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The ω-Limit Set of a Family of Periodic Orbits

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To Jil.

Preface

This manuscript is the product of my master's thesis performed at the Swiss Federal Institute of Technology in Zurich under the supervision of Will J. Merry in the academic year 2019/2020. The work presented here is based on the fairly recent paper [4]. The goal was to generalise this paper to the case of families of periodic orbits on homotopies of regular energy surfaces in symplectic manifolds. For example, such families were showed to exist in the spatial lunar problem [3].

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Acknowledgements

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Contents

Chapter 1
Introduction

Celestial mechanics has a long and vivid history with impact on current research. Roughly speaking, celestial mechanics is concerned with describing the motion of *celestial bodies*, that is, planets, under their mutual influences. In this chapter we are mainly interested in the case of three bodies under their gravitational force. Describing their motion is known as the *three-body problem*. A very geometric introduction to the realm of celestial mechanics is given in [17] and a more sophisticated one in either the classic [1] or the more modern approach via holomorphic curves in [15].

Let $n \in \mathbb{N}$, $n \geq 2$, and $m_1, \ldots, m_n > 0$. Consider the Riemannian manifold $\mathbb{R}^{3n}$ with coordinates $\left(x^i, \xi_i\right)$, where $x^i = \left(x_1^i, x_2^i, x_3^i\right) \in \mathbb{R}^3$ and $\xi_i = \left(\xi_i^1, \xi_i^2, \xi_i^3\right) \in \mathbb{R}^3$ with Riemannian metric (m_{ij}) defined by the diagonal matrix

$$(m_{ij}) := \operatorname{diag}(m_1, m_1, m_1, \ldots, m_n, m_n, m_n),$$

where we identify $T^*\mathbb{R}^{3n} \cong T\mathbb{R}^{3n} \cong \mathbb{R}^{3n} \times \mathbb{R}^{3n}$. Moreover, define the ***collision set C*** by

$$C := \bigcup_{1 \leq i < j \leq n} C_{ij} \subseteq \mathbb{R}^{3n},$$

where

$$C_{ij} := \left\{(x^1, \ldots, x^n) \in \mathbb{R}^{3n} : x^i = x^j\right\}.$$

Then C is closed in $\mathbb{R}^{3n}$ and thus $M := \mathbb{R}^{3n} \setminus C$ is open in $\mathbb{R}^{3n}$. The mechanical Hamiltonian $H \in C^\infty(T^*M)$ with potential energy $V \in C^\infty(M)$, called the potential energy of the ***gravitational force***, defined by

$$V(x) := -\sum_{1 \leq i < j \leq n} \frac{m_i m_j}{|x^i - x^j|_{\mathbb{R}^3}},$$

is given by

$$H(x, \xi) := \frac{1}{2} \sum_{i=1}^n \frac{1}{m_i} |\xi_i|^2_{\mathbb{R}^3} - \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{|x^i - x^j|_{\mathbb{R}^3}}. \tag{1.1}$$

Then one checks that the corresponding Hamiltonian vector field X_H, implicitly defined by the condition

$$i_{X_H}\omega_0 = -dH,$$

where $\omega_0 = d\lambda$ denotes the standard symplectic form on a cotangent bundle with λ denoting the Liouville form given locally by $\lambda = \xi_i dx^i$, in these coordinates is given by

$$X_H = \sum_{i=1}^n \left(\frac{\partial H}{\partial \xi_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial \xi_i} \right) = \sum_{i=1}^n \frac{\xi_i}{m_i}\frac{\partial}{\partial x^i} - \sum_{i=1}^n \sum_{\substack{j=1 \\ j\neq i}} m_i m_j \frac{x^i - x^j}{|x^i - x^j|^3_{\mathbb{R}^3}}\frac{\partial}{\partial \xi_i}$$

The *problem* of the n-body problem now consists in analysing the quantitative and qualitative behaviour of the dynamics of the Hamiltonian vector field X_H, that is, solutions of Hamilton's equations

$$\dot{\gamma}(t) = \frac{\partial H}{\partial \xi}\big(\gamma(t), \xi(t)\big) \qquad \text{and} \qquad \dot{\xi}(t) = -\frac{\partial H}{\partial x}\big(\gamma(t), \xi(t)\big)$$

The problem can be solved explicitly for the case $n = 2$, since one can reduce the two-body problem to the *Kepler problem*, which can be solved analytically. See for example [17, p. 51]. However, in the case $n \geq 3$ it is believed that one cannot solve the n-body problem in the same sense as in the very first case (see the history section in the introduction of [33]). One can still ask for a qualitative behaviour. Namely, one can ask for *periodic orbits*. As an illustrative example how aribtrary the orbits of point masses can be, see figure 1.1.

The spatial restricted three-body problem in a rotating coordinate system consists of the following data: Suppose we are given two celestial bodies, e and m, called the *earth* and the *moon*, respectively, with respective normalised masses $1-\mu$ and μ for $\mu \in [0, 1]$, and a third massless body, called the *satellite*. We place the earth e at $(-\mu, 0, 0)$ and the moon m at $(1-\mu, 0, 0)$. For describing the motion of the satellite we consider the Hamiltonian function $H \in C^\infty\big(T^*(\mathbb{R}^3 \setminus \{e, m\})\big)$ given by (see [15, p. 61])

$$H(x,\xi) := \frac{1}{2}|\xi|^2 - \frac{\mu}{|x-m|} - \frac{1-\mu}{|x-e|} + x^1\xi_2 - x^2\xi_1.$$

In [3] it is shown that in Hill's lunar problem, that is, a special case of the restricted three body problem, there exists a special family of periodic orbits lying on prescribed regular energy surfaces. In the qualitative theory of topological dynamical systems one is interested in the asymptotic limit behaviour of such a family of periodic orbits. A convenient way of describing such a behaviour is provided by the notion of the ω-limit set. More precisely, given a continuous topological dynamical system (X, θ), that is, a topological space X together with a flow $\theta \in C^0(\mathbb{R} \times X, X)$, the ***ω-limit set of θ at x ∈ X***, written $\omega_\theta(x)$, consists of all points $y \in X$ such that there exists a sequence $(t_k) \subseteq \mathbb{R}$ with $t_k \to +\infty$ and $\theta_{t_k}(x) \to y$ as $k \to \infty$. It is well-known that for all $x \in X$ in a compact metric space X the ω-limit set $\omega_\theta(x)$ is nonempty,

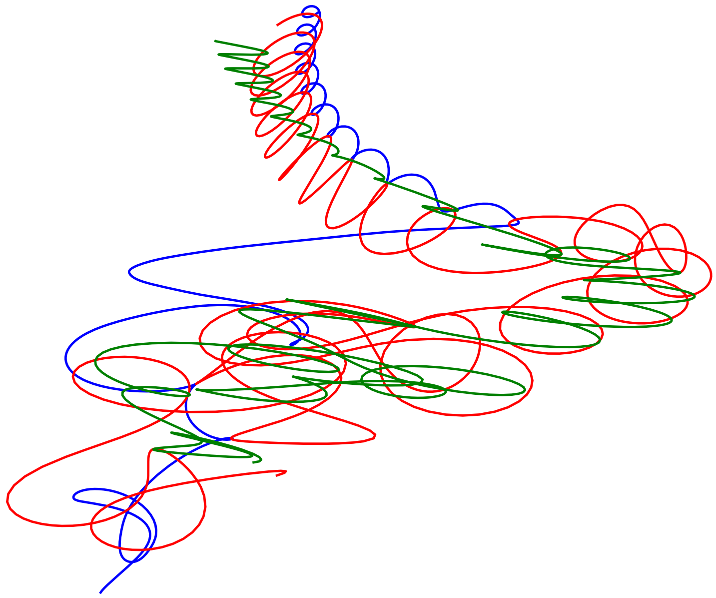


Fig. 1.1: Sample trajectories of a 3-body problem.

compact, connected and completely invariant under θ, that is, $\theta_t\left(\omega_\theta(x)\right) = \omega_\theta(x)$ for all $t \in \mathbb{R}$ and $x \in X$. For details see [20, p. 217]. A point $x \in X$ is said to be a ***recurrent point***, iff $x \in \omega_\theta(x)$, and is said to be ***periodic***, iff there exists $\tau > 0$ such that $\theta_\tau(x) = x$. Obviously, we have that any periodic point is recurrent. A point $x \in X$ is said to be a ***fixed point for*** $\boldsymbol{\theta}$, iff $\theta_t(x) = x$ for all $t \in \mathbb{R}$. We write $\mathrm{Fix}(\theta)$ for the set of fixed point for θ. A vast majority of problems in celestial mechanics are concerned with the search for periodic orbits. For an historical overview see [15, p. 93–95]. In this thesis we are not concerned with the existence problem of periodic orbits but rather with dynamical properties of such families which are assumed to exist. Our first main result is the following.

Theorem. *The ω-limit set of a family of periodic orbits on a compact stable homotopy of energy surfaces is nonempty, compact and connected.*

For a precise statement see corollary 2.88. For the restricted contact type case we define and use local Rabinowitz–Floer homology to study the behaviour of such families of periodic orbits near the ω-limit set.

Theorem. *If the ω-limit set of a family of maximally nondegenerate periodic orbits on a compact homotopy of energy surfaces of restricted contact type is isolated and cannot be extended above, then there exists another unparametrised periodic orbit sufficiently close to the ω-limit set from below.*

For the precise statement we refer to theorem 3.56.

Prerequisites: Classical mechanics is modelled using the language of symplectic geometry and the calculus of variations. Roughly speaking, symplectic geometry is the skew-symmetric analogue of Riemannian geometry and the calculus of variations is concerned with the analysis of extremal points of functionals defined on infinite dimensional spaces. For an excellent introduction to symplectic geometry see [40] and for a more extensive treatment [29]. For the functional analytic treatment of the calculus of variations see [41] and for an introduction to critical point theory in Lagrangian mechanics [27]. However, in order to properly understand the results in this thesis, we expect the reader to be familiar with most of the basics of (differential) topology, Riemannian geometry and functional analysis. An excellent introduction to the former two subjects is provided by the trilogy [24], [25], [26], and for an introduction to the latter see [28]. Moreover, for defining Rabinowitz–Floer homology one needs to be familiar with the basics of algebraic topology. A solid introduction is provided by [34].

Preliminaries: We assume that all manifolds are finite-dimensional and smooth, that is, of class C^∞, and unless otherwise specified without boundary. Given a smooth function $f \in C^\infty(M)$ on a smooth manifold M, we denote by

$$\operatorname{Crit}(f) := \{x \in M : df_x = 0\}$$

the set of ***critical points of f***. A ***symplectic manifold*** is defined to be a tuple (M, ω), where M is a smooth manifold and $\omega \in \Omega^2(M)$ is closed and nondegenerate, that is, for every $x \in M$ the induced map

$$\widehat{\omega}_x \colon T_x M \to T_x^* M, \qquad \widehat{\omega}_x(v) := i_v \omega_x$$

is an isomorphism. Any symplectic manifold (M, ω) is necessarily of even dimension and orientable with volume form ω^n. The prototypical example of a symplectic manifold is the cotangent bundle T^*M of a smooth manifold M with the standard symplectic form $\omega_0 := d\lambda$, where $\lambda \in \Omega^1(T^*M)$ is the ***Liouville form*** defined by

$$\lambda_{(x,\xi)}(v) := \xi\left(D\pi_{(x,\xi)}(v)\right), \qquad \forall (x, \xi) \in T^*M, v \in T_{(x,\xi)}T^*M,$$

where $\pi \in C^\infty(T^*M, M)$ denotes the canonical projection. Given $H \in C^\infty(M)$, we can uniquely define the Hamiltonian vector field $X_H \in \mathfrak{X}(M)$ by

$$X_H := -\widehat{\omega}^{-1}(dH) \tag{1.2}$$

Equivalently, X_H is the unique vector field satisfying $i_{X_H}\omega = -dH$. Indeed, we compute

$$i_{X_H}\omega = \widehat{\omega}(X_H) = -\widehat{\omega}\left(\widehat{\omega}^{-1}(dH)\right) = -dH.$$

More generally, a vector field $X \in \mathfrak{X}(M)$ on a symplectic manifold (M, ω) is said to be ***symplectic***, iff $i_X\omega$ is closed. The tuple $\big((M, \omega), H\big)$ is called a ***Hamiltonian system*** and H is called a ***Hamiltonian function on M***. For convenience, we usually write (M, ω, H) for a Hamiltonian system. Every symplectic manifold (M, ω) admits an ***ω-compatible almost complex structure***, that is, a smooth tensor field

$$J \in \Gamma\left(T^{(1,1)}TM\right) = \Gamma\left(\mathrm{End}(TM)\right)$$

such that

$$J \circ J = \mathrm{id}_{TM} \qquad \text{and} \qquad m_J := \omega \circ (J \times \mathrm{id}_{TM})$$

defines a Riemannian metric on M.

Let R be a ring with identity and denote by ${}_R\mathsf{Mod}$ the category of unital left R-modules. A ***$\mathbb{Z}$-graded chain complex in ${}_R\mathsf{Mod}$*** is defined to be a tuple $(C_\bullet, \partial_\bullet)$, consisting of a $\mathbb{Z}$-sequence

$$\ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots$$

of objects and morphisms in ${}_R\mathsf{Mod}$ such that

$$\partial_n \circ \partial_{n+1} = 0 \qquad \forall n \in \mathbb{Z}.$$

A chain complex $(C_\bullet, \partial_\bullet)$ is said to be ***nonnegative***, iff $C_n = 0$ for all $n < 0$. There is a category $\mathrm{Ch}({}_R\mathsf{Mod})$ of $\mathbb{Z}$-graded chain complexes in ${}_R\mathsf{Mod}$. Moreover, there exists a $\mathbb{Z}$-family of functors, called the ***homology functors***,

$$\mathrm{H}_\bullet\colon\ \mathrm{Ch}({}_R\mathsf{Mod}) \to {}_R\mathsf{Mod}$$

defined on objects by

$$\mathrm{H}_n(C_\bullet, \partial_\bullet) := \ker \partial_n / \operatorname{im} \partial_{n+1}, \qquad \forall n \in \mathbb{Z}.$$

Chapter 2
Blue Sky Catastrophes

In this section, we study the limit behaviour of a family of periodic orbits on homotopies of regular energy surfaces in symplectic manifolds. In order to do so, in the first section we will recall standard properties of regular energy surfaces situated in Hamiltonian systems. In the second section, we will give a variational approach for detecting such periodic orbits and in the third and last section we will prove that under certain reasonable assumptions, the ω-limit set of such families of periodic orbits is nonempty, compact and connected. The second and third section is based on the fairly recent paper [4].

2.1 Regular Energy Surfaces

Following [20], the existence problem we briefly describe originates in the search for periodic solutions in celestial mechanics initiated by Poincaré. The main theorem of this section, that is, *Poincaré's recurrence theorem*, is the corner stone of Hamiltonian dynamics. In order to prove this result we need to recall two fundamental properties of Hamiltonian systems.

Proposition 2.1 ([40, p. 127]). *Let (M, ω, H) be a Hamiltonian system and denote by $\theta \in C^\infty(\mathcal{D}, M)$ the flow of X_H. Then $(\theta_t^*\omega)_x = \omega_x$ for all $(t, x) \in \mathcal{D}$.*

Proof. Using Fisherman's formula[1] [25, Proposition 22.14] together with Cartan's magic formula [25, Theorem 14.35] we compute

$$\frac{d}{dt}\theta_t^*\omega = \theta_t^*\mathcal{L}_{X_H}\omega = \theta_t^*\left(i_X d\omega + d i_X\omega\right) = \theta_t^*\left(-(d \circ d)H\right) = 0.$$

Thus $\theta_t^*\omega$ is constant and since $\theta_0^*\omega = \omega$ the statement follows. □

[1] In [21], the Lie derivative is referred to as the *fisherman's derivative*. Thanks to Ana Cannas da Silva for pointing this out to me.

Remark 2.2. Proposition 2.1 illustrates well why we require a symplectic form to be both closed and nondegenerate.

Proposition 2.3 (Preservation of Energy, [15, p. 12]). *Let (M, ω, H) be a Hamiltonian system and let $\theta\colon \mathcal{D} \to M$ denote the flow of X_H. Then*

$$H\left(\theta_t(x)\right) = H(x)$$

for all $(t, x) \in \mathcal{D}$.

Proof. Let $x \in M$. Then $\mathcal{D}^{(x)} := \{t \in \mathbb{R} : (t, x) \in \mathcal{D}\}$ is an open interval containing 0 by definition of the flow domain $\mathcal{D}$. We compute

$$\begin{aligned}
\frac{d}{dt} H\left(\theta_t(x)\right) &= dH_{\theta_t(x)}\left(\frac{d}{dt}\theta_t(x)\right)\\
&= \left(i_{X_H}\omega\right)_{\theta_t(x)}\left(\frac{d}{dt}\theta_t(x)\right)\\
&= i_{X_H|_{\theta_t(x)}}\left(\omega_{\theta_t(x)}\right)\left(\frac{d}{dt}\theta_t(x)\right)\\
&= \omega_{\theta_t(x)}\left(X_H|_{\theta_t(x)}, \frac{d}{dt}\theta_t(x)\right)\\
&= \omega_{\theta_t(x)}\left(X_H|_{\theta_t(x)}, X_H|_{\theta_t(x)}\right)\\
&= 0
\end{aligned}$$

for all $t \in \mathcal{D}^{(x)}$ and thus

$$H\left(\theta_t(x)\right) = H\left(\theta_0(x)\right) = H(x).$$

$\square$

Proposition 2.3 has a geometric interpretation. Let (M, ω, H) be a Hamiltonian system and suppose that $\Sigma := H^{-1}(0)$. Then by preservation of energy, Σ is invariant under the flow $\theta\colon \mathcal{D} \to M$ of X_H, that is, if $x \in \Sigma$ then also $\theta_t(x) \in \Sigma$ for all $t \in \mathcal{D}^{(x)}$. If $\operatorname{Crit}(H) \cap \Sigma = \varnothing$, then Σ is an embedded hypersurface by the implicit function theorem for manifolds [25, p. 106], that is, Σ is a codimension 1 embedded submanifold of M. Moreover, we have that

$$D\iota_x\left(T_x\Sigma\right) = \ker dH_x$$

for all $x \in \Sigma$, where $\iota\colon \Sigma \hookrightarrow M$ denotes inclusion by exercise 5.40 [25, p. 118]. Thus X_H restricts to define a nowhere-vanishing vector field on Σ. Indeed, we have that

$$dH(X_H) = -i_{X_H}\omega(X_H) = -\omega\left(X_H, X_H\right) = 0,$$

and $X_H(x) \neq 0$ for all $x \in \Sigma$ as $X_H(x) = 0$ would imply that $x \in \operatorname{Crit}(H)$. Moreover, $X_H|_\Sigma$ spans the line distribution $\ker \omega|_\Sigma$ as for $x \in \Sigma$ and $v \in T_x\Sigma$ we compute

$$i_{X_H(x)}\omega(v) = -dH_x(v) = 0.$$

In the following, we will tacitly assume the convention in [25, p. 116] regarding the tangent space of an embedded hypersurface $\Sigma \subseteq M$, that is, we identify $T_x\Sigma$ as a linear subspace of T_xM for all $x \in \Sigma$ via the injection $d\iota_x : T_x\Sigma \to T_xM$.

Definition 2.4 (Regular Energy Surface, [20, p. 105]). A ***regular energy surface in a Hamiltonian system*** (M, ω, H) is defined to be an embedded hypersurface $\Sigma = H^{-1}(0)$ such that $\mathrm{Crit}(H) \cap \Sigma = \varnothing$, that is, 0 is a regular value of H. In this setting, the Hamiltonian function H is said to be a ***defining Hamiltonian for the energy surface*** Σ.

Remark 2.5. Let (M, ω, H) be a Hamiltonian system such that $E \in \mathbb{R}$ is a regular value of H, then 0 is a regular value of $\widetilde{H} \in C^\infty(M)$ defined by $\widetilde{H} := H - E$. Thus $\Sigma = \widetilde{H}^{-1}(0) = H^{-1}(E)$ is a regular energy surface in the shifted Hamiltonian system $(M, \omega, \widetilde{H})$. Moreover, we have that $X_H = X_{\widetilde{H}}$ as

$$X_{\widetilde{H}} = \widehat{\omega}^{-1}\left(d\widetilde{H}\right) = \widehat{\omega}^{-1}\left(dH - dE\right) = \widehat{\omega}^{-1}(dH) = X_H.$$

Remark 2.6 ([20, p. 113–114]). An embedded hypersurface $\Sigma \subseteq M$ in a symplectic manifold (M, ω) admits a local defining Hamiltonian, that is, a defining Hamiltonian in a neighbourhood of Σ, if and only if the line bundle $\ker \omega|_\Sigma \to \Sigma$ is orientable.

Remark 2.7. Let Σ be a regular energy surface in a Hamiltonian system (M, ω, H) such that H is a proper map. Then Σ is compact.

Remark 2.8 ([20, p. 21–22]). Let Σ be a compact regular energy surface in a Hamiltonian system (M, ω, H) and suppose that $\widetilde{H} \in C^\infty(M)$ is another defining Hamiltonian function for Σ. Then both X_H and $X_{\widetilde{H}}$ span the line distribution $\ker \omega|_\Sigma$ and as X_H as well as $X_{\widetilde{H}}$ are nowhere-vanishing on Σ we have that $X_{\widetilde{H}} = fX_H$ for $f \in C^\infty(\Sigma)$ nowhere-vanishing. It follows that the flows of X_H and $X_{\widetilde{H}}$ coincide modulo reparametrisation, and thus in particular also the periodic orbits.

Example 2.9. Let (M, m) be a Riemannian manifold. Consider the Hamiltonian system (T^*M, ω_0, H) where H is the mechanical Hamiltonian defined by

$$H(x, \xi) := \frac{1}{2}|\xi|^2_{m^*},$$

where m^* denotes the induced dual Riemannian metric on T^*M given by

$$m^*_x(\xi, \eta) := g_x\left(\widehat{g}^{-1}(\xi), \widehat{g}^{-1}(\eta)\right)$$

for all $x \in M$ and $\xi, \eta \in T^*_xM$, denoting by $\widehat{g} \in C^\infty(TM, T^*M)$ the tangent-cotangent bundle isomorphism [25, p. 341]. Let $\sigma > 0$ and define $\Sigma_\sigma := H^{-1}(\sigma)$. Then $\Sigma_\sigma \cong UT^*M$ for all $\sigma > 0$ after rescaling with $\sqrt{2\sigma}$, where

$$UT^*M := \{(x, \xi) : |\xi|_{g^*} = 1\}$$

denotes the unit cotangent bundle.

Example 2.10 (Jacobi Metric, [20, p. 130]). Let (M, m) be a Riemannian manifold. Consider the Hamiltonian system (T^*M, ω_0, H) where H is the mechanical Hamiltonian defined by kinetic energy plus potential energy $V \in C^\infty(M)$

$$H(x, \xi) := K(x, \xi) + V(x),$$

with

$$K(x, \xi) := \frac{1}{2} |\xi|^2_{m^*},$$

where m^* denotes the dual Riemannian metric on T^*M as in example 2.9. Suppose $E \in \mathbb{R}$ is a regular value of H and $E > \sup_{x \in \Sigma_E} V(x)$, where $\Sigma_E := H^{-1}(E)$. Then we can define another Hamiltonian, called the ***Jacobi metric***, by

$$\widetilde{H}(x, \xi) := \frac{K(x, \xi)}{E - V(x)} \in C^\infty(T^*M).$$

Then $\Sigma_E = \widetilde{H}^{-1}(1)$ and

$$X_H = K X_{\widetilde{H}}$$

on Σ_E, that is, the Hamiltonian vector fields X_H and $X_{\widetilde{H}}$ are positively parallel on Σ_E. Indeed, in local coordinates (x^i, ξ_i) on T^*M we have that

$$X_H = \frac{\partial H}{\partial \xi_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial \xi_i}$$

and

$$\begin{aligned} X_{\widetilde{H}} &= \frac{1}{c - V} \frac{\partial H}{\partial \xi_i} \frac{\partial}{\partial x^i} - \frac{1}{c - V} \frac{\partial K}{\partial x^i} \frac{\partial}{\partial \xi_i} - \frac{K}{(c - V)^2} \frac{\partial V}{\partial x^i} \frac{\partial}{\partial \xi_i} \\ &= \frac{1}{c - V} \left(\frac{\partial H}{\partial \xi_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial \xi_i} \right) \\ &= \frac{1}{c - V} X_H \end{aligned}$$

by the fact that $K/(c - V) = 1$ on Σ_E.

Let Σ be a regular energy surface in a Hamiltonian system (M, ω, H). By definition, there is a single tangential direction not contained in the tangent space of Σ. There is a convenient description of this single non-tangential direction in terms of the surrounding structures. Since $\operatorname{Crit}(H) \cap \Sigma = \varnothing$ there exists an open neighbourhood U of Σ in M such that $\operatorname{Crit}(H) \cap U = \varnothing$. Indeed, $\Sigma = H^{-1}(0)$ is closed by definition and $dH_x \neq 0$ for all $x \in \Sigma$. Suppose that J is an ω-compactible almost complex structure on M. Denote by m_J the Riemannian metric induced by the almost complex structure J and define the normalised gradient vector field $X \in \mathfrak{X}(U)$ by

$$X := \operatorname{grad}_{m_J} H / |\operatorname{grad}_{m_J} H|^2_{m_J}.$$

Note that X is well-defined as $\operatorname{grad}_{m_J} H \neq 0$ on U. Then we have two splittings

$$TM|_\Sigma = T\Sigma \oplus X \qquad \text{and} \qquad TM|_\Sigma = T\Sigma \oplus JX_H.$$

Indeed, the first follows readily from the definition of X since $dH(X) = 1$ and thus X does not belong to $T\Sigma$. For the second splitting it is enough to show that $dH(JX_H)$ is nowhere-vanishing on Σ for dimensional reasons. On one hand we compute

$$\omega(X, X_H) = -\omega(X_H, X) = dH(X) = 1,$$

and on the other

$$\omega(X, X_H) = \omega(JX, JX_H) = m_J(X, JX_H) = dH(JX_H)/\left|\operatorname{grad}_{m_J} H\right|^2_{m_J}.$$

This implies

$$dH(JX_H) = \left|\operatorname{grad}_{m_J} H\right|^2_{m_J} \neq 0,$$

which yields the desired splitting. This was to be expected since

$$dH(JX) = \left|\operatorname{grad}_{m_J} H\right|^2_{m_J} m_J(X, JX) = \left|\operatorname{grad}_{m_J} H\right|^2_{m_J} \omega(JX, JX) = 0.$$

By

$$dH(JX) = -\omega(X_H, JX) = \omega(JX, X_H) = m_J(X, X_H)$$

we also deduce that X and X_H are orthogonal on Σ with respect to the Riemannian metric m_J and JX does span the line distribution $\ker \omega|_\Sigma$ for

$$i_{JX}\omega|_\Sigma = i_X m_J = dH.$$

But in general X and X_H are not orthonormal since

$$\begin{aligned} |X_H|^2_{m_J} &= m_J(X_H, X_H) \\ &= \omega(JX_H, X_H) \\ &= -\omega(X_H, JX_H) \\ &= dH(JX_H) \\ &= \left|\operatorname{grad}_{m_J} H\right|^2_{m_J}. \end{aligned}$$

We now turn on measure theoretical considerations regarding regular energy hypersurfaces in Hamiltonian systems, that is, we consider them as measure-theoretical dynamical systems. Let X be a topological space and consider the convex space of probability measures on $\big(X, \mathscr{B}(X)\big)$, that is, Borel measures μ such that $\mu(X) = 1$. Given a topological dynamical system (X, F), that is, $F \in C^0(X, X)$, we say that a probability measure μ is ***F-invariant***, iff

$$\mu\left(F^{-1}(A)\right) = \mu(A), \qquad \forall A \in \mathscr{B}(X).$$

Equivalently, we can reformulate this condition in terms of the affine pushforward map

$$F_*\mu(A) := \mu\left(F^{-1}(A)\right), \qquad \forall A \in \mathscr{B}(X).$$

Clearly, the pushforward measure $F_*\mu$ is again a probability measure on X. For more details see [38, p. 50]. Moreover, for every Borel-measurable function $u\colon X \to \mathbb{R}$ we have that

$$\int_X (u \circ F) d\mu = \int_X u d(F_*\mu). \tag{2.1}$$

Proposition 2.11. *Let (M, η) be a compact oriented smooth manifold of positive dimension and suppose that $F \in \mathrm{Diff}(M)$ such that $F^*\eta = \eta$. Then there exists a unique regular F-invariant probability measure μ_η such that*

$$\int_M f\eta = \int_M f d\mu_\eta$$

holds for all $f \in C^\infty(M)$. The measure μ_η is called the ***measure associated to the volume form η****.*

Proof. Define a nonnegative normalised linear functional $I_\eta \in \left(C^\infty(M)\right)^*$ by

$$I_\eta(f) := \int_M f\widetilde{\eta}, \qquad \text{where} \qquad \widetilde{\eta} := \eta / \textstyle\int_M \eta.$$

Then I_η is well-defined by positivity [25, p. 407]. Because $C^\infty(M)$ is dense in $C^0(M)$ by the Stone-Weierstrass theorem [39, p. 27], we can extend I_η uniquely to a nonnegative linear functional $I_\eta \in \left(C^0(M)\right)^*$. Thus by the Riesz Representation theorem [10, p. 192] there exists a unique regular probability measure μ_η such that

$$I_\eta(f) = \int_M f d\mu_\eta$$

holds for all $f \in C^0(M)$. Since $F^*\eta = \eta$, we have that F is an orientation preserving diffeomorphism, and thus by diffeomorphism invariance [25, p. 407] we compute

$$I_\eta(f) = \int_M f\widetilde{\eta} = \int_M F^*(f\widetilde{\eta}) = \int_M (f \circ F)F^*\widetilde{\eta} = \int_M (f \circ F)\widetilde{\eta} = I_\eta(f \circ F)$$

for all $f \in C^\infty(M)$ and by density also for all $f \in C^0(M)$. In particular, we have that

$$\int_M f d\mu_\eta = \int_M (f \circ F) d\mu_\eta = \int_M f d(F_*\mu_\eta)$$

for all $f \in C^0(M)$ by 2.1. But $F_*\mu_\eta$ is a regular Borel measure by proposition 7.2.3 [10, p. 190], so by the uniqueness part of the Riesz representation theorem we conclude that

$$F_*\mu_\eta = \mu_\eta.$$

□

Lemma 2.12. *Let M be a smooth manifold. Suppose that $\eta \in \Omega^1(M)$ is nowhere vanishing and $\xi \in \Omega^k(M)$. Then $\eta \wedge \xi = 0$ if and only if there exists $\zeta \in \Omega^{k-1}(M)$ such that $\xi = \eta \wedge \zeta$.*

Proof. Suppose $\eta \wedge \xi = 0$. It is enough to show the existence of $X \in \mathfrak{X}(M)$ such that $\eta(X) = 1$. Indeed, set $\zeta := i_X(\xi)$. Then we compute

$$\eta \wedge \zeta = \eta \wedge i_X(\xi) = i_X(\eta) \wedge \xi - i_X(\eta \wedge \xi) = \eta(X)\xi = \xi.$$

To show that such a vector field X exists, let m be any Riemannian metric on M. Moreover, denote by $\widehat{m} \in C^\infty\left(\mathfrak{X}(M), \Omega^1(M)\right)$ the induced tangent-cotangent bundle isomorphism, that is

$$\widehat{m}(X)(Y) = m(X, Y) \qquad \text{for } X, Y \in \mathfrak{X}(M).$$

Define

$$X := \widehat{m}^{-1}(\eta) / \left|\widehat{m}^{-1}(\eta)\right|_m^2 .$$

Then $X \in \mathfrak{X}(M)$ since η is nowhere vanishing and

$$\begin{aligned}
\eta(X) &= \widehat{m}\left(\widehat{m}^{-1}(\eta)\right)(X) \\
&= m\left(\widehat{m}^{-1}(\eta), X\right) \\
&= m\left(\widehat{m}^{-1}(\eta), \widehat{m}^{-1}(\eta)\right) / \left|\widehat{m}^{-1}(\eta)\right|_m^2 \\
&= 1.
\end{aligned}$$

The other direction is immediate. □

Proposition 2.13. *Let Σ be a compact regular energy surface in a Hamiltonian system (M^n, ω, H). Denote by $\theta \in C^\infty(\mathbb{R} \times \Sigma, \Sigma)$ the flow of X_H on Σ. Then there exists a unique regular θ-invariant probability measure μ_Σ on Σ, that is, we have that*

$$(\theta_t)_* \mu_\Sigma = \mu_\Sigma, \qquad \forall t \in \mathbb{R}.$$

Proof. By proposition 2.11, it is enough to construct a θ-invariant volume form on Σ. We proceed similar to the proof of lemma 2.12. Since $dH \neq 0$ on Σ, we find a neighbourhood U of Σ in M such that $dH \neq 0$ on U. We claim that there exists $\alpha \in \Omega^{2n-1}(U)$ such that

$$\omega^n = dH \wedge \alpha. \tag{2.2}$$

Indeed, let m be any Riemannian metric on U. Define $X \in \mathfrak{X}(U)$ by

$$X := \operatorname{grad}_m H / \left|\operatorname{grad}_m H\right|_m^2 .$$

Then X is nowhere tangent to Σ because $dH(X) = 1$. Set $\alpha := i_X(\omega^n)$. Hence we compute

$$dH \wedge \alpha = dH \wedge i_X(\omega^n) = i_X(dH) \wedge \omega^n - i_X(dH \wedge \omega^n) = dH(X)\omega^n = \omega^n$$

since $dH \wedge \omega^n \in \Omega^{2n+1}(M) = \{0\}$. Denote by $\iota_\Sigma \colon \Sigma \hookrightarrow M$ the inclusion of Σ in M. By proposition 15.21 [25, p. 384–385], $\iota_\Sigma^* \alpha$ is a volume form on Σ. Moreover, this form is uniquely determined by the requirement (2.2). Indeed, suppose that there exists $\beta \in \Omega^{2n-1}(U)$ such that

$$\omega^n = dH \wedge \alpha = dH \wedge \beta.$$

Then

$$dH \wedge (\alpha - \beta) = 0,$$

and thus by lemma 2.12 there exists $\gamma \in \Omega^{2n-2}(U)$ such that

$$\alpha - \beta = dH \wedge \gamma.$$

But then

$$\begin{aligned}
\iota_\Sigma^* \alpha &= \iota_\Sigma(dH) \wedge \iota_\Sigma^* \gamma + \iota_\Sigma^* \beta \\
&= d\left(\iota_\Sigma^* H\right) \wedge \iota_\Sigma^* \gamma + \iota_\Sigma^* \beta \\
&= d\left(H \circ \iota_\Sigma\right) \wedge \iota_\Sigma^* \gamma + \iota_\Sigma^* \beta \\
&= \iota_\Sigma^* \beta
\end{aligned}$$

because $H \circ \iota_\Sigma$ is constant. Using proposition 2.1 and preservation of energy 2.3 we compute

$$\begin{aligned}
\omega^n &= \left(\theta_t^* \omega\right)^n \\
&= \theta_t^*(\omega^n) \\
&= \theta_t^*(dH \wedge \alpha) \\
&= d\left(\theta_t^* H\right) \wedge \theta_t^* \alpha \\
&= d\left(H \circ \theta_t\right) \wedge \theta_t^* \alpha \\
&= dH \wedge \theta_t^* \alpha
\end{aligned}$$

for all $t \in \mathbb{R}$. Hence by uniqueness

$$\iota_\Sigma^* \alpha = \iota_\Sigma^* \theta_t^* \alpha = \left(\theta_t \circ \iota_\Sigma\right)^* \alpha = \left(\iota_\Sigma \circ \theta_t\right)^* \alpha = \theta_t^* \iota_\Sigma^* \alpha$$

for all $t \in \mathbb{R}$. Consequently, $\iota_\Sigma^* \alpha$ is a θ-invariant volume form on Σ. □

Theorem 2.14 (Poincaré's Recurrence Theorem, [20, p. 20]). *Let Σ be a compact regular energy surface in a Hamiltonian system and denote by $\theta \in C^\infty(\mathbb{R} \times \Sigma, \Sigma)$ the flow of X_H on Σ. Then for almost every $x \in \Sigma$ there holds $x \in \omega_\theta(x)$ with respect to the probability measure μ_Σ defined in proposition 2.13.*

Proof. Let $T := \theta_1 \in \mathrm{Diff}(\Sigma)$. First, we show that

$$\mu\left(A \cap \bigcap_{k \geq 0} \bigcup_{l \geq k} T^{-l}(A)\right) = \mu(A) \tag{2.3}$$

holds for every Borel measurable set $A \subseteq \Sigma$, that is, almost every point in A returns infinitely often to A via iterations of T. For $k \in \mathbb{N}_0$, define

$$A_k := \bigcup_{l \geq k} T^{-l}(A).$$

Then $A_k \supseteq A_{k+1}$ and $T^k(A_k) = A_0$ for all $k \in \mathbb{N}_0$. Thus by θ-invariance we have that $\mu(A_k) = \mu(A_0)$. Since $\mu(A_0) \leq \mu(\Sigma) < +\infty$, we compute

$$\mu\left(\bigcap_{k\geq 0} A_k\right) = \lim_{k\to\infty} \mu(A_k) = \mu(A_0).$$

By monotonicity of μ, we have that

$$\mu\left(A \cup \bigcap_{k\geq 0} A_k\right) \geq \mu\left(\bigcap_{k\geq 0} A_k\right) = \mu(A_0).$$

Moreover, since $A \subseteq A_0$ we compute

$$\mu\left(A \cup \bigcap_{k\geq 0} A_k\right) \leq \mu(A \cup A_0) \leq \mu(A_0).$$

Thus

$$\mu\left(A \cup \bigcap_{k\geq 0} A_k\right) = \mu(A_0)$$

and so

$$\mu\left(A \cap \bigcap_{k\geq 0} A_k\right) = \mu(A) + \mu(A_0) - \mu\left(A \cup \bigcap_{k\geq 0} A_k\right) = \mu(A)$$

as claimed.

