Yannis Bähni

Mathematical Aspects of Classical Mechanics

Semester Thesis

April 13, 2019

Springer Nature

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

To Jil.

Preface

These notes are the product of two semester projects done at *ETH Zurich* in the academic year 2018/2019 under the supervision of *Prof. Dr. Ana Cannas da Silva*. The aim of these notes is to give a thoughtful introduction to the mathematical methods used in the realm of classical mechanics and their strong connection to differential topology and differential geometry, especially *symplectic geometry*. I will roughly follow the first chapter of the book *Quantum Mechanics for Mathematicians* by *Leon A. Takhtajan* [19], which serves as an introduction to classical mechanics. As the title already suggests, this is not a treatment of the physical part of classical mechanics, but rather a mathematical one. Finally, I make use of the book *Lectures on Symplectic Geometry* by *Ana Cannas da Silva* [15].

Happy reading!

Winterthur, *Yannis Bähni*
September 2018

Acknowledgements

I would like to thank first of all my supervisor Prof. Dr. Cannas da Silva for granting me this opportunity of writing these notes, and also for introducting me to symplectic geometry back in the autumn semester 2017. Moreover, I would like to thank *Prof. Dr. Will J. Merry*, whose brilliant lectures on *Algebraic Topology* as well as *Differential Geometry* helped me alot in understanding this and related subjects. Also, he was a great help in answering questions and clarifying concepts.

A big help was also the marvelous trilogy of books from *John M. Lee* ([5], [6] and [4]), which clear, thoughtful and highly formal exposition of the subject give an in-depth understanding of the matter. I won't deny the obvious: My style of writing and even the typeset of this document is highly inspired, sometimes even copied, from the style used by Jack Lee. The simple reason is, that I appreciate his work very much and try to achieve the same fineness. A prominent indicator of this fact is also the numerous citations of his books in these notes.

Contents

Chapter 1
Lagrangian Mechanics

Classical mechanics models systems of finitely many interacting particles, that is, material bodies whose dimensions may be neglected by describing their motion.

We begin by giving an example of an variational problem coming from the realm of partial differential equations to motivate the methods used later in this chapter. They are guided by a variational principle, which is of equal importance in both physics and mathematics.

Then we introduce the basic notions of Lagrangian mechanics, that are Lagrangian systems and the action functional associated to them. We derive the equations of motion of general Lagrangian systems, the Euler-Lagrange equations.

Next we introduce the dual notion of a Lagrangian function, that is the associated Hamiltonian function which is obtained via a Legendre transformation.

Then we introduce the most important theorem in Lagrangian mechanics concerning symmetries: Noether's theorem.

We conclude by giving a criterion for determining whether a certain Legendre transform is a diffeomorphism or not, since this is crucial for the dualisation process.

1.1 Introduction

Classical mechanics deals with ordinary differential equations originating from extremals of ***functionals***, that is functions defined on an infinite-dimensional function space. The study of such extremality properties of functionals is known as the ***calculus of variations***. To illustrate this fundamental principle, let us consider the *variational formulation* of second order elliptic operators in divergence form based on [18, 167–168].

For convention, unless explicietly stated otherwise, we will assume that all manifolds are smooth, that is of class C^∞, finite-dimensional, Hausdorff and paracompact with at most countably many connected components. Moreover, we use the Einstein summation convention.

Let $n \in \mathbb{N}$, $n \geq 1$, and $\Omega \subseteq\subseteq \mathbb{R}^n$ such that $\overline{\Omega}$ is a smooth manifold with boundary.

Moreover, let $H_0^1(\Omega)$ denote the Sobolev space $W_0^{1,2}(\Omega)$ with inner product

$$\langle u, v\rangle_{H_0^1(\Omega)} = \int_\Omega uv + \int_\Omega \nabla u \nabla v.$$

Suppose $a^{ij} \in C^\infty(\bar{\Omega})$ symmetric, $f \in C^\infty(\bar{\Omega})$ and consider the second order homogenous Dirichlet problem

$$\begin{cases} -\dfrac{\partial}{\partial x^j}\left(a^{ij}\dfrac{\partial u}{\partial x^i}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

Suppose $u \in C^\infty(\bar{\Omega})$ solves (1.1). Then integration by parts (see [6, 436]) yields

$$\begin{aligned} \int_\Omega fv &= -\int_\Omega \frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial u}{\partial x^i}\right) v \\ &= -\int_\Omega \operatorname{div}(X) v \\ &= \int_\Omega \langle X, \nabla v\rangle \\ &= \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^j} \end{aligned}$$

for any $v \in C_c^\infty(\Omega)$, where $X := \left(a^{ij}\frac{\partial u}{\partial x^i}\right)_j$. Thus we say that $u \in H_0^1(\Omega)$ is a *weak solution* of (1.1) iff

$$\forall v \in C_c^\infty(\Omega)\colon \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^j} = \int_\Omega fv.$$

If $(a^{ij})_{ij}$ is *uniformly elliptic*, i.e. there exists $\lambda > 0$ such that

$$\forall x \in \Omega \forall \xi \in \mathbb{R}^n\colon\ a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2,$$

then (1.1) admits a unique weak solution $u \in H_0^1(\Omega)$ (in fact $u \in C^\infty(\Omega)$ using *regularity theory*, for more details see [18, 175]). Indeed, observe that

$$\langle\cdot,\cdot\rangle_a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

defined by

$$\langle u, v\rangle_a := \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^j} \tag{1.2}$$

is an inner product on $H_0^1(\Omega)$ with induced norm equivalent to the standard one on $H_0^1(\Omega)$ due to Poincaré's inequality [18, 107]. Applying the Riesz Representation theorem [18, 49–50] yields the result. Moreover, this solution can be characterized by a *variational principle*, i.e. if we define the *energy functional* $E : H_0^1(\Omega) \to \mathbb{R}$

$$E(v) := \frac{1}{2}\|v\|_a^2 - \int_\Omega f v,$$

for any $v \in H_0^1(\Omega)$, where $\|\cdot\|_a$ denotes the norm induced by the inner product (1.2), then $u \in H_0^1(\Omega)$ solves (1.1) if and only if

$$E(u) = \inf_{v \in H_0^1(\Omega)} E(v). \tag{1.3}$$

Indeed, suppose $u \in H_0^1(\Omega)$ is a solution of (1.1). Let $v \in H_0^1(\Omega)$. Then $u = v + w$ for $w := u - v \in H_0^1(\Omega)$ and we compute

$$\begin{aligned} E(v) &= E(u+w) \\ &= \frac{1}{2}\|u\|_a^2 + \langle u, w\rangle_a + \frac{1}{2}\|w\|_a^2 - \int_\Omega f(u+w) \\ &= E(u) + \frac{1}{2}\|w\|_a^2 \\ &\geq E(u) \end{aligned}$$

with equality if and only if $u = v$ a.e. Conversely, suppose the infimum is attained by some $u \in H_0^1(\Omega)$. Thus by elementary calculus

$$0 = \left.\frac{d}{dt}\right|_{t=0} E(u+tv) = \langle u, v\rangle_a - \int_\Omega f v \tag{1.4}$$

for all $v \in H_0^1(\Omega)$.

Suppose now that $u \in C^\infty(\bar{\Omega})$ with $u|_{\partial\Omega} = 0$ solves the variational formulation (1.3). Then again integration by parts yields

$$\langle u, v\rangle_a - \int_\Omega f v = -\int_\Omega \operatorname{div}(X) v - \int_\Omega f v = \int_\Omega \left(-\frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial u}{\partial x^i}\right) - f\right) v$$

for all $v \in C_c^\infty(\Omega)$ and where $X := \left(a^{ij}\frac{\partial u}{\partial x^i}\right)_j$. Hence (1.4) implies

$$\forall v \in C_c^\infty(\Omega) : \int_\Omega \left(-\frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial u}{\partial x^i}\right) - f\right) v = 0.$$

We might expect that this implies

$$-\frac{\partial}{\partial x^j}\left(a^{ij}\frac{\partial u}{\partial x^i}\right) = f.$$

That this is indeed the case, is guaranteed by a foundational result in the *calculus of variations* (therefore the name).

Proposition 1.1 (Fundamental Lemma of Calculus of Variations [18, 40]). *Let $\Omega \subseteq \mathbb{R}^n$ open and $f \in L^1_{\mathrm{loc}}(\Omega)$. If*

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega f\varphi = 0,$$

then $f = 0$ *a.e.*

Thus we recovered a second order partial differential equation from the variational formulation. In fact, this is exactly the boundary value problem (1.1) from the beginning of our exposition. This technique, and in particular the fundamental lemma of calculus of variations 1.1 will play an important role in our treatment of classical mechanics. However, since we are concerned with smooth manifolds only, we use a version of the fundamental lemma of calculus of variations 1.1, which is fairly easy to prove and hence really deserves the terminology "lemma".

Lemma 1.2 (Fundamental Lemma of Calculus of Variations, Smooth Version). *Let* $\Omega \subseteq \mathbb{R}^n$ *open and* $f \in C^\infty(\Omega)$. *If*

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega f\varphi = 0,$$

then $f = 0$.

Proof. Towards a contradiction, assume that $f \neq 0$ on Ω. Thus there exists $x_0 \in \Omega$, such that $f(x_0) \neq 0$. Without loss of generality, we may assume that $f(x_0) > 0$, since otherwise, consider $-f$ instead of f. The smoothness of f implies the continuity of f on Ω. Thus there exists $\delta > 0$, such that $f(x) \in B_{f(x_0)/2}\big(f(x_0)\big)$ holds for all $x \in B_\delta(x_0)$ or equivalently, $f(x) > f(x_0)/2 > 0$ for all $x \in B_\delta(x_0)$. By lemma 2.22 [6, 42], there exists a smooth bump function φ supported in $B_\delta(x_0)$ and $\varphi = 1$ on $\overline{B}_{\delta/2}(x_0)$. In particular, $\varphi \in C_c^\infty(\Omega)$. Therefore we have

$$0 = \int_\Omega f\varphi = \int_{B_\delta(x_0)} f\varphi \geq \int_{B_{\delta/2}(x_0)} f\varphi > \frac{1}{2} f(x_0)\,|B_{\delta/2}(x_0)| > 0,$$

which is a contradiction. □

Exercise 1.3. [1] Let $\Omega \subseteq\subseteq \mathbb{R}^n$, $2 \leq p < \infty$ and define $\mathcal{B} := \{v \in C^\infty(\bar{\Omega}) : v|_{\partial\Omega} = 0\}$. Moreover, define $E_p : \mathcal{B} \to \mathbb{R}$ by $E_p(v) := \int_\Omega |\nabla v|^p$. Derive the partial differential equation satisfied by minimizers $u \in \mathcal{B}$ of the variational problem $E(u) = \inf_{v \in \mathcal{B}} E(v)$.

1.2 Lagrangian Systems and the Principle of Least Action

Mechanical systems, for example a pendulum, are modelled using the language of differential geometry. Thus it is necessary to introduce the relevant physical counterparts.

[1] This is exercise $1.2.(b)$ from exercise sheet 1 of the course *Functional Analysis II* taught by *Prof. Dr. A. Carlotto* at ETHZ in the spring of 2018, which can be found here.

Definition 1.4 (Configuration Space). A ***configuration space*** is defined to be a finite-dimensional smooth manifold.

Definition 1.5 (Motion). A ***motion in a configuration space*** $\boldsymbol{M}$ is defined to be a path $\gamma \in C^\infty(J, M)$, where $J \subseteq \mathbb{R}$ is an interval.

Definition 1.6 (State). A ***state of the configuration space*** is defined to be an element of the tangent bundle of the configuration space, called the ***state space***.

One should think of a state (x, v) of a configuration space as follows: x gives the position of the mechanical system and v its velocity at this position. The fundamental principle governing motions of mechanical systems is the following.

Axiom 1.7 (Newton-Laplace Determinacy Principle). A motion in a configuration space is completely determined by a state at some instant of time.

The Newton-Laplace determinacy principle 1.7 motivates our main definition of this chapter.

Definition 1.8 (Lagrangian System). A ***Lagrangian system*** is defined to be a tuple (M, L) consisting of a smooth manifold M and a function $L \in C^\infty(TM \times \mathbb{R})$, called a ***Lagrangian function***.

Example 1.9 (Lagrangian System). Let $T \in C^\infty(TM \times \mathbb{R})$ and $V \in C^\infty(M \times \mathbb{R})$. Define $L \in C^\infty(TM \times \mathbb{R})$ by $L := T - V$. In this situation, T is called the ***kinetic energy*** and V is called the ***potential energy***.

Definition 1.10 (Path Space). Let M be a smooth manifold. For $x_0, x_1 \in M$ and $t_0, t_1 \in \mathbb{R}$ with $t_0 \leq t_1$, define the ***path space of M connecting*** $\boldsymbol{(x_0, t_0)}$ ***and*** $\boldsymbol{(x_1, t_1)}$ to be the set

$$\mathcal{P}(M)^{x_0,t_0}_{x_1,t_1} := \left\{\gamma \in C^\infty\left([t_0, t_1], M\right) : \gamma(t_0) = x_0 \text{ and } \gamma(t_1) = x_1\right\}. \tag{1.5}$$

Remark 1.11. For the sake of simplicity, we will just use the terminology *path space* for $\mathcal{P}(M)^{x_0,t_0}_{x_1,t_1}$ and simply write $\mathcal{P}(M)$. However, we implicitely assume the conditions of definition 1.10.

Definition 1.12 (Variation). Let $\mathcal{P}(M)$ be a path space and $\gamma \in \mathcal{P}(M)$. A ***variation of*** $\boldsymbol{\gamma}$ is defined to be a morphism $\Gamma \in C^\infty\left([t_0, t_1] \times [-\varepsilon_0, \varepsilon_0], M\right)$ for some $\varepsilon_0 > 0$ and such that

- $\Gamma(t, 0) = \gamma$ for all $t \in [t_1, t_0]$.
- $\Gamma(t_0, \varepsilon) = x_0$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.
- $\Gamma(t_1, \varepsilon) = x_1$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

Remark 1.13. If Γ is a variation of $\gamma \in \mathcal{P}(M)$, we write $\gamma_\varepsilon(\cdot) := \Gamma(\cdot, \varepsilon)$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. With this notation, $\gamma_\varepsilon \in \mathcal{P}(M)$ for all $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$.

Example 1.14 (Perturbation of a Path along a Single Direction). Let M^n be a smooth manifold, (U, φ) a chart on M and suppose that γ is a path in U. With respect to this chart, we can write the coordinate representation of γ as

$$\gamma(t) = \left(\gamma^1(t), \dots, \gamma^n(t)\right)$$

for any $t \in [t_0, t_1]$. Let $f \in C_c^\infty(t_0, t_1)$. Consider the family

$$\Gamma : [t_0, t_1] \times [-\varepsilon_0, \varepsilon_0] \to M$$

defined by

$$\Gamma(t, \varepsilon) := (\iota \circ \varphi^{-1}) \left(\gamma^1(t), \dots, \gamma^i(t) + \varepsilon f(t), \dots, \gamma^n(t)\right)$$

where $\iota : U \hookrightarrow M$ denotes inclusion and $\varepsilon_0 > 0$ is to be determined. Suppose $\|f\|_\infty \neq 0$. By exercise 1.15, there exists $\delta > 0$ such that

$$U_\delta := \left\{x \in \mathbb{R}^n : \operatorname{dist}\left(x, \gamma([t_0, t_1])\right) < \delta\right\} \subseteq \varphi(U).$$

Choose $\varepsilon_0 > 0$ such that $0 < \varepsilon_0 < \delta / \|f\|_\infty$. Then in coordinates

$$\operatorname{dist}\left(\gamma_\varepsilon(t), \gamma([t_0, t_1])\right) \leq |\gamma_\varepsilon(t) - \gamma(t)| \leq |\varepsilon| \|f\|_\infty \leq \varepsilon_0 \|f\|_\infty < \delta$$

for all $t \in [t_0, t_1]$. Hence $\gamma_\varepsilon(t) \in U_\delta$ and thus $\gamma_\varepsilon(t) \in \varphi(U)$. Therefore, Γ is indeed well-defined. Moreover, it is easy to show that the properties of definition 1.12 holds, therefore, Γ is a variation of γ. In fact, this example shows, that any path γ contained in a single chart admits infinitely many variations. An example of such a variation is shown in figure 1.1.

Exercise 1.15. Let (X, d) be a metric space and $A \subseteq U \subseteq X$ where U is open in X and A is closed in X. Then there exists $\delta > 0$ such that

$$U_\delta := \{x \in X : \operatorname{dist}(x, A) < \delta\} \subseteq U.$$

Definition 1.16 (Action Functional). Let (M, L) be a Lagrangian system and $\mathcal{P}(M)$ be a path space. The morphism $S : \mathcal{P}(M) \to \mathbb{R}$ defined by

$$S(\gamma) := \int_{t_0}^{t_1} L\left(\gamma(t), \dot{\gamma}(t), t\right) dt$$

is called the ***action functional associated to the Lagrangian system*** $(\boldsymbol{M}, \boldsymbol{L})$.

Motions of Lagrangian systems are characterized by an axiom.

Axiom 1.17 (Hamilton's Principle of Least Action). Let (M, L) be a Lagrangian system and $\mathcal{P}(M)$ be a path space. A path $\gamma \in C^\infty([t_0, t_1], M)$ describes a motion of (M, L) between (x_0, t_0) and (x_1, t_1) if and only if

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\gamma_\varepsilon) = 0 \tag{1.6}$$

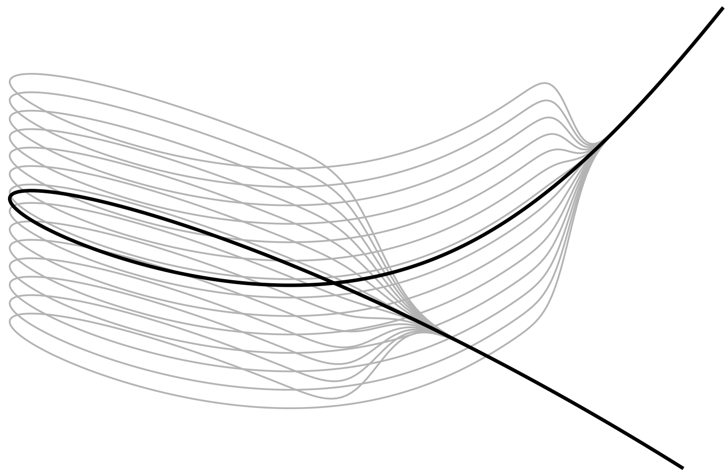


Fig. 1.1: Example of a variation of the path $\gamma(t) = (\gamma^1(t), \gamma^2(t))$ in $\mathbb{R}^2$ defined by $\gamma(t) := (t^2 + \sin(t)\cos(t), t^3 - t)$ for $t \in [-\frac{3}{2}, \frac{3}{2}]$ along the second coordinate using a smooth bump function as in [6, 42].

for all variations γ_ε of γ.

Definition 1.18 (Extremal). A motion of a Lagrangian system between two points is called an ***extremal of the action functional S***.

The Newton-Laplace determinacy principle 1.7 implies that motions of mechanical systems can be described as solutions of second order ordinary differential equations. That this is indeed the case, is shown by the next theorem. But first, let us fix some notation. Let M^n be a smooth manifold and (U, φ) be a chart on M with coordinates (x^i). In what follows, we will use the abbreviation

$$\frac{\partial}{\partial x} := \left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right),$$

where as usual $\frac{\partial}{\partial x^i} : U \to TM$ denotes the i*-th coordinate vector field*, that is

$$\frac{\partial f}{\partial x^i}(x) := \left.\frac{\partial}{\partial x^i}\right|_x f = \partial_i (f \circ \varphi^{-1}) \big(\varphi(x)\big),$$

for all $i = 1, \dots, n$, $x \in U$ and $f \in C^\infty(M)$. Also recall, that on this chart

$$df_x = \frac{\partial f}{\partial x^i}(x) dx^i|_x \tag{1.7}$$

holds for all $x \in U$ (see [6, 281]). Additionally, we need the following proposition.

Proposition 1.19 (Derivative of a Function along a Curve [6, 283]). *Suppose M is a smooth manifold, $J \subseteq \mathbb{R}$ an interval, $\gamma \in C^\infty(J, M)$ a curve on M and $f \in C^\infty(M)$. Then for all $t \in J$ holds*

$$(f \circ \gamma)'(t) = df_{\gamma(t)}\left(\gamma'(t)\right).$$

Theorem 1.20 (Euler-Lagrange Equations). *Let (M^n, L) be a Lagrangian system. A path $\gamma \in C^\infty([t_0, t_1], M)$ describes a motion of (M, L) between (x_0, t_0) and (x_1, t_1) if and only if with respect to all charts (U, x^i)*

$$\frac{\partial L}{\partial x}\left(\gamma(t), \dot{\gamma}(t), t\right) = \frac{d}{dt}\frac{\partial L}{\partial v}\left(\gamma(t), \dot{\gamma}(t), t\right) \tag{1.8}$$

holds, where (x^i, v^i) denotes the standard coordinates on TM. The system of equations (1.8) is referred to as the ***Euler-Lagrange equations****.*

Proof. By Hamilton's principle of least action 1.17, we may assume that γ is an extremal of the action functional S. The proof is divided into two steps.
Step 1: Suppose that γ is contained in a chart domain U. Let $t \in [t_0, t_1]$ and abreviate $x_t := (\gamma(t), \dot{\gamma}(t), t)$. Suppose $\Gamma : [t_0, t_1] \times [-\varepsilon_0, \varepsilon_0] \to M$ is a variation of γ. Then there exists a rectangle $\mathcal{R}$ such that

$$[t_0, t_1] \times \{0\} \subseteq \mathcal{R} \subseteq [t_0, t_1] \times [-\varepsilon_0, \varepsilon_0]$$

and $\Gamma(\mathcal{R}) \subseteq U$. Indeed, Γ is continuous since Γ is smooth and so $\Gamma^{-1}(U)$ is open in $[t_0, t_1] \times [-\varepsilon_0, \varepsilon_0]$. Since γ is a path in U, we get

$$[t_0, t_1] \times \{0\} \subseteq \Gamma^{-1}(U)$$

by the definition of a variation. By exercise 2.4. (c) [5, 22], the standard Euclidean metric and the *maximum metric* $|\cdot|_\infty$ generate the same topology, thus for all $t \in [t_0, t_1]$ there exists $r_t > 0$ such that

$$B_{r_t}(t, 0) := \left\{(x, \varepsilon) \in [t_0, t_1] \times [-\varepsilon_0, \varepsilon_0] : \max\{|x - t|, |\varepsilon|\} < r_t\right\} \subseteq \Gamma^{-1}(U).$$

Since $[t_0, t_1] \times \{0\}$ is compact in $[t_0, t_1] \times [-\varepsilon_0, \varepsilon_0]$, we find $m \in \mathbb{N}$ such that

$$[t_0, t_1] \times \{0\} \subseteq \bigcup_{i=1}^m B_{r_i}(t_i, 0).$$

Set $r := \min_{i=1,\ldots,m} r_i$ and define $\mathcal{R} := [t_0, t_1] \times (-r, r)$. Then if $(t, \varepsilon) \in \mathcal{R}$ we get that there exists some index i such that $(t, 0) \in B_{r_i}(t_i, 0)$. Hence $|t - t_i| < r_i$ and so

$$|(t, \varepsilon) - (t_i, 0)|_\infty = \max\{|t - t_i|, |\varepsilon|\} < r_i.$$

Thus $(t, \varepsilon) \subseteq B_{r_i}(t_i, 0) \subseteq \Gamma^{-1}(U)$ and so $\Gamma(\mathcal{R}) \subseteq U$. Hence we can write

$$\gamma_\varepsilon(t) = \left(\gamma_\varepsilon^1(t), \ldots, \gamma_\varepsilon^n(t)\right)$$

and

$$\dot{\gamma}_\varepsilon(t) = \left(\dot{\gamma}_\varepsilon^1(t), \ldots, \dot{\gamma}_\varepsilon^n(t)\right)$$

for all $(x, \varepsilon) \in \mathcal{R}$, where the dot denotes a derivative with respect to time.
Using the formula for the derivative of a function along a curve 1.19, we compute

$$\begin{aligned}\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} L\left(\gamma_\varepsilon(t), \dot{\gamma}_\varepsilon(t), t\right) &= dL_{x_t}\left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\gamma_\varepsilon(t), \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\dot{\gamma}_\varepsilon(t), 0\right) \\ &= dL_{x_t}\left(\frac{d\gamma_\varepsilon^j(t)}{d\varepsilon}(0)\frac{\partial}{\partial x^j}\bigg|_{\gamma(t)}, \frac{d\dot{\gamma}_\varepsilon^j(t)}{d\varepsilon}(0)\frac{\partial}{\partial v^j}\bigg|_{\dot{\gamma}(t)}, 0\right).\end{aligned}$$

for all variations γ_ε of γ in U. Moreover, using the formula for the differential of a function on coordinates (1.7) yields

$$dL_{x_t} = \frac{\partial L}{\partial x^i}(x_t)dx^i|_{x_t} + \frac{\partial L}{\partial v^i}(x_t)dv^i|_{x_t} + \frac{\partial L}{\partial t}(x_t)dt|_{x_t}.$$

Therefore

$$\begin{aligned}0 =& \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} S(\gamma_\varepsilon) \\ =& \int_{t_0}^{t_1} \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} L\left(\gamma_\varepsilon(t), \dot{\gamma}_\varepsilon(t), t\right) dt \\ =& \int_{t_0}^{t_1} dL_{x_t}\left(\frac{d\gamma_\varepsilon^j(t)}{d\varepsilon}(0)\frac{\partial}{\partial x^j}\bigg|_{\gamma(t)}, \frac{d\dot{\gamma}_\varepsilon^j(t)}{d\varepsilon}(0)\frac{\partial}{\partial v^j}\bigg|_{\dot{\gamma}(t)}, 0\right) \\ =& \int_{t_0}^{t_1} \frac{\partial L}{\partial x^i}(x_t)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt + \int_{t_0}^{t_1} \frac{\partial L}{\partial v^i}(x_t)\frac{d\dot{\gamma}_\varepsilon^i(t)}{d\varepsilon}(0)dt \\ =& \int_{t_0}^{t_1} \frac{\partial L}{\partial x^i}(x_t)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt + \int_{t_0}^{t_1} \frac{\partial L}{\partial v^i}(x_t)\left(\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)\right)' dt \\ =& \int_{t_0}^{t_1} \frac{\partial L}{\partial x^i}(x_t)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt + \frac{\partial L}{\partial v^i}(x_t)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)\bigg|_{t_0}^{t_1} \\ &- \int_{t_0}^{t_1} \frac{d}{dt}\frac{\partial L}{\partial v^i}(x_t)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt \\ =& \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^i}(x_t) - \frac{d}{dt}\frac{\partial L}{\partial v^i}(x_t)\right)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt\end{aligned}$$

since $\gamma_\varepsilon^i(t_0)$ and $\gamma_\varepsilon^i(t_1)$ are constant by definition of a variation. Let $f \in C_c^\infty(t_0, t_1)$, $j = 1, \ldots, n$ and γ_ε be the variation of γ defined in example 1.14 along the j-th direction. Above computation therefore yields

$$0 = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x^j}(x_t) - \frac{d}{dt}\frac{\partial L}{\partial v^j}(x_t)\right) f(t)dt$$

for all $f \in C_c^\infty(t_0, t_1)$. Hence the fundamental lemma of calculus of variations 1.2 implies

$$\frac{\partial L}{\partial x^j}(x_t) - \frac{d}{dt}\frac{\partial L}{\partial v^j}(x_t) = 0$$

for all $j = 1, \dots, n$.
Conversly, if we assume that the Euler-Lagrange equations (1.8) hold, above computation yields

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\gamma_\varepsilon) = \int_{t_0}^{t_1}\left(\frac{\partial L}{\partial x^i}(x_t) - \frac{d}{dt}\frac{\partial L}{\partial v^i}(x_t)\right)\frac{d\gamma_\varepsilon^i(t)}{d\varepsilon}(0)dt = 0$$

for every variation γ_ε of γ.
Step 2: Suppose that γ is an arbitrary extremal of S. The key technical result used here is the following lemma.

Lemma 1.21 (Lebesgue Number Lemma [5, 194]). *Every open cover of a compact metric space admits a Lebesgue number, i.e. a number $\delta > 0$ such that every subset of the metric space with diameter less than δ is contained in a member of the family.* □

Let $(U_\alpha)_{\alpha\in A}$ be the smooth structure on M, i.e. the maximal smooth atlas. Since γ is continuous, $\left(\gamma^{-1}(U_\alpha)\right)_{\alpha\in A}$ is an open cover for $[t_0, t_1]$. By the Lebesgue number lemma 1.21, this open cover admits a Lebesgue number $\delta > 0$. Let $N \in \mathbb{N}$ such that $(t_1 - t_0)/N < \delta$ and define

$$t_i := \frac{i}{N}(t_1 - t_0) + t_0$$

for all $i = 0, \dots, N$. Then for all $i = 1, \dots, N$, $\gamma|_{[t_{i-1}, t_i]}$ is contained in U_α for some $\alpha \in A$. Let us extend the construction of example 1.14. Suppose $f \in C_c^\infty(t_{i-1}, t_i)$. Then we can define a variation $\Gamma : [t_0, t_1] \times [-\varepsilon_0, \varepsilon_0] \to M$ as follows: Define

$$\Gamma : ([t_0, t_1] \setminus \operatorname{supp} f) \times [-\varepsilon_0, \varepsilon_0] \to M$$

by $\Gamma(t, \varepsilon) := \gamma(t)$, and $\Gamma : (t_{i-1}, t_i) \times [-\varepsilon_0, \varepsilon_0] \to M$ to be the map defined in example 1.14. Since both definitions agree on the overlap $(t_{i-1}, t_i) \setminus \operatorname{supp} f$, an application of the gluing lemma for smooth maps [6, 35] yields the existence of a variation Γ of γ on M. Therefore, step 1 implies the Euler-Lagrange equations (1.8). The converse direction is content of problem 1.80 □

Due to the Newton-Laplace Determinacy Principle 1.7, the motions on a Lagrangian system are inherently characterized by the Lagrangian function and locally by the Euler-Lagrange equations (1.8). Hence any motion satisfies locally a system of second order ordinary differential equations. This system bears its own name.

Definition 1.22 (Equations of Motion). The Euler-Lagrange equations (1.8) of a Lagrangian system are called the ***equations of motion***.

Example 1.23. Motions on Riemannian Manifolds Let (M^n, g) be a Riemannian manifold and consider the Lagrangian L on M defined in example 1.9 with kinetic energy

$$T(x, v, t) := \frac{1}{2} g_x(v, v) = \frac{1}{2} |v|_g^2$$

and potential energy $V(x, t) := 0$ for $x \in M$, $v \in T_x M$ and $t \in \mathbb{R}$. Let (U, x^i) be a chart on M. We compute

$$\begin{aligned} L(x, v, t) &= \frac{1}{2} g_x\,(v, v) \\ &= \frac{1}{2} g_x \left(v^i \frac{\partial}{\partial x^i}\bigg|_x, v^j \frac{\partial}{\partial x^j}\bigg|_x \right) \\ &= \frac{1}{2} g_x \left(\frac{\partial}{\partial x^i}\bigg|_x, \frac{\partial}{\partial x^j}\bigg|_x \right) v^i v^j \\ &= \frac{1}{2} g_{ij}(x) v^i v^j, \end{aligned}$$

where $g_{ij}(x) := g_x \left(\frac{\partial}{\partial x^i}\big|_x, \frac{\partial}{\partial x^j}\big|_x \right)$. Thus

$$\frac{\partial L}{\partial x^l}(x, v, t) = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^l}(x) v^i v^j$$

and in particular

$$\frac{\partial L}{\partial x^l} \left(\gamma(t), \dot{\gamma}(t), t \right) = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^l} \left(\gamma(t) \right) \dot{\gamma}^i(t) \dot{\gamma}^j(t),$$

for all $l = 1, \ldots, n$. Moreover

$$\frac{\partial L}{\partial v^l}(x, v, t) = \frac{1}{2} g_{ij}(x) \delta_l^i v^j + \frac{1}{2} g_{ij}(x) v^i \delta_l^j = \frac{1}{2} g_{lj}(x) v^j + \frac{1}{2} g_{il}(x) v^i$$

implies

$$\begin{aligned} \frac{d}{dt} \frac{\partial L}{\partial v^l} \left(\gamma, \dot{\gamma}, t \right) &= \frac{1}{2} \frac{d}{dt} g_{lj}(\gamma) \dot{\gamma}^j + \frac{1}{2} g_{lj}(\gamma) \ddot{\gamma}^j + \frac{1}{2} \frac{d}{dt} g_{il}(\gamma) \dot{\gamma}^i + \frac{1}{2} g_{il}(\gamma) \ddot{\gamma}^i \\ &= \frac{1}{2} d g_{lj}(\dot{\gamma}) \dot{\gamma}^j + \frac{1}{2} g_{lj}(\gamma) \ddot{\gamma}^j + \frac{1}{2} d g_{il}(\dot{\gamma}) \dot{\gamma}^i + \frac{1}{2} g_{il}(\gamma) \ddot{\gamma}^i \\ &= \frac{1}{2} \frac{\partial g_{lj}}{\partial x^k} \dot{\gamma}^k \dot{\gamma}^j + \frac{1}{2} g_{lj}(\gamma) \ddot{\gamma}^j + \frac{1}{2} \frac{\partial g_{il}}{\partial x^k} \dot{\gamma}^k \dot{\gamma}^i + \frac{1}{2} g_{il}(\gamma) \ddot{\gamma}^i \\ &= \frac{1}{2} \frac{\partial g_{jl}}{\partial x^k} \dot{\gamma}^k \dot{\gamma}^j + \frac{1}{2} g_{jl}(\gamma) \ddot{\gamma}^j + \frac{1}{2} \frac{\partial g_{il}}{\partial x^k} \dot{\gamma}^k \dot{\gamma}^i + \frac{1}{2} g_{il}(\gamma) \ddot{\gamma}^i \\ &= g_{il} \ddot{\gamma}^i + \frac{1}{2} \frac{\partial g_{jl}}{\partial x^i} \dot{\gamma}^i \dot{\gamma}^j + \frac{1}{2} \frac{\partial g_{il}}{\partial x^j} \dot{\gamma}^i \dot{\gamma}^j. \end{aligned}$$

Therefore the Euler-Lagrange equations (1.8) read

$$0 = \frac{d}{dt} \frac{\partial L}{\partial v^l} - \frac{\partial L}{\partial x^l} = g_{il} \ddot{\gamma}^i + \frac{1}{2} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right) \dot{\gamma}^i \dot{\gamma}^j,$$

for all $l = 1, \dots, n$. Multiplying both sides by g^{kl} yields

$$\ddot{\gamma}^k + \Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0, \tag{1.9}$$

for all $k = 1, \dots, n$, where

$$\Gamma^k_{ij} := \frac{1}{2} g^{kl} \left(\frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

are the ***Christoffel symbols*** with respect to the choosen chart (see [4, 70]). The system of equations (1.9) is called ***geodesic equations*** (see [4, 58]). Hence extremals γ of the action functional satisfy the geodesic equation and are therefore geodesics on the Riemannian manifold M.

Lemma 1.24. *Let (M, L) be a Lagrangian system and define $L + df \in C^\infty(TM \times \mathbb{R})$ by*

$$(L + df)(x, v, t) := L(x, v, t) + df_x(v)$$

for any $f \in C^\infty(M)$. Then (M, L) and $(M, L + df)$ admit the same equations of motion.

