Notes on Smooth Manifolds

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CHAPTER 1

General Topology

1. Definitions and Basic Notions

Definition 1.1. *Let X be a set. A **topology** on X is a collection $\mathcal{T}$ of subsets of X satisfying the following properties:*

(*i*) $\varnothing, X \in \mathcal{T}$.
(*ii*) *If* $U, V \in \mathcal{T}$, *then* $U \cap V \in \mathcal{T}$.
(*iii*) *If* $(U_\alpha)_{\alpha \in A}$ *is a family of elements of* $\mathcal{T}$, *then* $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$.

*A **topological space** is a tuple $(X, \mathcal{T})$, where $\mathcal{T}$ is a topology on X. Elements of $\mathcal{T}$ are called **open sets**.*

Example 1.1 (Topologies). Let X be a set. The reader may verify that the the following collections are indeed topologies on X.

(a) The collection $\mathcal{P}(X)$ is a topology on X, called the ***discrete topology***.
(b) The collection $\{\varnothing, X\}$ is a topology on X, called the ***trivial topology***.
(c) Let $\mathcal{T}$ be a topology on X and let $S \subseteq X$. Then the collection $\{S \cap U : U \in \mathcal{T}\}$ is a topology on S, called the ***subspace topology***.
(d) The collection

$$\mathcal{T}_d := \{A \subseteq M : \forall x \in A \exists r > 0 \text{ such that } B_r(x) \subseteq A\} \tag{1}$$

is a topology on M, called the ***metric topology induced by the metric d***.

Definition 1.2. *Let X be a topological space and $x \in X$. An open set U is called a **neighbourhood of x** if $x \in U$.*

Definition 1.3. *Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$. The **closure of A in X**, denoted by $\overline{A}$, is defined by*

$$\overline{A} := \bigcap \{B \subseteq X : A \subseteq B, B^c \in \mathcal{T}\}. \tag{2}$$

*The **interior of A in X**, denoted by $\operatorname{Int} A$, is defined by*

$$\operatorname{Int} A := \bigcup \{C \subseteq X : C \subseteq A, C \in \mathcal{T}\}. \tag{3}$$

Definition 1.4. *Let X be a topological space. X is called a **Hausdorff space** if given $p, q \in X$ with $p \neq q$ we find neighbourhoods U and V of p and q, respectively, such that $U \cap V = \varnothing$*

Definition 1.5. *Let $(X, \mathcal{T})$ be a topological space. A collection $\mathcal{B}$ of subsets of X is called a **basis for the topology of X** if the following two conditions hold:*

(*i*) $\mathcal{B} \subseteq \mathcal{T}$.
(*ii*) *For any* $U \in \mathcal{T}$ *we have* $U = \bigcup_{\alpha \in A} B_\alpha$ *where* $B_\alpha \in \mathcal{B}$ *for any* $\alpha \in A$.

Definition 1.6. *Let X be a topological space. X is called* ***second countable*** *if there exists a countable basis for the topology of X.*

Definition 1.7. *Let X and Y be two topological spaces and $f : X \to Y$. The map f is said to be* ***continuous*** *if for any open set $U \subseteq Y$ we have that $f^{-1}(U)$ is open in X.*

Definition 1.8. *Let X be a topological space. If $X = U \cup V$ for some disjoint, nonempty, open sets $U, V \subseteq X$, X is called* ***disconnected****, otherwise X is said to be* ***connected****.*

Definition 1.9. *Let X be a topological space. An* ***open cover*** *of X is a family $(U_\alpha)_{\alpha \in A}$ of open subsets of X such that $X = \bigcup_{\alpha \in A} U_\alpha$. A* ***subcover*** *of $(U_\alpha)_{\alpha \in A}$ is a subfamily $(U_\beta)_{\beta \in B}$, $B \subseteq A$, such that $X = \bigcup_{\beta \in B} U_\beta$.*

Definition 1.10. *A topological space X is said to be* ***compact*** *if every open cover has a finite subcover.*

Definition 1.11. *Let X be a topological space. X is said to be* ***locally Euclidean of dimension n*** *if every point of X has a neighbourhood in X that is homeomorphic to an open subset of $\mathbb{R}^n$.*

Definition 1.12 (Topological Manifold). *An* ***n-dimensional topological manifold*** *is a second countable Hausdorff space that is locally Euclidean of dimension n.*

Definition 1.13. *Let M be a topological n-manifold. If (U, φ), (V, ψ) ae two charts such that $U \cap V \neq \varnothing$, the composite map*

$$\psi \circ \varphi : \varphi(U \cap V) \to \psi(U \cap V) \tag{4}$$

is called the ***transition map from φ to ψ****.*

Definition 1.14. *Let M be a topological n-manifold. A collection of charts $\{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ is said to be an* ***atlas of class $\mathscr{C}$ for M*** *if*

(i) $M = \bigcup_{\alpha \in A} U_\alpha$.
(ii) $\varphi_\alpha \circ \varphi_\beta \in \mathscr{C}$ *for all* $\alpha, \beta \in A$.

