Blue Sky Catastrophes

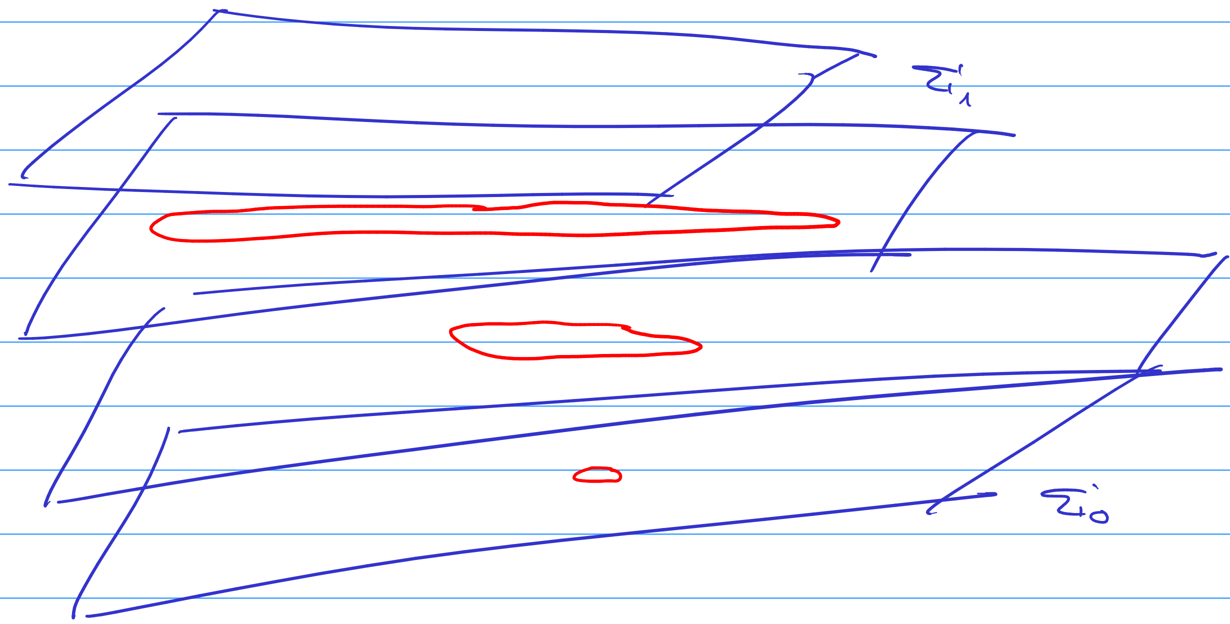
Let $(M, d\lambda)$ be an exact symplectic manifold and $H \in C^\infty(M \times [0,1])$ such that

(i) 0 is a regular value of $H_\sigma := H(\cdot, \sigma) \in C^\infty(M)$ for $\sigma \in [0,1]$.

(ii) $H^{-1}(0)$ is compact.

(iii) (Contact Condition) Set $\Sigma_\sigma := H_\sigma^{-1}(0)$,
$$dH_\sigma(X)|_{\Sigma_\sigma} > 0 \quad \forall \sigma \in [0,1],$$
where $X \in \mathfrak{X}(M)$ denotes the Liouville vector field defined by $i_X d\lambda = \lambda$.

(this just means that each $(\Sigma_\sigma, \lambda|_{\Sigma_\sigma})$ is a hypersurface of restricted contact type).



Suppose we are given a smooth family $(\gamma_\sigma, \tau_\sigma)_{\sigma \in [0,1)}$ of parametrised periodic orbits, that is,

(i) $(\gamma_\sigma, \tau_\sigma) \in \mathcal{L}\Sigma_\sigma \times (0, +\infty)$ for all $\sigma \in [0,1)$.

(ii) $\dot\gamma_\sigma(t) = \tau_\sigma X_{H_\sigma}(\gamma_\sigma(t)) \quad \forall t \in S^1$.

[$\mathcal{L}\Sigma_\sigma := C^\infty(S^1, \Sigma_\sigma)$

Can define $\gamma_\tau(t) := \gamma(t/\tau)$, $t \in \tau S^1$

Then γ_τ is an integral curve of X_H.]

Ex. Define $\overline{H} \in C^\infty(\mathcal{H} \times [0,1])$ by energy shift of a fixed Hamiltonian $H \in C^\infty(\mathcal{H})$:

$$\overline{H}(x,\sigma) := H(x) - \sigma.$$

If you have a nondegenerate periodic orbit on Σ_0, you can extend it to a family of periodic orbits locally.

