INTERPOLATION OF LINEAR OPERATORS(1)

BY

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The aim of this paper is to prove a generalization of a well-known convexity theorem of M. Riesz [8]. The Riesz theorem was originally deduced by "real-variable" techniques. Later, Thorin [10], Tamarkin and Zygmund [9], and Thorin [11] introduced convexity properties of analytic functions in their study of Riesz's theorem. These ideas were put in especially suggestive form by A. P. Calderón and A. Zygmund [3]. It is the last mentioned approach to the Riesz theorem which is our starting point. In the interpolation theorem we shall prove, we vary not only over the Lebesgue spaces in question, but we also vary the linear operators in question. An exact statement will be found in Theorem 1.

Part II contains the first application of the interpolation theorem. We shall consider "Bochner-Riesz" summability of multiple Fourier series and Fourier integrals; we prove that we have L_p norm convergence (for $1<p<\infty$) for the Bochner-Riesz means *below the critical index*. These results are contained in Theorems 3 and 4.

A second application will be found in Part III. We shall show that a theorem of Pitt for Fourier Series may be proved for all uniformly bounded orthonormal systems. The fact that Pitt's theorem may hold in general circumstances was suggested by Professor A. Zygmund to the author. The last result is interesting when reapplied to the case of Fourier series via the familiar device of rearrangements. The result contains well-known inequalities of F. Riesz and R. E. A. C. Paley, as well as an inequality recently proved by I. I. Hirschman(2).

PART I. INTERPOLATION THEOREMS

1. The "three-lines lemma." The following fact is basic to the proof of the M. Riesz theorem as given in [3]:

Let $\Phi(z)$ be analytic in the strip $0<R(z)<1$, and suppose that $\Phi(z)$ is bounded there. Let $M_t=\sup_{-\infty<y<\infty}|\Phi(t+iy)|$. Then $\log M_t$ *is a convex function of t, $0\leqq t\leqq 1$.*

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(1) This work was done while the author held a National Science Foundation Fellowship.

(2) *Note added in proof.* After the acceptance of this paper for publication, a special case of Theorem 5, and a result closely related to Theorem 2 appeared in a paper by I. I. Hirschman (Pacific Journal of Mathematics vol. 6 (1956) pp. 47–56).

We shall need a generalization of the above fact due to I. I. Hirschman [5], which he used in another connection involving "interpolation." We shall use the following definition: A function $\Phi(z)$ analytic in the open strip $0 < R(z) < 1$, and continuous in the closed strip, will be called of *admissible growth* if

$$\sup_{|y| \leqq r} \sup_{0 \leqq x \leqq 1} \log |\Phi(x + iy)| \leqq Ae^{ar}, \qquad a < \pi. \tag{1.1}$$

LEMMA OF HIRSCHMAN. *Let $\Phi(z)$ be analytic in the strip $0 < R(z) < 1$, continuous in the closed strip, and of admissible growth there. Let*

$$\log |\Phi(iy)| \leqq \alpha_0(y), \qquad \log |\Phi(1 + iy)| \leqq \alpha_1(y)$$

then:

$$\log |\Phi(t)| \leqq \int_{-\infty}^{+\infty} \omega(1 - t, y)\alpha_0(y)dy + \int_{-\infty}^{+\infty} \omega(t, y)\alpha_1(y)dy \tag{1.2}$$

where $0 \leqq t \leqq 1$, and

$$\omega(t, y) = \frac{1/2 \tan (\pi t/2)}{[\tan^2 (\pi t/2) + \tanh^2 (\pi y/2)] \cosh^2 (\pi y/2)}.$$

The proof of this lemma may be found in [5, p. 210].

2. **The interpolation theorem.** Suppose that we are given two measure spaces M and N, with measures dm and dn respectively. We shall be interested in a family of linear transformations T_z (depending on the complex parameter z). We shall call such a family an *analytic family of operators*, if it has the following properties:

(i) for each z, T_z is a linear transformation of "simple" functions(3) on M to measurable functions on N.

(ii) If ψ is a simple function on M, and ϕ is a simple function on N, then $\Phi(z) \equiv \int T_z(\psi)\phi dn$ is analytic in $0 < R(z) < 1$, and continuous in $0 \leqq R(z) \leqq 1$.

