Morse Homology, the Symplectic and the Rabinowitz Action Functional

Dominik Xaver Hörl

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Overview

Sketch of Morse Homology

- Fix a *Morse* function $f \in C^\infty(M, \mathbb{R})$ on a compact smooth manifold M, meaning that its second derivative (as a quadratic form) is nondegenerate at all critical points

$$\text{for all } p \in \mathrm{Crit}(f) : D^2 f_p(v, v) = 0 \implies v = 0$$

(these will then be isolated by the Morse lemma) and define its index $\mathrm{ind}(p)$ at $p \in \mathrm{Crit}(f)$ to be the number of negative eigenvalues of $D^2 f_p$.

- Choose a Riemannian metric g on M and consider the chain complex freely generated by the critical points of f.
Its grading is given by the index, and boundary maps $C_k(f, g) \xrightarrow{\partial} C_{k-1}(f, g)$ count the number of unparametrized gradient flow lines (wrt. g) between them (modulo 2):

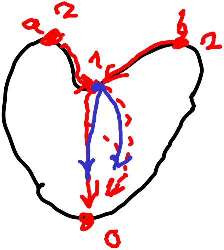
$$\partial p := \sum_{\substack{q \in \mathrm{Crit}(f): \\ \mathrm{ind}(q) = \mathrm{ind}(p) - 1}} q \cdot (\# \{\text{flow lines from } p \text{ to } q\} \mod 2)$$

Sketch of Morse Homology

By a gradient flow line we mean a solution of $\dot{\gamma} = -\nabla_g f \circ \gamma$, but these may be reparametized by time shift. Modding by this $\mathbb{R}$-action gives the space of unparametrized flow lines.
One proves $\partial^2 = 0$, so this is actually a complex (this is hard!). Then the Morse homology $H(M, f, g)$ is the homology of this complex.
Note:

- Morse homology is "independent" of the choice of f and g, meaning that the construction is functorial in them and for two choices f_1, g_1 and f_2, g_2 we get an isomorphism $H(M, f_1, g_1) \xrightarrow{\sim} H(M, f_2, g_2)$.
- It is an ordinary homology theory. So it computes the singular homology of M.
- The Morse condition is satisfied by "most" f. Similarly, one needs a Morse-Smale condition on (f, g) (will be explained later). This is also satisfied by "most" (f, g).

Sketch of Morse Homology: The Heart



$$0 \to \mathbb{Z}_2^2 \to \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to 0$$

a, b

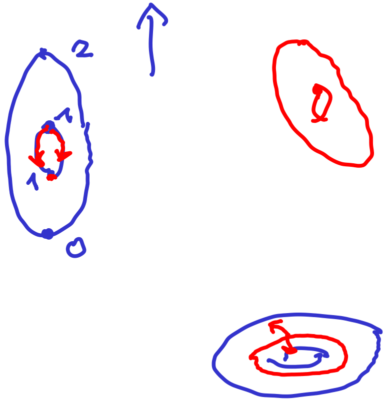
$$\mathbb{Z}_2 \to 0 \to \mathbb{Z}_2$$

Sketch of Morse Homology

Further Notes:

- The Morse-Smale condition demands that the stable and unstable manifolds associated to the critical points intersect transversally. Consider the naive height function on the standing torus for an instructive example where this goes wrong.
- The theory can be made to work even if the function is only Morse-Bott instead of Morse.
  This means that the critical points are no longer isolated, but they come as critical submanifolds, such that the differential is degenerate in directions normal to those submanifolds.
  A good example for this situation is the torus lying as a donut on your desk, with the usual height function.

Sketch of Morse Homology: The Torus



Sketch of Morse Homology: Compactness

The proof of $\partial^2 = 0$ uses compactness of M. It consists of careful analysis of the convergence of families of gradient flowlines, and uses the Arzela-Ascoli theorem to produce convergent subsequences.

Can we replicate this in an infinite dimensional setting (where compactness fails)?

Floer Homology: The Arnold Conjecture

Surprisingly the answer is positive. Floer gave a proof of the famous Arnold Conjecture under some assumptions using this theory.