Idea Study limit behaviour of the family $(\gamma_\sigma, \tau_\sigma)_{\sigma \in [0,1)}$ as $\sigma \to 1$.

For example, it might be the case that

$$\tau_\sigma \to +\infty \quad \text{as} \quad \sigma \to 1.$$

This is referred to as a blue sky catastrophe. A convenient way for studying such a limit behavior is by introducing the ω-limit set $\omega(\gamma_\sigma, \tau_\sigma)$ of the family $(\gamma_\sigma, \tau_\sigma)$, consisting of all

$$(\gamma_1, \tau_1) \in \mathcal{L}\Sigma_1 \times (0, +\infty)$$

such that there exists a sequence $(\sigma_k) \subseteq [0,1)$ with

$$\sigma_k \to 1 \quad \text{and} \quad (\gamma_{\sigma_k}, \tau_{\sigma_k}) \xrightarrow{C^\infty} (\gamma_1, \tau_1)$$

as $k \to \infty$.

Theorem (Beltrano/Frauenfelder/Koert) The ω-limit set $\omega(\gamma_\sigma, \tau_\sigma)$ is nonempty, compact and connected.

Proof. Compactness and connectedness are straightforward but tedious to prove. Note, that in order to prove $\omega(\gamma_\sigma, \tau_\sigma) \neq \emptyset$, it is enought to show that τ_σ is uniformly bounded from below and above, i.e. there exists

$C > 0$ such that

$$\boxed{0 < \frac{1}{C} \leq \tau_\sigma \leq C \quad \forall \sigma \in [0,1].}$$

Proof of $\square$.

<u>Idea</u> Recall that $(\gamma_\sigma, \tau_\sigma) \in \mathrm{Crit}\, \mathcal{A}^{H_\sigma}$ $\forall \sigma \in [0,1]$.

Note, without loss of generality, we may assume

$$(*) \qquad X_{H_\sigma}|_{\Sigma_\sigma} = R_\sigma \quad \forall \sigma \in [0,1].$$

Indeed, note that both X_{H_σ} and R_σ belong to the characteristic distribution $\ker \omega|_{\Sigma_\sigma}$.

One can check that

$$X_{H_\sigma}|_{\Sigma_\sigma} = dH_\sigma(X)|_{\Sigma_\sigma} R_\sigma \quad \forall \sigma \in [0,1].$$

So replace H by $f H$, where f_σ is an extension of $1/dH_\sigma(X)$, $f \in C^\infty(\mathcal{H}, (0,+\infty))$.

Hence we have the <u>period-action</u> equality

$$\mathcal{A}^{H_\sigma}(\gamma_\sigma, \tau_\sigma) = \int_0^1 \gamma_\sigma^* \lambda - \tau_\sigma \underbrace{\cancel{\int_0^1 H_\sigma \circ \gamma_\sigma}}_{=0}$$

$$= \int_0^1 \gamma_\sigma^* \lambda$$

$$= \int_0^1 \lambda(\dot{\gamma}_\sigma)$$

$$\overset{(*)}{=} \tau_\sigma \int_0^1 \lambda(R_\sigma \circ \gamma_\sigma)$$

$$= \tau_\sigma$$

Reeb fields satisfy $i_{R_\sigma} d\lambda = 0$ and $\lambda(R_\sigma) = 1$.

Now we can compute further

$$\partial_\sigma \tau_\sigma = \partial_\sigma (\mathcal{A}^{H_\sigma}(\gamma_\sigma, \tau_\sigma))$$

$$= (\partial_\sigma \mathcal{A}^{H_\sigma})(\gamma_\sigma, \tau_\sigma) + \cancel{d\mathcal{A}^{H_\sigma}|_{(\gamma_\sigma,\tau_\sigma)}(\partial_\sigma(\gamma_\sigma, \tau_\sigma))}$$

because $(\gamma_\sigma, \tau_\sigma)$ is a critical point of $\mathcal{A}^{H_\sigma}$.