Proof. Let us denote the action function corresponding to $L + df$ by $\widetilde{S}$ and suppose γ_ε is a variation of γ in M. Using the formula for the derivative of a function along a curve [6, 283] we compute

$$\begin{aligned}
\widetilde{S}(\gamma_\varepsilon) &= \int_{t_0}^{t_1} L(\gamma_\varepsilon(t), \dot{\gamma}_\varepsilon(t), t)dt + \int_{t_0}^{t_1} df_{\gamma_\varepsilon(t)}\left(\dot{\gamma}_\varepsilon(t)\right)dt \\
&= S(\gamma_\varepsilon) + \int_{t_0}^{t_1} (f \circ \gamma_\varepsilon)'(t)dt \\
&= S(\gamma_\varepsilon) + f\left(\gamma_\varepsilon(t_1)\right) - f\left(\gamma_\varepsilon(t_0)\right) \\
&= S(\gamma_\varepsilon) + f(x_1) - f(x_0).
\end{aligned}$$

In particular

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \widetilde{S}(\gamma_\varepsilon) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} S(\gamma_\varepsilon).$$

Remark 1.25. Lemma 1.24 implies, that the Lagrangian of a mechanical system can only be determined up to differentials of smooth functions. Actually, in coordinates, also up to total time derivatives. Hence a *law of motion*, that is a Lagrangian describing a certain mechanical system, is in fact an equivalence class of Lagrangian functions.

1.3 Legendre Transform

In this section we *dualize* the notion of a Lagrangian function, that is, to each Lagrangian function $L \in C^\infty(TM)$ we will associate a *dual function* $L^* \in C^\infty(T^*M)$. It turns out, that in this dual formulation, the equations of motion take a very symmetric form. To simplify the notation and illuminating the main concept, we consider Lagrangian functions of a special type.

Definition 1.26 (Autonomous System). An ***autonomous Lagrangian system*** is defined to be a tuple (M, L) consitsing of a smooth manifold M and a function $L \in C^\infty(M)$.

Let (M^n, L) be an autonomous Lagrangian system and (U, x^i) a chart on M. Moreover, let (x^i, v^i) denote standard coordinates on TM, that is $v^i := dx^i$ for all $i = 1, \ldots, n$. Expanding the Euler-Lagrange equations (1.8) yields

$$\begin{aligned}\frac{\partial L}{\partial x^j}\big(\gamma(t), \dot\gamma(t)\big) &= \frac{d}{dt}\frac{\partial L}{\partial v^j}\big(\gamma(t), \dot\gamma(t)\big) \\ &= \frac{\partial^2 L}{\partial x^i \partial v^j}\big(\gamma(t), \dot\gamma(t)\big)\,\dot\gamma^i(t) + \frac{\partial^2 L}{\partial v^i \partial v^j}\big(\gamma(t), \dot\gamma(t)\big)\,\ddot\gamma^i(t)\end{aligned}$$

for all $j = 1, \ldots, n$. In order to solve above system of second order ordinary differential equations for $\ddot\gamma^i(t)$ and all initial conditions in the chart on TU, the matrix $\mathcal{H}_L(x, v)$ defined by

$$\mathcal{H}_L(x, v) := \left(\frac{\partial^2 L}{\partial v^i \partial v^j}(x, v)\right)^i_j \tag{1.10}$$

must be invertible on TU.

Definition 1.27 (Nondegenrate System). An autonomous Lagrangian system (M, L) is said to be ***nondegenerate***, iff for all coordinate charts U on M, $\det \mathcal{H}_L(x, v) \neq 0$ holds on TU.

Example 1.28 (Nondegenrate System on a Riemannian Manifold). Let (M, g) be a Riemannian manifold. Consider the Lagrangian $T - V$ with kinetic energy $T \in C^\infty(TM)$ defined by $T(v) := \frac{1}{2}|v|^2$ and potential energy $V \in C^\infty(M)$. Then the computation performed in example 1.23 yields

$$\mathcal{H}_{T-V}(x, v) = \big(g_{ij}(x)\big)^i_j$$

on every chart since $\frac{\partial V}{\partial v^i} = 0$ for every i, and so this Lagrangian system is nondegenerate.

The nondegeneracy of an autonomous Lagrangian system is intrinsically connected to a certain differential form in $\Omega^1(TM)$, which we will construct now.

Proposition 1.29. *Let* (M, L) *be an autonomous Lagrangian system. For every* $(x, v) \in TM$ *we can define a covector* $\lambda_L|_{(x,v)} \in T^*_{(x,v)}TM$ *by setting*

$$\lambda_L|_{(x,v)} := \frac{\partial L}{\partial v^i}(x, v) dx^i|_{(x,v)}. \tag{1.11}$$

in induced coordinates $\left(x^i, v^i\right)$ *about* (x, v) *on* TM*. Then* $\lambda_L \in \Omega^1(TM)$.

Proof. We have to show that the chartwise definition (1.11) does not depend on the choice of coordinates. Let $\left(\widetilde{U}, \widetilde{x}^i\right)$ be another chart on M such that $U \cap \widetilde{U} \neq \varnothing$. Denote the induced coordinates on TM by $\left(\widetilde{x}^i, \widetilde{v}^i\right)$. Then for $(x, v) \in U \cap \widetilde{U}$ we have that

$$\left.\frac{\partial}{\partial x^j}\right|_{(x,v)} = \frac{\partial \widetilde{x}^k}{\partial x^j}(x, v)\left.\frac{\partial}{\partial \widetilde{x}^k}\right|_{(x,v)} + \frac{\partial \widetilde{v}^k}{\partial x^j}(x, v)\left.\frac{\partial}{\partial \widetilde{v}^k}\right|_{(x,v)},$$

and

$$\left.\frac{\partial}{\partial \widetilde{v}^i}\right|_{(x,v)} = \frac{\partial x^j}{\partial \widetilde{v}^i}(x, v)\left.\frac{\partial}{\partial x^j}\right|_{(x,v)} + \frac{\partial v^j}{\partial \widetilde{v}^i}(x, v)\left.\frac{\partial}{\partial v^j}\right|_{(x,v)} = \frac{\partial v^j}{\partial \widetilde{v}^i}(x, v)\left.\frac{\partial}{\partial v^j}\right|_{(x,v)}.$$

We compute

$$d\widetilde{x}^i|_{(x,v)}\left(\left.\frac{\partial}{\partial x^j}\right|_{(x,v)}\right) = \frac{\partial \widetilde{x}^i}{\partial x^j}(x, v)$$

and

$$d\widetilde{x}^i|_{(x,v)}\left(\left.\frac{\partial}{\partial v^j}\right|_{(x,v)}\right) = 0.$$

Thus

$$d\widetilde{x}^i|_{(x,v)} = \frac{\partial \widetilde{x}^i}{\partial x^j}(x, v) dx^j|_{(x,v)}.$$

Observe that

$$\frac{\partial \widetilde{x}^i}{\partial x^j}(x, v) = \frac{\partial \widetilde{x}^i}{\partial x^j}(x).$$

This can be seen directly by using the definitions and the coordinate structure on TM. Finally, we have that

$$dx^j|_x(v) = \frac{\partial x^j}{\partial \widetilde{x}^i}(x) d\widetilde{x}^i|_x(v),$$

or equivalently

$$v^j(x, v) = \frac{\partial x^j}{\partial \widetilde{x}^i}(x)\widetilde{v}^i(x, v).$$

Hence we compute

$$\lambda_L|_{(x,v)} = \frac{\partial L}{\partial \widetilde{v}^i}(x, v) d\widetilde{x}^i|_{(x,v)}$$

$$
\begin{aligned}
&= \frac{\partial L}{\partial v^j}(x,v)\frac{\partial v^j}{\partial \tilde{v}^i}(x,v)\frac{\partial \tilde{x}^i}{\partial x^k}(x)dx^k|_{(x,v)} \\
&= \frac{\partial L}{\partial v^j}(x,v)\frac{\partial x^j}{\partial \tilde{x}^i}(x)\frac{\partial \tilde{x}^i}{\partial x^k}(x)dx^k|_{(x,v)} \\
&= \frac{\partial L}{\partial v^j}(x,v)\delta^j_k dx^k|_{(x,v)} \\
&= \frac{\partial L}{\partial v^j}(x,v)dx^j|_{(x,v)}.
\end{aligned}
$$

Therefore λ_L is independent of the choice of coordinates and so $\lambda_L \in \Omega^1(TM)$. □

Corollary 1.30. *Let (M, L) be an autonomous Lagrangian system. Then the map $D^{\mathcal{F}}L : TM \to T^*M$ defined in coordinates $\left(x^i, v^i\right)$ about $(x, v) \in TM$ by*

$$D^{\mathcal{F}}L_{(x,v)} := \frac{\partial L}{\partial v^i}(x,v)dx^i|_x$$

is well-defined.

Proof. This follows immediately from the proof of proposition 1.29. Indeed, for different coordinates $\left(\tilde{x}^i, \tilde{v}^i\right)$ we compute

$$
\begin{aligned}
D^{\mathcal{F}}L_{(x,v)} &= \frac{\partial L}{\partial \tilde{v}^i}(x,v)d\tilde{x}^i|_x \\
&= \frac{\partial L}{\partial v^j}(x,v)\frac{\partial x^j}{\partial \tilde{x}^i}(x)\frac{\partial \tilde{x}^i}{\partial x^k}(x)dx^k|_x \\
&= \frac{\partial L}{\partial v^j}(x,v)dx^j|_x.
\end{aligned}
$$

□

Definition 1.31 (Associated Form). Let (M, L) be an autonomous Lagrangian system. Then the form λ_L defined in proposition 1.29 is called the ***associated form***.

Definition 1.32 (Fibrewise Derivative). Let (M, L) be an autonomous Lagrangian system. The map $D^{\mathcal{F}}L : TM \to T^*M$ defined in corollary 1.30 is called the ***fibrewise derivative***.

Example 1.33 (Fibrewise Derivative on a Riemannian Manifold). Consider the autonomous Lagrangian system as defined in example 1.28. Then the computation performed in example 1.23 yields

$$D^{\mathcal{F}}(T - V)_{(x,v)} = g_{ij}(x)v^i dx^j$$

on every chart since $\frac{\partial V}{\partial v^j} = 0$ for all j.

Definition 1.34 (Nondegenrate Tensor). Let V be a finite-dimensional real vector space. A tensor $\omega \in \Lambda^2(V^*)$ is said to be ***nondegenrate***, iff the map $\widehat{\omega} : V \to V^*$ defined by $\widehat{\omega}(v) := i_v\omega$ is an isomorphism.

Lemma 1.35. *Let V be a finite-dimensional real vector space and let $\omega \in \Lambda^2(V^*)$. Then the following statements are equivalent:*

(a) *ω is nondegenerate.*
(b) *With respect to any basis for V, the matrix representing $\widehat{\omega}$ is invertible.*
(c) *If $\omega(v, u) = 0$ for all $u \in V$, then $v = 0$.*
(d) *If $v \neq 0$, then there exists some $u \in V$ such that $\omega(v, u) \neq 0$.*
(e) *The matrix representing ω in any basis of V is invertible.*

Definition 1.36 (Nondegenerate Form). Let M be a smooth manifold and $\omega \in \Omega^2(M)$. Then ω is said to be ***nondegenerate***, iff ω_x is nondegenerate for every $x \in M$.

Proposition 1.37. *An autonomous Lagrangian system (M, L) is nondegenerate if and only if $d\lambda_L$ is nondegenerate.*

Proof. Using the computation performed in [6, 363], we get

$$d\lambda_L = d\left(\frac{\partial L}{\partial v^j} dx^j\right) = \frac{\partial^2 L}{\partial x^i \partial v^j} dx^i \wedge dx^j + \frac{\partial^2 L}{\partial v^i \partial v^j} dv^i \wedge dx^j.$$

Moreover, using part (e) of properties of the wedge product [6, 356], we compute

$$\begin{aligned}
d\lambda_L\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right) =& \frac{\partial^2 L}{\partial x^i \partial v^j} \det \begin{pmatrix} dx^i\left(\frac{\partial}{\partial x^k}\right) & dx^j\left(\frac{\partial}{\partial x^k}\right) \\ dx^i\left(\frac{\partial}{\partial x^l}\right) & dx^j\left(\frac{\partial}{\partial x^l}\right) \end{pmatrix} \\
&+ \frac{\partial^2 L}{\partial v^i \partial v^j} \det \begin{pmatrix} dv^i\left(\frac{\partial}{\partial x^k}\right) & dx^j\left(\frac{\partial}{\partial x^k}\right) \\ dv^i\left(\frac{\partial}{\partial x^l}\right) & dx^j\left(\frac{\partial}{\partial x^l}\right) \end{pmatrix} \\
=& \frac{\partial^2 L}{\partial x^i \partial v^j}(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) \\
=& \frac{\partial^2 L}{\partial x^k \partial v^l} - \frac{\partial^2 L}{\partial x^l \partial v^k}
\end{aligned}$$

for all $k, l = 1, \ldots, n$. Similarly, we compute

$$d\lambda_L\left(\frac{\partial}{\partial v^k}, \frac{\partial}{\partial x^l}\right) = \frac{\partial^2 L}{\partial v^k \partial v^l} \qquad \text{and} \qquad d\lambda_L\left(\frac{\partial}{\partial v^k}, \frac{\partial}{\partial v^l}\right) = 0,$$

and using skew-symmetry, we also deduce

$$d\lambda_L\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial v^l}\right) = -\frac{\partial^2 L}{\partial v^k \partial v^l}.$$

Therefore, the matrix representing $d\lambda_L$ with respect to the standard basis is given by the block matrix

$$d\lambda_L = \left(\begin{array}{c|c} * & -\mathcal{H}_L \\ \hline \mathcal{H}_L & 0 \end{array}\right),$$

where $\mathcal{H}_L$ is the matrix defined in (1.10). Thus

$$\det\left(d\lambda_L\right) = (-1)^n (\det \mathcal{H}_L)^2$$

Hence the matrix representation of $d\lambda_L$ is invertible if and only if $\mathcal{H}_L$ is invertible, and the conclusion follows. ☐

So far, we have associated to each Lagrangian system (M, L) a 1-form on TM, the associated form λ_L. In order to get closer to our goal of dualizing the concept of a Lagrangian function, we need also a 1-form on T^*M. Suppose (U, x^i) is a chart on M. The induced standard coordinates on the cotangent bundle T^*M of M are given by $\left(x^i, \xi_i\right)$, where $\xi_i := \frac{\partial}{\partial x^i}$, considered as an element of the double dual $T^{**}U$. On this chart, define a one 1-form α by $\lambda := \xi_i dx^i$. Suppose $\left(\widetilde{x}^i, \widetilde{\xi}_i\right)$ are other coordinates. Then from the computations performed at the beginning of the previous section, we have that

$$\widetilde{\xi}_i = \frac{\partial x^j}{\partial \widetilde{x}^i}\xi_j \qquad \text{and} \qquad d\widetilde{x}^i = \frac{\partial \widetilde{x}^i}{\partial x^k}dx^k.$$

Thus

$$\lambda = \widetilde{\xi}_i d\widetilde{x}^i = \frac{\partial x^j}{\partial \widetilde{x}^i}\xi_j \frac{\partial \widetilde{x}^i}{\partial x^k}dx^k = \xi_j \delta^j_k dx^k = \xi_j dx^j,$$

and so, λ is independen of the choice of coordinates.

Definition 1.38 (Tautological Form). Let M be a smooth manifold. The ***tautological form on T^*M***, denoted by λ, is the form $\alpha \in \Omega^1(T^*M)$ defined locally by

$$\lambda := \xi_i dx^i,$$

where $\left(x^i, \xi_i\right)$ denotes the standard coordinates on T^*M.

Remark 1.39. The preceeding discussion showed, that the tautological form α is well-defined.

Recall, that if $F \in C^\infty(M, N)$ for some smooth manifolds M and N, and $l \in \mathbb{N}$, we can define a mapping $F^* : \Gamma\left(T^{(0,l)}TN\right) \to \Gamma\left(T^{(0,l)}TM\right)$, called the *pullback by F*, by

$$(F^*A)_x(v_1, \ldots, v_l) := A_{F(x)}\left(dF_x(v_1), \ldots, dF_x(v_l)\right)$$

for all $x \in M$ and $v_1, \ldots, v_l \in T_xM$ (see [6, 320]).

Definition 1.40 (Legendre Transform). A ***Legendre transform of an autonomous Lagrangian system (M, L)*** is defined to be a fibrewise mapping $\tau_L \in C^\infty(TM, T^*M)$ such that

$$\lambda_L = \tau_L^*(\lambda).$$

Example 1.41. Legendre Transform on a Riemannian Manifold Let (M, L) be a Lagrangian system. Then the morphism $\tau_L : TM \to T^*M$ defined by

$$\tau_L(x, v) := \left(x, D^{\mathcal{F}} L_{(x,v)}\right) \tag{1.12}$$

is a Legendre transform. In particular, if we consider the Lagrangian system defined in example 1.28, we get that the above defined Legendre transform is a diffeomorphism. Indeed, suppose that $\tau_{T-V}(x, v) = \tau_{T-V}(\widetilde{x}, \widetilde{v})$. Then $x = \widetilde{x}$ and

$$g_{ij}(x)v^i dx^j = g_{ij}(x)\widetilde{v}^i dx^j$$

using example 1.33. So we must have

$$g_{ij}(x)v^i = g_{ij}(x)\widetilde{v}^i$$

for all j. Multiplying both sides by $g^{kj}(x)$ yields $v^k = \widetilde{v}^k$ for every k and hence $v = \widetilde{v}$. Thus τ_{T-V} is injective. Let $\xi \in T_x^*M$ be given by $\xi_i dx^i|_x$. Then $\tau_{T-V}(x, v) = (x, \xi)$, where v is given in coordinates by $v^k := g^{ki}(x)\xi_i$.

Since the nondegenracy of a Lagragian system (M, L) is inherently connected to the nondegenracy of the form $d\lambda_L$ and the definition of the Legendre transform invokes the form λ_L, one would expect a connection between the nondegeneracy of the Lagrangian system and a local property of Legendre transform. Moreover, the proof shows that any Legendre transform has the form from example 1.41.

Lemma 1.42. *A Legendre transform on a Lagrangian system is a local diffeomorphism if and only if the Lagrangian system is nondegenrate.*

Proof. Denote the Lagrangian system by (M, L). Let (U, x^i) be a chart on M and denote by $\left(x^i, v^i\right)$ and $\left(x^i, \xi_i\right)$ the induced standard coordinates on TM and T^*M, respectively. Then we compute

$$\tau_L^*(\lambda) = \tau_L^*\left(\xi_j dx^j\right) = (\xi_j \circ \tau_L) d\left(x^j \circ \tau_L\right),$$

which must coincide with

$$\lambda_L = \frac{\partial L}{\partial v^j} dx^j.$$

Thus

$$\tau_L(x, v) = D^{\mathcal{F}} L_{(x,v)}, \tag{1.13}$$

and so

$$D\tau_L|_{(x,v)} = \left(\begin{array}{c|c} I & 0 \\ \hline * & \mathcal{H}_L \end{array}\right)$$

at every $(x, v) \in TM$. Hence

$$\det\left(D\tau_L|_{(x,v)}\right) = \det \mathcal{H}_L.$$

If τ_L is a local diffeomorphism, by definition, we have that some restriction of τ_L to some neighbourhood of (x, v) is a diffeomorphism, and so, by properties of differentials (d) [6, 55], we have that $D\tau_L|_{(x,v)}$ is an isomorphism. Conversely, if the Lagrangian system is nondegenerate, we conclude using the inverse function theorem for manifolds [6, 79], that τ_L is a local diffeomorphism. □

Corollary 1.43. *Let (M, L) be an autonomous Lagrangian system with Legendre transform $\tau_L : TM \to T^*M$. Then*

$$\tau_L(x, v) = D^{\mathcal{F}} L_{(x,v)}.$$

Definition 1.44 (Energy). The ***energy of an autonomous Lagrangian system*** $\boldsymbol{(M, L)}$ is defined to be the function $E_L \in C^\infty(TM)$ given by

$$E_L(x, v) := D^{\mathcal{F}} L_{(x,v)}(v) - L(x, v)$$

for $(x, v) \in TM$.

Example 1.45 (Energy on a Riemannian Manifold). Consider the Lagrangian system defined in example 1.28. Then the computation performed in example 1.33 yields

$$\begin{aligned} E_{T-V}(x, v) &= \frac{\partial T}{\partial v^k} v^k - \frac{\partial V}{\partial v^k} v^k - T(v) + V(x) \\ &= \frac{1}{2} g_{ij} \delta^i_k v^j v^k + \frac{1}{2} g_{ij} v^i \delta^j_k v^k - T(v) + V(x) \\ &= g_{ij} v^i v^j - T(v) + V(x) \\ &= T(v) + V(x) \end{aligned}$$

for every $(x, v) \in TM$. Hence the energy of this Lagrangian system is given by *kinetic energy plus potential energy*.

Definition 1.46 (Hamiltonian Function). Let (M, L) be an autonomous Lagrangian system and τ_L a diffeomorphic Legendre transform. The morphism $H_L \in C^\infty(T^*M)$ defined by

$$H_L := E_L \circ \tau_L^{-1}$$

is called the ***Hamiltonian function associated to the Lagrangian function*** $\boldsymbol{L}$.

Example 1.47. Hamiltonian function on a Riemannian Manifold Consider the Lagrangian system defined in example 1.28. By example 1.41 the Legendre transform τ_{T-V} is a diffeomorphism. Using example 1.45, we compute

$$\begin{aligned} H_{T-V}(x, \xi) &= E_{T-V}\left(\tau^{-1}_{T-V}(x, \xi)\right) \\ &= E_{T-V}\left(x, v\right) \\ &= T(v) + V(x) \\ &= \frac{1}{2} g_{ij}(x) v^i v^j + V(x) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} g_{ij}(x) g^{ik}(x) \xi_k g^{jl}(x) \xi_l + V(x) \\
&= \frac{1}{2} \delta^k_j \xi_j g^{jl}(x) \xi_l + V(x) \\
&= \frac{1}{2} g^{kl}(x) \xi_k \xi_l + V(x)
\end{aligned}
$$

where $v = \left(g^{ki}\right)^k_i \xi$.

Theorem 1.48 (Hamilton's Equations). *Let γ be a motion on an autonomous Lagrangian system (M^n, L) and suppose that τ_L is a diffeomorphic Legendre transform. Then γ satisfies the Euler-Lagrange equations in every chart if and only if the path*

$$
\big(\gamma(t), \xi(t)\big) := \tau_L \left(\gamma(t), \dot{\gamma}(t)\right)
$$

satisfies the following system of first order ordinary differential equations in every chart:

$$
\dot{\gamma}(t) = \frac{\partial H_L}{\partial \xi} \big(\gamma(t), \xi(t)\big) \qquad \textit{and} \qquad \dot{\xi}(t) = -\frac{\partial H_L}{\partial x} \big(\gamma(t), \xi(t)\big) \tag{1.14}
$$

The equations (1.14) are called ***Hamilton's equations****.*

Proof. First we compute H_L in standard coordinates (x^i, ξ_i) on T^*M. By corollary 1.43 we have that

$$
\tau_L(x, v) = D^{\mathcal{F}} L_{(x,v)}. \tag{1.15}
$$

Since τ_L is a diffeomorphism by assumption, in particular it is a local diffeomorphism (see [6, 80]). Hence by lemma 1.42, the Lagrangian system (M, L) is nondegenerate. So considering $\tau_L^{-1}(x, \xi)$, we can apply the implicit function theorem [6, 661] to obtain v implicitely from the equation

$$
\xi = \frac{\partial L}{\partial v}(x, v).
$$

Hence in coordinates

$$
H_L(x, \xi) = \left(\frac{\partial L}{\partial v^i} v^i - L(x, v) \right) \Bigg|_{\xi = \frac{\partial L}{\partial v}}.
$$

Therefore

$$
\frac{\partial H_L}{\partial \xi^j} = \frac{\partial}{\partial \xi_j} \left(\xi_i v^i - L(x, v) \right) \big|_{\xi = \frac{\partial L}{\partial v}} = \delta^j_i v^i = v^j.
$$

Hence

$$
\frac{\partial H_L}{\partial \xi^j} \big(\gamma(t), \xi(t)\big) = \dot{\gamma}^j(t),
$$

for all $j = 1, \ldots, n$. Moreover, we have that

$$\frac{\partial H_L}{\partial x^j} = \frac{\partial}{\partial x^j}\left(\frac{\partial L}{\partial v^i}v^i - L(x,v)\right)\bigg|_{\xi=\frac{\partial L}{\partial v}} = -\frac{\partial L}{\partial x^j}(x,v)\bigg|_{\xi=\frac{\partial L}{\partial v}},$$

and so

$$\frac{\partial H_L}{\partial x^j}\left(\gamma(t),\xi(t)\right) = -\frac{\partial L}{\partial x^j}\left(\gamma(t),\dot\gamma(t)\right),$$

for all $j = 1,\dots,n$. If the Euler-Lagrange equations (1.8) hold, then we get

$$\frac{\partial H_L}{\partial x^j}\left(\gamma(t),\xi(t)\right) = -\frac{d}{dt}\frac{\partial L}{\partial v^j}\left(\gamma(t),\dot\gamma(t)\right) = -\dot\xi_j(t),$$

and thus the Hamilton's equations (1.14) hold. Conversly, if we suppose that Hamilton's equations (1.14) hold, we get that

$$-\frac{d}{dt}\frac{\partial L}{\partial v^j}\left(\gamma(t),\dot\gamma(t)\right) = -\dot\xi_j(t) = \frac{\partial H_L}{\partial x^j}\left(\gamma(t),\xi(t)\right) = -\frac{\partial L}{\partial x^j}\left(\gamma(t),\dot\gamma(t)\right),$$

and so the Euler-Lagrange equations (1.8) are satisfied. □

1.4 Conservation Laws and Noether's Theorem

Definition 1.49 (Conservation Law). A ***conservation law for a Lagrangian system*** $\boldsymbol{(M, L)}$ is defined to be a function $I \in C^\infty(TM)$ such that

$$\frac{d}{dt}I\left(\gamma(t),\dot\gamma(t)\right) = 0$$

for all extremals of the action functional (1.16).

Proposition 1.50 (Conservation of Energy). *The energy of an autonomous Lagrangian system is a conservation law.*

Proof. By definition of the fibrewise derivative 1.32 we have that

$$D^{\mathcal{F}}L_{(\gamma,\dot\gamma)}(\dot\gamma) = \frac{\partial L}{\partial v^i}(\gamma,\dot\gamma)\,dx^i\left(\dot\gamma^j\frac{\partial}{\partial x^j}\right) = \frac{\partial L}{\partial v^i}(\gamma,\dot\gamma)\,\dot\gamma^j\delta^i_j = \frac{\partial L}{\partial v^i}(\gamma,\dot\gamma)\,\dot\gamma^i.$$

Thus by definition of the energy 1.44 and the Euler-Lagrange equations 1.20 we compute

$$\begin{aligned}
\frac{d}{dt}E\left(\gamma,\dot{\gamma}\right) &= \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\left(\gamma,\dot{\gamma}\right)\dot{\gamma}^i\right) - \frac{d}{dt}L\left(\gamma,\dot{\gamma}\right) \\
&= \frac{d}{dt}\frac{\partial L}{\partial v^i}\left(\gamma,\dot{\gamma}\right)\dot{\gamma}^i + \frac{\partial L}{\partial v^i}\left(\gamma,\dot{\gamma}\right)\ddot{\gamma}^i - \frac{\partial L}{\partial x^i}(\gamma,\dot{\gamma})\dot{\gamma}^i - \frac{\partial L}{\partial v^i}(\gamma,\dot{\gamma})\ddot{\gamma}^i \\
&= \frac{d}{dt}\frac{\partial L}{\partial v^i}\left(\gamma,\dot{\gamma}\right)\dot{\gamma}^i - \frac{\partial L}{\partial x^i}(\gamma,\dot{\gamma})\dot{\gamma}^i \\
&= \frac{\partial L}{\partial x^i}\left(\gamma,\dot{\gamma}\right)\dot{\gamma}^i - \frac{\partial L}{\partial x^i}(\gamma,\dot{\gamma})\dot{\gamma}^i \\
&= 0.
\end{aligned}$$

Recall, that for a smooth manifold M, we define the *set of diffeomorphisms on* M by

$$\mathrm{Diff}(M) := \{\varphi \in C^\infty(M, M) : \varphi \text{ is a diffeomorphism}\}.$$

In fact $\mathrm{Diff}(M)$ constitutes a group under ordinary composition of maps. Thus we define a *one-parameter group of diffeomorphisms of* M to be a group homomorphisms

$$(\mathbb{R}, +) \to \mathrm{Diff}(M)$$

Explicitely, given any one-parameter group $\theta : (\mathbb{R}, +) \to \mathrm{Diff}(M)$, we define $\theta_s := \theta(s)$ for all $s \in \mathbb{R}$ and we can therefore write $(\theta_s)_{s\in\mathbb{R}}$ for the one-parameter group θ of diffeomorphisms of M. Since θ is a homomorphism of groups, we have that

$$\theta_{s+t} = \theta_s \circ \theta_t \qquad \text{and} \qquad \theta_0 = \mathrm{id}_M$$

for all $s, t \in \mathbb{R}$. We say that the one-parameter group $(\theta_s)_{s\in\mathbb{R}}$ of diffeomorphisms of M is smooth, iff the corresponding map $\theta : \mathbb{R} \times M \to M$ defined by $(s, x) \mapsto \theta_s(x)$ is smooth. If $F \in C^\infty(M, N)$ for two smooth manifolds M and N, for $x \in M$ we define the *differential of* F *at* x to be the mapping $DF_x : T_x M \to T_{F(x)}N$, given by $DF_x(v)(f) := v(f \circ F)$ for all $f \in C^\infty(N)$. These fibrewise mappings can be assembled to the *global differential of* F, defined to be the mapping $DF : TM \to TN$ given by $DF(x, v) := \big(F(x), DF_x(v)\big)$. The global differential is a smooth map (see [6, 68]) and has the following properties.

Proposition 1.51 (Properties of the Global Differential [6, 68]). *Let* M, N, P *be smooth manifolds,* $F \in C^\infty(M, N)$ *and* $G \in C^\infty(N, P)$*. Then:*

1. $D(G \circ F) = DG \circ DF$.
2. $D(\mathrm{id}_M) = \mathrm{id}_{TM}$.
3. *If* F *is a diffeomorphism, then* DF *is a diffeomorphism with* $(DF)^{-1} = D\left(F^{-1}\right)$.

Remark 1.52. In a more sophisticated language, proposition 1.51 says that the global differential is a functor $D : \mathsf{Man} \to \mathsf{Man}$, where Man denotes the category of finite-dimensional smooth manifolds.

Lemma 1.53. *Let* $(\theta_s)_{s\in\mathbb{R}}$ *be a smooth one-parameter group of diffeomorphisms of a smooth manifold* M*. Then* $(D\theta_s)_{s\in\mathbb{R}}$ *is a smooth one-prameter group of diffeomorphisms of* TM*.*

Proof. Part (c) of the properties of the global differential 1.51 implies that $D\theta_s$ is a diffeomorphism for all $s \in \mathbb{R}$. Moreover, by part (c) of the properties of the global differential 1.51 we compute

$$D\theta_{s+t} = D(\theta_s \circ \theta_t) = D\theta_s \circ D\theta_t$$

for all $s, t \in \mathbb{R}$. Lastly, part (b) of the properties of the global differential 1.51 implies

$$D\theta_0 = D(\mathrm{id}_M) = \mathrm{id}_{TM} .$$

Given a one-parameter group $(\theta_s)_{s\in\mathbb{R}}$ of diffeomorphisms of a smooth manifold M, we can define a vector field V by

$$V_x := \left.\frac{d}{ds}\right|_{s=0} \theta_s(x)$$

for all $x \in M$. This vector field is actually smooth by [6, 210] and is called the *infinitesimal generator of* θ.

Definition 1.54 (Symmetry). A ***symmetry of an autonomous Lagrangian system*** $\boldsymbol{(M, L)}$ is defined to be a diffeomorphism $F \in \mathrm{Diff}(M)$, such that

$$(DF)^* L = L.$$

A ***symmetry group of*** $\boldsymbol{(M, L)}$ is defined to be a Lie group G, such that there exists a left action $\theta : G \times M \to M$ and such that θ_g is a symmetry of (M, L) for all $g \in G$.

Recall, that if $k \in \mathbb{N}$ and $X \in (M)$ for a smooth manifold M, we can define a mapping $i_X : \Omega^{k+1}(M) \to \Omega^k(M)$, called *interior multiplication*, by

$$(i_X\omega)_x(v_1, \ldots, v_k) := \omega_x\left(X|_x, v_1, \ldots, v_k\right)$$

for all $x \in M$ and $v_1, \ldots, v_k \in T_xM$. One-parameter groups of symmetries of autonomous Lagrangian systems give rise to conservation laws.

Theorem 1.55 (Noether's Theorem, Lagrangian Version). *Let $(\theta_s)_{s\in\mathbb{R}}$ be a smooth one-parameter group of symmetries of an autonomous Lagrangian system. Then $i_V(\lambda_L)$ is a conservation law, where V denotes the infinitesimal generator of the one-parameter group $(D\theta_s)_{s\in\mathbb{R}}$ of diffeomorphisms of TM. The conservation law $i_V(\lambda_L)$ is called the* ***Noether integral****.*

Proof. Let $\left(TU, (x^i, v^i)\right)$ be a chart on TM. First we compute the infinitesimal generator V of the one-parameter group $(\theta_s)_{s\in\mathbb{R}}$ in the chart $\left(U, (x^i)\right)$. Let $x \in U$. Then

$$V_x = \left.\frac{d}{ds}\right|_{s=0} \theta_s(x) = \frac{d\theta_s^i(x)}{ds}(0)\left.\frac{\partial}{\partial x^i}\right|_{\theta_0(x)} = \frac{d\theta_s^i(x)}{ds}(0)\left.\frac{\partial}{\partial x^i}\right|_x .$$

Thus

$$V_x = V^i(x)\frac{\partial}{\partial x^i}\bigg|_x$$

where $V^i : U \to \mathbb{R}$ are given by

$$V^i(x) := \frac{d\theta_s^i(x)}{ds}(0).$$

Next consider the infinitesimal generator V of the one-parameter group $(D\theta_s)_{s\in\mathbb{R}}$. For $(x, v) \in TU$, where $v = v^i\frac{\partial}{\partial x^i}$, we compute

$$\begin{aligned}
V_{(x,v)} &= \frac{d}{ds}\bigg|_{s=0}\big(\theta_s(x), D\theta_s|_x(v)\big)\\
&= \frac{d}{ds}\bigg|_{s=0}\left(\theta_s(x), v^j\frac{\partial\theta_s^i}{\partial x^j}(x)\frac{\partial}{\partial x^i}\bigg|_{\theta_s(x)}\right)\\
&= \frac{d\theta_s^i(x)}{ds}(0)\frac{\partial}{\partial x^i}\bigg|_{(x,v)} + v^j\frac{\partial^2\theta^i}{\partial s\partial x^j}(0,x)\frac{\partial}{\partial v^i}\bigg|_{(x,v)}\\
&= V^i(x)\frac{\partial}{\partial x^i}\bigg|_{(x,v)} + v^j\frac{\partial^2\theta^i}{\partial x^j\partial s}(0,x)\frac{\partial}{\partial v^i}\bigg|_{(x,v)}\\
&= V^i(x)\frac{\partial}{\partial x^i}\bigg|_{(x,v)} + v^j\frac{\partial}{\partial x^j}\frac{d\theta_s^i(x)}{ds}(0)\frac{\partial}{\partial v^i}\bigg|_{(x,v)}\\
&= V^i(x)\frac{\partial}{\partial x^i}\bigg|_{(x,v)} + v^j\frac{\partial V^i}{\partial x^j}(x)\frac{\partial}{\partial v^i}\bigg|_{(x,v)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
i_V(\lambda_L)(x,v) &= \lambda_L|_{(x,v)}\big(V_{(x,v)}\big)\\
&= \frac{\partial L}{\partial v^i}(x,v)dx^i|_{(x,v)}\big(V_{(x,v)}\big)\\
&= \frac{\partial L}{\partial v^i}(x,v)V^i(x).
\end{aligned}$$

For $(x, v) \in TM$ set $\gamma(s) := d\theta_s(x, v)$. If $f \in C^\infty(TM)$, the definition of the velocity of a curve and of the differential yields

$$(Vf)(x,v) = V_{(x,v)}f = \left(\frac{d}{ds}\bigg|_{s=0}\gamma(s)\right)f = D\gamma\left(\frac{d}{ds}\bigg|_{s=0}\right)f = \frac{d}{ds}\bigg|_{s=0}(f\circ\gamma).$$

So using the Euler-Lagrange equations 1.20 and the assumption that θ_s is a symmetry of (M, L) for all $s \in \mathbb{R}$, we get

$$\frac{d}{dt}i_V(\lambda_L)(\gamma,\dot\gamma) = \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}(\gamma,\dot\gamma)V^i(\gamma)\right)$$

$$
\begin{aligned}
&= \frac{d}{dt}\frac{\partial L}{\partial v^i}(\gamma,\dot\gamma)V^i(\gamma) + \frac{\partial L}{\partial v^i}(\gamma,\dot\gamma)\frac{d}{dt}V^i(\gamma) \\
&= \frac{\partial L}{\partial x^i}(\gamma,\dot\gamma)V^i(\gamma) + \frac{\partial L}{\partial v^i}(\gamma,\dot\gamma)\frac{d}{dt}V^i(\gamma) \\
&= \frac{\partial L}{\partial x^i}(\gamma,\dot\gamma)V^i(\gamma) + \frac{\partial L}{\partial v^i}(\gamma,\dot\gamma)\frac{\partial V^i}{\partial x^j}(\gamma)\dot\gamma^j \\
&= V_{(\gamma,\dot\gamma)}L \\
&= \left.\frac{d}{ds}\right|_{s=0}\big(L\circ D\theta_s\big)(\gamma,\dot\gamma) \\
&= \left.\frac{d}{ds}\right|_{s=0} L(\gamma,\dot\gamma) \\
&= 0.
\end{aligned}
$$

Thus $i_V(\lambda_L)$ is a conservation law. ☐

1.5 Tonelli Lagrangians

In order to associate to a Lagrangian system a Hamiltonian function, we need that the Legendre transform is a diffeomorphism. So far, we discussed no conditions when this is the case. We follow [9, 7–8]. First of all, we give an invariant characterisation of the fibrewise derivative 1.32. Let $\pi : E \to N$ be a fibre bundle with fibre F and $\varphi \in C^\infty(M, N)$. Define

$$
\varphi^*E := \big\{(x,p) \in M \times E : \varphi(x) = \pi(p)\big\}
$$

Then the following diagram commutes

$$
\begin{tikzcd}
\varphi^*E \arrow[r, "\pi^2"] \arrow[d, "\pi^1"'] & E \arrow[d, "\pi"] \\
M \arrow[r, "\varphi"'] & N.
\end{tikzcd}
$$

Moreover, $\pi^1 : \varphi^*E \to M$ is a fibre bundle with fibre F, and if $\pi : E \to M$ admits a structure group G, then φ^*E admits a Lie subgroup of F as structure group. The fibre bundle φ^*E is called the ***pullback bundle of E by φ***.