For f a Morse function on M, we have the *Morse Inequalities*

$$\# \operatorname{Crit}(f) = \sum_k \dim C_k \geq \sum_k \dim H_k(M, f, g)$$

since the Morse complex is generated by the critical points and we only loose dimensions when passing to homology.
Since Morse Homology computes "the" Homology of M, the right hand side is also known as the sum of all Betti numbers $\beta_k = \dim H_k(M)$ of M.
Arnold's conjecture claims that a similiar statement holds for the fixpoints of a Hamiltonian symplectomorphism.

(time - 1 flow of X_H

Floer Homology: The Arnold Conjecture

Floer's proof of the Arnold Conjecture uses a reformulation of Hamiltonian mechanics in terms of a functional on the space of paths on M (closed loops actually in this case).
This is closely related to what is known in the physics world as the Lagrangian / action formulation of mechanics.

Before we explain this, we need a small technical lemma.

Homotopy Version of Cartan's Magic Formula

$\mathcal{L}_X \omega = d\iota_X\omega + \iota_X d\omega$

Let $I \times M \xrightarrow{f} N$ be a smooth homotopy and denote by x the vector field $\partial_t f : I \times M \to TN$ over f. Then for $\alpha \in \Omega(N)$:

$$\frac{d}{dt} f_t^* \alpha = f_t^{\#} \iota_{x_t}(d\alpha \circ f_t) + d\, f_t^{\#} \iota_{x_t}(\alpha \circ f_t)$$

$x \xrightarrow{h} y$

$h^*\omega(x) = \omega_{h(x)}(dh_x \cdot, \ldots)$

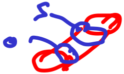
Where given some $g\colon M \to N$ we define $g^{\#} := \alpha \mapsto \Lambda Dg^v \circ \alpha$ and $\Lambda Dg^v \colon \Lambda g^* TN^v \to \Lambda TM^v$ is induced from the bundle morphism $Dg\colon TM \to g^* TN$.

So pullback of forms would arise as the composition $g^*\alpha = g^{\#}(\alpha \circ f)$, and for $v \in \Gamma(TM)$, its pushforward $g_* v \colon M \to TN$ and a form $\beta\colon M \to TN$ (both along g) we see that $\iota_v g^{\#}\beta = g^{\#}\iota_{g_* v}\, \beta$.

Homotopy Version of Cartan's Magic Formula

Homotopy Version of Cartan's Magic Formula

Over any point $p \in M$ pick a ball $B \subset M$ of dimension $deg(\alpha)$. Then for $s \in I$:



$I \times M \xrightarrow{f} N$ — α

$\cup$

B

$$\int_B f_s^*\alpha - f_0^*\alpha$$

$$= \boxed{\int_{[0,s]\times B} df^*\alpha} + \int_{[0,s]\times\partial B} f^*\alpha$$

$$= \int_B \int_0^s \iota_{\partial_t} i_t^* f^* d\alpha \; dt + \int_{\partial B}\int_0^s \iota_{\partial_t} i_t^* f^*\alpha \; dt$$

$$= \int_0^s \left(\int_B f_t^{\#}\iota_{X_t}(d\alpha \circ f_t) + \int_B d \; f_t^{\#}\iota_{X_t}(\alpha \circ f_t) \right)$$

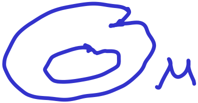
$\partial(A \times B)$

$= \partial A \times B$

$+ \; A \times \partial B$

Differentiating wrt. s and averaging over successively smaller balls B produces the theorem.

The Symplectic Action Functional



(M, ω, H) an exact Hamiltonian System with $\omega = d\alpha$, then

$$A_H\colon C^\infty(S^1, M) \to \mathbb{R}, \gamma \mapsto \int_{S^1} \gamma^*\alpha - \int_{S^1} H \circ \gamma$$

is called the symplectic action functional.

