Lecture 1: Review of Differential Topology

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April 15, 2021

Sheaf-theoretic view

$\mathcal{F}: \mathcal{O}(X) \longrightarrow \mathrm{Vect}_{\mathbb{R}}$.

$\mathcal{F} = C^k$, X is a C^k-manifold

$\mathcal{F}(U) := C^k(\cancel{X}U)$, if $U \subseteq V$, then define

$\mathcal{F}(U \hookrightarrow V): C^k(V) \longrightarrow C^k(U)$

$f \longmapsto f|_U$.

Sheaves

Definition (Presheaf)

Let $\mathscr{C}$ be a category and X a topological space. A ***presheaf on X with values in $\mathscr{C}$*** is a contravariant functor $\underline{\mathcal{O}(X)} \to \mathscr{C}$, where $\mathcal{O}(X)$ denotes the poset category of open subsets of X.

$\mathscr{C} = \mathrm{Vect}_{\mathbb{R}}$ a category has objects and morphisms.

real vector spaces

linear maps

$$\begin{tikzcd} \mathbb{R}^n \arrow[loop left, "\mathrm{id}"] \arrow[r, bend left, "f"] \arrow[r, bend right] \arrow[rr, bend left=40, "g \circ f"] & P_m \arrow[r, bend left, "g"] & C^\infty(M) \end{tikzcd}$$

$U \subseteq U$

$$\begin{tikzcd} U \arrow[loop left] \arrow[r] & V \arrow[r] & W \end{tikzcd}$$

if and only if $U \subseteq V \subseteq W$

U, W, V open subsets of X.

Definition (Sheaf)

Let X be a topological space. A ***sheaf on X*** is defined to be a presheaf

$$F : \mathcal{O}(X) \to \mathsf{Vect}$$

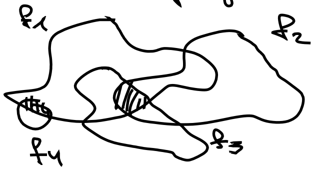
satisfying the following ***gluing condition*** for all $U \in \mathcal{O}(X)$.
Given any open cover $(U_\alpha)_{\alpha \in A}$ of U together with $f_\alpha \in F(U_\alpha)$ for all $\alpha \in A$ such that

$$f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta} \qquad \forall \alpha, \beta \in A,$$

then there exists a unique $f \in F(U)$ such that $f|_{U_\alpha} = f_\alpha$ for all $\alpha \in A$.

$C^k : \mathcal{O}(X) \to \mathsf{Vect}_{\mathbb{R}}$ $\quad 0 \leq k \leq \infty$

Sheaf needs to fulfill gluing condition.



Then $\exists!\, f \in C^k(U)$
s.t. $f|_{U_\alpha} = f_\alpha$.

dimension of M

Let $\pi\colon E \to M^n$ be a smooth vector bundle. Then

C^∞ $\qquad$ $\sigma : U \to \pi^{-1}(U)$

σ

$$\pi \circ \sigma = \mathrm{id}_M \qquad \Gamma_E\colon \mathcal{O}(M) \to \mathsf{Vect}, \qquad \Gamma_E(U) := \Gamma(U, E)$$

infinite dimensional

is a sheaf on M. In this talk we will focus on the sheaves

$$\mathcal{T}_M := \bigoplus_{k,l \geq 0} \mathcal{T}_M^{k,l} \qquad \text{and} \qquad \Omega_M := \bigoplus_{0 \leq k \leq n} \Omega_M^k$$

on M, the ***total sheaf of tensor fields on M*** and the ***total sheaf of differential forms on M***, respectively, where

$$\mathcal{T}_M^{k,l} := \Gamma_{T^{(k,l)}TM} \qquad \text{and} \qquad \Omega_M^k := \Gamma_{\Lambda^k(M)}.$$

alternating $T^{(0,k)}TM$

tensors

More concretely, for every $U \in \mathcal{O}(M)$ there is a canonical identification between $\Omega^k_M(U)$ and alternating $C^\infty(U)$-multilinear maps

$\omega(fX,Y) = f\omega(X,Y)$, $f \in C^\infty(U)$

$\omega \in \Omega^2(U)$

$$\underbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}_{k} \to C^\infty(U)$$

$\omega(X,Y)|_x := \omega_x(X_x, Y_x)$
$\forall x \in U$, $X, Y \in \mathfrak{X}(U)$.