$$= -\tau_\sigma \int_0^1 (\partial_\sigma H_\sigma) \circ \gamma_\sigma .$$

In particular

$$|\partial_\sigma \tau_\sigma| \leq k \tau_\sigma \quad \forall \sigma \in [0,1),$$

where k is such that

$$\|\partial_\sigma H_\sigma\|_{L^\infty(H^{-1}(0))} \leq k.$$

Integrating / Growall's inequality yields

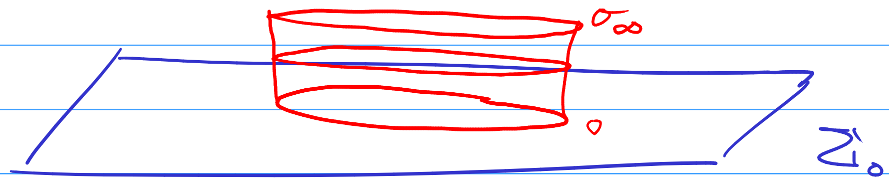
$$\boxed{\tau_0 e^{-k} \leq \tau_\sigma \leq \tau_0 e^{k} \quad \forall \sigma \in [0,1).}$$

Local Rabinowitz-Floer Homology

One can apply local Rabinowitz-Floer homology to the ω-limit set $\omega(\gamma_\sigma, \tau_\sigma)$ as it is compact and connected in the contact (stable) case.

To simplify the discussion, assume that $H \in C^\infty(M \times [0,1])$ is parametrised by energy.

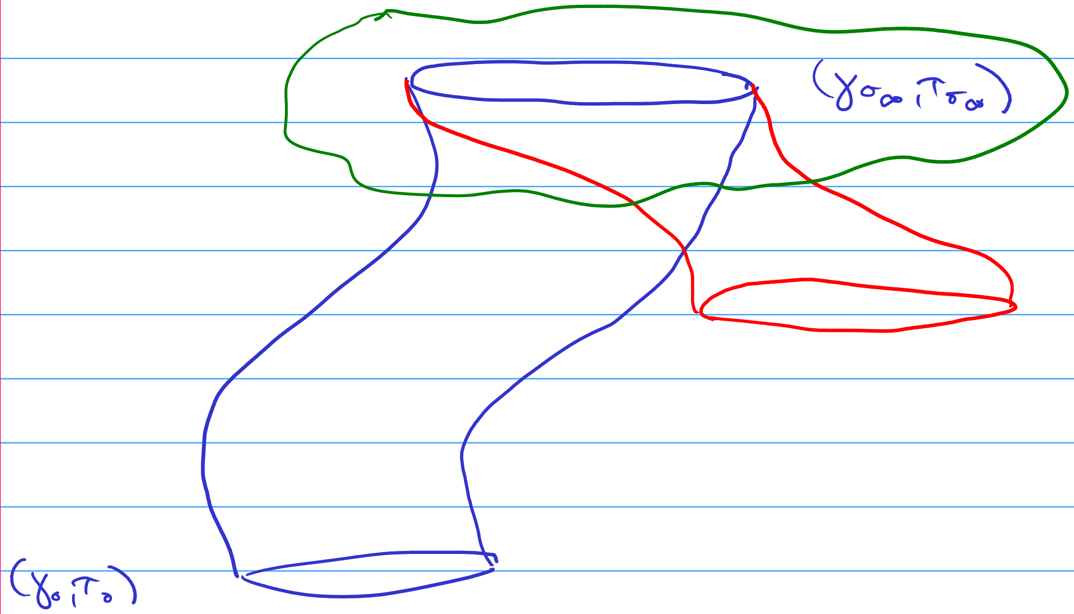
Assume (γ_0, τ_0) is a nondegenerate parametrised periodic orbit on Σ_0. By regular orbit cylinder theorem, there exists $0 < \sigma_\infty \leq 1$ and a smooth family of nondegenerate orbits $(\gamma_\sigma, \tau_\sigma)_{\sigma \in [0,\sigma_\infty)}$.



Roughly speaking, there are two options for the family.

1. The family extends across σ_∞.
2. $(\gamma_{\sigma_\infty}, \tau_{\sigma_\infty})$ is degenerate and there exists another family of parametrised periodic orbits with the same ω-limit set.

Forcing Theorem



For a construction of local RFH see the thesis by Kathrin Naef (ETH), 2018.
There must exist such a second family "killing" the first one as

$$RFH^{loc}(\Sigma_{\sigma_\infty}, \mathcal{H}|_{\Sigma_{\sigma_\infty}}, \omega(\gamma_\sigma, \tau_\sigma)) = 0.$$

Floer chain complex is generated by 1-periodic orbits
Rabinowitz chain complex is generated by τ-periodic orbits on fixed energy hypersurface.