ESTIMATES OF THE MODELING ERROR GENERATED BY HOMOGENIZATION OF AN ELLIPTIC BOUNDARY VALUE PROBLEM

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Abstract.

Contents

1. The Homogenization Problem

We follow [JKO94, pp. 1–30].

Definition 1.1. *Let $n \in \mathbb{Z}$, $n \geq 1$, and $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. A vector $v \in (L^1(\Omega))^n$ is said to be the* ***gradient of a function*** *$u \in L^1(\Omega)$ if*

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}\lambda = -\int_\Omega v_i \varphi \,\mathrm{d}\lambda \tag{1}$$

for all $\varphi \in \mathscr{C}_0^\infty(\Omega)$ and $i = 1, \ldots, n$. The gradient v of u is denoted by ∇u.

Define the Sobolev space $H^1(\Omega)$ by

$$H^1(\Omega) := \{u \in L^2(\Omega) : \nabla u \in (L^2(\Omega))^n\}. \tag{2}$$

$H^1(\Omega)$ equipped with the inner product

$$\langle u_1, u_2 \rangle_{H^1(\Omega)} := \int_\Omega u_1 u_2 \,\mathrm{d}\lambda + \sum_{k=1}^n \int_\Omega \frac{\partial u_1}{\partial x_k} \frac{\partial u_2}{\partial x_k} \,\mathrm{d}\lambda \tag{3}$$

is then a Hilbert space. We are mainly interested in the subspace

$$H_0^1(\Omega) := \overline{\mathscr{C}_0^\infty(\Omega)} \subseteq H^1(\Omega). \tag{4}$$

The dual of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$. We have that $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$.

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Definition 1.2. *Let $n \in \mathbb{Z}$, $n \geq 1$, $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and $p \in (L^2(\Omega))^n$. The* ***divergence of*** *p, written $\operatorname{div} p$, is defined to be an element of $H^{-1}(\Omega)$ satisfying*

$$\langle \operatorname{div} p, \varphi \rangle_{H^1(\Omega)} = -\int_\Omega \langle p, \nabla \varphi \rangle_{\mathbb{R}^n} \, \mathrm{d}\lambda \tag{5}$$

for all $\varphi \in H_0^1(\Omega)$.

Definition 1.3. *Let $n \in \mathbb{Z}$, $n \geq 1$. A constant positive definite matrix A_0 is said to be the* ***homogenized matrix*** *for a periodic matrix $A \in M_n(L^\infty(\mathbb{R}^n))$ satisfying the condition of ellipticity*

$$a_{ij}(x) y_i y_j \geq c\, |y|^2 \qquad \forall x, y \in \mathbb{R}^n \tag{6}$$

and some $c > 0$, if for any bounded domain $\Omega \subseteq \mathbb{R}^n$ and any $f \in H^{-1}(\Omega)$ the solutions u_ε of the Dirichlet problem

$$\operatorname{div}(A_\varepsilon \nabla u_\varepsilon) = f \in H^{-1}(\Omega), \qquad u_\varepsilon \in H_0^1(\Omega), \tag{7}$$

where $A_\varepsilon(x) := A(x/\varepsilon)$ possess the following propertiy of convergence:

$$u_\varepsilon \xrightharpoonup{H_0^1(\Omega)} u_0, \qquad A_\varepsilon \nabla u_\varepsilon \xrightharpoonup{L^2(\Omega)} A_0 \nabla u_0, \tag{8}$$

as $\varepsilon \to 0$, where u_0 is the solution of the Dirichlet problem

$$\operatorname{div}(A_0 \nabla u_0) = f, \qquad u_0 \in H_0^1(\Omega). \tag{9}$$

Theorem 1.1. *Let $A \in M_n(L^\infty(\mathbb{R}^n))$ be a periodic matrix satisfying the ellipticity condition (6). Consider the auxiliary periodic problem*

$$\operatorname{div}(Av) = 0, \qquad v \in L^2_{\text{pot}}(\square), \qquad \langle v \rangle = \alpha \in \mathbb{R}^n \tag{10}$$

and define A_0 by

$$\langle Av \rangle = A_0 \alpha. \tag{11}$$

Then A_0 is a homogenized matrix for A.

