \in X$ with $p \neq p'$ we find neighbourhoods U and U' of p and p' , respectively, such that $U \cap U' = \emptyset$.

1.3. Bases and Countability.

Definition 1.4. Let (X, \mathcal{T}) be a topological space. A collection \mathcal{B} of subsets of X is called a **basis for the topology of X** if the following two conditions hold:

- (i) $\mathcal{B} \subseteq \mathcal{T}$.
- (ii) For any $U \in \mathcal{T}$ we have $U = \bigcup_{\alpha \in A} B_\alpha$ where $B_\alpha \in \mathcal{B}$ for any $\alpha \in A$.

As we shall see later, a topology \mathcal{T} on a set X may have several bases but topologies having the same basis, are equal.

Corollary 1.1. *If X is a set, \mathcal{T} and \mathcal{T}' are topologies on X and \mathcal{B} is a basis for each of the topologies \mathcal{T} and \mathcal{T}' , then $\mathcal{T} = \mathcal{T}'$.*

Proof. This is immediate by the definition of a basis for a topology 1.4. \square

Proposition 1.2 (Basis Criterion). *Let X be a topological space and \mathcal{B} be a basis for the topology on X . Then U is open in X if and only if for each $p \in U$ there exists $B \in \mathcal{B}$ such that $p \in B \subseteq U$.*

Proof. Assume U is open. Since \mathcal{B} is a basis, we have $U = \bigcup_{\alpha \in A} B_\alpha$ where $B_\alpha \in \mathcal{B}$ for each $\alpha \in A$. Thus for each $p \in U$ we have $p \in \bigcup_{\alpha \in A} B_\alpha$ and so $p \in B_\alpha$ for some $\alpha \in A$. But since also $\bigcup_{\alpha \in A} B_\alpha \subseteq U$ we have $B_\alpha \subseteq U$. Conversely, we can write $U = \bigcup_{p \in U} B_p$ for some $U \subseteq X$ where for each $p \in U$ we have $B_p \in \mathcal{B}$. Since each basis element is open, U is open as a union of open sets. \square

Definition 1.5. *Let X be a set and \mathcal{B} be a collection of subsets of X . Then \mathcal{B} is a basis for some topology on X if and only if it satisfies the following two conditions:*

- (i) $\bigcup_{B \in \mathcal{B}} B = X$.
- (ii) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists an element $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Example 1.2. Let $(X_\alpha)_{\alpha \in A}$ be a family of topological spaces. The **product topology** on $\prod_{\alpha \in A} X_\alpha$ is defined to be the topology generated by the basis consisting of all subsets of $\prod_{\alpha \in A} X_\alpha$ of the form $\prod_{\alpha \in A} U_\alpha$ where U_α is open in X_α for any $\alpha \in A$ and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in A$. The reader may verify that this is indeed a basis for a topology.

Definition 1.6. *Let X be a topological space. X is called **second countable** if there exists a countable basis for the topology of X .*

1.4. Continuity and Convergence.

Definition 1.7. *Let X and Y be two topological spaces and $f : X \rightarrow Y$. The map f is said to be **continuous** if for any open set $U \subseteq Y$ we have that $f^{-1}(U)$ is open in X .*

Exercise 1.1. Prove that a mapping $f : X \rightarrow Y$ between two topological spaces X and Y is continuous if and only if for all $A \subseteq Y$ closed, $f^{-1}(A)$ is closed in X . *Hint:* Use that for $A, B \subseteq Y$ we have $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

Definition 1.8. *Let X be a topological space, $(x_n)_{n \in \mathbb{N}}$ be a sequence in X and $x \in X$. The sequence $(x_n)_{n \in \mathbb{N}}$ is said to **converge to x** if for every neighbourhood U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for any $n > N$.*

Corollary 1.2. *Let X be a topological space and $A \subseteq X$. If $(x_n)_{n \in \mathbb{N}}$ is a convergent sequence in A , i.e. $x_n \in A$ for any $n \in \mathbb{N}$, then its limit belongs to \overline{A} .*

Proof. This is immediate by the characterization of proposition 1.1. \square

Proposition 1.3. *Let $(X_\alpha)_{\alpha \in A}$ be a family of topological spaces and $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\prod_{\alpha \in A} X_\alpha$. Then $\lim_{n \rightarrow \infty} x_n = x$ if and only if $\lim_{n \rightarrow \infty} \pi_\alpha(x_n) = \pi_\alpha(x)$ for any $\alpha \in A$.*

Proof. Assume $\lim_{n \rightarrow \infty} x_n = x \in \prod_{\alpha \in A} X_\alpha$. Fix some $\alpha \in A$ and consider some neighbourhood U of $\pi_\alpha(x)$. Then $\prod_{\beta \in A} U_\beta$ where $U_\beta = X_\beta$ for any $\beta \neq \alpha$ and $U_\beta = U$ for $\beta = \alpha$ is a neighbourhood of x in $\prod_{\alpha \in A} X_\alpha$. Thus there exists some $N \in \mathbb{N}$ such that $x_n \in \prod_{\beta \in A} U_\beta$ whenever $n > N$. But then $\pi_\alpha(x_n) \in U$ for any

$n > N$ and thus $\lim_{n \rightarrow \infty} \pi_\alpha(x_n) = \pi_\alpha(x)$. Conversely, suppose $\lim_{n \rightarrow \infty} \pi_\alpha(x_n) = \pi_\alpha(x)$ for any $\alpha \in A$. Let U be some neighbourhood of x . Then by the basis criterion 1.2 we find some basis element $B = \prod_{\alpha \in A} B_\alpha$, where B_α is open in X_α and $B_\alpha = X_\alpha$ for all but finitely many $\alpha \in A$, such that $x \in B \subseteq U$. If $B_\alpha \neq X_\alpha$, define $N_\alpha \in \mathbb{N}$ to be the number such that $n > N_\alpha$ implies $\pi_\alpha(x_n) \in B_\alpha$, otherwise let $N_\alpha := 1$. Thus $N := \max \{N_\alpha : \alpha \in A\}$ is bounded above and is therefore well defined. Hence $x_n \in U$ whenever $n > N$ and thus we have convergence. \square

1.5. Connectedness and Compactness.

Definition 1.9. Let X be a topological space. If $X = U \cup V$ for some disjoint open sets $U, V \neq \emptyset$, X is called **disconnected**, otherwise X is said to be **connected**.