Suppose now that $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$ are two vector bundles, $f \in C^\infty(M_1, M_2)$ and $F \in C^\infty(E_1, E_2)$ such that the diagram

$$
\begin{tikzcd}
E_1 \arrow[r, "F"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\
M_1 \arrow[r, "f"'] & M_2
\end{tikzcd}
$$

commutes. For each $x \in M_1$ we get an induced map

$$F_x := F|_{E_1|_x} : E_1|_x \to E_2|_{f(x)}.$$

Its derivative is a map

$$D(F_x)_p : T_p E_1|_x \to T_{F(p)} E_2|_{f(x)}$$

for every $p \in E_1|_x$ and since $E_1|_x$ and $E_2|_{f(x)}$ are vector spaces, we get a map

$$\widetilde{D}(F_x)_p := \Phi_{F(p)}^{-1} \circ D(F_x)_p \circ \Phi_p : E_1|_x \to E_2|_{f(x)}, \tag{1.16}$$

where Φ_p and $\Phi_{F(p)}$ are the isomorphisms from lemma F.39. Consider the vector bundle $\widetilde{\pi}\colon \operatorname{Hom}(E_1, f^*E_2) \to M_1$. Then $p \mapsto \widetilde{D}(F_x)_p$ defines a smooth map

$$D^{\mathcal{F}} F : E_1 \to \operatorname{Hom}(E_1, f^*E_2).$$

If $M_1 = M_2 = M$, $f = \mathrm{id}_M$, $E_1 = TM$ and $E_2 = M \times \mathbb{R}$, then any $F \in C^\infty(TM, M \times \mathbb{R})$ can be identified with a function $L \in C^\infty(TM)$. Moreover, $D^{\mathcal{F}} L$ defined above coincides with the fibrewise derivative of L defined in 1.32. Indeed, let $(x, v) \in TM$ with $v = v^i \frac{\partial}{\partial x^i}\big|_x$, and $w = w^i \frac{\partial}{\partial x^i}\big|_x$ for some local coordinates (x^i) about x, we compute

$$\begin{aligned}
D^{\mathcal{F}} L_{(x,v)}(w) &= dL\left(\frac{d}{dt}\bigg|_{t=0} \left(v^i + tw^i\right) \frac{\partial}{\partial x^i}\bigg|_x\right)\\
&= dL\left(w^i \frac{\partial}{\partial v^i}\bigg|_{(x,v)}\right)\\
&= \frac{\partial L}{\partial v^i}(x, v) w^i\\
&= \frac{\partial L}{\partial v^i}(x, v) dx^i|_x(w).
\end{aligned}$$

Proposition 1.56. *Let (M, L) be an autonomous Lagrangian system with symmetry group G and corresponding action θ. Denote by $f \in C^\infty(TM)$ the function $f(x, v) := D^{\mathcal{F}} L_{(x,v)}(v)$. Then*

$$(D\theta_g)^* f = f$$

for all $g \in G$.

Proof. By definition of the pullback, we need to show that $f \circ D\theta_g = f$ holds for all $g \in G$. Let $(x, v) \in TM$. Then we compute (identifying the derivative with the differential)

$$(f \circ D\theta_g)(x, v) = \left(d(L_{\theta_g(x)})_{D\theta_g(x,v)} \circ \Phi_{D\theta_g(x,v)}\right)\left(D\theta_g(x, v)\right)$$

$$\begin{aligned}
&= d(L_{\theta_g(x)})_{D\theta_g(x,v)} \left(\frac{d}{dt}\bigg|_{t=0} (1+t)D(\theta_g)_x(v) \right) \\
&= \frac{d}{dt}\bigg|_{t=0} L_{\theta_g(x)} \big((1+t)D(\theta_g)_x(v)\big) \\
&= \frac{d}{dt}\bigg|_{t=0} L_{\theta_g(x)} \big(D(\theta_g)_x \big((1+t)v\big)\big) \\
&= \frac{d}{dt}\bigg|_{t=0} L_x \big((1+t)v\big) \\
&= d(L_x)_{(x,v)} \left(\frac{d}{dt}\bigg|_{t=0} (1+t)v \right) \\
&= d(L_x)_{(x,v)} \big(\Phi_v(v)\big) \\
&= f(x,v).
\end{aligned}$$

□

In what follows, we need to recall the rudiments of *convex analysis*. We do state the full results as encountered in [13].

Definition 1.57 (Convex Subset). Let V be a real vector space. A subset $A \subseteq V$ is said to be ***convex***, iff for all $x, y \in A$, we have that

$$(1-t)x + ty \in A$$

for all $t \in I$.

Definition 1.58 (Convex Function). Let V be a real vector space and $A \subseteq V$ convex. A function $f : A \to \mathbb{R}$ is said to be

- ***convex***, iff
$$f\big((1-t)x + ty\big) \le (1-t)f(x) + tf(y)$$
holds for all $x, y \in A$ and $t \in I$.
- ***strictly convex***, iff
$$f\big((1-t)x + ty\big) < (1-t)f(x) + tf(y)$$
holds for all $x, y \in A$, $x \neq y$, and $t \in (0,1)$.

Lemma 1.59. *Let E be a real Banach space, $U \subseteq E$ convex and $f : U \to \mathbb{R}$ convex. Then every local minimiser of f is a global minimiser. If f is strictly convex, the set of global minimiser is either a empty or a singleton.*

Lemma 1.60 (K. Weierstrass). *Let X be a topological space and $f : X \to \mathbb{R}$ satisfying the following condition: For all $\alpha \in \mathbb{R}$, the sublevel set $\{x \in X : f(x) \le \alpha\}$ is compact. Then f is uniformly bounded from below and attains its infimum.*

The condition of having bounded level sets is a growth condition.

Lemma 1.61. *Let E be a Banach space and $f: E \to \mathbb{R}$. Then all sublevel sets of f are bounded if and only if f is* ***coercive****, that is*

$$\lim_{|x|\to\infty} f(x) = +\infty.$$

Proof. Suppose that all level sets of f are bounded. Let $\alpha \in \mathbb{R}$. Then there exists $R = R(\alpha)$ such that $K_\alpha \subseteq B_R(0) \subseteq E$. Thus

$$\{x \in E : f(x) > \alpha\} = E \setminus K_\alpha \supseteq E \setminus B_R(0),$$

which implies $\lim_{|x|\to\infty} f(x) = +\infty$.
Conversly, suppose that f is coercive and that for some $\alpha \in \mathbb{R}$ the sublevel set K_α is unbounded. Thus we can construct a sequence $(x_n)_{n\in\mathbb{N}} \subseteq E$ with $|x_n| \to \infty$ as $n \to \infty$. But

$$\lim_{n\to\infty} f(x_n) \leq \alpha,$$

contradicting coercivity. □

Corollary 1.62. *Every coercive continuous (strictly) convex function $f: E \to \mathbb{R}$ on a finite-dimensional real normed space E admits a (unique) global minimiser.*

Proof. Lemma 1.61 implies that all sublevel sets of f are bounded. Since they are closed by definition and continuity of f, we get that each sublevel set is compact by Heine-Borel. Now apply 1.60 and conlude with lemma 1.59. □

Let us recall the notion of *directional derivatives* or *Gâteaux differentiability*.

Definition 1.63 (Gâteaux Derivative). Let E and F be two real Banach spaces. Suppose that $U \subseteq E$ is open and $x_0 \in U$. A function $f : U \to F$ is said to be ***Gâteaux differentiable at x_0***, iff

$$f'(x_0; x) := \lim_{t\to 0} \frac{f(x_0 + tx) - f(x_0)}{t}$$

exists for all $x \in E$ and such that $f'(x_0) : E \to F$ given by $f'(x_0)(x) := f'(x_0; x)$ is a continuous linear operator.

Lemma 1.64 (First Derivative Test of Convexity). *Let E be a real Banach space and $U \subseteq E$ open and convex. Then a Gâteaux differentiable function $f: U \to \mathbb{R}$ is convex if and only if*

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$

holds for every $x, x_0 \in U$.

Definition 1.65 (Second Gâteaux Derivative). Let E be a real Banach spaces. Suppose that $U \subseteq E$ is open and $x_0 \in U$. A Gâteaux differentiable function $f : U \to \mathbb{R}$ is said to be ***twice Gâteaux differentiable at x_0***, iff

$$f''(x_0; x, y) := \lim_{t \to 0} \frac{f'(x_0 + ty)(x) - f'(x_0)(x)}{t}$$

exists for all $x, y \in E$ and such that $f''(x_0) : E \times E \to F$ given by $f''(x_0)(x, y) := f'(x_0; x, y)$ is a continuous bilinear form.

Remark 1.66. If $E = \mathbb{R}^n$, then the twice Gâteaux differentiability simply means the existence of the *Hessian matrix*

$$\mathrm{Hess}_{x_0} f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j}(x_0) \right)_j^i$$

of f at x_0.

Lemma 1.67 (Second Derivative Test of Convexity). *Let E be a real Banach space and $U \subseteq E$ open and convex. Suppose that $f : U \to \mathbb{R}$ is twice Gâteaux differentiable on U.*

(a) *If*

$$f''(x; v, v) \geq 0$$

for all $x \in U$ and $v \in E$, then f is convex on U.

(b) *If the inequality in (a) is strict for $v \neq 0$, then f is strictly convex.*

(c) *Every convex function satisfies the inequality in (a).*

A stronger condition than Gâteaux differentiability is Fréchet differentiability.

Definition 1.68 (Fréchet Derivative). Let E and F be two real Banach spaces. Suppose that $U \subseteq E$ is open and $x_0 \in U$. A function $f : U \to F$ is said to be ***Fréchet differentiable at*** $\boldsymbol{x_0}$, iff there exists a continuous linear operator $Df(x_0) \in \mathrm{L}(E, F)$, such that

$$\lim_{x \to x_0} \frac{|f(x) - f(x_0) - Df(x_0)(x - x_0)|}{|x - x_0|} = 0.$$

Clearly, Fréchet differentiability implies Gâteaux differentiability and also the equality of the respective derivatives $f'(x_0) = Df(x_0)$. Moreover, the twice Fréchet differentiability of a function $f : U \to \mathbb{R}$ means the differentiability of both, f and Df, and one can show that

$$D^2 f(x_0)(x, y) = f''(x_0; x, y)$$

holds for all $x_0 \in U$ and $x, y \in E$.

Proposition 1.69. *Let $L \in C^\infty(\mathbb{R}^n)$ be convex. Then $DL : \mathbb{R}^n \to (\mathbb{R}^n)^*$ is a diffeomorphism if and only if L is* ***supercoersive****, that is*

$$\lim_{|x| \to \infty} \frac{L(x)}{|x|} = +\infty, \tag{1.17}$$

and the Hessian of L is everywhere positive-definite.

Remark 1.70. Note that supercoercivity implies coercivity. The converse might not be true, however. Consider for example the absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}$.

Proof. Suppose that DL is a diffeomorphism. Then the Hessian of L is invertibe. Since L is convex, the second derivative test of convexity 1.67 implies Hess $L \geq 0$. So Hess $L > 0$ and Hess L is positive-definite. For every $R > 0$ set

$$S_R := \{x \in \mathbb{R}^n : \|DL_x\| = R\}.$$

Then S_R is compact. Indeed, we have that

$$S_R = DL^{-1}\left(\mathbb{S}^{n-1}_R\right),$$

where

$$\mathbb{S}^{n-1}_R := \{\varphi \in (\mathbb{R}^n)^* : \|\varphi\| = R\}$$

denotes the sphere of radius R in $(\mathbb{R}^n)^*$. Since $(\mathbb{R}^n)^* \cong \mathbb{R}^n$ is finite-dimensional, Heine-Borel implies that $\mathbb{S}^{n-1}_R$ is compact because it is closed and bounded. Thus S_R is compact as the image of a compact set under a continuous function. Moreover, for every $x \in \mathbb{R}^n$, there exists a unique $x_0 = x_0(R, x) \in \mathbb{R}^n$ such that

$$DL_{x_0} = \frac{R}{|x|}\langle x, \cdot\rangle \in (\mathbb{R}^n)^*,$$

due to the assumption that DL is a diffeomorphism. We claim that $x_0 \in S_R$. Using the Cauchy-Schwarz inequality yields

$$\|DL_{x_0}\| = \sup_{y \in \mathbb{R}^n, |y|=1} |DL_{x_0}(y)| = \sup_{y \in \mathbb{R}^n, |y|=1} \frac{R}{|x|}\,|\langle x, y\rangle| \leq \sup_{y \in \mathbb{R}^n, \|y\|=1} \frac{R}{|x|}|x||y|,$$

and thus $\|DL_{x_0}\| \leq R$. But

$$\|DL_{x_0}\| \geq |DL_{x_0}(x/|x|)| = R,$$

and $x_0 \in S_R$. Using the first derivative test of convexity 1.64, we compute

$$\begin{aligned} L(x) &\geq L(x_0) + DL_{x_0}(x) - DL_{x_0}(x_0) \\ &= L(x_0) + R|x| - DL_{x_0}(x_0) \\ &\geq R|x| + \min_{y \in S_R}\left(L(y) - DL_y(y)\right), \end{aligned}$$

because S_R is compact. Set

$$C_R := \min_{y \in S_R}\left(L(y) - DL_y(y)\right) \in \mathbb{R}.$$

Then by the previous estimate we have that

$$\lim_{|x|\to\infty}\frac{L(x)}{|x|}\geq R+\lim_{|x|\to\infty}\frac{C_R}{|x|}=R.$$

Since $R>0$ was arbitrary, we conclude (1.17).

Conversely, suppose that L is supercoercive and that the Hessian of L is everywhere positive-definite. Let $\varphi\in(\mathbb{R}^n)^*$. Define $L_\varphi\in C^\infty(\mathbb{R}^n)$ by $L_\varphi:=L-\varphi$. Then $\operatorname{Hess}L_\varphi=\operatorname{Hess}L$, and thus by part (b) of the second derivative test of convexity 1.67 we get that L_φ is strictly convex. Moreover, L_φ is supercoercive. Indeed, since $\varphi\in(\mathbb{R}^n)^*$, there exists $C>0$ such that $|\varphi(x)|\leq C|x|$ holds for all $x\in\mathbb{R}^n$. Thus we estimate

$$L_\varphi(x)=L(x)-\varphi(x)\geq L(x)-|\varphi(x)|\geq L(x)-C\,|x|$$

for every $x\in\mathbb{R}^n$ and so

$$\lim_{|x|\to\infty}\frac{L_\varphi(x)}{|x|}=\lim_{|x|\to\infty}\frac{L(x)}{|x|}-C=+\infty.$$

Hence by corollary 1.62, L_φ admits a unique minimiser $x_0=x_0(\varphi)$. By elementary calculus

$$0=D(L_\varphi)_{x_0}=DL_{x_0}-\varphi.$$

So $DL_{x_0}=\varphi$. Hence DL is surjective. Moreover, uniqueness of x_0 implies injectivity of DL. Indeed, suppose that there exists $x_0'\in\mathbb{R}^n$ such that $DL_{x_0'}=\varphi$. Then x_0' is a critical point for L_φ. Since $\operatorname{Hess}L_\varphi$ is positive-definite, we have that x_0' is a minimiser of L_φ, in particular $x_0'=x_0$ by uniqueness. Hence DL is bijective. Since $\operatorname{Hess}L$ is positive definite, DL is a local diffeomorphism, thus a diffeomorphism.□

Definition 1.71 (Tonelli Lagrangian). Let (M,L) be an autonomous Lagrangian system. Fix a Riemannian metric g on M. The Lagrangian L is said to be ***Tonelli***, iff the following conditions are satisfied:

(T1) The fibrewise Hessian of L is positive-definite, that is,

$$\frac{\partial^2 L}{\partial v^i\partial v^j}(x,v)u^iu^j>0$$

for all $(x,v)\in TM$ and $u:=u^i\frac{\partial}{\partial x^i}\in T_xM$ such that $u\neq 0$.

(T2) L is fibrewise supercoersive, that is,

$$\lim_{|v|_g\to\infty}\frac{L(x,v)}{|v|_g}=+\infty$$

for all $x\in M$.

Example 1.72 (Tonelli Lagrangian on a Riemannian Manifold). Let (M,g) be a Riemannian manifold. For $V\in C^\infty(M)$, define $L\in C^\infty(TM)$ by

$$L(x,v):=\frac{1}{2}|v|_g^2-V(x).$$

Then by the computation performed in example 1.28, the fibrewise Hessian of L is positive-definite. Moreover, L is supercoercive since

$$\lim_{|v|_g\to\infty} \frac{L(x,v)}{|v|_g} = \lim_{|v|_g\to\infty} \frac{1}{2}|v|_g - \lim_{|v|_g\to\infty} \frac{V(x)}{|v|_g} = \lim_{|v|_g\to\infty} \frac{1}{2}|v|_g = +\infty$$

for all $x \in M$. Thus L is Tonelli.

Proposition 1.73. *Let (M^n, L) be a Lagrangian system such that L is fibrewise convex. Then the Legendre transform is a diffeomorphism if and only if L is Tonelli.*

Proof. Let $x \in M$. Then by equation 1.16, we have that

$$D^{\mathcal{F}} L|_{T_xM} : T_xM \to T_x^*M$$

is given by

$$D^{\mathcal{F}} L_v = d(L_x)_v \circ \Phi_v.$$

By proposition F.40, this is just the Fréchet derivative of L_x at $v \in T_xM$. Under the noncanonical identification $T_xM \cong \mathbb{R}^n$ the result follows from proposition 1.69. □

1.6 Legendre-Fenchel Duality

In this final section we come back to the terminology established in the section on the Legendre transform, namely, the notion of dualisation. Again, we make use of concepts established in the field of convex analysis and use them to show that the Legendre transform can be seen as a more concrete case of an abstract dualisation process, that is exchanging a normed space E by its dual E^*.

Definition 1.74 (Legendre-Fenchel Transform). Let E be a real Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$, $f \neq +\infty$. Then the ***Legendre-Fenchel transform of f***, written f^*, is defined to be the function $f^* : E^* \to \mathbb{R} \cup \{+\infty\}$, given by

$$f^*(\varphi) := \sup_{x\in E} \{\varphi(x) - f(x)\}. \tag{1.18}$$

Lemma 1.75. *Let E be a real Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$, $f \neq +\infty$. Then the Legendre-Fenchel transform $f^* : E^* \to \mathbb{R} \cup \{+\infty\}$ is convex.*

Proof. Let $\varphi, \psi \in E^*$ and $t \in I$. Then we compute

$$\begin{aligned} f^*\big((1-t)\varphi + t\psi\big) &= \sup_{x\in E} \big\{(1-t)\varphi(x) + t\psi(x) - f(x)\big\} \\ &= \sup_{x\in E} \big\{(1-t)\varphi(x) + t\psi(x) - (1-t)f(x) - tf(x)\big\} \\ &\leq (1-t) \sup_{x\in E} \{\varphi(x) - f(x)\} + t \sup_{x\in E} \{\psi(x) - f(x)\} \\ &= (1-t)f^*(\varphi) + tf^*(\psi). \end{aligned}$$

□

Remark 1.76. Note that the Legendre-Fenchel transform f^* of f is always convex by means of lemma 1.75, no matter what the nature of f is.

Proposition 1.77 (The Fenchel-Young Inequality). *Let E be a real Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$, $f \not\equiv +\infty$, lower semicontinuous and convex. Then*

$$f(x) + f^*(\varphi) \geq \varphi(x) \tag{1.19}$$

holds for all $x \in E$ and $\varphi \in E^*$.

Proposition 1.78 (The Classical Legendre Transform). *Let $f \in C^2(\mathbb{R}^n)$ with* Hess $f > 0$. *Then*

(a) *The map $\nabla : \mathbb{R}^n \to \mathbb{R}^n$ given by $x \mapsto \nabla f(x)$ is a homeomorphism.*
(b) *$f^*(x) = \langle x, (\nabla f)^{-1}(x) \rangle_{\mathbb{R}^n} - f\left((\nabla f)^{-1}(x)\right)$ for all $x \in \mathbb{R}^n$.*
(c) *$f^* \in C^1(\mathbb{R}^n)$ and $\nabla f^* = (\nabla f)^{-1}$.*
(d) *$\operatorname{Hess}_x f$ and $\operatorname{Hess}_{\nabla f(x)} f^*$ are inverse to each other for all $x \in \mathbb{R}^n$.*

Proposition 1.79. *Let $L \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that $L_x \in C^\infty(\mathbb{R}^n)$ defined by $L_x(y) := L(x, y)$ for all $x \in \mathbb{R}^n$ is convex. Suppose that the Legendre transform is a diffeomorphism. Then*

$$H_L(x, y) = (L_x)^*(y)$$

for all $x, y \in \mathbb{R}$.

Proof. By assumption, L is fibrewise convex and a diffeomorphism, thus L is Tonelli by proposition 1.73, and thus in particular Hess $L_x > 0$ for all $x \in \mathbb{R}^n$. Using part (b) of proposition 1.78, we compute

$$\begin{aligned}
H_L(x, y) &= \left(E_L \circ \tau_L^{-1}\right)(x, y) \\
&= E_L\left(x, (\nabla L_x)^{-1}(y)\right) \\
&= D^{\mathcal{F}} L_{(x,(\nabla L_x)^{-1}(y))}\left((\nabla L_x)^{-1}(y)\right) - L_x\left((\nabla L_x)^{-1}(y)\right) \\
&= \langle \nabla L_x\left((\nabla L_x)^{-1}(y)\right), (\nabla L_x)^{-1}(y) \rangle_{\mathbb{R}^n} - L_x\left((\nabla L_x)^{-1}(y)\right) \\
&= \langle y, (\nabla L_x)^{-1}(y) \rangle_{\mathbb{R}^n} - L_x\left((\nabla L_x)^{-1}(y)\right) \\
&= (L_x)^*(y)
\end{aligned}$$

for all $x, y \in \mathbb{R}^n$. □

1.7 Problems

1.80. Adopt the theory developed in the section on the *Legendre Transform* to the non-autonomous case, that is to the case of a Lagrangian system where the Lagrangian function can depend on time.

1.81. Complete the proof of theorem 1.20 about the Euler-Lagrange equations. *Hint:* Use the generalized notion of a *fibrewise differential* established in problem 1.80

Chapter 2
Hamiltonian Mechanics

Hamiltonian mechanics serves the same aim as Lagrangian mechanics, that is to describe systems of finitely many interacting particles. However, in the Hamiltonian case, we investigate a dual notion of a Lagrangian system, called Hamiltonian system, which has much more underlying structure. We begin this chapter by defining what this additional structure is, namely a symplectic structure. Then we give two important theorems in this new setting which are often used: the tangent-cotangent bundle isomorphism theorem and the Moser theorem.

Finally, we state the definitions governing Hamiltonian mechanics and prove an analogue of Noether's theorem for this case. Moreover, we point out the connection between the two versions of this theorem.

2.1 Symplectic Geometry

A profound difference between the tangent bundle TM and the cotangent bundle T^*M of a smooth manifold M is that on the latter there exists a natural 1-form, the tautological form α defined in definition 1.38.

A concise introduction to the very basics of symplectic geometry can be found in the last chapter of [6]. A more extensive treatment is given in [15] or [10].

2.1.1 Linear Symplectic Geometry

Recall the notion of a nondegenrate tensor 1.34.

Definition 2.1 (Symplectic Vector Space). A ***symplectic vector space*** is defined to be a tuple (V, ω), where V is a finite-dimensional real vector space and $\omega \in \Lambda^2(V^*)$ is nondegenerate, called a ***linear symplectic structure on V***.

Example 2.2. Let V be a finite-dimensional real vector space with $\dim V = 2n$. Let (a_i, b_i) be a basis for V and denote by (α^i, β^i) the corresponding dual basis. Then

$$\omega := \sum_{i=1}^{n} \alpha^i \wedge \beta^i$$

is a linear symplectic structure on V. Indeed, it is easy to see that the matrix representing $\widehat{\omega}$ is given by

$$\left(\widehat{\omega}^i_j\right) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Definition 2.3 (Symplectic Complement). Let (V, ω) be a symplectic vector space and $S \subseteq V$ a linear subspace. Define the ***symplectic complement of S in V with respect to ω***, written S^ω, to be the linear subspace of V given by

$$S^\omega := \{v \in V : \omega(v, u) = 0 \text{ for all } u \in S\}.$$

Lemma 2.4 (Dimension Formula for the Symplectic Complement). *Let (V, ω) be a symplectic vector space and $S \subseteq V$ a linear subspace. Then*

$$\dim S + \dim S^\omega = \dim V.$$

Proof. Define $\Phi : V \to S^*$ by $\Phi(v) := (i_v\omega)|_S$. Then clearly $\ker \Phi = S^\omega$ and moreover, Φ is surjective. Indeed, let $\varphi \in S^*$. Extend φ to $\widetilde{\varphi} \in V^*$ by setting

$$\widetilde{\varphi}(v) := \begin{cases} \varphi(v) & v \in S, \\ 0 & v \notin S. \end{cases}$$

Since $i_v\omega$ is an isomorphism, we find $v \in V$ such that $i_v\omega = \widetilde{\varphi}$. In particular, $(i_v\omega)|_S = \varphi$. Hence Φ is surjective and the usual rank-nullity theorem yields

$$\dim V = \dim S^* + \dim S^\omega = \dim S + \dim S^\omega.$$

□

Lemma 2.5. *Let $S \subseteq V$ be a subspace of a symplectic vector space (V, ω). Then*

$$\left(S^\omega\right)^\omega = S.$$

Proof. Using the dimension formula for the symplectic complement 2.4 twice, we get that $\dim \left(S^\omega\right)^\omega = \dim S$. Thus it is enough to show the inclusion $\supseteq$ only. Let $u \in S$. Then for any $v \in S^\omega$ we have that

$$\omega(u, v) = -\omega(v, u) = 0$$

by definition of S^ω. So $S \subseteq \left(S^\omega\right)^\omega$. □

Definition 2.6. A subspace $S \subseteq V$ of a symplectic vector space (V, ω) is said to be

• ***symplectic***, iff $S \cap S^{\omega} = \{0\}$.
• ***isotropic***, iff $S \subseteq S^{\omega}$.
• ***coisotropic***, iff $S^{\omega} \subseteq S$.
• ***Lagrangian***, iff $S = S^{\omega}$.

Lemma 2.7. *Let $S \subseteq V$ be a subspace of a symplectic vector space (V, ω). Then the following conditions are equivalent:*

(a) *S is symplectic.*
(b) *S^{ω} is symplectic.*
(c) *$\omega|_S \in \Lambda^2(S)$ is nondegenerate.*
(d) *$V = S \oplus S^{\omega}$.*

Proof. For proving (a)⇔(b), simply observe that

$$S^{\omega} \cap \left(S^{\omega}\right)^{\omega} = S^{\omega} \cap S$$

by lemma 2.5. For proving (a)⇔(c), suppose that $v \in S$ and $\omega(v, u) = 0$ for all $u \in S$. Then $v \in S^{\omega}$ and by assumption $v = 0$. Conversly, if $v \in S \cap S^{\omega}$, then $\omega(v, u) = \omega|_S(v, u) = 0$ for all $u \in S$. Since $\omega|_S$ is nondegenrate by assumption, we have that $v = 0$. Finally, for proving (a)⇔(d), we compute

$$\dim(S + S^{\omega}) = \dim S + \dim S^{\omega} - \dim(S \cap S^{\omega}) = \dim S + \dim S^{\omega} = \dim V$$

using lemma 2.4. Hence $V = S + S^{\omega}$. The converse is just the definition of the direct sum. □

The symplectic vector space given in example 2.2 turns out to be the standard model of any symplectic vector space.

Proposition 2.8 (Canonical Form Theorem for Symplectic Vector Spaces). *Let (V, ω) be a symplectic vector space. Then $\dim V = 2n$ and there exists a basis (a_i, b_i) of V such that*

$$\omega = \sum_{i=1}^{n} \alpha^i \wedge \beta^i$$

where $\left(\alpha^i, \beta^i\right)$ denotes the dual basis of (a_i, b_i).

Proof. We induct over the dimension $\dim V$. If $\dim V = 0$, there is nothing to show. So assume that the statement is true for all symplectic vector spaces with dimension strictly less than $\dim V \geq 1$. Since $\dim V \geq 1$, there exists $a_1 \in V$, such that $a_1 \neq 0$. Moreover, there exists $v \in V$ such that $\omega(a_1, v) \neq 0$ since ω is nondegenerate. Hence $\dim V \geq 2$. Set $b_1 := v/\omega(a_1, v)$. Then $\omega(a_1, b_1) = 1$ and antisymmetry of ω implies that (a_1, b_1) is linearly independent. Indeed, assume that $b_1 = \lambda a_1$ for some $\lambda \in \mathbb{R}$. Then $\omega(a_1, b_1) = \lambda\omega(a_1, a_1) = 0$. We claim that $S := \operatorname{span}_{\mathbb{R}}\{a_1, b_1\} \subseteq V$ is a symplectic subspace. It is immediate that $\omega|_S$ is nondegenerate, thus by lemma 2.7, we have that S is symplectic. Moreover, $V = S \oplus S^{\omega}$ and S^{ω} is symplectic. Again by lemma 2.7, this means that $\left(S^{\omega}, \omega|_{S^{\omega}}\right)$

is a symplectic vector space. But the dimension formula for symplectic complements 2.4 yields

$$\dim S^\omega = \dim V - \dim S = \dim V - 2.$$

Thus we can apply the induction assumption to $\left(S^\omega, \omega|_{S^\omega}\right)$. □

Definition 2.9 (Symplectomorphism). Let (V, ω) and $\left(\widetilde{V}, \widetilde{\omega}\right)$ be two symplectic vector spaces. An isomorphism $A : V \to \widetilde{V}$ is said to be a ***symplectomorphism***, iff $A^*\widetilde{\omega} = \omega$, where $A^*\widetilde{\omega}(\cdot,\cdot) := \widetilde{\omega}(A\cdot, A\cdot)$.

Let $n \in \mathbb{N}$, $n \geq 1$, and consider the real vector space $\mathbb{R}^{2n}$ with its standard basis (e_i). Setting $a_i := e_i$ for $1 \leq i \leq n$ and $b_i := e_i$ for $n+1 \leq i \leq 2n$, we get from example 2.2 the linear symplectic structure

$$\omega_0(v, u) := \sum_{i=1}^{n} \varepsilon^i \wedge \varepsilon^{n+i}(v, u) = v^t J_0 u,$$

where

$$J_0 := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathrm{Mat}(2n).$$

Proposition 2.10. *Let $n \in \mathbb{N}$, $n \geq 1$. Define*

$$\mathrm{Sp}(2n) := \mathrm{Sp}(\mathbb{R}^{2n}, \omega_0) := \{A \in \mathrm{GL}(2n) : A^*\omega_0 = \omega_0\}.$$

Then $\mathrm{Sp}(2n)$ *is a Lie group, called the* ***symplectic linear group*** *of dimension* $2n^2 + n$ *with associated Lie algebra*

$$\mathfrak{sp}(2n) = \{A \in \mathrm{Mat}(2n) : J_0 A + A^t J_0 = 0\}.$$

Proof. Let $A, B \in \mathrm{Sp}(2n)$. Then

$$\left(AB^{-1}\right)^* \omega_0 = \left(B^{-1}\right)^* A^*\omega_0 = \left(B^{-1}\right)^* \omega_0 = \left(B^{-1}\right)^* B^*\omega_0 = \mathrm{id}^*_{\mathbb{R}^{2n}} \omega_0 = \omega_0.$$

Thus $\mathrm{Sp}(2n)$ is a subgroup of $\mathrm{GL}(2n)$. Moreover, it is easy to check that by definition

$$\mathrm{Sp}(2n) = \{A \in \mathrm{GL}(2n) : A^t J_0 A = J_0\}.$$

We show that $\mathrm{Sp}(2n)$ is a regular level set of some smooth function. Observe that J_0 as well as $A^t J_0 A$ are antisymmetric. If $\mathfrak{o}(2n)$ denotes the real vector space of antisymmetric matrices in $\mathrm{Mat}(2n)$, we define $F : \mathrm{GL}(2n) \to \mathfrak{o}(2n)$ by

$$F(A) := A^t J_0 A.$$

Then $\mathrm{Sp}(2n) = F^{-1}(J_0)$ and we claim that J_0 is a regular value of F. Suppose $A_0 \in F^{-1}(J_0)$. We want to calculate $DF_{A_0} : T_{A_0}\mathrm{GL}(2n) \to T_{F(A_0)}\mathfrak{o}(2n)$. Since $\mathrm{GL}(2n)$ is an open subset of $\mathrm{Mat}(2n)$, we can identify $T_{A_0}\mathrm{GL}(2n)$ with $T_{A_0}\mathrm{Mat}(2n)$. By lemma F.39 the latter is given by $\mathrm{Mat}(2n)$ and for sufficiently small intervals

$\gamma_A(t) := A_0 + tA$ takes values in $\mathrm{GL}(2n)$ for $A \in \mathrm{Mat}(2n)$. Indeed, this follows from an application of the Neumann-series [18, 23]. Using proposition F.38, we compute

$$\begin{aligned} DF_{A_0}\left(\gamma_A'(0)\right) &= (F \circ \gamma_A)'(0) \\ &= \frac{d}{dt}\bigg|_{t=0} \left(J_0 + t(A_0^t J_0 A + A^t J_0 A_0) + t^2 A^t J_0 A\right) \\ &= A_0^t J_0 A + A^t J_0 A_0 . \end{aligned}$$

Let $B \in \mathfrak{o}(2n)$. Then one can check that $DF_{A_0}\left(\gamma_A'(0)\right) = B$ for $A = -\frac{1}{2} J_0 (A_0^t)^{-1} B$. Hence J_0 is a regular value and by the implicit function theorem for manifolds F.58, $\mathrm{Sp}(2n)$ is an embedded submanifold of $\mathrm{GL}(2n)$ of dimension

$$\dim \mathrm{Sp}(2n) = \dim \mathrm{GL}(2n) - \dim \mathfrak{o}(2n) = 4n^2 - n(2n-1) = 2n^2 + n .$$

So by proposition F.95, $\mathrm{Sp}(2n)$ is a Lie subgroup of $\mathrm{GL}(2n)$ and thus itself a Lie group. Using proposition F.59 we finally compute

$$\mathfrak{sp}(2n) = T_I \,\mathrm{Sp}(2n) = \ker DF_I = \{A \in \mathrm{Mat}(2n) : J_0 A + A^t J_0 = 0\} .$$

Exercise 2.11. Let $A \in \mathrm{GL}(n)$ and $B \in \mathrm{Mat}(n)$. Show that there exists an open interval $J \subseteq \mathbb{R}$ containing 0 such that $A + tB \in \mathrm{GL}(n)$ for all $t \in J$. *Hint:* Use the Neumann series.

