Lecture 2: Symplectic Manifolds

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April 22, 2021

Definition (Symplectic Vector Space)

A ***symplectic vector space*** is defined to be a tuple (V, ω), where V is a finite-dimensional real vector space and $\omega\colon V \times V \to \mathbb{R}$ is a nondegenerate skew-symmetric bilinear form.

Theorem

Let (V, ω) *be a symplectic vector space. Then there exists a basis* (a_i, b_i) *of* V *such that*

$$\omega(b_i, a_j) = \delta_{ij} \qquad \text{and} \qquad \omega(a_i, a_j) = \omega(b_i, b_j) = 0$$

for all i, j*. Any such basis is called a* ***symplectic basis for*** $\boldsymbol{V}$.

Corollary

Every symplectic vector space (V, ω) *is of even dimension* $2n$ *and*

$$\omega = \sum_{i=1}^{n} \beta^i \wedge \alpha^i$$

in the dual basis (α^i, β^i) *of a symplectic basis* (a_i, b_i) *for* V.

Corollary

A skew-symmetric bilinear form ω on a real vector space V^{2n} is nondegenerate if and only if $\omega^n \neq 0$.

The Category of Symplectic Manifolds

Definition (Symplectic Manifold)

A ***symplectic manifold*** is defined to be a tuple (M, ω) where M is a smooth finite-dimensional manifold and $\omega \in \Omega^2(M)$ is closed and nondegenerate.

Definition (Symplectomorphism)

Let (M, ω) and $(\widetilde{M}, \widetilde{\omega})$ be two symplectic manifolds. A ***symplectomorphism*** is a diffeomorphism $\varphi \colon M \to \widetilde{M}$ such that $\varphi^*\widetilde{\omega} = \omega$.

Lemma

Let (M, ω) be a compact symplectic manifold. Then $H^2_{\mathrm{dR}}(M) \neq 0$.

Corollary

No even sphere $\mathbb{S}^{2n}$ admits a symplectic form for $n \geq 2$.

Lemma

Let M be a smooth manifold. Then $(T^*M, d\lambda)$ is a symplectic manifold with the ***Liouville form***

$$\lambda_{(x,\xi)}(v) := \xi(D\pi(v)) \qquad \forall (x,\xi) \in T^*M, v \in T_{(x,\xi)}T^*M,$$

where $\pi\colon T^*M \to M$ denotes the cotangent bundle projection. The symplectic form $d\lambda$ on T^*M is called the ***canonical symplectic form****.*

Lemma

*Let $\varphi\colon M \to \widetilde{M}$ be a diffeomorphism between smooth manifolds M and $\widetilde{M}$. Then the **cotangent lift***

$$D\varphi^\dagger\colon T^*M \to T^*\widetilde{M}, \qquad D\varphi^\dagger_{(x,\xi)}(\varphi(x), v) := \xi(D\varphi^{-1}(v))$$

is an exact symplectomorphism.

Definition (Lagrangian Submanifold)

Let (M, ω) be a symplectic manifold. An embedded submanifold L of M is said to be ***Lagrangian***, iff $\dim L = \frac{1}{2}\dim M$ and $\iota_L^*\omega = 0$.

Lemma

Let $\varphi\colon M \to \widetilde{M}$ *be a diffeomorphism between symplectic manifolds* (M, ω) *and* $(\widetilde{M}, \widetilde{\omega})$. *Then* φ *is a symplectomorphism if and only if its graph*

$$\Gamma_\varphi \subseteq (M \times \widetilde{M}, -\omega \oplus \widetilde{\omega})$$

is a Lagrangian submanifold.

Theorem (Darboux Theorem)

Let M^{2n} be a smooth manifold and $\omega \in \Omega^2(M)$ nondegenerate. Then ω is closed if and only if for every $x \in M$ there exists a chart $\big(U, (x^i, y^i)\big)$ about x such that

$$\omega|_U = \sum_{i=1}^{n} dy^i \wedge dx^i.$$