Definition 1.10. Let X be a topological space. An **open cover** of X is a family $(U_\alpha)_{\alpha \in A}$ of open subsets of X such that $X = \bigcup_{\alpha \in A} U_\alpha$. A **subcover** of $(U_\alpha)_{\alpha \in A}$ is a subfamily $(U_\alpha)_{\alpha \in A'}$, $A' \subseteq A$.

Definition 1.11. A topological space X is said to be **compact** if any open cover has a finite subcover.

Theorem 1.1 (Main Theorem on Compactness). Let X and Y be topological spaces and $f : X \rightarrow Y$ be continuous. If X is compact, then $f(X)$ is compact.

Proposition 1.4. Every closed, bounded interval in \mathbb{R} is compact.

Theorem 1.2 (Extreme Value Theorem). If X is a compact space and $f : X \rightarrow \mathbb{R}$ is continuous, then f is bounded and attains its maximum and minimum values on X .

Exercises

2. Metric Spaces

2.1. Basic Definitions and Properties. We follow [Lee11, pp. 396 – 398].

Definition 2.1. Let M be a set. A **metric** on M is a function

$$d : M \times M \rightarrow \mathbb{R} \quad (3)$$

having the following properties:

- (i) $d(x, y) \geq 0$ for all $x, y \in M$.
- (ii) $d(x, y) = 0$ if and only if $x = y$.
- (iii) $d(x, y) = d(y, x)$ for all $x, y \in M$.
- (iv) (Triangle Inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$.

If on a set M a metric d has been specified, the tuple (M, d) is called a **metric space**.

Definition 2.2. Let (M, d) be a metric space. For any $x \in M$ and $r > 0$, the **open ball of radius r around x** is the set

$$B_r(x) := \{y \in M : d(y, x) < r\}. \quad (4)$$

Proposition 2.1. Let (M, d) be a metric space.

- (a) The collection

$$\mathcal{T}_d := \{A \subseteq M : \forall x \in A \exists r > 0 \text{ such that } B_r(x) \subseteq A\} \quad (5)$$

is a topology on M , called the **metric topology induced by the metric d** .

- (b) For each $x \in M$ and $r > 0$ we have $B_r(x) \in \mathcal{T}_d$.
- (c) $A \subseteq M$ is in \mathcal{T}_d if and only if A can be written as a union of some open balls.

Proof. First we prove (a). Obviously, $\emptyset, M \in \mathcal{T}_d$. Consider a family $(U_\alpha)_{\alpha \in A} \in \mathcal{T}_d$ and let $x \in \bigcup_{\alpha \in A} U_\alpha$. Thus $x \in U_\alpha$ for some $\alpha \in A$. Since $U_\alpha \in \mathcal{T}_d$, we find $r > 0$ such that $B_r(x) \subseteq U_\alpha$. Hence $B_r(x) \subseteq \bigcup_{\alpha \in A} U_\alpha$ and so $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_d$. Now assume $U_1, \dots, U_n \in \mathcal{T}_d$ and let $x \in U_1 \cap \dots \cap U_n$. Since each $U_i \in \mathcal{T}_d$, we find $r_i > 0$ such that $B_{r_i}(x) \subseteq U_i$ for each $i = 1, \dots, n$. Setting $r := \min\{r_1, \dots, r_n\}$ we have that $B_r(x) \subseteq U_1 \cap \dots \cap U_n$ and thus $U_1 \cap \dots \cap U_n \in \mathcal{T}_d$. To prove (b), let $y \in B_r(x)$. Then for $z \in B_{r-d(x,y)}(y)$ we have

$$d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r - d(x, y) = r \quad (6)$$

and hence $z \in B_r(x)$. To prove (c), let $A \in \mathcal{T}_d$. Then for any $x \in A$ we find r_x such that $B_{r_x}(x) \subseteq A$. But then $\bigcup_{x \in A} B_{r_x}(x) = A$. Conversely, assume $A = \bigcup_{\alpha \in A} B_{r_\alpha}(x_\alpha)$. By (b) we have that $B_{r_\alpha}(x_\alpha) \in \mathcal{T}_d$ for each $\alpha \in A$. Thus by (a) we have that $\bigcup_{\alpha \in A} B_{r_\alpha}(x_\alpha) \in \mathcal{T}_d$. \square

Proposition 2.2. *Let (M, d) and (M', d') be metric spaces and $f : M \rightarrow M'$. The mapping f is continuous if and only if the following condition holds: For any $x \in M$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$ for every $y \in M$.*

Proof. Assume f is continuous. Let $x \in M$ and $\varepsilon > 0$. Then $f^{-1}(B_\varepsilon(f(x)))$ is open in M since f is continuous and thus we find $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$. Conversely, let $U \subseteq M'$ be open. For any

Proposition 2.3 (Sequence Criterion for Continuity). *Let (M, d) and (M', d') be metric spaces. A mapping $f : M \rightarrow M'$ is continuous if and only if the following criterion holds: If $(x_n)_{n \in \mathbb{N}}$ is a sequence in M which converges to some $x \in M$, then $\lim_{n \rightarrow \infty} f(x_n) = f(x)$.*

Definition 2.3. *Two metrics d and d' on a set M are said to be **equivalent** if $\mathcal{T}_d = \mathcal{T}_{d'}$.*

A useful criterion to determine whether two metrics d and d' are equivalent or not is stated in the following proposition.