Proof. See [JKO94, p. 19]. ☐

From the proof of theorem 1.1 further follows that

$$A_0 = \langle A(I + \nabla N) \rangle \tag{12}$$

where ∇N is the matrix with columns $\nabla N_1, \ldots, \nabla N_n$ which are solutions of

$$\operatorname{div}(A(e_k + \nabla N_k)) = 0, \qquad N_k \in H^1(\square) \tag{13}$$

for $k = 1, \ldots, n$.

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Example 1.1. Let $n := 1$ and $\Omega := (0,1)$. Furthermore, for $\varepsilon > 0$ define

$$A_\varepsilon(x) := 2 + \cos\left(\frac{2\pi x}{\varepsilon}\right) \tag{14}$$

and let

$$f(x) := e^{10x}. \tag{15}$$

From (14) we deduce

$$A(x) = A_\varepsilon(\varepsilon x) = 2 + \cos(2\pi x). \tag{16}$$

Now we have to solve

$$(AN')' = A' \tag{17}$$

for the periodic solution N such that $\int_0^1 N = 0$. Integrating (17) yields

$$N'(x) = 1 + \frac{c}{A(x)} \tag{18}$$

for some constant $c \in \mathbb{R}$. Hence

$$N(x) = \int_0^x N'(t)\,\mathrm{d}t = \int_0^x \left(1 + \frac{c}{A(t)}\right)\mathrm{d}t \tag{19}$$

and from the periodicity requirement $N(0) = N(1)$ we get that

$$1 = -c\int_0^1 \frac{\mathrm{d}x}{A(x)}. \tag{20}$$

Using [FL03, p. 170] yields

$$\int_0^1 \frac{\mathrm{d}x}{A(x)} = \int_0^1 \frac{\mathrm{d}x}{2+\cos(2\pi x)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}y}{2+\cos(y)} = 2\sum_{|\zeta|<1} \operatorname{res}_\zeta\left(\frac{1}{z^2+4z+1}\right) = \frac{1}{\sqrt{3}}.$$

Hence

$$\boxed{N'(x) = 1 - \frac{\sqrt{3}}{2+\cos(2\pi x)}.} \tag{21}$$

and

$$\boxed{N(x) = \int_0^x \left(1 - \frac{\sqrt{3}}{2+\cos(2\pi t)}\right)\mathrm{d}t.} \tag{22}$$

since

$$\int_0^1 N(x)\,\mathrm{d}x = \frac{1}{2}\int_0^1 \left[\int_0^1 \left(1 - \frac{\sqrt{3}}{2+\cos(2\pi t)}\right)\mathrm{d}t\right]\mathrm{d}x = 0 \tag{23}$$

by the invariance of the integrand under the mapping $t \mapsto 1-t$. Furthermore

$$A_0 = \int_0^1 A(x)\,\mathrm{d}x - \int_0^1 A(x)N'(x)\,\mathrm{d}x = \int_0^1 A(x)\,\mathrm{d}x - \int_0^1 A(x)\left(1 - \frac{\sqrt{3}}{A(x)}\right)\mathrm{d}x = \sqrt{3}.$$

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Then we have to solve

$$(A_0 u_0')' = -f. \tag{24}$$

Integration with respect to x yields

$$u_0'(x) = -\frac{1}{\sqrt{3}}\frac{1}{10}e^{10x} + c_1 \tag{25}$$

for some constant $c_1 \in \mathbb{R}$ and thus

$$u_0(x) = -\frac{1}{\sqrt{3}}\frac{1}{100}e^{10x} + c_1 x + c_2 \tag{26}$$

for some constant $c_2 \in \mathbb{R}$. Since $u_0 \in H_0^1(\Omega)$, we have that $u_0(0) = u_0(1) = 0$ and thus we get the linear system

$$-\frac{1}{\sqrt{3}}\frac{1}{100} + c_2 = 0 \qquad\qquad -\frac{1}{\sqrt{3}}\frac{1}{100}e^{10} + c_1 + c_2 = 0. \tag{27}$$

Solving (27) yields

$$\boxed{u_0(x) = \frac{1}{\sqrt{3}}\frac{1}{100}\left((e^{10}-1)x + 1 - e^{10x}\right).} \tag{28}$$

Since $\partial\Omega = \{0,1\}$ it is easily seen that

$$\psi_\varepsilon(x) = \min\left(1, \varepsilon^{-1}\min(x, 1-x)\right) \tag{29}$$

and so

$$\boxed{w_\varepsilon^1(x) = u_0(x) - \varepsilon \min\left(1, \varepsilon^{-1}\min(x, 1-x)\right) N(\varepsilon^{-1}x) u_0'(x) \qquad x \in \Omega.} \tag{30}$$