Alternating: $\omega(Y,X) = -\omega(X,Y)$

and likewise between $\mathcal{T}^{k,l}_M(U)$ and $C^\infty(U)$-multilinear maps

$$\underbrace{\Omega^1(U) \times \cdots \times \Omega^1(U)}_{k} \times \underbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}_{l} \to C^\infty(U).$$

$(T^*M)^* \cong TM$

$T^{(k,l)}TM = \underbrace{TM \otimes \cdots \otimes TM}_{k} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{l}$

The Lie Derivative

The Lie derivative generalises the directional derivative of a function to arbitrary tensor fields on a smooth manifold. Let M be a smooth manifold and $A \in \mathcal{T}^{k,l}(M)$. Then for any $X \in \mathfrak{X}(M)$ we define the ***Lie derivative of A with respect to X*** to be the tensor field $\mathcal{L}_X A \in \mathcal{T}^{k,l}(M)$ given by

$$\mathcal{L}_X A := \left.\frac{d}{dt}\right|_{t=0} \theta_t^* A.$$

Flow of X

Definition (Tensor Derivation)

A ***tensor derivation on a smooth manifold M*** is defined to be a sheaf morphism $\mathcal{D} : \mathcal{T}_M \to \mathcal{T}_M$ that preserves type and satisfies: $\mathcal{D}(A) \in \mathcal{T}^{k,l}$ when $A \in \mathcal{T}^{k,l}$

- For all $U \in \mathcal{O}(M)$, $\mathcal{D}_U$ commutes with all contractions of $\mathcal{T}_M(U)$.
- For all $U \in \mathcal{O}(M)$, $\mathcal{D}_U$ is a derivation, that is

natural transformation

$$\mathcal{D}_U(A \otimes B) = \mathcal{D}_U A \otimes B + A \otimes \mathcal{D}_U B$$

holds for all $A, B \in \mathcal{T}(U)$.

Sheaf morphism:

$$\begin{array}{ccc} \mathcal{O}(M) & \xrightarrow{\mathcal{T}_M} & \mathrm{Vect} \\ & & \big\downarrow \mathcal{L}_X \\ \mathcal{O}(M) & \xrightarrow[\mathcal{T}_M]{} & \mathrm{Vect} \end{array}$$

$$\underbrace{\left(\mathcal{T}^{k,l}(U) \xrightarrow{\mathcal{D}} \mathcal{T}^{k,l}(U)\right)}_{U \in \mathcal{O}(M)}$$

$U \subseteq V$, then $\mathcal{L}_X A|_U = (\mathcal{L}_X A)|_U$

Lemma (Contraction Lemma)

Let $\mathfrak{D}$ be a tensor derivation, $U \in \mathcal{O}(M)$ and $A \in \mathcal{T}^{k,l}(U)$. Then for all $\omega^1, \ldots, \omega^k \in \Omega^1(U)$ and $X_1, \ldots, X_l \in \mathfrak{X}(U)$ we have that

$$
\begin{aligned}
\mathfrak{D}_U(A)\left(\omega^1, \ldots, \omega^k, X_1, \ldots, X_l\right) = \mathfrak{D}_U\left(A\left(\omega^1, \ldots, \omega^k, X_1, \ldots, X_l\right)\right) \\
- \sum_{i=1}^{k} A\left(\omega^1, \ldots, \mathfrak{D}_U\left(\omega^i\right), \ldots, \omega^k, X_1, \ldots, X_l\right) \\
- \sum_{i=1}^{l} A\left(\omega^1, \ldots, \omega^k, X_1, \ldots, \mathfrak{D}_U(X_i), \ldots, X_l\right).
\end{aligned}
$$

Observe: $\mathfrak{D}$ only depends on functions, vector fields and covector fields.

