THE KEPLER PROBLEM

Mathematical Aspects of Classical Mechanics

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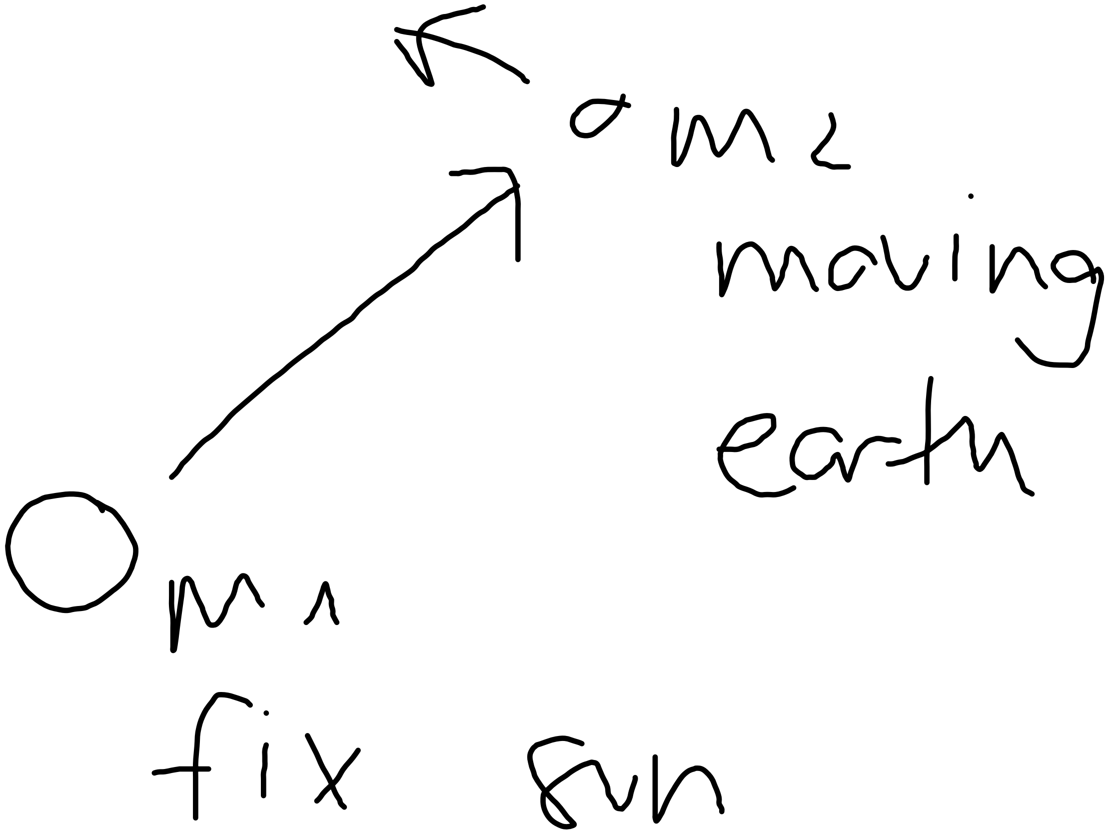
5 May 2021

What is the Kepler problem?



2-body problem

What is the Kepler problem?

For $M^n := \mathbb{R}^n \setminus \{0\}$ the Hamiltonian system $(T^*M^n, dp \wedge dq, H)$ with

momentum $H(q,p) := \frac{1}{2}|p|^2 - \frac{1}{|q|}$

position

kinetic potential

with $(q,p) \in M^n \times \mathbb{R}^n \cong T^*M^n$ is called the Kepler problem.

The case $n = 2$ is the planar Kepler problem,
whereas the case $n = 3$ is called the spatial Kepler problem.

$Sp(2n) := \{A \in Mat(2n, \mathbb{R}) : \underline{A^T J_0 A} = J_0\}$ with $J_0 := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ is a Lie group with Lie algebra $\mathfrak{sp}(2n)$.

