Hamiltonian Systems

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Very classical mechanics

For a particle in 3-space, we have the position $\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = r(t) \in \mathbb{R}^3$

$$\vec{v} = \frac{\mathrm{d}}{\mathrm{d}t} r, \quad \vec{a} = \frac{\mathrm{d}}{\mathrm{d}t} \vec{v}, \quad \vec{p} = m\vec{v}.$$

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The particle may be subject to a force $\vec{F}(r, \vec{v}, t)$, obeying Newton's laws:

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Several particles in one "vector":

$$R = \begin{bmatrix} x_1 \\ y_1 \\ \vdots \\ y_l \\ z_l \end{bmatrix}, \quad \begin{bmatrix} m_1 & & & & \\ & m_1 & & & \\ & & \ddots & & \\ & & & m_l & \\ & & & & m_l \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \vdots \\ \ddot{y}_l \\ \ddot{z}_l \end{bmatrix} \overset{!}{=} \begin{bmatrix} F_{1,x} \\ F_{1,y} \\ \vdots \\ F_{l,y} \\ F_{l,z} \end{bmatrix}$$

Motivation

Consider l particles in 3-space on holonomic constraints:

$$\underline{q} = (q^i) \mapsto R = \left[R^\lambda\right] \in N = f^{-1}[\{0\}] \cap \text{some open set} \overset{\text{sM}}{\subseteq} \mathbb{R}^{3l}.$$

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The masses m_λ give the (extrinsic) momentum P on a trajectory as a Riemannian metric m as on the last slide. This descents to N:

$$\overline{P} = m(\overline{\dot{R}}) \Rightarrow \overleftharpoon{p} = m(\vec{\dot{R}}) \doteq (p_i) = (m_{ij}\dot{q}^j)$$

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Kinetic energy:

$$2T = \sum_\lambda m_\lambda (v^\lambda)^2 = \|\dot{R}\|_m^2 = \|P\|_{m^{-1}}^2 = m_{ij}v^i v^j = m^{ij} p_i p_j$$

Generalized forces

The parallel force $\overleftarrow{Q} = Q_i \varepsilon^i = g(\vec{F}_{\|})$ may later depend on velocity, but here we consider conservative forces with the potential $V \in C^\infty(N, \mathbb{R})$

$$\vec{F} = -\vec{\nabla} V = -g^{-1}\left(\frac{\partial V}{\partial r}\right), \quad \text{i.e.} \quad \overleftarrow{Q} = -\overleftarrow{\mathrm{d}V} = -\frac{\partial V}{\partial q^i}\,\mathrm{d}q^i$$

Newton's equation becomes:

$$\overleftarrow{Q} = m\left(\frac{\mathrm{D}}{\mathrm{d}t}\vec{v}\right) = (\dot{v}^j)\vec{e}_j + v^i v^j \vec{\Gamma}_{ij}$$

The potential and kinetic energy constitute the **Hamiltonian** or total energy function

$$H(q^i, p_i, t) = T(q^i, p_i, t) + V(q^i, p_i, t)$$

Example 1 - l-body problem

Unrestricted motion of l celestial bodies with gravitation, i.e. the forces

$$\vec{F}_i = \sum_{j \neq i} -\frac{m_i m_j}{4\pi |r_i - r_j|^2} \vec{e}_{r_i - r_j}$$

generated by the shared potential

$$4\pi V(r_1, \ldots, r_l) = -\frac{m_1 \cdot m_2}{|r_1 - r_2|} - \cdots - \frac{m_1 \cdot m_l}{|r_1 - r_l|} - \frac{m_2 \cdot m_3}{|r_2 - r_3|} - \ldots$$

This has the Hamiltonian

$$H(r_1, \ldots, r_l, \vec{p}_1, \ldots, \vec{p}_l) = \sum_i \frac{|\vec{p}_i|^2}{2m_i} - \sum_{i<j} \frac{m_i m_j}{4\pi |r_i - r_j|}$$

We have to remove the collision set $\{R \,|\, V = \infty\}$, so $N = \operatorname{dom} V$.

