

CLASSICAL FOURIER ANALYSIS: INTERPOLATION OF L^p SPACES

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Suppose $(p_0, q_0), (p_1, q_1) \in [1, \infty] \times [1, \infty]$ are two pairs of indices and assume that the estimates

$$\|T(f)\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}} \quad \text{and} \quad \|T(f)\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}}$$

hold, where T is an appropriately chosen operator. Does this imply that

$$\|T(f)\|_{L^q} \leq M \|f\|_{L^p} \quad \text{for other pairs } (p, q) \in [1, \infty]?$$

Those and similar questions will be answered by a tool called *interpolation*, in our case interpolation of L^p spaces. Using interpolation it is possible to reduce difficult estimates to endpoint estimates and so interpolation can (but not always does) simplify matters. Among the numerous applications of interpolation is by far the shortest proof of *Young's inequality for convolutions* [Gra14, pp. 22–23]. There is not *the* interpolation theorem, merely a family of theorems which can be roughly divided into two main categories: *real* and *complex* interpolation methods. Real methods use so called *cut-off* functions to divide the functions in the domain of the operator T into a bounded and unbounded part and then establish bounds on each of those parts whereas complex interpolation theorems are based upon standard results in complex analysis and are more restrictive on the operator T in question but yield more natural bounds (even continuous estimates) and will therefore be considered in this task. First we need a rigorous idea of what an appropriately chosen operator means in the context of Lebesgue spaces.

DEFINITION 1.1. Let (X, μ) and (Y, ν) be measure spaces. Further let T be an operator defined on a linear space of complex-valued measurable functions on X and taking values in the set of all complex-valued, finite almost everywhere, measurable functions on Y . Then T is called linear if for all functions f and g in the domain of T and all $z \in \mathbb{C}$

$$T(f + g) = T(f) + T(g) \quad T(zf) = zT(f)$$

hold and quasi-linear if

$$|T(f + g)| \leq K(|T(f)| + |T(g)|) \quad |T(zf)| = |z| |T(f)|$$

hold for some constant $K > 0$. If $K = 1$, T is called sublinear.

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For two measure spaces (X, μ) , (Y, ν) let Σ_X and Σ_Y denote the set of all finitely simple functions on X , Y respectively.

THEOREM 1.1. (Riesz-Thorin interpolation theorem) *Suppose that (X, μ) , (Y, ν) are measure spaces and $1 \leq p_0, p_1, q_0, q_1 \leq \infty$. If $q_0 = q_1 = \infty$, suppose also that ν is semifinite. Let T be a linear operator defined on Σ_X and taking values in the set of measurable functions on Y , such that for some $0 < M_0, M_1 < \infty$ the estimates*

$$\|T(f)\|_{L^{q_0}} \leq M_0 \|f\|_{L^{p_0}} \quad \text{and} \quad \|T(f)\|_{L^{q_1}} \leq M_1 \|f\|_{L^{p_1}} \quad (1)$$

hold for all $f \in \Sigma_X$. Then for all $0 \leq \theta \leq 1$ we have

$$\|T(f)\|_{L^q} \leq M_0^{1-\theta} M_1^\theta \|f\|_{L^p} \quad (2)$$

for all $f \in \Sigma_X$, where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

The proof of the Riesz-Thorin interpolation theorem heavily relies on Hadamard's three lines lemma which is itself based on a restatement of the maximum modulus theorem (see [Rud87, p. 212]) found in [Rud87, p. 253]. To do so, we have first to establish some common terminology. A complex-valued function f is said to be *holomorphic* in $\Omega \subseteq \mathbb{C}$ open, if $f'(z)$ exists for any $z \in \Omega$. By a region we shall mean a nonempty connected open subset of the complex plane. The restatement reads as follows.

