GEOMETRY/TOPOLOGY I - SUMMARY

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1. Topology

DEFINITION 1.1. *Let X be a set. A* topology on X *is a collection $\mathscr{T}$ of subsets of X satisfying the following properties:*

(i) $X, \varnothing \in \mathscr{T}$.
(ii) *If* $U_1, \ldots, U_n \in \mathscr{T}$, *then* $U_1 \cap \cdots \cap U_n \in \mathscr{T}$.
(iii) *If* $(U_\alpha)_{\alpha \in A}$ *is a family of elements of* $\mathscr{T}$, *then* $\cup_{\alpha \in A} U_\alpha \in \mathscr{T}$.

DEFINITION 1.2. *Let X be a set, and suppose $\mathscr{B}$ is a collection of subsets of X. Then $\mathscr{B}$ is a basis for some topology on X if and only if it satisfies the following two conditions:*

(i) $\cup_{B \in \mathscr{B}} B = X$.
(ii) *If* $B_1, B_2 \in \mathscr{B}$ *and* $x \in B_1 \cap B_2$, *there exists an element* $B_3 \in \mathscr{B}$ *such that* $x \in B_3 \subseteq B_1 \cap B_2$.

If so, there is a unique topology on X for which $\mathscr{B}$ is a basis, called the topology generated by $\mathscr{B}$.

DEFINITION 1.3. *If d is a metric on the set X, then the collection of all ε-balls $B_\varepsilon(x)$, for $x \in X$ and $\varepsilon > 0$, is a basis for a topology on X, called the* metric topology *induced by d.*

DEFINITION 1.4. *If X and Y are topological spaces, a map $f : X \to Y$ is said to be* continuous *if for every open subset $U \subseteq Y$, its preimage $f^{-1}(U)$ is open in X.*

PROPOSITION 1.1. *Let X be a Hausdorff space.*

(a) *Every finite subset of X is closed.*
(b) *If a sequence (p_i) in X converges to a limit $p \in X$, the limit is unique.*

DEFINITION 1.5. *A topological space is said to be* second countable *if it admits a countable basis for its topology.*

DEFINITION 1.6. *A topological space M is said to be* locally Euclidean of dimension n *if every point of M has a neighbourhood in M that is homeomorphic to an open subset of $\mathbb{R}^n$.*

DEFINITION 1.7. *An* n-dimensional topological manifold *is a second countable Hausdorff space that is locally Euclidean of dimension n.*

DEFINITION 1.8. *An* n-dimensional topological manifold with boundary *is a second countable Hausdorff space in which every point has a neighbourhood homeomorphic either to an open subset of $\mathbb{R}^n$, or to an open subset of $\mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$, considering $\mathbb{H}^n$ as a topological space with its Euclidean topolog..*

DEFINITION 1.9. *Let $(X, \mathscr{T})$ be a topological space and $S \subseteq X$. The topology*

$$\mathscr{T}_S := \{S \cap U : U \in \mathscr{T}\} \tag{1}$$

is called the subspace topology *on S.*

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DEFINITION 1.10. *Suppose $(X_1, \mathscr{T}_1), \dots, (X_n, \mathscr{T}_n)$ are arbitrary topological spaces. On $X_1 \times \dots \times X_n$ we define the* product topology *to be the topology generated by the following basis:*

$$\mathscr{B} := \{U_1 \times \dots \times U_n : U_i \in \mathscr{T}_i, i = 1, \dots, n\} \tag{2}$$

EXAMPLE 1.1. *A particularly important example of a product manifold is $\mathbb{T}^n := \mathbb{S}^1 \times \dots \times \mathbb{S}^1$, which is an n-dimensional manifold called the* n-torus.

2. Geometry

DEFINITION 2.1. *Let $c : (a,b) \to \mathbb{R}^2$ be a regular curve. The* planar curvature κ of c *is defined to be*

$$\kappa(t) := \frac{\det\left(c'(t), c''(t)\right)}{\|c'(t)\|^3} \tag{3}$$

PROPOSITION 2.1. *Let $c : (a,b) \to \mathbb{R}^3$ be a regular curve. Then there is a reparametrization of c that is a unit speed curve.*

Proof. Pick some point $t_0 \in (a,b)$. Define a function $q : (a,b) \to \mathbb{R}$ by $q(t) := \int_{t_0}^{t} \|c'(s)\| \,\mathrm{d}s$. The image of q will be the interval (d,e) where $d := \int_{t_0}^{a} \|c'(s)\| \,\mathrm{d}s$ and $e := \int_{t_0}^{b} \|c'(s)\| \,\mathrm{d}s$. Let $h : (d,e) \to (a,b)$ be the inverse function of q. The unit speed reparametrization $\tilde{c} : (d,e) \to \mathbb{R}^3$ is now given by $\tilde{c} := c \circ h$. □

DEFINITION 2.2. *Let $c : (a,b) \to \mathbb{R}^3$ be a smooth curve. The* length *of c is defined to be*

$$\mathrm{Length}(c) := \int_a^b \|c'(s)\| \,\mathrm{d}s \tag{4}$$

DEFINITION 2.3. *Let $c : (a,b) \to \mathbb{R}^3$ be a smooth curve. For each $t \in (a,b)$ such that $\|c'(t)\| \neq 0$ the* unit tangent vector *to the curve at t is the vector*

$$T(t) := \frac{c'(t)}{\|c'(t)\|} \tag{5}$$

Furthermore if c is regular, for each $t \in (a,b)$ we define the unit normal vector *and the* unit binormal vector *by*

$$N(t) := \frac{T'(t)}{\|T'(t)\|} \qquad \textit{and} \qquad B(t) := T(t) \times N(t) \tag{6}$$

respectively whenever $\|T'(t)\| \neq 0$.