Abstraction to the phase space

Lagrangian mechanics

- $M = TN \ni \vec{v}@R \doteq \underline{v}@\underline{q} \stackrel{!}{=} \underline{\dot{q}}@\underline{q}$
- Lagrangian: $L = T - V \in \mathscr{F}(M)$
- We have the canonical momenta $p_i = m_{ij}v^j = \frac{\partial}{\partial \dot{q}^i}T = \frac{\partial}{\partial \dot{q}^i}L$.
- Lagrange equation: $\dot{q}^j = v^j, \quad \dot{p}_j = Q_j + \frac{\partial T}{\partial q^j} = \frac{\partial L}{\partial q^j}$

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Hamiltonian mechanics

- $M = T^*N \ni \overleftarrow{p}@R \doteq \underline{p}@\underline{q}$
- Hamiltonian: $H = \overleftarrow{p}\vec{v} - L = T + V \in \mathscr{F}(M)$
- $v^j = m^{ij} p_i = \frac{\partial}{\partial p_j} T = \frac{\partial}{\partial p_j} H$
- Hamilton equations: $\dot{q}^j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q^j}$

They are duals via a Legendre transformation.

Example 2 - Central potential

If $V(R) = V\left(\begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix}\right) = V(r,\varphi)$ only depends on r, we have

$$F = \frac{\partial V}{\partial r} \cdot (-\vec{e}_r) \quad \text{or} \quad (Q_r, Q_\varphi) = \underline{g}\begin{pmatrix} -\partial_r V \\ 0 \end{pmatrix} = (-\partial_r V, 0)$$

The canonical momenta are

$$\begin{pmatrix} p_r \\ p_\varphi \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & mr^2 \end{pmatrix} \begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix}$$

We can identify $L = p_\varphi = mr^2\dot{\varphi}$ as the angular momentum.

$$H = \frac{1}{2} m^{ij} p_i p_j + V(\underline{q}) = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + V(r)$$

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Hamiltonian equations:

$$\begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix} = \frac{\partial H}{\partial \underline{p}} = \begin{pmatrix} \frac{p_r}{m} \\ \frac{p_\varphi}{mr^2} \end{pmatrix}$$

$$(\dot{p}_r, \dot{p}_\varphi) = -\frac{\partial H}{\partial \underline{q}} = \left(-\frac{\partial V}{\partial r} + \frac{p_\varphi^2}{mr^3}, 0\right)$$

Notice the "fictional" centrifugal force $m\dot{\varphi}^2 r$. p_φ is conserved, because φ is a **cyclic coordinate**.

Generalization

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$$\dot{\boldsymbol{U}} \doteq \dot{\underline{u}} = \begin{pmatrix} \dot{\underline{q}} \\ \dot{\underline{p}} \end{pmatrix} \overset{!}{=} \begin{pmatrix} \frac{\partial H}{\partial p_j} \\ -\frac{\partial H}{\partial q^i} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}}_{(J^{\mu\nu})} \frac{\partial H}{\partial \underline{u}}^{\mathsf{T}} = \underline{\underline{J}}(\underline{\mathrm{D}H}) \doteq -\Omega(\mathrm{d}H) =: \boldsymbol{X}_H$$

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Here $\Omega = \omega^{-1}$, where $\omega = \mathrm{d}\lambda = \sum_i \mathrm{d}p^i \wedge \mathrm{d}x^i$ with the Liouville 1-form

$$\lambda(\boldsymbol{z}) = \underbrace{\pi_{TT^*N \longrightarrow T^*N}(\boldsymbol{z})}_{\text{Basepoint } \overleftarrow{p}} \underbrace{(T\pi_{T^*N \longrightarrow N}(\boldsymbol{z}))}_{\text{spatial part of } \boldsymbol{z}} = \overleftarrow{p}(\vec{v}) \Leftrightarrow \lambda_{\overleftarrow{p}} = \sum_i p^i \,\mathrm{d}x^i$$

which has the matrix representation $\omega_{\mu\nu} = J^{\mu\nu} = -\Omega^{\mu\nu}$