The Symplectic Action Functional

Some variants of this:

- For non closed orbits one uses paths with fixed endpoints $p, q \in M$ and A_H will be of type $C^{\infty}{}_{p}^{q}(I, M) \to \mathbb{R}$.
- If the symplectic form ω is not exact, use as source space the contractible paths ΛM and choose for every path $\gamma \in \Lambda M$ a filling $D \xrightarrow{\overline{\gamma}} M$, meaning that $\overline{\gamma} \circ (S^1 \hookrightarrow D) = \gamma$. Then write:

$$A_H \colon C^{\infty}(S^1, M) \to \mathbb{R}, \gamma \mapsto \int_D \overline{\gamma}^* \omega - \int_{S^1} H \circ \gamma$$

  For this to be well-defined, we need M to be symplectically aspherical, i.e. $\int_{S^2} i^* \omega = 0$ for every smooth map $S^2 \xrightarrow{i} \omega$.

Critical Points of the Action Functional

Critical points of the action (in any of these variants) correspond to trajectories of the Hamiltonian System:

$$dA_H|_\gamma = 0 \iff \dot{\gamma} = -X_H \circ \gamma$$

Critical Points of the Action Functional

What is dA_H?

The space $C^\infty(S^1, M)$ is a Fréchet manifold, so its tangent space $T_\gamma C^\infty(S^1, M)$ at γ is given by ([ ] denotes first order tangency)

$$\{[\sigma] \mid \sigma\colon (-\epsilon, \epsilon) \to C^\infty(S^1, M), \sigma(0) = \gamma\}$$

or equivalently (this is actually a theorem!)

$$\{[\sigma] \mid \sigma\colon C^\infty((-\epsilon, \epsilon) \times S^1, M), \sigma_0 = \gamma\}$$

For mapping spaces like ours, it admits a simpler description as vector fields along γ, which we denote

$$\mathfrak{X}(\gamma) := \Gamma(\gamma^* TM \to S^1)$$

$$\begin{tikzcd} \gamma^*TM \arrow[d] & TM \arrow[d] \\ S^1 \arrow[r, "\gamma"] & M \end{tikzcd}$$

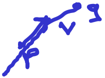
Critical Points of the Action Functional

On the equivalence of tangent spaces:
Pick a Riemannian metric on M, then a vector field $x \in \mathfrak{X}(\gamma)$ gives rise to a map $(-\epsilon, \epsilon) \times S^1 \to M, (t, s) \mapsto \exp(t \cdot x_s)$ via the Riemannian exponential / geodesic flow $D \subset TM \overset{\exp}{\to} M$, for D a suitable neighbourhood of the zero section.

Critical Points of the Action Functional

This is also how the charts on $C^\infty(S^1, M)$ arise. Choose D such that $(\pi, \exp)$ yields a diffeomorphism $D \xrightarrow{\sim} \psi(D) \subset M \times M$.
Fix $f \in C^\infty(S^1, M)$, then the chart centered around f is constructed as follows:
Given some $g \in C^\infty(S^1, M)$ such that (f, g) has image in $\psi(D)$ we have

$$\begin{tikzcd}
f^*D \subset f^*TM \arrow[r] \arrow[d] & D \subset TM \arrow[d, "\psi = (\pi, exp)", "\wr"'] \\
S^1 \arrow[r, "(f,g)"] & \psi(D) \subset M \times M
\end{tikzcd}$$



$p \quad q$

and $\psi^{-1} \circ (f, g)$ yields a section in f^*D. This association gives the chart map $\left\{ g \in C^\infty(S^1, M) \mid g(s) \in \mathrm{im}(\exp_{f(s)}) \text{ for all } s \in S^1 \right\} \to \Gamma(f^*D)$.

Next lets compute the critical points for the Floer case!