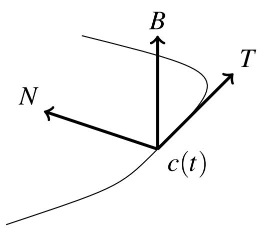

FIGURE 1. Frenet frame of a strongly regular curve $c : (a,b) \to \mathbb{R}^3$ at some $t \in (a,b)$.

DEFINITION 2.4. *A regular curve $c : (a,b) \to \mathbb{R}^3$ is* strongly regular *if one of the following equivalent conditions hold for all $t \in (a,b)$:*

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(i) $\|T'(t)\| \neq 0$;
(ii) $\{c'(t), c''(t)\}$ *is linearly independent;*
(iii) $c'(t) \times c''(t) \neq 0$.

THEOREM 2.1. (Frenet-Serret) *Let* $c : (a,b) \to \mathbb{R}^3$ *be a strongly regular unit speed curve. Then*

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \tag{7}$$

LEMMA 2.1. *Let* $c : (a,b) \to \mathbb{R}^3$ *be a strongly regular curve. Then*

$$B = \frac{c' \times c''}{\|c' \times c''\|} \qquad N = B \times T \qquad \kappa = \frac{\|c' \times c''\|}{\|c'\|^3} \qquad \tau = \frac{\det(c', c'', c''')}{\|c' \times c''\|^2} \tag{8}$$

DEFINITION 2.5. *Let* $U \subseteq \mathbb{R}^2$ *be open. A map* $x \in C^\infty(U, \mathbb{R}^3)$ *is a* coordinate patch *if it is injective and if* $x_1 \times x_2 \neq 0$ *at all points of* U.

DEFINITION 2.6. *Let* $M \subseteq \mathbb{R}^3$ *be a smooth surface and let* $x : U \to M$ *be a coordinate patch. The functions* $g_{ij} : U \to \mathbb{R}$ *defined by* $g_{ij} := \langle x_i, x_j \rangle$ *for* $i, j = 1,2$ *are called the* metric coefficients *of* M *with respect to the coordinate patch* x.

DEFINITION 2.7. *Let* $M \subseteq \mathbb{R}^3$ *be a smooth surface, let* $x : U \to M$ *be a coordinate patch and let* $S \subseteq x(U)$ *be a set. The* area *of* S *is defined to be*

$$\operatorname{Area}(S) := \int_{x^{-1}(S)} \sqrt{\det(g_{ij})}\, \mathrm{d}s\, \mathrm{d}t \tag{9}$$

LEMMA 2.2. *Let* $M \subseteq \mathbb{R}^3$ *be a smooth surface, let* $x : U \to M$ *be a coordinate patch, and let* $c : (a,b) \to x(U)$ *be a smooth curve. Then the length of* c *is given by*

$$\operatorname{Length}(c) = \int_a^b \sqrt{(c_1'(t))^2 g_{11}(\overline{c}(t)) + 2c_1'(t)c_2'(t) g_{12}(\overline{c}(t)) + (c_2'(t))^2 g_{22}(\overline{c}(t))}\, \mathrm{d}t \tag{10}$$

For example, if $c(t) = x(t, bt^2/2)$, then $\overline{c}(t) = (t, bt^2/2)$ and thus $c_1(t) = t$ and $c_2(t) = bt^2/2$.
Let $x : U \to \mathbb{R}^3$ be a coordinate patch. The *Gauss map* in a point $p = x(s,t)$ is given by

$$N(p) := \frac{\frac{\partial x}{\partial s} \times \frac{\partial x}{\partial t}}{\|\frac{\partial x}{\partial s} \times \frac{\partial x}{\partial t}\|} \tag{11}$$

and we define

$$l_{11} := \langle N(p), \frac{\partial^2 x}{\partial s^2} \rangle \qquad l_{12} := l_{21} := \langle N(p), \frac{\partial^2 x}{\partial s \partial t} \rangle \qquad l_{22} := \langle N(p), \frac{\partial^2 x}{\partial t^2} \rangle \tag{12}$$

LEMMA 2.3. *Let* $M \subseteq \mathbb{R}^3$ *be a smooth surface and let* $x : U \to M$ *be a coordinate patch. For all* $i, j = 1,2$, *we have*

$$\begin{pmatrix} \Gamma_{ij}^1 \\ \Gamma_{ij}^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial g_{j1}}{\partial u_i} + \frac{\partial g_{i1}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_1} \\ \frac{\partial g_{j2}}{\partial u_i} + \frac{\partial g_{i2}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_2} \end{pmatrix}. \tag{13}$$

The matrix representation for the *Weingarten map* $L : T_pM \to T_pM$ is given by

$$(L_{ij}) = (g_{ij})^{-1} (l_{ij})^t \tag{14}$$

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DEFINITION 2.8. *Let $M \subseteq \mathbb{R}^3$ be a smooth surface. We define two functions $K, H : M \to \mathbb{R}$ as follows. For each point $p \in M$ let L be the Weingarten map of M at p, and define*

$$K(p) := \det L = \frac{\det(l_{ij})}{\det(g_{ij})} \qquad \textit{and} \qquad H(p) := \frac{1}{2}\operatorname{tr} L = \frac{1}{2}\frac{g_{11}l_{22} - 2g_{12}l_{12} + g_{22}l_{11}}{\det(g_{ij})} \tag{15}$$

The number $K(p)$ is called the Gaussian curvature *at p, and the quantity $H(p)$ is called the* mean curvature *at p.*