Lecture 10: The Poincaré Recurrence Theorem

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Ergodic Theory

the study of measure theoretical ds

Definition (Dynamical System)

A ***dynamical system*** on a probability space $(X, \mathcal{A}, \mu)$ is a measurable transformation $T\colon X \to X$ such that $T_*\mu = \mu$.

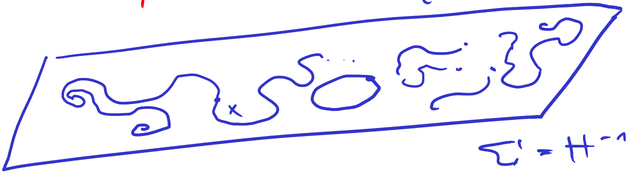
(M, ω, H)

discrete dynamical system

μ is a T-invariant measure

$\theta_t^{X_H}$

continuous dynamical system



$\Sigma = H^{-1}(0)$

$T := \theta_1^{X_H}$

Lemma

Let $T : X \to X$ be a dynamical system on a probability space $(X, \mathcal{A}, \mu)$. Given $A \in \mathcal{A}$, set

$$E := \bigcap_{n=0}^{\infty} E_n \qquad \text{where} \qquad E_n := \bigcup_{k=n}^{\infty} T^{-k} A.$$

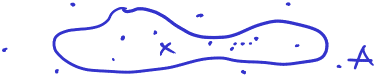
Then $\underline{\mu(A \cap E) = \mu(A)}$. here it is crucial that T preserves μ!

Proof. A good exercise in measure theory. $\square$

We have that $x \in A \cap E$ if and only if there exists a sequence $(k_n) \subseteq \mathbb{N}$ s.t.

$$k_n \to \infty \quad \text{and} \quad T^{k_n}(x) \in A$$

$\forall\, n \in \mathbb{N}$.

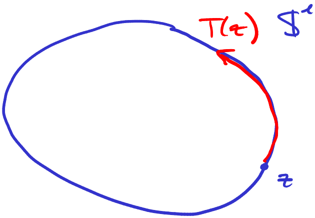


Theorem (Poincaré Recurrence Theorem in ET)

Let $T\colon X \to X$ be a dynamical system on a probability space $(X, \mathcal{A}, \mu)$. Let $A \in \mathcal{A}$ be measurable with $\mu(A) > 0$. Then almost every point of A returns infinitely often to A under T.

Proof. See lemma before. $\square$

Ex.



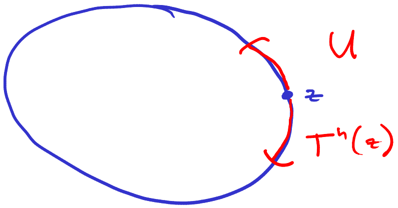
$$T_\theta\colon S^1 \to S^1$$
$$z \mapsto e^{2\pi i\theta} z$$
$\theta \in \mathbb{R}$.

- $\theta \in \mathbb{Q}$, that is $\theta = \frac{m}{n}$, so $T^n(z) = z$.

- $\theta \notin \mathbb{Q}$, then the orbit

$$\{T^n(z) : n \in \mathbb{N}\} \subseteq S^1$$

is dense by the above Theorem.



There must be some iterate in U, because otherwise

$$U, \ T^{-1}U, \ T^{-2}U, \ T^{-3}U, \ \dots \qquad n \in \mathbb{N}$$

are disjoint. But as $\mu(U) > 0$, $\rightarrow \underline{\mu(S^1) = \infty}$.

submanifold of codimension 1

Definition (Regular Energy Surface)

A ***regular energy surface in a Hamiltonian system*** (M, ω, H) is defined to be an embedded hypersurface $\Sigma = H^{-1}(0)$ such that $\mathrm{Crit}(H) \cap \Sigma = \varnothing$. Any such function is called a ***defining Hamiltonian function for*** Σ.

this just means that 0 is a regular value of H

Since 0 is a regular value of H, the preimage $H^{-1}(0)$ is an embedded hypersurface by the implicit function theorem. Moreover,

$$T_x\Sigma = \ker dH_x \quad \forall x \in \Sigma.$$

<u>Claim.</u> X_H is tangent to Σ.

$$dH_x(X_H(x)) = -\omega(X_H(x), X_H(x)) = 0.$$

for all $x \in \Sigma$.