We shall also say that the analytic family T_z is of *admissible growth*, if $\Phi(z) = \int T_z(\psi)\phi dn$ is of admissible growth in the sense of (1.1). Here, however, the constants A and a in (1.1) may depend on ψ and ϕ.

The following is the main result of this part.

THEOREM 1. *Let T_z be an analytic family of linear operators of admissible growth defined in the strip $0 \leqq R(z) \leqq 1$. Suppose that $1 \leqq p_1, p_2, q_1, q_2 \leqq \infty$, and that $1/p = (1-t) \cdot 1/p_1 + t/p_2$, $1/q = (1-t) \cdot 1/q_1 + t/q_2$, where $0 \leqq t \leqq 1$. Finally suppose*

$$\|T_{iy}(f)\|_{q_1} \leqq A_0(y)\|f\|_{p_1}, \tag{2.1}$$

and

(3) A simple function takes on only a finite number of nonzero values on sets of finite measure.

(2.2) $$\|T_{1+iy}(f)\|_{q_2} \leqq A_1(y)\|f\|_{p_2}$$

for any simple f. We also assume that:

(2.3) $$\log |A_i(y)| \leqq Ae^{a|y|}, \qquad a < \pi \text{ for } i = 0, 1.$$

Then we may conclude that

(2.4) $$\|T_t(f)\|_q \leqq A_t\|f\|_p$$

where

$$\log A_t = \int_{-\infty}^{+\infty} \omega(1-t, y) \log A_0(y)dy + \int_{-\infty}^{+\infty} \omega(t, y) \log A_1(y)dy.$$

3. **Proof of Theorem 1.** Let ψ and ϕ be arbitrary simple functions on M and N respectively, subject only to the restriction that $\|\psi\|_p = 1$, and $\|\phi\|_{q'} = 1$. Then it is sufficient to show that:

(3.1) $$\left|\int T_t(\psi)\phi dn\right| \leqq A_t.$$

Now let $\alpha(z) = (1-z)\cdot 1/p_1 + z/p_2, \beta(z) = (1-z)\cdot 1/q_1 + z/q_2$, also $\alpha = 1/p, \beta = 1/q$, and let

$$F_z = |\psi|^{\alpha(z)/\alpha} \operatorname{sign}(\psi) \qquad \text{if } \alpha \neq 0,$$

$$F_z = \psi \qquad \text{if } \alpha = 0,$$

also

$$G_z = |\phi|^{(1-\beta(z))/(1-\beta)} \operatorname{sign}(\phi) \qquad \text{if } \beta \neq 1,$$

$$= \phi \qquad \text{if } \beta = 1.$$

Then certainly: $F_t = \psi$, and $G_t = \phi$. Finally let:

(3.2) $$\Phi(z) = \int T_z(F_z)G_z dn.$$

Then using the linear character of $T_z(\cdot)$, and its analytic dependence on z, it is an easy matter to verify that:

(i) $\Phi(z)$ is analytic in the strip $0 < R(z) < 1$ and continuous on the closed strip.

(ii) $\Phi(z)$ is of admissible growth in the strip $0 \leqq R(z) \leqq 1$. Moreover,

$$\begin{aligned} |\Phi(iy)| &\leqq \|T_{iy}(F_{iy})\|_{q_1}\|G_{iy}\|_{q_1'} \\ &\leqq A_0(y)\|F_{iy}\|_{p_1}\|G_{iy}\|_{q_1'} \\ &\leqq A_0(y), \end{aligned}$$

since $\|F_{iy}\|_{p_1} = 1$, and $\|G_{iy}\|_{q_1}' = 1$.

Similarly:

$$|\Phi(1+iy)| \leqq A_1(y).$$

Therefore, making use of Hirschman's lemma we get

$$\log|\Phi(t)| \leqq \int_{-\infty}^{+\infty} \omega(1-t, y)\log A_0(y)dy + \int_{-\infty}^{+\infty} \omega(t, y)\log A_1(y)dy, \tag{3.3}$$

that is,

$$\left|\int T_t(\psi)\phi dn\right| \leqq A_t. \tag{3.4}$$

The proof of Theorem 1 is therefore complete.

4. **A special case of Theorem 1.** From Theorem 1 we may deduce the following interpolation theorem. In this theorem we vary the measures on our measure spaces together with the exponents of the Lebesgue classes.