Proposition 2.12. *Let V be a $2n$-dimensional real vector space and $\omega \in \Lambda^2(V^*)$. Then (V, ω) is a symplectic vector space if and only if $\omega^n \neq 0$.*

Proof. Suppose that (V, ω) is a symplectic vector space. By the canonical form theorem for symplectic vecto spaces 2.8, there exists a basis (a_i, b_i) of V such that

$$\omega = \sum_{i=1}^{n} \alpha^i \wedge \beta^i .$$

Using the multinomial theorem, we compute

$$\begin{aligned} \omega^n &= \sum_{i_1+\cdots+i_n=n} \binom{n}{i_1, \ldots, i_n} \left(\alpha^1 \wedge \beta^1\right)^{i_1} \wedge \cdots \wedge \left(\alpha^n \wedge \beta^n\right)^{i_n} \\ &= n!\left(\alpha^1 \wedge \beta^1 \wedge \cdots \wedge \alpha^n \wedge \beta^n\right), \end{aligned}$$

which is clearly nondegenerate. Conversly, suppose that ω is degenerate. Then there exists $v \in V$ such that $i_v \omega = 0$. Since i_v is a graded derivation, we have that $i_v(\omega^n) = n(i_v\omega) \wedge \omega^{n-1} = 0$. Extend v to a basis $(e_1, \ldots, e_{2n})$ of V with $e_1 = v$. Then

$$\omega^n(e_1, \ldots, e_{2n}) = \left(i_v(\omega^n)\right)(e_2, \ldots, e_{2n}) = 0.$$

Hence $\omega^n = 0$. □

From the definition of $\mathrm{Sp}(2n)$ it is easy to show that $\det A = \pm 1$. In fact, even more is true.

Lemma 2.13. *Let* $A \in \mathrm{Sp}(2n)$. *Then* $\det A = 1$.

Proof. By proposition 2.12, we have that $\omega_0^n \neq 0$. Moreover, we compute

$$\omega_0^n = \left(A^* \omega_0\right)^n = (\det A)\omega_0^n.$$

But then $\det A = 1$. □

2.1.2 The Category of Symplectic Manifolds

Definition 2.14 (Symplectic Manifold). A ***symplectic manifold*** is defined to be a tuple (M, ω) consisting of a smooth manifold M and a closed nondegenerate 2-form $\omega \in \Omega^2(M)$, called a ***symplectic form on*** M.

Lemma 2.15. *Let* (M, ω) *be a symplectic manifold. Then* $\dim M$ *is even and* M *is orientable.*

Proof. Let $x \in M$. Then $(T_x M, \omega_x)$ is a symplectic vector space. By proposition F.33, we have that $\dim M = \dim T_x M$. But by the canonical form theorem for symplectic vector spaces 2.8, we have that $\dim T_x M$ is even.
To show that M is orientable, it suffices to show the existence of a volume form. However, this immediately follows from proposition 2.12, since $\omega_x^n \neq 0$ for all $x \in M$ and thus ω^n is a volume form on M. □

Lemma 2.16. *Let* (M, ω) *be a compact symplectic manifold. Then* $H^2_{\mathrm{dR}}(M) \neq 0$.

Proof. Suppose ω is exact. Hence $\omega = d\eta$ for some $\eta \in \Omega^1(M)$. But then

$$\omega^n = (d\eta)^n = d\left(\eta \wedge \omega^{n-1}\right).$$

Using positivity together with Stokes theorem F.239, we compute

$$0 < \int_M \omega^n = \int_M d\left(\eta \wedge \omega^{n-1}\right) = \int_{\partial M} \eta \wedge \omega^{n-1} = 0.$$

So ω cannot be exact. But ω is closed, so $[\omega] \neq 0$ in $H^2_{\mathrm{dR}}(M)$. □

Corollary 2.17. $\mathbb{S}^{2n}$ *does not admit a symplectic form for all* $n \geq 2$.

Proof. Suppose that $\mathbb{S}^{2n}$ admits a symplectic form for $n \geq 2$. Then by lemma 2.16, we have that $H^2_{\mathrm{dR}}(M) \neq 0$. But by [6, 450]

$$\widetilde{H}^k_{\mathrm{dR}}(\mathbb{S}^n) = \begin{cases} \mathbb{R} & k = n, \\ 0 & k \neq n, \end{cases}$$

for all $n \geq 0$. □

Example 2.18 ($\mathbb{R}^{2n}$). Consider $\mathbb{R}^{2n}$ with coordinates (x^i, y^i). Then

$$\omega_0 := \sum_{i=1}^{n} dx^i \wedge dy^i$$

is a symplectic form on $\mathbb{R}^{2n}$.

Example 2.19 ($\mathbb{C}^n$). A more concise notation for the symplectic manifold given in example 2.18 is via the identification $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and

$$\omega_0 = \frac{i}{2} \sum_{k=1}^{n} dz^k \wedge d\overline{z}^k$$

where $\left(z^k\right)$ denote the standard coordinates on $\mathbb{C}^n$ with $z^k = x^k + iy^k$.

Example 2.20 (Orientable Surfaces). Let Σ be an orientable surface. Then any volume form on Σ is also a symplectic form because it is closed for dimensional reasons and nondegenerate since it is nowhere vanishing.

Example 2.21 (The Cotangent Bundle). Let M be a smooth manifold. Define $\lambda \in \Omega^1(T^*M)$ as follows: for $(x, \xi) \in T^*M$, define

$$\lambda_{(x,\xi)}(v) := \xi \left(D\pi_{(x,\xi)}(v)\right) \tag{2.1}$$

for all $v \in T_{(x,\xi)}T^*M$, where $\pi : T^*M \to M$ denotes the canonical projection. Of course, we need to check that λ is smooth. Let $\left(x^i, \xi_i\right)$ denote local coordinates on T^*M and $\xi = \xi_i dx^i$. Then we compute

$$\lambda\left(\frac{\partial}{\partial x^i}\right) = \xi_j dx^j \left(\frac{\partial \pi^k}{\partial x^i}\frac{\partial}{\partial x^k}\right) = \xi_j \delta_i^k \delta_k^j = \xi_i,$$

and similarly

$$\lambda\left(\frac{\partial}{\partial \xi_i}\right) = \xi_j dx^j \left(\frac{\partial \pi^k}{\partial \xi_i}\frac{\partial}{\partial x^k}\right) = 0.$$

Hence $\lambda = \xi_i dx^i$, which is smooth. The form $\lambda \in \Omega^1(T^*M)$ is called the ***tautological* 1*-form*** or ***Liouville* 1*-form***. Now set

$$\omega := -d\lambda \in \Omega^2(T^*M). \tag{2.2}$$

Then ω is clearly closed and moreover, we compute

$$\omega = -d\left(\xi_i dx^i\right) = -\frac{\partial \xi_i}{\partial x^j} dx^j \wedge dx^i - \frac{\partial \xi_i}{\partial \xi_j} d\xi_j \wedge dx^i = dx^i \wedge d\xi_i$$

in any local coordinates $\left(x^i, \xi_i\right)$. Hence ω is nondegenerate and thus a symplectic form, called the ***canonical symplectic form on*** $\boldsymbol{T^*M}$.

Example 2.22 ($\mathbb{S}^2$). By (F.5), we can identify $T_x\mathbb{S}^2$ with $x^\perp$ for every $x \in \mathbb{S}^2$. Thus we can define $\omega \in \Omega^2(\mathbb{S}^2)$ by

$$\omega_x(v, w) := \langle x, v \times w \rangle = \det(x, v, w)$$

for any $x \in \mathbb{S}^2$ and $v, w \in x^\perp$. Then ω is non-degenerate, since for $v \neq 0$ choose for example $w = x \times v$. To show that ω is smooth, we deduce a coordinate representation for it. Since $\mathbb{S}^2$ is an embedded hypersurface in $\mathbb{R}^3$, [6, 384] yields that

$$\iota^*_{\mathbb{S}^2}\left(i_N(dx \wedge dy \wedge dz)\right)$$

is an area form on $\mathbb{S}^2$ where N is the nowhere tangent vector field along $\mathbb{S}^2$ given by (this follows immediately from (F.5))

$$N := x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

Then

$$i_N(dx \wedge dy \wedge dz) = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$$

which is easily seen to be the same as ω. Moreover, in cylindrical coordinates (θ, h) on $\mathbb{S}^2$ given by

$$\left(\sqrt{1-h^2}\cos\theta, \sqrt{1-h^2}\sin\theta, h\right) \quad \text{for} \quad (\theta, h) \in (0, 2\pi) \times (-1, 1)$$

a short computation yields that

$$\omega = d\theta \wedge dh.$$

Proposition 2.23. *Let M be a smooth manifold and $F \in \mathrm{Diff}(M)$. Then*

$$\left(DF^\dagger\right)^* \lambda = \lambda.$$

Proof. Let $(x, \xi) \in T^*M$ and $v \in T_{(x,\xi)}T^*M$. We compute

$$\begin{aligned}
\left(\left(DF^\dagger\right)^* \lambda\right)_{(x,\xi)}(v) &= \lambda_{(DF^\dagger)(x,\xi)}\left(D\left(DF^\dagger\right)_{(x,\xi)}(v)\right) \\
&= \lambda_{(F(x), \xi\circ(DF_x)^{-1})}\left(D\left(DF^\dagger\right)_{(x,\xi)}(v)\right) \\
&= \left(\xi \circ (DF_x)^{-1}\right)\left(D\pi_{(F(x), \xi\circ(DF_x)^{-1})}\left(D\left(DF^\dagger\right)_{(x,\xi)}(v)\right)\right) \\
&= \xi\left((DF_x)^{-1} \circ D\left(\pi \circ DF^\dagger\right)_{(x,\xi)}(v)\right) \\
&= \xi\left((DF_x)^{-1} \circ D\left(F \circ \pi\right)_{(x,\xi)}(v)\right) \\
&= \xi\left((DF_x)^{-1} \circ DF_x \circ D\pi_{(x,\xi)}(v)\right) \\
&= \xi\left(D\pi_{(x,\xi)}(v)\right) \\
&= \lambda_{(x,\xi)}(v).
\end{aligned}$$

□

Definition 2.24. A morphism $F : (M, \omega) \to (\widetilde{M}, \widetilde{\omega})$ between two symplectic manifolds (M, ω) and $(\widetilde{M}, \widetilde{\omega})$ is defined to be a morphism $F \in C^\infty(M, \widetilde{M})$ such that $F^*\widetilde{\omega} = \omega$.

Exercise 2.25. Consider as objects symplectic manifolds and as morphisms the ones from definition 2.24. Show that they do form a category, the ***category of symplectic manifolds***.

Definition 2.26 (Symplectomorphism [10, 96]). A ***symplectomorphism*** is defined to be an isomorphism in the category of symplectic manifolds. Moreover, for (M, ω) a symplectic manifold, define the ***group of symplectomorphisms on*** $\boldsymbol{(M, \omega)}$, written $\mathrm{Symp}(M, \omega)$, by

$$\mathrm{Symp}(M, \omega) := \left\{F \in \mathrm{Diff}(M) : F^*\omega = \omega\right\}.$$

2.1.3 The Tangent-Cotangent Bundle Isomorphism

As in Riemannian geometry, one very important feature of a symplectic manifold (M, ω) is that there is a canonical identification of the tangent bundle TM and the cotangent bundle T^*M (for the Riemannian case see [6, 341]). But first we recall some basic facts from the tensor calculus on smooth manifolds.

Lemma 2.27 (Vector Bundle Chart Lemma [6, 253]). *Let M be a smooth manifold, $k \in \mathbb{N}$ and suppose that for all $x \in M$ we are given a real vector space E_x of dimension k. Let $E := \coprod_{x \in M} E_x$ and let $\pi : E \to M$ be given by $\pi(x, v) := x$. Moreover, suppose that we are given the following data:*

(i) *An open cover $(U_\alpha)_{\alpha \in A}$ of M.*
(ii) *For all $\alpha \in A$ a bijection $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ such that the restriction $\Phi_\alpha|_{E_x} : E_x \to \{x\} \times \mathbb{R}^k \cong \mathbb{R}^k$ is an isomorphism of vector spaces for all $x \in M$.*
(iii) *For all $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \varnothing$, a smooth mapping $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ such that the mapping $\Phi_\alpha \circ \Phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^k \to (U_\alpha \cap U_\beta) \times \mathbb{R}^k$ is of the form $\Phi_\alpha \circ \Phi_\beta^{-1}(x, v) = \big(x, \tau_{\alpha\beta}(x)v\big)$.*

Then E admits a unique topology and a smooth structure making it into a smooth manifold and a smooth vector bundle $\pi : E \to M$ of rank k with local trivializations $(U_\alpha, \Phi_\alpha)_{\alpha \in A}$.

Let M^n be a smooth manifold and let $k, l \in \mathbb{N}$. For all $x \in M$ define the space of ***mixed tensors of type*** $\boldsymbol{(k, l)}$ ***on*** $\boldsymbol{T_x M}$ by

$$T^{(k,l)}(T_x M) := \underbrace{T_x M \otimes \cdots \otimes T_x M}_{k} \otimes \underbrace{T_x^* M \otimes \cdots \otimes T_x^* M}_{l}.$$

By proposition 12.10 [6, 311] we have that

$$T^{(k,l)}(T_xM) \cong \mathrm{L}\big(\underbrace{T_x^*M, \ldots, T_x^*M}_{k}, \underbrace{T_xM, \ldots, T_xM}_{l}; \mathbb{R}\big)$$

since $(T_x^*M)^* \cong T_xM$ canonically (T_xM is finite-dimensional) where the latter denotes the space of all $\mathbb{R}$-valued multilinear forms on

$$\underbrace{T_x^*M \times \cdots \times T_x^*M}_{k} \times \underbrace{T_xM \times \cdots \times T_xM}_{l}.$$

We will always think of mixed tensors as multilinear forms. Let (U, x^i) be a chart about x. Then using corollary 12.12 [6, 313] we get that a basis for $T^{(k,l)}(T_xM)$ is given by all elements

$$\left.\frac{\partial}{\partial x^{i_1}}\right|_x \otimes \cdots \otimes \left.\frac{\partial}{\partial x^{i_k}}\right|_x \otimes dx^{j_1}|_x \otimes \cdots \otimes dx^{j_l}|_x$$

for all $1 \le i_1, \ldots, i_k, j_1, \ldots, j_l \le n$. Consequently, $\dim T^{(k,l)}(T_xM) = n^{k+l}$ and a particular tensor $A \in T^{(k,l)}(T_xM)$ expressed in this basis is given by

$$A = A^{i_1 \ldots i_k}_{j_1 \ldots j_l} \left.\frac{\partial}{\partial x^{i_1}}\right|_x \otimes \cdots \otimes \left.\frac{\partial}{\partial x^{i_k}}\right|_x \otimes dx^{j_1}|_x \otimes \cdots \otimes dx^{j_l}|_x \tag{2.3}$$

where

$$A^{i_1 \ldots i_k}_{j_1 \ldots j_l} := A\left(dx^{i_1}|_x, \ldots, dx^{i_k}|_x, \left.\frac{\partial}{\partial x^{j_1}}\right|_x, \ldots \left.\frac{\partial}{\partial x^{j_l}}\right|_x\right). \tag{2.4}$$

Next we want to "glue" together the different spaces of mixed tensors.

Proposition 2.28. *Let M be a smooth manifold and let $k, l \in \mathbb{N}$. Then*

$$T^{(k,l)}TM := \coprod_{x \in M} T^{(k,l)}(T_xM)$$

admits a unique topology and a smooth structure making it into a smooth manifold and a smooth vector bundle $\pi : T^{(k,l)}TM \to M$ of rank n^{k+l}. This smooth vector bundle is called the ***bundle of mixed tensors of type (k, l) on M****.*

Proof. This is an application of the vector bundle chart lemma 2.27. For all $x \in M$ define $E_x := T^{(k,l)}(T_xM)$. By the preceeding discussion, $\dim E_x = n^{k+l}$. Let $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ denote the smooth structure on M. Then clearly $(U_\alpha)_{\alpha \in A}$ is an open cover for M. For each $\alpha \in A$, define

$$\Phi_\alpha : \begin{cases} \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^{n^{k+l}} \\ (x, A) \mapsto \big(x, (A^{i_1 \ldots i_k}_{j_1 \ldots j_l})\big) \end{cases}$$

where we expressed A as in (2.3). Observe, that this map strongly depends on the coordinate functions. Clearly, the inverse is given by

$$\Phi_\alpha^{-1} : \begin{cases} U_\alpha \times \mathbb{R}^{n^{k+l}} \to \pi^{-1}(U_\alpha) \\ \big(x, (A^{i_1\ldots i_k}_{j_1\ldots j_l})\big) \mapsto (x, A) \end{cases} .$$

Hence each Φ_α is bijective. Now we have to check, that $\Phi_\alpha|_{E_x}$ is an isomorphism for all $x \in M$. By elementary linear algebra it is enough to show that Φ_α is linear. So let $\lambda \in \mathbb{R}$ and $A, B \in E_x$. Then

$$\begin{aligned} \Phi_\alpha|_{E_x}(x, A + \lambda B) &= \big(x, (A + \lambda B)^{i_1\ldots i_k}_{j_1\ldots j_l})\big) \\ &= \big(x, (A^{i_1\ldots i_k}_{j_1\ldots j_l}) + \lambda(B^{i_1\ldots i_k}_{j_1\ldots j_l})\big) \\ &= \Phi_\alpha|_{E_x}(x, A) + \lambda\Phi_\alpha|_{E_x}(x, B). \end{aligned}$$

Lastly, let $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$ and coordinates (x^i_α) and (x^i_β), respectively. Then for $x \in U_\alpha \cap U_\beta$ we have that

$$\left.\frac{\partial}{\partial x^i_\alpha}\right|_x = \frac{\partial x^j_\beta}{\partial x^i_\alpha}(x)\left.\frac{\partial}{\partial x^j_\beta}\right|_x \qquad \text{and} \qquad dx^i_\alpha|_x = \frac{\partial x^i_\alpha}{\partial x^j_\beta}(x)dx^j_\beta|_x.$$

So if $A^{i_1\ldots i_k}_{j_1\ldots j_l}$ are coordinates of a mixed tensor with respect to the basis induced by (x^i_α), we compute

$$\begin{aligned} A^{i_1\ldots i_k}_{j_1\ldots j_l} &= A\left(dx^{i_1}_\alpha|_x, \ldots, dx^{i_k}_\alpha|_x, \left.\frac{\partial}{\partial x^{j_1}_\alpha}\right|_x, \ldots \left.\frac{\partial}{\partial x^{j_l}_\alpha}\right|_x\right) \\ &= \frac{\partial x^{i_1}_\alpha}{\partial x^{p_1}_\beta}(x)\cdots\frac{\partial x^{i_k}_\alpha}{\partial x^{p_k}_\beta}(x)\frac{\partial x^{q_1}_\beta}{\partial x^{j_1}_\alpha}(x)\cdots\frac{\partial x^{q_l}_\beta}{\partial x^{j_l}_\alpha}(x)A^{p_1\ldots p_k}_{q_1\ldots q_l} \end{aligned}$$

Thus define $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n^{k+l}, \mathbb{R})$ by

$$\tau_{\alpha\beta}(x) := \left(\frac{\partial x^{i_1}_\alpha}{\partial x^{p_1}_\beta}(x)\cdots\frac{\partial x^{i_k}_\alpha}{\partial x^{p_k}_\beta}(x)\frac{\partial x^{q_1}_\beta}{\partial x^{j_1}_\alpha}(x)\cdots\frac{\partial x^{q_l}_\beta}{\partial x^{j_l}_\alpha}(x)\right).$$

Then $\tau_{\alpha\beta}$ is clearly smooth and moreover

$$\Phi_\alpha \circ \Phi_\beta^{-1}\left(x, (A^{p_1\ldots p_k}_{q_1\ldots q_l})\right) = \left(x, (A^{i_1\ldots i_k}_{j_1\ldots j_l})\right) = \left(x, \tau_{\alpha\beta}(x)(A^{p_1\ldots p_k}_{q_1\ldots q_l})\right).$$

Therefore, conditions (i)-(iii) in the vector bundle chart lemma 2.27 are satisfied and the statement follows. □

Remark 2.29. There is a much more abstract approach for constructing vector bundles[1] than the explicit one used for the bundle of mixed tensors in proposition 2.28. Let us first formulate a *metatheorem*:

"Anything one can do with vector spaces, one can also do with vector bundles."

We make this precise now. Let Vect denote the category of finite-dimensional real vector spaces. A functor

$$\mathcal{F} : \underbrace{\mathsf{Vect} \times \cdots \times \mathsf{Vect}}_{k} \to \mathsf{Vect}$$

which is either contravariant or covariant in its arguments, is said to be *smooth*, iff for all vector spaces $V_1, \dots, V_k, W_1, \dots, W_k \in \mathsf{Vect}$ the induced map

$$\bigoplus_{i=1}^{k} \widetilde{\mathrm{L}}(V_i, W_i) \to \mathrm{L}\left(\mathcal{F}(V_1, \dots, V_k), \mathcal{F}(W_1, \dots, W_k)\right)$$

where

$$\widetilde{\mathrm{L}}(V_i, W_i) := \begin{cases} \mathrm{L}(V_i, W_i) & \mathcal{F} \text{ is covariant in the } i\text{-th argument,} \\ \mathrm{L}(W_i, V_i) & \mathcal{F} \text{ is contravariant in the } i\text{-th argument,} \end{cases}$$

is a smooth map. The formal statement of the metatheorem can now be phrased as follows. If $\mathcal{F} : \mathsf{Vect} \times \cdots \times \mathsf{Vect} \to \mathsf{Vect}$ is a smooth functor as above and $\pi_i : E_i \to M$ are k vector bundles, then $\pi : \mathcal{F}(E_1, \dots, E_k) \to M$ is a vector bundle where

$$\mathcal{F}(E_1, \dots, E_k) := \coprod_{x \in M} \mathcal{F}(E_1|_x, \dots, E_k|_x)$$

and $\pi(x, v) := x$.

Recall, that in a category $\mathcal{C}$, a *section* of a morphism $f : X \to Y$ is a morphism $\sigma : Y \to X$ such that $f \circ \sigma = \mathrm{id}_Y$.

Definition 2.30 (Tensor Field). Let M be a smooth manifold and $k, l \in \mathbb{N}$. A ***smooth tensor field of type*** $\boldsymbol{(k, l)}$ ***on*** $\boldsymbol{M}$ is defined to be a section of $\pi : T^{(k,l)}TM \to M$. The space of all smooth tensor fields of type (k, l) on M is denoted by $\mathcal{T}^{k,l}(M) := \Gamma\left(T^{(k,l)}TM\right)$.

Example 2.31. Vector Field and Covector Field Let M be a smooth manifold. Of particular importance are the tensor fields such that $k + l = 1$. If $k = 1$, such tensor fields are called ***vector fields*** and we write $\mathfrak{X}(M) := \Gamma\left(T^{(1,0)}TM\right)$. Likewise, if $l = 1$, we call such tensor fields ***covector fields*** and write $\mathfrak{X}^*(M) := \Gamma\left(T^{(0,1)}TM\right)$.

[1] See lecture 14 from the lecture notes of the course *Differential Geometry I* taught by *Will J. Merry* at the *ETH Zurich* in the autumn semester 2018.

Let $\big(U,(x^i)\big)$ be a chart on M and $A : M \to T^{(k,l)}TM$ such that $A_x \in T^{(k,l)}(T_xM)$ for all $x \in M$. From (2.3) we get that

$$A_x = A^{i_1\dots i_k}_{j_1\dots j_l}(x)\frac{\partial}{\partial x^{i_1}}\bigg|_x \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}}\bigg|_x \otimes dx^{j_1}|_x \otimes \cdots \otimes dx^{j_l}|_x$$

for all $x \in U$ where $A^{i_1\dots i_k}_{j_1\dots j_l} : U \to \mathbb{R}$ are given as in (2.4). We will call these functions the ***component functions of A***. Recall, that a map $F : M \to N$ between two smooth manifolds M and N is said to be *smooth*, iff for every $x \in M$ there exists a chart (U,φ) about x on M and a chart (V,ψ) about $F(x)$ on N such that $U \cap F^{-1}(V)$ is open in M and $\psi \circ F \circ \varphi^{-1} : \varphi\big(U \cap F^{-1}(V)\big) \to \psi(V)$ is smooth. Moreover, if $A \subseteq U \subseteq M$, where U is open and A is closed in M, a function $\psi \in C^\infty(M)$ is said to be a *smooth bump function for A supported in U*, iff $0 \le \psi \le 1$, $\psi|_A = 1$ and $\operatorname{supp}\psi \subseteq U$. The paracompactness condition guarantees that smooth bump functions exist in great abundance.

Proposition 2.32 (Existence of Smooth Bump Functions [6, 44]). *Let M be a smooth manifold and $A \subseteq U \subseteq M$, where U is open and A is closed in M. Then there exists a smooth bump function for A supported in U.*

Proposition 2.33 (Smoothness Criteria for Tensor Fields [6, 317]). *Let M be smooth manifold, $k,l \in \mathbb{N}$ and $A : M \to T^{(k,l)}TM$ such that $A_x \in T^{(k,l)}T_xM$ for all $x \in M$. Then the following conditions are equivalent:*

(a) $A \in \Gamma\left(T^{(k,l)}TM\right)$.
(b) *In every smooth coordinate chart, the component functions of A are smooth.*
(c) *Each point of M is contained in a chart in which A has smooth component functions.*
(d) *For all $\omega^1,\dots,\omega^k \in \mathfrak{X}^*(M)$ and $X_1,\dots,X_l \in \mathfrak{X}(M)$, the function*

$$\mathcal{A}(\omega^1,\dots,\omega^k,X_1,\dots,X_l) : M \to \mathbb{R}$$

defined by

$$\mathcal{A}\left(\omega^1,\dots,\omega^k,X_1,\dots,X_l\right)(x) := A_x\left(\omega^1_x,\dots,\omega^k_x,X_1|_x,\dots,X_l|_x\right) \tag{2.5}$$

is smooth.
(e) *Let $U \subseteq M$ be open. If $\omega^1,\dots,\omega^k \in \mathfrak{X}^*(U)$ and $X_1,\dots,X_l \in \mathfrak{X}(U)$, then $\mathcal{A}$ defined by (2.5) belongs to $C^\infty(U)$.*

Proof. We prove (a) $\Leftrightarrow$ (b) and (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (b).
To prove (a) $\Leftrightarrow$ (b), let $x \in M$ and $\big(U,(x^i)\big)$ be a smooth chart on M about x. Proposition 2.28 yields a map $\Phi_U : \pi^{-1}(U) \to U \times \mathbb{R}^{n^{k+l}}$, and the proof of the vector bundle chart lemma implies, that the corresponding chart on $T^{(k,l)}TM$ is given by $\left(\pi^{-1}(U),\widetilde{\varphi}\right)$, where

$$\widetilde{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^{n^{k+l}}$$

is defined by

$$\widetilde{\varphi} := \left(\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}}\right) \circ \Phi_U.$$

Since $A_x \in T^{(k,l)} T_x M$ for all $x \in M$, we have that

$$A^{-1}\left(\pi^{-1}(U)\right) = (\pi \circ A)^{-1}(U) = \mathrm{id}_M(U) = U.$$

Hence $U \cap A^{-1}\left(\pi^{-1}(U)\right) = U$, which is open in M, and

$$\widetilde{\varphi} \circ A \circ \varphi^{-1} : \varphi(U) \to \widetilde{\varphi}\left(\pi^{-1}(U)\right)$$

is given by

$$\begin{aligned}
\left(\widetilde{\varphi} \circ A \circ \varphi^{-1}\right)\left(\varphi(y)\right) &= \left(\varphi \times \mathrm{id}_{\mathbb{R}^{n^{k+l}}}\right)\left(\Phi_U(A_y)\right) \\
&= \left(\varphi(y), (A^{i_1 \ldots i_k}_{j_1 \ldots j_l})(y)\right) \\
&= \left(\varphi(y), \left((A^{i_1 \ldots i_k}_{j_1 \ldots j_l}) \circ \varphi^{-1}\right)\left(\varphi(y)\right)\right)
\end{aligned}$$

for all $y \in U$. Thus $\widetilde{\varphi} \circ A \circ \varphi^{-1}$ is smooth if and only if $(A^{i_1 \ldots i_k}_{j_1 \ldots j_l}) \circ \varphi^{-1}$ is smooth, which is equivalent to $A^{i_1 \ldots i_k}_{j_1 \ldots j_l}$ being smooth.
The implication (b) $\Rightarrow$ (c) is immediate.
To prove (c) $\Rightarrow$ (d), suppose $x \in M$ and let $(U, (x^i))$ be a chart about x such that the component functions of A are smooth. By example 2.31 and the equivalence (a) $\Leftrightarrow$ (b) we have

$$\omega^i = \omega^i_j dx^j \qquad \text{and} \qquad X_i = X_i^j \frac{\partial}{\partial x^j}$$

on U for smooth functions ω^i_j and X_i^j. Thus for any $y \in U$ we compute

$$\begin{aligned}
\mathcal{A}\left(\omega^1, \ldots, \omega^k, X_1, \ldots, X_l\right)(y) &= A_x\left(\omega^1_x, \ldots, \omega^k_x, X_1|_x, \ldots, X_l|_x\right) \\
&= \omega^1_{i_1}(y) \cdots \omega^k_{i_k}(y) X_1^{j_1}(y) \cdots X_l^{j_l}(y) A^{i_1 \ldots i_k}_{j_1 \ldots j_l}(y)
\end{aligned}$$

and so $\mathcal{A}\left(\omega^1, \ldots, \omega^k, X_1, \ldots, X_l\right)$ is smooth.
To prove (d) $\Rightarrow$ (e), we use the fact that smoothness is a local property. Let $x \in U$ and suppose (V, φ) is a chart on U centered at x. Then $\varphi(V) \subseteq \mathbb{R}^n$ is open and so we find $\varepsilon > 0$ such that $B_\varepsilon(0) \subseteq \varphi(V)$. Set $A := \varphi^{-1}\left(\overline{B}_{\varepsilon/2}(0)\right) \subseteq U$. Then A is closed in U and by proposition 2.32 there exists a smooth bump function $\psi \in C^\infty(U)$ for A supported in U. Define $\widetilde{\omega}^i : M \to T^*M$ and $\widetilde{X}_i : M \to TM$ by

$$\widetilde{\omega}^i_x := \begin{cases} \psi(x)\omega^i_x & x \in U, \\ 0_x & x \in M \setminus \operatorname{supp}\psi, \end{cases}$$

and

$$\widetilde{X}_i|_x := \begin{cases} \psi(x) X_i|_x & x \in U, \\ 0_x & x \in M \setminus \operatorname{supp}\psi. \end{cases}$$

Then $\widetilde{\omega}^i \in \mathfrak{X}^*(M)$ and $\widetilde{X}_i \in \mathfrak{X}(M)$ by the gluing lemma for smooth maps (see [6, 35]). Moreover, on $\varphi^{-1}\left(B_{\varepsilon/2}(0)\right)$ we have that $\widetilde{\omega}^i = \omega^i$ and $\widetilde{X}_i = X_i$. But then also

$$\mathcal{A}\left(\widetilde{\omega}^1, \dots, \widetilde{\omega}^k, \widetilde{X}_1, \dots, \widetilde{X}_l\right) = \mathcal{A}\left(\omega^1, \dots, \omega^k, X_1, \dots, X_l\right)$$

on this neighbourhood, and so since the former is smooth by assumption, so is the latter. Finally, to prove (e) $\Rightarrow$ (b), let $(U, (x^i))$ be a chart about $x \in M$. Consider $\omega^i \in \mathfrak{X}^*(U)$ and $X_i \in \mathfrak{X}(U)$ defined by

$$\omega^i := \delta^i_j dx^j \qquad \text{and} \qquad X_i := \delta^j_i \frac{\partial}{\partial x^j}.$$

Then it is easy to verify that

$$\mathcal{A}\left(\omega^{i_1}, \dots, \omega^{i_k}, X_{j_1}, \dots, X_{j_l}\right) = A^{i_1 \dots i_k}_{j_1 \dots j_l}$$

holds on U. Thus by assumption, each component function is smooth. $\square$

Part (d) of the smoothness criteria for tensor fields 2.33 implies that for any tensor field $A \in \Gamma\left(T^{(k,l)}TM\right)$ there is a mapping

$$\mathcal{A} : \underbrace{\mathfrak{X}^*(M) \times \dots \times \mathfrak{X}^*(M)}_{k} \times \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{l} \to C^\infty(M)$$

defined by

$$\left(\omega^1, \dots, \omega^k, X_1, \dots, X_l\right) \mapsto \mathcal{A}\left(\omega^1, \dots, \omega^k, X_1, \dots, X_l\right).$$

We will call this mapping the ***map induced by the tensor field A***.

Proposition 2.34 (Tensor Field Characterisation Lemma [6, 318]). *Let M be a smooth manifold and $k, l \in \mathbb{N}$. A mapping*

$$\mathcal{A} : \underbrace{\mathfrak{X}^*(M) \times \dots \times \mathfrak{X}^*(M)}_{k} \times \underbrace{\mathfrak{X}(M) \times \dots \times \mathfrak{X}(M)}_{l} \to C^\infty(M)$$

is induced by a (k, l)-tensor field if and only if $\mathcal{A}$ is multilinear over $C^\infty(M)$.

