Hamiltonian Manifolds and the Regular Orbit Cylinder Theorem

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The Poincaré section map and the regular orbit cylinder theorem

Given a vector field X on the $2n$-dimensional manifold M. Assume that $x : I \to M$ is the trajectory of the vector field X, which is also periodic with fundamental period $T > 0$. Then we can intersect x at the point $p = x(0)$ with a $(2n-1)$-dimensional hypersurface Σ to which the vector field is nowhere tangent. This means that in a small neighbourhood around p we have

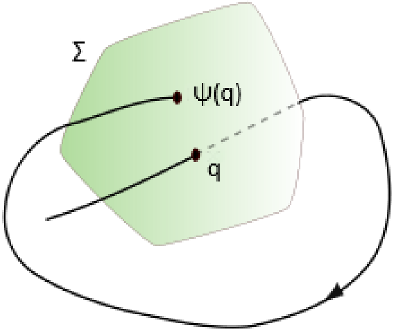
$$T_q M = T_q \Sigma \oplus \operatorname{span}\{X(q)\},$$

as long as q is also in Σ.

Let $\phi : M \times I \to M$ denote the flow of the vector field and since the flow is smooth in both components we can define a smooth map $\psi : U_p \subset \Sigma \to \Sigma$ by following an initial point $q \in U_p$ along its solution $\phi(q, t)$ until it meets Σ again at time $\tau(q)$, i.e.

$$\psi(q) = \phi(q, \tau(q)). \tag{1}$$

This map is called the Poincaré section map.



Lemma

$\mathrm{d}\phi^T(p)$ has 1 as an eigenvalue with eigenvector $X(p)$ and the remaining eigenvalues are those of $\mathrm{d}\psi(p)$.

Proof.

The first part is a straight forward computation:

$$\begin{aligned}
\mathrm{d}\phi^T(p)(X(p)) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\phi(\phi(p,t),T) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\phi(p,t+T) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\phi(p,t) \\
&= X(p)
\end{aligned}$$

Proof.

For the second part we differentiate ψ at p and obtain for $\zeta \in T_p\Sigma$

$$\begin{aligned}
\mathrm{d}\psi(p)\zeta &= \mathrm{d}\phi^{\tau(\,\cdot\,)}(\,\cdot\,)\Big|_p \zeta \\
&= \mathrm{d}\phi^T(p)\zeta + \frac{\mathrm{d}}{\mathrm{d}t}\phi^t\Big|_{t=T} \cdot \mathrm{d}\tau(p)\zeta \\
&= \mathrm{d}\phi^T(p)\zeta + (\mathrm{d}\tau(p))(\zeta)X(p).
\end{aligned}$$

With respect to the splitting $T_pM = \operatorname{span}\{X(p)\} \oplus T_p\Sigma$, the linear map $\mathrm{d}\phi^T(p)$ is therefore of the form:

$$\mathrm{d}\phi^T(p) = \left(\begin{array}{c|c} 1 & -(\mathrm{d}\tau(p)) \\ \hline 0 & \\ \vdots & \mathrm{d}\psi(p) \\ 0 & \end{array}\right)$$

$\square$

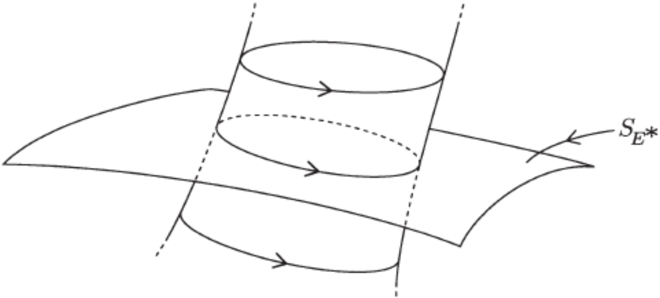
Theorem (the regular orbit cylinder theorem)

Let M be a symplectic manifold with Hamiltonian H and the Hamiltonian vector field X_H. Assume for a periodic solution $x(t, E^*)$ of X_H on M having energy $E^* = H(x(t, E^*))$ and period T^* the linear map $\mathrm{d}\phi^{T^*}(x(0, E^*))$ has exactly two eigenvalues equal to one. Then there exists a unique and smooth one parameter family $x(t, E)$ of periodic solutions with period $T(E)$ close to T^* and lying on the energy hypersurfaces

$$H(x(t, E)) = E$$

for $|E - E^*|$ sufficiently small.

Periodic solutions on energy surfaces



Proof.