Lastly we have to solve

$$(A_\varepsilon u_\varepsilon')' = -f. \tag{31}$$

Integration with respect to x yields

$$u_\varepsilon'(x) = \frac{1}{A_\varepsilon(x)}\left(c - \frac{1}{10}e^{10x}\right) \tag{32}$$

for some $c \in \mathbb{R}$. Since $u_\varepsilon \in H_0^1(\Omega)$, we have that

$$u_\varepsilon(x) = \int_0^x \frac{1}{A_\varepsilon(t)}\left(c - \frac{1}{10}e^{10t}\right) \mathrm{d}t \tag{33}$$

and from $u_\varepsilon(1) = 0$ we deduce

$$\boxed{u_\varepsilon(x) = \frac{1}{10}\int_0^x \frac{1}{A_\varepsilon(t)}\left(\int_0^1 \frac{e^{10s}}{A_\varepsilon(s)}\,\mathrm{d}s \left(\int_0^1 \frac{\mathrm{d}s}{A_\varepsilon(s)}\right)^{-1} - e^{10t}\right)\mathrm{d}t.} \tag{34}$$

In figure 1 we see the approximations of u_ε by w_ε^1. One observes that the approximation is even good for a relatively large choice of ε.

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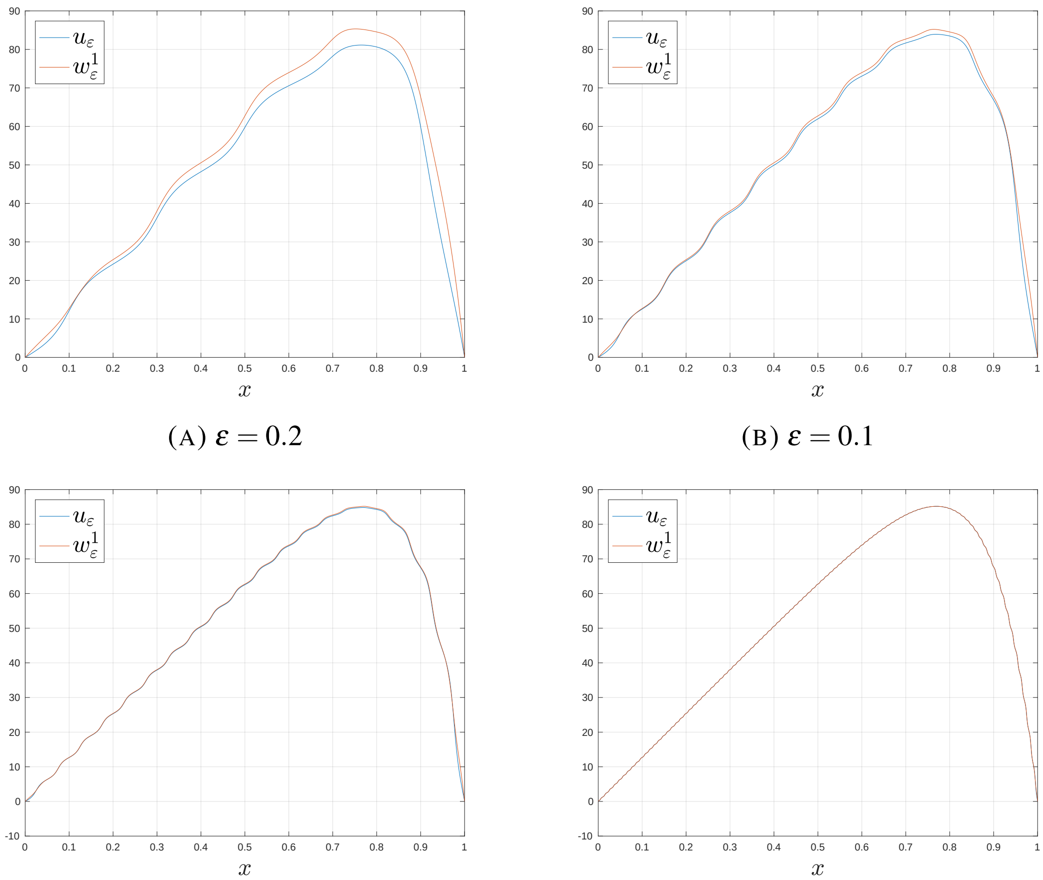

(A) $\varepsilon = 0.2$ (B) $\varepsilon = 0.1$

(C) $\varepsilon = 0.05$ (D) $\varepsilon = 0.01$

FIGURE 1. Plot of the approximations w_ε^1 of u_ε for certain values of ε.