Proposition 2.4. *Let d and d' be two metrics on a set M . Then d and d' are equivalent if and only if the following condition is satisfied: for every $x \in M$ and every $r > 0$ there exist $r_1, r_2 > 0$ such that $B_{r_1}^{(d')}(x) \subseteq B_r^{(d)}(x)$ and $B_{r_2}^{(d)}(x) \subseteq B_r^{(d')}(x)$.*

Proof. Assume $\mathcal{T}_d = \mathcal{T}_{d'}$. Then it is obvious that the condition is satisfied with $r_1 = r_2 = r$. Conversely, assume $U \in \mathcal{T}_d$. \square

Definition 2.4. *Two metrics d and d' on a set M are said to be **strongly equivalent** if there are $c, c' > 0$ such that for all $x, y \in M$*

$$d(x, y) \leq c' d'(x, y) \quad \text{and} \quad d'(x, y) \leq c d(x, y). \quad (7)$$

Corollary 2.1. *Strongly equivalent metrics are equivalent.*

Proof. This follows immediately from proposition 2.4 by setting $r_1 := r/c'$ and $r_2 := r/c$. \square

Proposition 2.5. *The product topology on $\mathbb{R}^n = \mathbb{R} \times \dots \times \mathbb{R}$, where $(\mathbb{R}, \mathcal{T}_{|\cdot|})$, is the same as the one induced by the metric $|\cdot|$.*

Exercises

Exercise 2.1. Let (M, d) be a metric space. Show that M is a Hausdorff space.

Exercise 2.2. In this exercise we show that $(\mathbb{R}^n, \mathcal{T}_{|\cdot|})$ is second countable.

- (a) For $a < b$ show that $(a, b) \subseteq \mathbb{R}$ contains a rational number.
- (b) Show that \mathbb{Q} is dense in \mathbb{R} . *Hint:* Prove that for any real point there is a rational sequence converging to it and use corollary 1.2.
- (c) Show that \mathbb{Q}^n is dense in \mathbb{R}^n . *Hint:* Use proposition 1.3 and 2.5.
- (d) Show that the collection consisting of all open balls $B_p(q) \subseteq \mathbb{R}^n$ where $p \in \mathbb{Q}$ and $q \in \mathbb{Q}^n$ is a countable basis of $(\mathbb{R}^n, \mathcal{T}_{|\cdot|})$.

Exercise 2.3. Let (M, d) be a metric space, $(x_n)_{n \in \mathbb{N}}$ a sequence in M and $x \in M$. The sequence $(x_n)_{n \in \mathbb{N}}$ converges to x if and only if the following condition is satisfied: For any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ whenever $n > N$.

Exercise 2.4. Let (M, d) be a metric space and $(x_n)_{n \in \mathbb{N}}$. Then $\lim_{n \rightarrow \infty} x_n = x$ if and only if $\lim_{n \rightarrow \infty} d(x_n, x) = 0$, i.e. the sequence $(d(x_n, x))_{n \in \mathbb{N}}$ converges to zero in $(\mathbb{R}, |\cdot|)$.

Exercise 2.5. Prove proposition 2.3.

Exercise 2.6. Let M be a set. Show that equivalence of metrics on M is an equivalence relation on the set of all metrics on M .

Exercise 2.7. Let M be a set. Show that strong equivalence of metrics on M is an equivalence relation on the set of all metrics on M .

Exercise 2.8. Let X be a topological space. A subset of X is called an **F_σ -set** if it is a countable union of closed sets and a **G_δ -set** if it is a countable intersection of open sets (see [OB98, p. 61]). Assume we are given a metric d on X . For a nonempty subset A of X we define the real valued **distance function** ρ_A by $\rho_A(x) := \inf \{d(x, a) : a \in A\}$ for any $x \in X$.

- (a) Show that ρ_A is continuous. *Hint:* Show that ρ_A is in fact Lipschitz continuous.
- (b) Show that $\rho_A^{-1}(\{0\}) = \overline{A}$. *Hint:* Use corollary 1.2.
- (c) Show that any closed subset of X is a G_δ -set.
- (d) Show that any open subset of X is an F_σ -set.

3. Normed Spaces

Definition 3.1. Let X be a real or complex vector space. A mapping

$$\|\cdot\| : X \rightarrow \mathbb{R} \quad (8)$$

is called **norm**, if

- (i) $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{C}$ and $x \in X$.
- (ii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.
- (iii) $\|x\| = 0$ implies $x = 0$.

If only (i) and (ii) hold, $\|\cdot\|$ is called a **seminorm**. If a norm has been specified on a vector space X , the tuple $(X, \|\cdot\|)$ is called a **normed space**.

Definition 3.2. Two norms $\|\cdot\|$ and $\|\cdot\|'$ are said to be **equivalent** if the induced metrics are strongly equivalent.

There is a useful result in the finite dimensional case (see [Wer11, p. 26]).

Theorem 3.1. Let X be a finite dimensional real or complex vector space. Then any two norms are equivalent.

Proof. Let $\{e_1, \dots, e_n\}$ be a basis of X , $x := \sum_{k=1}^n x_k e_k \in X$ and $\|\cdot\|$ be a norm on X . We show that $\|\cdot\|$ is equivalent to the euclidean norm. Hölder's inequality for series (see [Els11, p. 224]) yields

$$\|x\| \leq \left(\sum_{k=1}^n |x_k|^2 \right)^{1/2} \left(\sum_{k=1}^n \|e_k\|^2 \right)^{1/2} \leq \sqrt{n} \max \{\|e_1\|, \dots, \|e_n\|\} \|x\|_2 \quad (9)$$

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $(X, \|\cdot\|_2)$ which converges to $x \in X$. By estimate (9) and the reverse triangle inequality (see exercise 3.3) we get

$$0 \leq \|x_n\| - \|x\| \leq \|x_n - x\| \leq \sqrt{n} \max\{\|e_1\|, \dots, \|e_n\|\} \|x_n - x\|_2 \quad (10)$$

which implies by double application of exercise 2.4

$$\lim_{n \rightarrow \infty} \|x_n\| = \|x\|. \quad (11)$$

Hence $\|\cdot\|$ is a continuous function on $(X, \|\cdot\|_2)$. Furthermore, $\mathbb{S}^{n-1} = \|\cdot\|_2^{-1}(\{1\})$ and \mathbb{S}^{n-1} is clearly bounded since $\mathbb{S}^{n-1} \subseteq \overline{B}_1(0)$. Therefore \mathbb{S}^{n-1} is compact by Heine-Borel and by the extreme value theorem 1.2 we get that $\|\cdot\|$ attains its minimum value $\|x_0\|$ on \mathbb{S}^{n-1} . Since $\|\cdot\|$ is a norm, we must have $\|x_0\| > 0$ and since $x/\|x\|_2 \in \mathbb{S}^{n-1}$ for $x \neq 0$ we get

$$\|x_0\| \|x\|_2 \leq \|x\|. \quad (12)$$

The case $x = 0$ holds trivially. \square

Theorem 3.1 yields a particularly important result together with the following proposition.