General Definition of Hamiltonian Systems

Let (M, ω) be a symplectic manifold, with tangent-cotangent isomorphism

$$\omega : TM \xrightarrow{\cong} T^*M, z \mapsto \omega(z, \cdot) \quad \text{and} \quad \Omega = \omega^{-1}.$$

The triple (M, ω, H) is called a **Hamiltonian system** where $H \in \mathscr{F}(M)$ is the **Hamiltonian** or energy function.
Canonical coordinates $\underline{u} = (u^\mu) = (x^1, \dots, x^n, p^1, \dots, p^n)^\mathsf{T}$ are such that

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Important Definition

The **Hamiltonian vector field** associated to $f \in \mathscr{F}(M)$ is

$$X_f := -\Omega(\mathrm{d}f) \quad \text{i.e.} \quad \omega(X_f, z) = -\mathrm{d}f(z) = -\partial_z f.$$

Hamiltonian flow

The Hamiltonian equations now are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma = -\Omega(\mathrm{d}H)\circ\gamma = X_H\circ\gamma$$

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These exist at least locally (first order ODE):

$$U_0 \mapsto \gamma_{U_0} : t \mapsto \gamma_{U_0}(t) \leftarrow\!\shortmid U_0 : \psi^t_{X_H} \leftarrow\!\shortmid t$$

This is the **Hamiltonian flow** (flow of X_H).

Special vector fields

A vector field $\boldsymbol{z} \in \mathscr{X}(M)$ is ...

- **symplectic** iff ω is preserved along the integral curves of $\boldsymbol{z}$: $\psi_{\boldsymbol{z}}^{t\,*}\omega = \omega$, i.e. $\mathscr{L}_{\boldsymbol{z}}\omega = 0$

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That follows from the fact that the Lie-derivative wrt z commutes with the flow of z: (at point $P' = \psi_z^t(P) = \gamma_P(t)$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_z^{t*}(\omega) = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}(\psi_z^{t+s})^*(\omega) = \frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\psi_z^{t*}(\psi_z^{s*}\omega) = \psi_z^{t*}(\mathscr{L}_z\omega)$$

So if $\mathscr{L}_z\omega = 0$, $\psi_z^{t*}(\omega(\gamma_P(t)))$ must be constant wrt t; $\psi_z^{0*}(\omega(\gamma_P(0))) = \omega(P)$. If not, then there is some place where $\psi_z^{t*}(\omega(\gamma_P(t))) \neq \psi_z^{0*}(\omega(P))$

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Relation between these definitions

z symplectic $\Leftrightarrow$ $\omega(z)$ closed $\Leftrightarrow$ $\omega(z)$ is locally exact $\Leftrightarrow$ z locally Hamiltonian.

$$\mathscr{L}_z\omega \overset{\text{Cartan}}{=} \mathrm{d}(\iota_z\omega) + \iota_z\,\mathrm{d}\omega = \mathrm{d}(\omega(z))$$

Poisson bracket

One also defines the **Poisson bracket** of $g, f \in \mathscr{F}(M)$:

$$\{g, f\} := \omega(X_g, X_f) = \mathrm{d}f\,(X_g) = \partial_{X_g} f = -\Omega(\mathrm{d}g, \mathrm{d}f) = -\Omega^{\mu\nu} \cdot \partial_\mu g \cdot \partial_\nu f$$

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A quantity f is conserved iff it Poisson commutes with H, because on a trajectory γ:

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma) = \partial_{X_H} f \circ \gamma = \{H, f\} \circ \gamma$$

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Canonical commutation relations: $\{u^\mu, u^\nu\} = -\Omega^{\kappa\iota} \partial_\kappa u^\mu \partial_\iota u^\nu = -\Omega^{\mu\nu} = J^{\mu\nu}$

Hamiltonian equations: $\frac{\mathrm{d}}{\mathrm{d}t} u^\mu \circ \gamma = \{H, u^\mu\} \circ \gamma = -\Omega^{\kappa\mu} \frac{\partial H}{\partial u^\kappa} = X_H^\mu$

Commutators and Poisson Brackets

$$X_{\{f,g\}} = \left[X_f, X_g\right]$$

The Poisson bracket generalizes to the commutator (divided by $i\hbar$) in quantum mechanics.