Let m be a Riemannian metric on Σ. Then (Σ, d_m) is a metric space with metric topology coinciding with the manifold topology of Σ. Because σ is compact, for every $n \in \mathbb{N}$ there exists a finite index set I_n such that $\left(B_{1/n}(x_{i,n})\right)_{i\in I_n}$ is an open cover for Σ. Define

$$N := \bigcup_{n\in\mathbb{N}} \bigcup_{i\in I_n} \left(B_{1/n}(x_{i,n}) \setminus \left(B_{1/n}(x_{i,n}) \cap \bigcap_{k\geq 0} \bigcup_{l\geq k} T^{-l}\left(B_{1/n}(x_{i,n})\right)\right)\right).$$

Then $N \in \mathscr{B}(\Sigma)$ and $\mu(N) = 0$. Indeed, we have that

$$\mu(N) \leq \sum_{n\in\mathbb{N}} \sum_{i\in I_n} \mu\left(B_{1/n}(x_{i,n}) \setminus \left(B_{1/n}(x_{i,n}) \cap \bigcap_{k\geq 0} \bigcup_{l\geq k} T^{-l}\left(B_{1/n}(x_{i,n})\right)\right)\right)$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{N}} \sum_{i\in I_n} \mu\left(B_{1/n}(x_{i,n})\right) - \mu\left(B_{1/n}(x_{i,n}) \cap \bigcap_{k\geq 0}\bigcup_{l\geq k} T^{-l}\left(B_{1/n}(x_{i,n})\right)\right)\\
&= 0,
\end{aligned}
$$

by (2.3). Moreover, every $x \in N^c$ is a recurrent point. Indeed, $x \in N^c$ means that for all $n \in \mathbb{N}$ and $i \in I_n$

$$
x \in \left(B_{1/n}(x_{i,n})\right)^c \qquad \text{or} \qquad x \in \bigcap_{k\geq 0}\bigcup_{l\geq k} T^{-l}\left(B_{1/n}(x_{i,n})\right).
$$

Since $\left(B_{1/n}(x_{i,n})\right)_{i\in I_n}$ is an open cover for S_c, we conclude that

$$
x \in \bigcap_{k\geq 0}\bigcup_{l\geq k} T^{-l}\left(B_{1/n}(x_{i,n})\right) \tag{2.4}
$$

for some $i \in I_n$. Consequently, for every $n \in \mathbb{N}$ there exists an index $i_n \in I_n$ such that (2.4) holds. □

Here is an important application of Poincaré's recurrence theorem 2.14. Let Σ be a compact regular energy surface in a Hamiltonian system (M, ω, H) and denote by $\theta \in C^\infty(\mathbb{R} \times \Sigma, \Sigma)$ the flow of X_H on Σ. For $A \in \mathscr{B}(\Sigma)$ define the ***Poincaré return map***

$$
P_A : A \to A, \qquad P_A(x) := T^{s_A(x)}(x),
$$

where

$$
s_A : \Sigma \to \mathbb{N} \cup \{\infty\}, \qquad s_A(x) := \inf\{k \geq 1 : T^k(x) \in A\}
$$

denotes the ***first return time to A*** and $T := \theta_1 \in \mathrm{Diff}(\Sigma)$. This map is well-defined as by the proof of the Poincaré recurrence theorem we have that $s_A < \infty$ holds almost everywhere on A with repsect to the probability measure μ_Σ.

2.2 The Rabinowitz Action Functional

In view of the recurrence theorem 2.14 it seems quite natural to search for periodic phenomena on regular energy surfaces (see [20, p. 21]). Periodic orbits of Hamiltonian vector fields on energy hypersurfaces can be detected variationally by means of a suitable functional, namely the Rabinowitz action functional. The Rabinowitz action functional is a functional defined on the free loop space of a appropriate symplectic manifold (M, ω). In order to understand the structure of the free loop space it is necessary to first understand its topology. Let M and N be two C^k-manifolds for some $0 \leq k < +\infty$. Then the ***compact-open topology on*** $\boldsymbol{C^k(M, N)}$ is defined as follows. Let $f \in C^k(M, N)$ and (U, φ), (V, ψ) be charts on M and N, respectively. Suppose $K \subseteq U$ is compact such that $f(K) \subseteq V$. For $\varepsilon > 0$, define

$$A^k\left(f,(U,\varphi),(V,\psi),K,\varepsilon\right) \subseteq C^k(M,N)$$

to consist of all maps $g \in C^k(M,N)$ such that $g(K) \subseteq V$ and

$$\sup_{x\in\varphi(K)} \left| D^r(\psi \circ f \circ \varphi^{-1})(x) - D^r(\psi \circ g \circ \varphi^{-1})(x) \right| < \varepsilon$$

for all $0 \leq r \leq k$. Define $\mathcal{A} \subseteq 2^{C^k(M,N)}$ to be the set of all such sets $A^k\left(f,(U,\varphi),(V,\psi),K,\varepsilon\right)$, where f, (U,φ), (V,ψ), K and ε vary. We equip $C^k(M,N)$ with the topology generated by $\mathcal{A}$, that is, the topology of which $\mathcal{A}$ is a subbasis (see [24, p. 46–47]). For every $0 \leq k < +\infty$, we have that $C^\infty(M,N) \subseteq C^k(M,N)$. Thus we define the ***compact-open topology on*** $\boldsymbol{C^\infty(M,N)}$ to be the union of the subspace topologies of $C^\infty(M,N)$ in $C^k(M,N)$ for all $k \geq 0$, that is, the initial topology with respect to the inclusions $C^\infty(M,N) \hookrightarrow C^k(M,N)$ for all $k \geq 0$. For an invariant definition of the compact-open topology via jets see [19, p. 58–64]. From this invariant definition it also follows that $C^\infty(M,N)$ with the above described topology admits a complete metric d_∞. Moreover, this topology on $C^\infty(M,N)$ can also be thought of the topology of uniform convergence of functions and all of its derivatives on compact subsets, that is, of compact convergence.

Definition 2.15 (Path Space). Let M be a smooth manifold. For $x_0, x_1 \in M$ and $t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$, define the ***path space of M connecting*** $\boldsymbol{(x_0,t_0)}$ ***and*** $\boldsymbol{(x_1,t_1)}$, written $C^\infty_{x_0,x_1}([t_0,t_1],M)$, by

$$C^\infty_{x_0,x_1}([t_0,t_1],M) := \left\{\gamma \in C^\infty\left([t_0,t_1],M\right) : \gamma(t_0) = x_0 \text{ and } \gamma(t_1) = x_1\right\}.$$

Remark 2.16. The path space $C^\infty_{x_0,x_1}([t_0,t_1],M)$ admits the structure of an infinite-dimensional Fréchet manifold with tangent space

$$T_\gamma\left(C^\infty_{x_0,x_1}([t_0,t_1],M)\right) = \{X \in \Gamma_\gamma(TM) : X_{t_0} = 0_{x_0} \text{ and } X_{t_1} = 0_{x_1}\}$$

for every $\gamma \in C^\infty_{x_0,x_1}([t_0,t_1],M)$, see [22, p. 439] and [22, p. 447]. Since being a Fréchet manifold is a rather poor structure, for example the implicit function theorem is in general wrong in Fréchet manifolds (see [1, p. 25]), by weakening the regularity to Sobolev regularity, the path space

$$W^{1,2}_{x_0,x_1}([t_0,t_1],M) := \left\{\gamma \in W^{1,2}\left([t_0,t_1],M\right) : \gamma(t_0) = x_0 \text{ and } \gamma(t_1) = x_1\right\}$$

admits the structure of a Hilbert-Riemannian manifold (see [27, p. 50–52]), where for any closed interval $I \subseteq \mathbb{R}$ we define

$$W^{1,2}(I,M) := \left\{\gamma \in W^{1,2}(I,\mathbb{R}^{2n}) : \gamma(I) \subseteq M\right\},$$

where $M^n \hookrightarrow \mathbb{R}^{2n+1}$ via the Whitney embedding theorem [25, p. 134] and the Hilbert space $W^{1,2}(I,\mathbb{R}^{2n})$ can be identified with the space of absolutely continuous functions on I. For a more thorough introduction to infinite-dimensional manifolds, in particular Banach manifolds, see [23].

Definition 2.17 (Variation with Fixed Ends). Let $\gamma \in C^{\infty}_{x_0,x_1}([t_0,t_1], M)$. A ***variation of γ with fixed ends*** is defined to be a morphism

$$\Gamma \in C^{\infty}\big((-\varepsilon_0,\varepsilon_0) \times [t_0,t_1], M\big)$$

for some $\varepsilon_0 > 0$, such that

$$\gamma_0 = \gamma \qquad \text{and} \qquad \gamma_\varepsilon \in C^{\infty}_{x_0,x_1}([t_0,t_1], M)$$

for all $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$, where $\gamma_\varepsilon \in C^{\infty}([t_0,t_1], M)$ is defined by

$$\gamma_\varepsilon(t) := \Gamma(\varepsilon, t), \qquad \forall t \in [t_0,t_1].$$

Remark 2.18. If Γ is a variation of $\gamma \in C^{\infty}_{x_0,x_1}([t_0,t_1], M)$, then

$$\left.\frac{d\gamma_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} \in T_\gamma\left(C^{\infty}_{x_0,x_1}([t_0,t_1], M)\right),$$

where $T_\gamma\left(C^{\infty}_{x_0,x_1}([t_0,t_1], M)\right)$ is defined as in (2.16) and for fixed $t \in [t_0,t_1]$ we consider $\gamma_\varepsilon(t) \in C^{\infty}((-\varepsilon_0,\varepsilon_0), M)$. This tangent vector is traditionally called an ***infinitesimal variation of γ*** and is denoted by $\delta\gamma$.

Remark 2.19. Likewise, there is a notion of a ***free path space*** on a smooth manifold M. It is simply the manifold $C^{\infty}([t_0,t_1], M)$ of smooth curves in M. In that case, the tangent space to a curve $\gamma \in C^{\infty}([t_0,t_1], M)$ is given by

$$T_\gamma\left(C^{\infty}([t_0,t_1], M)\right) = \Gamma_\gamma(TM) = C^{\infty}(\gamma^* TM).$$

Free path spaces do occur for example in Riemannian geometry (see [26, p. 152]) and variations of elements of free path spaces are accordingly called ***variations with free ends***. More generally, if M is assumed to be compact, one can consider the path space

$$C^{\infty}_V([t_0,t_1], M) := \left\{\gamma \in C^{\infty}([t_0,t_1], M) : \big(\gamma(t_0), \gamma(t_1)\big) \in V\right\}$$

for any compact smooth submanifold $V \subseteq M \times M$ without boundary (see [27, p. 51]). Then $C^{\infty}_V([t_0,t_1], M)$ is a submanifold of $C^{\infty}([t_0,t_1], M)$. One particular instance of this is choosing the diagonal $\Delta_M := \{(x,x) : x \in M\}$. Then $C^{\infty}_{\Delta_M}([t_0,t_1], M)$ becomes the ***free loop space on M***, that is, the set $C^{\infty}(\mathbb{S}^1, M)$, where $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ (see example 21.14 (a) [25, p. 550]). The tangent space to a loop $\gamma \in C^{\infty}(\mathbb{S}^1, M)$ is simply given by all 1-periodic vector fields along γ, that is all sections of the pullback bundle $\gamma^* TM \to \mathbb{S}^1$. We will call variations on the loop space ***periodic variations***.

Example 2.20 (The Exponential Variation). Let M be a smooth manifold and $C^{\infty}_{x_0,x_1}([t_0,t_1], M)$ a path space. Choose a Riemannian metric m on M and denote by $\exp \in C^{\infty}(\mathcal{E}, M)$, $\mathcal{E} \subseteq TM$, the exponential map induced by the Levi-Civita connection on the Riemannian manifold (M, m). Let $\gamma \in C^{\infty}_{x_0,x_1}([t_0,t_1], M)$ and

$X \in T_\gamma\left(C^\infty_{x_0,x_1}([t_0,t_1], M)\right)$. Since $\gamma([t_0,t_1]) \subseteq TM$ embedded as the zero section is compact, there exists $\varepsilon_0 > 0$ such that $\Gamma_X \in C^\infty\left((-\varepsilon_0,\varepsilon_0)\times[t_0,t_1], M\right)$ defined by

$$\Gamma_X(\varepsilon, t) := \exp_{\gamma(t)}(\varepsilon X_t)$$

is well-defined. Then Γ is a variation of γ and moreover, we have that

$$\left.\frac{d\gamma_\varepsilon(t)}{d\varepsilon}\right|_{\varepsilon=0} = D\left(\exp_{\gamma(t)}\right)_{0_{\gamma(t)}}\left(\Phi_{0_{\gamma(t)}}(X_t)\right) = X_t$$

for all $t \in [t_0,t_1]$ by the properties of the exponential map [26, p. 128], where for a finite-dimensional real vector space V, we define the canonical isomorphism

$$\Phi_x \colon V \to T_x V, \qquad \Phi_x(v) := \left.\frac{d}{dt}\right|_{t=0}(x + tv),$$

for all $x \in V$.

Theorem 2.21 (Poincaré, [43, p. 33–34]). *Let $(M, \omega = d\alpha, H)$ be an exact Hamiltonian system. Define a functional, called the* ***Hamiltonian action functional****,*

$$\mathscr{A}^H \colon C^\infty_{x_0,x_1}([t_0,t_1], M) \to \mathbb{R}$$

by

$$\mathscr{A}^H(\gamma) := \int_{\widetilde{\gamma}}\left(\mathrm{pr}_1^*\alpha - (H\circ \mathrm{pr}_1)\,\mathrm{pr}_2^*(dt)\right),$$

where $\widetilde{\gamma} \in C^\infty_{x_0,x_1}([t_0,t_1], M\times\mathbb{R})$ is defined by $\widetilde{\gamma} := (\gamma, \iota_{[t_0,t_1]})$ and the form $\mathrm{pr}_1^*\alpha - (H\circ\mathrm{pr}_1^*)\,\mathrm{pr}_2^*(dt) \in \Omega^1(M\times\mathbb{R})$ on the extended phase space $M\times\mathbb{R}$ is called the ***Poincaré-Cartan form****. Then $\gamma \in C^\infty_{x_0,x_1}([t_0,t_1], M)$ is an integral curve of X_H if and only if $\gamma \in \mathrm{Crit}(\mathscr{A}^H)$, where by definition $\gamma \in \mathrm{Crit}(\mathscr{A}^H)$ iff*

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\mathscr{A}^H(\gamma_\varepsilon) = 0$$

for all variations Γ of γ with fixed ends.

Remark 2.22. Note that an exact symplectic manifold cannot be compact. Indeed, suppose $(M^{2n}, \omega = d\alpha)$ is a compact exact symplectic manifold. Then ω^n is a volume form on M and thus using positivity [25, p. 407] and Stoke's theorem [25, p. 411] we compute

$$0 < \int_M \omega^n = \int_M d\alpha^n = \int_M d(\alpha\wedge\omega^{n-1}) = \int_{\partial M}\alpha\wedge\omega^{n-1} = 0.$$

In order to prove theorem 2.21 we need first a preliminary technical result.

Definition 2.23 (Partial Pullback, [11, p. 93]). Let M and N be two smooth manifolds, $F \in C^\infty(M,N)$ and $\eta \in \Gamma_F\left(T^{(0,k)}TN\right)$. Define the ***partial pullback of η***

by F to be the form $F^*\eta \in \Omega^k(M)$ by

$$(F^*\eta)_x(v_1, \dots, v_k) := \eta_x\left(DF_x(v_1), \dots, DF_x(v_k)\right)$$

for all $x \in M$ and $v_1, \dots, v_k \in T_x M$ if $k \geq 1$, and $F^* f := f$ in the case $k = 0$.

Lemma 2.24 (Generalised Cartan's Magic Formula, [11, p. 101]). *Let $I \subseteq \mathbb{R}$ an open interval, $t_0 \in I$ and $F \in C^\infty(I \times M, N)$. Moreover, let $X \in \Gamma_{F_{t_0}}(TN)$ be defined by*

$$X := \left.\frac{d}{dt}\right|_{t=t_0} F_t,$$

where as usual $F_t := F(t, \cdot) \in C^\infty(M, N)$. Then

$$\left.\frac{d}{dt}\right|_{t=t_0} F_t^*\eta = F_{t_0}^* i_X(d\eta \circ F_{t_0}) + dF_{t_0}^* i_X(\eta \circ F_{t_0})$$

for every $\eta \in \Omega^k(N)$, where the pullback on the right hand side is interpreted as a partial pullback 2.23.

Proof. Since if two graded derivations of the same degree coinciding on functions and exact 1-forms must necessarily be equal, it is enough to prove the formula for functions and exact 1-forms. Let $f \in C^\infty(N)$. Then we have that

$$\left.\frac{d}{dt}\right|_{t=t_0} F_t^* f = \left.\frac{d}{dt}\right|_{t=t_0} (f \circ F_t) = df_{F_{t_0}}(X) = F_{t_0}^* i_X(df \circ F_{t_0}),$$

and

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_0} F_t^* df &= \left.\frac{d}{dt}\right|_{t=t_0} dF_t^* f \\
&= \left.\frac{d}{dt}\right|_{t=t_0} d(f \circ F_t) \\
&= d\left(df_{F_{t_0}}(X)\right) \\
&= dF_{t_0}^* i_X(df \circ F_{t_0}).
\end{aligned}$$

□

Proof (Theorem 2.21). First we rewrite the expression for the Poincaré-Cartan form. We have that

$$\mathcal{A}^H(\gamma) = \int_\gamma \alpha - \int_{t_0}^{t_1} H \circ \gamma. \tag{2.5}$$

Indeed, we compute

$$\mathcal{A}^H(\gamma) = \int_{\widetilde{\gamma}} \mathrm{pr}_1^* \alpha - \int_{\widetilde{\gamma}} (H \circ \mathrm{pr}_1)\, \mathrm{pr}_2^*(dt)$$

$$
\begin{aligned}
&= \int_{t_0}^{t_1} \widetilde{\gamma}^* \operatorname{pr}_1^* \alpha - \int_{t_0}^{t_1} (H \circ \operatorname{pr}_1 \circ \widetilde{\gamma}) \widetilde{\gamma}^* \operatorname{pr}_2^*(dt) \\
&= \int_{t_0}^{t_1} \gamma^* \alpha - \int_{t_0}^{t_1} (H \circ \gamma) \iota_{[t_0, t_1]}^*(dt) \\
&= \int_{\gamma} \alpha - \int_{t_0}^{t_1} (H \circ \gamma).
\end{aligned}
$$

Let Γ be a variation of γ. Using (2.5), the generalised Cartan's magic formula 2.24 and Stoke's theorem [25, p. 411], we compute

$$
\begin{aligned}
\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{A}^H(\gamma_\varepsilon) &= \int_{t_0}^{t_1} \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \gamma_\varepsilon^* \alpha - \int_{t_0}^{t_1} dH_\gamma(\delta\gamma) \\
&= \int_{t_0}^{t_1} \big(\gamma^* i_{\delta\gamma}(\omega \circ \gamma) + d\gamma^* i_{\delta\gamma}(\alpha \circ \gamma) \big) - \int_{t_0}^{t_1} dH_\gamma(\delta\gamma) \\
&= \int_{t_0}^{t_1} \gamma^* i_{\delta\gamma}(\omega \circ \gamma) + i_{\delta\gamma}(\alpha \circ \gamma)|_{t_0}^{t_1} - \int_{t_0}^{t_1} dH_\gamma(\delta\gamma) \\
&= \int_{t_0}^{t_1} \gamma^* i_{\delta\gamma}(\omega \circ \gamma) - \int_{t_0}^{t_1} dH_\gamma(\delta\gamma) \\
&= \int_{t_0}^{t_1} \gamma^* i_{\delta\gamma}(\omega \circ \gamma) + \int_{t_0}^{t_1} \left(i_{X_H} \omega \right)_\gamma (\delta\gamma) \\
&= \int_{t_0}^{t_1} \omega_\gamma \left(\delta\gamma, \dot{\gamma} \right) + \int_{t_0}^{t_1} \omega_\gamma (X_H \circ \gamma, \delta\gamma) \\
&= \int_{t_0}^{t_1} \omega_\gamma \left(\delta\gamma, \dot{\gamma} - (X_H \circ \gamma) \right),
\end{aligned} \tag{2.6}
$$

since $\delta\gamma|_{t_0} = \delta\gamma|_{t_1} = 0$, where we used the notation of an infinitesimal variation $\delta\gamma$ of γ (see remark 2.18).

Suppose that γ is an integral curve of the Hamiltonian vector field X_H. Then (2.6) immediately implies that

$$
\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{A}^H(\gamma_\varepsilon) = \int_{t_0}^{t_1} \omega_\gamma \left(\delta\gamma, \dot{\gamma} - (X_H \circ \gamma) \right) = 0
$$

for all variations Γ of γ with fixed ends. In particular, $\gamma \in \operatorname{Crit}(\mathcal{A}^H)$ by definition of $\operatorname{Crit}(\mathcal{A}^H)$.

Conversely, suppose that $\gamma \in \operatorname{Crit}(\mathcal{A}^H)$. Let $X \in T_\gamma \left(C^\infty_{x_0, x_1}([t_0, t_1], M) \right)$, where by definition

$$
T_\gamma \left(C^\infty_{x_0, x_1}([t_0, t_1], M) \right) = \left\{ X \in \Gamma_\gamma(TM) : X_{t_0} = 0_{x_0} \text{ and } X_{t_1} = 0_{x_1} \right\},
$$

as in remark 2.16, and let Γ_X be the exponential variation from example 2.20. Then $\delta\gamma = X$ and so equation (2.6) reads

$$0 = \int_{t_0}^{t_1} \omega_\gamma \left(X, \dot{\gamma} - (X_H \circ \gamma)\right)$$

for all $X \in T_\gamma \left(C^\infty_{x_0,x_1}([t_0, t_1], M)\right)$. Suppose that there exists $t \in [t_0, t_1]$ such that $\dot{\gamma}(t) \neq X_H|_{\gamma(t)}$. By continuity, there exists an open interval $I \subseteq [t_0, t_1]$ about t such that $\dot{\gamma} \neq X_H \circ \gamma$ on I. By nondegeneracy of ω we find $X_{\gamma(t)} \in T_{\gamma(t)}M$ such that

$$\omega_{\gamma(t)} \left(X_{\gamma(t)}, \dot{\gamma}(t) - X_H|_{\gamma(t)}\right) \neq 0.$$

Without loss of generality we may assume that the above expression is positive. Moreover, by shrinking I if necessary, we may extend $X_{\gamma(t)}$ to a vector field along γ supported in an open interval $J \subseteq I$ about t and such that the above expression is positive on J. But then

$$0 = \int_{t_0}^{t_1} \omega_\gamma \left(X, \dot{\gamma} - (X_H \circ \gamma)\right) = \int_J \omega_\gamma \left(X, \dot{\gamma} - (X_H \circ \gamma)\right) > 0,$$

and thus $\dot{\gamma} = X_H \circ \gamma$ on $[t_0, t_1]$, that is, γ is an integral curve of the Hamiltonian vector field X_H. $\square$

Remark 2.25 (Legendre Transform, [43, p. 28]). Let (M, L) be an autonomous Lagrangian system, that is, $L \in C^\infty(T^*M)$ and suppose that the fibrewise derivative $\tau_L \in C^\infty(TM, T^*M)$ defined by

$$\tau_L|_{(x,v)}(w) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L(x, v + \varepsilon w), \qquad w \in T_xM,$$

is a diffeomorphism. Then $\gamma \in C^\infty_{x_0,x_1}([t_0, t_1], M)$ is a critical point of the Lagrangian action functional

$$\mathcal{A}_L \colon C^\infty_{x_0,x_1}([t_0, t_1], M) \to \mathbb{R}, \qquad \mathcal{A}_L(\gamma) := \int_{\dot{\gamma}} L dt,$$

if and only if the lifted path $\tau_L \circ \dot{\gamma}$ is an integral curve of the Hamiltonian system (T^*M, ω_0, H_L), where ω_0 is the standard symplectic form on the cotangent bundle T^*M and $H_L \in C^\infty(T^*M)$ is the Hamiltonian function associated to the autonomous Lagrangian function L defined by $H_L := E_L \circ \tau_L^{-1}$, where $E_L \in C^\infty(TM)$ is the energy of the autonomous Lagrangian system defined by

$$E_L(x, v) := \tau_L|_{(x,v)}(v) - L(x, v).$$

In this setting the action of both $\tau_L \circ \dot{\gamma}$ and γ coincide as

$$\begin{aligned} \mathcal{A}^{H_L}(\tau_L \circ \dot{\gamma}) &= \int_{\tau_L \circ \dot{\gamma}} \lambda - \int_{t_0}^{t_1} H_L \circ \tau_L \circ \dot{\gamma} && \text{(by equation (2.5))} \\ &= \int_{\tau_L \circ \dot{\gamma}} \lambda - \int_{t_0}^{t_1} E_L \circ \dot{\gamma} && \text{(by definition of } H_L) \end{aligned}$$

$$
\begin{aligned}
&= \int_{t_0}^{t_1} (\tau_L \circ \dot\gamma)^*\lambda - \int_{t_0}^{t_1} E_L \circ \dot\gamma \\
&= \int_{t_0}^{t_1} \dot\gamma^* \tau_L^* \lambda - \int_{t_0}^{t_1} E_L \circ \dot\gamma \\
&= \int_{t_0}^{t_1} \dot\gamma^* \lambda_L - \int_{t_0}^{t_1} E_L \circ \dot\gamma && \text{(by definition of } \tau_L\text{)} \\
&= \int_{t_0}^{t_1} \tau_L|_{(\gamma,\dot\gamma)}(\dot\gamma) - \int_{t_0}^{t_1} E_L \circ \dot\gamma \\
&= \int_{t_0}^{t_1} L(\dot\gamma) && \text{(by definition of } E_L\text{)} \\
&= \mathscr{A}_L(\gamma),
\end{aligned}
$$

where $\lambda_L \in \Omega^1(TM)$ is the associated form defined by

$$
\lambda_L(v) := dL\left((\Phi \circ D\pi)(v)\right), \qquad v \in TTM,
$$

regarding $D\pi \in C^\infty(TTM, \pi^*TM)$ as a vector bundle homomorphism and denoting by

$$
VTM := \coprod_{p \in TM} \ker D\pi_p
$$

the vertical bundle of TM, $\Phi \in C^\infty(\pi^*TM, VTM)$ is the vector bundle isomorphism given by

$$
\Phi(p, q) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (p + \varepsilon q).
$$

Example 2.26 (Free Particle on a Pseudo-Riemannian Manifold, [15, p. 13]). Let (M, m) be a pseudo-Riemannian manifold and denote by $V \in \mathfrak{X}(TM)$ the geodesic vector field associated to the Levi–Civita connection (see [26, p. 129]). Then V gives rise to a vector field $V^* \in \mathfrak{X}(T^*M)$, called the ***cogeodesic vector field***, via

$$
\begin{array}{ccc}
TM & \xrightarrow{V} & TTM \\
{\scriptstyle \widehat{m}}\downarrow & & \downarrow{\scriptstyle D\widehat{m}} \\
T^*M & \xrightarrow[V^*]{} & TT^*M,
\end{array}
$$

where $\widehat{m}\colon TM \to T^*M$ denotes the tangent-cotangent bundle isomorphism defined by

$$
\widehat{m}(X)(Y) := m(X, Y), \qquad \forall X, Y \in \mathfrak{X}(M).
$$

Let $\theta^V \in C^\infty(\mathscr{D}, TM)$ denote the flow of V. Then we have that

$$
\theta_t^{V^*} = \widehat{m} \circ \theta_t^V \circ \widehat{m}^{-1}.
$$

Indeed, we compute

$$\begin{aligned}\frac{d}{dt}(\widehat{m} \circ \theta_t^V \circ \widehat{m}^{-1}) &= D\widehat{m} \circ \frac{d}{dt}\theta_t^V \circ \widehat{m}^{-1} \\ &= D\widehat{m} \circ V \circ \theta_t^V \circ \widehat{m}^{-1} \\ &= D\widehat{m} \circ V \circ \widehat{m}^{-1} \circ \widehat{m} \circ \theta_t^V \circ \widehat{m}^{-1} \\ &= V^* \circ \widehat{m} \circ \theta_t^V \circ \widehat{m}^{-1}.\end{aligned}$$

Consider the Hamiltonian system (T^*M, ω_0, H), where the mechanical Hamiltonian $H \in C^\infty(T^*M)$ is given by

$$H(x, \xi) := \frac{1}{2}m^*(\xi, \xi) := \frac{1}{2}m\left(\widehat{m}^{-1}(\xi), \widehat{m}^{-1}(\xi)\right).$$

Then one checks that $X_H = V^*$ and consequently, critical points of the Hamiltonian action functional $\mathcal{A}^H : C^\infty_{x_0,x_1}([t_0, t_1], M) \to \mathbb{R}$ are precisely the geodesics connecting x_0 and x_1 by the Poincaré theorem 2.21.