THEOREM. *Let $\Omega \subseteq \mathbb{C}$ be a bounded region and f be a continuous function on $\overline{\Omega}$ which is holomorphic in Ω . Then*

$$|f(z)| \leq \sup \{|f(z)| : z \in \partial\Omega\}$$

for every $z \in \Omega$. If equality holds at one point $z \in \Omega$, then f is constant.

LEMMA 1.1. (Hadamard's three lines lemma) *Let F be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, continuous and bounded on \overline{S} , such that $|F(z)| \leq B_0$ when $\operatorname{Re} z = 0$ and $|F(z)| \leq B_1$ when $\operatorname{Re} z = 1$, for some $0 < B_0, B_1 < \infty$. Then $|F(z)| \leq B_0^{1-\theta} B_1^\theta$ when $\operatorname{Re} z = \theta$, for any $0 < \theta < 1$.*

Proof. For $z \in \overline{S}$ define

$$G(z) := \frac{F(z)}{B_0^{1-z} B_1^z} \quad G_n(z) := G(z) e^{(z^2-1)/n}, \quad n \in \mathbb{N}_{>0}$$

$G(z)$ and $G_n(z)$ are holomorphic in S by

$$G'(z) = \frac{F'(z) - F(z) \log(B_1/B_0)}{B_0^{1-z} B_1^z} \quad G'_n(z) = G'(z) e^{(z^2-1)/n} + \frac{2}{n} z G_n(z)$$

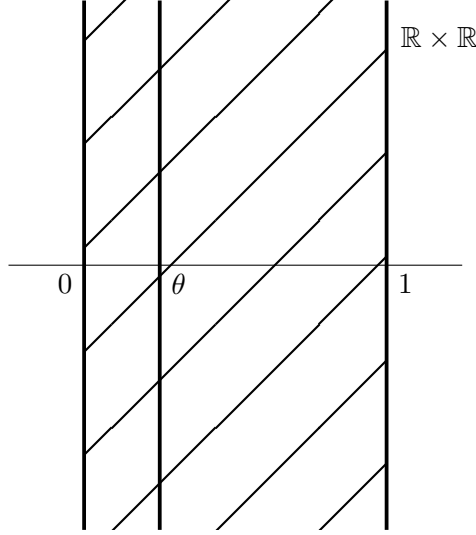


FIGURE 1. Sketch of the setting of Hadamard's three lines lemma.

and $e^z \neq 0$ for every $z \in \mathbb{C}$. Further, we have

$$|B_0^{1-z} B_1^z| = (B_0^{1-z} B_0^{1-\bar{z}} B_1^z B_1^{\bar{z}})^{1/2} = B_0^{1-\operatorname{Re} z} B_1^{\operatorname{Re} z}$$

Consider $0 \leq \operatorname{Re} z \leq 1$ and $B_0 \geq 1$. Then $B_0^{1-\operatorname{Re} z} \geq 1$ and $B_0^{1-\operatorname{Re} z} \geq B_0$ in the case $B_0 < 1$. Similarly, $B_1^{\operatorname{Re} z} \geq 1$ if $B_1 \geq 1$ and $B_1^{\operatorname{Re} z} \leq B_1$ if $B_1 < 1$. Hence

$$|B_0^{1-z} B_1^z| \geq \min\{1, B_0\} \min\{1, B_1\} > 0 \quad (3)$$

for all $z \in \bar{S}$. Since F is bounded on \bar{S} , we have $|F(z)| \leq L$ for some $L > 0$ and all $z \in \bar{S}$. Thus by (3)

$$|G(z)| = \frac{|F(z)|}{|B_0^{1-z} B_1^z|} \leq \frac{L}{\min\{1, B_0\} \min\{1, B_1\}} =: M$$

for every $z \in \bar{S}$. Fix $n \in \mathbb{N}_{>0}$ and write $z := x + iy \in \bar{S}$. Then