Critical Points of the Action Functional

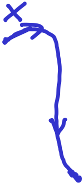
TM

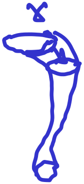
M

$\gamma : S^1 \to M$

t

$\gamma(t)$





Critical Points of the Action Functional

For $\sigma \colon (-\epsilon, \epsilon) \times S^1 \to M$ with $\sigma_0 = \gamma$ and $x = \partial_t \sigma|_{t=0} \in \mathfrak{X}(\gamma)$ and a chosen filling $\overline{\sigma} \colon (-\epsilon, \epsilon) \times D \to M$ along with $X = \partial_t\, \overline{\sigma}|_{t=0} \in \mathfrak{X}(\overline{\sigma}_0)$:

$$
\begin{aligned}
dA_H|_\gamma(\sigma) &= \frac{d}{dt}\bigg|_{t=0} \left(\int_D \overline{\sigma}_t^* \omega - \int_{S^1} H \circ \sigma_t \right) \\
&= \int_D \frac{d}{dt}\bigg|_{t=0} \overline{\sigma}_t^* \omega - \int_{S^1} \frac{d}{dt}\bigg|_{t=0} \sigma_t^* H \\
&= \int_D \overline{\sigma}_0^{\#} \iota_X (d\omega \circ \overline{\sigma}_0) + d\overline{\sigma}_0^{\#} \iota_X (\omega \circ \overline{\sigma}_0) - \int_{S^1} \iota_x dH \circ \gamma \\
&= \int_D d\overline{\sigma}_0^{\#} \iota_X (\omega \circ \overline{\sigma}_0) - \cancel{\int_{S^1} \sigma_0^{\#} \iota_x dH \circ \sigma_0} + \int_{S^1} \iota_x \iota_{X_H} \omega \circ \gamma \\
&= \int_{S^1} \gamma^{\#} \iota_x (\omega \circ \gamma) + \int_{S^1} \iota_x \iota_{X_H} \omega \circ \gamma \\
&= \int_{S^1} \omega(x, \dot{\gamma} + X_H \circ \gamma)
\end{aligned}
$$

So $dA_H|_\gamma = 0 \iff \dot{\gamma} = -X_H \circ \gamma$.

Critical Points of the Action Functional

Sketch of Floer Homology

(M, ω) a closed symplectic manifold with energy function $H_s, s \in S^1$, subject to some nondegeneracy condition.

As in the finite dimensional case, consider the chain complex generated by the critical points of the action functional A_H. Flow lines arise as follows: Fix a family of Riemmanian metrics g_s on M, $s \in S^1$, compatible with ω. The gradient $\nabla_g A_H$ is taken with respect to the induced L^2 inner product. So for $\gamma \in \Lambda M$ and $x, y \in T_\gamma(\Lambda M) \overset{\sim}{=} \mathfrak{X}(\gamma)$ we have $\langle x, y \rangle := \int_{S^1} g_s(x(s), y(s))\ ds$. This can be rewritten using the associated family of almost complex structures

$J_s^2 = -\mathrm{id}$

$$J_s := (TM \xrightarrow{\omega} TM^v \xrightarrow{g_s^{-1}} TM) \in \mathrm{End}(TM)$$

$i^2 = -1$

as

$$\langle x, y \rangle := \int_{S^1} \omega(x(s), J_s y(s))\ ds.$$

Sketch of Floer Homology: Flow lines

Then a flow line is a smooth map $u\colon \mathbb{R} \to \Lambda M$, which can be interpreted as a smooth map $\mathbb{R} \times S^1 \to M$, such that

$$\partial_t u = -\nabla_g A_H \circ u_t$$

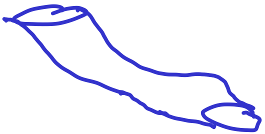
Using our calculation of dA_H and $dH_s = \iota_{X_{H_s}} \omega$ we compute for $x \in \mathfrak{X}(u_t)$:

$$\begin{aligned} -\langle \partial_t u, x \rangle &= dA_H\big|_{u_t} x \\ &= \int_{S^1} u_t^{\#} \iota_x (\omega \circ u_t) - \iota_x \iota_{X_{H_s}} (\omega \circ u_t) \\ &= -\langle J_s \partial_s u, x \rangle + \langle J_s X_{H_s} \circ u, x \rangle \end{aligned}$$

This leads us to the Floer equations:

$$\partial_t u + J_s \left(\partial_s u - X_{H_s} \circ u \right) = 0$$

Sketch of Floer Homology



Now we can forget about the loop space and consider flow lines as maps $\mathbb{R} \times S^1 \to M$ that satisfy the Floer equations and in the infinite time limit converge to fixed closed orbits. These maps are also called "Floer cylinders".