Remark 2.27 (Magnetic Hamiltonian System, [15, p. 18]). Let (M, m) be a pseuod-Riemannian manifold. For $\alpha \in \Omega^1(M)$ and $V \in C^\infty(M)$ define a Hamiltonian function $H \in C^\infty(T^*M)$, called a ***magnetic Hamiltonian***, by

$$H(x, \xi) := \frac{1}{2}m^*(\xi - \alpha_x, \xi - \alpha_x) + V(x).$$

The Hamiltonian system (T^*M, ω_0, H) is called a ***magnetic Hamiltonian system*** and generalised the notion of a mechanical Hamiltonian system where the Hamiltonian is simply given by kinetic plus potential energy.

Next we want to focus on the periodic case, that is, for a smooth manifold M we consider the free loop space

$$C^\infty(\mathbb{S}^1, M) \subseteq \{\gamma \in C^\infty([0, 1], M) : \gamma(0) = \gamma(1)\}$$

with tangent space

$$T_\gamma\left(C^\infty(\mathbb{S}^1, M)\right) = \Gamma_\gamma(TM) = \Gamma(\gamma^*TM),$$

for every $\gamma \in C^\infty(\mathbb{S}^1, M)$. Moreover, we do consider ***periodic variations***, that is, morphisms $\varGamma \in C^\infty\left((-\varepsilon_0, \varepsilon_0) \times \mathbb{S}^1, M\right)$ for some $\varepsilon_0 > 0$ such that $\gamma_0 = \gamma$. Recall, that a curve $\gamma \in C^\infty(\mathbb{R}, M)$ is said to be ***periodic***, iff there exists $\tau > 0$ such that $\gamma(t + \tau) = \gamma(t)$ holds for all $t \in \mathbb{R}$. In that case, we also say that ***γ is τ-periodic***. In terms of a flow $\theta : \mathcal{D} \to M$ of a vector field $X \in \mathfrak{X}(M)$, a point $x \in M$ is periodic iff there exists $\tau > 0$ such that $\theta_\tau(x) = x$, that is, x is a ***fixed point*** of the flow of X.

Let $(M, d\alpha, H)$ be an exact Hamiltonian system. Similar to Poincaré's theorem 2.21, we can consider a Hamiltonian action functional (2.5)

$$\mathcal{A}^H : C^\infty(\mathbb{S}^1, M) \to \mathbb{R}, \qquad \mathcal{A}^H(\gamma) = \int_\gamma \alpha - \int_0^1 H \circ \gamma.$$

An identical computation as in the proof of Poincaré's theorem yields

$$\operatorname{Crit}(\mathcal{A}^H) = \left\{\gamma \in C^\infty(\mathbb{S}^1, M) : \dot{\gamma} = X_H \circ \gamma\right\}.$$

In other words, *the set of critical points of the Hamiltonian action functional* $\mathcal{A}^H$ *coincides with the* 1*-periodic integral curves of the Hamiltonian vector field* X_H. However, we want to study periodic orbits of arbitrary periods.

Definition 2.28 (Parametrised Periodic Orbit, [15, p. 96], [36, p. 27]). Let Σ be a regular energy surface in a Hamiltonian system (M, ω, H). A ***parametrised periodic orbit of X_H on Σ*** is defined to be a tuple (γ, τ), where $\gamma \in C^\infty(\mathbb{S}^1, \Sigma)$ is a loop and $\tau \in \mathbb{R}$, with

$$\dot{\gamma}(t) = \tau X_H\left(\gamma(t)\right), \qquad \forall t \in \mathbb{S}^1.$$

We denote by $\mathcal{P}(X_H, \Sigma)$ the set of parametrised periodic orbits of X_H on Σ. Moreover, define by

$$\mathcal{P}^+(X_H, \Sigma) := \{(\gamma, \tau) \in \mathcal{P}(X_H, \Sigma) : \tau > 0\}$$

the ***set of positive parametrised periodic orbits of X_H on Σ***.

Remark 2.29 ([15, p. 96]). Let Σ be a regular energy surface in a Hamiltonian system (M, ω, H). Then clearly $(\Sigma \times \{0\}) \subseteq \mathcal{P}(X_H, \Sigma)$. If $(\gamma, \tau) \in \mathcal{P}^+(X_H, \Sigma)$, we can define a reparametrisation $\gamma_\tau \in C^\infty(\tau\mathbb{S}^1, \Sigma)$, where $\tau\mathbb{S}^1 := \mathbb{R}/\tau\mathbb{Z}$, of γ by setting $\gamma_\tau(t) := \gamma(t/\tau)$ for all $t \in \tau\mathbb{S}^1$. Then $\gamma_\tau(t) = \theta_t\left(\gamma(0)\right)$ for all $t \in \tau\mathbb{S}^1$, where $\theta \in C^\infty(\mathcal{D}, M)$ denotes the flow of the Hamiltonian vector field X_H. Indeed, we compute

$$\dot{\gamma}_\tau(t) = \frac{1}{\tau}\dot{\gamma}(t/\tau) = X_H\left(\gamma(t/\tau)\right) = X_H\left(\gamma_\tau(t)\right).$$

Thus τ is uniquely determined by γ and we refer to τ as the ***period of γ***. Moreover, there is a natural $\mathbb{S}^1$-action on $\mathcal{P}(X_H, \Sigma)$ defined by translation. Thus we may call elements $([\gamma], \tau) \in \mathcal{P}(X_H, \Sigma)/\mathbb{S}^1$ ***unparametrised periodic orbits of X_H on Σ***, for more details see [15, p. 98–99]. This $\mathbb{S}^1$-action is free on $\mathcal{P}^+(X_H, \Sigma)$.

Periodic orbits on regular energy hypersurfaces can be detected by two functionals defined on a free loop space (see remark 2.19).

Definition 2.30 (Rabinowitz Action Functional, [15, p. 96]). Let $(M, d\alpha, H)$ be an exact Hamiltonian system. The ***Rabinowitz action functional corresponding to the Hamiltonian function H***

$$\mathcal{R}^H : C^\infty(\mathbb{S}^1, M) \times \mathbb{R} \to \mathbb{R}$$

is defined by

$$\mathcal{R}^H(\gamma, \tau) := \int_\gamma \alpha - \tau \int_0^1 H \circ \gamma.$$

Remark 2.31. One might think of the Rabinowitz action functional as the Lagrange multiplier functional of the area functional under the constraint given by the mean value of the Hamiltonian function.

Definition 2.32 (Differential of the Rabinowitz Action Functional). The ***differential of the Rabinowitz action functional*** $\boldsymbol{\mathcal{R}^H}$ corresponding to an exact Hamiltonian system $(M, d\alpha, H)$ at a point $(\gamma, \tau) \in C^\infty(\mathbb{S}^1, M) \times \mathbb{R}$ is defined to be the map

$$d\mathcal{R}^H_{(\gamma,\tau)} \colon T_\gamma\left(C^\infty(\mathbb{S}^1, M)\right) \times \mathbb{R} \to \mathbb{R},$$

given by

$$d\mathcal{R}^H_{(\gamma,\tau)}(X, \widetilde{\tau}) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{R}^H\left(\gamma_\varepsilon, \tau + \varepsilon\widetilde{\tau}\right),$$

where Γ is a periodic variation of γ such that $\delta\gamma = X \in \Gamma_\gamma(TM)$. Note that these variations do always exists, since for example we can take the exponential variation introduced in example 2.20, which is a periodic variation of γ. We denote by $\mathrm{Crit}(\mathcal{R}^H)$ the set of critical points of the Rabinowitz action functional corresponding to H.

Proposition 2.33 ([15, p. 97], [36, p. 26–30]). *Let Σ be a regular energy surface in an exact Hamiltonian system $(M, \omega = d\alpha, H)$. Then we have that*

$$\mathrm{Crit}(\mathcal{R}^H) = \mathcal{P}(X_H, \Sigma).$$

Proof. Similar to the proof of Poincaré's theorem 2.21 we compute

$$d\mathcal{R}^H_{(\gamma,\tau)}(X, \widetilde{\tau}) = \int_0^1 \omega_\gamma\left(X, \dot{\gamma} - \tau(X_H \circ \gamma)\right) - \widetilde{\tau}\int_0^1 H \circ \gamma \tag{2.7}$$

for all $(\gamma, \tau) \in C^\infty(\mathbb{S}^1, M) \times \mathbb{R}$ and $(X, \widetilde{\tau}) \in T_\gamma\left(C^\infty(\mathbb{S}^1, M)\right) \times \mathbb{R}$.

Suppose that $(\gamma, \tau) \in \mathcal{P}(X_H, \Sigma)$. Then using (2.7), we get

$$d\mathcal{R}^H_{(\gamma,\tau)}(X, \widetilde{\tau}) = \int_0^1 \omega_\gamma\left(X, \dot{\gamma} - \tau(X_H \circ \gamma)\right) - \widetilde{\tau}\int_0^1 H \circ \gamma = 0$$

for all $(X, \widetilde{\tau}) \in T_\gamma\left(C^\infty(\mathbb{S}^1, M)\right) \times \mathbb{R}$ as $\Sigma = H^{-1}(0)$.

Conversely, suppose that $(\gamma, \tau) \in \mathrm{Crit}(\mathcal{R}^H)$. Then by (2.7), we get that

$$0 = d\mathcal{R}^H_{(\gamma,\tau)}(X, 0) = \int_0^1 \omega_\gamma\left(X, \dot{\gamma} - \tau(X_H \circ \gamma)\right)$$

for all $X \in T_\gamma\left(C^\infty(\mathbb{S}^1, M)\right)$. Arguing as in the proof of Poincaré's theorem 2.21, this implies by nondegeneracy of ω that $\dot{\gamma} = \tau(X_H \circ \gamma)$. On the other hand by (2.7) we also infer that

$$0 = d\mathcal{R}^H_{(\gamma,\tau)}(0, -\tau) = \tau\int_0^1 H \circ \gamma = \int_0^\tau H \circ \gamma_\tau,$$

where γ_τ is as in remark 2.29. Thus preservation of energy 2.3 yields $\gamma(\mathbb{S}^1) \subseteq \Sigma$, and so $\gamma \in C^\infty(\mathbb{S}^1, \Sigma)$ by restricting the codomain of a smooth map to an embedded submanifold [25, p. 113]. $\square$

Remark 2.34 ([14, p. 3–4]). Let (M, ω, H) be a Hamiltonian system and abbreviate by $\mathscr{L}(M)$ the free loop space $C^\infty(\mathbb{S}^1, M)$. Define $\mathfrak{a}_\omega \in \Omega^1(\mathscr{L}(M) \times \mathbb{R})$ by

$$\mathfrak{a}_\omega|_{(\gamma,\tau)}(X, \widetilde{\tau}) := \int_0^1 \gamma^* i_X(\omega \circ \gamma).$$

Moreover, define $\bar{H} \in C^\infty(\mathscr{L}(M) \times \mathbb{R})$ by

$$\bar{H}(\gamma, \tau) := \tau \int_0^1 H \circ \gamma.$$

Then $(\mathfrak{a}_\omega - d\bar{H})|_{(\gamma,\tau)} = 0$ if and only if γ is a parametrised τ-periodic orbit of X_H.

Given a regular energy surface Σ in a Hamiltonian system we can ask for additional structures on Σ. For dimensional reasons it is clear that Σ cannot admit a symplectic structure. However, there is a useful notion of an odd-dimensional analogue of a symplectic manifold.

Definition 2.35 (Hamiltonian Manifold, [15, p. 20], [9, p. 591]). A ***Hamiltonian manifold*** is defined to be a tuple (Σ, ω), where Σ is a smooth manifold of odd dimension and $\omega \in \Omega^2(\Sigma)$, called a ***Hamiltonian structure on*** $\boldsymbol{\Sigma}$, is closed and such that

$$\ker \omega := \{(x, v) \in T\Sigma : i_v \omega_x = 0\}$$

is a smooth line-distribution on Σ, called the ***Hamiltonian line field***.

Example 2.36 (Regular Energy Surface, [1, p. 373]). Let Σ be a regular energy surface in a Hamiltonian system (M, ω, H). Then $(\Sigma, \omega|_\Sigma)$ is a Hamiltonian manifold.

Proposition 2.37 (E. Cartan, [1, p. 371]). *Let (Σ, ω) be a Hamiltonian manifold. Then the line distribution* $\ker \omega$ *is integrable.*

Proof. Suppose that $X, Y \in \mathfrak{X}(\Sigma)$ belong to $\ker \omega$. Then so does $[X, Y]$, since

$$i_{[X,Y]}\omega = \mathscr{L}_X(i_Y\omega) - i_Y\mathscr{L}_X\omega = -i_Y\mathscr{L}_X\omega = -i_Y\left(i_X d\omega + d i_X\omega\right) = 0$$

by problem 12-12 (d) [25, p. 326] and Cartan's magic formula [25, p. 372]. $\square$

Since $\ker \omega$ is integrable by proposition 2.37, the global Frobenius theorem [25, p. 502] yields a unique foliation inducing $\ker \omega$ for every Hamiltonian manifold (Σ, ω).

Definition 2.38 (Characteristic Foliation, [32, p. 422], [8, p. 1773]). Let (Σ, ω) be a Hamiltonian manifold. The foliation inducing the integrable line distribution $\ker \omega$ is called the ***characteristic foliation of*** $\boldsymbol{\omega}$.

Definition 2.39 (Stable Hamiltonian Manifold, [8, p. 1773]). A ***stable Hamiltonian manifold*** is defined to be a tuple $\big((\Sigma,\omega),\alpha\big)$, where (Σ,ω) is a Hamiltonian manifold and $\alpha\in\Omega^1(\Sigma)$, called a ***stable Hamiltonian structure on*** $\boldsymbol{(\Sigma,\omega)}$, is such that

$$\alpha|_{\ker(\omega)} \text{ is nowhere-vanishing} \qquad \text{and} \qquad \ker\omega\subseteq\ker d\alpha.$$

A Hamiltonian manifold admitting at least one stable Hamiltonian structure is said to be ***stabilisable***.

Remark 2.40. Unlike the contact condition, stability is not an open condition as is shown in [7].

Proposition 2.41 (Stable Reeb Vector Field, [8, p. 1773]). *Let (Σ,ω,α) be a stable Hamiltonian manifold. Then there exists a unique vector field $R_\alpha\in\mathfrak{X}(\Sigma)$, called the **stable Reeb vector field associated to the stable Hamiltonian structure** $\boldsymbol{\alpha}$, such that*

$$i_{R_\alpha}\omega=0 \qquad \textit{and} \qquad \alpha(R_\alpha)=1.$$

Proof. The condition $i_{R_\alpha}\omega=0$ requires R_α to belong to $\ker\omega$. Suppose that there exists $R'_\alpha\in\mathfrak{X}(\Sigma)$ satisfying $i_{R'_\alpha}\omega=0$ and $\alpha(R'_\alpha)=1$. Linearity of α yields $\alpha(R_\alpha-R'_\alpha)=0$ wich implies $R_\alpha=R'_\alpha$. Indeed, since α is nowhere-vanishing on the line distribution $\ker\omega$, for every $x\in\Sigma$ there exists $v\in\ker(\omega_x)\setminus\{0_x\}$ such that $\alpha_x(v)\neq 0$. But this implies $\alpha_x(v)\neq 0$ for all $v\in\ker(\omega_x)\setminus\{0_x\}$ since $\ker\omega$ is a line distribution. Uniqueness follows.

Moreover, since $\ker\omega$ is a smooth line-distribution, there exists a locally spanning vector field X about every point of Σ. Thus set $R_\alpha:=\alpha(X)^{-1}X$ locally. This is possible since $\alpha|_{\ker(\omega)}$ is nowhere-vanishing. □

Proposition 2.42. *Let (Σ,ω,α) be a stable Hamiltonian manifold and denote by $\theta\in C^\infty(\mathcal{D},\Sigma)$ the flow of the stable Reeb vector field R_α. Then $(\theta_t^*\alpha)_x=\alpha_x$ for all $(t,x)\in\mathcal{D}$.*

Proof. The proof is very similar to the proof of proposition 2.1. Using Fisherman's formula [25, p. 571] together with Cartan's magic formula [25, p. 372] and proposition 11.20 (e) [25, p. 282] we compute

$$\frac{d}{dt}\theta_t^*\alpha=\theta_t^*\mathcal{L}_{R_\alpha}\alpha=\theta_t^*\left(i_{R_\alpha}d\alpha+d\,i_{R_\alpha}\alpha\right)=0,$$

since R_α belongs to the Hamiltonian line distribution $\ker\omega$ and thus also to $\ker d\alpha$ by the stable condition. Thus $\theta_t^*\alpha$ is constant and since $\theta_0^*\alpha=\alpha$ the statement follows. □

Proposition 2.43 (Equivalent Definition of Stability Condition, [40, p. 70]). *Let (Σ^{2n+1},ω) be a Hamiltonian manifold and $\alpha\in\Omega^1(\Sigma)$. Then (Σ,ω,α) is a stable Hamiltonian manifold if and only if $\alpha\wedge\omega^n$ is a volume form on Σ.*

Proof. Let $x\in\Sigma$ and write $T_x\Sigma=\ker\omega_x\oplus S_x$. By the standard form theorem for skew-symmetric bilinear maps [40, p. 3], there exists a basis (u,a_i,b_i) of $T_x\Sigma$

such that u spans $\ker \omega_x$ and (a_i, b_i) is a symplectic basis for S_x. Then the statement follows immediately from

$$\alpha_x \wedge \omega_x^n(u, a_i, b_i) = \alpha_x(u)\omega_x^n(a_i, b_i),$$

and the fact that ω_x is a symplectic tensor if and only if $\omega^n \neq 0$. □

A particularly important class of examples of stable Hamiltonian manifolds are given by certain hypersurfaces in symplectic manifolds.

Definition 2.44 (Contact Manifold, [15, p. 21]). A ***contact manifold*** is defined to be a tuple (Σ, α) such that $(\Sigma, d\alpha, \alpha)$ is a stable Hamiltonian manifold. We call α a ***contact form on $\boldsymbol{\Sigma}$***.

There exists a category having contact manifolds as its objects.

Definition 2.45 (Category of Contact Manifolds). Define the category of ***contact manifolds***, written Cont, to have as objects contact manifolds (Σ, α) and as morphisms $F\colon (\Sigma, \alpha) \to (\widetilde{\Sigma}, \widetilde{\alpha})$ maps $F \in C^\infty(\Sigma, \widetilde{\Sigma})$ such that there exists a positive function $f \in C^\infty(\Sigma)$ with $F^*\widetilde{\alpha} = f\alpha$.

In analogy to definition 2.45, denote by ESymp the category of exact symplectic manifolds, that is, ESymp has as objects exact symplectic manifolds $(M, d\alpha)$ and as morphisms $F\colon (M, d\alpha) \to (\widetilde{M}, d\widetilde{\alpha})$ maps $F \in C^\infty(M, \widetilde{M})$ such that $F^*\widetilde{\omega} = \omega$. Then we have the following result.

Proposition 2.46 (Extrinsic Symplectisation Functor). *There exists a functor*

$$S\colon \mathsf{Cont} \to \mathsf{ESymp},$$

called the ***extrinsic symplectisation functor****.*

Proof. First we define S on objects (Σ^{2n-1}, α) of Cont following [40, p. 76]. We claim that

$$\big(\mathbb{R} \times \Sigma, d(e^t\pi^*\alpha)\big)$$

is a symplectic manifold, called the ***extrinsic symplectisation of $(\boldsymbol{\Sigma}, \boldsymbol{\alpha})$***, where $\pi \in C^\infty(\mathbb{R} \times \Sigma, \Sigma)$ denotes the projection. Indeed, we compute

$$\begin{aligned}
\big(d(e^t\pi^*\alpha)\big)^n &= \big(de^t \wedge \pi^*\alpha + e^t d\pi^*\alpha\big)^n \\
&= \big(de^t \wedge \pi^*\alpha + e^t\pi^*(d\alpha)\big)^n \\
&= \big(e^t dt \wedge \pi^*\alpha + e^t\pi^*(d\alpha)\big)^n \\
&= e^{nt}\left(dt \wedge \pi^*\alpha + \pi^*(d\alpha)\right)^n \\
&= e^{nt} dt \wedge \pi^*\alpha \wedge \big(\pi^*(d\alpha)\big)^{n-1} \\
&= e^{nt} dt \wedge \pi^*\alpha \wedge \pi^*(d\alpha)^{n-1} \\
&= e^{nt} dt \wedge \pi^*\left(\alpha \wedge (d\alpha)^{n-1}\right).
\end{aligned}$$

Thus $\big(d(e^t\pi^*\alpha)\big)^n$ is a volume form on $\mathbb{R}\times\Sigma$ and hence $d(e^t\pi^*\alpha)$ is a symplectic form on $\mathbb{R}\times\Sigma$. Note that from the above computation it also follows that the induced orientation of the symplectic structure coincides with the product orientation [25, 382]

$$\pi_1^*(dt)\wedge\pi_2^*\left(\alpha\wedge(d\alpha)^{n-1}\right).$$

Based on exercise 3.5.26 (i) [29, p. 140] define S on morphisms F in Cont to be the morphism

$$S(F)\colon\ \big(\mathbb{R}\times\Sigma, d(e^t\alpha)\big)\to\big(\mathbb{R}\times\Sigma, d(e^t\widetilde{\alpha})\big)$$

given by

$$S(F)(t,x):=\big(t-\log f(x), F(x)\big)$$

where $f\in C^\infty(M)$ is positive such that $F^*\widetilde{\alpha}=f\alpha$. Indeed, we compute

$$S(F)^*d(e^t\widetilde{\alpha})=d\left(\big(e^t\circ S(F)\big)\,S(F)^*\widetilde{\alpha}\right)=d(e^te^{-\log f}F^*\widetilde{\alpha})=d(e^t\alpha).$$

Checking the functorial properties is straightforward and therefore left to the reader.□

Remark 2.47 (Intrinsic Symplectisation, [29, p. 139]). There exists also the notion of an intrinsic symplectisation. Let Σ be a manifold and $\alpha\in\Omega^1(\Sigma)$ nowhere-vanishing. Define $F_\alpha\in C^\infty(\mathbb{R}\times\Sigma, T^*\Sigma)$ by

$$F_\alpha(t,x):=(x,e^t\alpha_x).$$

Then (Σ,α) is a contact manifold if and only if $\big(\mathbb{R}\times\Sigma, F_\alpha^*(\omega_0)\big)$ is a symplectic manifold.

Via the extrinsic symplectisation 2.46, any contact manifold (Σ,α) can be realised as an embedded hypersurface in an exact symplectic manifold. Indeed, we have the composition of embeddings

$$(\Sigma,\alpha)\xhookrightarrow{\iota_0}\big(\{0\}\times\Sigma, d\alpha\big)\xhookrightarrow{\iota_{\{0\}\times\Sigma}}\big(\mathbb{R}\times\Sigma, d(e^t\alpha)\big).$$

Definition 2.48 (Contact Type Hypersurface, [20, p. 119], [36, p. 23]). Let (M,ω) be a symplectic manifold. An immersed hypersurface $\Sigma\subseteq M$ is said to be of ***contact type***, iff there exists $\alpha\in\Omega^1(\Sigma)$ such that

$$d\alpha=\iota_\Sigma^*\omega\qquad\text{and}\qquad\alpha|_{\ker\omega|_\Sigma}\text{ is nowhere-vanishing,}$$

where $\iota_\Sigma\colon\Sigma\hookrightarrow M$ denotes the inclusion. If $(M,d\alpha)$ is an exact symplectic manifold and $(\Sigma,\alpha|_\Sigma)$ is of contact type, we say that $(\Sigma,\alpha|_\Sigma)$ is of ***restricted contact type***.

Remark 2.49. The notion of a contact-type hypersurface in a symplectic manifold was introduced by Alan Weinstein in 1979. This notion is also part of an important cojecture due to Weinstein himself, namely that a compact hypersurface $\Sigma\subseteq(M,\omega)$ of contact typ and satisfying $H^1(\Sigma)=0$ carries a closed characteristic. For more details see [20, p. 120].

Example 2.50 (Mechanical Hamiltonian). Regular hypersurfaces of mechanical Hamiltonians are of contact type, see [20, p. 130].

An important class of examples of contact type hypersurfaces is given by the following construction.

Definition 2.51 (Liouville Vector Field, [16, p. 23]). Let (M, ω) be a symplectic manifold. A vector field $X \in \mathfrak{X}(M)$ is said to be a ***Liouville vector field for*** $\boldsymbol{\omega}$, iff

$$\mathcal{L}_X \omega = \omega.$$

Remark 2.52. If $X \in \mathfrak{X}(M)$ is a Liouville vector field in a symplectic manifold (M, ω), then

$$\theta_t^* \omega_x = e^t \omega_x \qquad \forall (t, x) \in \mathcal{D},$$

where $\theta \in C^\infty(\mathcal{D}, M)$ denotes the flow of X. Indeed, using Fisherman's formula [25, p. 571] we compute

$$\frac{d}{dt}\theta_t^* \omega = \theta_t^*(\mathcal{L}_X \omega) = \theta_t^* \omega.$$

Example 2.53 (Exact Symplectic Manifold, [16, p. 23]). Let $(M, \omega = d\alpha)$ be an exact symplectic manifold. Define $X \in \mathfrak{X}(M)$ by $X := \widehat{\omega}^{-1}(\alpha)$. Using Cartan's magic formula [25, p. 372] we compute

$$\mathcal{L}_X \omega = d i_X \omega + i_X d\omega = d i_X \omega = d\alpha = \omega.$$

Example 2.54 (Euler Vector Field, [25, p. 176]). Consider the standard symplectic manifold $\left(\mathbb{R}^{2n}, \omega_0\right)$. Define $X \in \mathfrak{X}(\mathbb{R}^{2n})$, called the ***Euler vector field***, by

$$X := \frac{1}{2}\left(x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i}\right).$$

Using Cartan's magic formula [25, p. 372] we compute

$$\begin{aligned}
\mathcal{L}_X \omega_0 &= d i_X \omega_0 + i_X d\omega_0 \\
&= d i_X \omega_0 \\
&= \sum_{i=1}^n d\left(i_X(dy^i) \wedge dx^i - dy^i \wedge i_X(dx^i)\right) \\
&= \frac{1}{2}\sum_{i=1}^n d\left(y^i dx^i - x^i dy^i\right) \\
&= \frac{1}{2}\sum_{i=1}^n \left(dy^i \wedge dx^i - dx^i \wedge dy^i\right) \\
&= \omega_0.
\end{aligned}$$

Thus the Euler vector field is a Liouville vector field for ω_0.

Example 2.55 (Extrinsic Symplectisation, [16, p. 24]). Let (Σ, α) be a contact manifold and $\big(\mathbb{R} \times \Sigma, d(e^t \pi^* \alpha)\big)$ its extrinsic symplectisation from proposition 2.46. Then ∂_t is a Liouville vector field on $\big(\mathbb{R} \times \Sigma, \omega := d(e^t \pi^* \alpha)\big)$. Indeed, using Cartan's magic formula [25, p. 372] we compute

$$\begin{aligned}
\mathcal{L}_{\partial_t} \omega &= d i_{\partial t} \omega + i_{\partial_t} d\omega \\
&= d i_{\partial_t} d(e^t \pi^* \alpha) \\
&= d \left(i_{\partial_t} \left(e^t dt \wedge \pi^* \alpha + e^t \pi^* (d\alpha) \right) \right) \\
&= d \left(e^t i_{\partial_t} dt \wedge \pi^* \alpha - e^t dt \wedge i_{\partial_t} \pi^* \alpha + e^t i_{\partial_t} \pi^* (d\alpha) \right) \\
&= d(e^t \pi^* \alpha) \\
&= \omega.
\end{aligned}$$

Proposition 2.56 ([16, p. 23], [20, p. 121]). *Let (M^{2n}, ω) be a symplectic manifold and suppose that $\Sigma \subseteq M$ is an immersed hypersurface with inclusion $\iota_\Sigma \colon \Sigma \hookrightarrow M$. Let X be a Liouville vector field in a neighbourhood of Σ in M which is nowhere tangent to Σ. Then $\big(\Sigma, \iota_\Sigma^* (i_X \omega)\big)$ is of contact type.*

Proof. Let $\alpha := \iota_\Sigma^*(i_X \omega)$. Then we compute

$$d\alpha = \iota_\Sigma^*(d i_X \omega) = \iota_\Sigma^*(\mathcal{L}_X \omega - i_X d\omega) = \iota_\Sigma^* \omega,$$

and

$$\alpha \wedge (d\alpha)^{n-1} = \iota_\Sigma^*(i_X \omega) \wedge (\iota_\Sigma^* \omega)^{n-1} = \iota_\Sigma^*(i_X \omega \wedge \omega^{n-1}) = \frac{1}{n} \iota_\Sigma^* \left(i_X(\omega^n) \right).$$

But $\iota_\Sigma^* \left(i_X(\omega^n) \right)$ is a volume form on Σ by proposition 15.21 [25, p. 384–385] and so is $\alpha \wedge (d\alpha)^{n-1}$. Thus (Σ, α) is of contact type by proposition 2.43. □

Remark 2.57. In fact, if the hypersurface of contact type is compact, the converse to proposition 2.56 is true. For details see [20, p. 120].

Unlike the symplectic case, there is no obstruction for odd-dimensional spheres to be contact manifolds (note that if (M, ω) is a compact symplectic manifold, then $H^2_{\mathrm{dR}}(M) \neq 0$ but $H^2_{\mathrm{dR}}(\mathbb{S}^{2n}) = 0$ for all $n \geq 2$ similar to remark 2.22).

Proposition 2.58 ([16, p. 54]). *Odd dimensional spheres $\mathbb{S}^{2n-1} \subseteq \left(\mathbb{R}^{2n}, \omega_0\right)$ are of restricted contact type.*

Proof. Consider the Euler vector field 2.54 on $\mathbb{R}^{2n}$ given by

$$X = \frac{1}{2} \left(x^i \frac{\partial}{\partial x^i} + y^i \frac{\partial}{\partial y^i} \right).$$

By exercise 5.40 [25, p. 118] we have that $T_x \mathbb{S}^{2n-1} = x^\perp$ for all $x \in \mathbb{S}^{2n-1}$. Thus X is a normal vector field to $\mathbb{S}^{2n-1}$ since

$$\langle x, X|_x\rangle = \frac{1}{2}\langle x, x\rangle = \frac{1}{2}|x|^2 = \frac{1}{2}$$

for all $x \in \mathbb{S}^{2n-1}$. In particular, X is nowhere tangent to $\mathbb{S}^{2n-1}$. Thus $\mathbb{S}^{2n-1}$ as an embedded hypersurface of $\mathbb{R}^{2n}$ admits a Liouville vector field which is nowhere tangent. Explicitely, we have that

$$\iota^*_{\mathbb{S}^{2n-1}}(i_X\omega_0) = \iota^*_{\mathbb{S}^{2n-1}}\left(\frac{1}{2}\sum_{i=1}^{n}\left(y^i dx^i - x^i dy^i\right)\right)$$

by example 2.54. This contact form is called the ***standard contact form on*** $\mathbb{S}^{2n-1}$.□

Remark 2.59. One might wonder, if the odd-dimensional tori $\mathbb{T}^{2n+1}$ also admit a contact structure. The answer to this question was unknown for a long time, but was answered to the affirmative in [5]. However, this result is highly nontrivial.

Proposition 2.58 implies that we can consider the extrinsic symplectisation 2.46 of $\mathbb{S}^{2n-1}$.

Proposition 2.60 (Extrinsic Symplectisation of $\mathbb{S}^{2n-1}$, [29, p. 139]). *The extrinsic symplectisation of* $\mathbb{S}^{2n-1}$ *is symplectomorphic to* $\left(\mathbb{R}^{2n}\setminus\{0\}, \omega_0\right)$.

Proof. Consider

$$\mathbb{R}\times\mathbb{S}^{2n-1} \xhookrightarrow{\mathrm{id}_{\mathbb{R}}\times\iota} \mathbb{R}\times\mathbb{R}^{2n} \xrightarrow{\Phi} \mathbb{R}^{2n}\setminus\{0\},$$

where $\iota\colon \mathbb{S}^{2n-1}\to\mathbb{R}^{2n}$ denotes the inclusion and $\Phi\colon \mathbb{R}\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}\setminus\{0\}$ is defined by

$$\Phi(t,x) := e^{t/2}x.$$

Then $\Phi\circ(\mathrm{id}_{\mathbb{R}}\times\iota)$ is a diffeomorphism. Indeed, the explicit inverse is given by

$$x \mapsto (2\log|x|, x/|x|).$$

We compute

$$\begin{aligned}
\left(\Phi\circ(\mathrm{id}_{\mathbb{R}}\times\iota)\right)^*(\omega_0) &= (\mathrm{id}_{\mathbb{R}}\times\iota)^*\Phi^*(\omega_0)\\
&= (\mathrm{id}_{\mathbb{R}}\times\iota)^*\left(\sum_{i=1}^{n} d\Phi^{n+i}\wedge d\Phi^i\right)\\
&= (\mathrm{id}_{\mathbb{R}}\times\iota)^*\left(e^t\sum_{i=1}^{n}\left(\frac{1}{2}y^i dt + dy^i\right)\wedge\left(\frac{1}{2}x^i dt + dx^i\right)\right)\\
&= (\mathrm{id}_{\mathbb{R}}\times\iota)^*\left(e^t dt\wedge\left(\frac{1}{2}\sum_{i=1}^{n}\left(y^i dx^i - x^i dy^i\right)\right) + e^t\omega_0\right)\\
&= e^t dt\wedge\iota^*\left(\frac{1}{2}\sum_{i=1}^{n}\left(y^i dx^i - x^i dy^i\right)\right) + e^t\iota^*(\omega_0)
\end{aligned}$$

which coincides with $d\left(e^t\pi^*(\alpha)\right)$, where

$$\alpha := \iota^*\left(\frac{1}{2}\sum_{i=1}^{n}\left(x^i dy^i - y^i dx^i\right)\right)$$

is the standard contact form on $\mathbb{S}^{2n-1}$ as in proposition 2.58. □

Returning to the setting of regular energy surfaces in Hamiltonian systems, we have the following result inspired by remark 2.8.