Let k_1, k_2 be two non-negative measurable functions on N. Also let u_1, u_2 be two non-negative measurable functions on M.

THEOREM 2. *Let T be a linear transformation defined on simple functions of M to measurable functions on N. Suppose $1 \leqq p_1, p_2, q_1, q_2 \leqq \infty$, and $1/p = (1-t)\cdot 1/p_1 + t\cdot 1/p_2$, $1/q = (1-t)\cdot 1/q_1 + t/q_2$, where $0 \leqq t \leqq 1$. Suppose that for simple f*(4)

$$\|(Tf)\cdot k_1\|_{q_1} \leqq M_1\|f\cdot u_1\|_{p_1}, \tag{4.1}$$

and

$$\|(Tf)\cdot k_2\|_{q_2} \leqq M_2\|f\cdot u_2\|_{p_2}. \tag{4.2}$$

Let $k = k_1^{1-t}\cdot k_2^t$, $u = u_1^{1-t}\cdot u_2^t$. Then we may conclude that T may be uniquely extended to a linear transformation on functions f, for which $\|f\cdot u\|_p < \infty$, so that

$$\|(Tf)\cdot k\|_q \leqq M_t\|f\cdot u\|_p \tag{4.3}$$

with $M_t = M_1^{1-t}\cdot M_2^t$.

5. **Proof of Theorem 2.** By (4.1) and (4.2) T has a unique extension to functions f, where either $\|f\cdot u_1\|_{p_1} < \infty$, or $\|f\cdot u_2\|_{p_2} < \infty$.

Now let E be any measurable subset of M, on which $u_1 \geqq \epsilon$, and $u_2 \geqq \epsilon$ ($\epsilon > 0$). Let T' be the linear operator defined on functions f on M which vanish outside E by $T'(f) = T(f)$, whenever the right side has meaning.

Now let $U_z(f) = k_1^{1-z}k_2^z T'(f\cdot u_1^{z-1}\cdot u_2^{-z})$.

Then it is an easy matter to verify that $U_z(\cdot)$ verifies the conditions of Theorem 1, and that

$$\|U_{iy}(f)\|_{q_1} \leqq M_1\|f\|_{p_1} \quad \text{and} \quad \|U_{1+iy}(f)\|_{q_2} \leqq M_2\|f\|_{p_2} \tag{5.1}$$

whenever f is a simple function vanishing outside E.

(4) The right-hand sides of (4.1) or (4.2) may be infinite for some f.

Applying Theorem 1, we obtain

$$\|U_t(f)\|_q \leqq M_t\|f\|_p, \text{ for simple } f. \tag{5.2}$$

This is clearly equivalent to

$$\|(Tf)\cdot k\|_q \leqq M_t\|f\cdot u\|_p \tag{5.3}$$

whenever $f\cdot u$ is simple, and $f=0$ outside E.

By the arbitrariness of E, (5.3) holds whenever $f\cdot u$ is simple, and $\|f\cdot u\|_p<\infty$. Finally, by a well-known argument, T has a unique extension, to those f's where $\|f\cdot u\|_p<\infty$, and the proof is complete.

6. **Remarks.**

REMARK 1. M. Riesz's theorem is of course contained in Theorem 1 in the case that the family $T_z(\cdot)$ does not depend on z.

REMARK 2. Theorem 2 was given in a special case by I. I. Hirschman [6, p. 49]. For the proof of Theorem 2 the full strength of Hirschman's lemma is not needed, but the classical three-line lemma would be enough.

PART II. MEAN CONVERGENCE BELOW THE CRITICAL INDEX

7. **Bochner-Riesz means for multiple Fourier integrals.** Let E_n denote the euclidean space of dimension $n\geqq 2$. Let $x=(x_1, x_2, \cdots, x_n)$, and $\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ be representative points in E_n. We denote, as usual, by dx the element of n-dimensional euclidean measure. Let also $x\cdot\xi=x_1\xi_1+x_2\xi_2\cdots+x_n\xi_n$ and $|x|=(x_1^2+x_2^2\cdots x_n^2)^{1/2}$.