$\square$

Lemma

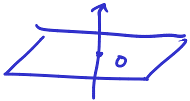
Every regular energy surface is orientable.

Proof. Let m be a Riemannian metric on M. Then consider the "normalised" gradient

$$X := \frac{\operatorname{grad}_m H}{\|\operatorname{grad}_m H\|_m^2} \in \mathfrak{X}(U)$$

where $U \subseteq M$ is an open neighborhood of Σ s.t. $dH \neq 0$ (this can be done if Σ is compact, but at the moment it suffices that the normalised gradient is defined on Σ). Volume on Σ: $\iota_X \omega^n|_\Sigma$.

$$\frac{d}{dt} H \circ \theta_t^X = \frac{dH(\operatorname{grad}_m H \circ \theta_t^X)}{\|\operatorname{grad}_m H \circ \theta_t^X\|^2} = 1. \qquad \square$$



$$\Rightarrow H \circ \theta_t^X = t \quad \forall t \in (-\varepsilon, \varepsilon).$$

Lemma

Let Σ^{2n-1} be an embedded hypersurface in a symplectic manifold (M^{2n}, ω). Then

$$\ker \omega|_{\Sigma} \to \Sigma$$

is a line bundle, called the ***characteristic line bundle of Σ***.

Ex. If Σ is a regular energy hypersurface, then $X_H|_\Sigma$ spans the charateristic line bundle. Indeed, for $x \in \Sigma$ and $v \in T_x\Sigma$ we compute

$$\omega(X_H(x), v) = -dH_x(v) = 0.$$

$\Rightarrow$ $X_H|_\Sigma$ belongs to the characteristic line bundle. But X_H never vanishes on Σ, so it does indeed span it.

But why line distribution?

$\omega|_\Sigma$ has even rank, so there exists $u_x \in \ker \omega_x|_\Sigma \setminus \{0\}$.

Then $\langle u_x \rangle = \ker \omega_x|_\Sigma$, because $\underbrace{\dim T_x\Sigma^\omega}_{\substack{= \\ \ker \omega|_\Sigma \\ 1}} + \underbrace{\dim T_x\Sigma}_{= 2n-1} = \underbrace{\dim M}_{= 2n}$

Lemma

Let $H, \widetilde{H} \in C^\infty(M)$ be two defining Hamiltonian functions for a regular energy surface Σ in a symplectic manifold (M, ω). Then there exists a nowhere-vanishing function $f \in C^\infty(\Sigma)$ such that

$$\boxed{X_{\widetilde{H}}|_\Sigma = f X_H|_\Sigma.}$$

immediate, because $X_{\hat{H}}|_{\Sigma}, X_H|_{\Sigma} \neq 0$.

Proof. We know from previous discussion that both $X_{\hat{H}}$ and X_H span the line distribution $\ker \omega|_\Sigma$. $\square$

This means that $\theta^{X_{\hat{H}}}$ is just a reparametrisation of θ^{X_H}!

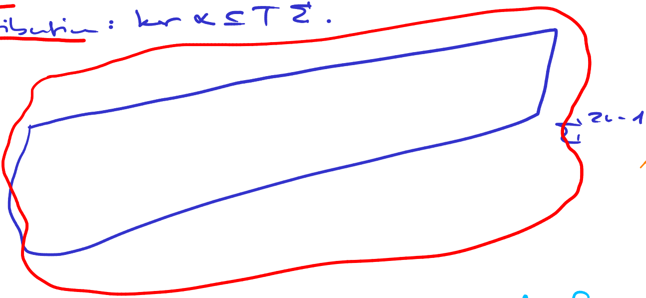
In particular, unparametrised periodic orbits coincide.

Symplectic Geometry

Contact form: $\alpha \in \Omega^1(\Sigma)$ $\alpha \wedge (d\alpha)^{n-1}$ is a volume on Σ.

Contact distribution: $\ker \alpha \subseteq T\Sigma$.

Contact Geometry



If α is a contact form so is $e^f \alpha$, $f \in C^\infty(\Sigma)$.

Moreover, they induce same contact distribution.