Now we want to numerically check the error estimate

$$\|u_\varepsilon - w_\varepsilon^1\|_{H^1(\Omega)} \leq c\sqrt{\varepsilon}. \tag{35}$$

For that we have to calculate $(w_\varepsilon^1)'$ which involves ψ_ε'. This is the only difficult task. Using the notion of weak derivatives and $\min(f,g) = f + g - |f - g|$ we get that

$$\boxed{\psi_\varepsilon'(x) = -\frac{1}{2\varepsilon}\operatorname{sgn}(2x-1)\left(1 + \operatorname{sgn}\left(1 - \frac{1}{2\varepsilon}(1 - |2x-1|)\right)\right).} \tag{36}$$

In figure 2 we see the approximation of u_ε' by $(w_\varepsilon^1)'$ for certain values of ε. One observes that the approximations are not as quite as good as the one of u_ε by w_ε^1.

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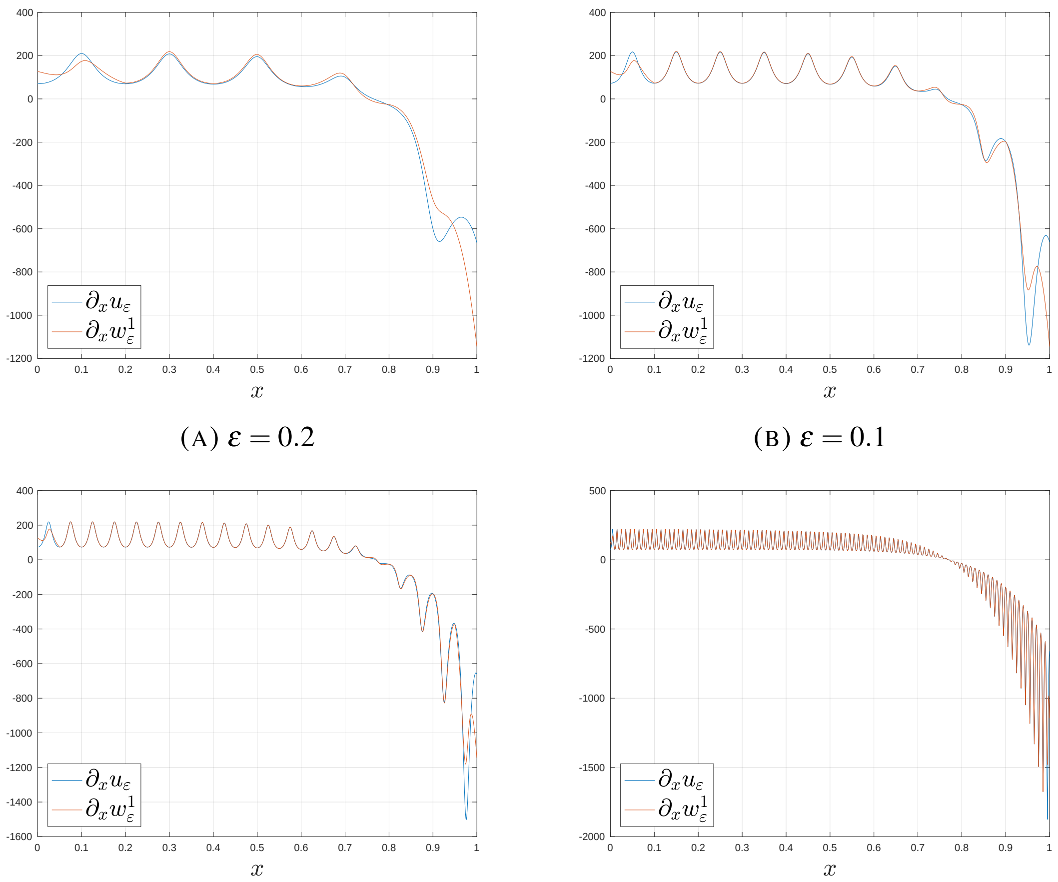

(A) $\varepsilon = 0.2$ (B) $\varepsilon = 0.1$

(C) $\varepsilon = 0.05$ (D) $\varepsilon = 0.01$

FIGURE 2. Plot of the approximations $(w_\varepsilon^1)'$ of u_ε' for certain values of ε.