Let $f(x)\in L_1(E_n)$, and let $F(\xi)=1/(2\pi)^{n/2}\int_{E_n}e^{-ix\cdot\xi}f(x)dx$. Finally let,

$$S_R^\delta(f) = \int_{|\xi|\leqq R}[1-(|\xi|/R)^2]^\delta e^{ix\cdot\xi}F(\xi)d\xi. \tag{7.1}$$

(7.1) has the equivalent form

$$S_R^\delta(f) = \int_{E_n} K_R^\delta(x-u)f(u)du \tag{7.2}$$

where(5)

$$K_R^\delta(x) = c_\delta R^n K^\delta(R|x|), \tag{7.3}$$

while:

$$K^\delta(\rho) = \frac{J_{\delta+n/2}(\rho)}{\rho^{\delta+n/2}}$$

and

(5) Formulae (7.1), (7.2) and (7.3) may be found in [2, pp. 176–177] for $\delta>(n-1)/2$. If $f(x)$ is simple, the formulae are meaningful for $R(\delta)\geqq 0$, and are valid by analytic continuation.

$$c_\delta = 2^{\delta - n/2+1}\Gamma(\delta + 1)/\Gamma(n/2).$$

We remark that if $f \in L_2(E_n)$ then $S_R^\delta(f)$, $\delta \geqq 0$, is always defined and $S_R^\delta(f) \in L_2(E_n)$.

The index $\delta = \kappa \equiv (n-1)/2$ is called, after Bochner, the *critical index*. Our aim is to prove the following theorem.

THEOREM 3. *Suppose that* $1 < p < 2$, *and that* R *is fixed, and that* $\delta > [(2/p) - 1]\kappa$. *Then* $S_R^\delta(f)$ *may be uniquely extended to a bounded operator taking* $L_p(E_n)$ *into itself. Furthermore,*

$$\|S_R^\delta(f)\|_p \leqq A_{p,\delta}\|f\|_p \tag{7.4}$$

where $A_{p,\delta}$ *does not depend on* R *and* f.

REMARK 1. If we now assume that $2 < p < \infty$, the conclusion of the theorem holds with $\delta > [(2/p') - 1]\kappa$. This may be proved directly as in Theorem 3, or may be deduced from Theorem 3 by noting that $S_R^\delta(f)$ is formally "self-adjoint."

REMARK 2. From Theorem 3 follows the following: Let $f \in L_p(E_n)$, $1 < p < 2$. Then $S_R^\delta(f) \rightarrow f$ in L_p mean as $R \rightarrow \infty$, whenever $\delta > [(2/p) - 1]\kappa$. Remark 2 follows from Theorem 3 by the Banach-Steinhaus "uniform boundedness" theorem [1, p. 79], and the fact that $S_R^\delta(f) \rightarrow f$ in L_p mean whenever f is smooth enough and vanishes outside a bounded set. An analogous statement holds if $2 < p < \infty$.

8. **Proof of Theorem 3.** For a fixed R, we shall consider the family of operators $S_R^\delta(f)$, depending on the parameter δ. Let us pick an $\epsilon > 0$, keep it fixed, and define

$$T_z(f) = S_R^{(\kappa+\epsilon)z}(f). \tag{8.1}$$

We notice first that

$$\|T_{iy}(f)\|_2 \leqq \|f\|_2. \tag{8.2}$$

In fact, (8.2) follows from the fact that $T_{iy}(f)$ is affected by multiplying the Fourier transform of f by $[1 - (|\xi|/R)^2]^{i(\kappa+\epsilon)y}$, which is bounded by 1 in absolute value.

Next we notice that $T_z(\cdot)$ is an analytic family; it is of admissible growth because

$$\left| \int T_z(\psi)\phi dy \right| \leqq \|\psi\|_2\|\phi\|_2, \qquad R(z) \geqq 0, \tag{8.3}$$

by Plancherel's theorem.

All that remains to be done in order to apply Theorem 1 to the family $T_z(\cdot)$ is to prove an estimate

$$\|T_{1+iy}(f)\|_1 \leqq A_\epsilon(y)\|f\|_1 \tag{8.4}$$

where the growth of $A_\epsilon(y)$ is sufficiently restricted. However

$$T_{1+iy}(f) = \int_{E_n} K_R^{1+iy}(u)f(x-u)du.$$

It is therefore sufficient to estimate

$$\int_{E_n} \left| K_R^{1+iy}(u) \right| du, \tag{8.5}$$

which by (7.3) is equal to

$$|c_1| \cdot \int_0^\infty \left| [J_{\sigma(y)}(\rho)]/\rho^{\sigma(y)} \right| \rho^{n-1}d\rho \tag{8.6}$$

where we have set:

$$\begin{aligned} \sigma(y) &= (\kappa + \epsilon)(1 + iy) + n/2, \\ c_1 &= \left\{2^{\sigma(y)-n+1}\Gamma\left[\sigma(y) - (n/2) + 1\right]\right\}/\Gamma(n/2). \end{aligned} \tag{8.7}$$

We now quote the following known estimate in the theory of Bessel functions(6).