$$|G_n(z)| \leq M \left(e^{(x^2+2ixy-y^2-1)/n} e^{(x^2-2ixy-y^2-1)/n} \right)^{1/2} = M e^{-y^2/n} e^{(x^2-1)/n} \leq M e^{-y^2/n}$$

for $0 \leq x \leq 1$. Thus

$$\lim_{y \rightarrow \pm\infty} \sup\{|G_n(z)| : 0 \leq x \leq 1\} = 0$$

by the pinching-principle. Hence there exist $C_0(n), C_1(n) \in \mathbb{R}$, such that

$$\sup\{|G_n(z)| : 0 \leq x \leq 1\} \leq 1$$

when $y > C_0(n)$ or $y < C_1(n)$. Letting

$$C(n) := \max \{|C_0(n)| + 1, |C_1(n)| + 1\}$$

we conclude $|G_n(z)| \leq 1$ for all $0 \leq x \leq 1$ when $|y| \geq C(n)$. Now consider the rectangle $R := (0, 1) \times (-C(n), C(n))$. We have $|G_n(z)| \leq 1$ on the lines $[0, 1] \times \{\pm C(n)\}$. By

$$|G_n(iy)| = \frac{|F(iy)|}{|B_0^{1-iy} B_1^{iy}|} e^{-(y^2+1)/n} \leq 1 \quad |G_n(1+iy)| = \frac{|F(1+iy)|}{|B_0^{-iy} B_1^{1+iy}|} e^{-y^2/n} \leq 1$$

we have $|G_n(z)| \leq 1$ on the lines $\{0\} \times [-C(n), C(n)]$, $\{1\} \times [-C(n), C(n)]$. Thus $|G_n(z)| \leq 1$ on ∂R . Since $|G_n(z)|$ is continuous on \overline{R} , holomorphic in R and R is a bounded region, the maximum modulus theorem implies

$$|G_n(z)| \leq \sup \{|G_n(z)| : z \in \partial R\} \leq 1$$

for every $z \in R$. Therefore $|G_n(z)| \leq 1$ on \overline{R} and so $|G_n(z)| \leq 1$ on \overline{S} . Since inequalities are preserved by limits and the modulus is a continuous function, we have that $|G(z)| = \lim_{n \rightarrow \infty} |G_n(z)| \leq 1$ for $z \in \overline{S}$. We conclude by

$$|F(\theta + it)| = |G(\theta + it)| \left| B_0^{1-\theta-it} B_1^{\theta+it} \right| \leq B_0^{1-\theta} B_1^\theta$$

whenever $0 < \theta < 1$, $t \in \mathbb{R}$. □

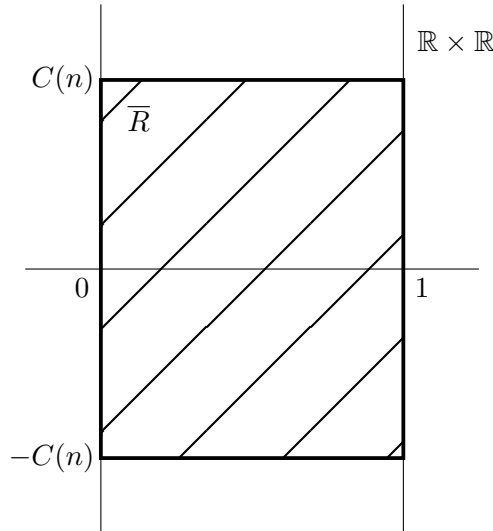


FIGURE 2. Sketch of the rectangle \overline{R} .