Exercises

Exercise 3.1. Show that if $\|\cdot\| : X \rightarrow \mathbb{R}$ is a norm, then $\|x\| \geq 0$ for any $x \in X$.

Exercise 3.2. Show that any norm $\|\cdot\|$ induces a metric $d_{\|\cdot\|}$ on X by setting $d_{\|\cdot\|}(x, y) := \|x - y\|$ for $x, y \in X$.

Exercise 3.3. Show that if $(X, \|\cdot\|)$ is a normed space, then the *reverse triangle inequality* $|\|x\| - \|y\|| \leq \|x - y\|$ holds for any $x, y \in X$. Deduce that $\|\cdot\|$ is a Lipschitz continuous function on $(X, \|\cdot\|)$.

CHAPTER 2

L^p Spaces

1. Interpolation of L^p Spaces

1.1. The Lemma of I. I. Hirschman and Hadamard's Three Lines Lemma.

Lemma 1.1 (I. I. Hirschman). *Let F be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and continuous on \overline{S} , such that for some $0 < A < \infty$ and $0 \leq \tau_0 < \pi$ we have $\log |F(z)| \leq Ae^{\tau_0 |\operatorname{Im} z|}$ for every $z \in \overline{S}$. Then*

$$|F(z)| \leq \exp \left(\frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log |F(it + iy)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log |F(1 + it + iy)|}{\cosh(\pi t) + \cos(\pi x)} \right] dt \right)$$

whenever $z := x + iy \in S$.

Proof. We will first prove the case $y = 0$. Assume F to be not identically zero (the case where F is identically zero is trivial). Let h be as in lemma (??) and let $\zeta := \rho e^{i\theta}$, $0 \leq \rho < 1$. Since $\zeta \in D$, we have $0 < \operatorname{Re} h(\zeta) < 1$ and thus the hypothesis on F and lemma (??) yields

$$\log |F(h(\zeta))| \leq Ae^{\frac{\tau_0}{\pi} |\log|(1+\zeta)/(1-\zeta)||} \leq Ae^{\tau_0/\pi} \frac{1}{|\cos(\theta/2)|^{\tau_0/\pi}} \frac{1}{|\sin(\theta/2)|^{\tau_0/\pi}} \quad (13)$$

for $1/(2e - 1) \leq \rho$. Since $0 < \tau_0 < \pi$, inequality (13) asserts, that $\log |F(h(\zeta))|$ is bounded from above by an integrable function of θ , independently of $\rho \geq 1/(2e - 1)$. Furthermore we have

$$M := \sup \{ \log |F(h(\zeta))| : \zeta \in \overline{B}_{1/(2e-1)} \} < \infty \quad (14)$$

since a upper semicontinuous function on a compact space attains its supremum (see lemma ??). Hence

$$\log |F(h(\rho e^{i\theta}))| \leq \max \left\{ M, Ae^{\tau_0/\pi} \frac{1}{|\cos(\theta/2)|^{\tau_0/\pi}} \frac{1}{|\sin(\theta/2)|^{\tau_0/\pi}} \right\} =: g(\theta) \quad (15)$$

for any $0 \leq \rho < 1$ where $g \in L^1[-\pi, \pi]$. Let $0 \leq \rho < R < 1$ and a_1, \dots, a_n denote the zeros of $F(h(\zeta))$ for $|\zeta| < R$ (since $F \circ h$ is holomorphic for $|\zeta| < 1$ there are indeed only finitely many ones) multiple zeros being repeated. Then for $F(h(\zeta)) \neq 0$ we have by the *Poisson-Jensen formula* (see [Ahl79, p. 208])

$$\log |F(h(\zeta))| = - \sum_{k=1}^n \log \left| \frac{R^2 - \overline{a}_k \zeta}{R(\zeta - a_k)} \right| + \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{Re} \left[\frac{Re^{it} + \zeta}{Re^{it} - \zeta} \right] \log |F(h(Re^{it}))| dt \quad (16)$$

Therefore by

$$\begin{aligned} \operatorname{Re} \left[\frac{Re^{it} + \zeta}{Re^{it} - \zeta} \right] &= \operatorname{Re} \left[\frac{R^2 - 2i \operatorname{Im} [\zeta Re^{-it}] - |\zeta|^2}{R^2 - 2 \operatorname{Re} [\zeta Re^{-it}] + |\zeta|^2} \right] \\ &= \operatorname{Re} \left[\frac{R^2 - 2i R \rho \sin(\theta - t) - \rho^2}{R^2 - 2R \rho \cos(\theta - t) + \rho^2} \right] \\ &= \frac{R^2 - \rho^2}{R^2 - 2R \rho \cos(\theta - t) + \rho^2} \end{aligned}$$

and since $(R^2 - |a_k|^2)(R^2 - \rho^2) \geq 0$ for all $k = 1, \dots, n$ implies $|R^2 - \bar{a}_k \zeta| \geq |R(\zeta - a_k)|$ the estimate

$$\log |F(h(\zeta))| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - \rho^2}{R^2 - 2R \rho \cos(\theta - t) + \rho^2} \log |F(h(Re^{it}))| dt \quad (17)$$

is valid for every $|\zeta| < R$. By [Rud87, p. 236] we have

$$\frac{R - \rho}{R + \rho} \leq \frac{R^2 - \rho^2}{R^2 - 2R \rho \cos(\theta - \varphi) + \rho^2} \leq \frac{R + \rho}{R - \rho} \quad (18)$$

for $0 \leq \rho < R$. Combining (15) and (18) yields

$$\frac{R^2 - \rho^2}{R^2 - 2R \rho \cos(\theta - t) + \rho^2} \log |F(h(\zeta))| \leq \frac{R + \rho}{R - \rho} g(\theta) =: G(\theta) \quad (19)$$

where $G \in L^1[-\pi, \pi]$. For $0 < R < 1$ let

$$f_R(\varphi) := \frac{R^2 - \rho^2}{R^2 - 2R \rho \cos(\theta - \varphi) + \rho^2} \log |F(h(Re^{i\varphi}))|$$

and for $\varphi \notin \{0, \pi\}$

$$f(\varphi) := \frac{1 - \rho^2}{1 - 2\rho \cos(\theta - \varphi) + \rho^2} \log |F(h(e^{i\varphi}))|$$