Proposition 2.61 ([4, p. 8]). *Let $(\Sigma, \alpha|_\Sigma)$ be a compact regular energy surface in an exact Hamiltonian system $(M, \omega = d\alpha, H)$ of restricted contact type. Then there exists a Hamiltonian function $\widetilde{H} \in C^\infty(M)$ such that $X_{\widetilde{H}}|_\Sigma = R$, where R denotes the Reeb vector field 2.41 associated to $\alpha|_\Sigma$, and*

$$\mathcal{P}(X_{\widetilde{H}}, \Sigma) = \mathcal{P}(X_H, \Sigma).$$

Proof. Let $X \in \mathfrak{X}(M)$ denote the Liouville vector field defined in example 2.53. Since Σ is of contact type, X is nowhere tangent to Σ. Indeed, we compute

$$dH(X) = -i_{X_H}\omega(X) = \omega(X, X_H) = i_X\omega(X_H) = \alpha(X_H).$$

But X_H spans $\ker\omega|_\Sigma$ and since $\alpha|_\Sigma$ is nowhere-vanishing on $\ker\omega|_\Sigma$ as Σ is of contact type, we conclude that $dH(X)$ is nowhere-vanishing on Σ. Moreover, we have that

$$X_H|_\Sigma = dH(X)|_\Sigma R, \tag{2.8}$$

since

$$\alpha|_\Sigma(X_H|_\Sigma) = dH(X)|_\Sigma = dH(X)|_\Sigma\alpha|_\Sigma(R) = \alpha|_\Sigma\left(dH(X)|_\Sigma R\right).$$

By compactness of Σ, there exist $C_0, C_1 \in \mathbb{R}$ such that

$$0 < C_0 \le |dH(X)|_\Sigma| \le C_1 < +\infty.$$

Thus by the extension lemma for smooth functions [25, p. 45], there exists a nowhere-vanishing smooth function $f \in C^\infty(M)$ such that $f|_\Sigma = 1/dH(X)|_\Sigma$ (note that Σ is closed in M). We compute

$$\begin{aligned} X_{fH} &= \widehat{\omega}^{-1}\left(d(fH)\right) \\ &= \widehat{\omega}^{-1}\left(Hdf + fdH\right) \\ &= H\widehat{\omega}^{-1}(df) + f\widehat{\omega}^{-1}(dH) \\ &= HX_f + fX_H. \end{aligned}$$

This implies that X_{fH} restricts to Σ since

$$X_{fH}|_x = H(x)X_f|_x + f(x)X_H|_x = f(x)X_H|_x = R|_x \in T_x\Sigma, \tag{2.9}$$

for all $x \in \Sigma$. Thus (2.8) and the definition of $f|_\Sigma$ yields

$$X_{fH}|_\Sigma = R. \tag{2.10}$$

We claim that

$$\mathcal{P}(X_{fH}, \Sigma) = \mathcal{P}(X_H, \Sigma). \tag{2.11}$$

Suppose that $(\gamma, \tau) \in \mathcal{P}(X_H, \Sigma)$. Then define $s \in C^\infty(\mathbb{S}^1)$ to be the solution of

$$\dot{s} = f \circ \gamma \circ s \qquad \text{with} \qquad s(0) = 0.$$

Then $(\gamma \circ s, \tau) \in \mathcal{P}(X_{fH}, \Sigma)$. Indeed, using (2.9) we compute

$$\begin{aligned}(\gamma \circ s)' &= (\dot{\gamma} \circ s)\dot{s} \\ &= \tau\left(X_H \circ \gamma \circ s\right)(f \circ \gamma \circ s) \\ &= \tau\left((fX_H) \circ \gamma \circ s\right) \\ &= \tau(X_{fH} \circ \gamma \circ s).\end{aligned}$$

The other inclusion is shown similarly. Thus set $\widetilde{H} := fH$. □

The terminology stable is justified by the following contact type-like result.

Proposition 2.62 ([9, p. 591]). *For a compact embedded hypersurface Σ in a symplectic manifold (M, ω), the following conditions are equivalent:*

(a) *There exists a tubular neighbourhood $(-\varepsilon_0, \varepsilon_0) \times \Sigma$ of $\Sigma \cong \{0\} \times \Sigma$ such that the Hamiltonian line fields on $\{\varepsilon\} \times \Sigma$ are independent of $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$.*
(b) *There exists a vector field X on a neighbourhood of Σ in M nowhere tangent to Σ such that* $\ker(\omega|_\Sigma) \subseteq \ker(\mathcal{L}_X\omega|_\Sigma)$.
(c) $(\Sigma, \omega|_\Sigma)$ *is a stabilisable Hamiltonian manifold.*

Proof. To show (a) implies (b), consider the vector field $X := \frac{\partial}{\partial\varepsilon}$ on $(-\varepsilon_0, \varepsilon_0) \times \Sigma$. If we denote the flow of X by θ, we have that $\theta_t^*\omega|_\Sigma$ is constant in t and thus $\ker\left(\theta_t^*\omega|_\Sigma\right) = \ker(\omega|_\Sigma)$ for all t. In particular, by definition of the Lie derivative we have that

$$\ker(\omega|_\Sigma) \subseteq \ker(\mathcal{L}_X\omega|_\Sigma).$$

To show (b) implies (c), set $\alpha := i_X\omega|_\Sigma$. Then since X is nowhere tangent to Σ, α is nowhere-vanishing on $\ker(\omega|_\Sigma)$. Moreover, using Cartan's magic formula [25, p. 372] we compute

$$d\alpha = \mathcal{L}_X\omega|_\Sigma - i_X d\omega_\Sigma = \mathcal{L}_X\omega|_\Sigma.$$

But then by assumption

$$\ker(\omega|_\Sigma) \subseteq \ker(\mathcal{L}_X\omega|_\Sigma) = \ker(d\alpha).$$

Finally, to show that (c) implies (a), suppose that $\alpha \in \Omega^1(\Sigma)$ is a stabilising form on Σ. Then Σ is orientable since $\alpha \wedge \omega|_\Sigma^{n-1}$ is a volume form on Σ by proposition

2.43. Thus by exercise 3.4.17 [29, p. 124], there exists a neighbourhood U of Σ in M symplectomorphic to

$$\big((-\varepsilon_0, \varepsilon_0) \times \Sigma, \omega|_\Sigma + d(\varepsilon\alpha)\big)$$

for $\varepsilon_0 > 0$ sufficiently small extending the embedding $\iota_\Sigma \colon \Sigma \hookrightarrow M$. Observe that the Reeb field R_α associated to the stabilising form α is contained in each Hamiltonian line field and thus the Hamiltonian line fields are independent of $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. $\square$

Motivated by proposition 2.62, we make the following definition.

Definition 2.63 (Stable Tubular Neighbourhood, [8, p. 1775]). Let (M, ω) be a symplectic manifold and $(\Sigma, \omega|_\Sigma, \alpha)$ a stable immersed hypersurface. A ***stable tubular neighbourhood of Σ*** is defined to be a tuple (ε_0, ψ), where $\varepsilon_0 > 0$ and $\psi \in C^\infty((-\varepsilon_0, \varepsilon_0) \times \Sigma, M)$ is an embedding such that

$$\psi|_{\{0\}\times\Sigma} = \iota_\Sigma \qquad \text{and} \qquad \psi^*\omega = \omega|_\Sigma + d(\varepsilon\alpha)$$

is a symplectic form.

Remark 2.64 ([8, p. 1775]). As proposition 2.62 shows, every compact embedded stable hypersurface admits a stable tubular neighbourhood.

Let $(\Sigma, \omega|_\Sigma, \alpha)$ be a compact stable regular energy surface in a Hamiltonian system (M, ω, H). In order to define the Rabinowitz action functional as in the exact case, we need to extend the stabilising form $\alpha \in \Omega^1(\Sigma)$ to the whole symplectic manifold (M, ω). Since a priori there is no connection between ω and α, for example, α might be closed and thus maximally degenerate, we need also to alter the defining Hamiltonian H for Σ. By proposition 2.64 there exists a stable tubular neighbourhood (ε_0, ψ) of Σ in M. In what follows, we need a standard technique used in modern analysis for approximating functions by smooth ones.

Definition 2.65 (Standard Mollifier, [12, p. 629]). Define $\varphi \in C_c^\infty(\mathbb{R}^n)$ by

$$\varphi(x) := \begin{cases} Ce^{(|x|^2-1)^{-1}} & |x| < 1, \\ 0 & |x| \geq 1, \end{cases}$$

where

$$C := \left(\int_{-1}^{1} e^{(|x|^2-1)^{-1}}\right)^{-1}.$$

For every $\varepsilon > 0$ define $\varphi_\varepsilon \in C^\infty(\mathbb{R}^n)$ by

$$\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n}\varphi(x/\varepsilon).$$

The family $(\varphi_\varepsilon)_{\varepsilon>0}$ is called the family of ***standard mollifiers***.

Remark 2.66. We have that $\varphi_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}(\varphi_\varepsilon) = \overline{B}_\varepsilon(0)$ and

$$\int_{\mathbb{R}^n} \varphi_\varepsilon(x)dx = \int_{B_\varepsilon(0)} \varphi_\varepsilon(x)dx = 1$$

for all $\varepsilon > 0$. See figure 2.1.

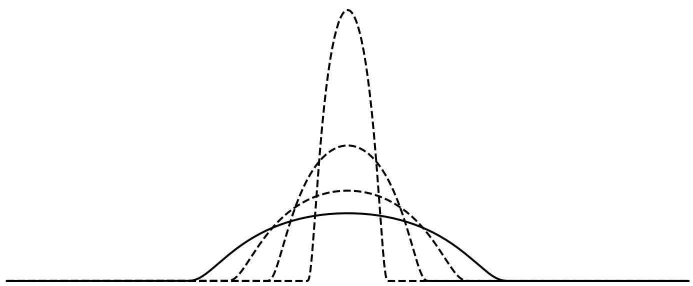


Fig. 2.1: Standard mollifiers $(\varphi_\varepsilon)_{\varepsilon>0}$.

Definition 2.67 (Mollification, [12, p. 629]). Let $U \subseteq \mathbb{R}^n$ open and $f \in L^1_{\mathrm{loc}}(U)$. For every $\varepsilon > 0$ define

$$U_\varepsilon := \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}$$

and define the ***mollification of f***, written f_ε, by

$$f_\varepsilon(x) := (\varphi_\varepsilon * f)(x) := \int_U \varphi_\varepsilon(x-y)f(y)dy = \int_{B_\varepsilon(0)} \varphi_\varepsilon(y)f(x-y)dy.$$

for all $x \in U_\varepsilon$.

Among the most important properties of mollifiers are the following (for details see [12, p. 630]). Mollifiers are always smooth, that is, $f_\varepsilon \in C^\infty(U_\varepsilon)$ for all $\varepsilon > 0$, no matter what the regularity of f is. We have that $f_\varepsilon \to f$ almost everywhere as $\varepsilon \to 0$ and if $f \in C(U)$, then $f_\varepsilon \to f$ uniformly on compact subsets of U as $\varepsilon \to 0$. Observe, that if f is an affine linear function on $\mathbb{R}$, that is, $f(x) = \lambda x + \mu$ for some $\lambda, \mu \in \mathbb{R}$, then $f_\varepsilon = f$ for all $\varepsilon > 0$. Indeed, we compute

$$\begin{aligned}
f_\varepsilon(x) &= (\varphi_\varepsilon * f)(x) \\
&= \lambda \int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(y)xdy - \lambda \int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(y)ydy + \mu \int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(y)dy \\
&= \lambda x + \mu \\
&= f,
\end{aligned}$$

for every $x \in \mathbb{R}$ since

$$\int_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(y) y \, dy = 0$$

as any integral of an odd function over a symmetric domain vanishes. In particular, given any piecewise linear continuous function, we can always build a smooth approximation by just "smoothing the corners".

Choose a function $f \in C^\infty(\mathbb{R})$ such that $f(\varepsilon) = \varepsilon + 1$ for all $\varepsilon \in [-\varepsilon_0/2, \varepsilon_0/2]$ and $\operatorname{supp} f \subseteq (-\varepsilon_0, \varepsilon_0)$. Such functions do exist as this condition can always be achieved by mollifying an appropriate piecewise-linear continuous function as in figure 2.2a. Define an extension $\widetilde{\alpha} \in \Omega^1_c(M)$ of the stabilising form $\alpha \in \Omega^1(\Sigma)$ by

$$\widetilde{\alpha}_p := \begin{cases} f(\varepsilon)\alpha_x & p = \psi(\varepsilon, x), \\ 0 & p \notin U, \end{cases} \tag{2.12}$$

where $U := \psi\left((-\varepsilon_0, \varepsilon_0) \times \Sigma\right)$. Thus we can define the ***modified Rabinowitz action functional***

$$\widehat{\mathcal{R}}^H : C^\infty(\mathbb{S}^1, M) \times \mathbb{R} \to \mathbb{R}, \quad \text{by} \quad \widehat{\mathcal{R}}^H(\gamma, \tau) := \int_\gamma \widetilde{\alpha} - \tau \int_0^1 H \circ \gamma. \tag{2.13}$$

Proposition 2.68 ([8, p. 1786]). *Let $(\Sigma, \omega|_\Sigma, \alpha)$ be a compact stable regular energy surface in a Hamiltonian system (M, ω, H). Then*

$$\mathcal{P}(X_H, \Sigma) \subseteq \operatorname{Crit}(\widehat{\mathcal{R}}^H).$$

Remark 2.69 ([29, p. 146–147]). The converse to proposition 2.68 needs not to be true. Indeed, it might be the case that the stabilising form is closed. For an explicit example, consider any symplectic manifold (M^{2n}, ω). For every $f \in C^\infty(M \times \mathbb{S}^1)$

$$(M \times \mathbb{S}^1, \omega + dt \wedge df, dt)$$

is a stable Hamiltonian structure since

$$dt \wedge (\omega + dt \wedge df)^n = dt \wedge (n\omega^{n-1} \wedge dt \wedge df + \omega^n) = dt \wedge \omega^n$$

is a volume form.

Proof. The statement immediately follows from the identity

$$d\widehat{\mathcal{R}}^H|_{(\gamma,\tau)}(X, \widetilde{\tau}) = \widehat{m}\left(\operatorname{grad} \mathcal{R}^H|_{(\gamma,\tau)}, (X, \widetilde{\tau})\right), \tag{2.14}$$

where we define the bilinear form

$$\widehat{m}\left((X_1, \widetilde{\tau}_1), (X_2, \widetilde{\tau}_2)\right) := \int_0^1 d\widetilde{\alpha}_\gamma\left(JX_1, X_2\right) + \widetilde{\tau}_1\widetilde{\tau}_2$$

for all $(X_1, \widetilde{\tau}_1), (X_2, \widetilde{\tau}_2) \in T_\gamma C^\infty(\mathbb{S}^1, M) \times \mathbb{R}$ and $(\gamma, \tau) \in C^\infty(\mathbb{S}^1, M)$ and

$$\operatorname{grad}\mathcal{R}^H|_{(\gamma,\tau)} := \begin{pmatrix} J\left(\dot\gamma - \tau(X_H\circ\gamma)\right) \\ -\displaystyle\int_0^1 H\circ\gamma \end{pmatrix},$$

where J is an ω-compatible almost complex structure on M. We will later see, that in an appropriate setting, grad $\mathcal{R}^H$ is indeed the gradient of the Rabinowitz action functional $\mathcal{R}^H$ with respect to an appropriate metric on M. In order to prove (2.14), we actually need to construct another defining Hamiltonian $\widetilde{H} \in C^\infty(M)$ for Σ, which does in general not coincide with H. However, $\mathcal{P}(X_H, \Sigma)$ and $\mathcal{P}(X_{\widetilde{H}}, \Sigma)$ coincide modulo reparametrisation by remark 2.8. Note that Σ separates M for

$$M = H^{-1}(-\infty, 0) \cup \Sigma \cup H^{-1}(0, +\infty).$$

Let (ε_0, ψ) be a stable tubular neighbourhood of Σ and $h \in C^\infty(\mathbb{R})$ be a sufficiently small mollification of the piecewise-linear function

$$h(\varepsilon) = \begin{cases} \varepsilon & \varepsilon \in [-\varepsilon_0/3, \varepsilon_0/3], \\ \varepsilon_0/3 & \varepsilon \in [\varepsilon_0/3, +\infty), \\ -\varepsilon_0/3 & \varepsilon \in (-\infty, -\varepsilon_0/3], \end{cases}$$

as in figure 2.2b. Define $\widetilde{H} \in C^\infty(M)$ by

$$\widetilde{H}(p) := \begin{cases} h(\varepsilon) & p = \psi(\varepsilon, x), \\ \varepsilon_0/3 & p \in U^c \cap H^{-1}(0, +\infty), \\ -\varepsilon_0/3 & p \in U^c \cap H^{-1}(-\infty, 0). \end{cases}$$

Then $\widetilde{H}$ is a defining Hamiltonian function for the regular hypersurface Σ. The corresponding Hamiltonian vector field to the Hamiltonian function $\widetilde{H}$ is given by $X_{\widetilde{H}} = h'R$, where R denotes the Reeb vector field associated to the stable Hamiltonian manifold $(\Sigma, \omega|_\Sigma, \alpha)$. Indeed, by assumption $\psi^*\omega$ is symplectic and since $d\widetilde{H} = 0$ on $M \setminus U$ we compute

$$\begin{aligned} i_{h'(\varepsilon)R}\psi^*\omega &= i_{h'(\varepsilon)R}\left(\omega|_\Sigma + d(\varepsilon\alpha)\right) \\ &= i_{h'(\varepsilon)R}\left(\omega|_\Sigma + d\varepsilon\wedge\alpha + \varepsilon d\alpha\right) \\ &= h'(\varepsilon)i_R\omega|_\Sigma + \left(i_{h'(\varepsilon)R}d\varepsilon\right)\alpha - h'(\varepsilon)i_R\alpha d\varepsilon + \varepsilon h'(\varepsilon)i_R d\alpha \\ &= -h'(\varepsilon)d\varepsilon \\ &= -d\widetilde{H}, \end{aligned}$$

since $i_R\omega|_\Sigma = 0$ and also $i_R d\alpha = 0$ by the stable condition. Moreover we compute

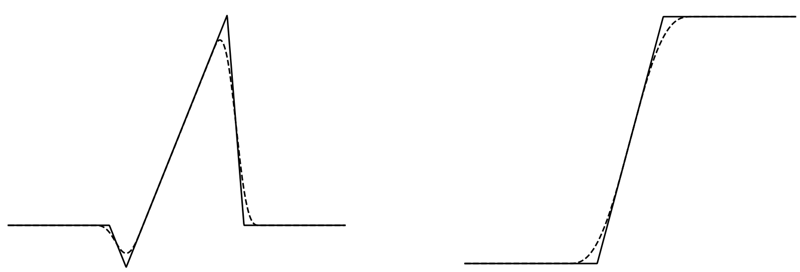
$$\begin{aligned} i_{X_{\widetilde{H}}}d\widetilde{\alpha} &= i_{X_{\widetilde{H}}}\left(f'(\varepsilon)d\varepsilon\wedge\alpha + f(\varepsilon)d\alpha)\right) \\ &= f'(\varepsilon)\left(i_{X_{\widetilde{H}}}d\varepsilon\right)\alpha - f'(\varepsilon)i_{X_{\widetilde{H}}}\alpha d\varepsilon + f(\varepsilon)i_{X_{\widetilde{H}}}d\alpha \end{aligned}$$

$$
\begin{aligned}
&= -f'(\varepsilon)h'(\varepsilon)d\varepsilon \\
&= -h'(\varepsilon)d\varepsilon \\
&= -d\widetilde{H},
\end{aligned}
$$

since $f' = 1$ on $[-\varepsilon_0/2, \varepsilon_0/2]$. Thus we compute

$$
\begin{aligned}
d\widehat{\mathcal{R}}^{\widetilde{H}}|_{(\gamma,\tau)}(X,\widetilde{\tau}) &= \int_0^1 d\widetilde{\alpha}_\gamma(X,\dot{\gamma}) - \tau \int_0^1 d\widetilde{H}_\gamma(X) - \widetilde{\tau}\int_0^1 \widetilde{H}\circ\gamma \\
&= \int_0^1 d\widetilde{\alpha}_\gamma(X,\dot{\gamma}) + \tau \int_0^1 d\widetilde{\alpha}_\gamma(X_{\widetilde{H}}\circ\gamma, X) - \widetilde{\tau}\int_0^1 \widetilde{H}\circ\gamma \\
&= \int_0^1 d\widetilde{\alpha}_\gamma\left(X, \dot{\gamma} - \tau(X_{\widetilde{H}}\circ\gamma)\right) - \widetilde{\tau}\int_0^1 H\circ\gamma \\
&= \widehat{m}\left(\operatorname{grad}\mathcal{R}^{\widetilde{H}}|_{(\gamma,\tau)}, (X,\widetilde{\tau})\right).
\end{aligned}
$$

□



(a) Possible choice of extension function f and its mollification.

(b) Mollification of the piecewise-linear continuous function h.

2.3 The ω-Limit Set of a Family of Periodic Orbits

In this section we consider smooth families of periodic orbits on smooth deformations of regular energy surfaces in symplectic manifolds.

Definition 2.70 (Homotopy of Energy Surfaces, [4, p. 9]). Let (M,ω) be a symplectic manifold. A ***homotopy of energy surfaces in*** (M,ω) is defined to be a family $(\Sigma_\sigma)_{\sigma\in[0,1]}$ of hypersurfaces in M such that there exists $H \in C^\infty(M\times[0,1])$ with the property that Σ_σ is a regular energy surface in the Hamiltonian system (M,ω,H_σ) with $H_\sigma := H(\cdot,\sigma)$, for all $\sigma\in[0,1]$. We say that a homotopy of energy surfaces in (M,ω) is ***compact***, iff $H^{-1}(0)$ is compact.

Given a compact homotopy of energy surfaces $(\Sigma_\sigma)_{\sigma\in[0,1]}$ in a symplectic manifold (M, ω), all hypersurfaces Σ_σ are diffeomorphic to each other. Indeed, this follows from the Ehresmann fibration theorem.

Proposition 2.71 ([15, p. 76]). *Let M be a smooth manifold and $F \in C^\infty(M\times[0, 1])$ such that $F^{-1}(0)$ is compact and 0 is a regular value of $F_\sigma := F(\cdot, \sigma) \in C^\infty(M)$ for all $\sigma \in [0, 1]$. Then all the hypersurfaces $F_\sigma^{-1}(0)$ are diffeomorphic to each other.*

Definition 2.72 (Family of Periodic Orbits, [4, p. 9]). Let $(\Sigma_\sigma)_{\sigma\in[0,1]}$ be a homotopy of energy surfaces in a symplectic manifold (M, ω). A ***family of periodic orbits on $(\boldsymbol{\Sigma_\sigma})_{\boldsymbol{\sigma\in[0,1]}}$*** is defined to be a family $(\gamma_\sigma, \tau_\sigma)_{\sigma\in[0,\sigma_\infty)}$ for some $\sigma_\infty \in (0, 1]$ such that there exist

$$\gamma \in C^\infty(\mathbb{S}^1 \times [0, \sigma_\infty)\,, M) \qquad \text{and} \qquad \tau \in C^\infty([0, \sigma_\infty)\,, (0, +\infty))$$

with $(\gamma_\sigma, \tau_\sigma) \in \mathscr{P}^+(X_{H_\sigma}, \Sigma_\sigma)$ for all $\sigma \in [0, \sigma_\infty)$, where as usual we abbreviate $\gamma_\sigma := \gamma(\cdot, \sigma) \in C^\infty(\mathbb{S}^1, M)$ and $\tau_\sigma := \tau(\sigma)$.

Remark 2.73. For examples of such families of periodic orbits on homotopies of energy surfaces and their evolution see [1, p. 595–616].

Given a family of periodic orbits $(\gamma_\sigma, \tau_\sigma)_{\sigma\in[0,\sigma_\infty)}$ on a homotopy of energy surfaces $(\Sigma_\sigma)_{\sigma\in[0,1]}$ in a symplectic manifold (M, ω), we may ask for the existence of a limit parametrised periodic orbit $(\gamma_{\sigma_\infty}, \tau_{\sigma_\infty})$ on Σ_{σ_∞}. It is illustrative, to consider the special case of an autonomous Hamiltonian function $H \in C^\infty(M)$ such that σ is a regular value of H for all $\sigma \in [0, 1]$. Then we can define a time-dependent Hamiltonian function $\widetilde{H} \in C^\infty(M \times [0, 1])$ by setting $\widetilde{H} := H - \sigma$. A smooth family $(\gamma_\sigma, \tau_\sigma)_{\sigma\in[0,\sigma_\infty)}$ of parametrised periodic orbits on the regular energy hypersurfaces $\Sigma_\sigma := H^{-1}(\sigma)$ can be depicted as an "orbit cylinder" as in figure 2.3. In this setting it may be the case that $\tau_\sigma \to +\infty$ as $\sigma \to \sigma_\infty$. The terminology for this situation goes back to Ralph Abraham.

Definition 2.74 (Blue Sky Catastrophe, [15, p. 121]). Let $(\gamma_\sigma, \tau_\sigma)_{\sigma\in[0,\sigma_\infty)}$ be a family of periodic orbits on a homotopy of energy surfaces $(\Sigma_\sigma)_{\sigma\in[0,1]}$ in a symplectic manifold (M, ω). We say that a ***blue sky catastrophe*** occurs, iff

$$\tau_\sigma \to +\infty \qquad \text{as} \qquad \sigma \to \sigma_\infty.$$

A convenient way of characterising the asymptotic behaviour of the smooth family $(\gamma_\sigma, \tau_\sigma)_{\sigma\in[0,\sigma_\infty)}$ is provided by the notion of the associated ω-limit set from the theory of dynamical systems.

Definition 2.75 (The $\boldsymbol{\omega}$-Limit Set of a Family of Periodic Orbits, [4, p. 10]). Given a family of periodic orbits $(\gamma_\sigma, \tau_\sigma)_{\sigma\in[0,\sigma_\infty)}$ on a homotopy of energy surfaces $(\Sigma_\sigma)_{\sigma\in[0,1]}$ in a symplectic manifold (M, ω), define the associated ***$\boldsymbol{\omega}$-limit set***, written $\omega(\gamma_\sigma, \tau_\sigma)$, to be the set consisting of all

$$(\gamma_{\sigma_\infty}, \tau_{\sigma_\infty}) \in C^\infty(\mathbb{S}^1, \Sigma_{\sigma_\infty}) \times (0, +\infty)$$

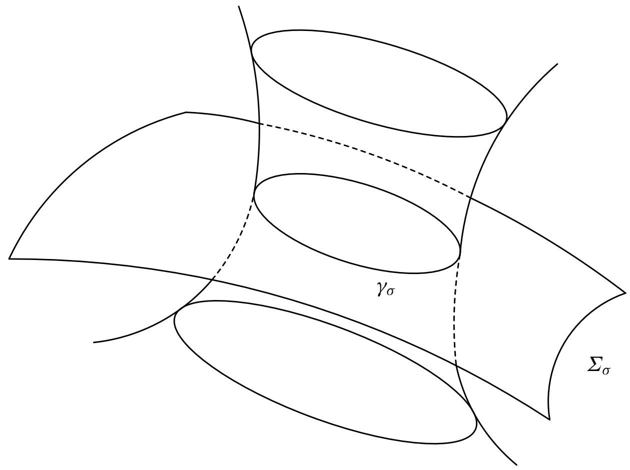


Fig. 2.3: An orbit cylinder.

such that there exists a sequence $(\sigma_k) \subseteq [0, \sigma_\infty)$ with

$$\lim_{k\to\infty} \sigma_k = \sigma_\infty \qquad \text{and} \qquad \lim_{k\to\infty} (\gamma_{\sigma_k}, \tau_{\sigma_k}) = (\gamma_{\sigma_\infty}, \tau_{\sigma_\infty}),$$

where the latter limit is with respect to the C^∞-topology.

Remark 2.76. If the ω-limit set associated to a family of periodic orbits on a homotopy of energy surfaces is non-empty, then no blue sky catastrophe can occur.

Here is the main result about the topological properties of the ω-limit set of a family of periodic orbits on a homotopy of energy surfaces. For the proof we need two elementary results from analysis and topology.

Proposition 2.77 (Ascolis Theorem, [35, p. 278]). *Let X be a compact topological space and (Y, d) a metric space such that all closed bounded subspaces are compact. Equip $C^0(X, Y)$ with the uniform topology, that is, the topology induced by the uniform metric ρ on $C^0(X, Y)$ given by*

$$\rho(f, g) := \sup_{x\in X} \min \left\{ d\left(f(x), g(x)\right), 1\right\}.$$

Then a family $\mathcal{F} \subseteq C^0(X, Y)$ has compact closure if and only if $\mathcal{F}$ is pointwise bounded and equicontinuous.

Proposition 2.78 ([42, p. 123]). *Let $\Omega \subseteq \mathbb{R}$ open and $(f_k) \subseteq C^1(\Omega, \mathbb{R}^n)$ such that*

$$f_k \to f \qquad \text{and} \qquad f_k' \to g$$

uniformly as $k \to \infty$. Then $f \in C^1(\Omega, \mathbb{R}^n)$ and $f' = g$.

Theorem 2.79 (The ω-Limit Set of a Family of Periodic Orbits, [4, p. 10]). *Let $(\gamma_\sigma, \tau_\sigma)_{\sigma \in [0,\sigma_\infty)}$ be a family of periodic orbits on a compact homotopy of energy surfaces $(\Sigma_\sigma)_{\sigma\in[0,1]}$ in a symplectic manifold (M, ω) such that $\tau_\sigma \leq C$ for all $\sigma \in [0, \sigma_\infty)$ for some $C \in \mathbb{R}$. Then $\omega(\gamma_\sigma, \tau_\sigma)$ is nonempty, compact and connected.*

Proof. We prove the result in three steps.