Proof. The idea is to bound the quantity (see [Fol99, p. 189])

$$M_q(T(f)) = \sup \left\{ \left| \int_Y T(f)g \, d\nu \right| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1 \right\}$$

appropriately. If either $\theta = 0$ or $\theta = 1$, the estimate (2) follows directly from the hypotheses (1) on T . Thus we may assume $0 < \theta < 1$. Furthermore, if $f \in \Sigma_X$, $\|f\|_{L^p} = 0$, then $f = 0$ μ -a.e. and either one of the hypotheses on T in (1) implies $T(f) = 0$ μ -a.e. and thus the estimate (2) holds trivially. Therefore we can assume $\|f\|_{L^p} \neq 0$. Fix $f \in \Sigma_X$, $g \in \Sigma_Y$ with representation

$$f = \sum_{j=1}^n a_j e^{i\alpha_j} \chi_{A_j} \quad g = \sum_{k=1}^m b_k e^{i\beta_k} \chi_{B_k}$$

where $a_j, b_k \neq 0$, $\alpha_j, \beta_k \in \mathbb{R}$ for any $j = 1, \dots, n$, $k = 1, \dots, m$, the sets A_j and B_k are each pairwise disjoint with $\mu(A_j), \nu(B_k) < \infty$ and so, that $\|g\|_{L^{q'}} \neq 0$ (recall $q' := q/(q-1)$). Define

$$P(z) := \frac{p}{p_0}(1-z) + \frac{p}{p_1}z \quad Q(z) := \frac{q'}{q'_0}(1-z) + \frac{q'}{q'_1}z$$

for $z \in \mathbb{C}$ (since either $p = \infty$ implies $p_0 = p_1 = \infty$ or $q = 1$ implies $q_0 = q_1 = 1$, the functions P, Q are well-defined). Further let

$$f_z := \sum_{j=1}^n a_j^{P(z)} e^{i\alpha_j} \chi_{A_j} \quad g_z := \sum_{k=1}^m b_k^{Q(z)} e^{i\beta_k} \chi_{B_k} \quad (4)$$

and

$$F(z) := \int_Y T(f_z)g_z \, d\nu \quad (5)$$

By (4), (5) and the linearity of the operator T we have

$$F(z) = \sum_{j=1}^n \sum_{k=1}^m a_j^{P(z)} b_k^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j}) \chi_{B_k} \, d\nu$$

Applying Hölder's inequality yields

$$\begin{aligned} \left| \int_Y T(\chi_{A_j}) \chi_{B_k} \, d\nu \right| &\leq \int_Y |T(\chi_{A_j}) \chi_{B_k}| \, d\nu \\ &= \|T(\chi_{A_j}) \chi_{B_k}\|_{L^1} \\ &\leq \|T(\chi_{A_j})\|_{L^{q_0}} \|\chi_{B_k}\|_{L^{q'_0}} \\ &\leq M_0 \|\chi_{A_j}\|_{L^{p_0}} \|\chi_{B_k}\|_{L^{q'_0}} \\ &\leq M_0 \mu(A_j)^{1/p_0} \nu(B_k)^{1/q'_0} \end{aligned} \quad (6)$$

for each $j = 1, \dots, n$, $k = 1, \dots, m$ (even in the cases where either $p_0 = \infty$ or $q'_0 = \infty$, or both, by observing that $\|\chi_A\|_{L^\infty} \leq 1$ for any measurable set A). Thus the function F is well-defined on \mathbb{C} . Let $t \in \mathbb{R}$. For $p, p_0 \neq \infty$

$$\begin{aligned} \|f_{it}\|_{L^{p_0}} &= \left(\sum_{j=1}^n \int_{A_j} |f_{it}|^{p_0} d\mu + \int_{X \setminus \bigcup_{j=1}^n A_j} |f_{it}|^{p_0} d\mu \right)^{1/p_0} \\ &= \left(\sum_{j=1}^n \left| a_j^{P(it)} e^{i\alpha_j} \right|^{p_0} \int_X \chi_{A_j} d\mu \right)^{1/p_0} \\ &= \left(\sum_{j=1}^n a_j^{p_0 \operatorname{Re} P(it)} \mu(A_j) \right)^{1/p_0} \\ &= \left(\sum_{j=1}^n a_j^p \mu(A_j) \right)^{p/(p_0 p)} \\ &= \|f\|_{L^p}^{p/p_0} \end{aligned}$$

