# Hamiltonian Systems

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# Very classical mechanics

For a particle in 3-space, we have the position $\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = r(t) \in \mathbb{R}^3$

$$\vec{v} = \frac{\mathrm{d}}{\mathrm{d}t} r, \quad \vec{a} = \frac{\mathrm{d}}{\mathrm{d}t} \vec{v}, \boxed{\vec{p} = m\vec{v}.}$$

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The particle may be subject to a force $\vec{F}(r, \vec{v}, t)$, obeying Newton's laws:

$$m \cdot \vec{a} = \frac{\mathrm{d}}{\mathrm{d}t} \vec{p} \overset{!}{=} \vec{F}$$

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$$\vec{p} = \begin{bmatrix} m_1 & & \\ & \ddots & \\ & & m_l \end{bmatrix} \vec{v}$$

Several particles in one "vector":

$$R = \begin{bmatrix} x_1 \\ y_1 \\ \vdots \\ y_l \\ z_l \end{bmatrix}, \quad \begin{bmatrix} m_1 & & & & \\ & m_1 & & & \\ & & \ddots & & \\ & & & m_l & \\ & & & & m_l \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \vdots \\ \ddot{y}_l \\ \ddot{z}_l \end{bmatrix} \stackrel{!}{=} \begin{bmatrix} F_{1,x} \\ F_{1,y} \\ \vdots \\ F_{l,y} \\ F_{l,z} \end{bmatrix}$$

# Motivation

Consider $l$ particles in 3-space on holonomic constraints:

$$\underline{q} = (q^i) \overset{\text{Chart}}{\mapsto} R = \left[R^\lambda\right] \in N = f^{-1}[\{0\}] \cap \text{some open set} \underset{}{\overset{\mathtt{sM}}{\subseteq}} \mathbb{R}^{3l}.$$

generalized coordinates

$$\begin{pmatrix} r \\ \varphi \end{pmatrix}$$

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The masses $m_\lambda$ give the (extrinsic) momentum $P$ on a trajectory as a Riemannian metric $m$ as on the last slide. This descents to $N$:

$$\overleftarrow{P} = m(\vec{\dot{R}}) \Rightarrow \underline{\overleftarrow{p}} = \underbrace{\overleftarrow{m}(\vec{\dot{R}})}_{m(\vec{\dot{R}},\,\cdot)} \doteq (\underline{p_i}) = (m_{ij}\dot{q}^j)$$

↑ depends on $q$

$g = \langle\ ,\ \rangle$

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Newton's equation (extrinsic):

$$\underline{F_\lambda = m_\lambda \dot{v}^\lambda}, \quad \text{i.e.} \quad \overleftarrow{F} = m\left(\frac{\mathrm{D}}{\mathrm{d}t}\vec{v}\right)$$

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Kinetic energy:

$$E_{kin} = \frac{1}{2} m v^2$$

inverse metric

$m_{ij} m^{jk} = \delta_i^k$

$$2T = \sum_\lambda m_\lambda (v^\lambda)^2 = \|\dot{R}\|_m^2 = \|P\|_{m^{-1}}^2 = \underline{m_{ij}v^i v^j} = \underline{m^{ij}p_i p_j}$$

# Generalized forces

$Q_i dq^i$ $\quad g_{\lambda\sigma} = \delta_{\lambda\sigma}$

The parallel force $\overleftarrow{Q} = Q_i \varepsilon^i = g(\vec{F}_{\parallel})$ may later depend on velocity, but here we consider conservative forces with the potential $V \in C^\infty(N, \mathbb{R})$

$Q_i = -\frac{\partial V}{\partial q^i}$

$$\vec{F} = -\vec{\nabla} V = -g^{-1}\left(\frac{\partial V}{\partial r}\right), \quad \text{i.e.} \quad \overleftarrow{Q} = -\overleftarrow{\mathrm{d}V} = -\frac{\partial V}{\partial q^i}\,\mathrm{d}q^i$$

Newton's equation becomes:


$$\overleftarrow{Q} = m\left(\frac{\mathrm{D}}{\mathrm{d}t}\vec{v}\right) \overset{m}{=} \left((\dot{v}^j)\vec{e}_j + v^i v^j \vec{\Gamma}_{ij}\right)$$

