Lecture 9: The Process of Measurement

working with measures

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Cohn "Measure Theory"

Until now, the *states* in a Hamiltonian system were simply points on the phase space. This can be generalised.

σ-algebra generated by the open/closed subsets on M

Lemma

Let M be a smooth manifold and denote by $\mathcal{B}(M)$ the Borel σ-algebra on M. Then the set of all probability measures $\mathcal{M}(M)$ on M is a convex space.

$\mu \in \mathcal{M}(M) \Leftrightarrow \mu(M) = 1,\ \mu(\emptyset) = 0$ and μ is countably additive.

Proof. Let $\mu, \nu \in \mathcal{M}(M)$.
For $\lambda \in [0,1]$ define the convex combination

$$(1-\lambda)\mu(A) + \lambda\nu(A), \quad A \in \mathcal{B}(M).$$

Then this is a probability measure on M.

$\square$

Why is this a generalisation of states?

Let M be a smooth manifold. Then there is an inclusion

Dirac measure

$$M \hookrightarrow \mathcal{M}(M), \qquad x \mapsto \delta_x. \qquad \delta_x(A) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

pure states

Representing states of a Hamiltonian system as probability measures on the phase space is well suited for *macroscopic* systems as encountered in classical statistical mechanics.

many particles $N \sim 10^{23}$

Assume M is compact. Then $\mathcal{M}(M)$ is itself a compact metric space with respect to the weak star topology, i.e.

$$\mu_k \to \mu \text{ in } \mathcal{M}(M) :\Longleftrightarrow \int_M f \, d\mu_k \to \int_M f \, d\mu$$

for all $f \in C(M)$.

This is not trivial, see Lecture 23 in Will Merry's dynamical systems notes.

With this metric, $M \hookrightarrow \mathcal{M}(M)$ is a top. embedding.

The ***expectation of a classical observable f in the state μ*** is given by

$\in C^\infty(M)$ $\in \mathcal{M}(M)$

$$\mathsf{E}_\mu(f) := \int_M f\,d\mu,$$

and the ***variance of the observable f in the state μ*** by

$$\mathsf{Var}_\mu(f) := \mathsf{E}_\mu\left((f - \mathsf{E}_\mu(f))^2\right) = \mathsf{E}_\mu(f^2) - \mathsf{E}_\mu(f)^2.$$

measures how f deviates from its expectation.

We always have that $\mathrm{var}_\mu(f) \geq 0$. Indeed, by Jensen's inequality we have

$$E_\mu(f)^2 = \left(\int_M f\,d\mu\right)^2 \underset{\mu(M)=1}{\leq} \int_M f^2\,d\mu = E_\mu(f^2).$$

Lemma

Pure states are the only states in which every observable has zero variance.

Proof. This follows from the Cauchy-Schwarz inequality. Indeed, we compute

$$\mathbb{E}_\mu(f)^2 = \mathbb{E}_\mu(f \cdot 1)^2$$

$$\leq \mathbb{E}_\mu(f^2) \cdot \mathbb{E}_\mu(1)$$

$$= \mathbb{E}_\mu(f^2).$$

But by CS, we have equality iff f and 1 are linearly dependent. Thus $f = \lambda \in \mathbb{R}$ a.e.

$\square$

Definition (Measurement)

A ***measurement*** in a Hamiltonian system (M, ω, H) is the map

$$C^\infty(M) \times \mathcal{M}(M) \to \mathcal{M}(\mathbb{R}), \qquad (f, \mu) \mapsto \mu_f := \underline{f_* \mu}.$$

For every $A \in \mathcal{B}(\mathbb{R})$, $\mu_f(A)$ is the probability that for a classical system in a state μ the result of a measurement of the observable f lies in A.

push-forward of μ, defined by

$$f_*\mu(I) := \mu(f^{-1}(I)) \quad \forall I \subset \mathbb{R} \text{ intervals.}$$

In general, if $T : X \to Y$ is measurable, then

$$T_*\mu(A) := \mu(T^{-1}(A)) \quad \forall A \in \mathcal{A}_Y.$$

Important formula:

$$\boxed{\int_Y f \, T_*\mu = \int_X \underbrace{(f \circ T)}_{T^* f} d\mu} \quad \forall f : Y \to \mathbb{R} \text{ measurable bounded}$$

Hamilton's Description of Dynamics

for few particles

The dynamics of a complete Hamiltonian system (M, ω, H) is described by

$$\frac{d\mu}{dt} = 0 \qquad \forall \mu \in \mathcal{M}(M),$$

States do not depend on time.

and ***Hamilton's equation*** (evolution equation)

$$\frac{df}{dt} = \{H, f\} \qquad \forall f \in C^\infty(M).$$

$$\boxed{f_t := f \circ \theta_t^{X_H} \quad \forall t \in \mathbb{R}}$$

The expectation is given by

$$\mathsf{E}_\mu(f_t) = \int_M f \circ \theta_t^{X_H}\, d\mu \qquad \forall t \in \mathbb{R}.$$

The Measure Associated with a Volume Form

Theorem (Stone–Weierstrass Theorem)

Let X be a locally compact Hausdorff space and $A \subseteq C_c^0(X)$ a subalgebra such that

i.e. a manifold

- *A separates the points of X.*
- *For every $x \in X$ there exists $f \in A$ such that $f(x) \neq 0$.*

Then A is uniformly dense in $C_c^0(X)$.

for all $x, y \in X$ s.t. $x \neq y$, there exists $f \in A$ s.t. $f(x) \neq f(y)$.