- $Sp(2n)$ Group structure

$$A, B \in Sp(2n) \longrightarrow AB, A^{-1}, A^T \in Sp(2n)$$

- smooth submanifold

$$f : \mathbb{R}^{2n \times 2n} \longrightarrow SO(2n) = \{A \in \mathbb{R}^{n \times n} : A^T = A\}$$

$$f(A) = A^T J_0 A$$

$$df(A)B = B^T J_0 A + A^T J_0 B$$

$A \in Sp(2n)$, $C \in SO(n)$ $\exists$ $B = -\frac{1}{2} A J_0 C$

sth $df(A)B = C$.

$\rightarrow$ J_0 is regular value

$f^{-1}(J_0) = Sp(2n)$ smooth subm.

$\rightarrow$ Lie group

$$sp(2n) = T_{\mathbb{1}} Sp(2n) = \ker df\, \mathbb{1}$$

$$= \{ A \in \mathbb{R}^{2n \times 2n} \mid A^T J_0 + J_0 A = 0 \}$$

$$\omega_0(v, u) = v^T J_0 u$$

$$A^* \omega_0(v, u) = \omega_0(v, u) \qquad A \in Sp(2n).$$

$A^T A = \mathbb{1}$

$SO(n) = \{A \in Mat(\mathbb{R}, n) : A^T = A^{-1}, det(A) = 1\}$ Lie group with

$\mathfrak{so}(n) = \{A \in Mat(\mathbb{R}, n) : A^T = -A\}$ Lie Algebra.

$f : \mathbb{R}^{n \times n} \longrightarrow S(n)$

$f(A) = A^T A$ $\qquad \mathbb{1}$ reg. value

A Lie Group G is a symmetry group of a Hamiltonian system (M, ω, H) iff there exists a weakly Hamiltonian action of G on (M, ω) such that $\theta_g^* H = H \ \forall g \in G$.

Proposition: SO(n) is a symmetry group of the Kepler problem.

We define the diagonal action $\theta : SO(n) \times T^*M^n \to T^*M^n$ by $A \cdot (q,p) := (Aq, Ap)$,

We see $\theta_A^* H = H$. $\theta_A^* H = H(A\cdot, A\cdot) = H$

$\forall A \in SO(n)$

$\theta_A^* \lambda = \lambda \quad \forall A \in SO(n)$

$= p\,dq$

Momentum

Lemma

$\mu(B) = i_{\tilde{B}}(\lambda) = \langle \cdot, B \cdot \rangle$

Poisson & Hamilton

$\mu : \mathfrak{so}(n) \to C^\infty(T^*\mathbb{R}^n)$ with $\underline{\mu(B)(q,p) := q^T Bp}$ is a Lie Algebra homomorphism.

Hamiltonian equations

$$\dot{p} = -\frac{\partial \mu(B)}{\partial q} = -Bp$$

$$\dot{q} = \frac{\partial \mu(B)}{\partial p} = (q^T B)^T = B^T q = -Bq$$

$$(q(t), p(t)) = (e^{-Bt} q, e^{-Bt} p) \;\hat{=}\; \text{action of } SO(n) \text{ on } T^*\mathbb{R}^n$$

We want to find integrals of the spatial Kepler problem $n=3$

$SO(3)$

Noether's Theorem: Let G be a symmetry group of a Hamiltonian system (M, ω, H). Then $\mu(\xi)$ is an integral of motion for all $\xi \in \mathfrak{g}$. $so(3)$: $A^T = -A$

$$\beta_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \beta_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \beta_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\mu(\beta_1)(q,p) = q^T \beta_1 p = q_1 p_2 - q_2 p_1$$

$$\mu(\beta_2)(q,p) = \quad = q_1 p_3 - q_3 p_1 \quad \{H, \mu(\beta_i)\} = 0$$

$$\mu(\beta_3)(q,p) = \quad = q_2 p_3 - q_3 p_2$$

We want to find integrals of the spatial Kepler problem

We define the angular momentum as function $L : T^*M^3 \to \mathbb{R}^3$ by $L(q, p) := q \times p$.

$L = (L_1, L_2, L_3)$ $\quad L_i = \mu(p_i) \to L_i$ is integral

$\to \{H, L_i\} = 0$

$\to \{L_i, L_j\} = -\varepsilon_{ijk} L_k$

For $(q, p) \in T^*M^3$ we define the Runge-Lenz vector as function $A : T^*M^3 \to \mathbb{R}^3$ by $A(q, p) := p \times L(q, p) - \frac{q}{|q|}$.