Step 1: The ω-limit set is nonempty. We make use of Ascoli's theorem 2.77. Define

$$\mathscr{F} \subseteq C^0\left(\mathbb{S}^1, H^{-1}(0) \times [0, C]\right) \qquad \text{by} \qquad \mathscr{F} := \{(\gamma_\sigma, \tau_\sigma) : \sigma \in [0, \sigma_\infty)\}\,.$$

Fix a Riemannian metric m on M. Then $\mathscr{F}$ is pointwise bounded since

$$\mathscr{F}_t := \{(\gamma_\sigma(t), \tau_\sigma) : \sigma \in [0, \sigma_\infty)\}$$

is contained in the compact, thus bounded, space $H^{-1}(0) \times [0, C]$ for all $t \in \mathbb{S}^1$. Moreover, the family $\mathscr{F}$ is equicontinuous. Indeed, for $t, t_0 \in \mathbb{S}^1$ we estimate

$$\begin{aligned} d_m\left(\gamma_\sigma(t_0), \gamma_\sigma(t)\right) &\leq L_m(\gamma_\sigma) \\ &= \int_{t_0}^t |\dot\gamma_\sigma(s)|_m\, ds \\ &= t_\sigma \int_{t_0}^t \left|X_{H_\sigma}\left(\gamma_\sigma(s)\right)\right|_m ds \\ &\leq C \int_{t_0}^t \left|X_{H_\sigma}\left(\gamma_\sigma(s)\right)\right|_m ds \\ &= C_0\, |t - t_0|\,, \end{aligned}$$

by definition of the metric d_m induced by the Riemannian metric and since

$$H^{-1}(0) \times [0, 1] \to TM, \qquad (x, \sigma) \mapsto X_{H_\sigma}|_x$$

is bounded. Because $H^{-1}(0) \times [0, C]$ is compact, every closed subset of it is automatically compact, thus an application of Ascoli's theorem yields that $\overline{\mathscr{F}}$ is compact. Let $(\sigma_k) \subseteq [0, \sigma_\infty)$ be a sequence such that $\sigma_k \to \sigma_\infty$ as $k \to \infty$. Then by compactness of $\overline{\mathscr{F}}$ there exists a subsequence of $(\gamma_{\sigma_k}, \tau_{\sigma_k})$, which can without loss of generality assumed to be indexed by $(\sigma_k) \subseteq [0, \sigma_\infty)$ as well, such that

$$\lim_{k\to\infty} (\gamma_{\sigma_k}, \tau_{\sigma_k}) = (\gamma_{\sigma_\infty}, \tau_{\sigma_\infty}) \in C^0(\mathbb{S}^1, H^{-1}(0) \times [0, C]).$$

In fact $\gamma_{\sigma_\infty} \in C^0(\mathbb{S}^1, \Sigma_{\sigma_\infty})$ since

$$H_{\sigma_\infty}(\gamma_{\sigma_\infty}) = \lim_{k\to\infty} H(\gamma_{\sigma_k}, \sigma_k) = \lim_{k\to\infty} H_{\sigma_k}(\gamma_{\sigma_k}) = 0.$$

Moreover

$$\lim_{k\to\infty}\dot{\gamma}_{\sigma_k} = \lim_{k\to\infty}\tau_{\sigma_k}(X_{H_{\sigma_k}}\circ\gamma_{\sigma_k}) = \tau_{\sigma_\infty}(X_{H_{\sigma_\infty}}\circ\gamma_{\sigma_\infty})$$

uniformly. Since smoothness is a local property, we may work in a chart. Thus by proposition 2.78, we conclude that $\gamma_{\sigma_\infty}\in C^1(\mathbb{S}^1,\Sigma_{\sigma_\infty})$ with

$$\dot{\gamma}_{\sigma_\infty} = \lim_{k\to\infty}\dot{\gamma}_{\sigma_k} = \tau_{\sigma_\infty}(X_{H_{\sigma_\infty}}\circ\gamma_{\sigma_\infty}).$$

We can now inductively improve the regularity of γ_{σ_∞}. This method is referred to as ***bootstrapping***. Indeed, we have that

$$\ddot{\gamma}_{\sigma_k} = \frac{d}{dt}\tau_{\sigma_k}(X_{H_{\sigma_k}}\circ\gamma_{\sigma_k}) = \tau_{\sigma_k}DX_{H_{\sigma_k}}(\dot{\gamma}_{\sigma_k}) = \tau_{\sigma_k}^2 DX_{H_{\sigma_k}}(X_{H_{\sigma_k}}\circ\gamma_{\sigma_k}).$$

Thus we can repeatedly apply proposition 2.78 to conclude $\gamma_{\sigma_\infty}\in C^\infty(\mathbb{S}^1,\Sigma_{\sigma_\infty})$. Lastly we claim that $\tau_{\sigma_\infty}\neq 0$. We proceed as in [20, p. 109]. Towards a contradiction, suppose that $\tau_{\sigma_\infty}=0$. Then $\gamma_{\sigma_\infty}=x\in\Sigma_{\sigma_\infty}$. Since Σ_{σ_∞} is a regular hypersurface, we have that $X_{H_{\sigma_\infty}}|_x\neq 0$. Working locally in a chart about x, we can estimate for k large enough that

$$\langle X_{H_{\sigma_k}}\circ\gamma_{\sigma_k}, X_{H_{\sigma_\infty}}|_x\rangle \geq (1-\varepsilon)\,|X_{H_{\sigma_\infty}}(x)|^2$$

uniformly in $t\in\mathbb{S}^1$ for $\varepsilon>0$ small enough, where we use the standard Euclidean inner product. But then

$$\begin{aligned}(1-\varepsilon)\,|X_{H_{\sigma_\infty}}(x)|^2 &\leq \int_0^1 \langle X_{H_{\sigma_k}}\left(\gamma_{\sigma_k}(t)\right), X_{H_{\sigma_\infty}}|_x\rangle dt\\ &= \frac{1}{\tau_k}\int_0^1\langle\dot{\gamma}_{\sigma_k}(t), X_{H_{\sigma_\infty}}|_x\rangle dt\\ &= 0,\end{aligned}$$

contradicting the fact that $X_{H_{\sigma_\infty}}|_x\neq 0$.

Step 2: The ω-limit set is compact. Since the free loop space is metrisable, it is enough to show that the omega-limit set is sequentially compact. Let (γ^ν,τ^ν) be a sequence in the omega-limit set of $(\gamma_\sigma,\tau_\sigma)$. By definition, for each $\nu\in\mathbb{N}$ there exists a sequence $(\sigma_k^\nu)\subseteq[0,\sigma_\infty)$, such that

$$\lim_{k\to\infty}\sigma_k^\nu = \sigma_\infty \qquad\text{and}\qquad \lim_{k\to\infty}(\gamma_{\sigma_k^\nu},\tau_{\sigma_k^\nu}) = (\gamma^\nu,\tau^\nu).$$

Set

$$x^\nu := (\gamma^\nu,\tau^\nu) \qquad\text{and}\qquad y_k^\nu := (\gamma_{\sigma_k^\nu},\tau_{\sigma_k^\nu}).$$

Define a sequence $(k_\nu)\subseteq\mathbb{N}$ inductively by setting $k_0:=0$ and

$$k_{\nu+1} := \min_{k\in\mathbb{N}}\left\{\sigma_\infty-\sigma_k^{\nu+1}\leq\frac{\sigma_\infty-\sigma_{k_\nu}^\nu}{2}, d\left(y_k^{\nu+1},x^\nu\right)\leq\frac{1}{1+\nu}\right\} > k_\nu.$$

Then we have that

$$\sigma_\infty - \sigma^\nu_{k_\nu} \leq \frac{\sigma_\infty - \sigma^0_0}{2^\nu} \to 0 \qquad \text{as } \nu \to \infty,$$

and thus $\lim_{\nu\to\infty} \sigma^\nu_{k_\nu} = \sigma_\infty$. By step 1 there exists a subsequence (ν_j) and a parametrised periodic orbit $x := (\gamma, \tau) \in C^\infty(\mathbb{S}^1, \Sigma_{\sigma_\infty})$ such that

$$\lim_{j\to\infty} y^{\nu_j}_{\sigma_{\nu_j}} = x.$$

In order to prove that $\omega(\gamma_\sigma, \tau_\sigma)$ is compact, it is enough to show that

$$\lim_{j\to\infty} x^{\nu_j} = x.$$

But this immediately follows from

$$d(x^{\nu_j}, x) \leq d(x^{\nu_j}, y^{\nu_j}_{k_{\nu_j}}) + d(y^{\nu_j}_{k_{\nu_j}}, x) \leq \frac{1}{1+\nu_j} + d(y^{\nu_j}_{k_{\nu_j}}, x) \to 0$$

as $j \to \infty$.

Step 3: The ω-limit set is connected. Suppose that $\omega(\gamma_\sigma, \tau_\sigma)$ is disconnected. Then we have that $\omega(\gamma_\sigma, \tau_\sigma) = A_1 \cup A_2$ for $A_1, A_2 \subseteq \omega(\gamma_\sigma, \tau_\sigma)$ nonempty open disjoint subsets. Both A_1 and A_2 are also closed since their complements are open. Thus by step 2 we have that A_1 and A_2 are compact as well. By lemma 4.34 [24, p. 95], there exist $U_1, U_2 \subseteq C^\infty(\mathbb{S}^1, M)$ open and disjoint such that $A_1 \subseteq U_1$ and $A_2 \subseteq U_2$ since the free loop space is Hausdorff as it is metrisable. Choose

$$x_1 := (\gamma_1, \tau_1) \in A_1 \qquad \text{and} \qquad x_2 := (\gamma_2, \tau_2) \in A_2.$$

There exist sequences $(\sigma^1_k), (\sigma^2_k) \subseteq [0, \sigma_\infty)$ such that

$$\sigma^1_k \to \sigma_\infty \qquad \text{and} \qquad \sigma^2_k \to \sigma_\infty$$

as $k \to \infty$ as well as

$$\lim_{k\to\infty} \left(\gamma_{\sigma^1_k}, \tau_{\sigma^1_k}\right) = x_1 \qquad \text{and} \qquad \lim_{k\to\infty} \left(\gamma_{\sigma^2_k}, \tau_{\sigma^2_k}\right) = x_2.$$

Set $k_1 := 0$ and inductively

$$k_{\nu+1} := \begin{cases} \min\{k : \sigma^2_k > \sigma^1_{k_\nu}\} & \nu \text{ even}, \\ \min\{k : \sigma^1_k > \sigma^2_{k_\nu}\} & \nu \text{ odd}. \end{cases}$$

Then $\sigma_{k_\nu} \to \sigma_\infty$ as $\nu \to \infty$ and for ν sufficiently large, one endpoint of the path

$$[\sigma_{k_\nu}, \sigma_{k_{\nu+1}}] \to C^\infty(\mathbb{S}^1, M) \times (0, +\infty), \qquad \sigma \mapsto (\gamma_\sigma, \tau_\sigma)$$

lies in U_1 whereas the other lies in U_2. Since $U_1 \cap U_2 = \emptyset$, we find $\sigma_\nu \in [\sigma_{k_\nu}, \sigma_{k_{\nu+1}}]$ such that

$$y_\nu := (\gamma_{\sigma_\nu}, \tau_{\sigma_\nu}) \in \big(C^\infty(\mathbb{S}^1, M) \times (0, +\infty)\big) \setminus (U_1 \cup U_2).$$

By construction, $\sigma_\nu \to \sigma_\infty$ as $\nu \to \infty$. Thus by step 1 there exists a subsequence $(\nu_j) \subseteq [0, \sigma_\infty)$ and $y \in \omega(\gamma_\sigma, \tau_\sigma)$ such that

$$\lim_{j\to\infty} y_{\nu_j} = y.$$

But

$$\big(C^\infty(\mathbb{S}^1, M) \times (0, +\infty)\big) \setminus (U_1 \cup U_2)$$

is closed, so

$$y \in \big(C^\infty(\mathbb{S}^1, M) \times (0, +\infty)\big) \setminus (U_1 \cup U_2) \subseteq \big(C^\infty(\mathbb{S}^1, M) \times (0, +\infty)\big) \setminus \omega(\gamma_\sigma, \tau_\sigma),$$

contradicting the fact that by construction $y \in \omega(\gamma_\sigma, \tau_\sigma)$. □

Remark 2.80 ([30, p. 363], [8, p. 1831]). In general, the ω-limit set of a family of periodic orbits may be empty. We recall a general construction. Let (M, m) be a compact connected Riemannian manifold. Then we can consider the twisted Hamiltonian system $(T^*M, \omega_0 + \pi^*\eta, H)$, where

$$H(x, \xi) = \frac{1}{2}|\xi|^2_{m^*} + V(x)$$

is a mechanical Hamiltonian and $\eta \in \Omega^2(M)$ is assumed to be weakly exact, that is, if $\Pi \in C^\infty(\widetilde{M}, M)$ is a smooth covering map (smooth covering manifolds do always exists since we can take the universal covering manifold [25, p. 94]), then $\Pi^*\eta$ is exact. Define the ***Mañé critical value associated to m, η, V and Π*** by

$$c(m, \eta, V, \Pi) := \inf_{\substack{\theta\in\Omega^1(\widetilde{M})\\ d\theta=\Pi^*\eta}} \sup_{x\in\widetilde{M}} \widetilde{H}(x, \theta_x).$$

For constructing the counterexample, let Γ be a cocompact lattice of $\mathrm{PSL}(2, \mathbb{R})$ acting transitively on the hyperbolic plane $\mathbb{H}^2$. Set $M := \mathbb{H}^2/\Gamma$ and let H be the kinetic energy. Then it can be showed that the Mañé critical value is given by $c = 1/2$ and that we have the following situation: If $\sigma > 1/2$, the dynamics are Anosov and conjugate (after rescaling) to the underlying geodesic flow and the energy levels Σ_σ are of contact type. If $\sigma = c$, then flow is the famous horocycle flow which has no closed orbits and the level set Σ_c is unstable. If $\sigma < 1/2$ all orbits are closed and contractible. Energy levels Σ_σ are of contact type but with opposite orientation.

Motivated by remark 2.80, it turns out, that requesting the homotopy of energy surfaces to be of restricted contact type is sufficient to prevent blue sky catastrophes. Again, we make use of a basic result from analysis. It is the key technical result giving lower and upper bounds for the periods.

Lemma 2.81 (Gronwall's Inequality). *Let $f \in C^1([a,b])$ and $\alpha, \beta \in C^0([a,b])$ such that*

$$f'(x) \leq \alpha(x) + \beta(x) f(x)$$

for all $x \in I$. Then

$$f(x) \leq f(a) e^{\int_a^x \beta(t) dt} + \int_a^x \alpha(t) e^{\int_t^x \beta(s) ds} dt.$$

Proposition 2.82 ([15, p. 121], [4, p. 8]). *Let $(\gamma_\sigma, \tau_\sigma)_{\sigma \in [0, \sigma_\infty)}$ be a family of periodic orbits on a compact homotopy of energy surfaces $(\Sigma_\sigma)_{\sigma \in [0,1]}$ in an exact symplectic manifold $(M, d\alpha)$ such that $(\Sigma_\sigma, \alpha|_{\Sigma_\sigma})$ is of restricted contact type for all $\sigma \in [0,1]$. Then there exist constants $C_0, C_1 \in \mathbb{R}$ independent of $\sigma \in [0, \sigma_\infty)$ such that*

$$0 < C_0 \leq \tau_\sigma \leq C_1 < +\infty \qquad \forall \sigma \in [0, \sigma_\infty)\,.$$

Moreover, the Rabinowitz action functional $\mathcal{R}^{H_{\sigma_\infty}}$ is constant on $\omega(\gamma_\sigma, \tau_\sigma)$.

Proof. Let $X \in \mathfrak{X}(M)$ denote the Liouville vector field associated to the exact symplectic manifold $(M, d\alpha)$ as in example 2.53 and define $f \in C^\infty\left(H^{-1}(0)\right)$ by

$$f(x, \sigma) := dH_\sigma(X)(x).$$

Denote by R_σ the Reeb vector field associated to the stabilising form $\alpha|_{\Sigma_\sigma}$ for all $\sigma \in [0,1]$. Arguing as in the proof of proposition 2.61, we have that

$$X_{H_\sigma}|_{\Sigma_\sigma} = f_\sigma R_\sigma, \qquad \forall \sigma \in [0,1]\,.$$

As $H^{-1}(0)$ is compact by assumption, there exists $C \in \mathbb{R}$ such that

$$\frac{1}{C} \leq |f_\sigma(x)| \leq C \qquad \text{and} \qquad |\partial_\sigma H_\sigma(x)| \leq C \tag{2.15}$$

for all $(x, \sigma) \in H^{-1}(0)$. Thus there exists an nowhere-vanishing extension $\overline{f} \in C^\infty(M \times [0,1])$ of $1/f$. Replacing H with $\overline{f} H \in C^\infty(M \times [0,1])$ guarantees that γ_σ is a τ_σ-periodic orbit of the Reeb vector field R_σ for all $\sigma \in [0, \sigma_\infty)$ modulo reparametrisation. Abbreviate the Rabinowitz action functional $\mathcal{R}^{H_\sigma}$ associated to H_σ by $\mathcal{R}_\sigma$. Then we have the period-action equality

$$\mathcal{R}_\sigma(\gamma_\sigma, \tau_\sigma) = \int_0^1 \gamma_\sigma^* \alpha - \tau_\sigma \int_0^1 H_\sigma \circ \gamma_\sigma = \int_0^1 \alpha(\dot{\gamma}_\sigma) = \tau_\sigma \int_0^1 \alpha(R_\sigma \circ \gamma_\sigma) = \tau_\sigma$$

for all $\sigma \in [0, \sigma_\infty)$. Using the fact that $\mathcal{P}(X_{H_\sigma}, \Sigma_\sigma) \subseteq \operatorname{Crit}(\mathcal{R}_\sigma)$ by proposition 2.33, we compute

$$\begin{aligned}
\partial_\sigma \tau_\sigma &= \partial_\sigma \left(\mathcal{R}_\sigma(\gamma_\sigma, \tau_\sigma)\right) \\
&= (\partial_\sigma \mathcal{R}_\sigma)(\gamma_\sigma, \tau_\sigma) + d\mathcal{R}_\sigma|_{(\gamma_\sigma, \tau_\sigma)}(\partial_\sigma \gamma_\sigma, \partial_\sigma \tau_\sigma) \\
&= (\partial_\sigma \mathcal{R}_\sigma)(\gamma_\sigma, \tau_\sigma)
\end{aligned}$$

$$= -\tau_\sigma \int_0^1 (\partial_\sigma H_\sigma) \circ \gamma_\sigma \tag{2.16}$$

for all $\sigma \in [0, \sigma_\infty)$. Invoking the bound (2.15), we conclude that $|\partial_\sigma \tau_\sigma| \leq C\tau_\sigma$ for all $\sigma \in [0, \sigma_\infty)$. Applying Gronwalls inequality 2.81 yields

$$0 < \tau_0 e^{-C\sigma} \leq \tau_\sigma \leq \tau_0 e^{C\sigma} < +\infty$$

for all $\sigma \in [0, \sigma_\infty)$. In particular, we have that

$$0 < \tau_0 e^{-C\sigma_\infty} \leq \tau_\sigma \leq \tau_0 e^{C\sigma_\infty} < +\infty \qquad \forall \sigma \in [0, \sigma_\infty)\,.$$

Finally, we show that $\mathcal{R}^{H_{\sigma_\infty}}$ is constant on $\omega(\gamma_\sigma, \tau_\sigma)$. From (2.16) and Gronwall's inequality 2.81 we deduce that

$$e^{-C(\sigma_1 - \sigma_0)} \mathcal{R}^{H_{\sigma_0}}(\gamma_{\sigma_0}, \tau_{\sigma_0}) \leq \mathcal{R}^{H_{\sigma_1}}(\gamma_{\sigma_1}, \tau_{\sigma_1}) \leq e^{C(\sigma_1 - \sigma_0)} \mathcal{R}^{H_{\sigma_0}}(\gamma_{\sigma_0}, \tau_{\sigma_0})$$

holds for all $0 \leq \sigma_0 < \sigma_1 < \sigma_\infty$. This inequality readily implies the claim. □

Corollary 2.83. *Let $(\gamma_\sigma, \tau_\sigma)_{\sigma \in [0, \sigma_\infty)}$ be a family of periodic orbits on a compact homotopy of energy surfaces $(\Sigma_\sigma)_{\sigma \in [0,1]}$ in an exact symplectic manifold $(M, d\alpha)$ such that $(\Sigma_\sigma, \alpha|_{\Sigma_\sigma})$ is of restricted contact type for all $\sigma \in [0, 1]$. Then the ω-limit set $\omega(\gamma_\sigma, \tau_\sigma)$ is nonempty, compact and connected.*

Proof. Immediate by proposition 2.82 and theorem 2.79. □

Next we want to generalise propostion 2.82 in an appropriate setting.

Definition 2.84 (Stable Homotopy of Energy Surfaces, [8, p. 1775]). Let (M, ω) be a symplectic manifold. A ***stable homotopy of energy surfaces in* (M, ω)** is defined to be a family $(\Sigma_\sigma, \alpha_\sigma)_{\sigma \in [0,1]}$ of stable hypersurfaces in M such that $(\Sigma_\sigma)_{\sigma \in [0,1]}$ is a homotopy of energy surfaces and there exists smooth time-dependent tensor field $\widetilde{\alpha} \in C^\infty(M \times [0, 1], T^*M)$, that is, $\widetilde{\alpha}(x, \sigma) \in T_x^*M$ for all $(x, \sigma) \in M \times [0, 1]$, such that $\widetilde{\alpha}_\sigma|_{\Sigma_\sigma} = \alpha_\sigma$ for all $\sigma \in [0, 1]$, where $\widetilde{\alpha}_\sigma := \widetilde{\alpha}(\cdot, \sigma) \in \Omega^1(M)$, and such that $(\Sigma_\sigma, \omega|_{\Sigma_\sigma}, \alpha_\sigma)$ is a stable Hamiltonian manifold for all $\sigma \in [0, 1]$.

Example 2.85 (Restricted Contact Type Homotopy of Energy Surfaces). Suppose that $(\Sigma_\sigma)_{\sigma \in [0,1]}$ is a homotopy of energy surfaces in an exact symplectig manifold $(M, d\alpha)$ such that $(\Sigma_\sigma, \alpha|_{\Sigma_\sigma})$ is of restricted contact type for all $\sigma \in [0, 1]$. Then $(\Sigma_\sigma, \alpha|_{\Sigma_\sigma})_{\sigma \in [0,1]}$ is a stable homotopy of energy surfaces. Indeed, every contact type hypersurface is a stable Hamiltonian manifold and we can define the time-dependent tensor field $\widetilde{\alpha}$ simply to be the constant tensor field α.

Proposition 2.86 ([8, p. 1775]). *Let $(\Sigma_\sigma, \alpha_\sigma)_{\sigma \in [0,1]}$ be a compact stable homotopy of energy surfaces in a symplectic manifold (M, ω). Then there exists a smooth family $(\varepsilon_0, \psi_\sigma)_{\sigma \in [0,1]}$ of stable tubular neighbourhoods, that is, there exists*

$$\psi \in C^\infty\left((-\varepsilon_0, \varepsilon_0) \times H^{-1}(0), M\right)$$

such that $(\varepsilon_0, \psi_\sigma)$ *is a stable tubular neighbourhood of* Σ_σ *for all* $\sigma \in [0, 1]$, *where we abbreviate* $\psi_\sigma(\cdot, x) := \psi(\cdot, x, \sigma)$ *for all* $(x, \sigma) \in H^{-1}(0)$.

Proof. Choose a smooth family $(X_\sigma)_{\sigma\in[0,1]}$ of vector fields on M such that

$$i_{X_\sigma}\omega = \widetilde{\alpha}_\sigma.$$

Since Σ_σ is compact, the flow θ^{X_σ} of X_σ exists locally near Σ_σ. Let $R_\sigma := R_{\alpha_\sigma}$ denote the Reeb vector field associated to the stabilising form α_σ. Then for all $\sigma \in [0, 1]$ there exists a nowhere-vanishing function $f_\sigma \in C^\infty(\Sigma_\sigma)$ such that $X_{H_\sigma} = f_\sigma R_\sigma$. Thus we compute

$$\begin{aligned} dH_\sigma(X_\sigma) &= -i_{X_{H_\sigma}}\omega(X_\sigma) \\ &= i_{X_\sigma}\omega(X_{H_\sigma}) \\ &= \widetilde{\alpha}_\sigma(X_{H_\sigma}) \\ &= f_\sigma\widetilde{\alpha}_\sigma(R_\sigma) \\ &= f_\sigma\alpha_\sigma(R_\sigma) \\ &= f_\sigma, \end{aligned}$$

implying that X_σ is nowhere tangent to Σ_σ. Hence by [25, Problem 9-8] we can define a smooth family of embeddings

$$\widetilde{\psi} \in C^\infty\left((-\widetilde{\varepsilon}_0, \widetilde{\varepsilon}_0) \times H^{-1}(0), M\right)$$

for some $\widetilde{\varepsilon}_\sigma > 0$ by

$$\widetilde{\psi}(\varepsilon, x, \sigma) := \theta^{X_\sigma}_\varepsilon(x).$$

Perhaps after shrinking $\widetilde{\varepsilon}_0$, we have that

$$\omega_\sigma := \omega|_{\Sigma_\sigma} + d(\varepsilon\alpha_\sigma)$$

is a symplectic form on $(-\widetilde{\varepsilon}_0, \widetilde{\varepsilon}_0) \times \Sigma_\sigma$. Note that ω_σ and $\widetilde{\psi}^*_\sigma\omega$ agree on $\{0\} \times \Sigma_\sigma$. Applying the Moser isotopy argument [29, p. 109], we find $\varepsilon_0 > 0$ and

$$\varphi \in C^\infty\left((-\varepsilon_0, \varepsilon_0) \times H^{-1}(0), (-\widetilde{\varepsilon}_0, \widetilde{\varepsilon}_0) \times H^{-1}(0)\right)$$

such that $\varphi(\varepsilon, x, \sigma) \in \Sigma_\sigma$ for all $(-\varepsilon_0, \varepsilon_0) \times H^{-1}(0)$, φ_σ is an embedding for all $\sigma \in [0, 1]$ and

$$\varphi_\sigma|_{\{0\}\times\Sigma_\sigma} = \iota_{\{0\}\times\Sigma_\sigma} \qquad \text{and} \qquad \varphi^*_\sigma\widetilde{\psi}^*_\sigma\omega = \omega_\sigma.$$

Set $\psi := \widetilde{\psi} \circ \varphi$. □

Theorem 2.87. *Let* $(\gamma_\sigma, \tau_\sigma)_{\sigma\in[0,\sigma_\infty)}$ *be a family of periodic orbits on a compact stable homotopy of energy surfaces* $(\Sigma_\sigma, \alpha_\sigma)_{\sigma\in[0,1]}$ *in a symplectic manifold* (M, ω). *Then there exist constants* $C_0, C_1 \in \mathbb{R}$ *independent of* $\sigma \in [0, \sigma_\infty)$ *such that*

$$0 < C_0 \leq \tau_\sigma \leq C_1 < +\infty \qquad \forall \sigma \in [0, \sigma_\infty) \,.$$

Moreover, the modified Rabinowitz action functional $\widehat{\mathcal{R}}^{H_{\sigma_\infty}}$ is constant on $\omega(\gamma_\sigma, \tau_\sigma)$.

Proof. By proposition 2.86, there exists a smooth family of stable tubular neighbourhoods $(\varepsilon_0, \psi_\sigma)_{\sigma \in [0,1]}$. Define a smooth time-dependent tensor field

$$\widetilde{\alpha} \in C^\infty(M \times [0,1]\,, T^*M)$$

by setting

$$\widetilde{\alpha}(p, \sigma) := \begin{cases} f(\varepsilon)\alpha_\sigma|_x & p = \psi_\sigma(\varepsilon, x), \\ 0 & p \notin U_\sigma, \end{cases}$$

where $f \in C_c^\infty(\mathbb{R})$ is (2.12) and

$$U_\sigma := \psi_\sigma((-\varepsilon_0, \varepsilon_0) \times \Sigma_\sigma), \qquad \forall \sigma \in [0,1]\,.$$

Moreover, replace H by $\widetilde{H} \in C^\infty(M \times [0,1])$ given by

$$\widetilde{H}(p, \sigma) := \begin{cases} h(\varepsilon) & p = \psi_\sigma(\varepsilon, x), \\ \varepsilon_0/3 & p \in U_\sigma^c \cap H_\sigma^{-1}(0, +\infty)\,, \\ -\varepsilon_0/3 & p \in U_\sigma^c \cap H_\sigma^{-1}(-\infty, 0)\,, \end{cases}$$

where $h \in C^\infty(\mathbb{R})$ is defined as in the proof of proposition 2.68. Abbreviate the modified Rabinowitz action functional $\widehat{\mathcal{R}}^{H_\sigma}$ associated to H_σ by $\widehat{\mathcal{R}}_\sigma$. Then again we have the period-action equality (recalling remark 2.8 and the fact that $X_{\widetilde{H}_\sigma} = h' R_\sigma$ from the proof of proposition 2.68) since

$$\begin{aligned} \widehat{\mathcal{R}}_\sigma(\gamma_\sigma, \tau_\sigma) &= \int_0^1 \gamma_\sigma^* \widetilde{\alpha}_\sigma - \tau_\sigma \int_0^1 H_\sigma \circ \gamma_\sigma \\ &= \int_0^1 \widetilde{\alpha}_\sigma(\dot{\gamma}_\sigma) \\ &= \int_0^1 \alpha_\sigma(\dot{\gamma}_\sigma) \\ &= \tau_\sigma \int_0^1 \alpha_\sigma(R_\sigma \circ \gamma_\sigma) \\ &= \tau_\sigma\,. \end{aligned}$$

Using $\mathcal{P}(X_{H_\sigma}, \Sigma_\sigma) \subseteq \operatorname{Crit}(\widehat{\mathcal{R}}_\sigma)$ by proposition 2.68 we compute

$$\begin{aligned} \partial_\sigma \tau_\sigma &= \partial_\sigma \left(\widehat{\mathcal{R}}_\sigma(\gamma_\sigma, \tau_\sigma) \right) \\ &= (\partial_\sigma \widehat{\mathcal{R}}_\sigma)(\gamma_\sigma, \tau_\sigma) + d\widehat{\mathcal{R}}_\sigma|_{(\gamma_\sigma, \tau_\sigma)}(\partial_\sigma \gamma_\sigma, \partial_\sigma \tau_\sigma) \\ &= (\partial_\sigma \widehat{\mathcal{R}}_\sigma)(\gamma_\sigma, \tau_\sigma) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 (\partial_\sigma \widetilde{\alpha}_\sigma)(\dot{\gamma}_\sigma) - \tau_\sigma \int_0^1 (\partial_\sigma H_\sigma) \circ \gamma_\sigma \\
&= \tau_\sigma \left(\int_0^1 (\partial_\sigma \widetilde{\alpha}_\sigma)(R_\sigma \circ \gamma_\sigma) - \int_0^1 (\partial_\sigma H_\sigma) \circ \gamma_\sigma \right). \qquad (2.17)
\end{aligned}
$$

The function

$$
H^{-1}(0) \to \mathbb{R}, \qquad (x, \sigma) \mapsto (\partial_\sigma \widetilde{\alpha}_\sigma)(R_\sigma)(x)
$$

is smooth, and since $H^{-1}(0)$ is compact by assumption, there exist $C \in \mathbb{R}$ such that

$$
|(\partial_\sigma \widetilde{\alpha}_\sigma)(R_\sigma)| \leq C \qquad \text{and} \qquad |\partial_\sigma H_\sigma| \leq C
$$

for all $\sigma \in [0, 1]$. Hence (2.17) yields the estimate $|\partial_\sigma \tau_\sigma| \leq 2C\tau_\sigma$ for all $\sigma \in [0, \sigma_\infty)$, and by Gronwalls inequality 2.81 we get

$$
0 < \tau_0 e^{-2C\sigma} \leq \tau_\sigma \leq \tau_0 e^{2C\sigma} < +\infty
$$

for all $\sigma \in [0, \sigma_\infty)$. In particular, we have that

$$
0 < \tau_0 e^{-2C\sigma_\infty} \leq \tau_\sigma \leq \tau_0 e^{2C\sigma_\infty} < +\infty \qquad \forall \sigma \in [0, \sigma_\infty).
$$

Finally, we show that $\mathcal{R}^{H_{\sigma_\infty}}$ is constant on $\omega(\gamma_\sigma, \tau_\sigma)$. From (2.17) and Gronwall's inequality 2.81 we deduce that

$$
e^{-2C(\sigma_1 - \sigma_0)} \widehat{\mathcal{R}}^{H_{\sigma_0}}(\gamma_{\sigma_0}, \tau_{\sigma_0}) \leq \widehat{\mathcal{R}}^{H_{\sigma_1}}(\gamma_{\sigma_1}, \tau_{\sigma_1}) \leq e^{2C(\sigma_1 - \sigma_0)} \widehat{\mathcal{R}}^{H_{\sigma_0}}(\gamma_{\sigma_0}, \tau_{\sigma_0})
$$

holds for all $0 \leq \sigma_0 < \sigma_1 < \sigma_\infty$. This inequality readily implies the claim. ☐

Corollary 2.88. *Let $(\gamma_\sigma, \tau_\sigma)_{\sigma \in [0, \sigma_\infty)}$ be a family of periodic orbits on a compact stable homotopy of energy surfaces $(\Sigma_\sigma, \alpha_\sigma)_{\sigma \in [0,1]}$ in a symplectic manifold (M, ω). Then the ω-limit set $\omega(\gamma_\sigma, \tau_\sigma)$ is nonempty, compact and connected.*

Proof. Immediate by theorem 2.87 and theorem 2.79. ☐

Chapter 3
Rabinowitz–Floer Homology

Roughly speaking, Rabinowitz–Floer homology is an infinite dimensional analogue of Morse theory based on the ideas pioneered by Andreas Floer. An excellent introduction to both Morse and Floer homology can be found in [37], and for Rabinowitz–Floer homology in [36] and the numerous references therein. Rabinowitz–Floer homology was introduced in 2009 by Kai Cieliebak and Urs Frauenfelder in [6].

We begin by recalling the basics of Morse theory, Floer theory and their local variants as they are essential in understanding Rabinowitz–Floer homology. Then we will introduce local Rabinowitz–Floer theory to study the behaviour of families of periodic orbits on regular energy surfaces near the ω-limit set.

3.1 The Morse–Smale–Witten Chain Complex

Let us consider as a model setting the embedded torus $\mathbb{T}^2 \subseteq \mathbb{R}^3$ as in figure 3.1, with height function h. As we go up from the bottom plane, the level set $h^{-1}(c)$, $c \in [0, +\infty)$, experiences a *change in topology*. Indeed, between the absolute minima and the first saddle point, the level set is simply a circle. Then passing by the first saddle point, we have that the level set is the disjoint union of two circles, hence disconnected. Passing above the second saddle point, between the absolute maxima, the level set is again a circle, in particular connected. In conclusion, passing through critical points alters the topological invariant connectedness. Thus there is an interplay between critical points of certain functions and the underlying topology of the manifold are. More precisely, the torus can be decomposed as a union of simpler "building blocks". For more details see problem 13-24 [25, p. 347]. The converse to above observation is not necessarily true in general. Indeed, consider the function $f \in C^\infty(\mathbb{R}^2)$ defined by $f(x, y) := \exp(x) - y^2$. For $c < 0$, the level set $f^{-1}(c)$ is disconnected while for $c > 0$, the level set $f^{-1}(c)$ is connected. Still, f admits no critical points at all. The latter example shows that even in a finite dimensional setting some notion of compactness is needed. For details see

[27, p. 163–164]. However, we are only interested in the case where the manifold in question is actually compact.