LEMMA. *If ν and y are real, $\nu > -1/2$, and $\rho \geqq 1$, then:*

$$\left| J_{\nu+iy}(\rho) \right| \leqq A_\nu e^{2\pi|y|}/\rho^{1/2} \tag{8.8}$$

where A_ν does not depend on y and ρ.

Applying (8.8), we deduce that the integrand of (8.6) is bounded by

$$\left[A \cdot e^{2\pi(\kappa+\epsilon)|y|}\right]/\rho^{1+\epsilon} \tag{8.9}$$

if $\rho \geqq 1$.

While by Plancherel's theorem we clearly have

$$\int_0^\infty \left| J_{\sigma(y)}(\rho)/\rho^{\sigma(y)} \right|^2 \rho^{n-1}d\rho \leqq A. \tag{8.10}$$

Applying (8.10) when $0 \leqq \rho < 1$, and (8.9) when $\rho \geqq 1$, we get as an estimate for (8.6) [and therefore for (8.4)] the following:

$$A_\epsilon(y) \leqq (A/\epsilon) \cdot e^{2\pi(\kappa+\epsilon)|y|}. \tag{8.11}$$

Let us remember that ϵ was fixed, arbitrary, but $\epsilon > 0$.

Because of (8.11) the transformation $T_z(f)$ satisfies the conditions of Theorem 1 with $p_1 = q_1 = 2$, and $p_2 = q_2 = 1$. The result of applying Theorem 1 is:

(6) The proof of the lemma may be found in [12, pp. 217–218].

(8.12) $$\|T_t(f)\|_p \leqq A_t\|f\|_p$$

where $1/p=(1-t)\cdot 1/2+t$, and $1<p<2$.

However

$$T_t(f) = S_R^{(\kappa+\epsilon)t}(f), \qquad \text{while } \epsilon > 0,$$

but otherwise arbitrary. It is clear, therefore, that (8.12) contains exactly the statement of the theorem.

9. **Bochner-Riesz means for multiple Fourier series.** Let I_n denote the fundamental cube in E_n, i.e. the set of $x\in E_n$ so that $-\pi\leqq x_i\leqq\pi$. If $N=(N_1, \cdots, N_n)$ denotes an integral component vector we shall denote by $N\cdot x=N_1x_1+\cdots+N_nx_n$. As in the case of Fourier integrals we define

(9.1) $$S_R^\delta(f) = \sum_{|N|\leqq R} a_N[1-(|N|/R)^2]^\delta e^{iN\cdot x}$$

where

$$a_N = 1/(2\pi)^{n/2}\int_{I_n} e^{-iN\cdot x}f(x)dx.$$

Then the following theorem holds.

THEOREM 4.

$$\|S_R^\delta(f)\|_p \leqq A_{p,\delta}\|f\|_p$$

whenever $\delta>[(2/p)-1]\kappa$, *if* $1<p<2$, *or* $\delta>[(2/p')-1]\kappa$, *if* $2<p<\infty$. [*Again* $\kappa=(n-1)/2$.] $A_{p,\delta}$ *is independent of f and R. Moreover.* $S_R^\delta(f)\to f$ *in* L_p *mean, whenever* $f\in L_p$, *and the above conditions on* δ *are satisfied.*

The proof of Theorem 4 is exactly the same as the proof of Theorem 3.

PART III. PITT'S THEOREM FOR ORTHONORMAL SYSTEMS

10. **Pitt's theorem and its generalization.** Let $f(x)$ be defined on $(-\pi, +\pi)$ and $f(x)\in L_1(-\pi, +\pi)$; let

$$a_n = \int_{-\pi}^{+\pi} f(x)e^{-inx}dx.$$

Pitt has proved the following [7]:

THEOREM.

$$\left[\sum_{n=-\infty}^{n=+\infty} |a_n|^q(|n|+1)^{-\lambda q}\right]^{1/q} \leqq A\left[\int_{-\pi}^{+\pi}|f(x)|^p|x|^{\alpha p}dx\right]^{1/p}$$

whenever $0\leqq\alpha<1/p'$, $q\geqq p$, *and,* $\lambda=1/q+1/p-1+\alpha\geqq 0$.