Theorem

Let $\mathfrak{D}$ and $\mathfrak{D}'$ be two tensor derivations on a smooth manifold which agree on functions and vector fields. Then $\mathfrak{D} = \mathfrak{D}'$.

Proof. $\omega \in \Omega^1(U)$.

$$\mathfrak{D}_U(\omega)(X) = \mathfrak{D}_U(\underbrace{\omega(X)}_{\in C^\infty(U)}) - \omega(\underbrace{\mathfrak{D}_U(X)}_{\in \mathfrak{X}(U)})$$

$\square$

By theorem 5 the Lie derivative is the unique tensor derivation such that

$$\mathcal{L}_X f = Xf \qquad \text{and} \qquad \mathcal{L}_X Y = [X, Y]$$

for all $f \in C^\infty(M)$ and $X, Y \in \mathfrak{X}(M)$.

$$[X,Y] = XY - YX$$

- $\mathcal{L}_X(fg) = (\mathcal{L}_X f)g + f\mathcal{L}_X g$
- $\mathcal{L}_X(fY) = (\mathcal{L}_X f)Y + f\mathcal{L}_X Y$

The Exterior Differential

Definition

Let M be a smooth manifold and $l \in \mathbb{Z}$. A ***graded derivation of degree l on M*** is defined to be a sheaf morphism $\mathcal{D} : \Omega_M \to \Omega_M$ satisfying:

- If $\omega \in \Omega^k(U)$, then $\mathcal{D}_U(\omega) \in \Omega^{k+l}(U)$.
- If $\omega \in \Omega^k(U)$ and $\eta \in \Omega(U)$, then

$$\mathcal{D}_U(\omega \wedge \eta) = \mathcal{D}_U(\omega) \wedge \eta + (-1)^{kl} \omega \wedge \mathcal{D}_U(\eta).$$

Lemma

Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. Then the Lie derivative $\mathcal{L}_X$ is a graded derivation of degree 0.

$$\mathcal{L}_X(\omega \wedge \eta) = \frac{d}{dt}\Big|_{t=0} \theta_t^*(\omega \wedge \eta)$$

$$= \frac{d}{dt}\Big|_{t=0} \theta_t^*\omega \wedge \theta_t^*\eta$$

$\theta_0 = \mathrm{id}$

$$= \frac{d}{dt}\Big|_{t=0} \theta_t^*\omega \wedge \eta + \omega \wedge \frac{d}{dt}\Big|_{t=0} \theta_t^*\eta$$

$$= \mathcal{L}_X\omega \wedge \eta + \omega \wedge \mathcal{L}_X\eta.$$

$\square$

Theorem

Let M be a smooth manifold and suppose that $\mathfrak{D}$ and $\mathfrak{D}'$ are two graded derivations on M of the same degree which coincide on functions and exact 1-forms. Then $\mathfrak{D} = \mathfrak{D}'$.

$$\omega = \omega_I \, dx^I \qquad I = (i_1, \dots, i_k)$$

in a chart $\qquad C^\infty(U) \qquad i_1 < \dots < i_k$

Theorem (The Exterior Differential)

Let M be a smooth manifold. Then there exists a unique graded derivation $d : \Omega_M \to \Omega_M$ of degree 1 such that

$$d_U(f) = df \qquad \textit{and} \qquad d \circ d = 0$$

*holds for all $f \in C^\infty(U)$. This graded derivation is called the **exterior differential**.*

$$d_U \omega := d\omega_I \wedge dx^I.$$

Check for yourself.

Theorem (Cartan's Magic Formula)

Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. Then

$$\mathcal{L}_X = d \circ i_X + i_X \circ d.$$

Proof. Observe that both sides are graded derivations of degree 0.