Proof. Suppose $\mathcal{A}$ is induced by a (k, l)-tensor field A. Let $\omega^1, \dots, \omega^k, \widetilde{\omega}^i \in \mathfrak{X}^*(M)$ and $X_1, \dots, X_l \in \mathfrak{X}(M)$ as well as $f \in C^\infty(M)$. Then for any $x \in M$ we compute

$$\begin{aligned}
\mathcal{A}\left(\dots, \omega^i + f\widetilde{\omega}^i, \dots\right)(x) &= A_x\left(\dots, \omega^i_x + f(x)\widetilde{\omega}^i_x, \dots\right) \\
&= A_x\left(\dots, \omega^i_x, \dots\right) + f(x)A_x\left(\dots, \widetilde{\omega}^i_x, \dots\right) \\
&= \mathcal{A}\left(\dots, \omega^i, \dots\right)(x) + f(x)\mathcal{A}\left(\dots, \widetilde{\omega}^i, \dots\right)(x) \\
&= \left(\mathcal{A}\left(\dots, \omega^i, \dots\right) + f\mathcal{A}\left(\dots, \widetilde{\omega}^i, \dots\right)\right)(x).
\end{aligned}$$

Thus $\mathcal{A}$ is $C^\infty(M)$-multilinear with respect to the first k arguments. Similarly, $\mathcal{A}$ is $C^\infty(M)$-multilinear with repect to the last l arguments.
Conversly, suppose that

$$\mathcal{A} : \underbrace{\mathfrak{X}^*(M) \times \cdots \times \mathfrak{X}^*(M)}_{k} \times \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{l} \to C^\infty(M)$$

is $C^\infty(M)$-multilinear. We wish to define a (k,l)-tensor field A that induces $\mathcal{A}$. That this is indeed possible, is the observation that $\mathcal{A}\left(\omega^1, \dots, \omega^k, X_1, \dots, X_l\right)(x)$ only depends on $\omega^1_x, \dots, \omega^k_x, X_1|_x, \dots, X_l|_x$. Thus we divide the remaining proof into three steps.
Step 1: $\mathcal{A}\left(\omega^1, \dots, \omega^k, X_1, \dots, X_l\right)$ *acts locally.* That is, if either some ω^i or X_i vanish on an open set U, then so does $\mathcal{A}\left(\omega^1, \dots, \omega^k, X_1, \dots, X_l\right)$. Let $x \in U$ and $\psi \in C^\infty(M)$ be a smooth bump function for $\{x\}$ supported in U. Then $\psi\omega^i = 0$ on M and by $C^\infty(M)$-multilinearity

$$0 = \mathcal{A}\left(\dots, \psi\omega^i, \dots\right) = \psi(x)\mathcal{A}\left(\dots, \omega^i, \dots\right)(x) = \mathcal{A}\left(\dots, \omega^i, \dots\right)(x).$$

An analogous argument works if some X_i vanishes on U.
Step 2: $\mathcal{A}\left(\omega^1, \dots, \omega^k, X_1, \dots, X_l\right)$ *acts pointwise.* Thats is, if ω^i_x or $X_i|_x$ vanish for some $x \in M$, then so does $\mathcal{A}\left(\omega^1, \dots, \omega^k, X_1, \dots, X_l\right)$. Let $(U, (x^i))$ be a chart about x. Then $\omega^i = \omega^i_j dx^j$ on U. Let $\psi \in C^\infty(U)$ denote the smooth bump function used in the proof of part (d) $\Rightarrow$ (e) of the smoothness criteria for tensor fields 2.33. Define

$$\varepsilon^j := \begin{cases} \psi(x) dx^j|_x & x \in U, \\ 0_x & x \in M \setminus \operatorname{supp}\psi, \end{cases}$$

and

$$f^i_j := \begin{cases} \psi(x)\omega^i_j(x) & x \in U, \\ 0_x & x \in M \setminus \operatorname{supp}\psi. \end{cases}$$

Then $\omega^i = f^i_j \varepsilon^j$ on a neighbourhood of x and so by multilinearity and step 1, we have that

$$\mathcal{A}\left(\dots, \omega^i, \dots\right) = f^i_j \mathcal{A}\left(\dots, \varepsilon^j, \dots\right)$$

on a neighbourhood of x. But since ω^i_x vanishes so does each $\omega^i_j(x)$. Hence

$$\mathcal{A}\left(\dots, \omega^i, \dots\right)(x) = f^i_j(x)\mathcal{A}\left(\dots, \varepsilon^j, \dots\right)(x) = \omega^i_j(x)\mathcal{A}\left(\dots, \varepsilon^j, \dots\right)(x) = 0.$$

An analogous argument works if some $X_i|_x$.
Step 3: Definition of the (k,l)*-tensor field* A *inducing* $\mathcal{A}$. Let $x \in M$, $\omega^1, \dots, \omega^k \in T^*_x M$ and $v_1, \dots, v_l \in T_x M$. Suppose that $\widetilde{\omega}^1, \dots, \widetilde{\omega}^k \in \mathfrak{X}^*(M)$ and $\widetilde{X}_1, \dots, \widetilde{X}_l \in \mathfrak{X}(M)$ are any extensions, respectively. That is, $\widetilde{\omega}^i_x = \omega^i$ and $\widetilde{X}_i|_x = v_i$. They do always exist, since in a chart $(U, (x^i))$ we may write

$$\omega^i = \omega^i_j dx^j|_x \qquad \text{and} \qquad v_i = v_i^j \left.\frac{\partial}{\partial x^j}\right|_x$$

and so using a smooth bump function for $\{x\}$ supported in U we can construct global maps as in step 2 if we consider the components as constant functions. Now define

$$A_x\left(\omega^1, \dots, \omega^k, v_1, \dots, v_l\right) := \mathcal{A}\left(\widetilde{\omega}^1, \dots, \widetilde{\omega}^k, \widetilde{X}_1, \dots, \widetilde{X}_l\right)(x). \tag{2.6}$$

This is well-defined by step 2. Now if $\omega^1, \dots, \omega^k \in \mathfrak{X}^*(M)$ and $X_1, \dots, X_l \in \mathfrak{X}(M)$, we have that

$$\mathcal{A}\left(\omega^1, \dots, \omega^k, X_1, \dots, X_l\right)(x) = A_x\left(\omega^1_x, \dots, \omega^k_x, X_1|_x, \dots, X_l|_x\right),$$

since ω^i and X_i are extensions of ω^i_x and $X_i|_x$, respectively, for all $x \in M$. So the assumption that $\mathcal{A}$ takes values in the space of smooth functions $C^\infty(M)$ together with part (d) of the smoothness criteria for tensor fields 2.33 yields that A is a smooth (k, l)-tensor field which moreover induces $\mathcal{A}$. $\square$

Proposition 2.35 (Bundle Homomorphism Characterisation Lemma [6, 262]). *Let $\pi : E \to M$ and $\widetilde{\pi} : \widetilde{E} \to M$ be smooth vector bundles over a smooth manifold M. A map $\mathcal{F} : \Gamma(E) \to \Gamma(\widetilde{E})$ is linear over $C^\infty(M)$ if and only if there exists a smooth bundle homomorphism $F : E \to \widetilde{E}$ over M such that $\mathcal{F}(\sigma) = F \circ \sigma$ for all $\sigma \in \Gamma(E)$.*

Theorem 2.36 (Tangent-Cotangent Bundle Isomorphism). *Let (M, ω) be a symplectic manifold. Define $\Omega : TM \to T^*M$ by*

$$\Omega(v)(w) := \omega_x(v, w) \tag{2.7}$$

*for all $x \in M$ and $v, w \in T_xM$. Then Ω is a well-defined smooth bundle isomorphism. The morphism Ω is called the **tangent-cotangent bundle isomorphism**.*

Proof. Using the tensor field characterisation lemma 2.34, ω induces a map

$$\omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$$

which is $C^\infty(M)$-multilinear. Thus for $X \in \mathfrak{X}(M)$ we define $\Omega_X : \mathfrak{X}(M) \to C^\infty(M)$ by

$$\Omega_X(Y) := \omega(X, Y).$$

Since ω is multilinear over $C^\infty(M)$, so is Ω_X, and thus again by the tensor field characterisation lemma 2.34, Ω_X belongs to $\mathfrak{X}^*(M)$. Hence we get a map $\Omega : \mathfrak{X}(M) \to \mathfrak{X}^*(M)$ by $\Omega(X) := \Omega_X$ which is also multilinear over $C^\infty(M)$. Finally, by the bundle homomorphism characterisation lemma 2.35, there exists a smooth vector bundle homomorphism $\Omega : TM \to T^*M$ such that $\Omega_X = \Omega \circ X$ for all $X \in \mathfrak{X}(M)$. Let $x \in M$, $v, w \in T_xM$ and $V, W \in \mathfrak{X}(M)$ be extensions of v and w, respectively (see step 3 in the proof of the tensor field characterisation lemma 2.34). We compute

$$\Omega_V|_x(w) = \Omega_V(W)(x) = \omega(V, W)(x) = \omega_x(V|_x, W|_x) = \omega_x(v, w)$$

and since $(\Omega \circ V)|_x(w) = \Omega(V|_x)(w) = \Omega(v)(w)$, we have that Ω coincides with the map defined in (2.7). Next we show that Ω is injective. Let $v, \widetilde{v} \in TM$ such that $\Omega(v) = \Omega(\widetilde{v})$. Since Ω is a fibrewise mapping, we must have that $v, \widetilde{v} \in T_xM$ for some $x \in M$. Moreover, by definition we have that $\omega_x(v - \widetilde{v}, w) = 0$ for every $w \in T_xM$. By nondegeneracy, it follows that $v = \widetilde{v}$. Moreover, since T_xM is finite-dimensional, we get that Ω is also surjective, thus bijective. Since any bijective smooth bundle homomorphism over M is automatically a smooth bundle isomorphism by [6, 262], Ω is a smooth bundle isomorphism. □

Remark 2.37. In what follows, we will denote both the smooth bundle isomorphism $\Omega : TM \to T^*M$ as well as the induced $C^\infty(M)$-linear morphism $\Omega : \mathfrak{X}(M) \to \mathfrak{X}^*(M)$ by the same letter Ω. However, as a subtle distinction between those two maps, we will write Ω_X for the evaluation of the latter at some $X \in \mathfrak{X}(M)$.

Proposition 2.38. *Let (M, ω) be a symplectic manifold and $\eta \in \Omega^1(M)$. Then there exists a unique vector field $X \in \mathfrak{X}(M)$ such that*

$$i_X\omega = \eta.$$

Proof. Using the tangent-cotangent bundle isomorphism 2.36, set

$$X := \Omega^{-1}(\eta).$$

Then for any $x \in M$ and $v \in T_xM$ we compute

$$\eta_x(v) = \left(\Omega_X\right)_x(v) = \Omega(X|_x)(v) = \omega_x\left(X|_x, v\right) = (i_X)_x(v).$$

□

2.1.4 The Darboux Theorem

This section deals with a nonlinear analogue of the canonical form theorem for a symplectic vector space 2.8. The main theorem of this section illustrates the most dramatic difference between symplectic structures and Riemannian ones: unlike in the Riemannian case, there is no local obstruction to a symplectic structure being locally equivalent to the standard flat model $\left(\mathbb{R}^{2n}, \omega_0\right)$.

Definition 2.39 (Time-Dependent Vector Field [6, 236]). Let M be a smooth manifold. A ***time-dependent vector field on*** $\boldsymbol{M}$ is a smooth map $X : J \times M \to TM$, where $J \subseteq \mathbb{R}$ is an interval, such that $X(t, x) \in T_xM$ for all $(t, x) \in J \times M$. An ***integral curve of a time-dependent vector field*** $\boldsymbol{X}$ is defined to be a curve $\gamma \in C^\infty(J_0, M)$, where $J_0 \subseteq J$ is an interval, such that

$$\gamma'(t) = X\left(t, \gamma(t)\right)$$

holds for all $t \in J_0$.

Proposition 2.40 (Fundamental Theorem of Time-Dependent Flows [6, 237]). *Let M be a smooth manifold, $J \subseteq \mathbb{R}$ an open interval and $X : J \times M \to TM$ a time-dependent vector field. Then there exists an open subset $\mathcal{D} \subseteq J \times J \times M$ together with a map $\psi \in C^\infty(\mathcal{D}, M)$, called the* ***time-dependent flow of X***, *such that:*

(a) *For all $t_0 \in J$ and $x \in M$, the set $\mathcal{D}^{(t_0,x)} := \{t \in J : (t, t_0, x) \in \mathcal{D}\}$ is an open interval containing t_0 and the curve $\psi^{(t_0,x)}(t) := \psi(t, t_0, x)$ is the unique maximal integral curve of X with initial condition $\psi^{(t_0,x)}(t_0) = x$.*
(b) *If $t_1 \in \mathcal{D}^{(t_0,x)}$ and $y = \psi^{(t_0,x)}(t_1)$, then $\mathcal{D}^{(t_1,y)} = \mathcal{D}^{(t_0,x)}$ and $\psi^{(t_1,y)} = \psi^{(t_0,x)}$.*
(c) *For each $(t_1, t_0) \in J \times J$, the set $M_{t_1,t_0} := \{x \in M : (t_1, t_0, x) \in \mathcal{D}\}$ is open in M, and the map $\psi_{t_1,t_0} : M_{t_1,t_0} \to M$ defined by $\psi_{t_1,t_0}(x) := \psi(t_1, t_0, x)$ is a diffeomorphism from M_{t_1,t_0} onto M_{t_0,t_1} with inverse ψ_{t_0,t_1}.*
(d) *If $x \in M_{t_1,t_0}$ and $\psi_{t_1,t_0}(x) \in M_{t_2,t_1}$, then $x \in M_{t_2,t_0}$ and*

$$\psi_{t_2,t_1} \circ \psi_{t_1,t_0}(x) = \psi_{t_2,t_0}(x).$$

In [2], the Lie derivative is referred to as the *fisherman's derivative*.

Proposition 2.41 (Fisherman's Formula [6, 571]). *Let M be a smooth manifold and suppose that $X : J \times M \to TM$ is a time-dependent vector field with time-dependent flow $\psi : \mathcal{D} \to M$. For any form $\omega \in \Omega^k(M)$ and any $(t_1, t_0, x) \in \mathcal{D}$*

$$\left.\frac{d}{dt}\right|_{t=t_1} \psi^*_{t,t_0}\omega = \psi^*_{t_1,t_0}\left(\mathcal{L}_{X_{t_1}}\omega\right),$$

holds, where $X_{t_1} := X(t_1, \cdot) \in \mathfrak{X}(M)$.

Proof. Applying $\psi^*_{t_0,t_1}$ to above equation yields

$$\left.\frac{d}{dt}\right|_{t=t_1} \psi^*_{t,t_1}\omega = \mathcal{L}_{X_{t_1}}\omega$$

using part (c) and (d) of the fundamental theorem of time-dependent flows 2.40. An appropriate modification of proposition F.191 shows that the left-hand-side is a graded derivation of degree zero. Thus it is enough to show that both sides coincide on functions and exact 1-forms by mean of proposition F.199. Let $f \in C^\infty(M)$. Then for any $x \in M$ we compute

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=t_1} \left(\psi^*_{t,t_1} f\right)_x &= \left.\frac{d}{dt}\right|_{t=t_1} f\left(\psi_{t,t_1}(x)\right)\\ &= \left(f \circ \psi^{(t_1,x)}\right)'(t_1)\\ &= \left(\psi^{(t_1,x)}\right)'(t_1) f\end{aligned}$$

$$
\begin{aligned}
&= X\left(t_1, \psi^{(t_1,x)}(t_1)\right) f \\
&= X_{t_1}(x) f \\
&= \left(\mathcal{L}_{X_{t_1}} f\right)_x
\end{aligned}
$$

using part (a) of the fundamental theorem of time-dependent flows 2.40. Moreover with proposition F.202 we compute

$$
\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_1} \psi^*_{t,t_1}(df) &= \left.\frac{d}{dt}\right|_{t=t_1} d\left(\psi^*_{t,t_1} f\right) \\
&= d\left(\left.\frac{d}{dt}\right|_{t=t_1} \psi^*_{t,t_1} f\right) \\
&= d\left(\mathcal{L}_{X_{t_1}} f\right) \\
&= \mathcal{L}_{X_{t_1}} df.
\end{aligned}
$$

□

We need the following adapted version of Fisherman's formula 2.41.

Proposition 2.42 ([6, 573]). *Let M be a smooth manifold and $J \subseteq \mathbb{R}$ an open interval. Suppose $X : J \times M \to TM$ is a time-dependent vector field with time-dependent flow $\psi : \mathcal{D} \to M$ and $\omega : J \times M \to \Lambda^k T^* M$ is a time-dependent differential k-form. Then for any $(t_1, t_0, x) \in \mathcal{D}$*

$$
\left.\frac{d}{dt}\right|_{t=t_1} \psi^*_{t,t_0}\omega_t = \psi^*_{t_1,t_0}\left(\mathcal{L}_{X_{t_1}}\omega_{t_1} + \left.\frac{d}{dt}\right|_{t=t_1}\omega_t\right) \tag{2.8}
$$

holds.

Proof. For $\varepsilon > 0$ sufficiently small, define

$$
F : (t_1 - \varepsilon, t_1 + \varepsilon) \times (t_1 - \varepsilon, t_1 + \varepsilon) \to \Lambda^k T^* M
$$

by

$$
F(u, v) := \psi^*_{u,t_0}\omega_v.
$$

Using the chain rule and Fisherman's formula 2.41 we compute

$$
\begin{aligned}
\left.\frac{d}{dt}\right|_{t=t_1} \psi^*_{t,t_0}\omega_t &= \left.\frac{d}{dt}\right|_{t=t_1} F(t,t) \\
&= \frac{\partial F}{\partial u}(t_1, t_1) + \frac{\partial F}{\partial v}(t_1, t_1) \\
&= \left.\frac{d}{du}\right|_{u=t_1} \psi^*_{u,t_0}\omega_{t_1} + \left.\frac{d}{dv}\right|_{v=t_1} \psi^*_{t_1,t_0}\omega_v \\
&= \psi^*_{t_1,t_0}\left(\mathcal{L}_{X_{t_1}}\omega_{t_1}\right) + \psi^*_{t_1,t_0}\left(\left.\frac{d}{dv}\right|_{v=t_1}\omega_v\right)
\end{aligned}
$$

$$= \psi_{t_1,t_0}^* \left(\mathcal{L}_{X_{t_1}} \omega_{t_1} + \frac{d}{dt}\bigg|_{t=t_1} \omega_t \right).$$

□

Proposition 2.43 (Moser's Trick [10, 108]). *Let M be a compact smooth manifold and suppose that we are given a smooth family of symplectic forms $(\omega_t)_{t\in J} \in \Omega^2(M)$ for some open interval $J \subseteq \mathbb{R}$ containing 0, such that there exists a smooth family $(\eta_t)_{t\in J} \in \Omega^1(M)$ with exact derivatives:*

$$\frac{d}{dt}\omega_t = d\eta_t.$$

Then there exists a family of diffeomorphisms $(\psi_t)_{t\in J} \in \mathrm{Diff}(M)$ such that

$$\psi_t^* \omega_t = \omega_0. \tag{2.9}$$

Proof. The key observation to achieve this is to represent the diffeomorphisms $(\psi_t)_{t\in J}$ as the time-dependent flow of a time-dependent vector field X, where $\psi_t := \psi_{t,0}$ in the terminology of proposition 2.40. This extremely useful argument is called ***Moser's argument*** or as ***Moser's trick***. Using the adapted version of Fisherman's formula 2.42 and Cartan's magic formula F.204 we compute

$$\begin{aligned}
\frac{d}{dt}\psi_t^*\omega_t &= \psi_t^* \left(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t \right) \\
&= \psi_t^* \left(i_{X_t}(d\omega_t) + d\left(i_{X_t}\omega_t\right) + \frac{d}{dt}\omega_t \right) \\
&= \psi_t^* \left(d\left(i_{X_t}\omega_t\right) + \frac{d}{dt}\omega_t \right) \\
&= \psi_t^* \left(d\left(i_{X_t}\omega_t\right) + d\eta_t \right)
\end{aligned}$$

In order to satisfy equation (2.9), we want $\psi_t^*\omega_t$ to be constant, so above computation yields

$$\psi_t^* \left(d\left(i_{X_t}\omega_t\right) + d\eta_t \right) = 0.$$

Since ψ_t^* is an isomorphism, we can equivalently solve

$$d\left(i_{X_t}\omega_t\right) + d\eta_t = 0$$

and because d is a sheaf morphism, it is sufficient to solve

$$i_{X_t}\omega_t + \eta_t = 0. \tag{2.10}$$

Equation (2.10) is called ***Moser's equation*** and can be solved by proposition 2.38 explicitly by

$$X_t = -\Omega_t^{-1}(\eta_t),$$

where Ω_t denotes the tangent-cotangent bundle isomorphism 2.36 induced by ω_t. Note that thus X_t varies smoothly in t. Hence by the fundamental theorem of time-dependent flows 2.40 together with the compactness of M yields the existence of a time-dependent flow $\psi : J \times J \times M \to M$. But then

$$\psi_t^* \omega_t = \psi_0^* \omega_0 = \mathrm{id}_M^* \, \omega_0 = \omega_0$$

as desired. □

Definition 2.44 (Tubular Neighbourhood [7, 133]). Let (M, g) be a Riemannian manifold, $S \subseteq M$ an embedded submanifold and denote by $\pi : NS \to S$ the normal bundle of S in M. Consider the restriction $\exp_S : \mathcal{E} \cap NS \to M$ the restriction of the exponential map of M with domain $\mathcal{E} \subseteq TM$. A neighbourhood U of S in M is called a ***tubular neighbourhood of S in M***, iff there exists a positive continuous function $\delta : S \to \mathbb{R}$ such that U is the diffeomorphic image under $\exp_S$ of a subset $V \subseteq \mathcal{E} \cap NS$ of the form

$$V = \left\{ (x, v) \in NS : |v|_g < \delta(x) \right\}.$$

If δ is constant, then U is called a ***uniform tubular neighbourhood of S in M***.

See figure 2.1 for an illustration of a tubular neighbourhood.

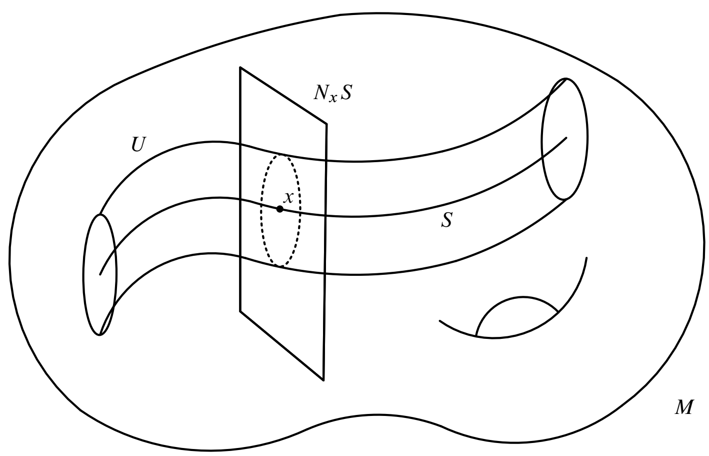


Fig. 2.1: A tubular neighbourhood U of an embedded submanifold S of a smooth manifold M.

Proposition 2.45 (The Tubular Neighbourhood Theorem [7, 133]). *Let (M, g) be a Riemannian manifold. Every embedded submanifold of M admits a tubular neighbourhood in M, and every compact submanifold admits a uniform tubular neighbourhood.*

In order to apply the tubular neighbourhood theorem 2.45 to any smooth manifold, we need the following basic result from Riemannian geometry.

Proposition 2.46 (Existence of Riemannian Metrics [6, 329]). *Every smooth manifold admits a Riemannian metric.*

Proposition 2.47 (Homotopy Formula [15, 45]). *Let U be a tubular neighbourhood of an embedded submanifold S of a smooth manifold M. Suppose that $\omega \in \Omega^k(U)$ is closed and $\iota^*\omega = 0$, where $\iota : S \hookrightarrow U$. Then there exists $\eta \in \Omega^{k-1}(U)$ such that $\omega = d\eta$. Moreover, we can choose η such that $\eta_x = 0$ for all $x \in S$.*

Proof. By definition of a tubular neighbourhood 2.44, there exists a positive continuous function $\delta : S \to \mathbb{R}$ such that $U = \exp_S(V)$, where

$$V = \left\{(x, v) \in NS : |v|_g < \delta(x)\right\}.$$

Let $t \in I$ and define $\psi_t : U \to U$ by

$$\psi_t\left(\exp_S(x, v)\right) := \exp_S(x, tv).$$

Then ψ_t is a diffeomorphism for $t > 0$ onto its image

$$\psi_t(U) = \exp_S(V_t)$$

where

$$V_t := \left\{(x, v) \in NS : |v|_g < t\delta(x)\right\},$$

since $\exp_S$ is injective and an explicit smooth inverse is given by

$$\exp_S(x, tv) \mapsto \exp_S(x, v).$$

Moreover, $\psi_t(U)$ is open in U because $\exp_S$ is a diffeomorphism, thus a homeomorphism and so in particular an open map. So we can restrict any form in $\Omega^k(U)$ to a form in $\Omega^k\left(\psi_t(U)\right)$. Also

$$\psi_1 = \mathrm{id}_U \qquad \text{and} \qquad \psi_0 = \iota \circ \pi \circ \exp_S^{-1}$$

where $\pi : NS \to S$ is the projection. Hence we are done if we show the existence of a map $H : \Omega^k(U) \to \Omega^{k-1}(U)$ such that

$$H \circ d + d \circ H = \psi_1^* - \psi_0^*, \tag{2.11}$$

since then by assumption $\omega = d(H\omega)$ and so we can choose $\eta := H(\omega)$. We claim that such an operator H is given by

$$H(\omega) := \int_0^1 \psi_t^*\left(i_{X_t}\omega\right) dt$$

for $\omega \in \Omega^k(U)$, where $X_t \in \mathfrak{X}\left(\psi_t(U)\right)$ is given by

$$X_t := \left(\frac{d}{dt}\psi_t\right) \circ \psi_t^{-1}$$

for $t > 0$. Or equivalently

$$H = \int_0^1 H_t dt$$

where $H_t : \Omega^k(U) \to \Omega^{k-1}(U)$ is defined by

$$(H_t\omega)_x(v_1, \ldots, v_{k-1}) := \omega_{\psi_t(x)}\left(\frac{d}{dt}\psi_t(x), D(\psi_t)_x(v_1), \ldots, D(\psi_t)_x(v_{k-1})\right)$$

for $x \in U$ and $v_1, \ldots, v_{k-1} \in T_xU$. Indeed, using Cartan's magic formula F.204 and Fisherman's formula 2.41 we compute

$$\begin{aligned} H(d\omega) + d(H\omega) &= \int_0^1 \psi_t^*\left(i_{X_t}(d\omega)\right) + d\left(\psi_t^*\left(i_{X_t}\omega\right)\right) dt \\ &= \int_0^1 \psi_t^*\left(i_{X_t}(d\omega) + d i_{X_t}\omega\right) dt \\ &= \int_0^1 \psi_t^*\left(\mathcal{L}_{X_t}\omega\right) dt \\ &= \int_0^1 \frac{d}{dt}\left(\psi_t^*\omega\right) dt \\ &= \psi_1^*\omega - \psi_0^*\omega, \end{aligned}$$

since ψ_t is the time-dependent flow of X_t. Moreover, we have that $\psi_t|_S = \mathrm{id}_S$ for all $t \in I$, so X_t vanishes on S and so does η. $\square$

Remark 2.48. Equation (2.11) is referred to as a ***Homotopy formula***, because a similar formula is used to show the homotopy invariance of the de Rham cohomology. See for example [6, 443–446].

Remark 2.49. Alternatively, one could also prove proposition 2.47 using the following result due to Élie Cartan:

Proposition 2.50 (Cartan [14, 104]). *Let M and N be smooth manifolds, $J \subseteq \mathbb{R}$ an interval and $F : I \times M \to N$ a smooth map. For all $t \in J$ define*

$$H_t : \Omega^k(N) \to \Omega^{k-1}(M)$$

by

$$(H_t\omega)_x(v_1, \ldots, v_{k-1}) := \omega_{F_t(x)}\left(\frac{d}{dt}F_t(x), D(F_t)_x(v_1), \ldots, D(F_t)_x(v_{k-1})\right)$$

for $x \in M$ and $v_1, \ldots, v_{k-1} \in T_xM$. Then

$$H_t(d\omega) + d(H_t\omega) = \frac{d}{dt} F_t^* \omega$$

for all $\omega \in \Omega^k(N)$ *and* $t \in J$.

Proposition 2.51 (Moser Isotopy [10, 109]). *Let* M^{2n} *be a smooth manifold and* $S \subseteq M$ *a compact submanifold. Suppose that* $\omega_0, \omega_1 \in \Omega^2(M)$ *are closed and such that:*

(i) $\omega_0|_x = \omega_1|_x$ *for all* $x \in S$.
(ii) $\omega_0|_x$ *and* $\omega_1|_x$ *are nondegenerate for all* $x \in S$.

Then there exist neighbourhoods U_0 *and* U_1 *of* S *in* M *and a diffeomorphism* $F : U_0 \to U_1$ *such that*

$$F|_S = \mathrm{id}_S \qquad \textit{and} \qquad F^*\left(\omega_1|_{U_1}\right) = \omega_0|_{U_0}.$$

Proof. In view of Moser's trick 2.43, we can argue as follows. Let U be a uniform tubular neighbourhood of S in M. Note that $\overline{U}$ is compact by construction. By proposition 2.47 there exists $\eta \in \Omega^1(U)$ such that

$$\omega_1 - \omega_0 = d\eta.$$

For $t \in \mathbb{R}$ set

$$\omega_t := \omega_0 + t(\omega_1 - \omega_0).$$

By shrinking U to a new neighbourhood U_0 of S in M if necessary, we may assume that ω_t is non-degenerate for t in some bounded open interval containing I (this is due to the fact that $\omega_t = \omega_0$ on S and non-degeneracy is an open condition). By definition

$$\frac{d}{dt}\omega_t = \omega_1 - \omega_0 = d\eta.$$

By Moser's trick and the fact that $\overline{U}_0 \subseteq \overline{U}$ is compact as a closed subset of a compact space, there exists a family of diffeomorphisms $(\psi_t)_{t\in J}$ such that

$$\psi_t^*\omega_t = \omega_0.$$

Thus set $F := \psi_1$ and $U_1 := F(U_0)$. Then by definition and since η vanishes on S, we have that $F|_S = \mathrm{id}_S$. ☐

Theorem 2.52 (The Darboux Theorem [6, 571]). *Let* $\left(M^{2n}, \omega\right)$ *be a symplectic manifold. For every* $x \in M$, *there exists a chart* $\left(U, (x^i, y^i)\right)$ *centred about* x *such that*

$$\omega|_U = \sum_{i=1}^n dx^i \wedge dy^i.$$

Any such chart is called a ***Darboux chart about x***.

Proof. By the canonical form theorem for symplectic vector spaces 2.8 there exists a basis (a_i, b_i) of $T_x M$ such that

$$\omega_x = \sum_{i=1}^{n} \alpha^i \wedge \beta^i$$

where $\left(\alpha^i, \beta^i\right)$ denotes the dual basis of (a_i, b_i). By proposition F.34, there exists a chart $\left(U, \widetilde{\varphi}\right)$ with associated coordinates $\left(\widetilde{x}^i, \widetilde{y}^i\right)$ centred about x such that

$$\left.\frac{\partial}{\partial \widetilde{x}^i}\right|_x = a_i \qquad \text{and} \qquad \left.\frac{\partial}{\partial \widetilde{y}^i}\right|_x = b_i .$$

In particular

$$\omega_x = \sum_{i=1}^{n} d\widetilde{x}^i|_x \wedge d\widetilde{y}^i|_x .$$

Set

$$\omega_0 := \omega|_U \qquad \text{and} \qquad \omega_1 := \sum_{i=1}^{n} d\widetilde{x}^i \wedge d\widetilde{y}^i .$$

Then ω_0 as well as ω_1 are symplectic forms on U. Applying the Moser isotopy 2.51 to the compact submanifold $\{x\}$ of U yields the existence of two neighbourhoods U_0 and U_1 of x in U and a diffeomorphism $F : U_0 \to U_1$ such that

$$F(x) = x \qquad \text{and} \qquad F^*\left(\omega_1|_{U_1}\right) = \omega_0|_{U_0} .$$

Define a new chart (U_0, φ) by $\varphi := \widetilde{\varphi}|_{U_1} \circ F$. Then the associated coordinates are given by

$$x^i = \widetilde{x}^i \circ F \qquad \text{and} \qquad y^i = \widetilde{y}^i \circ F.$$

Moreover $\varphi(x) = \widetilde{\varphi}(x) = 0$ and

$$\begin{aligned}
\omega|_{U_0} &= \omega_0|_{U_0} && \text{(by definition of } \omega_0\text{)}\\
&= F^*\left(\omega_1|_{U_1}\right) && \text{(by definition of } F\text{)}\\
&= F^*\left(\sum_{i=1}^{n} d\widetilde{x}^i \wedge d\widetilde{y}^i\right) && \text{(by definition of } \omega_1\text{)}\\
&= \sum_{i=1}^{n} F^*\left(d\widetilde{x}^i \wedge d\widetilde{y}^i\right) && \text{(since } F^* \text{ is linear)}\\
&= \sum_{i=1}^{n} F^*\left(d\widetilde{x}^i\right) \wedge F^*\left(d\widetilde{y}^i\right) && \text{(by lemma F.193)}\\
&= \sum_{i=1}^{n} d\left(F^*\widetilde{x}^i\right) \wedge d\left(F^*\widetilde{y}^i\right) && \text{(by proposition F.201)}
\end{aligned}$$

$$= \sum_{i=1}^{n} d\left(\tilde{x}^i \circ F\right) \wedge d\left(\tilde{y}^i \circ F\right) \qquad \text{(by definition of } F^*\text{)}$$

$$= \sum_{i=1}^{n} dx^i \wedge dy^i \qquad \text{(by definition of } x^i \text{ and } y^i\text{).}$$

Thus (U_0, φ) is our desired chart. □

2.2 Hamiltonian Systems

If the Legendre transform 1.40 is a diffeomorphism, we can define an associated Hamiltonian function by 1.46, that is a smooth function H on T^*M, where M is a smooth manifold. By example 2.21, we know that the cotangent bundle T^*M admits a canonical symplectic structure in terms of the tautological form 1.38. The tuple (T^*M, H) turns out to be the prototype of a much more general structure.

Definition 2.53 (Hamiltonian System). A ***Hamiltonian system*** is defined to be a tuple $\big((M, \omega), H\big)$ consisting of a symplectic manifold (M, ω), called a ***phase space***, and a function $H \in C^\infty(M)$, called a ***Hamiltonian function***.

Remark 2.54. In what follows, we will write simply (M, ω, H) for a Hamiltonian system instead of the more cumbersome $\big((M, \omega), H\big)$. The latter was choosen in the definition to emphasize the similarity to the definition of a Lagrangian system 1.8.

2.2.1 Hamiltonian Vector Fields

As in Riemannian geometry, a main advantage of the symplectic structure is to reinstate the definition of the gradient of a smooth function as a vector field instead of a covector field using the tangent-cotangent bundle isomorphism (for the Riemannian case see [6, 342–343]).

Definition 2.55 (Hamiltonian Vector Field). Let (M, ω, H) be a Hamiltonian system and denote by $\Omega : \mathfrak{X}(M) \to \mathfrak{X}^*(M)$ the tangent-cotangent bundle isomorphism from proposition 2.36. The vector field X_H defined by

$$X_H := \Omega^{-1}(dH) \tag{2.12}$$

is called the ***Hamiltonian vector field associated to the Hamiltonian system***.

Let (M, ω, H) be a Hamiltonian system and suppose $\big(U, (x^i, y^i)\big)$ is a Darboux chart (see theorem 2.52). In these coordinates write

$$X_H = X^i \frac{\partial}{\partial x^i} + Y^i \frac{\partial}{\partial y^i}.$$

Then we compute

$$\begin{aligned} i_{X_H}\omega &= i_{X_H}\left(\sum_{i=1}^{n} dx^i \wedge dy^i\right) \\ &= \sum_{i=1}^{n}\left(\left(i_{X_H}dx^i\right)\wedge dy^i - dx^i \wedge \left(i_{X_H}dy^i\right)\right) \\ &= \sum_{i=1}^{n}\left(X^i dy^i - Y^i dx^i\right). \end{aligned}$$

Comparing with

$$dH = \frac{\partial H}{\partial x^i}dx^i + \frac{\partial H}{\partial y^i}dy^i$$

yields

$$X^i = \frac{\partial H}{\partial y^i} \qquad \text{and} \qquad Y^i = -\frac{\partial H}{\partial x^i}.$$

Thus

$$X_H = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial y^i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial y^i}\right).$$

Definition 2.56 (Invariance). Let M be a smooth manifold, $X \in \mathfrak{X}(M)$ a complete vector field with flow θ. A tensor field $A \in \mathcal{T}^{k,l}(M)$ is said to be ***invariant under the flow θ of X***, iff

$$\theta_t^* A = A$$

for all $t \in \mathbb{R}$.

Recall, that a tensor field A is invariant under the flow of a vector field X if and only if $\mathcal{L}_X A = 0$ (see [6, 324]). The next proposition is a prime example why we require a symplectic structure to be both closed and nondegenerate. For the proof, we need one more preliminary result from the calculus of differential forms.