Lemma

Let Σ be a compact regular energy surface in a Hamiltonian system (M, ω, H). Denote by θ the flow of X_H on Σ. Then there exists a unique regular θ-invariant probability measure μ_Σ on Σ, that is, we have that

$$\theta_t^* \mu_\Sigma = \mu_\Sigma \qquad \forall t \in \mathbb{R}.$$

$$\int_{\Sigma} f \, \Omega_{\Sigma_i} = \int_{\Sigma} f \, \mu_{\Sigma_i} \quad \forall f \in C(\Sigma_i).$$

$\omega^n = dH \wedge \alpha, \quad \alpha \in \Omega^{2n-1}(U)$

Uniqueness:

$\iota_{\Sigma_i}^* \alpha = \iota_{\Sigma}^* \beta$ for all β s.t. $\omega^n = dH \wedge \beta$.

Lemma

Let (M, Ω) be a compact oriented smooth manifold of positive dimension and suppose that $\varphi \in \operatorname{Diff}(M)$ such that $\varphi^*\Omega = \Omega$. Then there exists a unique regular φ-invariant probability measure μ_Ω such that

$$\int_M f\Omega = \int_M f d\mu_\Omega \qquad \forall f \in C^0(M).$$

Proof. Clear from Lecture 8, as we compute

$$\int_M f\Omega = \int_M \varphi^*(f\Omega) = \int_M (f\circ\varphi)\varphi^*\Omega$$

$$= \int_M (f\circ\varphi)\Omega \qquad \forall f \in C^\infty(M)$$

$$= \int_M (f\circ\varphi)\,d\mu_\Omega$$

$$= \int_M f\, d(\varphi_*\mu_\Omega). \qquad \square$$

Lemma

Let M be a smooth manifold. Suppose that $\eta \in \Omega^1(M)$ is nowhere-vanishing and $\xi \in \Omega^k(M)$. Then $\eta \wedge \xi = 0$ if and only if there exists $\zeta \in \Omega^{k-1}(M)$ such that $\xi = \eta \wedge \zeta$.

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Found: $\omega^n = dH \wedge \alpha$.

If $\beta \in \Omega^{2n-1}(U)$ is another such form, then

$$dH \wedge (\alpha - \beta) = 0$$

find $\gamma \in \Omega^{2n-2}(U)$ s.t.

$$\alpha - \beta = dH \wedge \gamma.$$

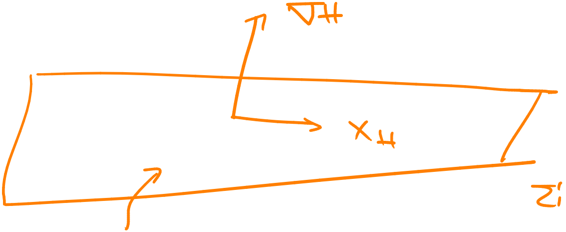
$$\iota_{\Sigma_i}^* \alpha = \iota_{\Sigma}^* (dH \wedge \gamma) + \iota_{\Sigma_i}^* \beta = \cancel{d(H \circ \iota_\Sigma)} \wedge \iota_{\Sigma_i}^* \gamma + \iota_{\Sigma_i}^* \beta$$
$$= \iota_{\Sigma_i}^* \beta.$$

$X = \frac{\nabla H}{\|\nabla H\|^2}$. $\quad dH(X) = \frac{\langle \nabla H, \nabla H \rangle}{\|\nabla H\|^2} = 1$. $\quad$ 1

$$0 = i_X(\underbrace{dH \wedge \omega^n}_{\in\, \Omega^{2n+1}(M) = \{0\}}) = i_X dH\, \omega^n - dH \wedge i_X \omega^n$$

$$\qquad\qquad = \omega^n - dH \wedge i_X \omega^n$$

$$\Rightarrow \omega^n = dH \wedge i_X \omega^n$$



$\iota^*_{\Sigma_i} i_X \omega^n$ volume form.

Hofer + Zehnder: Hamiltonian Dynamics and SG

Theorem (Poincaré's Recurrence Theorem)

Let Σ be a compact regular energy surface in a Hamiltonian system (M, ω, H). Then for almost every $x \in \Sigma$, with respect to the probability measure μ_Σ, there exists a sequence $(t_k) \subseteq \mathbb{R}$ such that

$$t_k \to +\infty \qquad \text{and} \qquad \lim_{k\to\infty} \theta_{t_k}^{X_H}(x) = x.$$

not defined if Σ is not compact in general

i.e. almost every point in Σ is a recurrent point.

Proof. This is up to details the Poincaré recurrence theorem from ergodic theory. $\square$

this crucially uses that I can cover my hypersurface with a countable number of metric balls (Σ admits a metric structure as a Riemannian manifold)