holds. Let $p_0 = \infty$, $p \neq \infty$. Then $\|f_{it}\|_{L^\infty} = 1$ since $|a_j^{P(it)}| = a_j^{p/p_0} = 1$ and that there exists some index j , such that $\mu(A_j) \neq 0$. If $p = \infty$, then $p_0 = p_1 = \infty$ and thus $P(it) = 1$. By the same considerations we have $\|g_{it}\|_{L^{q'_0}} = \|g\|_{L^{q'}}^{q'/q'_0}$. Hence

$$\begin{aligned} |F(it)| &\leq \int_Y |T(f_{it})g_{it}| d\nu \\ &= \|T(f_{it})g_{it}\|_{L^1} \\ &\leq \|T(f_{it})\|_{L^{q_0}} \|g_{it}\|_{L^{q'_0}} \\ &\leq M_0 \|f_{it}\|_{L^{p_0}} \|g_{it}\|_{L^{q'_0}} \\ &= M_0 \|f\|_{L^p}^{p/p_0} \|g\|_{L^{q'}}^{q'/q'_0} \end{aligned}$$

by Hölder's inequality. In an analogous manner we derive

$$\|f_{1+it}\|_{L^{p_1}} = \|f\|_{L^p}^{p/p_1} \quad \|g_{1+it}\|_{L^{q'_1}} = \|g\|_{L^{q'}}^{q'/q'_1}$$

and thus

$$|F(1+it)| \leq M_1 \|f\|_{L^p}^{p/p_1} \|g\|_{L^{q'}}^{q'/q'_1}$$

Further by estimate (6)

$$\begin{aligned}
|F(z)| &\leq \sum_{j=1}^n \sum_{k=1}^m \left| a_j^{P(z)} \right| \left| b_k^{Q(z)} \right| \left| \int_Y T(\chi_{A_j}) \chi_{B_k} d\nu \right| \\
&\leq M_0 \sum_{j=1}^n \sum_{k=1}^m a_j^{\operatorname{Re} P(z)} b_k^{\operatorname{Re} Q(z)} \mu(A_j)^{1/p_0} \nu(B_k)^{1/q'_0} \\
&\leq M_0 \sum_{j=1}^n \sum_{k=1}^m \max \left\{ 1, a_j^{p/p_0 + p/p_1} \right\} \max \left\{ 1, b_k^{q'/q'_0 + q'/q'_1} \right\} \mu(A_j)^{1/p_0} \nu(B_k)^{1/q'_0}
\end{aligned}$$

Hence F is bounded on \bar{S} by some constant depending on f and g only. By

$$\begin{aligned}
F'(z) &= \sum_{j=1}^n \sum_{k=1}^m a_j^{P(z)} \log(a_j) \left(\frac{p}{p_1} - \frac{p}{p_0} \right) b_k^{Q(z)} e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j}) \chi_{B_k} d\nu \\
&\quad + \sum_{j=1}^n \sum_{k=1}^m a_j^{P(z)} b_k^{Q(z)} \log(b_k) \left(\frac{q'}{q'_1} - \frac{q'}{q'_0} \right) e^{i\alpha_j} e^{i\beta_k} \int_Y T(\chi_{A_j}) \chi_{B_k} d\nu
\end{aligned}$$

it is immediate, that F is an entire function and thus holomorphic in S and continuous on \bar{S} . Therefore, Hadamard's three lines lemma yields

$$|F(z)| \leq \left(M_0 \|f\|_{L^p}^{p/p_0} \|g\|_{L^{q'}}^{q'/q'_0} \right)^{1-\theta} \left(M_1 \|f\|_{L^p}^{p/p_1} \|g\|_{L^{q'}}^{q'/q'_1} \right)^\theta = M_0^{1-\theta} M_1^\theta \|f\|_{L^p} \|g\|_{L^{q'}}$$