Proposition 2.57. *Let (M, ω, H) be a Hamiltonian system such that the Hamiltonian vector field is complete. Then the symplectic form is invariant under the flow of the Hamiltonian vector field.*

Proof. By the previous discussion it is enough to show that $\mathcal{L}_{X_H}\omega = 0$. Using Cartan's magic formula F.204, closedness of ω together with proposition 2.38 we compute

$$\mathcal{L}_{X_H}\omega = i_{X_H}(d\omega) + d(i_{X_H}\omega) = d(i_{X_H}\omega) = (d \circ d)H = 0.$$

□

2.2.2 Poisson Brackets

Definition 2.58 (Poisson Bracket). Let (M, ω) be a symplectic manifold. Define a mapping

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

by

$$\{f, g\} := \omega(X_f, X_g)$$

where X_f and X_g are Hamiltonian vector fields associated to the Hamiltonian systems (M, ω, f) and (M, ω, g), respectively. The mapping $\{\cdot, \cdot\}$ is called the ***Poisson bracket on $C^\infty(M)$***.

Recall, that if $f \in C^\infty(M)$ for a smooth manifold M, the *differential of f* is defined to be the covector field given by $df_x(v) := vf$ for $x \in M$ and $v \in T_xM$. This is indeed a smooth covector field by part (d) of the smoothness criteria for tensor fields 2.33 since

$$df(X)(x) = df_x(X|_x) = X|_x f = (Xf)(x) \tag{2.13}$$

for any $X \in \mathfrak{X}(M)$ and $x \in M$, and Xf is smooth by [6, 180] (proving this is analogous to the proof of the smoothness criteria for tensor fields 2.33).

Lemma 2.59. *Let (M, ω) be a symplectic manifold. Then $\{f, g\} = X_g f$ holds for all $f, g \in C^\infty(M)$.*

Proof. Using proposition 2.38 and equation (2.13), we compute

$$\{f, g\} = \omega(X_f, X_g) = (i_{X_f}\omega)(X_g) = df(X_g) = X_g f.$$

Definition 2.60 (Integral of Motion). Let (M, ω, H) be a Hamiltonian system. A function $f \in C^\infty(M)$ is said to be an ***integral of motion for the Hamiltonian system (M, ω, H)***, iff $\{H, f\} = 0$.

2.2.3 Lie Group Actions and Noether's Theorem

Let us recall some basic facts from the theory of Lie groups and Lie algebras. A *Lie group* is defined to be a group $(G, \cdot)$, such that G is a smooth manifold and the multiplication $\cdot$ as well as the inversion map $\cdot^{-1} : G \to G$ defined by $g \mapsto g^{-1}$ are smooth. If G is a Lie group, we can associate to G its *Lie algebra* $\mathfrak{g}$ defined to be $\mathfrak{g} := T_eG$, where e denotes the neutral element of G. It can be shown that $\mathfrak{g} \cong \mathfrak{X}_L(G)$ as real vector spaces, where $\mathfrak{X}_L(G) \subseteq \mathfrak{X}(G)$ denotes the space of *left invariant vector fields on G*, that is, the vector fields $X \in \mathfrak{X}(G)$ satisfying $(L_g)_*X = X$, where L_g is the diffeomorphism $L_g : G \to G$ defined by $L_g(h) := gh$ and $(L_g)_*$ is the *pushforward of X* defined to be the vector field $\big((L_g)_*X\big)_h := d(L_g)_{g^{-1}h}X|_{g^{-1}h}$

for $h \in G$. Most importantly, any left invariant vector field on G is complete and so we can define the *exponential map* $\exp : \mathfrak{g} \to G$ by

$$\exp v := \gamma(1),$$

where $\gamma \in C^\infty(\mathbb{R}, G)$ is the integral curve of the *left invariant vector field* X_v *associated to* v *on* G, that is $X_v|_g := d(L_g)_e(v)$, with starting point $\gamma(0) = e$. Then we have that $\gamma(t) = \exp tv$ and $(\exp tv)^{-1} = \exp(-tv)$ for all $v \in \mathfrak{g}$ and $t \in \mathbb{R}$.

The most important applications of Lie groups to smooth manifold theory involve actions by Lie groups on manifolds. Let G be a Lie group and M be a smooth manifold. A map in $C^\infty(G \times M, M)$ given by $(g, x) \mapsto g \cdot x$, is said to be a *left action of* G *on* M iff

$$g \cdot (h \cdot x) = (gh) \cdot x \qquad \text{and} \qquad e \cdot x = x$$

holds for all $g, h \in G$ and $x \in M$. Similarly, a *right action of* G *on* M is defined to be a map in $C^\infty(M \times G, M)$ given by $(x, g) \mapsto x \cdot g$ satisfying

$$(x \cdot g) \cdot h = x \cdot (gh) \qquad \text{and} \qquad x \cdot e = x$$

for all $g, h \in G$ and $x \in M$. Note that any left action of G on M can be transformed into a right action of G on M by defining $x \cdot g := g^{-1} \cdot x$ for all $g \in G$ and $x \in M$, and similarly every right action of G on M can be transformed into a left action of G on M.

Suppose we are given a right action of a Lie group G on a smooth manifold M. Then each element $v \in \mathfrak{g}$ determines a global flow on M by

$$(t, x) \mapsto x \cdot \exp tv.$$

Define $\widehat{v} \in \mathfrak{X}(M)$ by

$$\widehat{v}_x := \left.\frac{d}{dt}\right|_{t=0} x \cdot \exp tv$$

for all $x \in M$. This is the *infinitesimal generator* associated to the above flow (see [6, 210]). Hence we get a map $\mathfrak{g} \to \mathfrak{X}(M)$ defined by $v \mapsto \widehat{v}$. By [6, 526], this map is actually a *Lie algebra homomorphism*. This is the main reason we are working with right actions rather than left actions.

Lemma 2.61 (Computing the Differential Using a Velocity Vector [6, 70]). *Let* $F \in C^\infty(M, N)$ *for two smooth manifolds* M *and* N, $x \in M$ *and* $v \in T_x M$. *Then*

$$dF_x(v) = (F \circ \gamma)'(0)$$

for any path $\gamma \in C^\infty(J, M)$, *where* $J \subseteq \mathbb{R}$ *is an interval such that* $0 \in J$, $\gamma(0) = x$ *and* $\gamma'(0) = v$.

Proposition 2.62. *Suppose we are given a right action of a Lie group* G *on a smooth manifold* M. *Then for each* $v \in \mathfrak{g}$, *the infinitesimal generator* $\widehat{v}$ *associated to the* *flow generated by* v *satisfies*

$$(\hat{v}f)(x) = \left.\frac{d}{dt}\right|_{t=0} f(x \cdot \exp tv)$$

for all $x \in M$ *and* $f \in C^\infty(M)$.

Proof. Let $x \in M$ and denote by $\theta : M \times G \to M$ the right action of G on M. Define $\theta^x : G \to M$ by $\theta^x(g) := x \cdot g$. Then θ^x is smooth since θ^x is the composition

$$G \cong \{x\} \times G \hookrightarrow M \times G \xrightarrow{\theta} M$$

where the first two maps steem from [6, 100]. Set $\gamma(t) := \exp tv$ for all $t \in \mathbb{R}$. Then it is immediate, that

$$x \cdot \exp tv = \theta^x\left(\gamma(t)\right).$$

Thus we compute

$$\begin{aligned}
(\hat{v}f)(x) &= \hat{v}_x f \\
&= \left.\frac{d}{dt}\right|_{t=0} \theta^x\left(\gamma(t)\right) f \\
&= d(\theta^x)_e(v) f && \text{(by lemma 2.61)} \\
&= v(f \circ \theta^x) && \text{(by definition of } d\theta^x) \\
&= d(f \circ \theta^x)_e(v) && \text{(by definition of } d(f \circ \theta^x)) \\
&= (f \circ \theta^x \circ \gamma)'(0) && \text{(by lemma 2.61)} \\
&= \left.\frac{d}{dt}\right|_{t=0} f(x \cdot \exp tv).
\end{aligned}$$

Remark 2.63. From now on, we will consider left actions of Lie groups G on smooth manifolds M only instead of right actions, since they are more common. This is however no drawback, since any left action can be converted into a right action. Hence if $v \in \mathfrak{g}$, the corresponding infinitesimal generator V is given by

$$\hat{v}_x = \left.\frac{d}{dt}\right|_{t=0} \exp(-tv) \cdot x.$$

Proposition 2.64. *Let* $\theta : G \times M \to M$ *be a Lie group action. Then for all* $g \in G$ *and* $v \in \mathfrak{g}$ *we have that*

$$\widehat{\mathrm{Ad}_{g^{-1}}(v)} = \theta_g^* \hat{v}.$$

Proof. By proposition F.109 we have a commutative diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad}_g"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "\mu_g"'] & G, \end{tikzcd}$$

where $\mu_g(h) := ghg^{-1}$ denotes the conjugation action of G on itself. Let $x \in M$. We compute

$$\begin{aligned}
\left(\theta_g^* \widehat{v}\right)_x &= \left(\left(\theta_{g^{-1}}\right)_* \widehat{v}\right)_x \\
&= D\left(\theta_{g^{-1}}\right)_{\theta_g(x)} \left(\widehat{v}|_{\theta_g(x)}\right) \\
&= D\left(\theta_{g^{-1}}\right)_{\theta_g(x)} \left(\left.\frac{d}{dt}\right|_{t=0} \theta_{\exp(-tv)}\left(\theta_g(x)\right)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \theta_{g^{-1}} \circ \theta_{\exp(-tv)}\left(\theta_g(x)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \theta_{g^{-1}\exp(-tv)g}(x) \\
&= \left.\frac{d}{dt}\right|_{t=0} \theta_{\exp(-t\,\mathrm{Ad}_{g^{-1}}(v))}(x) \\
&= \widehat{\mathrm{Ad}_{g^{-1}}(v)}|_x
\end{aligned}$$

for all $g \in G$ and $v \in \mathfrak{g}$. □

Definition 2.65 (Action by Symplectomorphisms). A left action θ of a Lie group G on a symplectic manifold (M, ω) is said to be an ***action by symplectomorphisms***, iff $\theta_g \in \mathrm{Symp}(M, \omega)$ for all $g \in G$.

Example 2.66 (Cotangent Lift). Let θ be a left action of a Lie group G on a smooth manifold M. Define a left action $\widetilde{\theta}$ of G on T^*M by

$$\widetilde{\theta}_g := D\left(\theta_g^{-1}\right)^\dagger$$

for all $g \in G$. Using proposition 2.23 we compute

$$\widetilde{\theta}_g^* \omega = -\widetilde{\theta}_g^*(d\lambda) = -d\left(\widetilde{\theta}_g^* \lambda\right) = -d\lambda = \omega$$

for all $g \in G$.

Definition 2.67 (Equivariant Action [6, 164]). Let G be a Lie group acting on smooth manifolds M and N on the left. A map $F \in C^\infty(M, N)$ is said to be ***G-equivariant***, iff

$$F(g \cdot x) = g \cdot F(x)$$

holds for all $g \in G$ and $x \in M$.

Definition 2.68 (Weakly Hamiltonian Action and Hamiltonian Action). A left action of a Lie group G on a symplectic manifold (M, ω) by symplectomorphisms is said to be

- ***weakly Hamiltonian***, iff for each $v \in \mathfrak{g}$, there exists a Hamiltonian system (M, ω, H_v), such that $X_{H_v} = \widehat{v}$.
- ***Hamiltonian***, iff the action is weakly Hamiltonian and additionally the induced mapping $\mathfrak{g} \to C^\infty(M)$ defined by $v \mapsto H_v$ is G-equivariant with respect to the adjoint action of G on its associated Lie algebra $\mathfrak{g}$ (see F.120) and the induced action of G on $C^\infty(M)$, that is

$$H_{\mathrm{Ad}_{g^{-1}}(v)} = H_v \circ \theta_g$$

holds for all $g \in G$ and $v \in \mathfrak{g}$.

Lemma 2.69. *Let $F : M \to N$ be a diffeomorphism, $X \in \mathfrak{X}(N)$ and $\omega \in \Omega^1(M)$. Then*

$$i_{F^*X}(\omega) = i_X \left(F_*\omega\right) \circ F.$$

Proof. By definition we have that $F^* = \left(F^{-1}\right)_*$, so for $x \in M$ we compute

$$\begin{aligned}\left(i_{F^*X}(\omega)\right)(x) &= \left(i_{(F^{-1})_*X}(\omega)\right)(x) \\ &= \omega_x\left(D\left(F^{-1}\right)_{F(x)}\left(X_{F(x)}\right)\right) \\ &= \left(\left(F^{-1}\right)^*\omega\right)_{F(x)}\left(X_{F(x)}\right) \\ &= i_X(F_*\omega)\left(F(x)\right).\end{aligned}$$

□

Proposition 2.70. *Let θ be a left action of a Lie group G on an exact symplectic manifold $\left(M, -d\eta\right)$ such that $\theta_g^*\eta = \eta$ for all $g \in G$ holds. Then the action θ is Hamiltonian with*

$$H_v = i_{\widehat{v}}(\eta)$$

for all $v \in \mathfrak{g}$.

Proof. Step 1: θ is an action by symplectomorphisms. Let $g \in G$. Using proposition F.201 we compute

$$-\theta_g^*(d\eta) = -d\left(\theta_g^*\eta\right) = -d\eta.$$

Step 2: θ is a weakly Hamiltonian action. Let $v \in \mathfrak{g}$. We want to prove that $X_{H_v} = \widehat{v}$. By proposition 2.38, X_{H_v} is the unique vector field such that $-i_{X_{H_v}}(d\eta) = dH_v$. Thus it is enough to show that $-i_{\widehat{v}}(d\eta) = dH_v$ holds for all $v \in \mathfrak{g}$. Using Cartan's magic formula F.204 we compute

$$-i_{\widehat{v}}(d\eta) = d\left(i_{\widehat{v}}\eta\right) - \mathcal{L}_{\widehat{v}}\eta$$

$$
\begin{aligned}
&= d\left(i_{\widehat{v}}\eta\right) - \left.\frac{d}{dt}\right|_{t=0} \theta^*_{\exp(-tv)}\eta \\
&= d\left(i_{\widehat{v}}\eta\right) - \left.\frac{d}{dt}\right|_{t=0} \eta \\
&= d\left(i_{\widehat{v}}\eta\right) \\
&= dH_v.
\end{aligned}
$$

Step 3: θ is a Hamiltonian action. Left to show is that θ_g is G-equivariant with respect to the adjoint action of G on $\mathfrak{g}$ and the induced action of G on $C^\infty(M)$, that is, we have to show

$$H_{\mathrm{Ad}_g(v)} = H_v \circ \theta_g$$

for all $g \in G$ and $v \in \mathfrak{g}$. We compute

$$
\begin{aligned}
H_{\mathrm{Ad}_{g^{-1}}(v)} &= i_{\widehat{\mathrm{Ad}_{g^{-1}}(v)}}\eta \\
&= i_{\theta_g^* \widehat{v}}\eta && \text{(by proposition 2.64)} \\
&= i_{\widehat{v}}\left(\theta^*_{g^{-1}}\eta\right) \circ \theta_g && \text{(by lemma 2.69)} \\
&= i_{\widehat{v}}\eta \circ \theta_g \\
&= H_v \circ \theta_g.
\end{aligned}
$$

Thus θ is a Hamiltonian action. □

Definition 2.71 (Symmetry Group). A Lie group G is said to be a ***symmetry group of a Hamiltonian system*** (M, ω, H), iff there exists a weakly Hamiltonian action of G on (M, ω), such that

$$\theta_g^* H = H$$

holds for all $g \in G$.

Theorem 2.72 (Noether's Theorem, Hamiltonian Version). *Let G be a symmetry group of a Hamiltonian system (M, ω, H). Then for each $v \in \mathfrak{g}$, the function $H_v \in C^\infty(M)$ such that $X_{H_v} = \widehat{v}$ is an integral of motion.*

Proof. Let $x \in M$. We compute

$$
\begin{aligned}
\{H, H_v\}(x) &= \left(X_{H_v} H\right)(x) && \text{(by lemma 2.59)} \\
&= (\widehat{v} H)(x) \\
&= \left.\frac{d}{dt}\right|_{t=0} H\left(\exp(-tv) \cdot x\right) && \text{(by proposition 2.62)} \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\theta^*_{\exp(-tv)} H\right)(x) \\
&= \left.\frac{d}{dt}\right|_{t=0} H(x) \\
&= 0.
\end{aligned}
$$

□

If $F \in C^\infty(M, N)$, then the derivative of F can be interpreted as a vector bundle homomorphism $DF : TM \to F^*TN$. Indeed, define

$$DF(x, v) := \big(x, \big(F(x), DF_x(v)\big)\big)$$

for any $(x, v) \in TM$. If $\pi : E \to M$ is a fibre bundle, we can define

$$VE := \coprod_{p \in E} \ker D\pi_p.$$

Then VE with the usual footpoint projection is a vector bundle over E, called the ***vertical bundle of E***. Moreover, one can show that VE is isomorphic to π^*E. Explicitely, the isomorphism $\Phi : \pi^*E \to VE$ is given by

$$\Phi(p, q) := \left.\frac{d}{dt}\right|_{t=0} (p + tq). \tag{2.14}$$

Proposition 2.73 (Invariant Definition of the Associated Form). *Let (M, L) be an autonomous Lagrangian system. Then*

$$\lambda_L(v) = dL\left((\Phi \circ D\pi)v\right) \tag{2.15}$$

for all $v \in TTM$, where $\pi : TM \to M$ is the projection and $\Phi : \pi^*TM \to VTM$ is the vector bundle isomorphism (2.14).

Proof. Let $u \in T_{(x,v)}TM$ be given by

$$u := \lambda^i \left.\frac{\partial}{\partial x^i}\right|_{(x,v)} + \mu^i \left.\frac{\partial}{\partial v^i}\right|_{(x,v)}.$$

Then we compute

$$\begin{aligned}
dL\left((\Phi \circ D\pi)u\right) &= dL\left(\Phi\left((x, v), \left(x, D\pi_{(x,v)}(u)\right)\right)\right) \\
&= dL_{(x,v)}\left(\left.\frac{d}{dt}\right|_{t=0} \left(v + tD\pi_{(x,v)}(u)\right)\right) \\
&= dL_{(x,v)}\left(\left.\frac{d}{dt}\right|_{t=0} v + t\left(\lambda^i \frac{\partial \pi^j}{\partial x^i}(x, v) + \mu^i \frac{\partial \pi^j}{\partial v^i}(x, v)\right) \left.\frac{\partial}{\partial x^j}\right|_x\right) \\
&= dL_{(x,v)}\left(\left.\frac{d}{dt}\right|_{t=0} v + t\lambda^i \left.\frac{\partial}{\partial x^i}\right|_x\right) \\
&= dL_{(x,v)}\left(\lambda^i \left.\frac{\partial}{\partial v^i}\right|_{(x,v)}\right) \\
&= \frac{\partial L}{\partial v^i}(x, v)\lambda^i
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial L}{\partial v^i}(x, v) dx^i|_{(x,v)}(u) \\
&= \lambda_L(u).
\end{aligned}
$$

□

Corollary 2.74. *Let G be a symmetry group of a Lagrangian system (M, L). Then*

$$
(D\theta_g)^* \lambda_L = \lambda_L
$$

for all $g \in G$.

Proof. Using proposition 2.73 we compute

$$
\begin{aligned}
(D\theta_g)^* \lambda_L(v) &= (D\theta_g)^* dL \left((\Phi \circ D\pi)v\right) \\
&= d\left((D\theta_g)^* L\right)\left((\Phi \circ D\pi)v\right) \\
&= dL\left((\Phi \circ D\pi)v\right) \\
&= \lambda_L(v).
\end{aligned}
$$

for $v \in T(TM)$. □

Proposition 2.75. *Let (M, L) be an autonomous Lagrangian system with symmetry group G and such that the Legendre transform is a diffeomorphism. Then G is a symmetry group of the corresponding Hamiltonian system $\left(T^*M, \omega, E_L \circ \tau_L^{-1}\right)$, where ω denotes the canonical symplectic form on the cotangent bundle, with*

$$
H_v = i_{\widetilde{v}}(\lambda_L) \circ \tau_L^{-1}
$$

for all $v \in \mathfrak{g}$, where $\widetilde{v} \in \mathfrak{X}(TM)$ is defined by

$$
\widetilde{v} := \left.\frac{d}{dt}\right|_{t=0} D\theta_{\exp(-tv)}.
$$

Moreover, the action is Hamiltonian.

Proof. Define a left action $\widetilde{\theta}$ of G on T^*M by

$$
\widetilde{\theta}_g := \tau_L \circ D\theta_g \circ \tau_L^{-1}
$$

for all $g \in G$. This action preserves λ, that is $\theta_g^* \lambda = \lambda$ for all $g \in G$, and preserves the Hamiltonian function $H_L = E_L \circ \tau_L^{-1}$. Indeed, using corollary 2.74 we compute

$$
\begin{aligned}
\widetilde{\theta}_g^* \lambda &= \left(\tau_L \circ D\theta_g \circ \tau_L^{-1}\right)^* \lambda \\
&= \left(\tau_L^{-1}\right)^* (D\theta_g)^* \tau_L^* \lambda \\
&= \left(\tau_L^{-1}\right)^* (D\theta_g)^* \lambda_L \\
&= \left(\tau_L^{-1}\right)^* \lambda_L
\end{aligned}
$$

$$= \lambda,$$

and denoting by $f \in C^\infty(TM)$ the function

$$f(x,v) := D^{\mathcal{F}} L_{(x,v)}(v),$$

proposition 1.56 yields

$$\begin{aligned}
\widetilde{\theta}_g^* H_L &= \widetilde{\theta}_g^* \left(E_L \circ \tau_L^{-1}\right) \\
&= \widetilde{\theta}_g^* \left(\tau_L^{-1}\right)^* E_L \\
&= \left(\tau_L^{-1} \circ \widetilde{\theta}_g\right)^* E_L \\
&= \left(D\theta_g \circ \tau_L^{-1}\right)^* E_L \\
&= \left(\tau_L^{-1}\right)^* (D\theta_g)^* E_L \\
&= \left(\tau_L^{-1}\right)^* (D\theta_g)^* (f - L) \\
&= \left(\tau_L^{-1}\right)^* \left((D\theta_g)^* f - (D\theta_g)^* L\right) \\
&= \left(\tau_L^{-1}\right)^* (f - L) \\
&= \left(\tau_L^{-1}\right)^* E_L \\
&= E_L \circ \tau_L^{-1} \\
&= H_L
\end{aligned}$$

for all $g \in G$. Hence by proposition 2.70, the action $\widetilde{\theta}$ is Hamiltonian with

$$H_v = i_{\widehat{v}} \eta$$

for all $v \in \mathfrak{g}$. But

$$\begin{aligned}
H_v(x,\xi) &= \left(i_{\widetilde{v}}(\lambda_L) \circ \tau_L^{-1}\right)(x,\xi) \\
&= \left(i_{\widetilde{v}}\left(\tau_L^* \lambda\right) \circ \tau_L^{-1}\right)(x,\xi) \\
&= \left(\tau_L^* \lambda\right)_{\tau_L^{-1}(x,\xi)} \left(\widetilde{v}|_{\tau_L^{-1}(x,\xi)}\right) \\
&= \lambda_{(x,\xi)} \left(D\tau_L \left(\widetilde{v}|_{\tau_L^{-1}(x,\xi)}\right)\right) \\
&= \lambda_{(x,\xi)} \left(D\tau_L \left(\left.\frac{d}{dt}\right|_{t=0} D\theta_{\exp(-tv)} \left(\tau_L^{-1}(x,\xi)\right)\right)\right) \\
&= \lambda_{(x,\xi)} \left(\left.\frac{d}{dt}\right|_{t=0} \tau_L \circ D\theta_{\exp(-tv)} \left(\tau_L^{-1}(x,\xi)\right)\right) \\
&= \lambda_{(x,\xi)} \left(\left.\frac{d}{dt}\right|_{t=0} \widetilde{\theta}_{\exp(-tv)}(x,\xi)\right) \\
&= \lambda_{(x,\xi)} \left(\widehat{v}|_{(x,\xi)}\right) \\
&= \left(i_{\widehat{v}} \lambda\right)(x,\xi)
\end{aligned}$$

for all $(x, \xi) \in T^*M$. $\square$

2.2.4 Moment Maps

Definition 2.76 (Moment Map [10, 205]). A ***moment map for a weakly Hamiltonian action of a Lie group G on a symplectic manifold (M, ω)*** is defined to be a G-equivariant map

$$\mu : M \to \mathfrak{g}^*$$

with respect to the coadjoint action Ad^* of G on $\mathfrak{g}^*$, such that for all $v \in \mathfrak{g}$ we have that $\widehat{v} = X_{H_v}$ where $H_v \in C^\infty(M)$ is defined by

$$H_v(x) := \mu(x)v.$$

Proposition 2.77. *Let $\mu : M \to \mathfrak{g}^*$ be a moment map for a weakly Hamiltonian action of a Lie group G on a symplectic manifold (M, ω). Then the action is Hamiltonian.*

Proof. Denote $\Phi : \mathfrak{g} \to C^\infty(M)$ the map given by $\Phi(v) := H_v$. Then we must show that

$$\Phi\left(\mathrm{Ad}_g(v)\right)(x) = H_v(g \cdot x)$$

holds for all $g \in G$, $v \in \mathfrak{g}$ and $x \in M$. We compute

$$\begin{aligned}
\Phi\left(\mathrm{Ad}_g(v)\right)(x) &= H_{\mathrm{Ad}_g(v)}(x) && \text{(by definition of } \Phi\text{)}\\
&= \mu(x)\left(\mathrm{Ad}_g(v)\right) && \text{(by definition of } H_v\text{)}\\
&= \mathrm{Ad}_g^*\left(\mu(x)\right)v && \text{(by definition of } \mathrm{Ad}_g^*\text{)}\\
&= \mu(g \cdot x)v && \text{(by } G\text{-equivariance of } \mu\text{)}\\
&= H_v(g \cdot x) && \text{(by definition of } H_v\text{).}
\end{aligned}$$

$\square$

By virtue of proposition 2.77, every weakly Hamiltonian action admitting a moment map is in fact Hamiltonian. Thus the action being Hamiltonian is a necessary condition for a moment map to exist.

Definition 2.78 (Hamiltonian G-Space). Let $\mu : M \to \mathfrak{g}^*$ be a moment map for a Hamiltonian action of a Lie group G on a symplectic manifold (M, ω). Then the tuple (M, ω, G, μ) is called a ***Hamiltonian G-space***.

2.3 Problems

Appendix A
Basic Category Theory

A short introduction to the rudiments of category theory can be found in [8]. A more extensive treatment is given in the classic [3].

A.1 Categories

Definition A.1 (Category). A ***category*** $\mathcal{C}$ consists of

- A class $\text{ob}(\mathcal{C})$, called the ***objects of*** $\mathcal{C}$.
- A class $\text{mor}(\mathcal{C})$, called the ***morphisms of*** $\mathcal{C}$.
- Two functions $\text{dom} : \text{mor}(\mathcal{C}) \to \text{ob}(\mathcal{C})$ and $\text{cod} : \text{mor}(\mathcal{C}) \to \text{ob}(\mathcal{C})$, which assign to each morphism f in $\mathcal{C}$ its ***domain*** and ***codomain***, respectively.
- For each $X \in \text{ob}(\mathcal{C})$ a function $\text{ob}(\mathcal{C}) \to \text{mor}(\mathcal{C})$ which assigns a morphism id_X such that $\text{dom}\,\text{id}_X = \text{cod}\,\text{id}_X = X$.
- A function

  $$\circ : \{(g, f) \in \text{mor}(\mathcal{C}) \times \text{mor}(\mathcal{C}) : \text{dom}\, g = \text{cod}\, f\} \to \text{mor}(\mathcal{C}) \tag{A.1}$$

  mapping (g, f) to $g \circ f$, called ***composition***, such that $\text{dom}(g \circ f) = \text{dom}\, f$ and $\text{cod}(g \circ f) = \text{cod}\, g$.

Subject to the following axioms:

- ***(Associativity Axiom)*** For all $f, g, h \in \text{mor}(\mathcal{C})$ with $\text{dom}\, h = \text{cod}\, g$ and $\text{dom}\, g = \text{cod}\, f$, we have that
  $$(h \circ g) \circ f = h \circ (g \circ f). \tag{A.2}$$
- ***(Unit Axiom)*** For all $f \in \text{mor}(\mathcal{C})$ with $\text{dom}\, f = X$ and $\text{cod}\, f = Y$ we have that

  $$f = f \circ \text{id}_X = \text{id}_Y \circ f. \tag{A.3}$$

Remark A.2. Let $\mathcal{C}$ be a category. For $X, Y \in \text{ob}(\mathcal{C})$ we will abreviate

$$\mathcal{C}(X, Y) := \{f \in \mathrm{mor}(\mathcal{C}) : \mathrm{dom}\, f = X \text{ and } \mathrm{cod}\, f = Y\}.$$

Moreover, $f \in \mathcal{C}(X, Y)$ is depicted as

$$f : X \to Y. \tag{A.4}$$

Example A.3. Let $*$ be a single, not nearer specified object. Consider as morphisms the class of all cardinal numbers and as composition cardinal addition. By [**?**, 112–113], cardinal addition is associative and $\varnothing$ serves for the identity id_*.

Definition A.4 (Locally Small, Hom-Set). A category $\mathcal{C}$ is said to be ***locally small*** if for all $X, Y \in \mathcal{C}$, $\mathcal{C}(X, Y)$ is a set. If $\mathcal{C}$ is locally small, $\mathcal{C}(X, Y)$ is called a ***hom-set*** for all $X, Y \in \mathcal{C}$.

Definition A.5 (Monic). Let $\mathcal{C}$ be a category. A morphism $f \in \mathcal{C}(X, Y)$ is said to be ***monic***, iff for all objects $A \in \mathcal{C}$ and morphisms $g, h \in \mathcal{C}(A, X)$

$$f \circ g = f \circ h \Rightarrow g = h$$

holds.

Exercise A.6. In Set, show that a morphism is monic if and only if it is injective.

Definition A.7 (Epic). Let $\mathcal{C}$ be a category. A morphism $f \in \mathcal{C}(X, Y)$ is said to be ***epic***, iff f is monic in $\mathcal{C}^{\mathrm{op}}$.

Exercise A.8. In Set, show that a morphism is epic if and only if it is surjective.

Definition A.9 (Isomorphism). Let $\mathcal{C}$ be a category. An ***isomorphism in $\boldsymbol{\mathcal{C}}$*** is a morphism $f \in \mathcal{C}(X, Y)$, such that there exists a morphism $g \in \mathcal{C}(Y, X)$ with

$$g \circ f = \mathrm{id}_X \qquad \text{and} \qquad f \circ g = \mathrm{id}_Y\,.$$

Exercise A.10. Let $\mathcal{C}$ be a category. Show that any isomorphism is both monic and epic.

Exercise A.11. In Set, show that any monic and epic morphism is an isomorphism.

In the definition of an isomorphism A.9, a morphism is forced to admit a two-sided inverse. However, in reality, often only one-sided inverses do exist. Since they are particularly useful, they get they own terminology.

Definition A.12 (Section). Let $\mathcal{C}$ be a category and $f \in \mathcal{C}(X, Y)$. A morphism $\sigma \in \mathcal{C}(Y, X)$ is called a ***section of $\boldsymbol{f}$***, iff $f \circ \sigma = \mathrm{id}_Y$.

Exercise A.13. Let $\mathcal{C}$ be a category. Show that any morphism admitting a section is epic.

Exercise A.14. In Set, show that any epic morphism admits a section (observe the subtle use of the axiom of choice!).

Definition A.15 (Retraction). Let $\mathcal{C}$ be a category and $f \in \mathcal{C}(X, Y)$. A morphism $\rho \in \mathcal{C}(Y, X)$ is called a ***retraction of $\boldsymbol{f}$***, iff $\rho \circ f = \mathrm{id}_X$.

Exercise A.16. Let $\mathcal{C}$ be a category. Show that any morphism admitting a retraction is monic.

In algebraic topology, there is a very useful construction on categories.

Definition A.17 (Congruence). Let $\mathcal{C}$ be a category. A ***congruence on $\mathcal{C}$*** is an equivalence relation $\sim$ on $\mathrm{mor}(\mathcal{C})$ such that

(a) If $f \in \mathcal{C}(X, Y)$ and $f \sim g$, then $g \in \mathcal{C}(X, Y)$.
(b) If $f_0 : X \to Y$ and $g_0 : Y \to Z$ such that $f_0 \sim f_1$ and $g_0 \sim g_1$, then $g_0 \circ f_0 \sim g_1 \circ f_1$.

Exercise A.18. Let $\mathcal{C}$ be a category. Show that for any congruence on $\mathcal{C}$, there exists a category $\mathcal{C}'$, called ***quotient category***, with $\mathrm{ob}(\mathcal{C}') = \mathrm{ob}(\mathcal{C})$, for any objects $X, Y \in \mathcal{C}'$

$$\mathcal{C}'(X, Y) = \{[f] : f \in \mathcal{C}(X, Y)\},$$

and pointwise composition.

A.2 Functors

Definition A.19 (Functor). Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A ***functor $F : \mathcal{C} \to \mathcal{D}$*** is a pair of functions (F_1, F_2), $F_1 : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$, called the ***object function*** and $F_2 : \mathrm{mor}(\mathcal{C}) \to \mathrm{mor}(\mathcal{D})$, called the ***morphism function***, such that for every morphism $f : X \to Y$ we have that $F_2(f) : F_1(X) \to F_1(Y)$ and (F_1, F_2) is subject to the following ***compatibility conditions***:

- For all $X \in \mathrm{ob}(\mathcal{C})$, $F_2(\mathrm{id}_X) = \mathrm{id}_{F_1(X)}$.
- For all $f \in \mathcal{C}(X, Y)$ and $g \in \mathcal{C}(Y, Z)$ we have that $F_2(g \circ f) = F_2(g) \circ F_2(f)$.

Remark A.20. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. It is convenient to denote the components F_1 and F_2 also with F.

Appendix B
Basic Point-Set Topology

A short but formal introduction to the basics of point-set topology is given in the first four chapters of [5]. A more extensive treatment can be found in the classic [11].

Appendix C
Review of Analysis

C.1 Normed Spaces

Definition C.1 (Weak Convergence). Let $(X, \|\cdot\|_X)$ be a normed space and $x \in X$. A sequence $(x_k)_{k\in\mathbb{N}} \subseteq X$ is said to ***converge weakly to x***, written $x_k \xrightarrow{w} x$ as $k \to \infty$, iff for all $\varphi \in X^*$ we have that

$$\varphi(x_k) \to \varphi(x)$$

as $k \to \infty$.

Definition C.2 (Reflexivity). A normed space $(X, \|\cdot\|)_X$ is said to be ***reflexive***, iff the map $\Phi : X \to X^{**}$, defined by $\Phi(x)(\varphi) := \varphi(x)$ is surjective (Φ is already a linear isometry).

Theorem C.3 (Eberlein-Šmulyan). *Let $(X, \|\cdot\|_X)$ be reflexive and $(x_k)_{k\in\mathbb{N}}$ bounded. Then there exists $x \in X$ and a subsequence $\Lambda \subseteq \mathbb{N}$ such that*

$$x_k \xrightarrow{w} x$$

as $k \to \infty$, $k \in \Lambda$.

C.2 Differentiability

Definition C.4 (Carathéodory Differentiability). Let $(V, |\cdot|_V)$ and $(W, |\cdot|_W)$ be finite-dimensional vector spaces, $U \subseteq V$ open and $x_0 \in U$. A map $F : U \to W$ is said to be ***differentiable at x_0***, iff there exists a map $\varphi : U \to \mathrm{L}(V, W)$ such that φ is continuous at x_0 and

$$F(x) - F(x_0) = \varphi(x)(x - x_0) \tag{C.1}$$

holds for all $x \in U$.

Example C.5 (Linear Map). Let $(V,|\cdot|_V)$ and $(W,|\cdot|_W)$ be finite-dimensional vector spaces and $L \in \mathrm{L}(V, W)$. Then L is differentiable at every $x_0 \in V$ since

$$L(x) - L(x_0) = L(x - x_0) = \varphi(x)(x - x_0)$$

holds, where $\varphi : V \to \mathrm{L}(V, W)$ is given by $\varphi(x) := L$.

