Lecture 4: The Poisson Algebra of Observables

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Lie Algebras and Poisson Algebras

$\mathfrak{g}$

Definition (Lie Algebra)

A ***real*** ***Lie algebra*** is defined to be a real vector space $\mathfrak{g}$ admitting a bilinear map

$$[\cdot,\cdot]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$$

called a ***Lie bracket***, satisfying the following conditions:

- $[\cdot,\cdot]$ is skew-symmetric. $[Y,X]=-[X,Y]$
- $[\cdot,\cdot]$ satisfies the ***Jacobi identity***, that is (substitute for associativity)

$$[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]]=0 \qquad \forall X,Y,Z\in\mathfrak{g}.$$

Lemma

Let M be a smooth manifold. Then $(\mathfrak{X}(M), [\cdot,\cdot])$ is a Lie algebra.

$$[\cdot,\cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

$$[X,Y] := XY - YX$$ (Standard sign convention)

Every vector field is a $C^\infty(M)$-derivation,

$$X : C^\infty(M) \to C^\infty(M), \quad f \mapsto Xf$$

[$\mathrm{Diff}(M)$ is an infinite-dimensional Lie group with Lie algebra $T_{\mathrm{id}_M}\mathrm{Diff}(M) = \mathfrak{X}(M)$.]

McDuff + Salamon

Need to check:

- $[X,Y](fg) = f[X,Y]g + g[X,Y]f$.

  $\Rightarrow [X,Y] \in \mathfrak{X}(M)$.

- Jacobi identity

Definition (Algebra of Classical Observables)

Let (M, ω) be a symplectic manifold. Then the commutative real algebra $C^\infty(M)$ of smooth functions on M is called the ***algebra of classical observables***.

In QM: non-commutative algebra of bounded self-adjoint operators on a complex separable Hilbert space.

$(\mathfrak{p}, \{\cdot,\cdot\})$ is a Lie algebra

Definition (Poisson Algebra)

A ***Poisson algebra*** is defined to be a real commutative algebra $\mathfrak{p}$ together with a Lie bracket $\{\cdot,\cdot\}$ on $\mathfrak{p}$ satisfying the ***Leibniz rule***

$$\{f, gh\} = h\{f, g\} + g\{f, h\} \qquad \forall\, f, g, h \in \mathfrak{p}.$$

$\{\cdot,\cdot\}$ serves as a derivation for the algebra product.

Lemma

Let (M, ω) be a symplectic manifold. Then $(C^\infty(M), \{\cdot,\cdot\})$ is a Poisson algebra, where

$$\{f, g\} := \omega(X_f, X_g) \qquad \forall f, g \in C^\infty(M)$$

denotes the ***Poisson bracket of classical observables****.*

The proof is a routine exercise in Cartan calculus.

- $X_{\{f,g\}} = [X_f, X_g]$ for all $f, g \in C^\infty(M)$.

This means:

$$\Phi : (C^\infty(M), \{\cdot,\cdot\}) \longrightarrow (\mathfrak{X}(M), [\cdot,\cdot])$$
$$f \longmapsto X_f$$

is a Lie algebra homomorphism.

respects both Lie algebra structures

$$\Phi\{f,g\} = [\Phi(f), \Phi(g)].$$

- $X_{\{f,g\}}$ is the unique vector field s.t.

$$i_{X_{\{f,g\}}}\omega = -d\{f,g\}.$$

If I can show $i_{[X_f,X_g]}\omega$ $\overset{/\!/}{}$

Tangent-Cotangent bundle isomorphism

$$i_{[X_F, X_g]}\omega = \mathcal{L}_{X_g} i_{X_g}\omega - i_{X_g}\mathcal{L}_{X_g}\omega = -d\{f,g\}$$

Why?

<u>Claim</u>. For all $X, Y \in \mathfrak{X}(M)$ we have

$$i_{[X,Y]} = \mathcal{L}_X i_Y - i_Y \mathcal{L}_X .$$

Proof. Note that both sides are graded derivations of degree -1. So it is enough to show equality on smooth functions and exact 1-forms.

- Show the Leibniz rule and the Jacobi-identity.

  Exercise.

Lemma

Let (M, ω) *be a symplectic manifold and* $\varphi \in \mathrm{Symp}(M, \omega)$. *Then*

$$\varphi^*\{f, g\} = \{\varphi^* f, \varphi^* g\} \qquad \forall f, g \in C^\infty(M).$$

Proof. This means that φ^* is a Lie algebra homomorphism.

$$\{\varphi^* f, \varphi^* g\} = \omega(\underbrace{X_{\varphi^* f}}_{=\varphi^* X_f}, \underbrace{X_{\varphi^* g}}_{=\varphi^* X_g})$$

$$= \omega(\cancel{D\varphi^{-1}} \circ X_f \circ \varphi, \cancel{D\varphi^{-1}} \circ X_g \circ \varphi)$$

$$= \omega(X_f, X_g) \circ \varphi = \varphi^*\{f, g\}. \qquad \square$$

$\varphi^*\omega = \omega$

$\varphi_* := (\varphi^{-1})^*$

In general $\varphi^* X_f = X_{\varphi^* f}$.

This means $\theta_t^{X_{\varphi^* f}} = \varphi^{-1} \circ \theta_t^{X_f} \circ \varphi$.

i.e. $\theta^{X_{\varphi^* f}}$ is just the conjugated flow of X_f.

If $\varphi^* f = f \Rightarrow \varphi$ and θ^{X_f} commute, i.e.

$$\varphi \circ \theta_t^{X_f} = \theta_t^{X_f} \circ \varphi.$$

(Jacobi)

The Evolution Operator

Leon A. Takhtajan: QM for Mathematicians.