The potential and kinetic energy constitute the **Hamiltonian** or total energy function

$$H(q^i, p_i, t) = T(q^i, p_i, t) + V(q^i, p_i, t)$$

$$L(q^i, v^i, t) = T \qquad - V$$

# Example 1 - $l$-body problem

Unrestricted motion of $l$ celestial bodies with gravitation, i.e. the forces

$$\begin{bmatrix} F_{12} \\ -F_{12} \end{bmatrix}$$

$$\vec{F}_i = \sum_{j \neq i} -\frac{m_i m_j}{4\pi |r_i - r_j|^2} \vec{e}_{r_i - r_j} \qquad 4\pi G$$

generated by the shared potential $\qquad \frac{mM}{4\pi R^2}(-\vec{e}_r)$

$$4\pi V(r_1, \ldots, r_l) = \underbrace{-\frac{m_1 \cdot m_2}{|r_1 - r_2|}}_{\frac{1}{2} m |v|^2} - \cdots - \frac{m_1 \cdot m_l}{|r_1 - r_l|} - \frac{m_2 \cdot m_3}{|r_2 - r_3|} - \ldots$$

$$- \frac{mM}{4\pi R}$$

This has the Hamiltonian

$$H(\underline{r_1, \ldots, r_l}, \underline{\vec{p}_1, \ldots, \vec{p}_l}) = \sum_i \frac{|\vec{p}_i|^2}{2m_i} - \sum_{i<j} \frac{m_i m_j}{4\pi |r_i - r_j|}$$

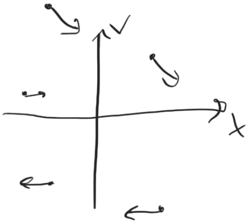
We have to remove the collision set $\{R \,|\, V = \infty\}$, so $\underline{N = \operatorname{dom} V}$.

# Abstraction to the phase space

## Lagrangian mechanics

- $\underline{M = TN} \ni \vec{v}@R \doteq \underline{v}@\underline{q} \overset{!}{=} \underline{\dot{q}}@\underline{q}$ $\quad v_i @ q^i$
- Lagrangian: $L = \underline{T - V} \in \mathscr{F}(M)$
- We have the canonical momenta $\boxed{p_i} = m_{ij}v^j = \frac{\partial}{\partial \dot{q}^i} T = \boxed{\frac{\partial}{\partial \dot{q}^i} L}.$
- Lagrange equation: $\dot{q}^j = v^j$, $\boxed{\dot{p}_j} = Q_j + \frac{\partial T}{\partial q^i} = \boxed{\frac{\partial L}{\partial q^i}}$

Cancels the $\Gamma_{ij}^{k}$




$\ddot{u} = F$

$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ F \end{pmatrix}$

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$L(v) \longrightarrow H(p) = pv - L$

$p = \frac{\partial L}{\partial v}$ ← has to be invertible

$T = m(v,v)/2$

## Hamiltonian mechanics

- $M = T^*N \ni \overleftarrow{p}@R \doteq \underline{p}@\underline{q}$
- Hamiltonian: $H = \overleftarrow{p}\vec{v} - L = T + V \in \mathscr{F}(M)$
- $v^j = m^{ij}p_i = \frac{\partial}{\partial p_j}T = \frac{\partial}{\partial p_j}H$
- Hamilton equations: $\dot{q}^j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q^j}$

$H(q^i, p_i) = p_i v^i - L(q^i, v^i)$

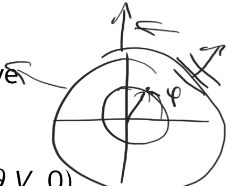
$T(v^i) \longmapsto T(p_i)$

$\dot{p}_\lambda = -\frac{\partial V}{\partial q^\lambda}$

← Defines what $p_i$ is (for general $H$)

They are duals via a Legendre transformation.