Since $\log |F(h(\zeta))|$ is upper semicontinuous on $\bar{D} \setminus \{\pm 1\}$ by lemma ?? we get

$$\begin{aligned} \limsup_{R \nearrow 1} f_R(\varphi) &= \limsup_{R \nearrow 1} \left[\frac{R^2 - \rho^2}{R^2 - 2R \rho \cos(\theta - \varphi) + \rho^2} \log |F(h(Re^{i\varphi}))| \right] \\ &= \lim_{R \nearrow 1} \frac{R^2 - \rho^2}{R^2 - 2R \rho \cos(\theta - \varphi) + \rho^2} \limsup_{R \nearrow 1} \log |F(h(Re^{i\varphi}))| \\ &= \frac{1 - \rho^2}{1 - 2\rho \cos(\theta - \varphi) + \rho^2} \log |F(h(e^{i\varphi}))| = f(\varphi) \end{aligned}$$

using [Bou95, p. 363] and proposition ?. The functions $G - f_R$ being non-negative, an application of Fatou's lemma yields

$$\int_{-\pi}^{\pi} \liminf_{R \nearrow 1} [G(\varphi) - f_R(\varphi)] d\varphi \leq \liminf_{R \nearrow 1} \int_{-\pi}^{\pi} [G(\varphi) - f_R(\varphi)] d\varphi$$

By [Bou95, p. 354], we get

$$\limsup_{R \nearrow 1} \int_{-\pi}^{\pi} [f_R(\varphi) - G(\varphi)] d\varphi \leq \int_{-\pi}^{\pi} \limsup_{R \nearrow 1} [f_R(\varphi) - G(\varphi)] d\varphi$$

and thus

$$\begin{aligned}
\limsup_{R \nearrow 1} \int_{-\pi}^{\pi} f_R(\varphi) d\varphi - \int_{-\pi}^{\pi} G(\varphi) d\varphi &= \limsup_{R \nearrow 1} \int_{-\pi}^{\pi} f_R(\varphi) d\varphi - \lim_{R \nearrow 1} \int_{-\pi}^{\pi} G(\varphi) d\varphi \\
&= \limsup_{R \nearrow 1} \int_{-\pi}^{\pi} [f_R(\varphi) - G(\varphi)] d\varphi \\
&\leq \int_{-\pi}^{\pi} \limsup_{R \nearrow 1} [f_R(\varphi) - G(\varphi)] d\varphi \\
&\leq \int_{-\pi}^{\pi} \limsup_{R \nearrow 1} f_R(\varphi) d\varphi - \int_{-\pi}^{\pi} \lim_{R \nearrow 1} G(\varphi) d\varphi \\
&= \int_{-\pi}^{\pi} \limsup_{R \nearrow 1} f_R(\varphi) d\varphi - \int_{-\pi}^{\pi} G(\varphi) d\varphi
\end{aligned}$$

by [Bou95, p. 358]. Hence

$$\limsup_{R \nearrow 1} \int_{-\pi}^{\pi} f_R(\varphi) d\varphi \leq \int_{-\pi}^{\pi} \limsup_{R \nearrow 1} f_R(\varphi) d\varphi$$

and so

$$\log |F(h(\zeta))| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - \rho^2}{1 - 2\rho \cos(\theta - \varphi) + \rho^2} \log |F(h(e^{i\varphi}))| d\varphi \quad (20)$$

The lemma now follows from (20) by a change of variables. By stipulating $x := h(\zeta)$ we obtain

$$\begin{aligned}
\zeta = h^{-1}(x) &= \frac{e^{\pi i x} - i}{e^{\pi i x} + i} = \frac{\cos(\pi x) + i \sin(\pi x) - i}{\cos(\pi x) + i \sin(\pi x) + i} \\
&= \frac{\cos(\pi x) + i \sin(\pi x) - i \cos(\pi x) - i \sin(\pi x) - i}{\cos(\pi x) + i \sin(\pi x) + i \cos(\pi x) - i \sin(\pi x) - i} \\
&= -i \frac{\cos(\pi x)}{1 + \sin(\pi x)} = \left(\frac{\cos(\pi x)}{1 + \sin(\pi x)} \right) e^{-i\pi/2} \quad (21)
\end{aligned}$$

by

$$\begin{aligned}
&(\cos(\pi x) + i \sin(\pi x) - i)(\cos(\pi x) - i \sin(\pi x) - i) \\
&= \cos^2(\pi x) - i \sin(\pi x) \cos(\pi x) - i \cos(\pi x) + i \sin(\pi x) \cos(\pi x) \\
&\quad + \sin^2(\pi x) + \sin(\pi x) - i \cos(\pi x) - \sin(\pi x) - 1 = -2i \cos(\pi x)
\end{aligned}$$

and

$$\begin{aligned}
&(\cos(\pi x) + i \sin(\pi x) + i)(\cos(\pi x) - i \sin(\pi x) - i) \\
&= \cos^2(\pi x) - i \sin(\pi x) \cos(\pi x) - i \cos(\pi x) + i \sin(\pi x) \cos(\pi x) \\
&\quad + \sin^2(\pi x) + \sin(\pi x) + i \cos(\pi x) + \sin(\pi x) + 1 = 2 + 2 \sin(\pi x)
\end{aligned}$$