$\dot{A} = 0 \quad \{H, A_i\} = 0 \quad \{A_i, L_j\} = -\varepsilon_{ijk} A_k$

Lemma: The spatial Kepler problem is completely integrable.

- $(H, L_i, |L|^2)$ $\qquad |L|^2 = |L_1|^2 + |L_2|^2 + |L_3|^2$

- (H, L_i, A_i)

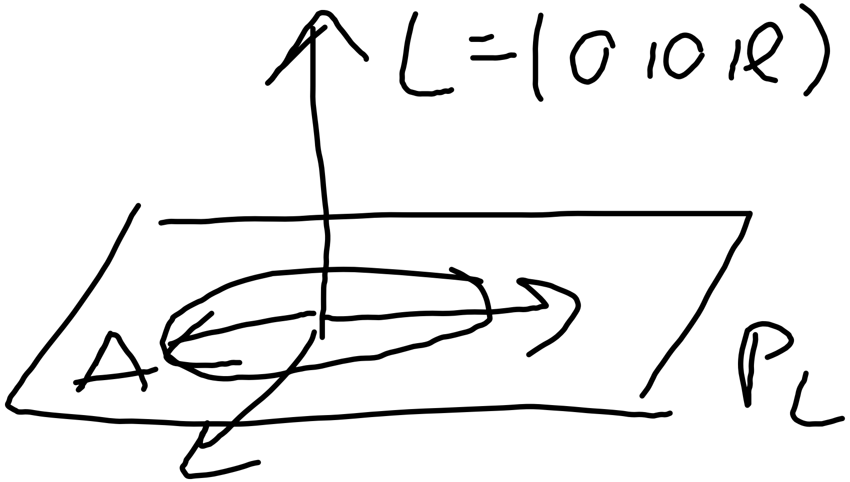
Corollary: The planar Kepler problem is completely integrable. H, L_3, A_1, A_2

Theorem: The solutions of the spatial Kepler problem are conic sections.

$$P_L = \{ v \in \mathbb{R}^3 \mid \langle v, L \rangle = 0 \}$$

$$p, q \in P_L \qquad A \in P_L$$

$$\langle A, L \rangle = \underbrace{\langle p \times L, L \rangle}_{=0} - \underbrace{\left\langle \frac{q}{|q|}, L \right\rangle}_{=0} = 0$$



$$A = (|A| \cos \vartheta, |A| \sin \vartheta, 0)$$

ϑ perihelion

$$\underline{|\underline{q}| + \langle A, \underline{q} \rangle} = \left\langle \frac{\underline{q}}{|\underline{q}|}, \underline{q} \right\rangle + \langle \underline{A}, \underline{q} \rangle$$

$$= \langle p \times L, \underline{q} \rangle = \langle q \times p, L \rangle = \underline{|L|^2}$$