for $\operatorname{Re} z = \theta$, $0 < \theta < 1$. We have

$$\{T(f) \neq 0\} = \bigcup_{n=1}^{\infty} \{|T(f)| > 1/n\}$$

and by Chebychev's inequality either

$$\nu(\{|T(f)| > 1/n\}) \leq n^{q_0} \|T(f)\|_{L^{q_0}}^{q_0}$$

or

$$\nu(\{|T(f)| > 1/n\}) \leq n^{q_1} \|T(f)\|_{L^{q_1}}^{q_1}$$

whenever $q_0 \neq \infty$ or $q_1 \neq \infty$. Therefore, the set $\{T(f) \neq 0\}$ is σ -finite unless $q_0 = q_1 = \infty$. Further we have $P(\theta) = Q(\theta) = 1$. Thus by

$$\begin{aligned}
M_q(T(f)) &= \sup \left\{ \left| \int_Y T(f) g d\nu \right| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1 \right\} \\
&= \sup \{ |F(\theta)| : g \in \Sigma_Y, \|g\|_{L^{q'}} = 1 \} \\
&\leq M_0^{1-\theta} M_1^\theta \|f\|_{L^p}
\end{aligned}$$

we conclude

$$\|T(f)\|_{L^q} = M_q(T(f)) \leq M_0^{1-\theta} M_1^\theta \|f\|_{L^p}$$

for any $f \in \Sigma_X$. □

REMARK. A more general version of the Riesz-Thorin interpolation theorem can be found in [Fol99, pp. 200–202]. There, a linear map

$$T : L^{p_0} + L^{p_1} \rightarrow L^{q_0} + L^{q_1}$$

is considered. This follows using a density argument from the current version of the theorem and will not be proven here.

REMARK. Using the previous remark, a standard application of the Riesz-Thorin interpolation theorem is to prove Young's inequality for convolutions [Gra14, pp. 22–23].

One disadvantage of the Riesz-Thorin interpolation theorem is, that it is applicable to linear operators only without a loss in the constant, but there is some advantage. Due to the complex nature of the proof, one may imagine to extend the theorem to an appropriate family of linear operators $(T_z)_{z \in \Omega}$, where $\Omega \subseteq \mathbb{C}$. This was done by Stein and Weiss in the fifties. We need some further concepts.

DEFINITION 1.2. (Analytic family, admissible growth) Let (X, μ) , (Y, ν) be two σ -finite measure spaces and for every $z \in \bar{S}$ we have an associated linear operator T_z which is defined on Σ_X and taking values in the space of all measurable functions on Y such that

$$\int_Y |T_z(\chi_A)\chi_B| d\nu < \infty$$

whenever $\mu(A), \nu(B) < \infty$. The family $(T_z)_{z \in \bar{S}}$ is said to be analytic if for all $f \in \Sigma_X$, $g \in \Sigma_Y$ we have that

$$z \mapsto \int_Y T_z(f)g d\nu$$

is analytic in S and continuous on \bar{S} . Further, an analytic family $(T_z)_{z \in \bar{S}}$ is called of admissible growth, if there is a constant $\tau_0 \in (0, \pi)$, such that for all $f \in \Sigma_X$, $g \in \Sigma_Y$ a constant $0 < C(f, g) < \infty$ exists with

$$\log \left| \int_Y T_z(f)g d\nu \right| \leq C(f, g) e^{\tau_0 |\operatorname{Im} z|}$$

for all $z \in \bar{S}$.