From equality (21) we deduce $\rho = \frac{\cos(\pi x)}{1 + \sin(\pi x)}$, $\theta = -\frac{\pi}{2}$ if $0 < x \leq \frac{1}{2}$ and $\rho = -\frac{\cos(\pi x)}{1 + \sin(\pi x)}$, $\theta = \frac{\pi}{2}$ if $\frac{1}{2} \leq x < 1$. Let $0 < x \leq \frac{1}{2}$. Then we have

$$\begin{aligned}
&\frac{1 - \rho^2}{1 - 2\rho \cos(\theta - \varphi) + \rho^2} \\
&= \frac{1 + 2 \sin(\pi x) + \sin^2(\pi x) - \cos^2(\pi x)}{1 + 2 \sin(\pi x) + \sin^2(\pi x) + 2 \cos(\pi x) \sin(\varphi)(1 + \sin(\pi x)) + \cos^2(\pi x)} \\
&= \frac{\sin(\pi x) + \sin^2(\pi x)}{1 + \sin(\pi x) + \cos(\pi x) \sin(\varphi)(1 + \sin(\pi x))} = \frac{\sin(\pi x)}{1 + \cos(\pi x) \sin(\varphi)}
\end{aligned}$$

and also for $\frac{1}{2} \leq x < 1$. Let Φ and Ψ be defined as in lemma (??). We have

$$\begin{aligned} e^{i\Phi(t)} &= h^{-1}(it) = \frac{e^{-\pi t} - i e^{-\pi t} - i}{e^{-\pi t} + i e^{-\pi t} - i} = \frac{e^{-2\pi t} - 2i e^{-\pi t} - 1}{e^{-2\pi t} + 1} = \frac{e^{-2\pi t} - 1}{e^{-2\pi t} + 1} - \frac{2i e^{-\pi t}}{e^{-2\pi t} + 1} \\ &= \frac{e^{-2\pi t} - 1}{e^{-2\pi t} + 1} - \frac{2i}{e^{-\pi t} + e^{\pi t}} = \frac{1 - e^{2\pi t}}{1 + e^{2\pi t}} - \frac{2i}{e^{-\pi t} + e^{\pi t}} = -\tanh(\pi t) - i \operatorname{sech}(\pi t) \end{aligned}$$

and thus

$$\begin{aligned} \sin(\Phi(t)) \cosh(\pi t) &= \sin(-i \log(-\tanh(\pi t) - i \operatorname{sech}(\pi t))) \cosh(\pi t) \\ &= \frac{1}{2i} \left[-\tanh(\pi t) - i \operatorname{sech}(\pi t) + \frac{1}{\tanh(\pi t) + i \operatorname{sech}(\pi t)} \right] \cosh(\pi t) \\ &= \frac{1}{2i} \left[\frac{\cosh(\pi t) - \tanh(\pi t) \sinh(\pi t) - 2i \tanh(\pi t) + \operatorname{sech}(\pi t)}{\tanh(\pi t) + i \operatorname{sech}(\pi t)} \right] \\ &= \frac{1}{2i} \left[\frac{\cosh^2(\pi t) - \sinh^2(\pi t) - 2i \sinh(\pi t) + 1}{\sinh(\pi t) + i} \right] \\ &= \frac{1 - i \sinh(\pi t)}{i \sinh(\pi t) - 1} = -1 \end{aligned}$$

Therefore the transformation formula yields

$$\frac{1}{2\pi} \int_{-\pi}^0 \frac{\sin(\pi x)}{1 + \cos(\pi x) \sin(\varphi)} \log |F(h(e^{i\varphi}))| d\varphi = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin(\pi x)}{\cosh(\pi t) - \cos(\pi x)} \log |F(it)| dt$$

and in a similar manner

$$\frac{1}{2\pi} \int_0^{\pi} \frac{\sin(\pi x)}{1 + \cos(\pi x) \sin(\varphi)} \log |F(h(e^{i\varphi}))| d\varphi = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin(\pi x)}{\cosh(\pi t) + \cos(\pi x)} \log |F(1 + it)| dt$$

holds since

$$\begin{aligned} \sin(\Psi(t)) \cosh(\pi t) &= \sin(-i \log(-\tanh(\pi t) + i \operatorname{sech}(\pi t))) \cosh(\pi t) \\ &= \frac{1}{2i} \left[-\tanh(\pi t) + i \operatorname{sech}(\pi t) - \frac{1}{-\tanh(\pi t) + i \operatorname{sech}(\pi t)} \right] \cosh(\pi t) \\ &= \frac{1}{2i} \left[\frac{-\cosh(\pi t) + \tanh(\pi t) \sinh(\pi t) - 2i \tanh(\pi t) - \operatorname{sech}(\pi t)}{-\tanh(\pi t) + i \operatorname{sech}(\pi t)} \right] \\ &= \frac{1}{2i} \left[\frac{-\cosh^2(\pi t) + \sinh^2(\pi t) - 2i \sinh(\pi t) - 1}{i - \sinh(\pi t)} \right] \\ &= \frac{1 + i \sinh(\pi t)}{1 + i \sinh(\pi t)} = 1 \end{aligned}$$

Thus the case $y = 0$ is proven.

The case $y \neq 0$ follows easily from the previous one. Fix $y \neq 0$ and define $G(z) := F(z + iy)$ for $z \in \bar{S}$. Then G is a holomorphic function in S and continuous on \bar{S} as a composition of continuous and holomorphic functions. Moreover, the hypothesis on F yields

$$\log |G(z)| = \log |F(z + iy)| \leq A e^{\tau_0 |\operatorname{Im} z + y|} \leq A e^{\tau_0 |\operatorname{Im} z|} e^{\tau_0 |y|} \quad (22)$$

for all $z \in \bar{S}$. The previous case yields for G with A replaced by $A e^{\tau_0 |y|}$

$$|G(x)| \leq \exp \left(\frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log |G(it)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log |G(1 + it)|}{\cosh(\pi t) + \cos(\pi x)} \right] dt \right) \quad (23)$$

Now, observing $G(x) = F(x + iy)$, $G(it) = F(it + iy)$ and $G(1 + it) = F(1 + it + iy)$ yields the desired result. \square

An almost immediate consequence of the Lemma of I. I. Hirschman 1.1 is the so called Hadamard's Three Lines Lemma.

