

## GEOMETRY/TOPOLOGY I - SUMMARY

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### 1. Topology

**DEFINITION 1.1.** Let  $X$  be a set. A topology on  $X$  is a collection  $\mathcal{T}$  of subsets of  $X$  satisfying the following properties:

- (i)  $X, \emptyset \in \mathcal{T}$ .
- (ii) If  $U_1, \dots, U_n \in \mathcal{T}$ , then  $U_1 \cap \dots \cap U_n \in \mathcal{T}$ .
- (iii) If  $(U_\alpha)_{\alpha \in A}$  is a family of elements of  $\mathcal{T}$ , then  $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$ .

**DEFINITION 1.2.** Let  $X$  be a set, and suppose  $\mathcal{B}$  is a collection of subsets of  $X$ . Then  $\mathcal{B}$  is a basis for some topology on  $X$  if and only if it satisfies the following two conditions:

- (i)  $\bigcup_{B \in \mathcal{B}} B = X$ .
  - (ii) If  $B_1, B_2 \in \mathcal{B}$  and  $x \in B_1 \cap B_2$ , there exists an element  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ .
- If so, there is a unique topology on  $X$  for which  $\mathcal{B}$  is a basis, called the topology generated by  $\mathcal{B}$ .

**DEFINITION 1.3.** If  $d$  is a metric on the set  $X$ , then the collection of all  $\varepsilon$ -balls  $B_\varepsilon(x)$ , for  $x \in X$  and  $\varepsilon > 0$ , is a basis for a topology on  $X$ , called the metric topology induced by  $d$ .

**DEFINITION 1.4.** If  $X$  and  $Y$  are topological spaces, a map  $f : X \rightarrow Y$  is said to be continuous if for every open subset  $U \subseteq Y$ , its preimage  $f^{-1}(U)$  is open in  $X$ .

**PROPOSITION 1.1.** Let  $X$  be a Hausdorff space.

- (a) Every finite subset of  $X$  is closed.
- (b) If a sequence  $(p_i)$  in  $X$  converges to a limit  $p \in X$ , the limit is unique.

**DEFINITION 1.5.** A topological space is said to be second countable if it admits a countable basis for its topology.

**DEFINITION 1.6.** A topological space  $M$  is said to be locally Euclidean of dimension  $n$  if every point of  $M$  has a neighbourhood in  $M$  that is homeomorphic to an open subset of  $\mathbb{R}^n$ .

**DEFINITION 1.7.** An  $n$ -dimensional topological manifold is a second countable Hausdorff space that is locally Euclidean of dimension  $n$ .

DEFINITION 1.8. An  $n$ -dimensional topological manifold with boundary is a second countable Hausdorff space in which every point has a neighbourhood homeomorphic either to an open subset of  $\mathbb{R}^n$ , or to an open subset of  $\mathbb{H}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$ , considering  $\mathbb{H}^n$  as a topological space with its Euclidean topolog..

DEFINITION 1.9. Let  $(X, \mathcal{T})$  be a topological space and  $S \subseteq X$ . The topology

$$\mathcal{T}_S := \{S \cap U : U \in \mathcal{T}\} \quad (1)$$

is called the subspace topology on  $S$ .

DEFINITION 1.10. Suppose  $(X_1, \mathcal{T}_1), \dots, (X_n, \mathcal{T}_n)$  are arbitrary topological spaces. On  $X_1 \times \dots \times X_n$  we define the product topology to be the topology generated by the following basis:

$$\mathcal{B} := \{U_1 \times \dots \times U_n : U_i \in \mathcal{T}_i, i = 1, \dots, n\} \quad (2)$$

EXAMPLE 1.1. A particularly important example of a product manifold is  $\mathbb{T}^n := \mathbb{S}^1 \times \dots \times \mathbb{S}^1$ , which is an  $n$ -dimensional manifold called the  $n$ -torus.

## 2. Geometry

DEFINITION 2.1. Let  $c : (a, b) \rightarrow \mathbb{R}^2$  be a regular curve. The planar curvature  $\kappa$  of  $c$  is defined to be

$$\kappa(t) := \frac{\det(c'(t), c''(t))}{\|c'(t)\|^3} \quad (3)$$

PROPOSITION 2.1. Let  $c : (a, b) \rightarrow \mathbb{R}^3$  be a regular curve. Then there is a reparametrization of  $c$  that is a unit speed curve.

*Proof.* Pick some point  $t_0 \in (a, b)$ . Define a function  $q : (a, b) \rightarrow \mathbb{R}$  by  $q(t) := \int_{t_0}^t \|c'(s)\| ds$ . The image of  $q$  will be the interval  $(d, e)$  where  $d := \int_{t_0}^a \|c'(s)\| ds$  and  $e := \int_{t_0}^b \|c'(s)\| ds$ . Let  $h : (d, e) \rightarrow (a, b)$  be the inverse function of  $q$ . The unit speed reparametrization  $\tilde{c} : (d, e) \rightarrow \mathbb{R}^3$  is now given by  $\tilde{c} := c \circ h$ .  $\square$

DEFINITION 2.2. Let  $c : (a, b) \rightarrow \mathbb{R}^3$  be a smooth curve. The length of  $c$  is defined to be

$$\text{Length}(c) := \int_a^b \|c'(s)\| ds \quad (4)$$

DEFINITION 2.3. Let  $c : (a, b) \rightarrow \mathbb{R}^3$  be a smooth curve. For each  $t \in (a, b)$  such that  $\|c'(t)\| \neq 0$  the unit tangent vector to the curve at  $t$  is the vector

$$T(t) := \frac{c'(t)}{\|c'(t)\|} \quad (5)$$

DEFINITION 2.4. Let  $U \subseteq \mathbb{R}^2$  be open. A map  $x \in C^\infty(U, \mathbb{R}^3)$  is a coordinate patch if it is injective and if  $x_1 \times x_2 \neq 0$  at all points of  $U$ .

DEFINITION 2.5. Let  $M \subseteq \mathbb{R}^3$  be a smooth surface, let  $x : U \rightarrow M$  be a coordinate patch and let  $S \subseteq x(U)$  be a set. The area of  $S$  is defined to be

$$\text{Area}(S) := \int_{x^{-1}(S)} \sqrt{\det(g_{ij})} \, ds \, dt \quad (6)$$

Let  $x : U \rightarrow \mathbb{R}^3$  be a coordinate patch. The Gauss map in a point  $p = x(s, t)$  is given by

$$N(p) := \frac{\frac{\partial x}{\partial s} \times \frac{\partial x}{\partial t}}{\left\| \frac{\partial x}{\partial s} \times \frac{\partial x}{\partial t} \right\|} \quad (7)$$

and we define

$$l_{11} := \left\langle N(p), \frac{\partial^2 x}{\partial s^2} \right\rangle \quad l_{12} := l_{21} := \left\langle N(p), \frac{\partial^2 x}{\partial s \partial t} \right\rangle \quad l_{22} := \left\langle N(p), \frac{\partial^2 x}{\partial t^2} \right\rangle \quad (8)$$

LEMMA 2.1. Let  $M \subseteq \mathbb{R}^3$  be a smooth surface and let  $x : U \rightarrow M$  be a coordinate patch. For all  $i, j = 1, 2$ , we have

$$\begin{pmatrix} \Gamma_{ij}^1 \\ \Gamma_{ij}^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial g_{j1}}{\partial u_i} + \frac{\partial g_{i1}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_1} \\ \frac{\partial g_{j2}}{\partial u_i} + \frac{\partial g_{i2}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_2} \end{pmatrix}. \quad (9)$$