The following special cases are noteworthy:

(i) when $\alpha=\lambda=0$. Then $q=p'$, and we have the inequality of Hausdorff-Young.

(ii) when $q=p$ and $\alpha=0$. Then we have the inequality of Hardy and Littlewood; similarly for $q=p$ and $\lambda=0$.

Our aim is to prove the following:

THEOREM 5. *Let $\{\phi_n(x)\}$ be an orthonormal system, taken over the interval $(0, h)$ with standard Lebesgue measure. Assume $|\phi_n(x)| \leqq M$. For any $f(x) \in L_1(0, h)$, set $c_n=\int_0^h f(x)\bar{\phi}_n(x)dx$. Then*

$$(10.1) \qquad \left[\sum_{n=1}^{\infty} |c_n|^q n^{-\lambda q}\right]^{1/q} \leqq A\left[\int_0^h |f(x)|^p x^{\alpha p} dx\right]^{1/p}$$

whenever $0\leqq\alpha<1/p'$, $q\geqq p$, and $\lambda=1/q+1/p-1+\alpha\geqq 0$. A depends only on M, p, q, λ, and α.

REMARK. The special case when $q=p$ and $\alpha=0$ is of course a well-known result of R. E. A. C. Paley; similarly when $q=p$ and $\lambda=0$([7]). The case when $q=p=2$ has recently been obtained by I. I. Hirschman [4]. We shall make use of Paley's results in proving Theorem 5.

We may strengthen (10.1) by increasing the left-hand side, and decreasing the right-hand side in the following familiar manner:

We first rearrange the index of the system $\{\phi_n(x)\}$ so that $\{|c_n|\}$ becomes a nonincreasing set. Next we subject the interval $(0, h)$ to a measure preserving transformation so that $|f(x)|$ becomes nonincreasing.

In this manner our orthonormal system $\{\phi_n(x)\}$ becomes transformed into another orthonormal system $\{\theta_n(x)\}$ and $f(x)$ into another function $\bar{f}(x)$. Applying Theorem 5 to $\{\theta_n(x)\}$ and $\bar{f}(x)$ we get

COROLLARY. *If c_n^* denotes a nonincreasing rearrangement of $|c_n|$, and $f^*(x)$ denotes the function equimeasurable with $|f(x)|$ and nonincreasing, then using the notation of Theorem 5, we have:*

$$(10.2) \qquad \left[\sum_{n=1}^{\infty} (c_n^*)^q n^{-\lambda q}\right]^{1/q} \leqq A\left[\int_0^h [f^*(x)]^p x^{\alpha p} dx\right]^{1/p}$$

where $0\leqq\alpha<1/p'$, $q\geqq p$ and $\lambda=1/q+1/p-1+\alpha\geqq 0$.

REMARK. We should point out that both (10.2) and (10.1) have analogues when the roles of $f(x)$ and c_n are interchanged. The proofs of these analogous statements are completely parallel with the proofs of (10.1) and (10.2) proper.

11. **Proof of Theorem 5.** In order to simplify the argument we shall break up the proof into several special cases. This piecemeal approach will also indicate the more interesting special cases involved. In what follows, expres-

([7]) For proofs of Paley's theorems, see [13, pp. 202–207].

sions like A_p will denote general constants depending only on the parameters indicated.