Lemma 1.2 (Hadamard's Three Lines Lemma). *Let F be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, continuous and bounded on \bar{S} , such that $|F(z)| \leq B_0$ when $\operatorname{Re} z = 0$ and $|F(z)| \leq B_1$ when $\operatorname{Re} z = 1$, for some $0 < B_0, B_1 < \infty$. Then $|F(z)| \leq B_0^{1-x} B_1^x$ when $\operatorname{Re} z = x$, for any $0 < x < 1$.*

Proof. Let $0 < x < 1$. Then

$$\begin{aligned} \frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \frac{1}{\cosh(\pi t) + \cos(\pi x)} dt &= \frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \frac{1}{\frac{1}{2}(e^{\pi t} + e^{-\pi t}) + \cos(\pi x)} dt \\ &= \frac{\sin(\pi x)}{\pi} \int_0^{\infty} \frac{1}{s^2 + 2\cos(\pi x)s + 1} ds \\ &= \frac{\sin(\pi x)}{\pi} \int_0^{\infty} \frac{1}{(s + \cos(\pi x))^2 + \sin^2(\pi x)} ds \\ &= \frac{1}{\pi \sin(\pi x)} \int_0^{\infty} \frac{1}{\left(\frac{s + \cos(\pi x)}{\sin(\pi x)}\right)^2 + 1} ds \\ &= \frac{1}{\pi} \int_{\cot(\pi x)}^{\infty} \frac{1}{u^2 + 1} du \\ &= \frac{1}{\pi} \left[\frac{\pi}{2} - \arctan(\cot(\pi x)) \right] \\ &= x \end{aligned}$$

and in the same manner

$$\frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \frac{1}{\cosh(\pi t) - \cos(\pi x)} dt = 1 - x$$

Assume that F is holomorphic in S , continuous and bounded on \bar{S} with $|F(z)| \leq B_0$ when $\operatorname{Re} z = 0$ and $|F(z)| \leq B_1$ when $\operatorname{Re} z = 1$ for some $0 < B_0, B_1 < \infty$. If $|F(z)| \leq M$ for $0 < M < \infty$, F satisfies the hypothesis with $A := \log(M)$ and $\tau_0 = 0$. Therefore

$$\begin{aligned} |F(z)| &\leq \exp \left(\frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log |F(it + iy)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log |F(1 + it + iy)|}{\cosh(\pi t) + \cos(\pi x)} \right] dt \right) \\ &\leq \exp \left(\frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log B_0}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log B_1}{\cosh(\pi t) + \cos(\pi x)} \right] dt \right) \\ &= \exp(x \log B_0 + (1 - x) \log B_1) \\ &= B_0^x B_1^{1-x} \end{aligned}$$

whenever $z := x + iy \in S$. □

1.2. The Stein-Weiss Interpolation Theorem for Analytic Families of Operators and the Riesz-Thorin Interpolation Theorem.

Definition 1.1 (Analytic family, admissible growth). *Let (X, μ) , (Y, ν) be two σ -finite measure spaces and $(T_z)_{z \in \bar{S}}$, where T_z is defined Σ_X and taking values in the space of all measurable functions on Y such that*

$$\int_Y |T_z(\chi_A) \chi_B| d\nu \quad (24)$$

whenever $\mu(A), \nu(B) < \infty$. The family $(T_z)_{z \in \bar{S}}$ is said to be analytic if for all $f \in \Sigma_X$, $g \in \Sigma_Y$ we have that

$$z \mapsto \int_Y T_z(f)g \, d\nu \quad (25)$$

is analytic on S and continuous on \bar{S} . Further, an analytic family $(T_z)_{z \in \bar{S}}$ is called of admissible growth, if there is a constant $\tau_0 \in [0, \pi)$, such that for all $f \in \Sigma_X$, $g \in \Sigma_Y$ a constant $C(f, g)$ exists with

$$\log \left| \int_Y T_z(f)g \, d\nu \right| \leq C(f, g) e^{\tau_0 |\operatorname{Im} z|} \quad (26)$$

for all $z \in \bar{S}$.

Theorem 1.1 (Stein-Weiss Interpolation Theorem of Analytic Families of Operators). *Let $(T_z)_{z \in \bar{S}}$ be an analytic family of admissible growth, $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and suppose that M_0, M_1 are positive functions on the real line such that for some $\tau_1 \in [0, \pi)$*

$$\sup_{-\infty < y < \infty} e^{-\tau_1 |y|} \log M_0(y) < \infty \quad \text{and} \quad \sup_{-\infty < y < \infty} e^{-\tau_1 |y|} \log M_1(y) < \infty. \quad (27)$$

Fix $0 < \theta < 1$ and define

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \quad (28)$$

Further suppose that for all $f \in \Sigma_X$ and $y \in \mathbb{R}$ we have

$$\|T_{iy}(f)\|_{L^{q_0}} \leq M_0(y) \|f\|_{L^{p_0}} \quad \text{and} \quad \|T_{1+iy}(f)\|_{L^{q_1}} \leq M_1(y) \|f\|_{L^{p_1}}. \quad (29)$$

Then for all $f \in \Sigma_X$ we have

$$\|T_\theta(f)\|_{L^q} \leq M(\theta) \|f\|_{L^p}$$

where for $0 < x < 1$

$$M(x) = \exp \left(\frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi x)} \right] dt \right).$$

Proof. Fix $0 < \theta < 1$ and $f \in \Sigma_X$, $g \in \Sigma_Y$ with $\|f\|_{L^p} = \|g\|_{L^{q'}} = 1$. Define f_z, g_z as in (??) and for $z \in \bar{S}$

$$F(z) := \int_Y T_z(f_z)g_z \, d\nu$$

Since the family $(T_z)_{z \in \bar{S}}$ is of admissible growth we have that there exist constants $c(\chi_{A_j}, \chi_{B_k})$ for any $j = 1, \dots, n$ and $k = 1, \dots, m$ such that

$$\log \left| \int_{B_k} T_z(\chi_{A_j}) \, d\nu \right| \leq c(\chi_{A_j}, \chi_{B_k}) e^{\tau_0 |\operatorname{Im} z|}$$