# Example 2 - Central potential




If $V(R) = V\left(\begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix}\right) = V(r,\varphi)$ only depends on $r$, we have

$$F = \frac{\partial V}{\partial r} \cdot (-\vec{e}_r) \quad \text{or} \quad (Q_r, Q_\varphi) = \underline{g}\begin{pmatrix} -\partial_r V \\ 0 \end{pmatrix} = (-\partial_r V, 0)$$

$$\begin{pmatrix} 1 & \\ & r^2 \end{pmatrix}$$

The canonical momenta are

$$\begin{pmatrix} p_r \\ p_\varphi \end{pmatrix} = \underbrace{\begin{pmatrix} m & 0 \\ 0 & mr^2 \end{pmatrix}}_{m_{ij} = m \cdot g_{ij}} \begin{pmatrix} \dot r \\ \dot\varphi \end{pmatrix} \qquad p_r = m\dot r$$

$e_\varphi$ $e_\varphi$ $e_\varphi$

We can identify $L = p_\varphi = mr^2\dot\varphi$ as the angular momentum.

$$H = \frac{1}{2} m^{ij} p_i p_j + V(\underline{q}) = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} + V(r)$$

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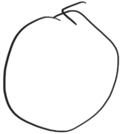
Hamiltonian equations:

$$\dot{q}^j = v^j = \begin{pmatrix} \dot{r} \\ \dot{\varphi} \end{pmatrix} = \frac{\partial H}{\partial \underline{p}} = \begin{pmatrix} \frac{p_r}{m} \\ \frac{p_\varphi}{mr^2} \end{pmatrix} = m^{ij} p_i$$



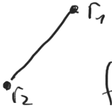
$$(\dot{p}_r, \dot{p}_\varphi) = -\frac{\partial H}{\partial \underline{q}} = \left( -\frac{\partial V}{\partial r} + \frac{p_\varphi^2}{mr^3}, 0 \right)$$

Notice the "fictional" centrifugal force $m\dot{\varphi}^2 r$. $p_\varphi$ is conserved, because $\varphi$ is a **cyclic coordinate**.

$\uparrow \partial_{q^i} H = 0$

# Generalization

We consider a system with $n$ degrees of freedom, described by a smooth $n$-dimensional manifold $N$, called the **configuration space** and its cotangent bundle $M = T^*N$, the **phase space** with a function $H$.




$f\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = |r_1 - r_2|^2 - \ell^2$

$S^1 \times \mathbb{R}$

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With canonical coordinates $\underline{u} = \begin{pmatrix} \underline{q} \\ \underline{p} \end{pmatrix} \overset{\text{chart}}{\mapsto} U \in M$ the Hamiltonian equations turn into:

antisymm. $T^{(2,0)}M$

$$\dot{\underline{U}} \doteq \dot{\underline{u}} = \begin{pmatrix} \dot{\underline{q}} \\ \dot{\underline{p}} \end{pmatrix} \overset{!}{\doteq} \begin{pmatrix} \frac{\partial H}{\partial p_j} \\ -\frac{\partial H}{\partial q^i} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}}_{(J^{\mu\nu})} \frac{\partial H}{\partial \underline{u}}^{\mathsf{T}} = \underline{\underline{J}}(\mathrm{D}H) \doteq -\tilde{\Omega}(\mathrm{d}H) =: X_H$$

$$\left( \frac{\partial H}{\partial q^i} \quad \frac{\partial H}{\partial p^i} \right)$$

$$\left( \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right) = J^{\mu\nu}$$

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Here $\Omega = \omega^{-1}$, where $\omega = \mathrm{d}\lambda = \sum_i \mathrm{d}p^i \wedge \mathrm{d}x^i$ with the Liouville 1-form

$\in TM = TT^*N$ // $\frac{1}{2}T$

$$\lambda(z) = \underbrace{\pi_{TT^*N \longrightarrow T^*N}(z)}_{\text{Basepoint } \vec{p}} \underbrace{(T\pi_{T^*N \longrightarrow N}(z))}_{\text{spatial part of } z} = \vec{p}(\vec{v}) \Leftrightarrow \lambda_{\vec{p}} = \sum_i p^i \, \mathrm{d}x^i$$

↖ That was the thing in $H = \vec{p}\vec{v} - L$

which has the matrix representation $\omega_{\mu\nu} = J^{\mu\nu} = -\Omega^{\mu\nu}$

# General Definition of Hamiltonian Systems

Let $(M, \omega)$ be a symplectic manifold, with tangent-cotangent isomorphism

← Phase space

$$\omega : TM \xrightarrow{\cong} T^*M, z \mapsto \omega(z, \cdot) \quad \text{and} \quad \Omega = \omega^{-1}. : T^*M \to TM$$

$$\omega(z) = \omega(z, \bullet) = i_z\omega = \iota_z\omega$$

The triple $(M, \omega, H)$ is called a **Hamiltonian system** where $H \in \mathscr{F}(M)$ is the **Hamiltonian** or energy function.