Proposition C.6. *Let $(V,|\cdot|_V)$ and $(W,|\cdot|_W)$ be finite-dimensional vector spaces, $U \subseteq V$ open and $x_0 \in U$. Suppose $\varphi, \psi : U \to \mathrm{L}(V, W)$ are continuous at x_0 such that*

$$F(x) - F(x_0) = \varphi(x)(x - x_0) \qquad \text{and} \qquad F(x) - F(x_0) = \psi(x)(x - x_0)$$

holds for all $x \in U$. Then $\varphi(x_0) = \psi(x_0)$.

Proof. Define $\Phi : U \to \mathrm{L}(V, W)$ by

$$\Phi(x) := \varphi(x) - \psi(x).$$

Then

$$\Phi(x)(x - x_0) = \varphi(x)(x - x_0) - \psi(x)(x - x_0) = 0$$

holds for all $x \in U$ and so

$$|\Phi(x_0)(x - x_0)|_W = \left|\big(\Phi(x_0) - \Phi(x)\big)(x - x_0)\right|_W \leq |\Phi(x_0) - \Phi(x)|_{\mathrm{op}}\, |x - x_0|_V$$

for all $x \in U$. Equivalently

$$\left|\Phi(x_0)\frac{x - x_0}{|x - x_0|_V}\right|_W \leq |\Phi(x_0) - \Phi(x)|_{\mathrm{op}} \tag{C.2}$$

for all $x \neq x_0$. Let $\varepsilon > 0$. Since Φ is continuous at x_0, there exists $0 < \delta < r$, where $B_r(x_0) \subseteq U$, wuch that for all $x \in \dot{B}_\delta(x_0)$

$$|\Phi(x_0) - \Phi(x)|_{\mathrm{op}} < \varepsilon$$

holds. Moreover

$$\left\{|\Phi(x_0)(x - x_0)/\, |x - x_0|_V| : x \in \dot{B}_\delta(a)\right\} = \left\{|\Phi(x_0)x|_W : |x|_V = 1\right\}.$$

Indeed, the inclusion $\subseteq$ is clear. Suppose that $|y|_V = 1$. Define $x := x_0 + \frac{\delta}{2}y$. Then

$$|x - x_0|_V \leq \frac{\delta}{2}\,|y|_V < \delta$$

and so $x \in \dot{B}_\delta(x_0)$. Also

$$\Phi(x_0)\frac{x - x_0}{|x - x_0|} = \Phi(x_0)\frac{\frac{\delta}{2}y}{\frac{\delta}{2}\,|y|_V} = \Phi(x_0)y.$$

Hence (C.2) yields

$$|\Phi(x_0)|_{\mathrm{op}} = \sup_{|x|_V=1} |\Phi(x_0)x| = \sup_{x\in \dot{B}_\delta(x_0)} \left|\Phi(x_0)\frac{x-x_0}{|x-x_0|}\right| < \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we have that $|\Phi(x_0)|_{\mathrm{op}} = 0$ and thus $\Phi(x_0) = 0$, that is $\varphi(x_0) = \psi(x_0)$. □

Definition C.7 (Differential). Let $(V, |\cdot|_V)$ and $(W, |\cdot|_W)$ be finite-dimensional vector spaces, $U \subseteq V$ open and $x_0 \in U$. If F is differentiable at x_0, define the ***differential of F at x_0***, written DF_{x_0}, by

$$DF_{x_0} := \varphi(x_0)$$

where φ is as in C.1.

Lemma C.8. *Let $U \subseteq \mathbb{R}$ open, $f : U \to \mathbb{R}^n$ and $x_0 \in U$. Then f is differentiable at x_0 if and only if*

$$\lim_{x\to x_0, x\in U} \frac{f(x) - f(x_0)}{x - x_0} \in \mathbb{R}. \tag{C.3}$$

Proof.

Definition C.9 (Derivative). Let $U \subseteq \mathbb{R}$ open and $f : U \to \mathbb{R}^n$ differentiable at $x_0 \in U$. Then the ***derivative of f at x_0***, written $f'(x_0)$, is defined by

$$f'(x_0) := \lim_{x\to x_0, x\in U} \frac{f(x) - f(x_0)}{x - x_0}.$$

Definition C.10 (Directional Derivative). Let $U \subseteq \mathbb{R}^n$ be open, $F : U \to \mathbb{R}^m$ and $v \in \mathbb{R}^n$. Define the ***directional derivative of F in direction v at x_0***, written $D_v F_{x_0}$, by

$$D_v F_{x_0} := \lim_{t\to 0, t\in\mathbb{R}} \frac{F(x_0 + tv) - F(x_0)}{t}$$

Definition C.11 (Partial Derivative). Let $U \subseteq \mathbb{R}^n$ open, $F : U \to \mathbb{R}^m$ and $x_0 \in U$. If F is differentiable at x_0, then define the ***i-th partial derivative of F at x_0***, written $D_i F(x_0)$, by

$$D_i F(x_0) := D_{e_i} F_{x_0},$$

where (e_i) denotes the standard basis of $\mathbb{R}^n$.

Proposition C.12. *Let $U \subseteq \mathbb{R}^n$ open, $F : U \to \mathbb{R}^m$ and $x_0 \in U$. If F is differentiable at x_0, then*

$$DF_{x_0}(v) = D_v F_{x_0}$$

for all $v \in \mathbb{R}^n$.

Proof. Consider the composition

$$t \xmapsto{f} x_0 + tv \xmapsto{F} F(x_0 + tv).$$

Then we compute

$$D_v F_{x_0} = (F \circ f)'(0) = D(F \circ f)_0 = DF_{x_0} \circ Df_0 = DF_{x_0} \circ f'(0) = DF_{x_0}(v).$$

C.3 The Inverse Function Theorem

Theorem C.13 (The Inverse Function Theorem). *Let $U \subseteq \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ smooth and $x \in U$. If Df_x is invertible, then there exists a neighbourhood $V \subseteq U$ of x such that $f : V \to f(V)$ is a diffeomorphism.*

C.4 The Implicit Function Theorem

Theorem C.14 (The Implicit Function Theorem). *Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$ be open, $\Phi : U \to \mathbb{R}^k$ smooth, $(x_0, y_0) \in U$ and $c := \Phi(x_0, y_0)$. If*

$$\det \left(D_j \Phi^i_{(x_0,y_0)} \right)^i_{j=n+1,\dots,n+k} \neq 0,$$

then there exist neighbourhoods $V_0 \subseteq \mathbb{R}^n$ of x_0 and $W_0 \subseteq \mathbb{R}^k$ of y_0 and a smooth function $F : V_0 \to W_0$ such that $\Phi^{-1}(c) \cap (V_0 \times W_0)$ is the graph of F.

C.5 Sobolev Spaces

In what follows, let $n \in \mathbb{N}$, $n \geq 1$, and $1 \leq p \leq \infty$.

Definition C.15 (Distributional and Weak Derivative). Let $\Omega \subseteq \mathbb{R}^n$ open and $u \in L^1_{\mathrm{loc}}(\Omega)$. For any multiindex α, the ***distributional derivative of order α of u***, written $D^\alpha u$, is defined to be the mapping $D^\alpha u : C^\infty_c(\Omega) \to \mathbb{R}$ defined by

$$\varphi \mapsto (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi dx.$$

Moreover, a function $D^\alpha u \in L^p(\Omega)$ is called ***weak derivative of order α of u with exponent p***, iff

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega D^\alpha u \varphi dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi dx.$$

Theorem C.16 (Fundamental Lemma of Variational Calculus). *Let $\Omega \subseteq \mathbb{R}^n$ open and $f \in L^1_{\text{loc}}(\Omega)$. If*

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega f \varphi dx = 0,$$

then $f = 0$ a.e.

Remark C.17. Let $\Omega \subseteq \mathbb{R}^n$ open. Then $L^p(\Omega) \subseteq L^1_{\text{loc}}(\Omega)$.

Remark C.18. From the fundamental lemma of variational calculus C.16 it follows that *weak derivatives, if they exist, are unique*.

Examples C.19 (Weak Derivatives).

(a) Suppose u is classically differentiable. Then u is weakly differentiable using integration by parts **??**.
(b) Consider $\Omega := (-1, 1)$ and $u := |x|$. Then $u' = \chi_{[0,1)} - \chi_{(-1,0)}$.
(c) Consider $\Omega := \mathbb{R}$ and $u := \chi_{(0,\infty)}$. Then the weak derivative u' does not exist. Indeed, the *Dirac distribution* is not representable as one may see by considering the smooth family $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ for $\varepsilon > 0$ defined by

$$\varphi_\varepsilon(x) := \begin{cases} e^{\varepsilon^2/(x^2-\varepsilon^2)} & |x| < \varepsilon, \\ 0 & |x| \geq \varepsilon. \end{cases}$$

(d) Let $\Omega := (0, 1)$ and consider the *Cantor function* $u : \Omega \to \Omega$. Then $u' = 0$ classically a.e. but the distributional derivative of u does not vanish.

Definition C.20 (Sobolev Space). Let $\Omega \subseteq \mathbb{R}^n$ open. For any $k \in \omega$, the ***Sobolev space of index (k, p)***, written $W^{k,p}(\Omega)$, is defined to be the space

$$W^{k,p}(\Omega) := \{ f \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ exists for all } |\alpha| \leq k \},$$

with norm

$$\|-\|_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha -\|_{L^p(\Omega)}.$$

Moreover, define

$$W_0^{k,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|-\|_{W^{k,p}(\Omega)}},$$

and $H^k(\Omega) := W^{k,2}(\Omega)$ as well as $H_0^k(\Omega) := W_0^{k,2}(\Omega)$.

Theorem C.21. *Let $\Omega \subseteq \mathbb{R}^n$ open. Then $W^{k,p}(\Omega)$ is*

(*a*) *a Banach space for all $1 \leq p \leq \infty$.*
(*b*) *separable for all $1 \leq p < \infty$.*
(*c*) *reflexive for all $1 < p < \infty$.*

Proof. The proof basically boils down to using the correponding properties of the Lebesgue spaces $L^p(\Omega)$.

(a) This follows from the fact that $L^p(\Omega)$ is a Banach space for all $1 \le p \le \infty$. Let $(f_i)_{i\in\omega}$ be a Cauchy sequence in $W^{k,p}$. By definition of the $W^{k,p}$-norm, $(D^\alpha f_i)_{i\in\omega}$ is a Cauchy sequence in L^p. Thus we get $D^\alpha f_i \to f_\alpha$ in L^p, in particular, $f_i \to f$ in L^p. Using Hölder's inequality we compute

$$\int_\Omega f_\alpha \varphi dx = \lim_{i\to\infty} \int_\Omega D^\alpha f_i \varphi dx = (-1)^{|\alpha|} \lim_{i\to\infty} \int_\Omega f_i D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi dx$$

for $\varphi \in C_c^\infty(\Omega)$.
(b) For simplicity, we consider $k = 1$ only. Consider $\iota : W^{1,p} \hookrightarrow (L^p)^{n+1}$ defined in the obvious way. Then ι is an isometry and the statement follows.
(c) Same argument as in part (b). □

In what follows, let $-\infty \le a < b \le \infty$ and $I := (a, b)$.

Lemma C.22 (Du Bois-Reymond). *Let $f \in L^1_{\text{loc}}(I)$ such that*

$$\forall \varphi \in C_c^\infty(I) : \int_I f\varphi' dx = 0.$$

Then f is almost everywhere constant.

Proof. Let $v := w - c_0\psi$ for $w, \psi \in C_c^\infty(I)$ such that $\int_I \psi = 1$ and $\int_I v = 0$. This implies $c_0 = \int_I w$. By the fundamental theorem of calculus, the function $\varphi : I \to \mathbb{R}$ defined by

$$\varphi(x) := \int_I v(t)dt$$

belongs to $C_c^\infty(I)$ with $\varphi' = v$. Thus we compute

$$0 = \int_I f\varphi' = \int_I fv = \int_I fw - c_0 \int_I f\psi = \int_I fw - \int_I w \int_I f\psi = \int_I (f - c)w,$$

where $c := \int_I f\psi$. Since w was arbitrary, we conclude by the fundamental lemma of variational calculus C.16. □

Lemma C.23. *Let $f \in L^1_{\text{loc}}(I)$ and $x_0 \in I$. Then $u : I \to \mathbb{R}$ defined by*

$$u(x) := \int_{x_0}^x f(t)dt$$

is absolutely continuous and belongs to $W^{1,1}_{\text{loc}}(I)$ with $u' = f$ a.e.

Proof. Absolute continuity follows from real analysis. Let $\varphi \in C_c^\infty(I)$. Then Fubini yields

$$
\begin{aligned}
\int_I u\varphi' &= \int_a^{x_0}\int_{x_0}^x f(t)\varphi'(x)dtdx + \int_{x_0}^b\int_{x_0}^x f(t)\varphi'(x)dtdx \\
&= -\int_a^{x_0}\int_x^{x_0} f(t)\varphi'(x)dtdx + \int_{x_0}^b\int_{x_0}^x f(t)\varphi'(x)dtdx \\
&= -\int_a^{x_0}\int_a^t f(t)\varphi'(x)dxdt + \int_{x_0}^b\int_t^b f(t)\varphi'(x)dxdt \\
&= -\int_a^{x_0} f(t)\varphi(t)dt - \int_{x_0}^b f(t)\varphi(t)dt \\
&= -\int_I f\varphi.
\end{aligned}
$$

Theorem C.24. *Let $u \in W^{1,p}(I)$. Then there exists an absolutely continuous representant $\widetilde{u}$ of u on $\overline{I}$, such that*

$$\widetilde{u}(x) = \widetilde{u}(x_0) + \int_{x_0}^x u'(t)dt$$

holds for all $x, x_0 \in I$. In particular, $\widetilde{u}$ is classically differentiable a.e. and $\widetilde{u}' = u'$.

Proof. By lemma C.23, the function $v(x) := \int_{x_0}^x u'(t)dt$ is in $W^{1,1}_{\mathrm{loc}}(I)$ with weak derivative u'. Moreover, for any $\varphi \in C_c^\infty(I)$ we compute

$$\int_I (u-v)\varphi' = \int_I u\varphi' - \int_I v\varphi' = -\int_I u'\varphi + \int_I u'\varphi = 0.$$

Thus lemma C.22 yields $u = c + v$, for some $c \in \mathbb{R}$. Set

$$\widetilde{u}(x) := c + \int_{x_0}^x u'(t)dt.$$

Then $\widetilde{u}(x_0) = c$ and thus the statement follows. □

Theorem C.25 (Characterization of $W^{1,p}(I)$). *Let $1 < p \le \infty$ and $u \in L^p(I)$. Then the following statements are equivalent:*

(a) $u \in W^{1,p}(I)$.
(b) There exists $C \ge 0$ such that

$$\forall \varphi \in C_c^\infty(I) : \left|\int_I u\varphi'\right| \le C\,\|\varphi\|_{L^q}.$$

(c) There exists $C \ge 0$ such that for all $I' \subseteq\subseteq I$ and $|h| < \operatorname{dist}(I', \partial I)$ holds

$$\|\tau_h u - u\|_{L^p(I')} \le C\,|h|,$$

where $\tau_h u(x) := u(x+h)$.

Proof. The implication $(a) \Rightarrow (b)$ follows immediately from Hölder's inequality. To prove $(b) \Rightarrow (a)$, we observe that $l : C_c^\infty(I) \to \mathbb{R}$ defined by

$$l(\varphi) := \int_I u\varphi'$$

is continuous. Since $C_c^\infty(I)$ is dense in $L^q(I)$, we get that $l \in (L^q(I))^*$. Hence we find $g \in L^p$, such that $\int_I g\varphi = l(\varphi)$ and so $u' = -g$.
Next we show $(a) \Rightarrow (c)$. By theorem C.24, we find an absolutely continuous representant $\widetilde{u}$ of u. Thus

$$\widetilde{u}(x+h) - \widetilde{u}(x) = h \int_0^1 u'(x+th)dt$$

Hence Jensen's inequality yields

$$\|\tau_h u - u\|_{L^p(I')} \leq |h| \int_0^1 \|u'(\cdot + th)\|_{L^p(I')}\, dt \leq |h|\, \|u'\|_{L^p(I)}\,.$$

Lastly, we prove $(c) \Rightarrow (b)$. Let $\varphi \in C_c^\infty(I)$. Then we may find $I' \subseteq\subseteq I$ such that $\operatorname{supp}\varphi \subseteq I'$. Hence we compute

$$\begin{aligned}
\left|\int_I u\varphi'\right| &= \lim_{h\to 0} \frac{1}{|h|}\left|\int_I u(x)\left(\varphi(x+h) - \varphi(x)\right)dx\right| \\
&= \lim_{h\to 0} \frac{1}{|h|}\left|\int_I \left(u(x-h) - u(x)\right)\varphi(x)dx\right| \\
&= \lim_{h\to 0} \frac{1}{|h|}\left|\int_I (\tau_{-h}u - u)\,\varphi\right| \\
&\leq \lim_{h\to 0} \frac{1}{|h|}\, \|\tau_{-h}u - u\|_{L^p(I')}\|\varphi\|_{L^q(I)} \\
&\leq C\,\|\varphi\|_{L^q(I)}\,.
\end{aligned}$$

Theorem C.26 (Sobolev Embedding). *There is a continuous embedding*

$$W^{1,p}(I) \hookrightarrow L^\infty(I).$$

Proof. First consider I bounded. By theorem C.24 we get that

$$\|u\|_{L^\infty} = \sup_{x\in I}|u(x)| \leq |u(y)| + \sup_{x\in I}\left|\int_y^x u'(t)dt\right| \leq |u(y)| + \|u'\|_{L^1}\,,$$

for any $y \in I$. Hence

$$\|u\|_{L^\infty} \leq \inf_{y\in I}|u(y)| + \|u'\|_{L^1} \leq \frac{1}{|I|}\int_I |u(y)| + \|u'\|_{L^1} \leq C\,\|u\|_{W^{1,1}} \leq C\,\|u\|_{W^{1,p}}\,.$$

Assume now that I is unbounded. Then we find $I' \subseteq\subseteq I$ such that

$$\|u\|_{L^\infty(I')} \geq \frac{1}{2}\|u\|_{L^\infty(I)}$$

and thus the claim follows by the previous computation. Indeed, note that by theorem C.24, we have that

$$|u(x)| \leq |u(y)| + \|u'\|_{L^1(I)}$$

for all $x \in I$ and fixed $y \in I$, and thus $u \in L^\infty(I)$. Moreover, there exists $x_0 \in I$ such that $|u(x_0)| > \frac{1}{2}\|u\|_{L^\infty(I)}$, if not, this would contradict the definition of the supremum norm. Since u is continuous by theorem C.24, we find $\delta > 0$ such that

$$|u(x) - u(x_0)| \leq |u(x_0)| - \frac{1}{2}\|u\|_{L^\infty(I)}$$

for all $x \in I$ such that $|x - x_0| < \delta$. Hence the reversed triangle inequality yields

$$\frac{1}{2}\|u\|_{L^\infty(I)} - |u(x_0)| \leq |u(x)| - |u(x_0)| \leq |u(x_0)| - \frac{1}{2}\|u\|_{L^\infty(I)}$$

and so

$$\frac{1}{2}\|u\|_{L^\infty(I)} \leq |u(x)|$$

for all $x \in I \cap (x_0 - \delta, x_0 + \delta) =: I'$. $\square$

Appendix D
Review of Algebraic Topology

A quick introduction to the rudiments of Algebraic Topology can be found in chapters 7-13 in [5]. A more extensive treatment can be found in [12]. However, we focus primarily on the excellent lecture notes of the course *Algebraic Topology I/II* given at the *ETH Zurich* in the autumn semester 2017 and spring semester 2018. These notes can be found here

https://www.merry.io/algebraic-topology/.

Appendix E
The Fundamental Group

The Fundamental Grupoid

π_0

Lemma E.1. *There exists a functor* $\mathsf{Top} \to \mathsf{Set}$*. Moreover, if* $f, g \in \mathsf{Top}(X, Y)$ *are freely homotopic, then* $\pi_0(f) = \pi_0(g)$*.*

Proof. On objects $X \in \mathrm{ob}(\mathsf{Top})$, define $\pi_0(X)$ to be the set of equivalence classes of X under path connectedness. On morphisms $f : X \to Y$, define $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ by $\pi_0(f)\,[x] := [f(x)]$. This is well defined since if $[x] = [y]$, there exists a path u from x to y in X and it is easy to check that $f \circ u$ is a path from $f(x)$ to $f(y)$. Checking that π_0 is indeed a functor is left as an exercise. Suppose $H : f \simeq g$ and let $x \in X$. Then $H(x, t)$ is a path from $f(x)$ to $g(x)$ and thus $\pi_0(f)\,[x] = [f(x)] = [g(x)] = \pi_0(g)\,[x]$. □

Exercise E.2. Check the functoriality of $\pi_0 : \mathsf{Top} \to \mathsf{Set}$.

Proposition E.3. *If* $X, Y \in \mathrm{ob}(\mathsf{Top})$ *have the same homotopy type, then* $|\pi_0(X)| = |\pi_0(Y)|$*, i.e.* X *and* Y *have the same number of path components.*

Proof. Since X and Y are of the same homotopy type, they are isomorphic in hTop. By lemma E.1, π_0 descends to hTop and since functors preserve isomorphisms, we have that $\pi_0(X) \cong \pi_0(Y)$. In Set, isomorphisms are bijections and thus the statement follows. □

Construction of the Fundamental Grupoid

Lemma E.4 (Gluing Lemma). *Let* $X, Y \in \mathrm{ob}(\mathsf{Top})$*,* $(X_\alpha)_{\alpha\in A}$ *a finite closed cover of* X *and* $(f_\alpha)_{\alpha\in A}$ *a finite family of maps* $f_\alpha \in \mathsf{Top}(X_\alpha, Y)$ *such that* $f_\alpha|_{X_\alpha\cap X_\beta} = f_\beta|_{X_\alpha\cap X_\beta}$ *for all* $\alpha, \beta \in A$*. Then there exists a unique* $f \in \mathsf{Top}(X, Y)$ *such that* $f|_{X_\alpha} = f_\alpha$ *for all* $\alpha \in A$*.*

Proof. Let $x \in X$. Since $(X_\alpha)_{\alpha \in A}$ is a cover of X, we find $\alpha \in A$ such that $x \in X_\alpha$. Define $f(x) := f_\alpha(x)$. This is well defined, since if $x \in X_\alpha \cap X_\beta$ for some $\beta \in A$, we have that $f(x) = f_\beta(x) = f_\alpha(x)$. Clearly $f|_{X_\alpha} = f_\alpha$ for all $\alpha \in A$ and f is unique. Let us show continuity. To this end, let $K \subseteq Y$ be closed. Then

$$\begin{aligned} f^{-1}(K) &= X \cap f^{-1}(K) \\ &= \bigcup_{\alpha \in A} X_\alpha \cap f^{-1}(K) \\ &= \bigcup_{\alpha \in A} \left(X_\alpha \cap f^{-1}(K)\right) \\ &= \bigcup_{\alpha \in A} \left(X_\alpha \cap f_\alpha^{-1}(K)\right). \end{aligned}$$

Since each f_α is continuous, $f_\alpha^{-1}(K)$ is closed in X_α for each $\alpha \in A$ and thus since X_α is closed, $f^{-1}(K)$ is closed as a finite union of closed sets. $\square$

Theorem E.5. *There is a functor* $\mathsf{Top} \to \mathsf{Grpd}$.

Proof. The proof is divided into several steps. Let us denote $\Pi : \mathsf{Top} \to \mathsf{Grpd}$ for the claimed functor.

Step 1: Definition of Π on objects. Let $X, Y \in \mathrm{ob}(\mathsf{Top})$, $f, g \in \mathsf{Top}(X, Y)$ and $A \subseteq X$. A map $F \in \mathsf{Top}(X \times I, Y)$ is called a ***homotopy from X to Y relative to A***, if

- $F(x, 0) = f(x)$, for all $x \in X$.
- $F(x, 1) = g(x)$, for all $x \in X$.
- $F(x, t) = f(x) = g(x)$, for all $x \in A$ and for all $t \in I$.

If there exists a homotopy between f and g relative to A we say that f and g are ***homotopic relative to A*** and write $f \simeq_A g$. If we want to emphasize the homotpoy relative to A, we write $F : f \simeq_A g$.

Lemma E.6. *Let $X, Y \in \mathrm{ob}(\mathsf{Top})$ and $A \subseteq X$. Then being homotopic relative to A is an equivalence relation on* $\mathsf{Top}(X, Y)$. $\square$

Proof. Define a binary relation $R_A \subseteq \mathsf{Top}(X, Y) \times \mathsf{Top}(X, Y)$ by

$$f R_A g \quad :\Leftrightarrow \quad f \simeq_A g.$$

Let $f \in \mathsf{Top}(X, Y)$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x, t) := f(x).$$

Then clearly $F : f \simeq_A f$. Hence R_A is reflexive.
Let $g \in \mathsf{Top}(X, Y)$ and assume that $f R_A g$. Thus $G : f \simeq_A g$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x, t) := G(x, 1 - t).$$

Then it is easy to check that $F : g \simeq_A f$ and so R_A is symmetric.
Finally, let $h \in \mathsf{Top}(X, Y)$ and suppose that $f R_A g$ and $g R_A h$. Hence $F_1 : f \simeq_A g$ and $F_2 : g \simeq_A h$. Define $F \in \mathsf{Top}(X \times I, Y)$ by

$$F(x,t) := \begin{cases} F_1(x, 2t) & 0 \leq t \leq \frac{1}{2}, \\ F_2(x, 2t-1) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Continuity of F follows by an application of the gluing lemma E.4. Then it is easy to check that $F : f \simeq_A h$ and hence R_A is transitive. □

Let $X \in \mathrm{ob}(\mathsf{Top})$ and u a path in X from p to q. Define the ***path class* $[u]$ *of* u** by $[u] := [u]_{R_{\partial I}}$. Define now

- $\mathrm{ob}\left(\Pi(X)\right) := X$.
- $\Pi(X)(p,q) := \{[u] : u \text{ is a path from } p \text{ to } q\}$ for all $p, q \in X$.
- Let $p \in X$. Then define $\mathrm{id}_p \in \Pi(X)(p,p)$ by $\mathrm{id}_p := [c_p]$, where c_p is the constant path defined by $c_p(s) := p$ for all $s \in I$.
- And $\Pi(X)(q,r) \times \Pi(X)(p,q) \to \Pi(X)(p,r)$ by

  $$([v], [u]) \mapsto [u * v]$$

  Where $u * v \in \mathsf{Top}(p, r)$ is the ***concatenated path of u and v***, defined by

  $$(u * v)(s) := \begin{cases} u(2s) & 0 \leq t \leq \frac{1}{2}, \\ v(2s-1) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

  Continuity follows again from the gluing lemma E.4 whereas well definedness follows from the next lemma.

Lemma E.7. *Suppose that* $[u_1], [u_2] \in \Pi(X)(p,q)$ *and* $[v_1], [v_2] \in \Pi(X)(q,r)$ *such that* $[u_1] = [u_2]$ *and* $[v_1] = [v_2]$. *Then* $[u_1 * v_1] = [u_2 * v_2]$. □

Proof. By assumption we have $G : u_1 \simeq_{\partial I} u_2$ and $H : v_1 \simeq_{\partial I} v_2$. Define $F \in \mathsf{Top}(I \times I, X)$ by

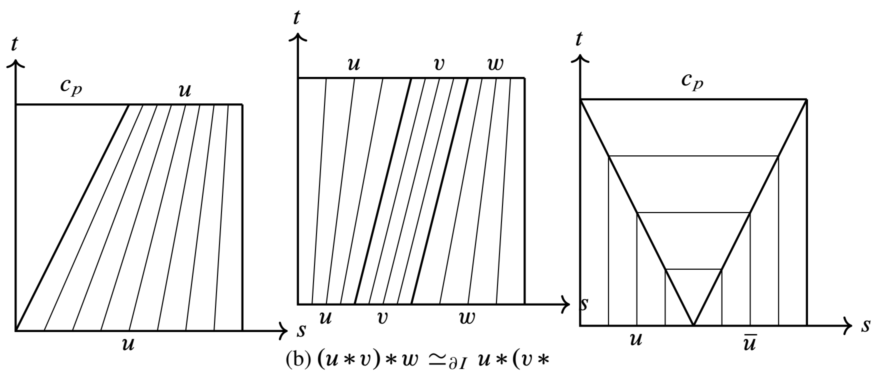
$$F(s,t) := \begin{cases} G(2s, t) & 0 \leq s \leq \frac{1}{2}, \\ H(2s-1, t) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Again, continuity follows from the gluing lemma E.4 and it is easy to check that $F : u_1 * v_1 \simeq_{\partial I} u_2 * v_2$. □

Let us now check that $\Pi(X)$ is indeed a category. Let $[u] \in \Pi(X)(p,q)$. We want to show that $u \simeq_{\partial I} c_p * u$. To this end, we consider figure E.1a and conclude that a suitable homotopy is given by $F \in \mathsf{Top}(I \times I, X)$ defined by

$$F(s,t) := \begin{cases} p & 0 \le 2s \le t, \\ u\left(\dfrac{2s-t}{2-t}\right) & t \le 2s \le 2. \end{cases}$$

Similarly, considering figure E.1b leads to $F \in \mathsf{Top}(I \times I, X)$ defined by



(a) $u \simeq_{\partial I} c_p * u$. (b) $(u * v) * w \simeq_{\partial I} u * (v * w)$. (c) $u * \overline{u} \simeq_{\partial I} c_p$.

Fig. E.1: Visualization of the proof that $\Pi(X)$ is a grupoid object.

$$F(s,t) := \begin{cases} u\left(\dfrac{4s}{t+1}\right) & -1 \le 4s-1 \le t, \\ v(4s-t-1) & t \le 4s-1 \le t+1, \\ w\left(\dfrac{4s-t-2}{4-t-2}\right) & t+1 \le 4s-1 \le 3. \end{cases}$$

Lastly, we check that $\Pi(X)$ is a grupoid. To this end, for a path u from p to q, define its ***reverse path*** $\overline{u}$ by

$$\overline{u}(s) := u(1-s).$$

We claim that $u * \overline{u} \simeq_{\partial I} c_p$. From figure E.1c we deduce that $F \in \mathsf{Top}(I \times I, X)$ is given by

$$F(s,t) := \begin{cases} u(2s) & 0 \le 2s \le 1-t, \\ u(1-t) & 1-t \le 2s \le t+1, \\ \overline{u}(2s-1) & t+1 \le 2s \le 2. \end{cases}$$

Step 2: Definition of Π on morphisms. Let $f \in \mathsf{Top}(X,Y)$. Then $\Pi(f)$ is a functor from $\Pi(X)$ to $\Pi(Y)$. Define $\Pi(f)$ as follows:

- Let $p \in \mathrm{ob}\big(\Pi(X)\big)$. Then define $\Pi(f)(p) := f(p) \in \mathrm{ob}\big(\Pi(Y)\big)$.

- Let $[u] \in \Pi(X)(p,q)$. Then define $\Pi(f)[u] := [f \circ u] \in$. We have to check that this definition is independent of the choice of the representative.

  Lemma E.8. *Let u and v be paths from p to q in X and suppose that $[u] = [v]$. Then for any $f \in \mathsf{Top}(X,Y)$ we also have that $[f \circ u] = [f \circ v]$.* □

  Proof. Suppose that $H : u \simeq_{\partial I} v$. Define $F \in \mathsf{Top}(I \times I, Y)$ by

  $$F(s,t) := (f \circ F)(s,t).$$

  Then $F : f \circ u \simeq_{\partial I} f \circ v$. □

Checking that Π satisfies the functorial properties is left as an exercise. □

Exercise E.9. Check that $\Pi : \mathsf{Top} \to \mathsf{Grpd}$ is indeed a functor.

Definition E.10 (Free Homotopy). Let $f, g \in \mathsf{Top}(X,Y)$. f and g are said to be ***freely homotopic*** if $f \simeq_{\varnothing} g$.

Example E.11 (Straight Line Homotopy). Let V be a real vector space. A subset $S \subseteq V$ is said to be ***convex***, if the line segment $\{(1-t)p + tq : 0 \leq t \leq 1\}$ is contained in S for all $p, q \in V$. Suppose now that V is finite dimensional and $S \subseteq V$ is convex. Moreover, let $f, g \in \mathsf{Top}(X,S)$ for some $X \in \mathrm{ob}(\mathsf{Top})$. Define $H : X \times I \to S$ by

$$H(x,t) := (1-t)f(x) + tg(x).$$

Then H is continuous and clearly $H : f \simeq g$. We call H the ***straight line homotopy between f and g***. Hence any two continuous maps defined on the same domain into a convex space are freely homotopic.

Remark E.12. We will also write $f \simeq g$ for a free homotopy.

Definition E.13 (Nullhomotopic). A mapping $f \in \mathsf{Top}(X,Y)$ is said to be ***nullhomotopic***, if f is freely homotopic to a constant map.

Definition E.14 (Contractible). A topological space X is said to be ***contractible***, if id_X is nullhomotopic.

Definition E.15 (Reparametrization). Let u be a path in a topological space X. A ***reparametrization*** of u is a path $u \circ \varphi$, where $\varphi \in \mathsf{Top}(I,I)$ fixing 0 and 1.

Lemma E.16. *let u be a path in a topological space x and $u \circ \varphi$ a reparametrization of u. Then $u \simeq_{\partial I} u \circ \varphi$.*

Proof. Since I is convex, we find a straight line homotopy $H : \mathrm{id}_I \simeq \varphi$. Now $u \circ H$ is the homotopy we are looking for. □

The Fundamental Group

Lemma E.17. *Let $\mathcal{G}$ be a locally small grupoid. Then for every $X \in \mathrm{ob}(\mathcal{G})$, $\mathcal{G}(X, X)$ can be equipped with a group structure.*

Proof. Since $\mathcal{G}$ is locally small, $\mathcal{G}(X, X)$ is a set for every $X \in \mathrm{ob}(\mathcal{G})$. Define a multiplication $\mathcal{G}(X, X) \times \mathcal{G}(X, X) \to \mathcal{G}(X, X)$ by $gh := h \circ g$. Clearly, this multiplication is associative. Moreover, the identity element is given by $\mathrm{id}_X \in \mathcal{G}(X, X)$ and since every $g \in \mathcal{G}(X, X)$ is an isomorphism, the multiplicative inverse is given by the inverse in $\mathcal{G}(X, X)$. □

Proposition E.18. *There is a functor* $\mathsf{Top}_* \to \mathsf{Grp}$.

Proof. Define $\pi_1 : \mathsf{Top}_* \to \mathsf{Grp}$ on objects $(X, p) \in \mathsf{Top}_*$ by

$$\pi_1(X, p) := \Pi(X)(p, p).$$

By theorem E.5 together with lemma E.17, $\pi_1(X, p)$ is actually a group, called the ***fundamental group of X with basepoint p***. On morphisms $f \in \mathsf{Top}_*\big((X, p), (Y, q)\big)$, define

$$\pi_1(f) := \Pi(f) : \Pi(X)(p, p) \to \Pi(Y)(q, q).$$

Let $[u], [v] \in \pi_1(X, p)$. Then

$$\begin{aligned} \pi_1([u]\,[v]) &= \Pi(f)([u]\,[v]) \\ &= \Pi(f)\,[u * v] \\ &= [f \circ (u * v)] \\ &= [(f \circ u) * (f \circ v)] \\ &= \Pi(f)\,[u]\,\Pi(f)\,[v] \\ &= \pi_1(f)\,[u]\,\pi_1(f)\,[v]\,. \end{aligned}$$

Thus $\pi_1(f)$ is a morphism in Grp. Functoriality of π_1 immediately follows from the functoriality of Π. □

Definition E.19 (Simply Connected). A path connected topologial space X is said to be ***simply connected***, if $\pi_1(X)$ is trivial.

