Lecture 5: Noether's Theorem

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Poisson Actions

Let $\theta \in C^{\infty}(G \times M, M)$ be a left action of a Lie group G on a smooth manifold M and denote by $\mathfrak{g} := \mathrm{Lie}(G)$ the corresponding Lie algebra. Each element $\xi \in \mathfrak{g}$ determines a smooth global flow on M by

$$(t,x) \mapsto \theta_{\exp(-t\xi)}(x).$$

Define $\hat{\xi} \in \mathfrak{X}(M)$ to be the infinitesimal generator of this flow, that is,

$$\hat{\xi}_x = \frac{d}{dt} \bigg|_{t=0} \theta_{\exp(-t\xi)}(x).$$

The map $\xi \mapsto \hat{\xi}$ is a Lie algebra homomorphism.

Definition (Weakly Hamiltonian Action)

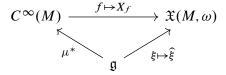
A left action of a Lie group G on a symplectic manifold (M,ω) by symplectomorphisms is said to be *weakly Hamiltonian*, iff for each $\xi \in \mathfrak{g}$ there exists $H_{\xi} \in C^{\infty}(M)$ such that

$$X_{H_{\xi}}=\widehat{\xi}.$$

Definition (Comoment Map)

Given a weakly Hamiltonian action of a Lie group G on a symplectic manifold (M, ω) , define its *comoment map* by

$$\mu^* : \mathfrak{g} \to C^{\infty}(M), \qquad \mu^*(\xi) := H_{\xi}.$$



Definition (Poisson Action)

A left action of a Lie group G on a symplectic manifold (M, ω) by symplectomorphisms is said to be **Poisson**, iff it is a weakly Hamiltonian action such that the corresponding comoment map

$$\mu^*: (\mathfrak{g}, [\cdot, \cdot]) \to (C^{\infty}(M), \{\cdot, \cdot\})$$

is a Lie algebra homomorphism.

The Momentum Lemma

Lemma (Momentum Lemma)

Let $\theta \in C^{\infty}(G \times M, M)$ be a Lie group action on an exact symplectic manifold $(M, d\lambda)$ such that $\theta_g^* \lambda = \lambda$ for all $g \in G$ holds. Then the action θ is Poisson with

$$\mu^*(\xi) = i_{\widehat{\xi}}(\lambda), \quad \forall \xi \in \mathfrak{g}.$$

Corollary

Let $\theta \in C^{\infty}(G \times M, M)$ be a Lie group action on a smooth manifold M. Then the lifted action $g \mapsto D\theta_g^{\dagger}$ on $(T^*M, d\lambda)$ is Poisson with

$$\mu^*(\xi)(q,p) = p(\widehat{\xi}).$$

Noether's Theorem

Definition (Symmetry Group)

A Lie group G is said to be a *symmetry group of a Hamiltonian system* (M, ω, H) , iff there exists a weakly Hamiltonian action θ of G on (M, ω) , such that $\theta_g^* H = H$ for all $g \in G$.

Theorem (Noether's Theorem)

Let G be a symmetry group of a Hamiltonian system (M, ω, H) . Then $\mu^*(\xi)$ is an integral of motion for all $\xi \in \mathfrak{g}$.

Lie Algebra Cohomology

Let g be a Lie algebra. Define

$$C^k := \Lambda^k \mathfrak{g}^*$$

and $d: \mathbb{C}^k \to \mathbb{C}^{k+1}$ by

$$df(X_0, ..., X_k) := \sum_{0 \le i < j \le k} (-1)^{i+j} f([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_k).$$

Then one checks that $d \circ d = 0$. The resulting nonnegative chain complex is called the *Chevalley–Eilenberg cochain complex*.

Definition (Lie Algebra Cohomology)

Let $\mathfrak g$ be a Lie algebra. Then the *k-th cohomology group of* $\mathfrak g$ is defined by

$$H^{k}(\mathfrak{g};\mathbb{R}) := \frac{\ker d: C^{k} \to C^{k+1}}{\operatorname{im} d: C^{k-1} \to C^{k}}.$$

Theorem

Let $\theta \in C^{\infty}(G \times M, M)$ be a weakly Hamiltonian action on a connected symplectic manifold (M, ω) . If $H^2(\mathfrak{g}; \mathbb{R}) = 0$, then the action is Poisson.

Theorem

Let $\theta \in C^{\infty}(G \times M, M)$ be a Poisson action on a connected symplectic manifold (M, ω) with comoment maps μ_1^* and μ_2^* . If $H^1(\mathfrak{g}; \mathbb{R}) = 0$, then

$$\mu_1^* = \mu_2^*.$$

Whitehead Lemmas

Lemma (Whitehead's First Lemma)

Let \mathfrak{g} be a semisimple Lie algebra. Then $H^1(\mathfrak{g};\mathbb{R})=0$.

Lemma (Whitehead's Second Lemma)

Let \mathfrak{g} be a semisimple Lie algebra. Then $H^2(\mathfrak{g}; \mathbb{R}) = 0$.