Appendix A. Listings

```
%Definitions
f = @(x) exp(10 * x);
A_epsilon = @(x, epsilon) 2 + cos(2 * pi * x/epsilon);
dN = @(x) 1 - sqrt(3)./(2 + cos(2 * pi * x));
N = @(x) integral(dN,0,x);
u_0 = @(x) 1/sqrt(3) * 1/100 * ((exp(10) - 1 ) * x + 1 - f(x));
du_0 = @(x) 1/sqrt(3) * 1/100 * (exp(10) - 1 - 10 * f(x));
ddu_0 = @(x) -1/sqrt(3) * f(x);
psi_epsilon = @(x, epsilon) min(1,min(x,1-x)/epsilon);
dpsi_epsilon = @(x, epsilon) -1/(2 * epsilon) * sign(2 * x - 1) * (1 + ...
    sign(1 - 1/(2 * epsilon) * (1 - abs(2 * x - 1))));
w_1_epsilon = @(x, epsilon) u_0(x) - epsilon * psi_epsilon(x, epsilon) ...
```

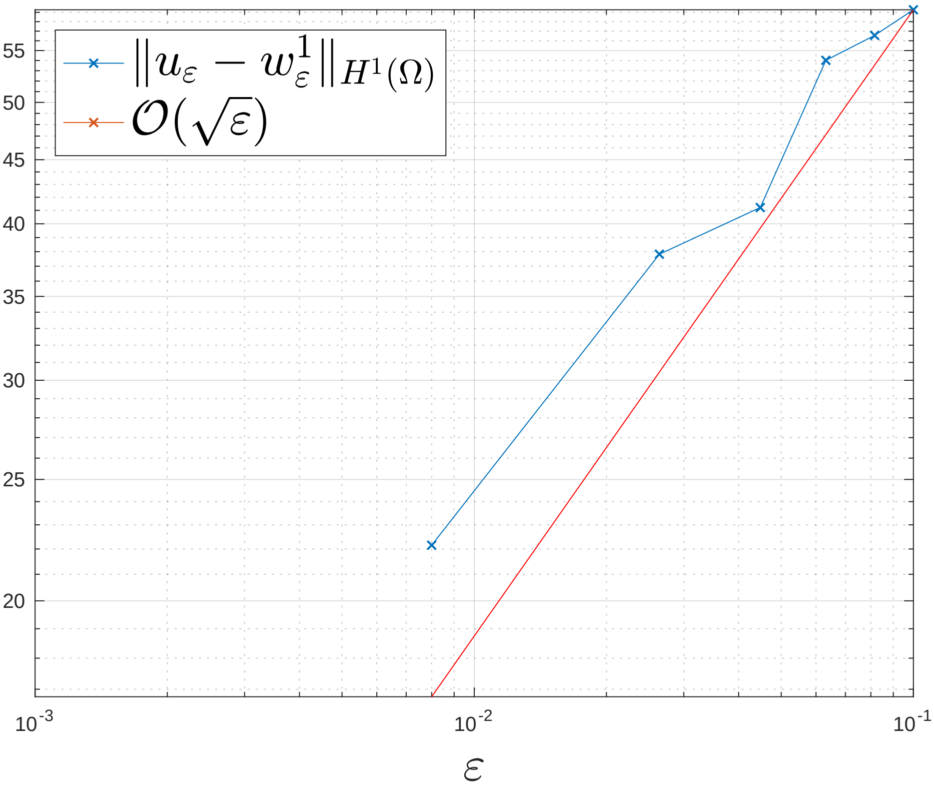

FIGURE 3. Plot of the error $\|u_\varepsilon - w_\varepsilon^1\|_{H^1(\Omega)}$.

```
        * N(x/epsilon) * du_0(x);
t_1 = @(x,epsilon) dpsi_epsilon(x,epsilon) * N(x/epsilon) * du_0(x);
t_2 = @(x,epsilon) 1/epsilon * psi_epsilon(x,epsilon) * dN(x/epsilon) * du_0(x)
    ;
t_3 = @(x,epsilon) psi_epsilon(x,epsilon) * N(x/epsilon) * ddu_0(x);
dw_1_epsilon = @(x,epsilon) du_0(x) - epsilon * (t_1(x,epsilon) + ...
    t_2(x,epsilon) + t_3(x,epsilon));
c = @(epsilon) integral(@(t) f(t)./A_epsilon(t,epsilon),0,1) * ...
    1/integral(@(t) 1./A_epsilon(t,epsilon),0,1);
u_epsilon = @(x,epsilon,c) 1/10 * integral(@(t) 1./A_epsilon(t,epsilon) * ...
    (c - f(t)),0,x,'ArrayValued',true);
du_epsilon = @(x,epsilon,c) 1/10 * 1/A_epsilon(x,epsilon) * (c - f(x));

%Parameters
x = linspace(0,1,1e+3);
epsilon = 1e-2;

%Windows
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```
f1 = figure;
f2 = figure;
f3 = figure;

%Plot of the approximation of u_eps by w_1_eps
c = c(epsilon);
y_1 = arrayfun(@(x) u_epsilon(x,epsilon,c),x);
y_2 = arrayfun(@(x) w_1_epsilon(x,epsilon),x);
figure(f1);
plot(x,y_1)
hold on;
plot(x,y_2)
grid on;
xlabel('$x$', 'interpreter', 'latex', 'fontsize', 24);
l1 = legend('$u_\varepsilon$','$w_\varepsilon^1$');
set(l1, 'fontsize', 24, 'interpreter', 'latex','Location','northwest');
fig1 = gcf;
set(fig1,'Units','centimeters');
pos1 = get(fig1,'Position');
set(fig1,'PaperPositionMode','Auto','PaperUnits','centimeters','PaperSize',[