Evidently the fixed points of ψ near $x(0, E^*)$ are the initial conditions of all the periodic solution near T^*. We shall use this observation in order to prove this theorem.
Claim: [Foundations of Mechanics, Theorem 8.2.2]
In our current setting we can introduce near $p = x(0, E^*)$ convenient local coordinates $(x_l, \ldots, x_{2n}) \in \mathbb{R}^{2n}$ such that p corresponds to $x^* = (E^*, 0, ..., 0)$ and such that $H(x_1, \ldots, x_{2n}) = x_1$ and moreover, such that $x_{2n} = 0$ is our hypersurface Σ.
Since $H(\phi^t(p)) = H(p)$ the section map ψ is in these coordinates expressed by

$$\psi : \begin{pmatrix} x' \\ x'' \end{pmatrix} \mapsto \begin{pmatrix} \psi'(x', x'') = x' \\ \psi''(x', x'') \end{pmatrix},$$

where the coordinates (x', x'') on Σ stand for $x' = x_1$ and $x'' = (x_2, \ldots, x_{2n-1})$.

Proof.

In order to find fixed points, we need to solve the equation

$$\psi''(x', x'') = x''.$$

In our chosen coordinates the Jacobian at x^* of ψ is given by

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline * & \dfrac{\partial}{\partial x''}\psi''(x^*) \end{array}\right).$$

By the previous Lemma the eigenvalues of $\mathrm{d}\phi^{T^*}(p)$ consist of 1 and the eigenvalue of $\mathrm{d}\psi(p)$ and since by assumption $\mathrm{d}\phi^{T^*}(p)$ only has two times the eigenvalue 1, $\frac{\partial}{\partial x''}\psi''(x^*)$ can't have the eigenvalue 1. Therefore

$$\frac{\partial}{\partial x''}\left(\psi''(x^*) - x''\right) = \frac{\partial}{\partial x''}\psi''(x^*) - \mathbb{1}$$

is bijective.

Theorem (Implicite Function Theorem)

Let $f : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a continuously differentiable function, and let $\mathbb{R}^{n+m}$ have coordinates (x, y). Fix a point $(a, b) = (a_1, \dots, a_n, b_1, \dots, b_m)$ with $f(a, b) = 0$, where $0 \in \mathbb{R}^m$ is the zero vector. If the Jacobian matrix

$$\left[\frac{\partial f}{\partial y_i}(a, b)\right]$$

is invertible, then there exists an open set $U \in \mathbb{R}^n$ containing a such that there exists a unique continuously differentiable function $g : U \to \mathbb{R}^m$ such that $g(a) = b$ and $f(x, g(x)) = 0$ for all $x \in U$.

Proof.

Therefore

$$\frac{\partial}{\partial x''}\left(\psi''(x^*) - x''\right) = \frac{\partial}{\partial x''}\psi''(x^*) - \mathbb{1}$$

is bijective. Hence by the implicit function theorem there exists a unique differentiable function $g : U \to \mathbb{R}^{2n-2}$ such that

$$\psi''(x', g(x')) - g(x') = 0.$$

This means we found a family of fixed points of ψ to which there exists a unique family of periodic orbits $x(t, E)$ with $x' = E$ and $x(0, E)$ corresponds to $(x', g(x'))$. □

Hamiltonian Manifolds

Definition

A Hamiltonian manifold is a pair (Σ, ω), where Σ is an odd-dimensional manifold and $\omega \in \Omega^2(\Sigma)$ is a closed two-form with the property that

$$\ker \omega := \bigsqcup_{p \in \Sigma} \left\{ v_p \in T_p\Sigma \mid \omega_p(v_p, \cdot\,) = 0 \in T_p^*\Sigma \right\}$$

defines a one-dimensional subbundle in $T\Sigma$.

Example

Let (M, ω) be a symplectic manifold and $H \in C^\infty(M, \mathbb{R})$ a Hamiltonian function with a regular value E_0. Then the energy hypersurface

$$\Sigma = H^{-1}(E_0) \subseteq M$$

with the restricted symplectic form $\omega\big|_\Sigma$ forms a Hamiltonian manifold $\left(\Sigma, \omega\big|_\Sigma\right)$:
Since for all $\zeta = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \gamma_\zeta(t) \in T_p\Sigma$

$$\omega_p(X_H(p), \zeta) = \mathrm{d}H_p(\zeta) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} H(\gamma_\zeta(t))$$

and H is by construction constant on Σ, we have

$$\omega_p(X_h(p), \zeta) = 0 \ \text{ for all } \zeta \in T_p\Sigma.$$

Therefore

$$\bigsqcup_{p\in\Sigma} \operatorname{span}\{X_H(p)\} \subseteq \ker \omega\big|_{\Sigma}$$

and because ω is non degenerate on M the kernel can only be one dimensional. Hence

$$\bigsqcup_{p\in\Sigma} \operatorname{span}\{X_H(p)\} = \ker \omega\big|_{\Sigma},$$

i.e. $\ker \omega\big|_{\Sigma}$ is a one-dimensional subbundle in $T\Sigma$.