Definition 1.15 (Manifolds of Class $\mathscr{C}$). *Let M be a topological manifold. A* ***$\mathscr{C}$-structure on M*** *is an atlas $\mathcal{A} := \{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ of class $\mathscr{C}$ for M that has the following* ***maximality property****: if (U, φ) is a chart such that $\varphi \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi^{-1}$ are of class $\mathscr{C}$ for all $\alpha \in A$, then $(U, \varphi) \in \mathcal{A}$. The tuple $(M, \mathcal{A})$, where $\mathcal{A}$ is a $\mathscr{C}$-structure on M, is called a* ***manifold of class $\mathscr{C}$****. An atlas for M which has the maximality property is called a* ***maximal atlas of class $\mathscr{C}$ for M****.*

Proposition 1.1. *Let M be a topological manifold. Every atlas $\mathcal{A}$ of class $\mathscr{C}$ for M is contained in a unique maximal atlas of the same class $\mathscr{C}$ for M, called the* ***$\mathscr{C}$-structure determined by $\mathcal{A}$****.*

Definition 1.16 (Tangent Space). *A linear map $v : C^\infty(M) \to \mathbb{R}$ is called a **derivation at p** if it satisfies*

$$v(fg) = f(p)v(g) + g(p)v(f) \qquad \forall f, g \in C^\infty(M). \tag{5}$$

*The set of all derivations of $C^\infty(M)$ at p, denoted by T_pM, is a vector space called the **tangent space to M at p**.*

Lemma 1.1 (Properties of Tangent Vectors on Manifolds). *Suppose M is a smooth manifold, $p \in M$, $v \in T_pM$ and $f, g \in C^\infty(M)$.*

(a) If f is a constant function, then $v(f) = 0$.
(b) If $f(p) = g(p) = 0$, then $v(fg) = 0$.

Proof. Property (b) is immediate and (a) follows from

$$v(1) = v(1 \cdot 1) = 2v(1) \tag{6}$$

which implies $v(1) = 0$ and thus by linearity

$$v(\lambda) = v(\lambda \cdot 1) = \lambda v(1) = 0 \tag{7}$$

where $\lambda \in \mathbb{R}$. □

Definition 1.17 (Tangent Bundle). *Let M be a smooth manifold. The **tangent bundle of M**, denoted by TM, is defined to be*

$$TM := \coprod_{p \in M} T_pM. \tag{8}$$

*The mapping $\pi : TM \to M$ which is defined by $\pi(p, v) := p$ is called **projection map**.*

Definition 1.18 (Vector Fields). *Let M be a smooth manifold. A **vector field on M** is a section of the map $\pi : TM \to M$, i.e. a vector field is a continuous map $X : M \to TM$, usually written $p \mapsto X_p$, with the property that*

$$\pi \circ X = \mathrm{id}_M \tag{9}$$

*or equivalently, $X_p \in T_pM$ for every $p \in M$. We say that a vector field X is a **smooth vector field** if $X : M \to TM$ is a smooth map. The **set of all smooth vector fields on M** is denoted by $\mathfrak{X}(M)$.*

Proposition 1.2 (Lipschitz Estimate for C^1-functions).

Theorem 1.1 (Inverse Function Theorem). *Suppose U and V are open subsets of $\mathbb{R}^n$, and $F : U \to V$ is a $\mathscr{C}^1$-function. If $DF(a)$ is invertible at some point $a \in U$, then there exist connected neighborhoods $U_0 \subseteq U$ of a and $V_0 \subseteq V$ of $F(a)$ such that $F|_{U_0} : U_0 \to V_0$ is a $\mathscr{C}^1$-diffeomorphism.*

Proof. The proof is based on Banach fixed point theorem (see [Lee13, p. 657]). Since U is open, we find $\varepsilon > 0$ such that $B_\varepsilon(a) \subseteq U$. Hence define $\varphi : B_\varepsilon(0) \to \mathbb{R}^n$ by

$$\varphi(x) := F(x + a) - F(a) \tag{10}$$

Observe, that $\varphi \in \mathscr{C}^1$ with $D\varphi(0) = DF(a)$. Also the composition $D\varphi(0)^{-1} \circ \varphi$ is $\mathscr{C}^1$. Furthermore

$$D(D\varphi(0)^{-1} \circ \varphi)(0) = D(D\varphi(0)^{-1})(\varphi(0)) \circ D\varphi(0) = D\varphi(0)^{-1} \circ D\varphi(0) = \mathrm{id}_{\mathbb{R}^n}$$

Therefore we can assume that $F : B_\varepsilon(0) \to \mathbb{R}^n$ fulfills $F(0) = 0$ and $DF(0) = \mathrm{id}_{\mathbb{R}^n}$. Since $\det(DF(x))$ is a continuous function, we can assume that $DF(x)$ is invertible for every $x \in U$. Define $\psi : U \to \mathbb{R}^n$ by

$$\psi(x) := x - F(x). \tag{11}$$

□

Theorem 1.2 (Implicit Function Theorem). *Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$ be an open subset, and let $(x, y) = (x^1, \dots, x^n, y^1, \dots, y^k)$ denote the standard coordinates on U. Suppose $\Phi : U \to \mathbb{R}^k$ is a $\mathscr{C}^1$-function, $(a, b) \in U$, and $c := \Phi(a, b)$. If the $k \times k$ matrix*

$$A := \left(\frac{\partial \Phi^i}{\partial y^j}(a, b)\right)_{1 \le i,j \le k} \tag{12}$$

is non-singular, i.e. $A \in \mathrm{GL}(k, \mathbb{R})$, then there exist neighbourhoods $V_0 \subseteq \mathbb{R}^n$ of a and $W_0 \subseteq \mathbb{R}^k$ of b and a $\mathscr{C}^1$-function $F : V_0 \to W_0$ such that $\Phi^{-1}(c) \cap (V_0 \times W_0)$ is the graph of F, that is, $\Phi(x, y) = c$ for $(x, y) \in V_0 \times W_0$ if and only if $y = F(x)$.

Proof. □

Bibliography

[Lee13] J.M. Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics. Springer New York, 2013. ISBN: 9780387217529.