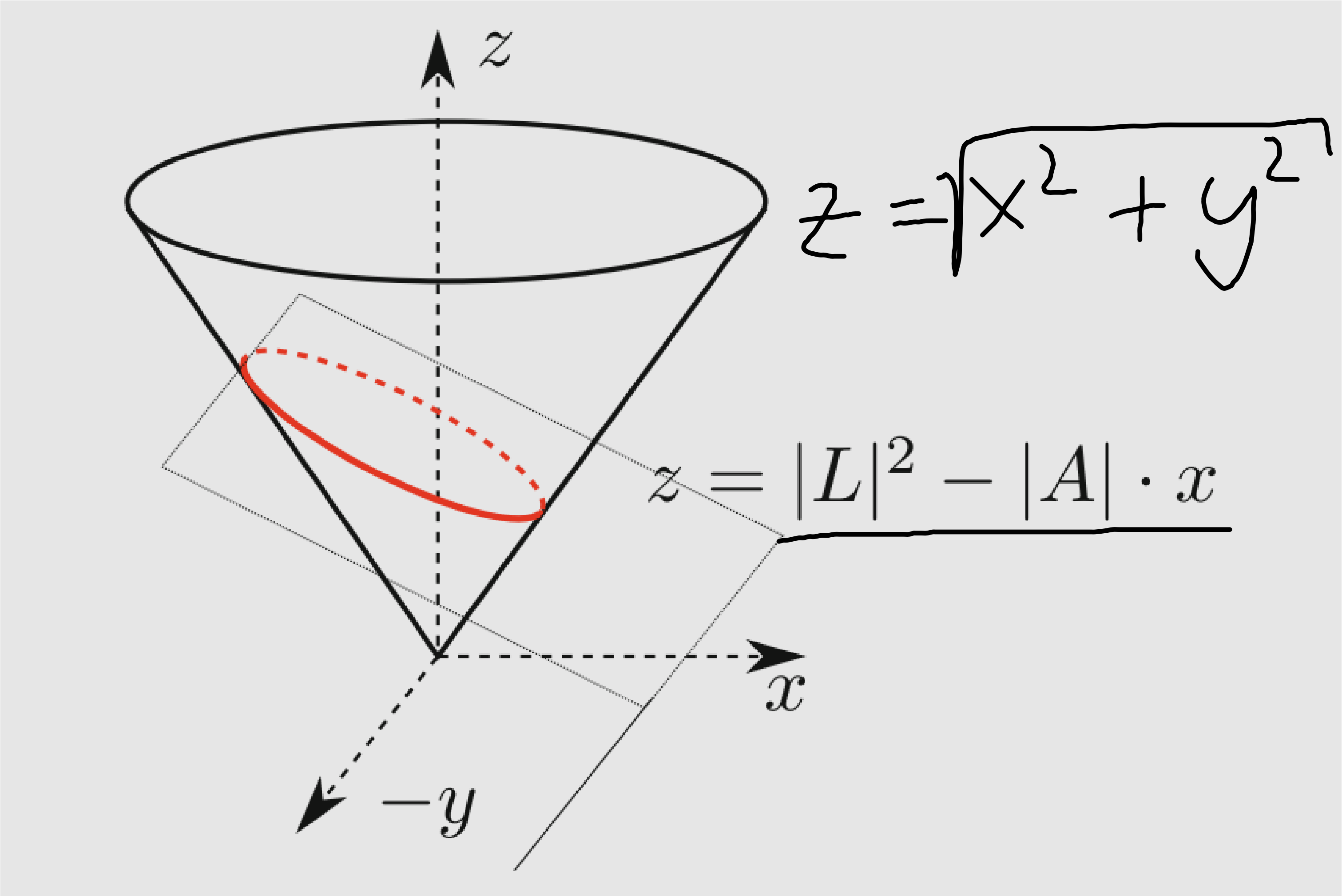
$$\underline{q} = (r \cos \varphi, r \sin \varphi, 0)$$