THEOREM 1.2. (Stein-Weiss theorem on interpolation of analytic families of operators) Let $(T_z)_{z \in \bar{S}}$ be an analytic family of admissible growth, $1 \leq p_0, p_1, q_0, q_1 \leq \infty$ and

suppose that M_0, M_1 are positive functions on the real line such that for some $\tau_1 \in (0, \pi)$

$$\sup \left\{ e^{-\tau_1|y|} \log M_0(y) : y \in \mathbb{R} \right\} < \infty \quad \sup \left\{ e^{-\tau_1|y|} \log M_1(y) : y \in \mathbb{R} \right\} < \infty \quad (7)$$

Fix $0 < \theta < 1$ and define

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \frac{1}{q} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \quad (8)$$

Further suppose that for all $f \in \Sigma_X$ and $y \in \mathbb{R}$ we have

$$\|T_{iy}(f)\|_{L^{q_0}} \leq M_0(y) \|f\|_{L^{p_0}} \quad \|T_{1+iy}(f)\|_{L^{q_1}} \leq M_1(y) \|f\|_{L^{p_1}} \quad (9)$$

Then for all $f \in \Sigma_X$ we have

$$\|T_\theta(f)\|_{L^q} \leq M(\theta) \|f\|_{L^p}$$

where for $0 < x < 1$

$$M(x) = \exp \left(\frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log M_0(t)}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log M_1(t)}{\cosh(\pi t) + \cos(\pi x)} \right] d\lambda(t) \right).$$

Let X be a topological space. A function $f : X \rightarrow [-\infty, \infty)$ is said to be *upper semicontinuous* if for every $x_0 \in X$ and $M > f(x_0)$ a neighbourhood U of x_0 exists such that $M > f(x)$ holds for every $x \in U$. An upper semicontinuous function on a compact topological space attains its supremum. In particular it is bounded from above.

LEMMA 1.2. (Hadamard's three lines lemma, extension) *Let F be a holomorphic function in the strip $S := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$ and continuous on \overline{S} , such that for some $0 < A < \infty$ and $\tau_0 \in (0, \pi)$ we have $\log |F(z)| \leq A e^{\tau_0 |\operatorname{Im} z|}$ for every $z \in \overline{S}$. Then*

$$|F(z)| \leq \exp \left(\frac{\sin(\pi x)}{2} \int_{-\infty}^{\infty} \left[\frac{\log |F(it + iy)|}{\cosh(\pi t) - \cos(\pi x)} + \frac{\log |F(1 + it + iy)|}{\cosh(\pi t) + \cos(\pi x)} \right] d\lambda(t) \right)$$

whenever $z := x + iy \in S$.

Proof. Consider the mapping

$$h(z) := \frac{1}{\pi i} \log \left(i \frac{1+z}{1-z} \right)$$

for $z \in \overline{B}_1(0) \setminus \{\pm 1\}$ where we are taking that continuous branch of $\log z$ in the complex plane slit along the negative imaginary axis, $\mathbb{C} \setminus (\{0\} \times [0, \infty))$. The inverse mapping is given by

$$h^{-1}(z) = \frac{e^{\pi i z} - i}{e^{\pi i z} + 1}$$

for $z \in \overline{S}$. Therefore the composition $\log |F \circ h|$ is subharmonic in $B_1(0)$ and upper semicontinuous on $\overline{B}_1(0) \setminus \{\pm 1\}$. For $1/(2e-1) \leq \rho < 1$ we have

$$\log |F(h(\rho e^{i\theta}))| \leq A e^{\tau_0/\pi} \frac{1}{|\cos(\theta/2)|^{\tau_0/\pi}} \frac{1}{|\sin(\theta/2)|^{\tau_0/\pi}}$$

and for $0 \leq \rho \leq 1/(2e-1)$, we have $\rho e^{i\theta} \in \overline{B}_{1/(2e-1)}(0)$ and by compactness, $\log |F \circ h|$ is bounded from above. Therefore

$$\log |F(h(\rho e^{i\theta}))| \leq g(\theta)$$

for all $0 \leq \rho < 1$, where $g \in L^1[-\pi, \pi]$. The maximum modulus principle for subharmonic functions and the Poisson integral formula yields