- let $f \in C^\infty(U)$.

$$\mathcal{L}_X f = Xf \qquad i_X df = df(X) = Xf.$$

- $$\mathcal{L}_X df = \frac{d}{dt}\Big|_{t=0} \theta_t^* df = \frac{d}{dt}\Big|_{t=0} d\theta_t^* f$$
$$= d(\mathcal{L}_X f) = d(X(f)) = d(i_X df)$$

$\square$

Fisherman's Format

$$\underset{\Omega^{k,l}(U)}{\uparrow}$$

$$\frac{d}{dt}\theta_t^* A = \frac{d}{ds}\Big|_{s=0} \theta^*_{s+t} A$$

$\downarrow$ flow property

$$= \frac{d}{ds}\Big|_{s=0} \theta_t^* \circ \theta_s^* A$$

$$= \theta_t^* \mathcal{L}_X A .$$

Theorem (Fisherman's Formula)

Let M be a smooth manifold and suppose that $X : I \times M \to TM$ is a time-dependent vector field with time-dependent flow $\psi : \mathcal{D} \to M$. Then

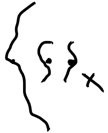
flow domain $\mathcal{D} \subseteq \mathbb{R} \times M$

$$\frac{d}{dt}\psi_t^*\omega = \psi_t^*\mathcal{L}_{X_t}\omega \qquad \forall \omega \in \Omega_M.$$

$\forall x \in M$ $\{t \in \mathbb{R} : (t,x) \in \mathcal{D}\}$ is an open interval containing 0

Proof. Check again on functions and exact 1-forms

- $\frac{d}{dt}\psi_t^* f = \frac{d}{dt} f \circ \psi_t = df\left(\frac{d}{dt}\psi_t\right) = df(X_t \circ \psi_t)$



- $\frac{d}{dt}\psi_t^* df = d\left(\frac{d}{dt}\psi_t^* f\right)$

The Tangent-Cotangent Bundle Isomorphism

Definition (Nondegenerate Bilinear Form)

Let V be a finite-dimensional real vector space. A skew-symmetric bilinear form $\omega\colon V \times V \to \mathbb{R}$ is said to be ***nondegenerate***, iff the map $\widehat{\omega}\colon V \to V^*$ defined by $\widehat{\omega}(v) := i_v\omega$ is an isomorphism.

$$V \overset{\wedge}{=}_\omega V^*$$

Lemma

Let V be a finite-dimensional real vector space and $\omega\colon V \times V \to \mathbb{R}$ skew-symmetric. Then the following statements are equivalent:

- ω *is symplectic*/ nondegenerate
- *With respect to any basis for V, the matrix representing $\hat{\omega}$ is invertible.*
- *If $\omega(v,u) = 0$ for all $u \in V$, then $v = 0$.* any nonzero vector has a friend
- *If $v \neq 0$, then there exists some $u \in V$ such that $\omega(v,u) \neq 0$.*
- *The matrix representing ω in any basis of V is invertible.*

Theorem (Tangent-Cotangent Bundle Isomorphism)

Let M be a smooth manifold and $\omega \in \Omega^2(M)$ nondegenerate. Define

$$\hat{\omega}: TM \to T^*M, \qquad \hat{\omega}(v)(w) := \omega_x(v,w) = i_v(\omega)$$

for all $x \in M$ and $v, w \in T_xM$. Then $\hat{\omega}$ is a bundle isomorphism. The morphism $\hat{\omega}$ is called the ***tangent-cotangent bundle isomorphism****.*

$$\hat{\omega}: \mathfrak{X}(M) \xrightarrow{\cong} \Omega^1(M)$$
$$X \mapsto i_X\omega$$

Can define gradient of a smooth function $f \in C^\infty(M)$:

$$\operatorname{grad}_\omega f := \hat{\omega}^{-1}(df) \in \mathfrak{X}(M)$$

Lemma

Let M be a smooth manifold, $\omega \in \Omega^2(M)$ nondegenerate and $\lambda \in \Omega^1(M)$. Then there exists a unique vector field $X \in \mathfrak{X}(M)$ such that

$$i_X\omega = \lambda.$$

Proof. $X := \hat{\omega}^{-1}(\lambda)$. Then

$$i_X \omega = \hat{\omega}(X) = \hat{\omega}(\hat{\omega}^{-1}(\lambda)) = \lambda.$$

$\square$

Riemannian case:

$\hat{m}$ is referred to as a musical isomorphism

$\#$ $\flat$

Riemannian metric

- know how to identify tangent and cotangent bundle.

- know definitions of Lie derivative
- know definition of exterior differential
- know how to apply uniqueness to prove formulas