Definition (Evolution Operator)

Let (M, ω, H) be a complete Hamiltonian system. Define the ***evolution operator*** $\theta : \mathbb{R} \times M \to M$

$$U_t : C^\infty(M) \to C^\infty(M), \qquad U_t(f) := f \circ \theta_t^{X_H} = (\theta_t^{X_H})^* f$$

for all $t \in \mathbb{R}$.

In QM. Let $\mathcal{H}$ be a complex separable Hilbert space. If H is self-adjoint, the the quantum evolution of a state $\psi \in \mathcal{H}$ is given by

$$\psi(t) = e^{-itH}\psi \quad \forall t \in \mathbb{R}.$$

not a matrix exponential!

so $\psi(t)$ satisfies the time-dependent Schrödinger eq. $i\frac{d\psi}{dt} = H\psi$.

Theorem

Let (M, ω, H) be a complete Hamiltonian system. Then

$$\frac{d}{dt}U_t(f) = U_t\{H, f\} \qquad \forall f \in C^\infty(M).$$

evolution equation

Proof.

Fisherman's formula

$$\frac{d}{dt}U_t(f) = \frac{d}{dt}(\theta_t^{X_H})^* f \overset{\downarrow}{=} (\theta_t^{X_H})^* \mathcal{L}_{X_H} f$$

$$= (\theta_t^{X_H})^* X_H f = (\theta_t^{X_H})^* \{H, f\} = U_t\{H, f\} \qquad \square$$

$$\{H, f\} = \omega(X_H, X_f) = df(X_H) = X_H f.$$

$$df = -i_{X_f}\omega(X_H) = -\omega(X_f, X_H)$$

Definition (Integral of Motion)

Let (M, ω, H) be a Hamiltonian system. An ***integral of motion*** is defined to be a function $I \in C^\infty(M)$ such that $\{H, I\} = 0$.

In particular, the Hamiltonian function H is an integral of motion, because

$$\{H, H\} = 0$$

due to skew-symmetry.

This is of course only true since H is assumed to be autonomous / time-independent.

Preservation of Energy:

$$H(\theta_t^{X_H}(x)) = H(x) \quad \forall (t,x) \in \mathcal{D}.$$

flow domain.

Energy hypersurfaces are preserved.

The Lie Algebra of a Lie Group

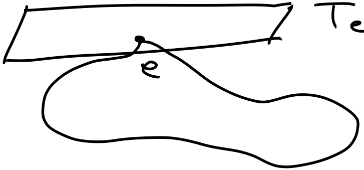
products smooth $\Rightarrow$ inverses smooth

Definition (Lie Group)

A ***Lie group*** is defined to be a group object in the category of finite-dimensional smooth manifolds.

A Lie group is a smooth manifold which is also a group.

- Matrix Lie groups: $GL(n)$, $Sp(n)$
  i.e. the Lie group of symplectic matrices.



Given a Lie group G, the tangent space $\mathfrak{g} := T_e G$ to the identity element $e \in G$ is a Lie algebra. Indeed, there is a canonical isomorphism

$X \in \mathfrak{X}_L(G)$ $:\Longleftrightarrow L_g^* X = X \ \forall g \in G$

$$T_e G \cong \mathfrak{X}_L(G),$$

$X \mapsto X(e)$.

where $\mathfrak{X}_L(G) \subseteq \mathfrak{X}(G)$ denotes the Lie subalgebra of all left-invariant vector fields on G.

If G is a Lie group, have canonical maps

$L_g, R_g : G \to G$, $L_g(h) := gh$, $R_g(h) := hg$.

As every left-invariant vector field is complete, we can define the ***exponential map***

$$\boxed{\exp\colon \mathfrak{g} \to G, \qquad \xi \mapsto \gamma_\xi(1),}$$

where $\gamma_\xi\colon \mathbb{R} \to G$ denotes the unique integral curve of $X_\xi \in \mathfrak{X}_L(G)$ with $X_\xi(e) = \xi$ such that $\gamma_\xi(0) = e$ and $\dot{\gamma}_\xi(0) = \xi$.

For matrix Lie groups:

$$e^A := \sum_{k\geq 0} \frac{A^k}{k!}, \quad A \in GL(n).$$

geodesic flow

- exp is the "usual" exponential in Riemannian geometry, if you pick a bi-invariant metric on the Lie group

Left and right invariant metrics always exist, but not bi-invariant ones. (for a compact Lie group there exist bi-invariant ones)

- Jost showed in 1970 that the symplectic structure can be encoded in Poisson structure of the algebra of classical observables.

algebraic $\Leftrightarrow$ smooth

More generally, a Poisson manifold is a smooth manifold s.t.

$$(C^\infty(M), \{\cdot, \cdot\})$$

is a Poisson algebra. given

nondegenerate Poisson manifolds correspond to symplectic manifolds

$\varphi \in \mathrm{Symp}(M,\omega) \iff \varphi \in \mathrm{Diff}(M)$
$\varphi^* \omega = \omega.$

In Riemannian geometry
$\varphi^* m = m$, φ isometry for Riemannian metric m.

The Lie algebra of $\mathrm{Symp}(M,\omega)$ is (M compact)

$$\mathfrak{X}(M,\omega) := \{X \in \mathfrak{X}(M) : d i_X \omega = 0\}$$

the Lie algebra of symplectic vector fields, i.e. s.t. $i_X\omega$ is closed.

Every Hamiltonian vector field is closed, since $i_{X_H}\omega = -dH$. Ex.