Basic to the proof are the following two inequalities of Paley.

$$(11.1)\qquad \left(\sum |c_n|^r n^{r-2}\right)^{1/r} \leqq A_r\left(\int_0^h |f(x)|^r dx\right)^{1/r}, \qquad 1 < r \leqq 2,$$

and

$$(11.2)\qquad \left(\sum |c_n|^s\right)^{1/s} \leqq A_s\left(\int_0^h |f(x)|^s x^{s-2} dx\right)^{1/s}, \qquad 2 \leqq s < \infty.$$

Case 1, $p=q=2$. Choose r and s, so that $1/r+1/s=1$, and $(2-r)/2r=\alpha$, where $0\leqq\alpha<1/2$. Now apply Theorem 2 to the case where $q_1=p_1=r$, $q_2=p_2=s$, $k_1=n^{(r-2)/r}$, $k_2=1$, $u_1=1$, $u_2=x^{(s-2)/s}$, and $t=1/2$. The result is

$$(11.3)\qquad \left(\sum_{n=1}^{\infty} |c_n|^2 n^{-2\alpha}\right)^{1/2} \leqq A_\alpha\left(\int_0^h |f(x)|^2 x^{2\alpha} dx\right)^{1/2}, \quad 0 \leqq \alpha < 1/2.$$

Case 2, $1<p=q\leqq 2$. Rewrite (11.3) as

$$(11.4)\qquad \left(\sum |c_n|^2 n^{-2\beta}\right)^{1/2} \leqq A_\beta\left(\int_0^h |f(x)|^2 x^{2\beta} dx\right)^{1/2}, \qquad 0 \leqq \beta < 1/2.$$

Suppose p, α, and λ are given, with $1<p\leqq 2$, $\alpha<1/p'$, and $\lambda=2/p-1+\alpha \geqq 0$.

Choose $1/r=(1/p)+\alpha$, $\beta=(1/p)+\alpha-1/2$, and $t=\alpha/(1/p+\alpha-1/2)$. Then since $(2/p)-1+\alpha\geqq 0$, we have $0\leqq\beta<1/2$, and $0\leqq t\leqq 1$. Now interpolate between (11.1) and (11.4). The result is

$$(11.5)\qquad \left(\sum |c_n|^p n^{-\lambda p}\right)^{1/p} \leqq A_{p,\alpha}\left(\int_0^h |f(x)|^p x^{\alpha p} dx\right)^{1/p}$$

for $1<p\leqq 2$, $\alpha<1/p'$, and $\lambda=(2/p)-1+\alpha\geqq 0$.

Case 3, $2\leqq p=q<\infty$. The argument is exactly parallel with that of Case 2, except we make use of (11.2) instead of (11.1).

Case 4, $q=p'$, $1\leqq p\leqq 2$. Here we make use of the inequality

$$(11.6)\qquad |c_n| \leqq M\int_0^h |f|\, dx.$$

Choose t, so that $1/p=(1-t)+t/2$ and $\beta t=\alpha$. Now interpolate between (11.6) and (11.4). The result is

$$(11.7)\qquad \left(\sum_{n=1}^{\infty} |c_n|^{p'} n^{-\alpha p'}\right)^{1/p'} \leqq A_{p,\alpha}\left(\int_0^h |f(x)|^p x^{\alpha p} dx\right)^{1/p}$$

where $1\leqq p\leqq 2$, $0\leqq\alpha<1/p'$.

CASE 5, $1<p\leqq 2$. Rewrite (11.5) as follows

$$\left(\sum |c_n|^p n^{-\mu p}\right)^{1/p} \leqq A_{p,\alpha}\left(\int_0^h |f(x)|^p x^{\alpha p} dx\right)^{1/r} \tag{11.8}$$

with $1<p\leqq 2$, $0\leqq\alpha<1/p'$, $\mu=(2/p)-1+\alpha\geqq 0$. Choose t, so that $1/q=(1-t)\cdot 1/p'+t/p$ and interpolate between (11.7) and (11.8). The result is

$$\left(\sum |c_n|^q n^{-\lambda q}\right)^{1/q} \leqq A_{p,q,\alpha}\left(\int_0^h |f(x)|^p x^{\alpha p} dx\right)^{1/p} \tag{11.9}$$

with $1<p\leqq 2$, $q\geqq p$, $0\leqq\alpha<1/p'$, and $\lambda=(1/p)+(1/q)-1+\alpha\geqq 0$.

CASE 6, $2\leqq p<\infty$. The proof is exactly parallel to that of Case 5, except that instead of (11.7) we use

$$\left(\sum |c_n|^q n^{-\lambda q}\right)^{1/q} \leqq A_{q,\lambda}\left(\int_0^h |f(x)|^{q'} x^{\lambda q'} dx\right)^{1/q'} \tag{11.10}$$

which is allowable since $q\geqq 2$.

Cases 5 and 6 together contain the full statement of Theorem 5.

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