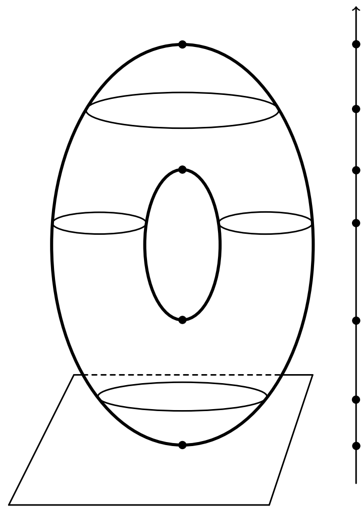


Fig. 3.1: Height function on the embedded torus $\mathbb{T}^2 \subseteq \mathbb{R}^3$.

Definition 3.1 (Hessian, [26, p. 100]). Let M be a smooth manifold and ∇ be a connection on M. For $f \in C^\infty(M)$ define the ***Hessian of f with respect to ∇*** or the ***covariant Hessian of f***, written $\mathrm{Hess}^\nabla(f)$, to be the $(0,2)$-tensor field defined by

$$\mathrm{Hess}^\nabla(f)(X,Y) := \nabla_X(df)(Y),$$

for all $X, Y \in \mathfrak{X}(M)$.

Lemma 3.2 ([2, p. 7–8]). *Let M be a smooth manifold and $f \in C^\infty(M)$. Moreover, suppose that ∇ and $\tilde{\nabla}$ are two connections on M. If $x \in \mathrm{Crit}(f)$, then*

$$\mathrm{Hess}^\nabla(f)|_x = \mathrm{Hess}^{\tilde{\nabla}}(f)|_x.$$

Moreover, if ∇ is torsion free, then

$$\mathrm{Hess}^\nabla(X,Y) = \mathrm{Hess}^\nabla(Y,X)$$

for all $X, Y \in \mathfrak{X}(M)$.

Proof. Let $X, Y \in \mathfrak{X}(M)$. Then we compute

$$\begin{aligned}\mathrm{Hess}^\nabla(f)(X,Y) &= \nabla_X(df)(Y) \\ &= X\big(Y(f)\big) - df(\nabla_X Y)\end{aligned}$$

$$= \mathrm{Hess}^{\widetilde{\nabla}}(f)(X,Y) + df\left(\widetilde{\nabla}_X Y - \nabla_X Y\right).$$

Thus if $v, w \in T_x M$ and $V, W \in \mathfrak{X}(M)$ denote any extensions of v and w, respectively, we have that

$$\begin{aligned} \mathrm{Hess}^{\nabla}(f)|_x(v,w) &= \mathrm{Hess}^{\widetilde{\nabla}}(f)|_x(v,w) + df_x\left(\widetilde{\nabla}_V W - \nabla_V W\right)\big|_x \\ &= \mathrm{Hess}^{\widetilde{\nabla}}(f)|_x(v,w), \end{aligned}$$

since $x \in \mathrm{Crit}(f)$ by assumption. Let ∇ denote any torsion-free connection on M. Then for all $X, Y \in \mathfrak{X}(M)$ we compute

$$\mathrm{Hess}^{\nabla}(f)(X,Y) = \mathrm{Hess}^{\nabla}(f)(Y,X) - df\left(T^{\nabla}(X,Y)\right) = \mathrm{Hess}^{\nabla}(f)(Y,X),$$

where

$$T^{\nabla}(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

denotes the torsion tensor with respect to the connection ∇. □

Corollary 3.3 (Pointwise Hessian). *Let M be a smooth manifold and $f \in C^{\infty}(M)$. For $x \in \mathrm{Crit}(f)$ define the* ***pointwise Hessian of f at x****, written $\mathrm{Hess}(f)|_x$, to be the bilinear form*

$$\mathrm{Hess}(f)|_x := \mathrm{Hess}^{\nabla}(f)|_x,$$

where ∇ is any connection on M. Then $\mathrm{Hess}(f)|_x$ is well-defined symmetric billinear form.

Proof. The pointwise Hessian $\mathrm{Hess}(f)|_x$ is well-defined by the first part of lemma 3.2, and $\mathrm{Hess}(f)|_x$ is symmetric because we can choose a torsion-free connection on M. Indeed, M admits a Riemannian metric and thus the Levi-Civita connection which is torsion-free. □

Definition 3.4 (Nondegenerate Critical Point, [2, p. 8]). Let M be a smooth manifold and $f \in C^{\infty}(M)$. A critical point $x \in \mathrm{Crit}(f)$ is said to be ***nondegenerate***, iff the pointwise Hessian $\mathrm{Hess}(f)|_x$ is nondegenerate, that is, the induced map

$$\widehat{\mathrm{Hess}(f)}|_x \colon T_x M \to T_x^* M, \qquad \widehat{\mathrm{Hess}(f)}|_x(u)(v) := \mathrm{Hess}(f)|_x(u,v)$$

is an isomorphism.

Definition 3.5 (Morse Function, [2, p. 8]). A smooth function on a smooth manifold is said to be a ***Morse function***, iff all its critical points are nondegenerate.

Remark 3.6. Morse functions exist in great abundance. Indeed, if M is a compact manifold, then the set of Morse functions on M is a dense open subset of $C^k(M)$ with respect to the compact-open topology introduced in section 2.2 for all $2 \leq k \leq \infty$ (see [19, p. 147]).

Proposition 3.7 (Morse Lemma, [2, p. 12]). *Let $f \in C^{\infty}(M^n)$ and $x \in \mathrm{Crit}(f)$ nondegenerate. Then there exists a chart (U, φ) about x such that*

$$f \circ \varphi^{-1}(x^1, \dots, x^n) = f(x) - \sum_{k=1}^{i} \left(x^k\right)^2 + \sum_{k=i+1}^{n} \left(x^k\right)^2 .$$

Any such chart is called a ***Morse chart*** *and i is called the* ***Morse index of f at x*** *and is denoted by* $\mathrm{ind}_f(x)$*. Denote the set of all critical points of index k by* $\mathrm{Crit}_k(f)$.

Remark 3.8. Let $f \in C^\infty(M)$ and $x \in \mathrm{Crit}(f)$. Then the Morse index $\mathrm{ind}_f(x)$ coincides with the maximal dimension of linear subspaces of $T_x M$ where the restriction of the pointwise Hessian $\mathrm{Hess}(f)|_x$ is negative-definite.

Remark 3.9. The Morse lemma 3.7 implies that Morse critical points are isolated in the set of critical points. In particular, every Morse function on a compact manifold has only finitely many critical points.

In order to motivate the theory, let us first recall some basic facts of the qualitative theory of topological dynamics. Recall that a (continuous) topological dynamical system is defined to be a topological space X together with a flow $\theta \in C^0(\mathbb{R} \times X, X)$, that is,

$$\theta_t \circ \theta_s = \theta_{t+s} \qquad \text{and} \qquad \theta_0 = \mathrm{id}_X ,$$

for all $t, s \in \mathbb{R}$ where we abreviate $\theta_t := \theta(t, \cdot) \in C^0(X, X)$ as usual.

Definition 3.10 (Lyapunov Function, [20, p. 218]). Let (X, θ) be a a topological dynamical system. A ***Lyapunov function for θ*** is defined to be a function $f \in C^0(X)$ such that f is strictly monotone decreasing along nonconstant flow lines, that is, if $x \notin \mathrm{Fix}(\theta)$, then

$$s < t \qquad \Rightarrow \qquad f\left(\theta_t(x)\right) < f\left(\theta_s(x)\right) .$$

Example 3.11 (Negative Gradient Flow, [20, p. 218]). Let (M, m) be a compact Riemannian manifold and denote by $\theta \in C^\infty(\mathbb{R} \times M, M)$ the flow of the negative gradient vector field $-\mathrm{grad}_m(f)$ for some $f \in C^\infty(M)$, where $\mathrm{grad}_m(f) \in \mathfrak{X}(M)$ denotes the gradient vector field of f with respect to the metric m implicitly defined by the condition

$$m\left(\mathrm{grad}_m(f)|_x, v\right) = df_x(v) \qquad \forall x \in M, v \in T_x M.$$

Then f is a Lyapunov function for θ. Indeed, we compute

$$\begin{aligned}
\frac{d}{dt} f\left(\theta_t(x)\right) &= df_{\theta_t(x)}\left(\frac{d}{dt}\theta_t(x)\right) \\
&= -df_{\theta_t(x)}\left(\mathrm{grad}_m(f)\left(\theta_t(x)\right)\right) \\
&= -(i_{\mathrm{grad}_m(f)}\widehat{m})_{\theta_t(x)}\left(\mathrm{grad}_m(f)\left(\theta_t(x)\right)\right) \\
&= -\left|\mathrm{grad}_m(f)\left(\theta_t(x)\right)\right|_m^2 \\
&\leq 0,
\end{aligned}$$

where equality holds if and only if $x \in \operatorname{Crit}(f)$. The statement follows by noting that $\operatorname{Fix}(\theta) = \operatorname{Crit}(f)$.

Proposition 3.12 ([20, p. 218]). *If $f \in C^0(X)$ is a Lyapunov function for a flow θ on a compact metric space X, then $\omega_\theta(x) \subseteq \operatorname{Fix}(\theta)$ for all $x \in X$.*

Proof. As f is by assumption monotone decreasing and X is compact we have that

$$\lim_{t\to+\infty} f\left(\theta_t(x)\right) = \inf_{t\geq 0} f\left(\theta_t(x)\right) =: C_x \geq \min_{x\in X} f(x) > -\infty.$$

Let $y \in \omega_\theta(x)$. Then by definition there exists a sequence $(t_k) \subseteq \mathbb{R}$ with $t_k \to +\infty$ as $k \to \infty$. Thus using continuity we compute

$$f\left(\theta_t(y)\right) = \lim_{k\to\infty} f\left(\theta_t\left(\theta_{t_k}(x)\right)\right) = \lim_{k\to\infty} f\left(\theta_{t+t_k}(x)\right) = C_x$$

for all $t \in \mathbb{R}$. Thus f cannot be a Lyapunov function unless $y \in \operatorname{Fix}(\theta)$. $\square$

Motivated by proposition 3.12 and example 3.11, we consider the following setting. Let (M, m) be a compact Riemannian manifold and $f \in C^\infty(M)$ a Morse function. If $\gamma \in C^\infty(\mathbb{R}, M)$ is a negative gradient flow line of f, that is

$$\dot\gamma(t) = -\operatorname{grad}_m(f)|_{\gamma(t)} \qquad \forall t \in \mathbb{R},$$

then there exist $x^\pm \in \operatorname{Crit}(f)$ such that

$$\lim_{t\to\pm\infty} \gamma(t) = x^\pm.$$

The proof relies heavily on the fact that there are only finitely many critical points by remark 3.9. Denote by $\theta \in C^\infty(\mathbb{R}\times M, M)$ the flow of $-\operatorname{grad}_m f$. For $x^\pm \in \operatorname{Crit}(f)$ we can define the ***stable and unstable manifolds*** by

$$W^s(x^+) := \left\{x \in M : \lim_{t\to+\infty} \theta_t(x) = x^+\right\}$$

and

$$W^u(x^-) := \left\{x \in M : \lim_{t\to-\infty} \theta_t(x) = x^-\right\},$$

respectively. Both the stable and the unstable manifolds are embedded submanifolds of M with $\dim W^u(x^-) = \operatorname{ind}_f(x^-)$ and $\operatorname{codim} W^s(x^+) = \operatorname{ind}_f(x^+)$. Indeed, one can interpret both the stable and the unstable manifolds as subsets of path spaces and then one can apply the infinite dimensional implicit function theorem [28, p. 541]. The dimension formula follows from the Fredholm index of a vertical differential. We say that (f, m) is a ***Morse–Smale pair***, iff the stable and unstable manifolds intersect transversely for all critical points, that is, for all $x^\pm \in \operatorname{Crit}(f)$ and every $x \in W^u(x^-) \cap W^s(x^+)$, we have that $T_x W^u(x^-)$ and $T_x W^s(x^+)$ span $T_x M$. This condition guarantees that the intersection

$$\mathcal{M}(x^-, x^+) := W^u(x^-) \cap W^s(x^+)$$

is an embedded submanifold of M of dimension $\operatorname{ind}_f(x^-) - \operatorname{ind}_f(x^+)$ for all $x^\pm \in \operatorname{Crit}(f)$. In particular, $\operatorname{ind}_f(x^+) \le \operatorname{ind}_f(x^-)$, that is, the Morse index decreases along gradient flow lines. This submanifold is called the ***trajectory space from x^- to x^+*** and

$$\mathcal{M}(x^-, x^+) = \{x \in M : \lim_{t\to\pm\infty} \theta_t(x) = x^\pm\}.$$

That is, $\mathcal{M}(x^-, x^+)$ is the set of negative gradient flow lines connecting x^- and x^+. We have that $\mathcal{M}(x^-, x^+)$ is compact modulo breaking. Moreover, $\mathbb{R}$ acts on $\mathcal{M}(x^-, x^+)$ via time translation, that is, $\lambda x := \theta_\lambda(x)$ for $\lambda \in \mathbb{R}$ and $x \in \mathcal{M}(x^-, x^+)$. If $x^- \neq x^+$, this action is free. Thus

$$\widehat{\mathcal{M}}(x^-, x^+) := \mathcal{M}(x^-, x^+)/\mathbb{R}$$

is a manifold of dimension

$$\dim \widehat{\mathcal{M}}(x^-, x^+) = \operatorname{ind}_f(x^-) - \operatorname{ind}_f(x^+) - 1.$$

An equivalence class in $\widehat{\mathcal{M}}(x^-, x^+)$ is an unparametrised negative gradient flow line connecting x^- and x^+. If $\operatorname{ind}_f(x^-) = \operatorname{ind}_f(x^+) + 1$, it can be showed that $\mathcal{M}(x^-, x^+)$ is a 0-dimensional compact manifold and thus of contains only finitely many gradient flow lines (see [2, p. 59]). Thus set $n(x^-, x^+) := \#_2\mathcal{M}(x^-, x^+)$. Then we can define a nonnegative chain complex $(\mathrm{CM}_\bullet(f), \partial_\bullet)$ of vector spaces over $\mathbb{Z}_2$, called the ***Morse–Smale–Witten chain complex***, by setting $\mathrm{CM}_k(f)$ to be the free $\mathbb{Z}_2$-vector space generated by elements of $\operatorname{Crit}_k(f)$ for every $k \in \mathbb{N}$, that is,

$$\mathrm{CM}_k(f) := \left\{\varphi \in \mathbb{Z}_2^{\operatorname{Crit}_k(f)} : \operatorname{supp}(\varphi) \text{ is finite}\right\},$$

where the latter set is equipped with pointwise addition and scalar multiplication, and define the boundary map

$$\partial_k \colon \mathrm{CM}_k(f) \to \mathrm{CM}_{k-1}(f)$$

by

$$\partial_k(x^-) := \sum_{x^+ \in \operatorname{Crit}_{k-1}(f)} n(x^-, x^+)x^+$$

on generators and then extending by linearity. Define the ***Morse homology of M with respect to f and $\mathbb{Z}_2$ coefficients***, written $\mathrm{HM}_\bullet(M, f; \mathbb{Z}_2)$ to be the homology of the Morse–Smale–Witten chain complex as it is standard in algebraic topology. It can be showed that Morse homology does not depend on neither the choice of Morse function nor the Riemannian metric. In particular, if $f, g \in C^\infty(M)$ are two Morse functions on a compact manifold M, then there is a canonical isomorphism

$$\mathrm{HM}_\bullet(M, f; \mathbb{Z}_2) \cong \mathrm{HM}_\bullet(M, g; \mathbb{Z}_2).$$

We sketch the argument. Let (f_0, m_0) and (f_1, m_1) be two Morse–Smale pairs. Choose a smooth family $(f_\sigma)_{\sigma\in\mathbb{R}} \subseteq C^\infty(M)$ and a smooth family of Riemannian metrics $(m_\sigma)_{\sigma\in\mathbb{R}}$ such that

$$f_\sigma = \begin{cases} f_0 & \sigma \le 0, \\ f_1 & \sigma \ge 1, \end{cases} \qquad \text{and} \qquad m_\sigma = \begin{cases} m_0 & \sigma \le 0, \\ m_1 & \sigma \ge 1. \end{cases}$$

Define a morphism of chain complexes $\Phi_0^1\colon \operatorname{CM}_\bullet(f_0) \to \operatorname{CM}_\bullet(f_1)$, called a ***continuation morphism***, as follows. Given $x_0 \in \operatorname{Crit}_k(f_0)$, set

$$\Phi_0^1(x_0) := \sum_{x_1\in\operatorname{Crit}_k(f_1)} n(x_0, x_1)x_1,$$

where $n(x_0, x_1)$ denotes the $\mathbb{Z}_2$-count of time-dependent negative gradient flow lines from x_0 to x_1, that is, $\gamma \in C^\infty(\mathbb{R}, M)$ satisfying

$$\dot\gamma(\sigma) = -\operatorname{grad}_{m_\sigma}(f_\sigma)|_{\gamma(\sigma)}, \qquad \forall \sigma \in \mathbb{R},$$

and attaining the asymptotics

$$\lim_{\sigma\to-\infty} \gamma(\sigma) = x_0 \qquad \text{and} \qquad \lim_{\sigma\to+\infty} \gamma(\sigma) = x_1.$$

Then one can show that Φ_0^1 induces an isomorphism

$$\operatorname{HM}_\bullet(f_0, m_0) \cong \operatorname{HM}_\bullet(f_1, m_1)$$

with inverse Φ_1^0.

Example 3.13 ($\mathbb{S}^n$). Consider the sphere $\mathbb{S}^n$ embedded into $\mathbb{R}^{n+1}$ with height function

$$(x^1, \dots, x^{n+1}) \mapsto x^{n+1}.$$

Then we get the Morse–Smale–Witten chain complex $\operatorname{CM}_\bullet(\mathbb{S}^n)$

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow 0 \longrightarrow \dots \longrightarrow 0 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

in the case $n > 1$ and

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \longrightarrow 0$$

in the case $n = 1$ since there is an even number of trajectories between the two critical points. Thus

$$\widetilde{\operatorname{HM}}_k(\mathbb{S}^n; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & k = n \\ 0 & k \ne n, \end{cases}$$

where $\widetilde{\operatorname{H}}_\bullet$ as usual denotes the reduced homology. Again, this result coincides with that of singular homology.

Example 3.14 (The Heart). Consider the heart-shaped submanifold $S \subseteq \mathbb{R}^3$ with the height function as depicted in figure 3.2.

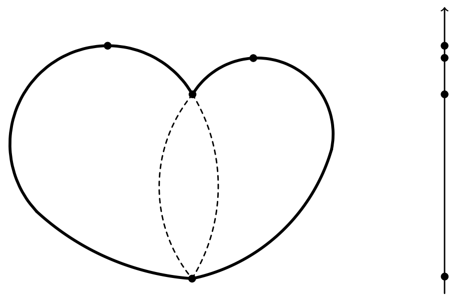


Fig. 3.2: Height function on the heart-shaped embedded submanifold of $\mathbb{R}^3$.

The corresponding Morse–Smale–Witten chain complex $\mathrm{CM}_\bullet(S)$ is given by

$$0 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{(1,1)} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \longrightarrow 0$$

Thus

$$\widetilde{\mathrm{HM}}_k(S;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & k = 2 \\ 0 & k \neq 2. \end{cases}$$

In particular, $\mathrm{HM}_\bullet(S;\mathbb{Z}_2) = \mathrm{HM}_\bullet(\mathbb{S}^2;\mathbb{Z}_2)$.

Example 3.15 ($\mathbb{T}^2$). Consider the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ embedded into $\mathbb{R}^3$ with the height function as in figure 3.1. Then we get the Morse–Smale–Witten chain complex $\mathrm{CM}_\bullet(\mathbb{T}^2)$

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \longrightarrow 0$$

since there is always an even number of trajectories between the critical points. Thus

$$\mathrm{HM}_k(\mathbb{T}^2;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & k = 0, 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & k = 1, \\ 0 & k \neq 0, 1, 2. \end{cases}$$

This result coincides with that of singular homology. Indeed, using induction and the Künneth formula one can show that

$$\mathrm{H}_k(\mathbb{T}^n;\mathbb{Z}) = \mathbb{Z}^{\binom{n}{k}}.$$

Using the universal coefficient theorem we get that

$$\mathrm{H}_k(\mathbb{T}^n;\mathbb{Z}_2) \cong \mathrm{H}_k(\mathbb{T}^n;\mathbb{Z}) \otimes \mathbb{Z}_2 \oplus \mathrm{Tor}\left(\mathrm{H}_{k-1}(\mathbb{T}^n), \mathbb{Z}_2\right)$$

and using the fact that $\mathrm{Tor}(A, B) = 0$ if either A or B are torsion-free abelian groups.

Remark 3.16. In order to guarantee the Smale-condition in the above example, one actually has to slightly tilt the torus. Basically, this boils down to the fact that in the non-tilted case, there are negative gradient flow lines between the two saddle points, which is awkward.

Remark 3.17. By replacing a Morse function on a manifold by its negative, one obtains Poincaré-duality.

As the two examples suggest, Morse homology can be showed to be a homology theory in the sense of Eilenberg–Steenrod and in particular, Morse homology coincides with singular homology with coefficients in $\mathbb{Z}_2$. For example, the dimension axiom is immediate. That is, for any one-point space $*$ we have that

$$\mathrm{HM}_k(*; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & k = 0, \\ 0 & k > 0. \end{cases}$$

Indeed, a Morse function $f \in C^\infty(*)$ is the same as a choice of a real number which is a critical point of index 0. Thus the Morse chain complex $\mathrm{CM}_\bullet(*)$ is of the form

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

which readily implies the statement. In the same manner and taking orientations of the unstable manifold into account, one can define integer-valued Morse homology. However, for us the most important result is the following.

Proposition 3.18 (Morse Inequalities, [2, p. 85]). *Let f be a Morse function on a compact smooth manifold M^n. Then*

$$\# \mathrm{Crit}(f) \geq \sum_{k=0}^{n} \dim \mathrm{HM}_k(M; \mathbb{Z}_2).$$

Proof. We estimate

$$\# \mathrm{Crit}(f) = \sum_{k=0}^{n} \dim \mathrm{CM}_k(f) \geq \sum_{k=0}^{n} \dim \mathrm{HM}_k(M; \mathbb{Z}_2).$$

□

Remark 3.19. Let M be a compact smooth manifold. For each $k \in \mathbb{N}$, the natural number

$$b_k := \dim \mathrm{HM}_k(M; \mathbb{Z}_2)$$

is called the ***k-th Betti number of M***.

3.2 Hamiltonian Floer Homology

Floer Homology is an infinite-dimensional analogue of Morse Homology. It is an essential tool in Symplectic Topoology and allowed Andreas Floer to make a significant progress on a longstanding difficult problem. We follow [37] and ocasionally rely [2] as well as [29].

Definition 3.20 (Symplectic Isotopy, [29, p. 101]). A ***symplectic isotopy on a symplectic manifold*** (M, ω) is defined to be a map $\psi \in C^\infty(M \times [0,1], M)$ such that $\psi_t := \psi(\cdot, t) \in \mathrm{Symp}(M, \omega)$ and $\psi_0 = \mathrm{id}_M$.

Remark 3.21. Let ψ be a symplectic isotopy on the symplectic manifold (M, ω). Then ψ is generated by a time-dependent vector $(X_t)_{t\in[0,1]} \subseteq \mathfrak{X}(M)$ by setting

$$X_t := \left(\frac{d}{dt}\psi_t\right) \circ \psi_t^{-1}.$$

That is, for $t_0 \in [0,1]$ and $x \in M$

$$X_{t_0}|_x = \left.\frac{d}{dt}\right|_{t=t_0} \psi_t\left(\psi_{t_0}^{-1}(x)\right) \in T_xM.$$

By proposition 3.1.5 [29, p. 97], we have that $(X_t)_{t\in[0,1]}$ is a family of symplectic vector fields.

Definition 3.22 (Hamiltonian Isotopy, [29, p. 102]). A symplectic isotopy ψ on a symplectic manifold (M, ω) is said to be a ***Hamiltonian isotopy***, iff there exists a time-dependent Hamiltonian function $H \in C^\infty(M \times [0,1])$ such that the vector fields

$$X_t := \left(\frac{d}{dt}\psi_t\right) \circ \psi_t^{-1}, \qquad t \in [0,1]$$

are the Hamiltonian vector fields of the Hamiltonian systems (M, ω, H_t).

Definition 3.23 (Hamiltonian Symplectomorphism, [29, p. 102]). A symplectomorphism $F \in \mathrm{Symp}(M, \omega)$ is said to be ***Hamiltonian***, iff there exists a Hamiltonian isotopy $(F_t)_{t\in[0,1]}$ such that $F = \psi_1$.

Definition 3.24 (Displaceability, [8, p. 1766]). A subset $A \subseteq (M, \omega)$ is said to be ***displaceable***, iff there exists a Hamiltonian symplectomorphisms F such that the corresponding time-dependent Hamiltonian H is compactly supported and

$$F(A) \cap A = \varnothing.$$

In what follows we will consider a special case of a Hamiltonian symplectomorphism. Let (M, ω) be a compact symplectic manifold and $H \in C^\infty(M \times \mathbb{S}^1)$ a 1-periodic Hamiltonian function. Define a time-dependent vector field $(X_t)_{t\in\mathbb{S}^1}$ by $i_{X_t}\omega = -dH_t$ for all $t \in \mathbb{S}^1$, that is, $X_t = X_{H_t}$. Let $(\psi_t)_{t\in\mathbb{S}^1}$ denote the time-dependent flow of $(X_t)_{t\in\mathbb{S}^1}$. Then $(\psi_t)_{t\in\mathbb{S}^1}$ is a 1-periodic Hamiltonian isotopy and the fixed points of the Hamiltonian symplectomotphism $F := \psi_1$ correspond to 1-periodic integral curves $\gamma \in C^\infty(\mathbb{S}^1, M)$ of

$$\dot{\gamma}(t) = X_t\left(\gamma(t)\right), \qquad \forall t \in \mathbb{S}^1.$$

A 1-periodic solution $\gamma \in C^\infty(\mathbb{S}^1, M)$ of (3.2) is said to be ***nondegenerate***, iff

$$\det\left(DF_{\gamma(0)} - \mathrm{id}_{T_{\gamma(0)}M}\right) \neq 0.$$

Now we are ready to state two versions of the famous Arnold conjecture.

Conjecture 3.25 (Arnold, [29, p. 35], [2, p. 152]). Let F be a Hamiltonian symplectomorphism of a compact symplectic manifold (M^{2n}, ω). Then

$$\#\,\mathrm{Fix}(F) \geq \mathrm{Crit}(M) := \min_{f \in C^\infty(M)} \#\,\mathrm{Crit}(f).$$

If all the fixed points of F are assumed to be nondegenerate, then

$$\#\,\mathrm{Fix}(F) \geq \sum_{k=0}^{2n} \beta_k = \sum_{k=0}^{2n} \dim \mathrm{HM}_k(M; \mathbb{Z}_2).$$

In the autonomous case the first part of the Arnold conjecture follows immediately and the second part from Morse theory.

Definition 3.26 (Nondegenerate Periodic Orbit, [2, p. 137]). Let (M, ω, H) be a Hamiltonian system. An integral curve $\gamma \in C^\infty(\mathbb{S}^1, M)$ of X_H is said to be ***nondegenerate***, iff

$$\dim\ker\left(D(\theta_1^H)_{\gamma(0)} - \mathrm{id}_{T_{\gamma(0)}M}\right) = 0,$$

where $\theta^H \in C^\infty(\mathcal{D}, M)$ denotes the flow of X_H.

Proposition 3.27 ([2, p. 137]). *Let (M, ω, H) be a Hamiltonian system. If a critical point of H is nondegenerate as a 1-periodic integral curve of X_H, then it is nondegenerate as a critical point of H.*

Proposition 3.28 ([2, p. 151]). *The number of 1-periodic integral curves of a compact Hamiltonian system (M, ω, H) such that all 1-periodic integral curves of X_H are nondegenerate is greater or equal to the sum of the Betti-numbers.*

Proof. If $x \in \mathrm{Crit}(H)$, the $\gamma \in C^\infty(\mathbb{S}^1, M)$ defined by $\gamma := x$ is a 1-periodic integral curve of X_H since $X_H|_x = 0$. By assumption, γ is nondegenerate and by proposition 3.27, x is nondegenerate as a critical point of H. Thus H is a Morse function and by the Morse inequalities 3.18 we get that

$$\#\{\text{1-periodic integral curves of } X_H\} \geq \#\,\mathrm{Crit}(H) \geq \sum_{k=0}^{\dim M} \beta_k.$$

□

To prove the time-dependent case of the Arnold conjecture is considerably harder and requires sophisticated methods going back to Andreas Floer. The second part of the Arnold conjecture has been proved in full generality [37, p. 4].

Definition 3.29 (Symplectic Asphericity, [2, p. 156]). A symplectic manifold (M,ω) is said to be ***symplectically aspherical***, iff

$$\forall F \in C^\infty(\mathbb{S}^2, M) : \int_{\mathbb{S}^2} F^*\omega = 0.$$

Remark 3.30. A sufficient condition for a symplectic manifold (M,ω) to be symplectically aspherical is $\pi_2(M) = 0$.

Let (M,ω) be a symplectically aspherical symplectic manifold. Denote by

$$\Lambda M := C^\infty_{\mathrm{contr}}(\mathbb{S}^1, M) \subseteq C^\infty(\mathbb{S}^1, M)$$

the free contractible loop space on M, that is, every $\gamma \in \Lambda M$ is continuously homotopic to a constant loop. For $H \in C^\infty(M \times \mathbb{S}^1)$ define the ***symplectic action functional***

$$\mathcal{A}^H : \Lambda M \to \mathbb{R}, \qquad \mathcal{A}^H(\gamma) := \int_{\mathbb{D}^2} \overline{\gamma}^*\omega - \int_0^1 H_t\left(\gamma(t)\right) dt,$$

where $\mathbb{D}^2 \subseteq \mathbb{C}$ denotes the closed complex unit disc and $\overline{\gamma} \in C^\infty(\mathbb{D}^2, M)$ is a smooth extension of γ to $\mathbb{D}^2$, that is, $\overline{\gamma}|_{\mathbb{S}^1} = \gamma$. Note that such extensions exists since γ is assumed to be contractible. The symplectic action functional is well-defined since (M,ω) is symplectically aspherical. Indeed, if $\overline{\gamma}_1, \overline{\gamma}_2 \in C^\infty(\mathbb{D}^2, M)$ are two extensions of γ, then

$$\int_{\mathbb{D}^2} \overline{\gamma}_1^*\omega - \int_{\mathbb{D}^2} \overline{\gamma}_2^*\omega = \int_{\mathbb{D}^2} \overline{\gamma}_1^*\omega + \int_{-\mathbb{D}^2} \overline{\gamma}_2^*\omega = \int_{\mathbb{S}^2} F^*\omega = 0.$$

See figure 3.3. If M is assumed to be ***symplectically atoroidal***, that is, if

$$\forall F \in C^\infty(\mathbb{T}^2, M) : \int_{\mathbb{T}^2} F^*\omega = 0,$$

the symplectic action functial $\mathcal{A}^H$ can actually be defined on the whole free loop space ΛM.