Contact Manifolds

Definition (contact form for a Hamiltonian manifold)

Assume that (Σ, ω) is a Hamiltonian manifold of dimension $2n-1$. A contact form for (Σ, ω) is a one-form $\lambda \in \Omega^1(\Sigma)$ which meets the following two assumption

1. $\mathrm{d}\lambda = \omega$,
2. $\lambda \wedge \omega^{n-1}$ is a volume form on Σ.

But in fact we don't need to to assume that Σ is a Hamiltonian manifold in order to define a contact form:

Definition (contact manifold)

Let Σ be a $2n-1$ dimensional manifold and $\lambda \in \Omega^1(\Sigma)$ a one-form such that

$$\lambda \wedge (\mathrm{d}\lambda)^{n-1} \tag{2}$$

is a volume form. Then we call (Σ, λ) a contact manifold with contact form λ.

Remark

Note that not every Hamiltonian manifold admits a contact form, in fact a necessary condition for the existence of a contact form would be $[\omega] = 0 \in H^2_{dR}(\Sigma)$. But on the other hand a contact manifold can always be made into a Hamiltonian manifold by setting $\omega := \mathrm{d}\lambda$.

On a contact manifold we can then define the following vector field:

Definition (Reeb vector field)

Let (Σ, λ) be a contact manifold, then we define the Reeb vector field $R \in \Gamma(T\Sigma)$ implicitly by demanding:

$$i_R \,\mathrm{d}\lambda = 0 \quad \text{and} \quad \lambda(R) = 1 \tag{3}$$

Remark

From this definition we immediately see that the Reeb vector field is a non-vanishing section of the line bundle $\ker(\mathrm{d}\lambda) = \ker \omega$. If further Σ arises as the level set of a Hamiltonian $\Sigma = H^{-1}(0)$ on a symplectic manifold, the Reeb vector field is just a multiple of the Hamiltonian vector field.

Liouville Domains

Definition (Liouville vector field)

Let (M, ω) be a symplectic manifold. We call a vector field X, which satisfies

$$\mathscr{L}_X \omega = \omega \tag{4}$$

a Liouville vector field

We can now use this Liouville vector field to define a one form by setting $\lambda = i_X \omega$.

Proposition

Suppose that X is a Liouville vector field defined on a neighbourhood of a hypersurface $\Sigma \subset M$. Assume that X is transverse to Σ, so $T_p\Sigma \bigoplus span\{X_p\} = T_pM$ for all $p \in \Sigma$. Then $(\Sigma, (i_X\omega)\big|_\Sigma)$ is a contact manifold with contact form $\lambda := (i_X\omega)\big|_\Sigma$.

Proof.

First note that with the help of Cartan's magic formula we get:

$$\mathrm{d}i_X\omega = \mathscr{L}_X\omega - i_x\,\mathrm{d}\omega = \omega - 0 = \omega$$

Given $p \in \Sigma$, choose a basis $\{v_1, \dots, v_{2n-1}\}$ of $T_p\Sigma$. We compute

$$\begin{aligned} \lambda \wedge (\mathrm{d}\lambda)^{n-1}(v_1, \dots, v_{2n-1}) =& i_X\omega \wedge \omega^{n-1}(v_1, \dots, v_{2n-1}) \\ =& \frac{1}{n}\omega^n(X_p, v_1, \dots, v_{2n-1}). \end{aligned}$$

By assumption $\{X_p, v_1, \dots, v_{2n-1}\}$ is a basis of T_pM and since ω is non-degenerate, it follows that

$$\omega^n(X_p, v_1, \dots, v_{2n-1}) \neq 0$$

and we see that λ is indeed a contact form on Σ. □

Definition (Liouville domain)

A Liouville domain is a compact exact symplectic manifold (M, λ) with the property that the Liouville vector field X, which is in this setting implicitly defined by $i_X \, \mathrm{d}\lambda = \lambda$, is transverse to the boundary.

Corollary

Assume that (M, λ) is a Liouville domain. Then $(\partial M, \lambda|_{\partial M})$ is a contact manifold.

Proof.

This would actually be a very good exercise, so I encourage you to give it a try. □

Literature

Helmut Hofer and Eduard Zehnder: Symplectic Invariants and Hamiltonian Dynamics. Birkhäuser (1994)

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