**Canonical coordinates** $\underline{u} = (u^\mu) = (x^1, ..., x^n, p^1, ..., p^n)^\mathsf{T}$ are such that

$q^i = x^i$

$$\omega = \sum_i dp^i \wedge dx^i, \quad \text{i.e.} \quad \omega(e_{p^i}) = \varepsilon_{x^i}.$$

$dx^i = dq^i$

Darboux-Coordinates

$$\omega(e_{x^i}) = -dp^i$$

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$$\omega = \sum_i \mathrm{d}p^i \wedge \mathrm{d}x^i, \quad \text{i.e.} \quad \omega(\boldsymbol{e}_{p^i}) = \boldsymbol{\varepsilon}_{x^i}.$$

## Important Definition

The **Hamiltonian vector field** associated to $f \in \mathscr{F}(M)$ is $\vec{X}_H = -\Omega(dH)$

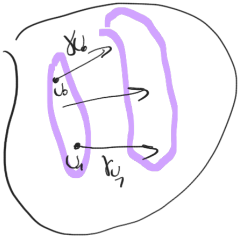
$$X_f := -\Omega(\mathrm{d}f) \quad \text{i.e.} \quad \omega(X_f, z) = -\mathrm{d}f(z) = -\partial_z f.$$

# Hamiltonian flow

The Hamiltonian equations now are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma = -\Omega(\mathrm{d}H)\circ\gamma = X_H\circ\gamma$$

The **trajectories** of the Hamiltonian system (solutions to the Hamiltonian equations) are exactly the integral curves of $X_H$.



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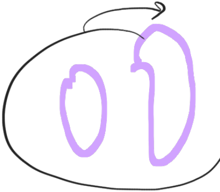
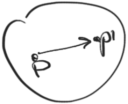
These exist at least locally (first order ODE):

$$U_0 \mapsto \gamma_{U_0} : \overset{0 \mapsto U_0}{t \mapsto \gamma_{U_0}(t)} \mapsfrom U_0 : \psi^t_{X_H} \mapsfrom t$$



This is the **Hamiltonian flow** (flow of $X_H$).

# Special vector fields




A vector field $z \in \mathscr{X}(M)$ is ...

- **symplectic** iff $\omega$ is preserved along the integral curves of $z$: $\psi_z^{t*}\omega = \omega$, i.e. $\mathscr{L}_z\omega = 0$

$$\omega_{p'}\left(T\varphi_z^t(e_i),\, T\varphi_z^t(e_j)\right)$$

$$\stackrel{!}{=} \omega_p(\vec{e_i}, \vec{e_j})$$

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- **symplectic** iff $\omega$ is preserved along the integral curves of $\boldsymbol{z}$: $\psi_z^{t*}\omega = \omega$, i.e. $\mathscr{L}_z\omega = 0$

That follows from the fact that the Lie-derivative wrt $\boldsymbol{z}$ commutes with the flow of $\boldsymbol{z}$: (at point $P' = \psi_z^t(P) = \gamma_P(t)$)

flow property $\psi_z^t \psi_z^s = \psi_z^{t+s}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_z^{t*}(\omega) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}(\psi_z^{t+s})^*(\omega) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0}\psi_z^{t*}(\psi_z^{s*}\omega) = \psi_z^{t*}(\mathscr{L}_z\omega)$$

So if $\mathscr{L}_z\omega = 0$, $\psi_z^{t*}(\omega(\gamma_P(t)))$ must be constant wrt $t$; $\psi_z^{0*}(\omega(\gamma_P(0))) = \omega(P)$.
If not, then there is some place where $\psi_z^{t*}(\omega(\gamma_P(t))) \neq \psi_z^{0*}(\omega(P))$

# Special vector fields

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- [locally] **Hamiltonian** iff it [locally] coincides with $X_f$ for some $f \in \mathscr{F}(M)$

A vector field $z \in \mathscr{X}(M)$ is …

- **symplectic** iff $\omega$ is preserved along the integral curves of $z$: $\psi_z^{t*}\omega = \omega$, i.e. $\mathscr{L}_z\omega = 0$
- [locally] **Hamiltonian** iff it [locally] coincides with $X_f$ for some $f \in \mathscr{F}(M)$ $\omega(z) = -df$ $= -\Omega(df)$

## Relation between these definitions

$z$ symplectic $\Leftrightarrow$ $\omega(z)$ closed $\Leftrightarrow$ $\omega(z)$ is locally exact $\Leftrightarrow$ $z$ locally Hamiltonian.