Similar to the proof of Poincaré's theorem 2.21 we compute

$$\begin{aligned} d\mathcal{A}^H|_\gamma(X) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathcal{A}^H(\gamma_\varepsilon) \\ &= \int_{\mathbb{D}^2} \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \overline{\gamma}_\varepsilon^*\omega - \int_0^1 dH_t|_\gamma(X) \end{aligned}$$

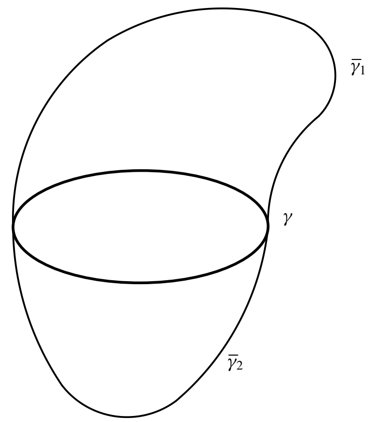


Fig. 3.3: A contractible loop γ with two extensions $\overline{\gamma}_1$ and $\overline{\gamma}_2$ to the disc $\mathbb{D}^2$.

$$
\begin{aligned}
&= \int_{\mathbb{D}^2} d\overline{\gamma}^* i_X(\omega \circ \gamma) + \int_0^1 \omega_\gamma(X_{H_t} \circ \gamma, X) \\
&= \int_{\mathbb{S}^1} \overline{\gamma}^* i_X(\omega \circ \gamma) + \int_0^1 \omega_\gamma(X_{H_t} \circ \gamma, X) \\
&= \int_0^1 \omega_\gamma(X, \dot{\gamma}) + \int_0^1 \omega_\gamma(X_{H_t} \circ \gamma, X) \\
&= \int_0^1 \omega_\gamma(X, \dot{\gamma} - X_{H_t} \circ \gamma)
\end{aligned} \tag{3.1}
$$

for all $\gamma \in \Lambda M$ and $X \in T_\gamma \Lambda M = \Gamma(\gamma^* TM)$. Thus $\gamma \in \mathrm{Crit}(\mathcal{A}^H)$ if and only if

$$\dot{\gamma}(t) = X_{H_t}\left(\gamma(t)\right), \qquad \forall t \in \mathbb{S}^1.$$

Let $(J_t)_{t \in \mathbb{S}^1}$ be a smooth family of ω-compactible almost complex structures on M. Define an L^2-inner product on ΛM by

$$\langle X, Y \rangle_{L^2(\Lambda M)} := \int_0^1 \omega_{\gamma(t)}\left(J_t\left(X(t)\right), Y(t)\right) dt.$$

With respect to this inner product, the gradient of the symplectic action functional $\mathcal{A}^H$ is the unique vector field $\operatorname{grad} \mathcal{A}^H \in \mathfrak{X}(\Lambda M)$ defined by the condition

$$\langle \operatorname{grad} \mathcal{A}^H, \cdot \rangle_{L^2(\Lambda M)} = d\mathcal{A}^H.$$

Let $\gamma \in \Lambda M$ and $X \in T_\gamma \Lambda M = \Gamma(\gamma^* TM)$. Then we have that

$$\langle \operatorname{grad} \mathcal{A}^H|_\gamma, X \rangle_{L^2(\Lambda M)} = \int_0^1 \omega_{\gamma(t)}\left(J_t \operatorname{grad} \mathcal{A}^H|_\gamma(t), X(t)\right) dt$$

$$= -\int_0^1 \omega_{\gamma(t)}\left(X(t), J_t \operatorname{grad}\mathcal{A}^H|_\gamma(t)\right) dt.$$

Comparing with (3.1) yields

$$\operatorname{grad}\mathcal{A}^H|_\gamma(t) = J_t\left(\dot\gamma(t) - X_{H_t}\left(\gamma(t)\right)\right). \tag{3.2}$$

In analogy to Morse homology we want to consider negative gradient flow lines, that is, maps $u \in C^\infty(\mathbb{R}, \Lambda M)$ such that

$$\dot u = -\operatorname{grad}\mathcal{A}^H|_u. \tag{3.3}$$

We claim that $\mathcal{A}^H$ is decreasing along gradient flow lines. Indeed, we compute

$$\begin{aligned}\frac{d}{ds}\mathcal{A}^H\left(u(s)\right) &= d\mathcal{A}^H|_u\left(\frac{d}{ds}u(s)\right)\\ &= d\mathcal{A}^H|_u\left(\operatorname{grad}\mathcal{A}^H|_u(s)\right)\\ &= -\left|\operatorname{grad}\mathcal{A}^H|_u(s)\right|^2_{L^2(\Lambda M)}\\ &\le 0.\end{aligned}$$

Now the ingenious idea by Floer was to consider the gradient flow line u as a map in $C^\infty(\mathbb{R}\times\mathbb{S}^1, M)$ via $u(s,t) := u(s)(t)$, rather than as a path in the contractible loop space ΛM. Thus negative gradient flow lines of the symplectic action functional are cylinders in the manifold M. See figure 3.4. Thus the ordinary differential equation (3.3) turns into a partial differential equation

$$\partial_s u(s,t) + J_t\left(\partial_t u(s,t) - X_{H_t}\left(u(s,t)\right)\right) = 0 \tag{3.4}$$

for all $(s,t) \in \mathbb{R}\times\mathbb{S}^1$. Equation (3.4) is known as the ***Floer equation***. Note that if $H = 0$, then the Floer equation (3.4) has an apparent similarity to pseudoholomorphic curves [29, p. 180–181]. One can rewrite the Floer equation (3.4) as

$$\partial_s u(s,t) + J_t\partial_t u(s,t) = \operatorname{grad}_{m_t} H_t|_{u(s,t)},$$

where we abbreviate $m_t := m_{J_t}$. Indeed, we have that

$$i_{J_t X_{H_t}} m_t = -i_{X_{H_t}}\omega = dH_t,$$

and thus

$$\operatorname{grad}_{m_t} H_t = J_t X_{H_t}.$$

Suppose that M is compact and define the ***energy*** of a solution $u \in C^\infty(\mathbb{R}\times\mathbb{S}^1, M)$ of the Floer equation (3.4) by

$$E(u) := \int_{-\infty}^{+\infty}\int_0^1 |\partial_s u|^2_{m_t}\, dt\, ds$$

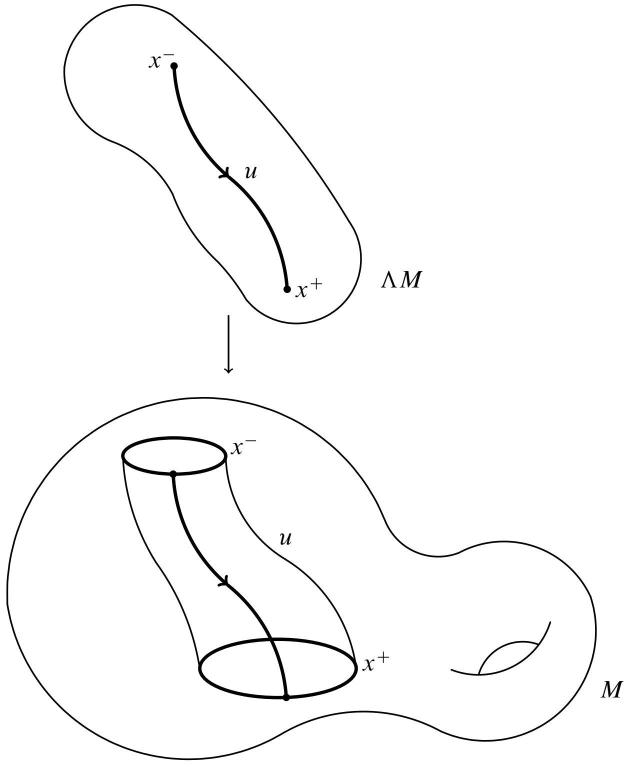


Fig. 3.4: Depiction of the two viewpoints of a negative gradient flow line of the symplectic action functional $\mathcal{A}^H$.

Finite energy solutions play a particularly important role in Floer theory because under a nondegeneracy assumption it is possible to connect critical points of $\mathcal{A}^H$. Using elliptic regularity one can show the following result. Assuming that all contractible 1-periodic integral curves of H are nondegenerate, every finite energy solution u of the Floer equation (3.4) satisfies

$$u_s \xrightarrow{C^\infty} x^\pm, \qquad s \to \pm\infty \tag{3.5}$$

for some $x^\pm \in \mathrm{Crit}(\mathcal{A}^H)$. Conversely, every solution u of the Floer equation such that (3.5) holds has finite energy. Indeed, we compute

$$\begin{aligned} \mathcal{A}^H(x^-) - \mathcal{A}^H(x^+) &= -\int_{-\infty}^{+\infty} \frac{d}{ds}\mathcal{A}^H(u_s)ds \\ &= -\int_{-\infty}^{+\infty} d\mathcal{A}^H|_{u_s}(\partial_s u)ds \end{aligned}$$

$$
\begin{aligned}
&= -\int_{-\infty}^{+\infty} \langle \operatorname{grad} \mathcal{A}^H|_{u_s}, \partial_s u \rangle_{L^2(\Lambda M)} ds \\
&= \int_{-\infty}^{+\infty} \langle \partial_s u, \partial_s u \rangle_{L^2(\Lambda M)} ds \\
&= \int_{-\infty}^{+\infty} \int_0^1 \omega\left(J_t \partial_s u(s,t), \partial_s u(s,t)\right) dt ds \\
&= \int_{-\infty}^{+\infty} \int_0^1 |\partial_s u|^2_{m_t}\, dt ds \\
&= E(u).
\end{aligned}
$$

In particular, the energy is decreasing along gradient flow lines. Given $x^{\pm} \in \operatorname{Crit}(\mathcal{A}^H)$, define the set $\mathcal{M}(x^-, x^+)$ to consist of all solutions $u \in C^\infty(\mathbb{R} \times \mathbb{S}^1, M)$ of the Floer equation (3.4) and such that $\lim_{s \to \pm\infty} u_s = x^{\pm}$. Similar to Morse theory, one needs to show that the moduli spaces $\mathcal{M}(x^-, x^+)$ are finite-dimensional manifolds. In order to achieve this, we use Fredholm theory and Floer–Gromov compactness.

Let us furthermore assume that the first Chern class $c_1(TM)$ vanishes on $\pi_2(M)$ and that H is generic. Since a critical point of $\mathcal{A}^H$ has an infinite Morse index and coindex, we need another notion of index in this setting to define a chain complex in the sense of Floer. It turns out that this index is the so-called Conley–Zehnder index in a suitable setting. Define a chain complex $(\mathrm{CF}_\bullet(H), \partial_\bullet)$, called the ***Floer chain complex***, by declaring that $\mathrm{CF}_k(H)$ is the free $\mathbb{Z}_2$-vector space with basis the critical points of $\mathcal{A}^H$ of Conley–Zehnder index k and the boundary maps $\partial\colon \mathrm{CF}_k(H) \to \mathrm{CF}_{k-1}(H)$ are in analogy to Morse homology given by

$$
\partial_k(x^-) := \sum_{\mu(x^+) = k-1} n(x^-, x^+) x^+,
$$

where $n(x^-, x^+) := \#_2 \mathcal{M}(x^-, x^+)/\mathbb{R}$.

3.3 Local Morse and Local Hamiltonian Floer Homology

In this subsection we consider a local version of the Morse homology and the Hamiltonian Floer homology. Local Hamiltonian Floer homology was used to partially solve the Conley conjecture in 2010, see [18]. Moreover, local Floer homology can be thought of a decomposition of the standard Floer homology, if defined. Let M be a smooth manifold and $f \in C^\infty(M)$. Suppose that $x \in \operatorname{Crit}(f)$ is isolated in $\operatorname{Crit}(f)$, that is, there exists a neighbourhood $U \subseteq M$ of x such that $U \cap \operatorname{Crit}(f) = \{x\}$. By shrinking U if necessary, we find a generic perturbation $\widetilde{f} \in C^\infty(U)$ of f in U, that is, $\widetilde{f}$ is Morse and C^1-close to f (consider remark 3.6). Then any negative gradient flow line connecting two critical points of $\widetilde{f}$ has image entirely contained in U. Moreover, this is also true for broken negative gradient flow lines. Thus we can define a local Morse–Smale–Witten chain complex in the neighbourhood U of x. One can show that the resulting local homology neither depends on the choice of generic perturbation $\widetilde{f}$ nor the choice of neighbourhood U. We denote the resulting local homology by $\mathrm{HM}^{\mathrm{loc}}_{\bullet}(f, p)$ and call it the ***local Morse homology of f at x***.

Example 3.31 (Nondegenerate Critical Point). Let $f \in C^\infty(M)$ and suppose that $x \in \mathrm{Crit}(f)$ is nondegenerate. Then by remark 3.9, x is isolated in $\mathrm{Crit}(f)$ and $f|_U$ is a generic perturbation of f in U. Thus we have that

$$\mathrm{HM}^{\mathrm{loc}}_k(f, x) = \begin{cases} \mathbb{Z}_2 & k = \mathrm{ind}_f(x), \\ 0 & \text{else.} \end{cases}$$

Now we turn on the local version of Hamiltonian Floer homology following [18]. Let (M, ω) be a compact symplectic manifold and let $H \in C^\infty(M \times \mathbb{S}^1)$ be a time-dependent Hamiltonian function. Suppose $\gamma \in C^\infty(\mathbb{S}^1, M)$ is an isolated 1-periodic orbit of H, that is, there exists a neighbourhood of γ in the free loop space $C^\infty(\mathbb{S}^1, M)$ equipped with the compact-open topology such that there exists no other 1-periodic orbit of H in this neighbourhood. See figure 3.5. Pick a sufficiently small tubular neighbourhood U of $\gamma(\mathbb{S}^1)$ in $M \times \mathbb{S}^1$ and a nondegenerate C^2-small perturbation $\widetilde{H}$ of H supported in U, that is, $\widetilde{H}$ is C^2-close to H, $\widetilde{H}|_{U^c} = H|_{U^c}$ and every 1-periodic orbit of $\widetilde{H}$ intersecting U are nondegenerate. Consider 1-periodic orbits of $\widetilde{H}$ contained in U. Then every negative gradient flow line connecting two such nondegenerate orbits is also contained in U provided the C^2-norm of the difference $\widetilde{H} - H$ and $\mathrm{supp}(\widetilde{H} - H)$ is sufficiently small (see [18, p. 1143] for references). One can show that the free $\mathbb{Z}_2$-vector space generated by the 1-periodic orbits of $\widetilde{H}$ in U can be turned into a Floer chain complex. The homology of this complex is independent of the choice of the tubular neighbourhood U and the perturbation $\widetilde{H}$. Thus we write $\mathrm{HF}^{\mathrm{loc}}_{\bullet}(H, \gamma)$ for this homology and call it the ***local Floer homology of H at γ***.

Example 3.32 (Nondegenerate 1-periodic Orbit). Let (M, ω) be a symplectic manifold and $H \in C^\infty(M \times \mathbb{S}^1)$. Suppose that $\gamma \in C^\infty(\mathbb{S}^1, M)$ is a nondegenerate 1-periodic orbit of H. Then

$$\mathrm{HF}^{\mathrm{loc}}_k(H, \gamma) = \begin{cases} \mathbb{Z}_2 & k = \mu_{\mathrm{CZ}}(\gamma), \\ 0 & \text{else.} \end{cases}$$

Local Floer homology are building blocks for filtered Floer homology. For simplicity suppose that $(M, \omega = d\alpha)$ is a compact exact symplectic manifold and let $c \in \mathbb{R}$ such that all $\gamma \in \mathrm{Crit}(\mathcal{A}^H)$, where $\mathcal{A}^H$ denotes the symplectic action functional, with symplectic action value $\mathcal{A}^H(\gamma) = c$ are isolated in $\mathrm{Crit}(\mathcal{A}^H)$. Consequently, we have that there are only finitely many such orbits, say $\gamma_1, \dots, \gamma_k$. Then for $\varepsilon > 0$ sufficiently small, we have that

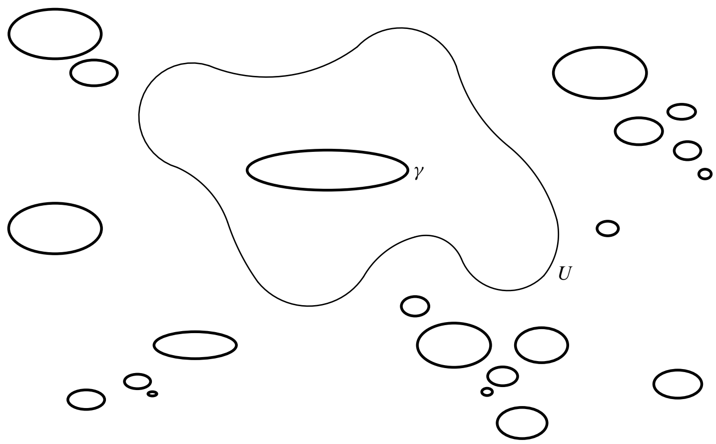


Fig. 3.5: An isolated 1-periodic orbit γ with isolating neighbourhood U.

$$\mathrm{HF}_\bullet^{(c-\varepsilon,c+\varepsilon)}(H) = \bigoplus_{i=1}^{k} \mathrm{HF}_\bullet^{\mathrm{loc}}(H, \gamma_i).$$

3.4 Local Rabinowitz–Floer Homology

Roughly speaking, local Rabinowitz–Floer homology is the local Floer homology of the Rabinowitz action functional. We follow the treatment in [36].

Definition 3.33 (Perturbed Rabinowitz Action Functional, [36, p. 26]). Given an exact Hamiltonian system $(M, d\alpha, H)$ and $F \in C^\infty(M \times \mathbb{S}^1)$, define the ***perturbed Rabinowitz action functional***

$$\mathcal{R}_F^H : C^\infty(\mathbb{S}^1, M) \times \mathbb{R} \to \mathbb{R}$$

by

$$\mathcal{R}_F^H(\gamma, \tau) := \int_\gamma \alpha - \tau \int_0^1 H\left(\gamma(t)\right) dt - \int_0^1 F\left(\gamma(t), t\right) dt.$$

As in the Floer setting, we want to calculate the gradient of the perturbed Rabinowitz action functional with respect to some distinguished metric. Let

$$J \in C^\infty(M \times \mathbb{S}^1 \times \mathbb{R}, T^{(1,1)}TM)$$

be a smooth family of ω-compatible almost complex structures on an exact symplectic manifold $(M, \omega = d\alpha)$, that is, $J(x, t, \tau) \in T^{(1,1)}(T_xM) = \mathrm{End}(T_xM)$ for all $(x, t, \tau) \in M \times \mathbb{S}^1 \times \mathbb{R}$ is such that $m_{J_{t,\tau}} := \omega(J_{t,\tau}\cdot, \cdot)$ defines a Riemannian metric on M for all $(x, \tau) \in \mathbb{S}^1 \times \mathbb{R}$, where we abbreviate $J_{t,\tau} := J(\cdot, t, \tau) \in \Gamma\left(\mathrm{End}(TM)\right)$.

The dependence of the family $J_{t,\tau}$ of ω-compatible almost complex structures on the Lagrange multiplier τ is needed in order to achieve transversality, that is, to show that the moduli spaces of local Floer trajectories are generically manifolds. For details see [36, p. 94]. Define a metric m on $C^\infty(\mathbb{S}^1, M) \times \mathbb{R}$ by

$$m\left((X_1, \widetilde{\tau}_1), (X_2, \widetilde{\tau}_2)\right) := \int_0^1 \omega_{\gamma(t)}\left(J_{t,\tau}\left(\gamma(t)\right) X_1(t), X_2(t)\right) dt + \widetilde{\tau}_1 \widetilde{\tau}_2,$$

for $(X_1, \widetilde{\tau}_1), (X_2, \widetilde{\tau}_2) \in T_\gamma C^\infty(\mathbb{S}^1, M) \times \mathbb{R}$. Using the computation in the proof of 2.33 we have that

$$d\mathcal{R}_F^H|_{(\gamma,\tau)}(X, \widetilde{\tau}) = \int_0^1 \omega_\gamma\left(X, \dot{\gamma} - \tau(X_H \circ \gamma) - X_{F_t} \circ \gamma\right) - \widetilde{\tau} \int_0^1 H \circ \gamma.$$

Consequently

$$\operatorname{grad}_m \mathcal{R}_F^H|_{(\gamma,\tau)}(t) = \begin{pmatrix} J_{t,\tau}\left(\dot{\gamma} - \tau(X_H \circ \gamma) - X_{F_t} \circ \gamma\right) \\ -\displaystyle\int_0^1 H \circ \gamma \end{pmatrix}. \tag{3.6}$$

From (3.6) we immediately deduce that $(\gamma, \tau) \in \operatorname{Crit}(\mathcal{R}_F^H)$ if and only if

$$\dot{\gamma} = \tau(X_H \circ \gamma) + X_{F_t} \circ \gamma \qquad \text{and} \qquad \int_0^1 H \circ \gamma = 0.$$

Negative gradient flow lines of the perturbed Rabinowitz action functional $\mathcal{R}_F^H$ are maps $(u, \tau) \in C^\infty(\mathbb{R} \times \mathbb{S}^1, M) \times C^\infty(\mathbb{R})$ such that

$$\dot{u} = -\operatorname{grad}_m \mathcal{R}_F^H|_u.$$

The corresponding Floer equations are thus given by

$$\begin{aligned} 0 &= \partial_s u(s,t) + J_{t,\tau(s)}\left(u(s,t)\right)\left(\partial_t u(s,t) - \tau(s) X_H\left(u(s,t)\right) - X_{F_t}\left(u(s,t)\right)\right), \\ 0 &= \partial_s \tau(s) - \int_0^1 H\left(u(s,t)\right) dt, \end{aligned} \tag{3.7}$$

for all $(s,t) \in \mathbb{R} \times \mathbb{S}^1$. A more involved computation along the lines of [13, p. 15–17] shows that

$$\nabla^2 \mathcal{R}_F^H|_{(\gamma,\tau)}(X, \widetilde{\tau}) = -\begin{pmatrix} J_{t,\tau}\left([X, \tau X_H + X_{F_t}] + \widetilde{\tau} X_H\right) \\ \displaystyle\int_0^1 dH(X) \end{pmatrix}$$

for $(\gamma, \tau) \in \operatorname{Crit}(\mathcal{R}_F^H)$. Moreover, one can show that $(X, \widetilde{\tau})$ belongs to the kernel of the Hessian $\nabla^2 \mathcal{R}_F^H|_{(\gamma,\tau)}$ if and only if

$$\dot{v} = \widetilde{\tau} X_H\left(\gamma(0)\right) + X_{F_t}\left(\gamma(0)\right) \qquad \text{and} \qquad \int_0^1 dH_{\gamma(0)} \circ v = 0,$$

where $v \in C^\infty([0,1], T_{\gamma(0)}M)$ is defined by

$$v(t) := D\theta^H_{-\tau t}|_{\gamma(t)}\left(X(t)\right).$$

Since the unperturbed Rabinowitz action functional $\mathcal{R}^H$ is invariant with respect to the natural $\mathbb{S}^1$-action, we cannot expect $\mathcal{R}^H$ to be a Morse function in general.

Definition 3.34 (Morse–Bott Function, [15, p. 101]). A function $f \in C^\infty(M)$ on a smooth manifold M is said to be a ***Morse–Bott function***, iff $\operatorname{Crit}(f)$ is a smooth manifold and

$$T_x \operatorname{Crit}(f) = \ker \operatorname{Hess}(f)|_x, \qquad \forall x \in \operatorname{Crit}(f).$$

Proposition 3.35 ([36, p. 29–30]). *The perturbed Rabinowitz action functional $\mathcal{R}^H_F$ is generically a Morse function and the unperturbed Rabinowitz action functional $\mathcal{R}^H$ is generically a Morse–Bott function.*

Definition 3.36 (Isolating Neighbourhood, [36, p. 87]). Let $(M, d\alpha, H)$ be an exact Hamiltonian system. A connected component $K \subseteq \operatorname{Crit}(\mathcal{R}^H)$ is said to be ***isolated***, iff there exists $U \subseteq\subseteq M$ and an open bounded interval $I \subseteq \mathbb{R}$ satisfying

$$\gamma(\mathbb{S}^1) \subseteq U \qquad \text{and} \qquad \tau \in I$$

for all $(\gamma, \tau) \in K$, and such that if $(\gamma, \tau) \in \operatorname{Crit}(\mathcal{R}^H)$ with $\gamma(\mathbb{S}^1) \cap U \neq \varnothing$ then $(\gamma, \tau) \in K$. If K is an isolated connected component of $\operatorname{Crit}(\mathcal{R}^H)$, then $U \times I$ is said to be an ***isolating neighbourhood of K***. An isolated connected component K is said to be ***action-constant***, iff

$$\mathcal{R}^H(\gamma_1, \tau_1) = \mathcal{R}^H(\gamma_2, \tau_2)$$

for all $(\gamma_1, \tau_1), (\gamma_2, \tau_2) \in K$.

Remark 3.37 ([36, p. 86]). In the setting of definition 3.36, if $K \subseteq \operatorname{Crit}(\mathcal{R}^H)$ is a connected component, we can always write $K = K' \times \{\tau\}$ for some $\tau \in \mathbb{R}$.

Lemma 3.38. *Let $(M, d\alpha, H)$ be an exact Hamiltonian system and $K \subseteq \operatorname{Crit}(\mathcal{R}^H)$ a connected component such that there exists a neighbourhood U of K in $C^\infty(\mathbb{S}^1, M) \times \mathbb{R}$ with*

$$U \cap \operatorname{Crit}(\mathcal{R}^H) = K.$$

Then K is isolated.

Proof. If $K = \varnothing$, then the statement is vacuously true. So suppose that $K \neq \varnothing$. Let $(\gamma, \tau) \in K$ and assume that there exists no isolating neighbourhood of K. Thus we find a sequence $(\gamma_k, \tau_k) \in \operatorname{Crit}(\mathcal{R}^H) \setminus K$ such that there exists a sequence $(t_k) \subseteq \mathbb{S}^1$ with $\gamma_k(t_k) \to \gamma(0)$ and $\tau_k \to \tau$ as $k \to \infty$. By Ascoli's theorem 2.77 we may assume that $(\gamma_k, \tau_k) \to (\gamma, \tau)$ modulo subsequences. Indeed, as $\gamma_k(t_k) \to \gamma(0)$ and uniqueness of integral curves we necessarily have that the limit of (γ_k, τ_k) is (γ, τ). Hence there cannot exist a neighbourhood U of K in $C^\infty(\mathbb{S}^1, M) \times \mathbb{R}$ with

$$U \cap \operatorname{Crit}(\mathcal{R}^H) = K.$$

$\square$

Definition 3.39 (Perturbation Space, [36, p. 87]). Let $(M, d\alpha, H)$ be an exact Hamiltonian system and $U \times I$ an isolating neighbourhood of an isolated connected component $K \subseteq \operatorname{Crit}(\mathcal{R}^H)$. Let $\delta > 0$. Define the ***perturbation space***, written $\mathcal{F}(U, \delta)$, by

$$\mathcal{F}(U, \delta) := \{F \in C_c^\infty(M \times \mathbb{S}^1) : \|F\|_{C^1(M \times \mathbb{S}^1)} < \delta \text{ and } \operatorname{supp} F \subseteq U \times \mathbb{S}^1\}.$$

Let $(M, d\alpha, H)$ be an exact Hamiltonian system and $F \in C^\infty(M \times \mathbb{S}^1)$. If $U \subseteq M$ and $I \subseteq \mathbb{R}$ is an interval, we write

$$\operatorname{Crit}(\mathcal{R}^H_F, U \times I) := \{(\gamma, \tau) \in \operatorname{Crit}(\mathcal{R}^H_F) : \gamma(\mathbb{S}^1) \subseteq U \text{ and } \tau \in I\}.$$

Lemma 3.40 ([36, p. 88]). *Let $(M, d\alpha, H)$ be an exact Hamiltonian system and $K \subseteq \operatorname{Crit}(\mathcal{R}^H)$ an action-constant isolated connected component with $U_0 \times I_0 \subseteq U \times I$ two isolating neighbourhoods for K. Then there exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ and all $F \in \mathcal{F}(U, \delta)$, if $(\gamma, \tau) \in \operatorname{Crit}(\mathcal{R}^H_F, U \times I)$, then $(\gamma, \tau) \in \operatorname{Crit}(\mathcal{R}^H_F, U_0 \times I_0)$.*

Proof. Suppose that no such $\delta_0 > 0$ exists. Thus we may find a sequence $\delta_k \to 0$ of positive real numbers and $F_k \in \mathcal{F}(U, \delta_k)$ together with $(\gamma_k, \tau_k) \in \operatorname{Crit}(\mathcal{R}^H_{F_k}, U \times I)$ such that there exists $t_k \in \mathbb{S}^1$ with $\gamma_k(t_k) \notin U_0$ or $\tau_k \notin I_0$ for all $k \in \mathbb{N}$. We proceed as in the proof of theorem 2.79. Define a family

$$\mathcal{F} := \{(\gamma_k, \tau_k) : k \in \mathbb{N}\} \subseteq C^0(\mathbb{S}^1, \overline{U \times I}).$$

Fix a Riemannian metric m on M. Then $\mathcal{F}$ is pointwise bounded and equicontinuous. Indeed, for $t, t_0 \in \mathbb{S}^1$ we estimate

$$\begin{aligned} d_m\left(\gamma_k(t_0), \gamma_k(t)\right) &\leq L_m(\gamma_\sigma) \\ &= \int_{t_0}^t |\dot\gamma_k(s)|_m \, ds \\ &\leq |\tau_k| \int_{t_0}^t \left|X_H\left(\gamma_k(s)\right)\right|_m ds + \int_{t_0}^t \left|X_{F_k}\left(\gamma_k(s), s\right)\right|_m ds \\ &= C\,|t - t_0| \end{aligned}$$

for all $k \in \mathbb{N}$. Hence by Ascoli's theorem 2.77, we have that

$$(\gamma_k, \tau_k) \to (\gamma_\infty, \tau_\infty) \in C^0(\mathbb{S}^1, \overline{U}) \times \overline{I}$$

modulo subsequences. But since $\|F_k\|_{C^1(M\times\mathbb{S}^1)} < \delta_k$ we have

$$0 = d\mathcal{R}^H_{F_k}(\gamma_k, \tau_k) \to d\mathcal{R}^H(\gamma_\infty, \tau_\infty) \qquad \text{as} \qquad k \to \infty.$$

Thus by bootstrapping $(\gamma_\infty, \tau_\infty) \in \mathrm{Crit}(\mathcal{R}^H)$. By shrinking $U \times I$ if necessary, we may assume that

$$\gamma_\infty(\mathbb{S}^1) \cap U \neq \varnothing \qquad \text{and} \qquad \tau_\infty \in I.$$

Since $U \times I$ is an isolating neighbourhood for K and

$$\mathcal{R}^H(\gamma_k, \tau_k) \to \mathcal{R}^H(\gamma_\infty, \tau_\infty) \qquad \text{as} \qquad k \to \infty,$$

we conclude $(\gamma_\infty, \tau_\infty) \in K$. But then $U_0 \times I_0$ cannot be an isolating neighbourhood for K. ☐

The locally perturbed Rabinowitz action functional $\mathcal{R}^H_F$ is generically Morse. In order to give a sketch of the proof of this result, we need a notion from functional analysis.

Definition 3.41 (Fredholm Operator, [28, p. 531]). Let X and Y be Banach spaces. A bounded linear operator $D \in \mathcal{L}(X, Y)$ is said to be a ***Fredholm operator***, iff $\ker(D)$ is finite dimensional, its image $D(X)$ is closed and the cokernel $Y/D(X)$ is finite dimensional as well. The ***index of a Fredholm operator D***, written $\mathrm{ind}(D)$, is defined by

$$\mathrm{ind}(D) := \dim(\ker(D)) - \dim(\mathrm{coker}(D)) \in \mathbb{Z}.$$

Fredholm operators as well as their index are stable under perturbations.

Proposition 3.42 (Stability of Fredholm Operators, [28, Theorem A.1.5]). *Let $D \in \mathcal{L}(X, Y)$ be a Fredholm operator between two Banach spaces X and Y. There exists $\varepsilon > 0$ such that for all $D' \in \mathcal{L}(X, Y)$ with $\|D' - D\| < \varepsilon$ we have that D' is a Fredholm operator with* $\mathrm{ind}(D') = \mathrm{ind}(D)$.

Regular level sets of differentiable operators with Fredholm derivatives give rise to submanifolds. Denote by $C^1(X, Y)$ the space of continuously Fréchet differentiable maps between two Banach manifolds X and Y. We say that $F \in C^1(X, Y)$ is ***Fredholm***, iff the derivative $DF_x \in \mathcal{L}(X, Y)$ is a Fredholm operator for every $x \in X$. Since the Fredholm index is stable under perturbations by proposition 3.42, $\mathrm{ind}(DF_x)$ is independent of the choice of $x \in X$, thus we simply write $\mathrm{ind}(F)$ for that index. Note that if $y \in Y$ is a regular value of F, that is, $DF_x \in \mathcal{L}(X, Y)$ is surjective for all $x \in F^{-1}(y)$, then $\mathrm{ind}(F) \geq 0$.

Proposition 3.43 (Implicit Function Theorem for Fredholm Maps, [28, Theorem A.3.3]). *Let X and Y be Banach spaces, $U \subseteq X$ open and $k \geq 1$. If $y \in Y$ is a regular value of a Fredholm map $F \in C^k(U, Y)$, then $M := F^{-1}(y) \subseteq X$ is an embedded Banach submanifold of class C^k with*

$$\dim M = \mathrm{ind}(F) \qquad \textit{and} \qquad T_x M = \ker DF_x$$

for all $x \in M$.

Moreover, we need the following infinite dimensional analogue of Sard's theorem.

Proposition 3.44 (Sard–Smale, [28, Theorem A.5.1]). *Let* X *and* Y *be two separable Banach spaces and* $U \subseteq X$ *open. Suppose that* $F \in C^k(U, Y)$ *is a Fredholm map with*

$$k \geq \max\{1, \operatorname{ind}(F) + 1\}.$$

Then the set of regular values of F *is residual in* Y *in the sense of Baire, that is, it contains a countable intersection of open dense subsets.*

We are now able to state the genericity result about the perturbed Rabinowitz action functional.