One also has a relative version of the Morse index, called the Conley-Zehnder index.

Finally to show $\partial^2 = 0$ one can then prove that a similar compactness result as in the finite dimensional case holds, by a procedure called bubbling analysis.

Sketch of Floer Homology: Bubbles and Compactness

By "bubbles" we mean holomorphic spheres $S^2 \xrightarrow{i} M$, that a priori could arise in the limit of a sequence of Floer cylinders (i.e. gradient flow lines) with fixed asymptotics. One then sees that for such spheres one has $\int_{S^2} i^*\omega \neq 0$, which is precisely precluded by symplectic asphericity.
In the absence of such bubbles the family of cylinders must then converge to a once broken cylinder, similar to what one has in finite dimensional morse theory.

Floer Homology computes Homology

Key points on Floer homology:

- As in Morse homology, this is "independent" of the Riemannian metric, but also of the choice of Hamiltonian (although these still need to be "generic").
- By choosing a time independent Hamiltonian one has as critical points of A_H at least the critical points of H. Actually if $H = 0$ one obtains precisely the points of M (embedded as constant loops). Analyzing this situation shows that Floer homology computes ordinary (Morse-)homology of M.
- Because the Floer complex is generated by periodic orbits, there must be at least as many of these as the cumulated Betti numbers of M. This is Floers proof of the Arnold Conjecture.

Rabinowitz Action: Varying the Period

Floer homology only detects (contractible) orbits of fixed period τ (in our presentation $\tau = 1$). A more physically meaningful situation would be to look at a hypersurface of fixed energy but look at orbits of any period. This is the domain of Rabinowitz-Floer Homology.

How to realise periodic orbits of varying period as maps $S^1 \to M$?

Rabinowitz Action Functional

Allow for the speed of trajectories to vary!
This is achieved by enlarging the source of the action functional to include another factor $\tau \in \mathbb{R}$ that will be the (generalized) period of the orbits. Since the speed of trajectories scales with the gradient of the Hamiltonian on the hypersurface, we must rescale the Hamiltonian appropriately by τ to $H_\tau := c + \tau \cdot (H - c)$.
For the exact case $\omega = d\alpha$ and a hypersurface $\Sigma = H^{-1}(c)$ we write down the Rabinowitz Action functional:

$$C^\infty(S^1, M) \times \mathbb{R} \to M$$

$$A^H(\gamma, \tau) = \int_{S^1} \gamma^* \alpha - \int_{S^1} \gamma^* H_\tau$$

The new Hamiltonian is chosen such that $dH_\tau = d(c + \tau(H - c) = \tau dH$, so the gradient of H (and thereby the speed of the trajectories) is scaled by τ, but for $\tau \neq 0$ we still have $H_\tau^{-1}(c) = H^{-1}(c) = \Sigma$.

Rabinowitz Action Functional: Critical Points

Computing the critical points of A^H yields:

1. $\int_{S^1} \gamma^*(H - c) = 0$ or $\int_{S^1} \gamma^* H = c$
2. $\omega(\dot\gamma, \ \cdot \ ) = -d(\tau H) \circ \gamma \iff \dot\gamma = -\tau X_H$

Since γ is a solution of the Hamilton equations above, its energy is constant, and because of the first equation equal to c, so γ lies in Σ.
Looking at a rescaling $\tilde\gamma := \gamma \circ (\cdot \frac{1}{\tau})$, this will obey the unmodified Hamilton equations and live on $\mathbb{R}/\tau\mathbb{Z}$, so has period τ.
It will still live on Σ obviously, so critical points (γ, τ) of A^H correspond precisely to periodic orbits on Σ of period τ.

Note: Usually we take $c = 0$ to simplify things. Since the orbits only depend on dH, we can always do this without losing anything.

Rabinowitz Floer Homology

Space for Writing 0

Space for Writing 1

Space for Writing 2