Poincaré

$$\mathscr{L}_z\omega \overset{\text{Cartan}}{=} d(\iota_z\omega) + \iota_z \underbrace{d\omega}_{0} = d(\omega(z))$$

Hamiltonian $\Leftrightarrow$ exact

# Poisson bracket

One also defines the **Poisson bracket** of $g, f \in \mathscr{F}(M)$:

$$\{g, f\} := \omega(X_g, X_f) = df(X_g) = \partial_{X_g} f = -\Omega(dg, df) = -\Omega^{\mu\nu} \cdot \partial_\mu g \cdot \partial_\nu f$$

$$-\underbrace{\omega(X_f)}_{df}(X_g)$$

$$\| $$

$$X_g(df) = -\Omega(dg)(df)$$

$$\uparrow J^{\mu\nu}$$

Physical definition

$$= \sum_i \frac{\partial g}{\partial x^i}\frac{\partial f}{\partial p^i} - \frac{\partial g}{\partial p^i}\frac{\partial f}{\partial q^i}$$

# Poisson bracket

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$$\underline{\{g, f\}} := \omega(X_g, X_f) = \mathrm{d}f\,(X_g) = \underline{\partial_{X_g} f} = -\Omega(\mathrm{d}g, \mathrm{d}f) = -\Omega^{\mu\nu} \cdot \partial_\mu g \cdot \partial_\nu f$$

A quantity $f$ is conserved iff it Poisson commutes with $H$, because on a trajectory $\gamma$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma) = \partial_{X_H} f \circ \gamma = \underline{\{H, f\}} \circ \gamma$$

Energy is conserved along trajectories, because $\{H, H\} = 0$.

Preservation of energy

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Canonical commutation relations: $\{u^\mu, u^\nu\} = -\Omega^{\kappa\iota} \partial_\kappa u^\mu \partial_\iota u^\nu = -\Omega^{\mu\nu} \overset{!}{=} J^{\mu\nu}$

Hamiltonian equations: $\frac{\mathrm{d}}{\mathrm{d}t} u^\mu \circ \gamma = \{H, u^\mu\} \circ \gamma = -\Omega^{\kappa\mu} \frac{\partial H}{\partial u^\kappa} = X_H^\mu$

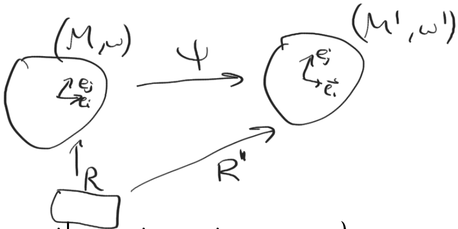
(if $-\Omega^{\kappa\mu} = J^{\kappa\mu}$

# Commutators and Poisson Brackets

$$X_{\{f,g\}} = \left[X_f, X_g\right]$$ Lie-Bracket

The Poisson bracket generalizes to the commutator (divided by $i\hbar$) in quantum mechanics.: Replace $f \mapsto \partial_{X_f} / i\hbar$

# Symplectomorphisms

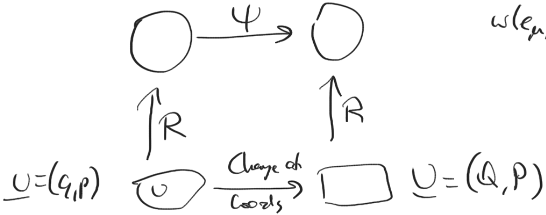



$$\left[(M,\omega) \underset{\mathtt{Symp}}{\longrightarrow} (M',\omega')\right] = \left\{\psi : M \underset{\mathtt{sM}}{\longrightarrow} M' \,\middle|\, \psi^*(\omega') = \omega' \circ T\psi = \omega\right\}$$

$$\Updownarrow$$

$$\mathsf{Symp}(M,\omega) := \mathsf{Aut}_{\mathtt{Symp}}(M,\omega) = \left\{\psi : M \longrightarrow M \,\middle|\, \omega_{\mu\nu} = \omega'_{\mu\nu}\right\}$$

