GEOMETRY I - SUMMARY

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1. Topology

Definition 1.1. *Let X be a set. A* topology *on X is a collection $\mathcal{T}$ of subsets of X satisfying the following properties:*

(i) $X, \varnothing \in \mathcal{T}$.
(ii) If $U_1, \dots, U_n \in \mathcal{T}$, *then* $U_1 \cap \cdots \cap U_n \in \mathcal{T}$.
(iii) If $(U_\alpha)_{\alpha \in A}$ *is a family of elements of* $\mathcal{T}$, *then* $\cup_{\alpha \in A} U_\alpha \in \mathcal{T}$.

Definition 1.2. *Let X be a set, and suppose $\mathcal{B}$ is a collection of subsets of X. Then $\mathcal{B}$ is a basis for some topology on X if and only if it satisfies the following two conditions:*

(i) $\cup_{B \in \mathcal{B}} B = X$.
(ii) If $B_1, B_2 \in \mathcal{B}$ *and* $x \in B_1 \cap B_2$, *there exists an element* $B_3 \in \mathcal{B}$ *such that* $x \in B_3 \subseteq B_1 \cap B_2$.

If so, there is a unique topology on X for which $\mathcal{B}$ is a basis, called the topology generated by $\mathcal{B}$.

Definition 1.3. *If d is a metric on the set X, then the collection of all ε-balls $B_\varepsilon(x)$, for $x \in X$ and $\varepsilon > 0$, is a basis for a topology on X, called the* metric topology *induced by d.*

Definition 1.4. *If X and Y are topological spaces, a map $f : X \to Y$ is said to be* continuous *if for every open subset $U \subseteq Y$, its preimage $f^{-1}(U)$ is open in X.*

Proposition 1.1. *Let X be a Hausdorff space.*

(a) Every finite subset of X is closed.
(b) If a sequence (p_i) in X converges to a limit $p \in X$, the limit is unique.

Definition 1.5. *A topological space is said to be* second countable *if it admits a countable basis for its topology.*

2. Geometry

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Definition 2.1. *Let $c : (a, b) \to \mathbb{R}^2$ be a regular curve. The* planar curvature κ of c *is defined to be*

$$\kappa(t) := \frac{\det\left(c'(t), c''(t)\right)}{\|c'(t)\|^3} \tag{1}$$

Proposition 2.1. *Let $c : (a, b) \to \mathbb{R}^3$ be a regular curve. Then there is a reparametrization of c that is a unit speed curve.*

Proof. Pick some point $t_0 \in (a, b)$. Define a function $q : (a, b) \to \mathbb{R}$ by $q(t) := \int_{t_0}^{t} \|c'(s)\| \, \mathrm{d}s$. The image of q will be the interval (d, e) where $d := \int_{t_0}^{a} \|c'(s)\| \, \mathrm{d}s$ and $e := \int_{t_0}^{b} \|c'(s)\| \, \mathrm{d}s$. Let $h : (d, e) \to (a, b)$ be the inverse function of q. The unit speed reparametrization $\tilde{c} : (d, e) \to \mathbb{R}^3$ is now given by $\tilde{c} := c \circ h$. □

Definition 2.2. *Let $c : (a, b) \to \mathbb{R}^3$ be a smooth curve. The* length *of c is defined to be*

$$\mathrm{Length}(c) := \int_a^b \|c'(s)\| \, \mathrm{d}s \tag{2}$$

Definition 2.3. *Let $c : (a, b) \to \mathbb{R}^3$ be a smooth curve. For each $t \in (a, b)$ such that $\|c'(t)\| \neq 0$ the* unit tangent vector *to the curve at t is the vector*

$$T(t) := \frac{c'(t)}{\|c'(t)\|} \tag{3}$$

Definition 2.4. *Let $U \subseteq \mathbb{R}^2$ be open. A map $x \in C^\infty(U, \mathbb{R}^3)$ is a* coordinate patch *if it is injective and if $x_1 \times x_2 \neq 0$ at all points of U.*

Lemma 2.1. *Let $M \subseteq \mathbb{R}^3$ be a smooth surface and let $x : U \to M$ be a coordinate patch. For all $i, j = 1, 2$, we have*

$$\begin{pmatrix} \Gamma_{ij}^1 \\ \Gamma_{ij}^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial g_{j1}}{\partial u_i} + \frac{\partial g_{i1}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_1} \\ \frac{\partial g_{j2}}{\partial u_i} + \frac{\partial g_{i2}}{\partial u_j} - \frac{\partial g_{ij}}{\partial u_2} \end{pmatrix}. \tag{4}$$