$$r = \frac{|L|^2}{1 + |A| \cos(\varphi - g)}$$

$|A|$ eccentricity

$\varphi - g$ true anomaly

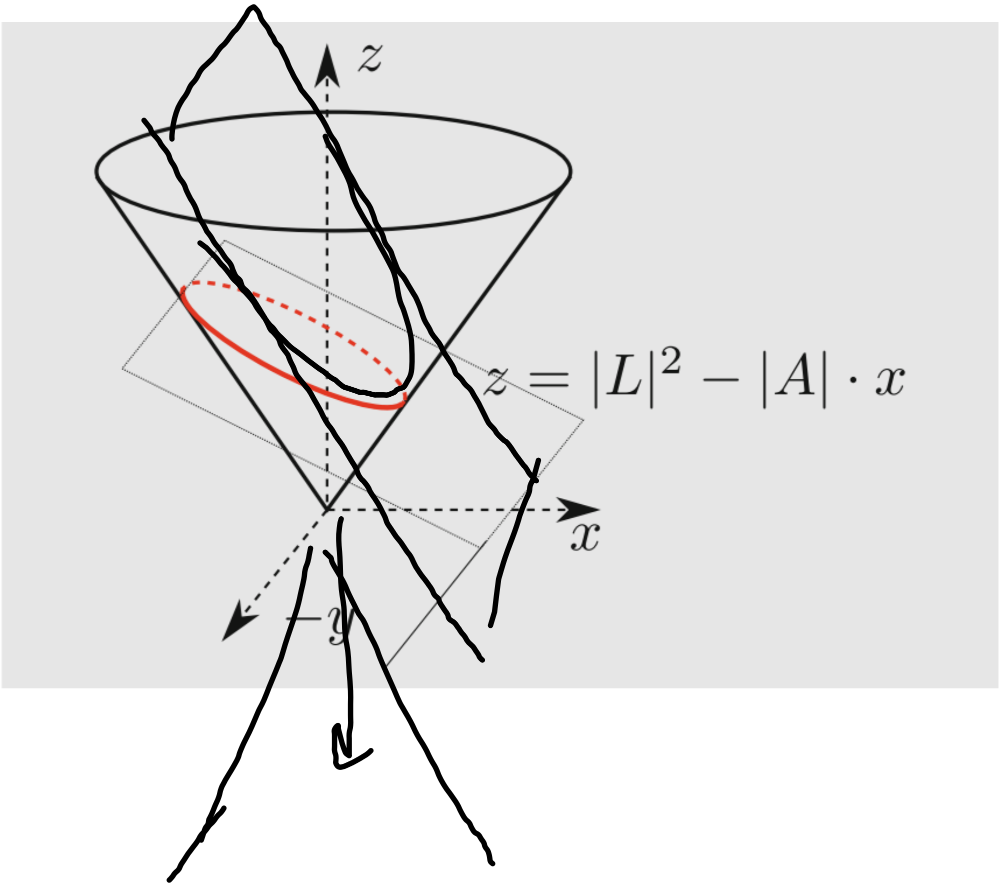
Solutions to the spatial Kepler Problem are conic sections.



$z = \sqrt{x^2 + y^2} = |q|$

$\langle A, q \rangle$
$= |A| r \cos \varphi$
$= A x$

Solutions to the spatial Kepler Problem are conic sections.