$\omega_{\mu\nu} = \omega(e_\mu, e_\nu)$, $\omega'_{\mu\nu} = \omega'(e'_\mu, e'_\nu)$



# Changes of coordinates

## Canonical transformations are symplectomorphisms

The canonical transformations are changes of coordinates $\begin{pmatrix} \underline{q} \\ \underline{p} \end{pmatrix} \mapsto \begin{pmatrix} \underline{Q} \\ \underline{P} \end{pmatrix}$ that leave the form of Hamilton's equations the same, i.e. such that the new coordinates are also canonical, i.e. symplectomorphisms:

$$-\{u^\mu, u^\nu\} = \Omega^{\mu\nu} \overset{!}{=} \tilde{\Omega}^{\mu\nu} = -\{\underbrace{\tilde{u}^\mu, \tilde{u}^\nu}_{}\}^{{}^{\text{"}}P^i, Q^j} = \frac{\partial \tilde{u}^\mu}{\partial u^\kappa} \frac{\partial \tilde{u}^\nu}{\partial u^\iota} \Omega^{\iota\kappa}$$

New Hamiltonian vector field: $\overset{X_{H\circ\psi}}{X_{\psi^* H}} = \psi^* X_H := T\psi^{-1} X_H$

$$\omega(X_{\psi^* H}) = -\mathrm{d}(\psi^* H) = -\psi^* \mathrm{d}H = \psi^* \omega(X_H) \underset{\uparrow \text{Symplecticness}}{=} \omega(\psi^* X_H)$$

# Symplectomorphisms and the flow


## Fact

If for $t \in \mathbb{R}$, the flow function $\Phi_H^t$ is well-defined, it is a symplectomorphism onto its image.

It is a diffeomorphism with smooth inverse $\Phi_H^{-t}$, which is defined on $\operatorname{im} \Phi_H^t$.
That it is symplectic is the fact that $X_H$ is symplectic.

follows from ODE

# Electromagnetism

Consider the vector potential of the electromagnetic force $(\phi, \vec{A})$ generating the Lorentz force

$$\vec{F}(r, \vec{v}, t) = q \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right) = q \cdot \left(-\vec{\nabla}\phi - \underbrace{\frac{\partial \vec{A}}{\partial t}}_{\text{ignore this}} + \vec{v} \times \vec{\nabla} \times \vec{A}\right).$$

This can be described by the Lagrangian

$H(q, p, t)$

$\frac{d}{dt} f = \{H, f\} + \frac{\partial f}{\partial t}$

$$L(r, \vec{v}) = \frac{m}{2}|\vec{v}|^2 - q \cdot (\phi(r) - \vec{v} \cdot \vec{A}(r)) = \frac{1}{2} m(\vec{v}, \vec{v}) - q\phi + q\overleftarrow{A}\vec{v}$$

$\overleftarrow{A} = g(\vec{A})$

giving the canonically conjugated momentum

$$\overleftarrow{p} = \frac{\partial L}{\partial \vec{v}} = m(\vec{v}) + q\overleftarrow{A} \Leftrightarrow \vec{v} = m^{-1}(\overleftarrow{p} - q\overleftarrow{A})$$

and the Hamiltonian $(\overleftarrow{p}\vec{v} - L)$

$-\frac{\partial H}{\partial q^i}$

$$H = \frac{1}{2} m(\vec{v}, \vec{v}) + q\phi = \frac{1}{2} m^{-1}(\overleftarrow{p} - q\overleftarrow{A}, \overleftarrow{p} - q\overleftarrow{A}) + q\phi$$

# Twisted contangent bundle

A **twisted cotangent bundle** is $T^*N$ as a symplectic manifold, but the symplectic form differs from $\mathrm{d}\lambda$:

$$\omega = \mathrm{d}\lambda + \sigma.$$

$d\rho = \omega(e_x) + \sigma(e_x)$

This gives rise to a kind of "Lorentz force" $Y : TN \xrightarrow{\text{linear}} TN$,

$$g(\vec{Y}(\vec{v})) = \sigma(\vec{v})$$

$\vec{v} \longmapsto \sigma\vec{v} \in T^*N$

$Y \leftarrow g$

This is an alternative approach to the magnetism problem from the last slide.

$Y = (\vec{B} \times \cdot)$