# GEOMETRY/TOPOLOGY I - SUMMARY

YANNIS BÄHNI

## 1. Topology

DEFINITION 1.1. *Let $X$ be a set. A* topology *on $X$ is a collection $\mathscr{T}$ of subsets of $X$ satisfying the following properties:*

*(i)* $X, \varnothing \in \mathscr{T}$.
*(ii) If $U_1, \dots, U_n \in \mathscr{T}$, then $U_1 \cap \dots \cap U_n \in \mathscr{T}$.*
*(iii) If $(U_\alpha)_{\alpha \in A}$ is a family of elements of $\mathscr{T}$, then $\cup_{\alpha \in A} U_\alpha \in \mathscr{T}$.*

DEFINITION 1.2. *Let $X$ be a set, and suppose $\mathscr{B}$ is a collection of subsets of $X$. Then $\mathscr{B}$ is a basis for some topology on $X$ if and only if it satisfies the following two conditions:*

*(i)* $\cup_{B \in \mathscr{B}} B = X$.
*(ii) If $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, there exists an element $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

*If so, there is a unique topology on $X$ for which $\mathscr{B}$ is a basis, called the* topology generated by $\mathscr{B}$.

DEFINITION 1.3. *If $d$ is a metric on the set $X$, then the collection of all $\varepsilon$-balls $B_\varepsilon(x)$, for $x \in X$ and $\varepsilon > 0$, is a basis for a topology on $X$, called the* metric topology *induced by $d$.*

DEFINITION 1.4. *If $X$ and $Y$ are topological spaces, a map $f : X \to Y$ is said to be* continuous *if for every open subset $U \subseteq Y$, its preimage $f^{-1}(U)$ is open in $X$.*

PROPOSITION 1.1. *Let $X$ be a Hausdorff space.*

*(a) Every finite subset of $X$ is closed.*
*(b) If a sequence $(p_i)$ in $X$ converges to a limit $p \in X$, the limit is unique.*

DEFINITION 1.5. *A topological space is said to be* second countable *if it admits a countable basis for its topology.*

DEFINITION 1.6. *A topological space $M$ is said to be* locally Euclidean of dimension $n$ *if every point of $M$ has a neighbourhood in $M$ that is homeomorphic to an open subset of $\mathbb{R}^n$.*

DEFINITION 1.7. *An $n$-*dimensional topological manifold *is a second countable Hausdorff space that is locally Euclidean of dimension $n$.*

(Yannis Bähni) UNIVERSITY OF ZURICH, RÄMISTRASSE 71, 8006 ZURICH
*E-mail address*: yannis.baehni@uzh.ch.

Yannis Bähni
yannis.baehni@uzh.ch

DEFINITION 1.8. *An $n$-dimensional* topological manifold with boundary *is a second countable Hausdorff space in which every point has a neighbourhood homeomorphic either to an open subset of $\mathbb{R}^n$, or to an open subset of $\mathbb{H}^n := \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_n \geq 0\}$, considering $\mathbb{H}^n$ as a topological space with its Euclidean topolog..*

DEFINITION 1.9. *Let $(X,\mathscr{T})$ be a topological space and $S \subseteq X$. The topology*

$$\mathscr{T}_S := \{S \cap U : U \in \mathscr{T}\} \tag{1}$$

*is called the* subspace topology *on $S$.*

DEFINITION 1.10. *Suppose $(X_1,\mathscr{T}_1),\dots,(X_n,\mathscr{T}_n)$ are arbitrary topological spaces. On $X_1 \times \cdots \times X_n$ we define the* product topology *to be the topology generated by the following basis:*

$$\mathscr{B} := \{U_1 \times \cdots \times U_n : U_i \in \mathscr{T}_i, i = 1,\dots,n\} \tag{2}$$

EXAMPLE 1.1. *A particularly important example of a product manifold is $\mathbb{T}^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$, which is an n-dimensional manifold called the $n$-*torus.

## 2. Geometry

DEFINITION 2.1. *Let $c : (a,b) \to \mathbb{R}^2$ be a regular curve. The* planar curvature $\kappa$ of $c$ *is defined to be*

$$\kappa(t) := \frac{\det\left(c'(t), c''(t)\right)}{\|c'(t)\|^3} \tag{3}$$

PROPOSITION 2.1. *Let $c : (a,b) \to \mathbb{R}^3$ be a regular curve. Then there is a reparametrization of $c$ that is a unit speed curve.*

*Proof.* Pick some point $t_0 \in (a,b)$. Define a function $q : (a,b) \to \mathbb{R}$ by $q(t) := \int_{t_0}^{t} \|c'(s)\|\,\mathrm{d}s$. The image of $q$ will be the interval $(d,e)$ where $d := \int_{t_0}^{a} \|c'(s)\|\,\mathrm{d}s$ and $e := \int_{t_0}^{b} \|c'(s)\|\,\mathrm{d}s$. Let $h : (d,e) \to (a,b)$ be the inverse function of $q$. The unit speed reparametrization $\tilde{c} : (d,e) \to \mathbb{R}^3$ is now given by $\tilde{c} := c \circ h$. □