$|A| = 0 \longrightarrow$ circle

$|A| \in (0,1) \longrightarrow$ ellipse

$|A| = 1 \longrightarrow$ parabola

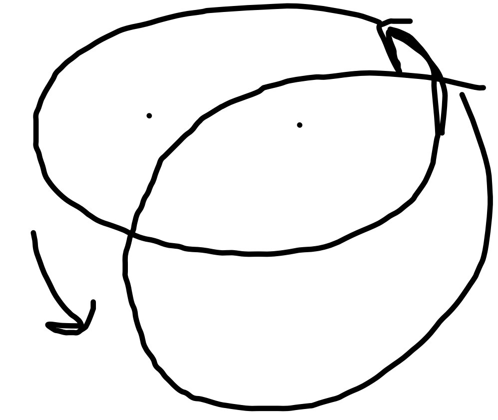
$|A| > 1 \longrightarrow$ hyperbola

$\psi - g$

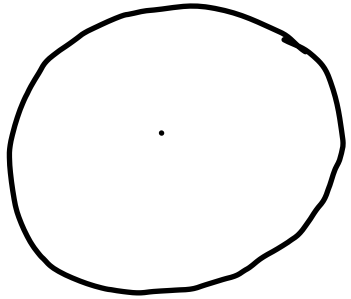
The planar Kepler problem

$n=2$

Liouville-Arnold

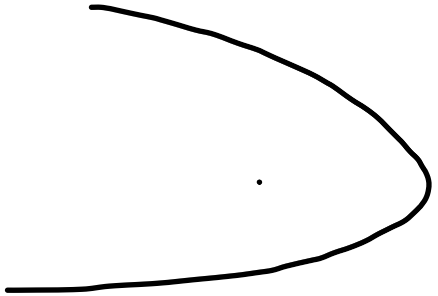


$S^1 \times S^1 = T^2$



regular value

→ no invariant tori

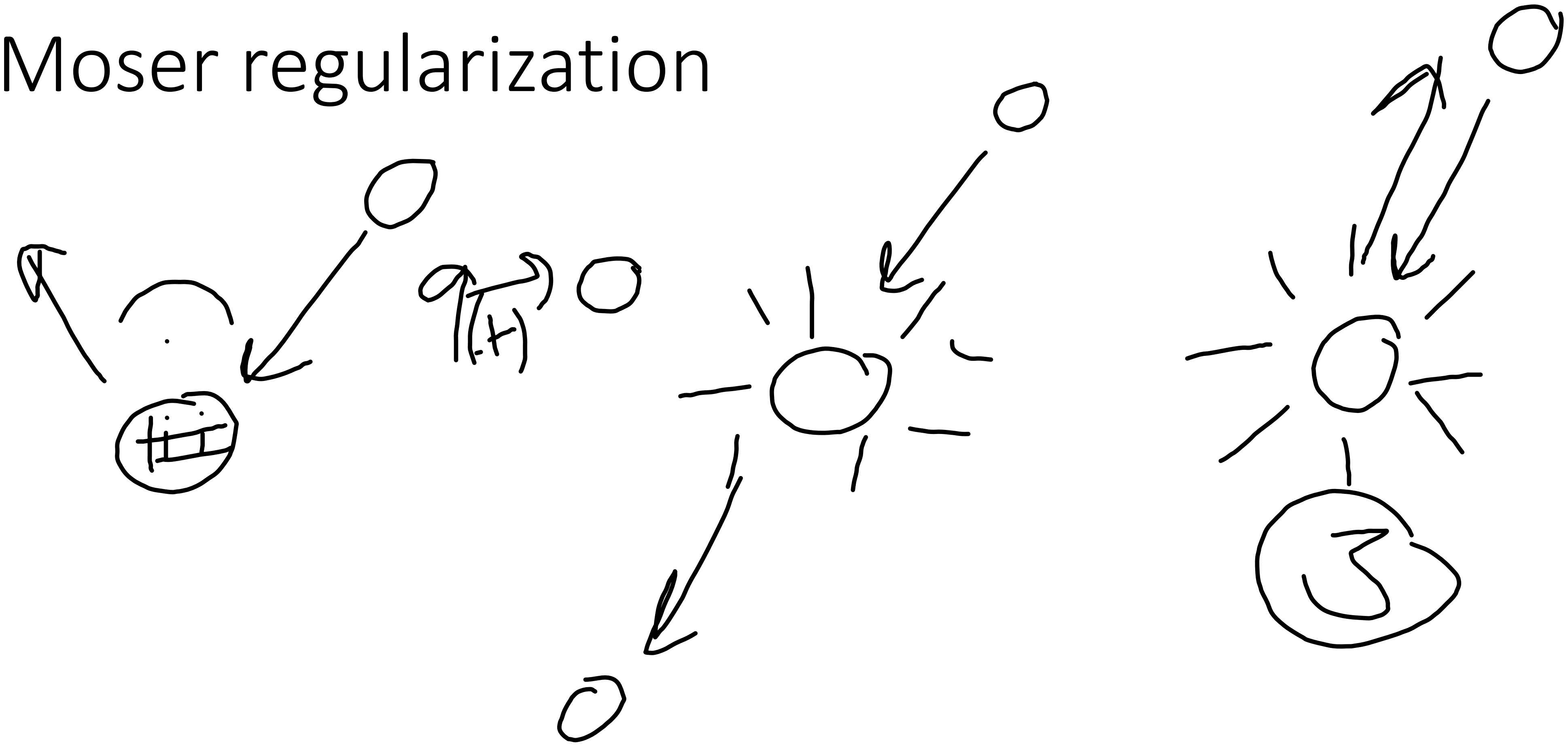


compact

→ no invariant tori

Collisions, Runge-Lenz Vector and hidden symmetries of H?

→ Moser regularization



Runge-Lenz Vector and hidden symmetries of H? ⟶ Moser regularization

We explain the Moser regularization for the planar Kepler problem for $E = -\frac{1}{2}$.

$$E(q,p) = \frac{1}{2}|p|^2 - \frac{1}{|q|} = -1/2 \qquad | \cdot |q|$$

$$\Longleftrightarrow \frac{|p|^2 |q|}{2} - 1 = -\frac{|q|}{2}$$

$$\Longleftrightarrow \underbrace{\frac{1}{2}\left(\frac{|p|^2 - 1}{2}\right)^2 |q|^2}_{K(p,-q) :=} = 1/2 \qquad |^2 \quad | \cdot 1/2$$