For shortness we will denote these constants simply by $c(A_j, B_k)$ and get

$$\begin{aligned}
\log |F(z)| &= \log \left| \sum_{j=1}^n \sum_{k=1}^m a_j^{P(z)} b_j^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T_z(\chi_{A_j})(y) \chi_{B_k}(y) d\nu(y) \right| \\
&\leq \log \left[\sum_{j=1}^n \sum_{k=1}^m \max \{1, a_j^{p/p_0+p/p_1}\} \max \{1, b_k^{q'/q'_0+q'/q'_1}\} \left| \int_{B_k} T_z(\chi_{A_j}) d\nu \right| \right] \\
&\leq \log \left[\sum_{j=1}^n \sum_{k=1}^m (1+a_j)^{p/p_0+p/p_1} (1+b_k)^{q'/q'_0+q'/q'_1} e^{c(A_j, B_k)} e^{\tau_0 |\operatorname{Im} z|} \right] \\
&\leq \log \left[\sum_{j=1}^n \sum_{k=1}^m e^{\log((1+a_j)^{p/p_0+p/p_1} (1+b_k)^{q'/q'_0+q'/q'_1}) + c(A_j, B_k)} e^{\tau_0 |\operatorname{Im} z|} \right] \\
&\leq \log (mne^{\sum_{j=1}^n \sum_{k=1}^m \log((1+a_j)^{p/p_0+p/p_1} (1+b_k)^{q'/q'_0+q'/q'_1}) + c(A_j, B_k)} e^{\tau_0 |\operatorname{Im} z|}) \\
&= \log(mn) + \sum_{j=1}^n \sum_{k=1}^m \log((1+a_j)^{p/p_0+p/p_1} (1+b_k)^{q'/q'_0+q'/q'_1}) + c(A_j, B_k) e^{\tau_0 |\operatorname{Im} z|}
\end{aligned}$$

since $\tau_0 \in [0, \pi)$ and thus $e^{\tau_0 |\operatorname{Im} z|} \geq 1$, F satisfies the hypotheses of the extension of Hadamard's three lines lemma ?? with

$$A = \log(mn) + \sum_{j=1}^n \sum_{k=1}^m \left(\frac{p}{p_0} + \frac{p}{p_1} \right) \log(1+a_j) + \left(\frac{q'}{q'_0} + \frac{q'}{q'_1} \right) \log(1+b_k) + c(A_j, B_k)$$

The same calculations as in the proof of the Riesz-Thorin interpolation theorem ?? yield for $y \in \mathbb{R}$

$$\|f_{iy}\|_{L^{p_0}} = \|f\|_{L^p}^{p/p_0} = 1 = \|g\|_{L^{q'}}^{q'/q'_0} = \|g_{iy}\|_{L^{q'_0}}$$

and

$$\|f_{1+iy}\|_{L^{p_1}} = \|f\|_{L^p}^{p/p_1} = 1 = \|g\|_{L^{q'}}^{q'/q'_1} = \|g_{1+iy}\|_{L^{q'_1}}$$

Further

$$\|F(iy)\| \leq \|T_{iy}(f_{iy})\|_{L^{q_0}} \|g_{iy}\|_{L^{q'_0}} \leq M_0(y) \|f_{iy}\|_{L^{p_0}} \|g_{iy}\|_{L^{q'_0}} = M_0(y)$$

and

$$\|F(1+iy)\| \leq \|T_{1+iy}(f_{1+iy})\|_{L^{q_1}} \|g_{1+iy}\|_{L^{q'_1}} \leq M_1(y) \|f_{1+iy}\|_{L^{p_1}} \|g_{1+iy}\|_{L^{q'_1}} = M_1(y)$$

by Hölder's inequality and the hypotheses on the analytic family $(T_z)_{z \in \overline{S}}$. Therefore the extension of Hadamard's three lines lemma ?? yields

$$|F(x)| \leq \exp \left(\frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi x)} \right] dt \right) = M(x)$$

for every $0 < x < 1$. Furthermore observe that

$$F(\theta) = \int_Y T_\theta(f) g d\nu$$

and thus by [Fol99, p. 189]

$$\begin{aligned}
M_q(T_\theta(f)) &= \sup \left\{ \left| \int_Y T_\theta(f) g \, d\nu \right| : g \in \Sigma_Y, \|g\|_{L^{q'}} \right\} \\
&= \sup \{ |F(\theta)| : g \in \Sigma_Y, \|g\|_{L^{q'}} \} \\
&\leq M(\theta)
\end{aligned}$$

Since $M(\theta)$ is an absolutely convergent integral (this is immediate by the growth conditions (27)) for any $0 < \theta < 1$, $M_q(T_\theta(f)) < \infty$ and thus $M_q(T_\theta(f)) = \|T_\theta(f)\|_{L^q}$. The general statement follows by replacing f with $f/\|f\|_{L^p}$ when $\|f\|_{L^p} \neq 0$. The theorem is trivially true when $\|f\|_{L^p} = 0$. \square

Theorem 1.2 (Riesz-Thorin interpolation theorem). *Suppose that (X, μ) , (Y, ν) are measure spaces and $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. If $q_0 = q_1 = \infty$, suppose also that ν is semifinite. Let T be a linear operator defined on Σ_X and taking values in the set of measurable functions on Y , such that for some $0 < M_0, M_1 < \infty$ the estimates*

$$\|T(f)\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}} \quad \text{and} \quad \|T(f)\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}} \quad (30)$$

hold for all $f \in \Sigma_X$. Then for all $0 \leq \theta \leq 1$ we have

$$\|T(f)\|_{L^q} \leq M_0^{1-\theta} M_1^\theta \|f\|_{L^p} \quad (31)$$

for all $f \in \Sigma_X$, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

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