DEFINITION 2.2. *Let $c : (a,b) \to \mathbb{R}^3$ be a smooth curve. The* length *of $c$ is defined to be*

$$\mathrm{Length}(c) := \int_a^b \|c'(s)\|\,\mathrm{d}s \tag{4}$$

DEFINITION 2.3. *Let $c : (a,b) \to \mathbb{R}^3$ be a smooth curve. For each $t \in (a,b)$ such that $\|c'(t)\| \neq 0$ the* unit tangent vector *to the curve at $t$ is the vector*

Yannis Bähni
yannis.baehni@uzh.ch

$$T(t) := \frac{c'(t)}{\|c'(t)\|} \tag{5}$$

*Furthermore if $c$ is regular, for each $t \in (a,b)$ we define the* unit normal vector *and the* unit binormal vector *by*

$$N(t) := \frac{T'(t)}{\|T'(t)\|} \qquad \text{and} \qquad B(t) := T(t) \times N(t) \tag{6}$$

*respectively whenever $\|T'(t)\| \neq 0$.*

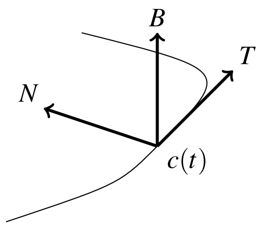


FIGURE 1. Frenet frame of a strongly regular curve $c : (a,b) \to \mathbb{R}^3$ at some $t \in (a,b)$.

DEFINITION 2.4. *A regular curve $c : (a,b) \to \mathbb{R}^3$ is* strongly regular *if one of the following equivalent conditions hold for all $t \in (a,b)$:*

*(i)* $\|T'(t)\| \neq 0$*;*
*(ii)* $\{c'(t), c''(t)\}$ *is linearly independent;*
*(iii)* $c'(t) \times c''(t) \neq 0$.

THEOREM 2.1. (Frenet-Serret) *Let $c : (a,b) \to \mathbb{R}^3$ be a strongly regular unit speed curve. Then*

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \tag{7}$$

LEMMA 2.1. *Let $c : (a,b) \to \mathbb{R}^3$ be a strongly regular curve. Then*

$$B = \frac{c' \times c''}{\|c' \times c''\|} \qquad N = B \times T \qquad \kappa = \frac{\|c' \times c''\|}{\|c'\|^3} \qquad \tau = \frac{\det(c', c'', c''')}{\|c' \times c''\|^2} \tag{8}$$

DEFINITION 2.5. *Let $U \subseteq \mathbb{R}^2$ be open. A map $x \in C^\infty(U, \mathbb{R}^3)$ is a* coordinate patch *if it is injective and if $x_1 \times x_2 \neq 0$ at all points of $U$.*

Yannis Bähni
yannis.baehni@uzh.ch

DEFINITION 2.6. *Let $M \subseteq \mathbb{R}^3$ be a smooth surface and let $x : U \to M$ be a coordinate patch. The functions $g_{ij} : U \to \mathbb{R}$ defined by $g_{ij} := \langle x_i, x_j \rangle$ for $i, j = 1, 2$ are called the* metric coefficients *of $M$ with respect to the coordinate patch $x$.*

DEFINITION 2.7. *Let $M \subseteq \mathbb{R}^3$ be a smooth surface, let $x : U \to M$ be a coordinate patch and let $S \subseteq x(U)$ be a set. The* area *of $S$ is defined to be*

$$\operatorname{Area}(S) := \int_{x^{-1}(S)} \sqrt{\det(g_{ij})}\, \mathrm{d}s\, \mathrm{d}t \tag{9}$$

Let $x : U \to \mathbb{R}^3$ be a coordinate patch. The *Gauss map* in a point $p = x(s,t)$ is given by

$$N(p) := \frac{\frac{\partial x}{\partial s} \times \frac{\partial x}{\partial t}}{\|\frac{\partial x}{\partial s} \times \frac{\partial x}{\partial t}\|} \tag{10}$$

and we define

$$l_{11} := \langle N(p), \frac{\partial^2 x}{\partial s^2} \rangle \qquad l_{12} := l_{21} := \langle N(p), \frac{\partial^2 x}{\partial s \partial t} \rangle \qquad l_{22} := \langle N(p), \frac{\partial^2 x}{\partial t^2} \rangle \tag{11}$$

LEMMA 2.2. *Let $M \subseteq \mathbb{R}^3$ be a smooth surface and let $x : U \to M$ be a coordinate patch. For all $i, j = 1, 2$, we have*

$$\begin{pmatrix} \Gamma_{ij}^1 \\ \Gamma_{ij}^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial g_{j1}}{\partial u_i} + \frac{\partial g_{i1}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_1} \\ \frac{\partial g_{j2}}{\partial u_i} + \frac{\partial g_{i2}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_2} \end{pmatrix}. \tag{12}$$

The matrix representation for the *Weingarten map* $L : T_pM \to T_pM$ is given by

$$(L_{ij}) = (g_{ij})^{-1}(l_{ij})^t \tag{13}$$