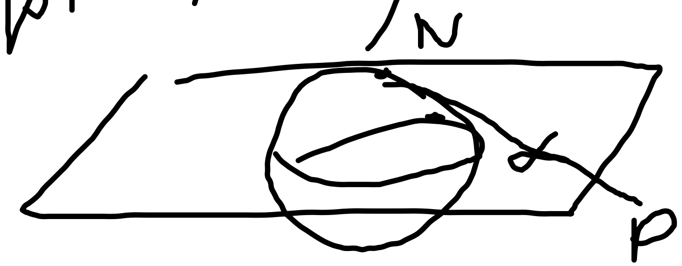
$$E^{-1}(-1/2) = K^{-1}(1/2)$$

Runge-Lenz Vector and hidden symmetries of H? $\longrightarrow$ Moser regularization

free motion $\hat{=}$ geodesic flow on the metric

$$\left(\frac{2}{|p|^2+1}\right)^2 \langle \cdot , \cdot \rangle$$

is the standard metric of S^2 with stereogr. projection



we conclude that K is the kinetic energy of the momentum $\cancel{\eta}$ p with respect to the round metric on S^2 in the chart obtained by stereographic projection.

Moreover one can show that on $T^*\mathbb{S}^2$ the $SO(3)$-symmetry of the round $\mathbb{S}^2$ is generated by (A_1, A_2, L_3). This explains the hidden symmetries of the planar Kepler problem and the definition of the Runge-Lenz vector A. These symmetries cannot be seen as they do not act on the conifguration space but rather on the phase space.

$$\boxed{\theta: G \times M \to M \text{ preserving } \lambda \rightsquigarrow D\theta^{t}: G \times T^*M \to T^*M}$$

Consider $\mathbb{R}^{2n}$ with standard symplectic form $\omega = d\lambda$. Then we have

$$\boxed{\omega(v,u) = v^t J_0 u}, \qquad J_0 := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

almost complex structure

$v,u \in T_{(x,y)}\mathbb{R}^{2n} \cong \mathbb{R}^{2n}$

$\langle v, J_0 u \rangle$

$\mathbb{R}^{2n}$ is a vector space and

$$\Phi_x \begin{cases} V \longrightarrow T_x V \\ v \longmapsto \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} x+tv \end{cases}, \quad x \in V$$

is a canonical isomorphism

does not depend on a choice of basis.

• Consequently, if $A \in SO(n)$, then

$$\theta_A^* \omega(v,u) = \omega(\theta_{A*}v, \theta_{A*}u) = (\theta_A v)^t J_0 (\theta_A u)$$
$$= (Av_1, Av_2)^t J_0 \begin{pmatrix} Au_1 \\ Au_2 \end{pmatrix}$$

diagonal action

$$= v^t \begin{pmatrix} 0 & -A^tA \\ A^tA & 0 \end{pmatrix} u$$
$$= v^t J_0 u$$
$$= \omega(v,u).$$

This is the reason why you do not let $SO(2n)$ act on $\mathbb{R}^{2n}$.

This shows also that you cannot just act on one factor.

• Also we have that

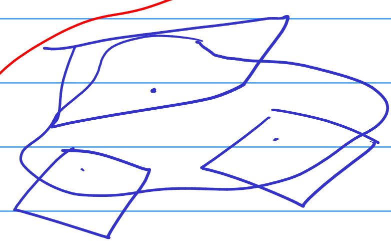
$$\theta_A^* \lambda = \lambda.$$

Why? First need a formula for λ.

$$\boxed{\lambda_{(x,y)}(v,u) = y^t v}$$

$(x,y) \in \mathbb{R}^{2n}$, $(v,u) \in \mathbb{R}^{2n}$

In canonical coordinates on phase space $\lambda = \xi_i dx^i$.



$$(\theta_A^* \lambda)_{(x,y)}(v,u)$$
$$= \lambda_{(Ax,Ay)}(Av, Au)$$
$$= (Ay)^t Av = y^t \underbrace{A^t A}_{=I} v = y^t v = \lambda_{(x,y)}(v,u).$$

Globally $\lambda_{(x,\xi)}(v) = \xi(D\pi_{T^*M}(v))$.