Symplectomorphisms

$$\left[(M,\omega) \underset{\mathtt{Symp}}{\longrightarrow} (M',\omega')\right] = \left\{\psi : M \underset{\mathtt{sM}}{\longrightarrow} M' \,\middle|\, \psi^*(\omega') = \omega' \circ T\psi = \omega\right\}$$

$$\mathrm{Symp}(M,\omega) := \mathrm{Aut}_{\mathtt{Symp}}(M,\omega) = \left\{\psi : M \longrightarrow M \,\middle|\, \omega_{\mu\nu} = \omega'_{\mu\nu}\right\}$$

Changes of coordinates

Canonical transformations are symplectomorphisms

The canonical transformations are changes of coordinates $\begin{pmatrix} q \\ p \end{pmatrix} \mapsto \begin{pmatrix} Q \\ P \end{pmatrix}$ that leave the form of Hamilton's equations the same, i.e. such that the new coordinates are also canonical, i.e. symplectomorphisms:

$$-\{u^\mu, u^\nu\} = \Omega^{\mu\nu} \stackrel{!}{=} \tilde{\Omega}^{\mu\nu} = -\{\tilde{u}^\mu, \tilde{u}^\nu\} = \frac{\partial \tilde{u}^\mu}{\partial u^\kappa} \frac{\partial \tilde{u}^\nu}{\partial u^\iota} \Omega^{\iota\kappa}$$

New Hamiltonian vector field: $X_{\psi^* H} = \psi^* X_H$:

$$\omega(X_{\psi^* H}) = \mathrm{d}(\psi^* H) = \psi^* \mathrm{d}H = \psi^* \omega(X_H) = \omega(\psi^* X_H)$$

Symplectomorphisms and the flow

Fact

If for $t \in \mathbb{R}$, the flow function Φ_H^t is well-defined, it is a symplectomorphism onto its image.

It is a diffeomorphism with smooth inverse Φ_H^{-t}, which is defined on $\operatorname{im} \Phi_H^t$. That it is symplectic is the fact that X_H is symplectic.

Electromagnetism

Consider the vector potential of the electromagnetic force $(\phi, \vec{A})$ generating the Lorentz force

$$\vec{F}(r, \vec{v}, t) = q \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right) = q \cdot \left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times \vec{\nabla} \times \vec{A}\right).$$

This can be described by the Lagrangian

$$L(r, \vec{v}) = \frac{m}{2}|\vec{v}|^2 - q \cdot (\phi(r) - \vec{v} \cdot \vec{A}(r)) = \frac{1}{2}m(\vec{v}, \vec{v}) - q\phi + \overleftarrow{A}\vec{v}$$

giving the canonically conjugated momentum

$$\overleftarrow{p} = \frac{\partial L}{\partial \vec{v}} = m(\vec{v}) + q\overleftarrow{A} \Leftrightarrow \vec{v} = m^{-1}(\overleftarrow{p} - q\overleftarrow{A})$$

and the Hamiltonian $(\overleftarrow{p}\vec{v} - L)$

$$H = \frac{1}{2}m(\vec{v}, \vec{v}) + q\phi = \frac{1}{2}m^{-1}(\overleftarrow{p} - q\overleftarrow{A}, \overleftarrow{p} - q\overleftarrow{A}) + q\phi$$

Twisted contangent bundle

A **twisted cotangent bundle** is T^*N as a symplectic manifold, but the symplectic form differs from $\mathrm{d}\lambda$:

$$\omega = \mathrm{d}\lambda + \sigma.$$

This gives rise to a kind of "Lorentz force" $Y : TN \longrightarrow TN$,

$$g(\vec{Y}(\vec{v})) = \sigma(\vec{v})$$

This is an alternative approach to the magnetism problem from the last slide.