- $(C_c^0(X), \|\cdot\|_\infty)$, i.e.

$$\|f\|_\infty := \sup_{x \in X} |f(x)|.$$

Lemma

Let M be a smooth manifold. Then $C_c^\infty(M)$ is uniformly dense in $C_c^0(M)$.

Proof.

- Let $x, y \in M$ s.t. $x \neq y$. As M is Hausdorff, there exists $U \subseteq M$ open s.t. $y \in U$ and $x \notin U$. Pick a smooth bump function $\beta \in C_c^\infty(M)$ s.t. $\beta(y) = 1$ and $\operatorname{supp} \beta \subseteq U$. Then

$$0 = \beta(x) \neq \beta(y) = 1$$

- Take the same β for any $y \in M$.

So conclude by Stone-Weierstrass.

$\square$

Lemma

Let X be a normed space and $A \subseteq X$ dense. If $I \in A^*$ is continuous, then there exists a unique continuous extension $\overline{I} \in X^*$.

Proof. Just define

$$\overline{I}(x) := \lim_{k\to\infty} I(x_k) \quad \forall\, x \in X,\ (x_k) \subseteq A,\ x_k \to x \text{ as } k \to \infty.$$

Need to show independence of sequence.

$\square$

Do the details for yourself!

not necessarily continuous functionals

Theorem (Riesz Representation Theorem)

Let X be a locally compact Hausdorff space and $I \in C_c^0(X)^*$ a positive linear functional. Then there is a unique regular Borel measure μ on X such that

$$I(f) = \int_X f \, d\mu \qquad \forall f \in C_c^0(X).$$

If I is normalised, i.e.

$$\|I(f)\| \leq \|f\|_\infty \quad \forall f \in C_c^0(X),$$

then we actually can get a probability measure.

Lemma

volume form on M

Let (M, Ω) be a smooth oriented manifold. Then there exists a unique regular Borel measure μ_Ω on M such that

$$\int_M f\,\Omega = \int_M f\,d\mu_\Omega \qquad \forall f \in C^0_c(M).$$

Proof. Let M be compact. Define

$$I(f) := \int_M f\,\Omega \quad \forall f \in C^\infty(M).$$

Extend to $C^0(M)$ by Lemma and apply Riesz Representation Theorem.

Serge Lang "Fundamentals of Differential Geometry"

Liouville's Description of Dynamics

many particles

The dynamics of a complete Hamiltonian system (M, ω, H) is described by

$$\frac{df}{dt} = 0 \qquad \forall f \in C^\infty(M),$$

observables do not depend on time.

and by ***Liouville's equation***

$$\frac{d\rho}{dt} = -\{H, \rho\} \qquad \forall \mu \in \cancel{C^\infty(M)} \, \mathcal{M}(M),$$

where $\rho = \frac{d\mu}{d\mu_\omega}$ is the Radon–Nikodym derivative.

μ_ω is the unique regular Borel measure associated with the volume form ω^n of the symplectic manifold (M^{2n}, ω).

Liouville's equation has to be interpreted in a distributional sense.

Definition (Distribution)

Let M be a smooth manifold. A ***distribution on M*** is a continuous linear functional on $C_c^\infty(M)$. We denote the vector space of all distributions on M by $\mathscr{D}(M)$.

<u>Example</u>. If $\mu \in \mathcal{M}(M)$, then define

$$T_\mu(f) := \int_M f \, d\mu \quad \forall f \in C_c^\infty(M).$$

$$|T_\mu(f)| = \left| \int_M f \, d\mu \right|$$

$$\leq \int_M |f| \, d\mu$$

$$\leq \|f\|_\infty \int_M d\mu$$

$$= \|f\|_\infty$$

Definition (Evolution Operator)

Let (M, ω, H) be a complete Hamiltonian system. Define an ***evolution operator on the space of states***

$$W_t \colon \mathcal{M}(M) \to \mathcal{M}(M), \qquad \mu \to \mu_t$$

for all $t \in \mathbb{R}$, where $\mu_t \in \mathcal{M}(M)$ satisfies

$$W_t(\mathsf{E}_\mu) = \mathsf{E}_{\mu_t}.$$

Have $\mu \in \mathcal{M}(M)$. Then get a distribution E_μ.
Define an evolution operator on the associated distributions E_μ, $\mu \in \mathcal{M}(M)$ by $\boxed{W_t(E_\mu)(f) := \int_M f \circ \theta_t^{X_H} \, d\mu.}$

By Riesz, there exists unique regular Borel measure s.t.

$$W_t(E_\mu) = E_{\mu_t}.$$

$$W_t(T)(f) := T(f \circ \theta_t^{X_H}).$$

The expectation of a classical observable is given by

$$\mathsf{E}_{\mu_t}(f) = \int_M f\rho_{-t}\,d\mu_\omega \qquad \forall t \in \mathbb{R}.$$

Consequently, we have that

Liouville's picture

Hamilton's picture

$$\mathsf{E}_{\mu}(f_t) = \mathsf{E}_{\mu_t}(f) \qquad \forall t \in \mathbb{R},$$

for all classical observables $f \in C^\infty(M)$ and states $\mu \in \mathcal{M}(M)$, implying that "Hamilton's and Liouville's description of classical dynamics are equivalent."

Lecture 24 in Merry's notes