    pos1(3), pos1(4)])
print(fig1,'u_eps_w_1_eps_4','-dpdf','-r0')

%Plot of the approximation of du_eps by dw_1_eps
dy_1 = arrayfun(@(x) du_epsilon(x,epsilon,c),x);
dy_2 = arrayfun(@(x) dw_1_epsilon(x,epsilon),x);
figure(f2);
plot(x,dy_1);
hold on;
plot(x,dy_2);
grid on;
xlabel('$x$', 'interpreter', 'latex', 'fontsize', 24);
l2 = legend('$\partial_x u_\varepsilon$','$\partial_x w_\varepsilon^1$');
set(l2, 'fontsize', 24, 'interpreter', 'latex','Location','southwest');
fig2 = gcf;
set(fig2,'Units','centimeters');
pos2 = get(fig2,'Position');
set(fig2,'PaperPositionMode','Auto','PaperUnits','centimeters','PaperSize',[
    pos2(3), pos2(4)])
print(fig2,'du_eps_dw_1_eps_4','-dpdf','-r0')

%Error
epsilon_array = linspace(1e-1,8e-3,6);
n = length(epsilon_array);
error = zeros(n);
for i = 1:n
norm_1 = integral(@(x) (u_epsilon(x,epsilon_array(i),c) - w_1_epsilon(x,
    epsilon_array(i)))^2,0,1,'ArrayValued',true);
norm_2 = integral(@(x) (du_epsilon(x,epsilon_array(i),c) - dw_1_epsilon(x,
    epsilon_array(i)))^2,0,1,'ArrayValued',true);
norm = sqrt(norm_1 + norm_2);
```

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```
error(i) = norm;
end

figure(f3);
loglog(epsilon_array,error,'-x');
hold on;
loglog(epsilon_array,error(1)/epsilon_array(1)^.5 * epsilon_array.^.5,'color','red')
grid on;
xlabel('$\varepsilon$', 'interpreter', 'latex', 'fontsize', 24);
l3 = legend('$\|u_\varepsilon - w_\varepsilon^1\|_{H^1(\Omega)}$','$\mathcal{O}(\sqrt{\varepsilon})$');
set(l3, 'fontsize', 24, 'interpreter', 'latex','Location','northwest');
fig3 = gcf;
set(fig3,'Units','centimeters');
pos3 = get(fig3,'Position');
set(fig3,'PaperPositionMode','Auto','PaperUnits','centimeters','PaperSize',[pos3(3), pos3(4)])
print(fig3,'error','-dpdf','-r0')
```

src/example.m

References

[FL03] Wolfgang Fischer and Ingo Lieb. *Funktionentheorie: Komplexe Analysis in einer Veränderlichen*. 8. Auflage. vieweg studium; Aufbaukurs Mathematik. Vieweg+Teubner Verlag, 2003.

[JKO94] V.V. Jikov, S.M. Kozlov, and O.A. Oleinik. *Homogenization of Differential Operators and Integral Functionals*. Springer-Verlag Berlin Heidelberg, 1994.