$$\begin{aligned} \log |F(h(\rho e^{i\theta}))| &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(h(R e^{i\varphi}))| \frac{R^2 - \rho^2}{R^2 - 2R\rho \cos(\theta - \varphi) + \rho^2} d\lambda(\varphi) \\ &\leq \frac{1}{2\pi} \frac{R - \rho}{R + \rho} \int_{-\pi}^{\pi} g(\varphi) d\lambda(\varphi) \end{aligned}$$

for $\rho < R < 1$. By defining

$$f_R(\varphi) := \log |F(h(R e^{i\varphi}))| \frac{R^2 - \rho^2}{R^2 - 2R\rho \cos(\theta - \varphi) + \rho^2}$$

and

$$f(\varphi) := \log |F(h(e^{i\varphi}))| \frac{1 - \rho^2}{1 - 2\rho \cos(\theta - \varphi) + \rho^2}$$

By the upper semicontinuity of $\log |F \circ h|$ we get

$$\limsup_{R \uparrow 1} f_R(\varphi) = f(\varphi)$$

Further by letting

$$G(\varphi) := \frac{1}{2\pi} \frac{R - \rho}{R + \rho} \int_{-\pi}^{\pi} g(\varphi) d\lambda(\varphi)$$

$G(\varphi) - f_R(\varphi)$ is non-negative. Thus an application of the reverse version of Fatou's lemma yields

$$\limsup_{R \uparrow 1} \int_{-\pi}^{\pi} f_R(\varphi) d\lambda(\varphi) \leq \int_{-\pi}^{\pi} \limsup_{R \uparrow 1} f_R(\varphi) d\lambda(\varphi)$$

The lemma now follows by a change of variables in the the integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |F(h(e^{i\varphi}))| \frac{1 - \rho^2}{1 - 2\rho \cos(\theta - \varphi) + \rho^2} d\lambda(\varphi)$$

□

Proof. As mentioned in [Terence Tao's blog](#), Fefferman once noted, that this proof can be obtained from that of the Riesz-Thorin theorem 1.1 simply by adding a single letter of the alphabet. Indeed, this is truly the case, since all hypotheses made in the theorem incorporate the same proof as in the Riesz-Thorin theorem. The only heavy and technical part is the proof of the extension of Hadamard's three lines lemma 1.2. \square

THEOREM 1.3. (The Marcinkiewicz interpolation theorem) *Let (X, μ) , (Y, ν) be two σ -finite measure spaces and $0 < p_0 < p_1 \leq \infty$. Further let T be a sublinear operator defined on*

$$L^{p_0} + L^{p_1} := \{f_0 + f_1 : f_0 \in L^{p_0}(X, \mu), f_1 \in L^{p_1}(X, \mu)\}$$

and taking values in the space of measurable functions on Y . Assume that there exist $0 < A_0, A_1 < \infty$ such that

$$\forall f \in L^{p_0}(X, \mu) \quad \|T(f)\|_{L^{p_0, \infty}} \leq A_0 \|f\|_{L^{p_0}} \quad (10)$$

$$\forall f \in L^{p_1}(X, \mu) \quad \|T(f)\|_{L^{p_1, \infty}} \leq A_1 \|f\|_{L^{p_1}} \quad (11)$$

Then for all $p_0 < p < p_1$ and for all $f \in L^p(X, \mu)$ we have the estimate

$$\|T(f)\|_{L^p} \leq A \|f\|_{L^p} \quad (12)$$

where

$$A := 2 \left(\frac{p}{p - p_0} + \frac{p}{p_1 - p} \right)^{1/p} A_0^{\frac{\frac{1}{p} - \frac{1}{p_1}}{\frac{1}{p_0} - \frac{1}{p_1}}} A_1^{\frac{\frac{1}{p_0} - \frac{1}{p}}{\frac{1}{p_0} - \frac{1}{p_1}}}. \quad (13)$$