Proposition 3.45 ([36, p. 89]). *Let* $(M, d\alpha, H)$ *be an exact Hamiltonian system and* $U \times I$ *an isolating neighbourhood of an action-constant isolated connected component* $K \subseteq \operatorname{Crit}(\mathcal{R}^H)$. *Then for all* $\delta > 0$ *the set* $\mathcal{F}_{\mathrm{reg}}(U \times I, \delta) \subseteq \mathcal{F}(U, \delta)$ *consisting of all* $F \in \mathcal{F}(U, \delta)$ *such that every critical point* $(\gamma, \tau) \in \operatorname{Crit}(\mathcal{R}^H_F, U \times I)$ *is nondegenerate, is of second category, that is,* $\mathcal{F}_{\mathrm{reg}}(U \times I, \delta)$ *cannot be expressed as a countable union of nowhere dense sets.*

Proof. We give only a sketch of the proof. Fix $k \geq 1$ and denote by $C^k_0(U \times \mathbb{S}^1)$ denote the Banach manifold of all $F \in C^k(U \times \mathbb{S}^1)$ such that for all $t \in \mathbb{S}^1$ we have that $\operatorname{supp}(F_t) \subseteq\subseteq U$. For a fixed smooth family of ω-compatible almost complex structures $J = (J_{t,\tau})_{t \in \mathbb{S}^1, \tau \in \mathbb{R}}$ consider the section

$$\sigma \colon W^{1,2}(\mathbb{S}^1, M) \times \mathbb{R} \times C^k_0(U \times \mathbb{S}^1) \to \mathcal{E}, \quad \sigma(\gamma, \tau, F) := \operatorname{grad}_{m_J} \mathcal{R}^H_F(\gamma, \tau),$$

where $\mathcal{E}$ denotes the Hilbert bundle whose fibre over $(\gamma, \tau) \in W^{1,2}(\mathbb{S}^1, M) \times \mathbb{R}$ is given by $L^2(\gamma^* TM)$ and the Hilbert manifold $W^{1,2}(\mathbb{S}^1, M)$ is defined as in [27, p. 50]. The main part of the proof consists of showing that $\sigma^{-1}(0)$ admits the structure of a Banach manifold. This is done via the implicit function theorem for Fredholm maps 3.43 and we omit the details. If we assume that $\sigma^{-1}(0)$ is a Banach manifold, then the projection

$$\pi \colon \sigma^{-1}(0) \to C^k_0(U \times \mathbb{S}^1), \qquad \pi(x, \tau, F) := F,$$

is a Fredholm map and its set of regular values $\mathcal{F}^k_{\mathrm{reg}}(U \times I)$ coincides with the set of all functions $F \in C^k_0(U \times \mathbb{S}^1)$ such that all critical points of $\mathcal{R}^H_F$ are nondegenerate. By the Sard–Smale theorem 3.44 we have that $\mathcal{F}^k_{\mathrm{reg}}(U \times I) \subseteq C^k_0(U \times \mathbb{S}^1)$ is dense. Consequently,

$$\left\{ F \in \bigcap_{k \geq 1} \mathcal{F}^k_{\mathrm{reg}}(U \times I) : \|F\|_{C^1(M \times \mathbb{S}^1)} < \delta \right\} \subseteq \mathcal{F}_{\mathrm{reg}}(U \times I, \delta)$$

is dense. $\square$

As in Morse homology, we consider moduli spaces of negative gradient flow lines.

Definition 3.46 (Moduli Space of Local Negative Gradient Flow Lines, [36, p. 91]). Let $(M, \omega = d\alpha, H)$ be an exact Hamiltonian system and $U \times I$ an isolating neighbourhood of an action-constant isolated connected component $K \subseteq \operatorname{Crit}(\mathcal{R}^H)$. Fix $F \in \mathcal{F}_{\text{reg}}(U \times I, \delta)$ for some $\delta > 0$ and let $(\gamma^\pm, \tau^\pm) \in \operatorname{Crit}(\mathcal{R}^H_F, U \times I)$. Moreover, let $J = (J_{t,\tau})_{t \in \mathbb{S}^1, \tau \in \mathbb{R}}$ be a smooth family of ω-compactible almost complex structures on M. Define the ***moduli space of local negative gradient flow lines***

$$\mathcal{M}\left((\gamma^-, \tau^-), (\gamma^+, \tau^+), \mathcal{R}^H_F, J, U \times I\right)$$

to consist of all negative gradient flow lines (u, τ) of the perturbed Rabinowitz action functional $\mathcal{R}^H_F$, that is, (u, τ) satisfies the perturbed Floer equations 3.7, such that

$$\lim_{s \to \pm\infty} \left(u_s, \tau(s)\right) = (\gamma^\pm, \tau^\pm) \qquad \text{and} \qquad u(\mathbb{R} \times \mathbb{S}^1) \subseteq U, \tau(\mathbb{R}) \subseteq I.$$

Recall, that if $(\Sigma, \alpha|_\Sigma)$ is a regular energy surface in an exact Hamiltonian system $(M, d\alpha, H)$, then $\operatorname{Crit}(\mathcal{R}^H) = \mathcal{P}(X_H, \Sigma)$ by proposition 2.33.

Lemma 3.47 ([36, p. 91]). *Let $(\Sigma, \alpha|_\sigma)$ be a compact regular energy surface of restricted contact type in an exact Hamiltonian system $(M, \omega = d\alpha, H)$ and suppose that $K \subseteq \operatorname{Crit}(\mathcal{R}^H)$ is an action-constant isolated connected component with $U_0 \times I_0 \subseteq U \times I$ two isolating neighbourhoods for K. Fix a smooth family $J = (J_{t,\tau})_{t \in \mathbb{S}^1, \tau \in \mathbb{R}}$ of ω-compatible almost complex structures on M. Then there exists $\delta_1 > 0$ such that for all $0 < \delta < \delta_1$ and all $F \in \mathcal{F}_{\text{reg}}(U \times I, \delta)$ given $(\gamma^\pm, \tau^\pm) \in \operatorname{Crit}(\mathcal{R}^H_F, U_0 \times I_0)$ then the moduli spaces of local negative gradient flow lines*

$$\mathcal{M}\left((\gamma^-, \tau^-), (\gamma^+, \tau^+), \mathcal{R}^H_F, J, U \times I\right)$$

and

$$\mathcal{M}\left((\gamma^-, \tau^-), (\gamma^+, \tau^+), \mathcal{R}^H_F, J, U_0 \times I_0\right)$$

coincide.

Proof. Let $\delta_0 > 0$ be as in lemma 3.40. Suppose that no such $\delta_1 > 0$ exists. Thus we find a sequence $(\delta_k) \subseteq (0, \delta_0)$ such that $\delta_k \to 0$ as $k \to \infty$, $F_k \in \mathcal{F}_{\text{reg}}(U \times I, \delta_k)$, $(\gamma_k^\pm, \tau_k^\pm) \in \operatorname{Crit}(\mathcal{R}^H_{F_k}, U_0 \times I_0)$, $(u_k, \tau_k) \in \mathcal{M}\left((\gamma_k^-, \tau_k^-), (\gamma_k^+, \tau_k^+), \mathcal{R}^H_{F_k}, J, U \times I\right)$ and a sequence $(s_k, t_k) \subseteq \mathbb{R} \times \mathbb{S}^1$ with $\left(u_k(s_k, t_k), \tau_k(s_k)\right) \notin U_0 \times I_0$ for all $k \in \mathbb{N}$. By translating u_k in the first argument we may assume that $s_k = 0$ for all $k \in \mathbb{N}$. Moreover, as $\mathbb{S}^1$ is compact, we may assume that $t_k \to t_\infty$ modulo subsequences. Arguing as in the proof of lemma 3.40, there exist $(\gamma_\infty^\pm, \tau_\infty^\pm) \in K$ such that

$$(\gamma_k^\pm, \tau_k^\pm) \xrightarrow{C^\infty} (\gamma_\infty^\pm, \tau_\infty^\pm) \qquad \text{as} \qquad k \to \infty.$$

Since $\mathcal{R}^H$ satisfies the period-action equality and K is action-constant we have that $\tau^+ = \tau^-$. Set $\tau := \tau^\pm$. Analogous to the Floer case define the ***energy of a local negative gradient flow line (u, τ)*** by

$$E(u,\tau) := \int_{-\infty}^{+\infty} \int_0^1 |\partial_s(u,\tau)|^2_{m_{J_{t,\tau(s)}}} \, dt ds.$$

By proposition 2.61 we may assume that $H = R_\alpha$, where R_α denotes the Reeb vector field associated to $\alpha|_\Sigma$. Then we estimate

$$\begin{aligned}
E(u_k,\tau_k) &= \mathcal{R}^H_{F_k}(\gamma_k^-,\tau_k^-) - \mathcal{R}^H_{F_k}(\gamma_k^+,\tau_k^+) \\
&= \tau_k^- + \int_0^1 \left(\alpha\left(X_{F_k(\gamma_k^-,t)}\right) - F_k(\gamma_k^-,t)\right) dt \\
&\quad - \tau_k^+ - \int_0^1 \left(\alpha\left(X_{F_k(\gamma_k^+,t)}\right) - F_k(\gamma_k^+,t)\right) dt \\
&\le |\tau_k^- - \tau_k^+| + 2\left(\|\alpha\|_{C^0(\bar{U})}\|F_k\|_{C^1(M\times\mathbb{S}^1)} + \|F_k\|_{C^0(M\times\mathbb{S}^1)}\right) \\
&\le |\tau_k^- - \tau_k^+| + 2\left(\|\alpha\|_{C^0(\bar{U})} + 1\right)\|F_k\|_{C^1(M\times\mathbb{S}^1)} \\
&< |\tau_k^- - \tau_k^+| + 2\left(\|\alpha\|_{C^0(\bar{U})} + 1\right)\delta_k
\end{aligned}$$

for all $k \in \mathbb{N}$. Consequently,

$$E(u_k,\tau_k) \to 0 \qquad \text{as} \qquad k \to \infty.$$

In particular, $\sup_{k\in\mathbb{N}} E(u_k,\tau_k) < +\infty$. Using that $\bar{U}$ is compact, by Gromov–Floer compactness [2, p. 167] there exists a local negative gradient flow line (u,τ) such that

$$(u_k,\tau_k) \xrightarrow{C^\infty_{\text{loc}}} (u,\tau),$$

modulo subsequences. Clearly $E(u,\tau) = 0$ thus (u,τ) is actually a constant negative gradient flow line. This constant gradient flow line is contained in K but not in $U_0 \times I_0$. Thus $U_0 \times I_0$ cannot be an isolating neighbourhood for K. □

We can finally define local Rabinowitz–Floer homology. Let $(\Sigma,\alpha|_\Sigma)$ be a compact regular energy surface of restricted contact type in an exact symplectic manifold $(M, d\alpha, H)$. Suppose that $U \times I$ is an isolating neighbourhood for an action-constant isolated connected component $K \subseteq \mathcal{P}(X_H,\Sigma)$. Fix $0 < \delta < \delta_1$, where $\delta_1 > 0$ is given in lemma 3.47, and $F \in \mathcal{F}_{\text{reg}}(U \times I, \delta)$. Let $\mathrm{CF}^{\text{loc}}(\Sigma,\alpha|_\Sigma, F, U \times I)$ denote the free $\mathbb{Z}_2$-vector space generated by $\mathrm{Crit}(\mathcal{R}^H_F, U \times I)$. For a generic choice of a family of ω-compatible almost complex structures $J = (J_{t,\tau})_{t\in\mathbb{S}^1,\tau\in\mathbb{R}}$ define

$$\partial^{\text{loc}} \colon \mathrm{CF}^{\text{loc}}(\Sigma,\alpha|_\Sigma, F, U\times I) \to \mathrm{CF}^{\text{loc}}(\Sigma,\alpha|_\Sigma, F, U\times I)$$

by

$$\partial^{\text{loc}}(\gamma^-,\tau^-) := \sum_{(\gamma^+,\tau^+)\in\mathrm{Crit}(\mathcal{R}^H_F, U\times I)} n\left((\gamma^-,\tau^-),(\gamma^+,\tau^+)\right)(\gamma^+,\tau^+),$$

where

$$n\left((\gamma^-,\tau^-),(\gamma^+,\tau^+)\right) := \#_2\mathcal{M}^0\left((\gamma^-,\tau^-),(\gamma^+,\tau^+),\mathcal{R}^H_F,J,U\times I\right)$$

and $\mathcal{M}^0\left((\gamma^-,\tau^-),(\gamma^+,\tau^+),\mathcal{R}^H_F,J,U\times I\right)$ denotes the 0-dimensional component of the reduced moduli space

$$\mathcal{M}\left((\gamma^-,\tau^-),(\gamma^+,\tau^+),\mathcal{R}^H_F,J,U\times I\right)/\mathbb{R}.$$

Then ∂^{loc} is well-defined by lemma 3.47 and one can show that the 1-dimensional component

$$\mathcal{M}^1\left((\gamma^-,\tau^-),(\gamma^+,\tau^+),\mathcal{R}^H_F,J,U\times I\right)$$

of the reduced moduli space is a compact manifold with boundary

$$\bigcup_{(\gamma,\tau)\in\mathrm{Crit}(\mathcal{R}^H_F,U\times I)} \mathcal{M}^0\left((\gamma^-,\tau^-),(\gamma,\tau)\right)\times\mathcal{M}^0\left((\gamma,\tau),(\gamma^+,\tau^+)\right)$$

supressing the additional dependences of the reduced moduli spaces. In order to show this one needs to invoke a technique called Floer's gluing method. For details see [2, Section 9]. Thus we compute

$$\begin{aligned}(\partial^{\text{loc}}\circ\partial^{\text{loc}})(\gamma^-,\tau^-) &= \sum_{(\gamma,\tau)\in\mathrm{Crit}(\mathcal{R}^H_F,U\times I)} n\left((\gamma^-,\tau^-),(\gamma,\tau)\right)\partial^{\text{loc}}(\gamma,\tau)\\ &= \sum_{(\gamma^+,\tau^+)\in\mathrm{Crit}(\mathcal{R}^H_F,U\times I)} n\left((\gamma^-,\tau^-),(\gamma,\tau),(\gamma^+,\tau^+)\right)(\gamma^+,\tau^+)\\ &= 0,\end{aligned}$$

where

$$n\left((\gamma^-,\tau^-),(\gamma,\tau),(\gamma^+,\tau^+)\right) := \#_2\partial\mathcal{M}^1\left((\gamma^-,\tau^-),(\gamma^+,\tau^+),\mathcal{R}^H_F,J,U\times I\right),$$

as any compact 1-dimensional manifold is a finite disjoint union of diffeomorphic copies of $\mathbb{S}^1$ and the unit interval $[0,1]$ by the classification theorem of smooth 1-manifolds [25, Problem 15-13 and 15-14]. Hence we obtain a local Floer homology

$$\mathrm{HF}^{\text{loc}}(\Sigma,\alpha|_\Sigma,F,U\times I) := \mathrm{HF}\left(\mathrm{CF}^{\text{loc}}(\Sigma,\alpha|_\Sigma,F,U\times I),\partial^{\text{loc}}\right) \tag{3.8}$$

which does not depend on the generic choice of the family of ω-compatible almost complex structures.

Proposition 3.48 ([36, p. 98]). *Let $(\Sigma,\alpha|_\Sigma)$ be a compact regular energy surface of restricted contact type in an exact symplectic manifold $(M,d\alpha,H)$. Suppose that $U\times I$ an isolating neighbourhood of an action-constant isolated connected component $K\subseteq\mathcal{P}(X_H,\Sigma)$. Then there exists $0<\delta_2<\delta_1$, where $\delta_1>0$ is as in lemma 3.47, such that if $F_0,F_1\in\mathcal{F}_{\text{reg}}(U\times I,\delta)$ for some $0<\delta<\delta_2$, then*

$$\mathrm{HF}^{\text{loc}}(\Sigma,\alpha|_\Sigma,F_0,U\times I)\cong\mathrm{HF}^{\text{loc}}(\Sigma,\alpha|_\Sigma,F_1,U\times I).$$

Proof. We give only a sketch as the essential idea is already contained in showing the independence of Morse homology of the choice of Morse–Smale pairs. The technical details can be found in [36, p. 98–101]. Let $(F_\sigma)_{\sigma\in\mathbb{R}} \subseteq \mathscr{F}(U \times I, \delta)$ be a smooth family with

$$F_\sigma = \begin{cases} F_0 & \sigma \le 0, \\ F_1 & \sigma \ge 1, \end{cases}$$

and such that no time-dependent negative gradient flow line leaves the isolating neighbourhood $U \times I$. Then define a continuation morphism

$$\Phi_0^1 \colon \mathrm{CF}^{\mathrm{loc}}(\Sigma, \alpha|_\Sigma, F_0, U \times I) \to \mathrm{CF}^{\mathrm{loc}}(\Sigma, \alpha|_\Sigma, F_1, U \times I)$$

by

$$\Phi_0^1(\gamma_0, \tau_0) := \sum_{(\gamma_1,\tau_1)\in \mathrm{Crit}(\mathscr{R}^H_{F_1}, U\times I)} n\left((\gamma_0, \tau_0), (\gamma_1, \tau_1)\right)(\gamma_1, \tau_1),$$

where

$$n\left((\gamma_0, \tau_0), (\gamma_1, \tau_1)\right) := \#_2\mathscr{M}^0\left((\gamma_0, \tau_0), (\gamma_1, \tau_1), \mathscr{R}^H_{F_\sigma}, J, U \times I\right),$$

denoting by $\mathscr{M}^0\left((\gamma_0, \tau_0), (\gamma_1, \tau_1), \mathscr{R}^H_{F_\sigma}, J, U \times I\right)$ the 0-dimensional component of the set of time-dependent negative gradient flow lines from (γ_0, τ_0) to (γ_1, τ_1).□

A consequence of proposition 3.48 is that the local Floer homology (3.8) is independent of the choice of perturbation F, thus we may write $\mathrm{HF}^{\mathrm{loc}}(\Sigma, \alpha|_\Sigma, U\times I)$.

Definition 3.49 (Local Rabinowitz–Floer Homology, [36, p. 98]). Let $(\Sigma, \alpha|_\Sigma)$ be a compact regular energy surface of restricted contact type in an exact Hamiltonian system $(M, d\alpha, H)$. Suppose $K \subseteq \mathscr{P}(X_H, \Sigma)$ is an action-constant isolated connected component. Define the ***local Rabinowitz–Floer homology of K*** by

$$\mathrm{RFH}^{\mathrm{loc}}(\Sigma, \alpha|_\Sigma, K) := \mathrm{HF}^{\mathrm{loc}}(\Sigma, \alpha|_\Sigma, U \times I),$$

where $U \times I$ is an isolating neighbourhood for K and $F \in \mathscr{F}_{\mathrm{reg}}(U \times I, \delta)$ for some $0 < \delta < \delta_2$, where $\delta_2 > 0$ is as in proposition 3.48.

Having defined local Rabinowitz–Floer homology, we can turn on the question whether one can extend a given family of parametrised periodic orbits on a homotopy of contact type energy surfaces in an exact symplectic manifold and how this family behaves locally. Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. We write $\mathscr{P}(X)$ for the set of all tuples (γ, τ), where $\tau \in \mathbb{R}$ and $\gamma \in C^\infty(\mathbb{S}^1, M)$ such that

$$\dot\gamma(t) = \tau X\left(\gamma(t)\right) \qquad \forall t \in \mathbb{S}^1.$$

Moreover, we define

$$\mathscr{P}^+(X) := \{(\gamma, \tau) \in \mathscr{P}(X) : \tau > 0\}.$$

Linearising the flow of X on parametrised periodic orbits yields useful invariants.

Definition 3.50 (Floquet Multiplier, [20, p. 110]). Let M be a smooth manifold, $X \in \mathfrak{X}(M)$ and $(\gamma, \tau) \in \mathcal{P}(X)$. A ***Floquet multiplier associated to γ*** is defined to be an eigenvalue of the linear map

$$D(\theta_\tau)_{\gamma(0)} \colon T_{\gamma(0)}M \to T_{\gamma(0)}M,$$

where $\theta \in C^\infty(\mathcal{D}, M)$ denotes the flow of X.

Proposition 3.51 ([20, p. 110]). *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. If $(\gamma, \tau) \in \mathcal{P}(X)$ with $\gamma(0) \notin \operatorname{Crit}(X)$ and $\tau \neq 0$, then 1 is a Floquet multiplier of γ.*

Proof. We claim that

$$D(\theta_\tau)_{\gamma(0)}\left(\dot\gamma(0)\right) = \dot\gamma(0).$$

Note that by assumption, $\dot\gamma(0) = \tau X\left(\gamma(0)\right) \neq 0$. We compute

$$\begin{aligned}
D(\theta_\tau)_{\gamma(0)}\left(\dot\gamma(0)\right) &= D(\theta_\tau)_{\gamma(0)}\left(\left.\frac{d}{dt}\right|_{t=0} \theta_t\left(\gamma(0)\right)\right)\\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\theta_\tau \circ \theta_t^H\right)\left(\gamma(0)\right)\\
&= \left.\frac{d}{dt}\right|_{t=0} \theta_{\tau+t}\left(\gamma(0)\right)\\
&= \left.\frac{d}{dt}\right|_{t=0} \theta_{t+\tau}\left(\gamma(0)\right)\\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\theta_t \circ \theta_\tau\right)\left(\gamma(0)\right)\\
&= \left.\frac{d}{dt}\right|_{t=0} \theta_t\left(\gamma(0)\right)\\
&= \dot\gamma(0),
\end{aligned}$$

since $\theta_\tau\left(\gamma(0)\right) = \gamma(0)$. □

We now define an analogue of the Poincaré return map defined at the end of section 2.1 in differentiable dynamics. Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. Suppose that $(\gamma, \tau) \in \mathcal{P}^+(X)$ with $\gamma(0) \notin \operatorname{Crit}(X)$. By the canonical form theorem near a regular point [25, Theorem 9.22], there exists a local embedded hypersurface $\Sigma \subseteq M$ with $\gamma(0) \in \Sigma$ and such that X is nowhere-tangent to Σ. Then we can define a smooth map

$$P(x) := \theta(s(x), x), \qquad s(x) := \inf\left\{s > 0 : \theta_s(x) \in \Sigma\right\},$$

locally near p, where $\theta \in C^\infty(\mathcal{D}, M)$ denotes the flow of X. This map is called the ***Poincaré section map of γ***. Evidently, the fixed points of the Poincaré section map P near $\gamma(0)$ are the initial conditions of all the parametrised periodic orbits of X in the vicinity of the reference solution γ. Moreover, there is a relation between the Floquet multipliers of γ and the eigenvalues of the linearised Poincaré section map

$$DP_{\gamma(0)} \colon T_{\gamma(0)}\Sigma \to T_{\gamma(0)}\Sigma.$$

Proposition 3.52 ([20, p. 111]). *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. Suppose that $(\gamma, \tau) \in \mathcal{P}^+(X)$ with $\gamma(0) \notin \operatorname{Crit}(X)$. Then*

$$\det\left(D(\theta_\tau)_{\gamma(0)} - \lambda\right) = (1-\lambda)\det\left(DP_{\gamma(0)} - \lambda\right).$$

Proof. Let $v \in T_{\gamma(0)}\Sigma$ and choose $\widetilde{\gamma} \in C^\infty((-\varepsilon, \varepsilon), \Sigma)$ such that $\widetilde{\gamma}(0) = \gamma(0)$ and $\widetilde{\gamma}'(0) = v$. Then we compute

$$\begin{aligned} DP_{\gamma(0)}(v) &= \frac{d}{dt}\bigg|_{t=0} P\left(\widetilde{\gamma}(t)\right) \\ &= \frac{d}{dt}\bigg|_{t=0} \theta\left(s\left(\widetilde{\gamma}(t)\right), \widetilde{\gamma}(t)\right) \\ &= D(\theta_\tau)_{\gamma(0)}(v) + \frac{d}{dt}\bigg|_{t=0} \theta_{s(\widetilde{\gamma}(t))}\left(\gamma(0)\right) \\ &= D(\theta_\tau)_{\gamma(0)}(v) + \theta_{s(\widetilde{\gamma}(t))}\left(\gamma(0)\right) \\ &= D(\theta_\tau)_{\gamma(0)}(v) + ds_{\gamma(0)}(v) X_{\gamma(0)}. \end{aligned}$$

As $X_{\gamma(0)} \neq 0_{\gamma(0)}$ by assumption, we have the splitting

$$T_{\gamma(0)}M = T_{\gamma(0)}\Sigma \oplus \langle X_{\gamma(0)}\rangle.$$

The above computation implies that

$$D(\theta_\tau)_{\gamma(0)} = \left(\begin{array}{c|c} DP_{\gamma(0)} & -ds_{\gamma(0)} \\ \hline 0 & 1 \end{array}\right)$$

with respect to this splitting, implying the claim. ☐

By means of the implicit function theorem, using the Poincaré section map P associated to a parametrised periodic orbit γ one can show that γ belongs to a whole smooth family of parametrised periodic orbits under some mild assumption on the Floquet multipliers associated to γ. This method is called the ***Poincaré continuation method***.

Proposition 3.53 (Regular Orbit Cylinder Theorem, [1, p. 576], [20, p. 110]). *Let Σ be a regular energy surface in a Hamiltonian system (M, ω, H) and suppose that $(\gamma, \tau) \in \mathcal{P}^+(X_H, \Sigma)$ is such that that γ admits exactly two Floquet multiplier equal to 1. Then there exists $\varepsilon > 0$ and a family $(\gamma_\sigma, \tau_\sigma)_{\sigma \in (-\varepsilon, \varepsilon)}$ of periodic orbits such that $\gamma_0 = \gamma$, $H \circ \gamma_\sigma = \sigma$ for all $\sigma \in (-\varepsilon, \varepsilon)$ and $H \circ \Gamma$ does not admit a critical point, where*

$$\Gamma \in C^\infty(\mathbb{S}^1 \times (-\varepsilon, \varepsilon), M), \qquad \Gamma(t, \sigma) := \gamma_\sigma(t).$$

Remark 3.54. The regular orbit cylinder theorem 3.53 should also be valid in the case of a parametrised periodic orbit on a regular energy surface belonging to a homotopy of regular energy surfaces. However, in the literature only the case of energy-shifts of a single Hamiltonian function is considered and even in that case the proof is highly technical.

Proof. We only give a sketch of the proof. Choose a chart $(U^{2n}, (x^i))$ about $\gamma(0)$ such that $H(x^1, \ldots, x^{2n}) = x^1$ and $\{x^{2n} = 0\}$ is a local transversal section. By preservation of energy 2.3, the Poincaré section map P in these coordinates is given by

$$P(x^1, \ldots, x^{2n-1}) = \left(x^1, \widetilde{P}(x^1, \ldots, x^{2n-1})\right).$$

Thus in order to find the fixed points of the Poincaré section map it is enough to solve

$$\widetilde{P}(x^1, \ldots, x^{2n-1}) = (x^2, \ldots, x^{2n-1}).$$

Set

$$\Phi(x^1, \ldots, x^{2n-1}) := \widetilde{P}(x^1, \ldots, x^{2n-1}) - (x^2, \ldots, x^{2n-1}).$$

Then clearly $\Phi\left(\gamma^1(0), \ldots, \gamma^{2n-1}(0)\right) = 0$ and by assumption together with proposition 3.52 we have that the matrix

$$D\widetilde{P}\left(\gamma^1(0), \ldots, \gamma^{2n-1}(0)\right) - I_{2n-2}$$

is invertible. Hence by the implicit function theorem [25, Theorem C.40], there exists a smooth map $x^1 \mapsto F(x^1)$ with $\Phi(x^1, F(x^1)) = 0$, that is,

$$\widetilde{P}(x^1, F(x^1)) = F(x^1)$$

for x^1 sufficiently close to $\gamma(0)$. Thus the required smooth family of parametrised periodic orbits is given by $\theta(\cdot, (x^1, F(x^1))$, where $\theta \in C^\infty(\mathcal{D}, M)$ denotes the flow of X_H. □

Motivated by the regular orbit cylinder theorem 3.53 we use the following notion of nondegeneracy of parametrised periodic orbits lying on prescribed energy surfaces.

Definition 3.55 (Nondegenerate Periodic Orbit). A parametrised periodic orbit $(\gamma, \tau) \in \mathcal{P}^+(X_H, \Sigma)$ on a regular energy surface Σ in a Hamiltonian system (M, ω, H) is said to be ***nondegenerate***, iff γ admits exactly two Floquet multipliers equal to 1.

Here is the main result of this chapter and of the whole thesis.

Theorem 3.56. *Let $(\gamma_\sigma, \tau_\sigma)_{\sigma \in [0, \sigma_\infty)}$ be a family of maximally nondegenerate periodic orbits on a compact restricted contact type homotopy of energy surfaces $(\Sigma_\sigma, \alpha|_{\Sigma_\sigma})_{\sigma \in [0,1]}$ in an exact symplectic manifold $(M, d\alpha)$ satisfying the following assumptions:*

(i) *The corresponding ω-limit set $\omega(\gamma_\sigma, \tau_\sigma)$ is isolated in $\mathcal{P}^+(X_{H_{\sigma_\infty}}, \Sigma_{\sigma_\infty})$, that is, there exists an open subset $U \subseteq C^\infty(\mathbb{S}^1, M) \times (0, +\infty)$ such that*

$$U \cap \mathcal{P}^+(X_{H_{\sigma_\infty}}, \Sigma_{\sigma_\infty}) = \omega(\gamma_\sigma, \tau_\sigma).$$

(ii) *The family of maximally nondegenerate periodic orbits* $(\gamma_\sigma, \tau_\sigma)_{\sigma\in[0,\sigma_\infty)}$ *does not extend to* 1*, that is,* $\sigma_\infty < 1$.

(iii) *There exists a sequence* $(\sigma_k) \subseteq (\sigma_\infty, 1]$ *such that* $\sigma_k \to \sigma_\infty$ *as* $k \to \infty$ *and*

$$\mathcal{P}^+(X_{H_{\sigma_k}}, \Sigma_{\sigma_k}) \cap U = \varnothing$$

for all $k \in \mathbb{N}$.

Then there exists $\delta > 0$*, such that*

$$\#\left(\mathcal{P}^+(X_{H_\sigma}, \Sigma_\sigma)/\mathbb{S}^1 \cap U\right) \geq 2,$$

for all $\sigma \in (\sigma_\infty - \delta, \sigma_\infty)$.

Remark 3.57. Theorem 3.56 guarantees the existence of a second unparametrised periodic orbit on Σ_σ for $\sigma < \sigma_\sigma$ sufficiently close to σ_∞, killing the local Rabinowitz–Floer homology of the ω-limit set $\omega(\gamma_\sigma, \tau_\sigma)$. See figure 3.6.

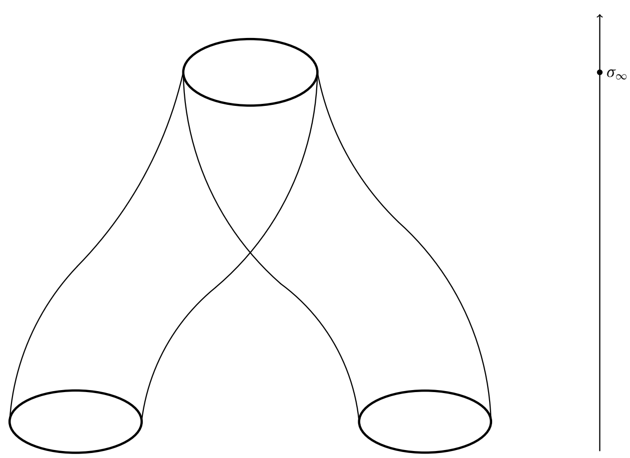


Fig. 3.6: A second family of unparametrised periodic orbits killing of the local Rabinowitz–Floer homology of $\omega(\gamma_\sigma, \tau_\sigma)$.

Proof. By 2.83 the ω-limit set $\omega(\gamma_\sigma, \tau_\sigma)$ is connected and assumption (i) together with lemma 3.38 implies that the ω-limit set $\omega(\gamma_\sigma, \tau_\sigma)$ is isolated. Moreover, by proposition 2.82 we have that $\omega(\gamma_\sigma, \tau_\sigma)$ is action-constant. Hence the local Rabinowitz–Floer homology of the ω-limit set $\omega(\gamma_\sigma, \tau_\sigma)$

$$\mathrm{RFH}^{\mathrm{loc}}\left(\Sigma_{\sigma_\infty}, \alpha|_{\Sigma_{\sigma_\infty}}, \omega(\gamma_\sigma, \tau_\sigma)\right)$$

is well-defined.

In view of remark 3.54 we proceed to give a sketch of the argument only in the special case where the homotopy of energy surfaces is modelled by energy-shifts of a single Hamiltonian function. As every parametrised periodic orbit in $\omega(\gamma_\sigma, \tau_\sigma)$ is necessarily degenerate as otherwise one can extend the maximally nondegenerate family of parametrised periodic orbits over σ_∞ by means of the regular orbit cylinder theorem 3.53 contradicting assumption (ii), the proof goes as follows. Suppose that no such $\delta > 0$ exists. Then by pertubing the Rabinowitz action functional slightly above Σ_{σ_∞} we get

$$\mathrm{RFH}^{\mathrm{loc}}\left(\Sigma_{\sigma_\infty}, \alpha|_{\Sigma_{\sigma_\infty}}, \omega(\gamma_\sigma, \tau_\sigma)\right) = 0$$

by assumption (iii). However, by slightly perturbing the Rabinowitz action functional below Σ_{σ_∞}, we get that (see [36, Proposition 3.4.2])

$$\mathrm{RFH}^{\mathrm{loc}}\left(\Sigma_{\sigma_\infty}, \alpha|_{\Sigma_{\sigma_\infty}}, \omega(\gamma_\sigma, \tau_\sigma)\right) \cong \mathrm{HF}^{\mathrm{loc}}(P_\sigma, \gamma_\sigma(0)) \otimes \mathrm{H}(\mathbb{S}^1; \mathbb{Z}_2) \neq 0,$$

where $P_\sigma := P_{\gamma_\sigma}$ denotes the Poincaré secton map associated to the parametrised periodic orbit γ_σ for σ sufficiently close to σ_∞. Note that the local Floer homology $\mathrm{HF}^{\mathrm{loc}}(P_\sigma, \gamma_\sigma(0))$ is well-defined as the Poincaré section map P_σ can be seen as a Hamiltonian diffeomorphism (see [31, Lemma 4.16]). $\square$

Remark 3.58. It should also be possible to deduce theorem 3.56 in the case of compact stable homotopies of energy surfaces, but in order to do so, one needs to extend the local Rabinowitz–Floer homology to stable Hamiltonian structures. This generalisation is definitely nontrivial as remarked in [8, p. 1768].

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