First Properties of the Fundamental Group

Lemma E.20. *Let $X \in \mathrm{ob}(\mathsf{Top})$, $p \in X$ and A be the path component of X containing p. Then $\pi_1(\iota)$, where $\iota : A \hookrightarrow X$ denotes the inclusion, is an isomorphism.*

Proof. Suppose $[u] \in \ker \pi_1(\iota)$. Then $[\iota \circ u] = [c_p]$ and Hence $F : \iota \circ u \simeq_{\partial I} c_p$. Since $I \times I$ is path connected and $p \in F(I \times I)$, it follows that $F(I \times I) \subseteq A$

and thus $F : u \simeq_{\partial I} c_p$ in A and hence $[u] = [c_p]$. To see that $\pi_1(\iota)$ is surjective, just observe that $u(I) \subseteq A$ for $[u] \in \pi_1(X, p)$ since $u(I)$ is path connected and $p \in u(I)$. □

Lemma E.21. *Let $X \in \mathrm{ob}(\mathsf{Top})$ be path connected and $p, q \in X$. Then*

$$\pi_1(X, p) \cong \pi_1(X, q).$$

Proof. Since X is path connected we find a path v from p to q in X. Define a mapping $\Phi_v : \pi_1(X, p) \to \pi_1(X, q)$

$$\Phi_v\,[u] := [\overline{v} * u * v]\,.$$

Clearly, Φ_v is invertible with inverse $\Phi_{\overline{v}}$. Moreover, for $[u]\,, [w] \in \pi_1(X, p)$ we have that

$$\begin{aligned}
\Phi_v([u]\,[w]) &= \Phi_v\,[u * w] \\
&= [\overline{v} * u * w * v] \\
&= [\overline{v} * u * v * \overline{v} * w * v] \\
&= [\overline{v} * u * v]\,[\overline{v} * w * v] \\
&= \Phi_v\,[u]\,\Phi_v\,[w]\,.
\end{aligned}$$

Lemma E.22 (Square Lemma). *Let $F \in \mathsf{Top}(I \times I, X)$. Then*

$$F(0, \cdot) * F(\cdot, 1) \simeq_{\partial I} F(\cdot, 0) * F(1, \cdot).$$

Proof. The idea is to consider first the case $F = \mathrm{id}_{I \times I}$. Hence define the paths f_0, f_1, g_0 and g_1 in $I \times I$ as indicated in figure E.2a. Then there is a straight line homotopy $H : I \times I \to I \times I$ between them as indicated in figure E.2b. Explicitly

$$H(s, t) := (1 - t)(f_0 * f_1)(s) + t(g_0 * g_1)(s).$$

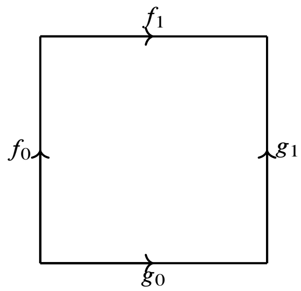
Then

$$(F \circ H)(s, t) = \begin{cases} F\big(2st, 2s(1 - t)\big) & 0 \le s \le \frac{1}{2}, \\ F\big(t + (1 - t)(2s - 1), 1 + 2t(s - 1)\big) & \frac{1}{2} \le s \le 1, \end{cases}$$

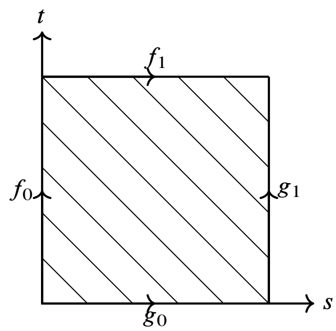
is the homotopy we are looking for.

Proposition E.23. *Let $f_0, f_1 \in \mathsf{Top}(X, Y)$ such that $F : f_0 \simeq f_1$. Moreover, let $p \in X$. Then the diagram*

$$\begin{tikzcd}
\pi_1(X, p) \arrow[r, "\pi_1(f_0)"] \arrow[dr, "\pi_1(f_1)"'] & \pi_1(Y, f_0(p)) \arrow[d, "\Phi_{F(p,\cdot)}"] \\
& \pi_1(Y, f_1(p))
\end{tikzcd}$$



(a) The paths f_0, f_1, g_0 and g_1 in $I \times I$.



(b) $f_0 * f_1 \simeq_{\partial I} g_0 * g_1$.

commutes, where Φ. denotes the isomorphism in lemma E.21.

Proof. Let $[u] \in \pi_1(X, p)$. We have that

$$\begin{aligned}\pi_1(f_1)\,[u] = \big(\Phi_{F(p,\cdot)} \circ \pi_1(f_0)\big)\,[u] &\Leftrightarrow [f_1 \circ u] = [\overline{F}(p,\cdot) * (f_0 \circ u) * F(p,\cdot)]\\ &\Leftrightarrow [F(p,\cdot) * (f_1 \circ u)] = [(f_0 \circ u) * F(p,\cdot)]\\ &\Leftrightarrow [F(u(0),\cdot) * F(u\cdot,1)] = [F(u\cdot,0) * F(u(1),\cdot)]\,,\end{aligned}$$

where the last equality is true by the square lemma E.22. $\square$

Homotopy Invariance of π_1

Lemma E.24. *Being freely homotopic is a congruence on* Top.

Proof. (a) is immediate so we only have to check (b). Suppose $f_0 \in \mathsf{Top}(X, Y)$ and $g_0 \in \mathsf{Top}(Y, Z)$ such that $F : f_0 \simeq f_1$ and $G : g_0 \simeq g_1$. Consider $H_1 : X \times I \to Z$ defined by $H_1 := g_0 \circ F$. Then clearly $H_1 : g_0 \circ f_0 \simeq g_0 \circ f_1$. Moreover, we define $H_2 : X \times I \to Z$ by $H_2 := G(f_1\cdot, \cdot)$. Then $H_2 : g_0 \circ f_1 \simeq g_1 \circ f_1$. And we conclude by transitivity. $\square$

Definition E.25 (hTop). The quotient category under the congruence of being freely homotopic is called the ***homotopy category***, and is denoted by hTop.

Definition E.26 (Homotopy Type). Two topological spaces X and Y are of the ***same homotopy type***, if they are isomorphic in hTop. An explicit choice of such an isomorphism is called a ***homotopy equivalence***.

Exercise E.27. Show that a topological space X has the same homotopy type as a one-point space if and only if X is contractible.

Theorem E.28 (Homotopy Invariance of π_1). *Suppose X and Y are of the same homotopy type with homotopy equivalence $f : X \to Y$. Then for any $p \in X$ we have that $\pi_1(f) : \pi_1(X, p) \to (Y, f(p))$ is an isomorphism.*

Proof. By assumption there exists $g \in \mathsf{Top}(Y, X)$ such that $F : g \circ f \simeq \mathrm{id}_X$ and $G : f \circ g \simeq \mathrm{id}_Y$. By the functoriality of π_1 and proposition E.23, the diagram

$$\begin{tikzcd}
 & \pi_1(Y, f(p)) \arrow[dr, "\pi_1(g)"] & \\
\pi_1(X, p) \arrow[ur, "\pi_1(f)"] \arrow[rr, "\pi_1(g\circ f)"] \arrow[dr, "\mathrm{id}_{\pi_1(X,p)}"'] & & \pi_1(X, g(f(p))) \arrow[dl, "\Phi_{F(p,\cdot)}"] \\
 & \pi_1(X, p) &
\end{tikzcd}$$

commutes. Since $\Phi_{F(p,\cdot)}$ is an isomorphism, $\pi_1(g \circ f)$ is an isomorphism, too. Hence $\pi_1(f)$ is injective. Using G instead of F and a similar argument yields that $\pi_1(f)$ is surjective. $\square$

Lemma E.29. *Let $G \in \mathrm{ob}(\mathsf{Grp})$, $S \in \mathsf{Set}$ and $\varphi : U(G) \to S$ a bijection. Then S can be given a group structure such that φ is an isomorphism.*

Proof. It is easy to show that $xy := \varphi\left(\varphi^{-1}(x)\varphi^{-1}(y)\right)$ defines a group structure on S with the requested property. $\square$

Proposition E.30. *Let $(X, p) \in \mathrm{ob}(\mathsf{Top}_*)$. Then $\pi_1(X, p) \cong \mathsf{hTop}_*\left((\mathbb{S}^1, 1), (X, p)\right)$.*

Proof. Let $u \in \Omega(X, p)$. Then u passes to the quotient $\widetilde{u} : (\mathbb{S}^1, 1) \to (X, p)$. Define now $\varphi\,[u] := [\widetilde{u}] \in \mathsf{hTop}_*\left((\mathbb{S}^1, 1), (X, p)\right)$. This is well defined, since if $H : u \simeq_{\partial I} v$, it is easy to see that $\widetilde{H} : \widetilde{u} \simeq_{\{1\}} \widetilde{v}$. Moreover, if $f \in \mathsf{hTop}_*\left(\mathbb{S}^1, 1), (X, p)\right)$, we define $\psi\,[f] := [f \circ \omega]$. Again, this is well defined since if $H : f \simeq_{\{1\}} g$, then $H \circ (\omega \times \mathrm{id}_I) : f \circ \omega \simeq_{\partial I} g \circ \omega$. It is easy to check that φ and ψ are inverses of each other and thus we have a bijection $\pi_1(X, p) \cong \mathsf{hTop}_*\left((\mathbb{S}^1, 1), (X, p)\right)$ of sets. Hence an application of lemma E.29 yields the result. $\square$

$\pi_1(\mathbb{S}^1)$

Definition E.31 (Exponential Quotient Map and Fundamental Loop). The mapping $\varepsilon : \mathbb{R} \to \mathbb{S}^1$ defined by

$$\varepsilon(x) := e^{2\pi i x} \tag{E.1}$$

is called the ***exponential quotient map***. Moreover, the ***fundamental loop*** $\boldsymbol{\omega}$ is defined to be the restriction $\omega := \varepsilon|_I$.

Proposition E.32 (Lifting Property of the Circle). *Let $n \in \mathbb{Z}$, $n \geq 0$, $X \subseteq \mathbb{R}^n$ compact and convex, $p \in X$, $f \in \mathsf{Top}_* \big((X, p), (\mathbb{S}^1, 1)\big)$ and $m \in \mathbb{Z}$. Then there exists a unique map $\widetilde{f} \in \mathsf{Top}_* \big((X, p), (\mathbb{R}, m)\big)$, called the **lifting of f**, such that*

$$\begin{tikzcd} & (\mathbb{R}, m) \arrow[d, "\varepsilon"] \\ (X, p) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (\mathbb{S}^1, 1) \end{tikzcd}$$

commutes.

Proof. We show first existence and then uniqueness.

Step 1: Existence. Since X is compact and f is continuous, f is uniformly continuous on X. Thus we find $\delta > 0$ such that $|f(x) - f(y)| < 2$, whenever $|x - y| < \delta$, i.e. $f(x)$ and $f(y)$ are not antipodal points. Moreover, since X is compact, X is bounded and hence we find $N \in \mathbb{N}$, such that $|x - y| < N\delta$ holds for all $x, y \in X$. Let $x \in X$. For $0 \leq k \leq N$, define $L_k : X \to X$ by

$$L_k(x) := \left(1 - \frac{k}{N}\right) p + \frac{k}{N} x.$$

Those are well defined functions since X is convex. Moreover, each L_k is continuous. Indeed, it is easy to check that L_k is Lipschitz. Also, for each $0 \leq k < N$, $f(L_k(x))$ and $f(L_{k+1}(x))$ are not antipodal for all $x \in X$. Indeed, it is easy to check that $|L_k(x) - L_{k+1}(x)| < \delta$ holds for all $x \in X$. For $0 \leq k < N$ define $g_k : X \to \mathbb{S}^1 \setminus \{-1\}$ by

$$g_k(x) := \frac{f(L_{k+1}(x))}{f(L_k(x))}.$$

Clearly g_k is well defined and continuous as a composition of continuous functions. Let $\mathrm{Log} : \mathbb{S}^1 \setminus \{-1\} \to \mathbb{C}$ denote the principal branch of the logarithm. Define $\widetilde{f} : X \to \mathbb{R}$ by

$$\widetilde{f}(x) := m + \frac{1}{2\pi i} \sum_{k=0}^{N-1} \mathrm{Log}(g_k(x)).$$

Clearly, $\widetilde{f}$ is continuous and moreover we have that $\widetilde{f} = m$ since $g_k(p) = 1$ for all $0 \leq k < N$. Finally, for any $x \in X$ we have that

$$(\varepsilon \circ \widetilde{f})(x) = \varepsilon(m) \prod_{k=0}^{N-1} g_k(x) = \frac{f(L_N(x))}{f(L_0(x))} = \frac{f(x)}{f(p)} = f(x).$$

Step 2: Uniqueness. Suppose $\widetilde{g} \in \mathsf{Top}_*\big((X, p), (\mathbb{R}, m)\big)$ is another such function. Define $\varphi \in \mathsf{Top}_*\big((X, p), (\mathbb{R}, 0)\big)$ by

$$\varphi(x) := \widetilde{f}(x) - \widetilde{g}(x).$$

Then clearly $\varepsilon \circ \varphi = 1$ and thus $\varphi(X) \subseteq \mathbb{Z}$. Since X is convex, X is connected and so $\varphi = 0$. □

Corollary E.33. *Let $u, v \in \Omega(\mathbb{S}^1, 1)$ such that $[u] = [v]$. If $\widetilde{u}, \widetilde{v} : (I, 0) \to (\mathbb{R}, 0)$ are the liftings of u and v, respectively, then $[\widetilde{u}] = [\widetilde{v}]$.*

Proof. Let $F : u \simeq_{\partial I} v$. By proposition E.32, we find $\widetilde{F} \in \mathsf{Top}_*\big((I \times I, (0,0)), (\mathbb{R}, 0)\big)$, such that $\varepsilon \circ \widetilde{F} = F$. We claim that $\widetilde{F} : \widetilde{u} \simeq_{\partial I} \widetilde{v}$. For $s \in I$ define $\widetilde{u}_0(s) := \widetilde{F}(s, 0)$. Then $\widetilde{u}_0(0) = \widetilde{F}(0,0) = 0$ and since $\widetilde{u}_0$ is continuous we have that $\widetilde{u}_0 \in \mathsf{Top}_*\big((I, 0), (\mathbb{R}, 0)\big)$. Moreover

$$(\varepsilon \circ \widetilde{u}_0)(s) = \varepsilon\left(\widetilde{F}(s, 0)\right) = F(s, 0) = u(s)$$

for all $s \in I$ and thus $\widetilde{u}_0$ is a lifting of u. But by proposition E.32, liftings are unique and thus $\widetilde{u}_0 = \widetilde{u}$. Next define $\widetilde{w}_0(t) := \widetilde{F}(0, t)$ for all $t \in I$. Then $\widetilde{w}_0(0) = \widetilde{F}(0,0) = 0$ and so $\widetilde{w}_0 \in \mathsf{Top}_*\big((I, 0), (\mathbb{R}, 0)\big)$. Moreover

$$(\varepsilon \circ \widetilde{w}_0)(t) = \varepsilon\left(\widetilde{F}(0, t)\right) = F(0, t) = u(0) = v(0) = 1.$$

for all $t \in I$. Thus

$$\begin{tikzcd} & (\mathbb{R}, 0) \arrow[d, "\varepsilon"] \\ (I, 0) \arrow[ur, "\widetilde{w}_0"] \arrow[r, "c_1"'] & (\mathbb{S}^1, 1) \end{tikzcd}$$

commutes. But also c_0 makes the above diagram commute. By uniqueness, $\widetilde{w}_0 = c_0$. Define $\widetilde{v}_0(s) := \widetilde{F}(s, 1)$ for all $s \in I$. Then $\widetilde{v}_0(0) = \widetilde{F}(0, 1) = \widetilde{w}_0(1) = 0$ and it is easy to check that $\widetilde{v}_0$ is a lift for v. Hence $\widetilde{v}_0 = \widetilde{v}$. Finally, define $\widetilde{w}_1(t) := \widetilde{F}(1, t)$ for all $t \in I$. Then $\widetilde{w}_1(0) = \widetilde{F}(1, 0) = \widetilde{u}(1)$ and thus $\widetilde{w}_1 \in \mathsf{Top}_*\big((I, 0), (\mathbb{R}, \widetilde{u}(1))\big)$. Moreover

$$(\varepsilon \circ \widetilde{w}_1)(t) = \varepsilon\left(\widetilde{F}(1, t)\right) = F(1, t) = v(1) = u(1) = 1$$

for all $t \in I$. By proposition E.32, we have again that $\widetilde{w}_1 = c_{\widetilde{u}(1)}$. So $F : \widetilde{u} \simeq_{\partial I} \widetilde{v}$. □

Definition E.34 (Degree). Let $u \in \Omega(\mathbb{S}^1, 1)$. The ***degree of u***, written $\deg u$, is defined by $\deg u := \widetilde{u}(1)$, where $\widetilde{u}$ is the unique lift of u such that $\widetilde{u}(0) = 0$.

Theorem E.35 (Fundamental Group of the Circle). $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.

Proof. Define $\deg : \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z}$ by $\deg [u] := \deg u$. This is well defined by corollary E.33, since if $[u] = [v]$, then $[\widetilde{u}] = [\widetilde{v}]$ and in particular $\widetilde{u}(1) = \widetilde{v}(1)$.

Step 1: $\deg \in \mathsf{Grp}\big(\pi_1(\mathbb{S}^1, 1), (\mathbb{Z}, +)\big)$. Let $[u], [v] \in \pi_1(\mathbb{S}^1, 1)$. Moreover, let $\widetilde{u}$ and $\widetilde{v}$ denote the unique liftings of u and v, respectively, such that $\widetilde{u}(0) = 0$ and $\widetilde{v}(0) = 0$. Define $\widetilde{w} : I \to \mathbb{R}$ by

$$\widetilde{w}(s) := \begin{cases} \widetilde{u}(2s) & 0 \leq s \leq \frac{1}{2}, \\ \deg u + \widetilde{v}(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Then $\widetilde{w}$ is continuous by the gluing lemma and $\widetilde{w}(0) = 0$. Hence $\widetilde{w} \in \mathsf{Top}_*\big((I, 0), (\mathbb{R}, 0)\big)$. Also we have that $\varepsilon \circ \widetilde{w} = u * v$ and thus $\widetilde{w}$ is the lift of $u * v$. But $\widetilde{w}(1) = \deg u + \deg v$ and so

$$\deg([u]\,[v]) = \deg\,[u * v] = \deg(u * v) = \widetilde{w}(1) = \deg u + \deg v = \deg\,[u] + \deg\,[v]\,.$$

Step 2: $\deg$ *is injective.* Suppose $\deg [u] = 0$. Then $\widetilde{u}(1) = 0$ and thus $\widetilde{u} \in \Omega(\mathbb{R}, 0)$. Since $\mathbb{R}$ is contractible, we have that $[\widetilde{u}] = [c_0]$ and thus

$$[u] = [\varepsilon \circ \widetilde{u}] = \pi_1(\varepsilon)\,[\widetilde{u}] = \pi_1(\varepsilon)\,[c_0] = [\varepsilon \circ c_0] = [c_1]\,.$$

Thus $\ker(\deg)$ is trivial.
Step 3: $\deg$ *is surjective.* Let $m \in \mathbb{Z}$. Then $\deg [\varepsilon^m] = \deg \varepsilon^m = \widetilde{\varepsilon^m}(1) = m$. $\square$

The Seifert-Van Kampen Theorem

Coproducts and Pushouts in Grp

Proposition E.36 (Coproducts in Grp). Grp *has all small coproducts.*

Proof. Let $A \in \mathrm{ob}(\mathsf{Set})$ and A be the small category defined as the discrete category with $\mathrm{ob}(\mathsf{A}) := A$, i.e.

$$\bullet \qquad \bullet \qquad \bullet \qquad \cdots \qquad \bullet \qquad \bullet \qquad \bullet$$

Let $D : \mathsf{A} \to \mathsf{Grp}$ be a functor. Hence we get a family $(G_\alpha)_{\alpha \in A}$ in Grp, where $G_\alpha := D(\alpha)$ for all $\alpha \in A$. A ***word*** in $(G_\alpha)_{\alpha \in A}$ is a finite sequence in $\coprod_{\alpha \in A} G_\alpha$. A word in $(G_\alpha)_{\alpha \in A}$ will simply be written as $(g_1, \ldots, g_n)$, where $g_k \in G_\alpha$ for some $\alpha \in A$. The ***empty word*** is denoted by $()$. Let $\mathcal{W}$ denote the set of all words in $(G_\alpha)_{\alpha \in A}$. On $\mathcal{W}$ define a multiplication by ***concatenation***

$$(g_1, \ldots, g_n)(h_1, \ldots, h_m) := (g_1, \ldots, g_n, h_1, \ldots, h_m).$$

An ***elementary reduction*** is an operation of one of the following forms:

- $(g_1, \ldots, g_k, g_{k+1}, \ldots, g_n) \mapsto (g_1, \ldots, g_k g_{k+1}, \ldots, g_n$, where $g_k, g_{k+1} \in G_\alpha$ for some $\alpha \in A$.
- $(g_1, \ldots, g_{k-1}, 1_\alpha, g_{k+1}, \ldots, g_n) \mapsto (g_1, \ldots, g_{k-1}, g_{k+1}, \ldots, g_n)$. □

Let $\sim$ denote the equivalence relation on $\mathcal{W}$ generated by elementary reductions.

Lemma E.37. *$\mathcal{W}/\sim$ together with concatenation of representatives is an element of* Grp.

Proof. Define

$$[(g_1, \ldots, g_n)]\,[(h_1, \ldots, h_m)] := [(g_1, \ldots, g_n, h_1, \ldots, h_m)]\,.$$

It is left to the reader to show that this is well defined and that $\mathcal{W}/\sim$ is indeed a group. □

The group defined in lemma E.37 will be denoted by $\ast_{\alpha \in A} G_\alpha$ and called the ***free product of*** $\boldsymbol{(G_\alpha)_{\alpha \in A}}$. Let us define a cocone on D. For this consider the inclusions $\iota_\alpha : G_\alpha \to \ast_{\alpha \in A} G_\alpha$ defined by

$$\iota_\alpha(g) := [(g)]$$

for all $\alpha \in A$. It is immediate from

$$\iota_\alpha(gh) = [(gh)] = [(g, h)] = [(g)]\,[(h)] = \iota_\alpha(g)\iota_\alpha(h)$$

for $g, h \in G_\alpha$, that ι_α is a morphism of groups. Since there are only the identity morphisms in A, $\big(\ast_{\alpha \in A} G_\alpha, (\iota_\alpha)_{\alpha \in A}\big)$ is a cocone on D. Let us show that this is in fact a universal cocone. To this end, suppose that $\big(C, (\varphi_\alpha)_{\alpha \in A}\big)$ is another cocone on D. Define a mapping $\bar{f} : \ast_{\alpha \in A} G_\alpha \to C$ by

$$\bar{f}\,[(g_1, \ldots, g_n)] := \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$$

where $g_k \in G_{\alpha_k}$. Then $\bar{f}$ is easily seen to be well defined since each φ_α is a morphism of groups. Moreover, if $g \in G_\alpha$, then

$$(\bar{f} \circ \iota_\alpha)(g) = \bar{f}\,[(g)] = \varphi_\alpha(g)$$

for all $\alpha \in A$. Suppose that $f : \ast_{\alpha \in A} G_\alpha \to C$ is another homomorphism of groups such that $f \circ \iota_\alpha = \varphi_\alpha$ for all $\alpha \in A$. Then for $[(g_1, \ldots, g_n)] \in \ast_{\alpha \in A} G_\alpha$ we have

$$\begin{aligned} f\,[(g_1, \ldots, g_n)] &= f([(g_1)] \cdots [(g_n)]) \\ &= f\,[(g_1)] \cdots f\,[(g_n)] \\ &= f\,(\iota_{\alpha_1}(g_1)) \cdots f\,(\iota_{\alpha_n}(g_n)) \\ &= \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n) \\ &= \bar{f}\,[(g_1, \ldots, g_n)]\,. \end{aligned}$$

Exercise E.38. Check that $\mathcal{W}/\sim$ is indeed a group with the declared group structure and that $\bar{f}$ is indeed well defined.

Proposition E.39 (Pushouts in Grp). Grp *has all pushouts.*

Proof. Consider the diagram $D : \mathsf{A} \to \mathsf{Grp}$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \quad \xrightarrow{D} \quad \begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \\ H_2 & \end{tikzcd}$$

and define N to be the normal subgroup of $H_1 * H_2$ generated by elements of the form $[(\varphi_1(g^{-1}), \varphi_2(g))]$ for $g \in G$. Let $K := (H_1 * H_2)/N$. Then

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \arrow[d, "\pi\circ\iota_1"] \\ H_2 \arrow[r, "\pi\circ\iota_2"'] & K \end{tikzcd}$$

commutes. Indeed, if $g \in G$, we have that $(\pi \circ \iota_1 \circ \varphi_1)(g) = [(\varphi_1(g))]\, N$ and similarly $(\pi \circ \iota_2 \circ \varphi_2)(g) = [(\varphi_2(g))]\, N$. Then

$$[(\varphi_1(g))]^{-1}\, [(\varphi_2(g))] = [(\varphi_1(g)^{-1})]\, [(\varphi_2(g))] = [(\varphi_1(g^{-1}))]\, [(\varphi_2(g))] \in N.$$

Suppose that we have another cocone on D:

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \arrow[d, "\psi_1"] \\ H_2 \arrow[r, "\psi_2"'] & C \end{tikzcd}$$

By proposition E.36, there exists a unique morphism of groups $f : H_1 * H_2 \to C$ and we thus get the following diagram:

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \arrow[d, "\iota_1"] \arrow[rrdd, bend left, "\psi_1"] & & \\ H_2 \arrow[r, "\iota_2"'] \arrow[rrrd, bend right, "\psi_2"'] & H_1 * H_2 \arrow[rd, "\pi"] \arrow[rrd, bend right, "f"'] & & \\ & & K \arrow[r, dashed, "\bar{f}"] & C \end{tikzcd}$$

To show that $N \subseteq \ker f$ is left as an exercise. Hence by the factorization theorem (see [1, 23]), f factors uniquely through π, i.e. there exists a unique morphism of groups $\overline{f} : K \to C$ such that $\overline{f} \circ \pi = f$. □

Exercise E.40. In the previous proposition, verify that $N \subseteq \ker f$.

Definition E.41 (Amalgamated Free Product). The pushout of a diagram

$$\begin{tikzcd} G \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & H_1 \\ H_2 & \end{tikzcd}$$

in Grp is called the ***amalgamated free product of H_1 and H_2 along $(G, \varphi_1, \varphi_2)$***, written $H_1 *_{(G,\varphi_1,\varphi_2)} H_2$.

The Seifert-Van Kampen Theorem and its Consequences

Theorem E.42 (Seifert-Van Kampen). *Let $X \in \mathrm{ob}(\mathsf{Top})$, (U, V) an open cover for X, such that U, V and $U \cap V$ are path connected. Moreover, let $p \in U \cap V$. Then*

$$\pi_1(X, p) \cong \pi_1(U, p) *_{(\pi_1(U \cap V, p), \pi_1(\iota_U), \pi_1(\iota_V))} \pi_1(V, p), \tag{E.2}$$

where $\iota_U : U \cap V \hookrightarrow U$ and $\iota_V : U \cap V \hookrightarrow V$ denote inclusion.

Proof. Let $\jmath_U : U \hookrightarrow X$ and $\jmath_V : V \hookrightarrow X$ denote inclusions. We will show that $\big(\pi_1(X, p), \pi_1(\jmath_U), \pi_1(\jmath_V)\big)$ is a pushout of the diagram

$$\begin{tikzcd} \pi_1(U \cap V, p) \arrow[r, "\pi_1(\iota_U)"] \arrow[d, "\pi_1(\iota_V)"'] & \pi_1(U, p) \\ \pi_1(V, p) & \end{tikzcd} \tag{E.3}$$

in Grp and hence by proposition E.39 and uniqueness, the statement follows. Clearly

$$\begin{tikzcd} \pi_1(U \cap V, p) \arrow[r, "\pi_1(\iota_U)"] \arrow[d, "\pi_1(\iota_V)"'] & \pi_1(U, p) \arrow[d, "\pi_1(\jmath_U)"] \\ \pi_1(V, p) \arrow[r, "\pi_1(\jmath_V)"'] & \pi_1(X, p) \end{tikzcd}$$

commutes. Suppose now that $(G, \varphi_U, \varphi_V)$ is another cocone for the diagram (E.3). We want to show that there exists a unique homomorphism $\Phi : \pi_1(X, p) \to G$ such that $\Phi \circ \pi_1(\jmath_U) = \varphi_U$ and $\Phi \circ \pi_1(\jmath_V) = \varphi_V$. Let $[u] \in \pi_1(X, p)$. Choose a partition $0 = x_0 < \cdots < x_n = 1$ of I such that $u(x_k) \in U \cap V$ for all $k = 0, \ldots, n$ and such that all $u|_{[x_{k-1},x_k]}$ take values either in U or in V for all $k = 1, \dots, n$. The existence of such a partition follows from an application of the Lebesgue number lemma on the open cover $(u^{-1}(U), u^{-1}(V))$ of I. Indeed, if $\delta > 0$ is the corresponding Lebesgue number of the cover, we find $n \in \omega$, $n > 0$, such that $1/n < \delta$. Thus $[(i-1)/n, i/n]$ is contained in either $u^{-1}(U)$ or $u^{-1}(V)$ for all $i = 1, \dots, n$. Now choose those i such that $u(i/n) \in U \cap V$. For $k = 1, \dots, n$, let $u_k : I \to X$ be defined by

$$u_k(s) := u\left((1-s)x_{k-1} + sx_k\right).$$

Moreover, for each $k = 1, \dots, n-1$ choose a path γ_k in $U \cap V$ from p to $u(x_k)$ and set $\gamma_0, \gamma_n := c_p$. Define now

$$\Phi\,[u] := \prod_{k=1}^{n} \varphi_\bullet\,[\gamma_{k-1} * u_k * \overline{\gamma}_k]\,, \tag{E.4}$$

where $\varphi_\bullet$ denotes either φ_U or φ_V depending on whether $\gamma_{k-1} * u_k * \overline{\gamma}_k$ is a loop in U or in V. If u is a loop in $U \cap V$, we can choose either φ_U or φ_V since $(G, \varphi_U, \varphi_V)$ is a cocone of the diagram (E.3). Now there are some things to check.
Φ is a function. Suppose $H : u \simeq_{\partial I} v$.
$\Phi\,[u]$ does not depend on the choice of γ_k. Fix some $k = 1, \dots, n-1$ and suppose that γ'_k is another path from p to $u(x_k)$ in $U \cap V$. Then we have that

$$\varphi_\bullet\,[\gamma_{k-1} * u_k * \overline{\gamma}_k] = \varphi_\bullet\,[\gamma_{k-1} * u_k * \overline{\gamma}'_k * \gamma'_k * \overline{\gamma}_k] = \varphi_\bullet\,[\gamma_{k-1} * u_k * \overline{\gamma}'_k]\,\varphi_\bullet\,[\gamma'_k * \overline{\gamma}_k]$$

and

$$\begin{aligned}\varphi_\bullet\,[\gamma_k * u_{k+1} * \overline{\gamma}_{k+1}] &= \varphi_\bullet\,[\gamma_k * \overline{\gamma}'_k * \gamma'_k * u_{k+1} * \overline{\gamma}_{k+1}] \\ &= \varphi_\bullet\,[\gamma_k * \overline{\gamma}'_k]\,\varphi_\bullet\,[\gamma'_k * u_{k+1} * \overline{\gamma}_{k+1}] \\ &= \left(\varphi_\bullet\,[\gamma'_k * \overline{\gamma}_k]\right)^{-1} \varphi_\bullet\,[\gamma'_k * u_{k+1} * \overline{\gamma}_{k+1}]\,.\end{aligned}$$

Since $\gamma'_k * \overline{\gamma}_k$ is a loop in $U \cap V$, we have that

$$\varphi_\bullet\,[\gamma_{k-1} * u_k * \overline{\gamma}_k]\,\varphi_\bullet\,[\gamma_k * u_{k+1} * \overline{\gamma}_{k+1}] = \varphi_\bullet\,[\gamma_{k-1} * u_k * \overline{\gamma}'_k]\,\varphi_\bullet\,[\gamma'_k * u_{k+1} * \overline{\gamma}_{k+1}]\,.$$

$\Phi\,[u]$ does not depend on the choice of a partition of I. Suppose $\mathcal{P}_1$ and $\mathcal{P}_2$ are both partitions of I, their union $\mathcal{P}_1 \cup \mathcal{P}_2$ is a refinement of both $\mathcal{P}_1$ and $\mathcal{P}_2$. If we can show that adding a single point to a partition $\mathcal{P}$ of I does not affect the value $\Phi\,[u]$, then so it does not on $\mathcal{P}_1 \cup \mathcal{P}_2$ and hence is independent of the choice of a partition. Suppose we add $x_{k-1} < y < x_k$. Let us denote by u_y the reparametrized restriction of u from $u(x_{k-1})$ to $u(y)$ and by u'_k the reparametrized restriction of u from $u(y)$ to $u(x_k)$. Moreover, let γ_y be a path from p to $u(y)$ in $U \cap V$. We compute

$$\begin{aligned}\varphi_\bullet\,[\gamma_{k-1} * u_y * \overline{\gamma}_y]\,\varphi_\bullet\,[\gamma_y * u'_k * \overline{\gamma}_k] &= \varphi_\bullet\,[\gamma_{k-1} * u_y * \overline{\gamma}_y * \gamma_y * u'_k * \overline{\gamma}_k] \\ &= \varphi_\bullet\,[\gamma_{k-1} * u_y * u'_k * \overline{\gamma}_k] \\ &= \varphi_\bullet\,[\gamma_{k-1} * u_k * \overline{\gamma}_k]\,,\end{aligned}$$

since $u_y * u'_k$ is a reparametrization of u_k and $\gamma_{k-1} * u_y * \overline{\gamma}_y$, $\gamma_y * u'_k * \overline{\gamma}_k$ are both loops either in U or in V.

Φ is a morphism of groups. Let $[u], [v] \in \pi_1(X, p)$. Let $0 = x_0 < \cdots < x_n = 1$ be a partition of I as above. By invariance under a change of partitions, we may assume that $0 = x_0 < \cdots < x_m = \frac{1}{2} < \cdots < x_n = 1$. Clearly $(u * v)(x_m) = p \in U \cap V$. Now both $0 = 2x_0 < \cdots < 2x_m = 1$ and $0 = 2x_m - 1 < \cdots < 2x_n - 1 = 1$ are partitions of I with $(u * v)_k = u_k$ for $k = 1, \ldots, m$ and $(u * v)_k = v_k$ for $k = m + 1, \ldots, n$. By using invariance of the choice of a partition again and invariance of the choice of the γ_k yields

$$\begin{aligned}
\Phi([u]\,[v]) &= \Phi\,[u * v] \\
&= \prod_{k=1}^{n} \varphi_\bullet\,[\gamma_{k-1} * (u * v)_k * \overline{\gamma}_k] \\
&= \prod_{k=1}^{m} \varphi_\bullet\,[\gamma_{k-1} * u_k * \overline{\gamma}_k] \prod_{k=m+1}^{n} \varphi_\bullet\,[\gamma_{k-1} * v_k * \overline{\gamma}_k] \\
&= \Phi\,[u]\,\Phi\,[v]\,.
\end{aligned}$$

Checking commutativity. We have to show that $\Phi \circ \pi_1(\jmath_U) = \varphi_U$ and $\Phi \circ \pi_1(\jmath_V) = \varphi_V$ hold. Let us show the first identity, the second is similar. Let $[u] \in \pi_1(U, p)$. Then we can choose the trivial partition $0 = x_0 < x_1 = 1$ of I and thus get

$$(\Phi \circ \pi_1(\jmath_U))\,[u] = \Phi\,[u] = \varphi_U\,[\gamma_0 * u_1 * \overline{\gamma}_1] = \varphi_U\,[u]\,.$$

Showing uniqueness of Φ. Suppose $\Psi : \pi_1(X, p) \to G$ is another map with the smae properties as Φ. Let $[u] \in \pi_1(X, p)$. They keypoint is to observe that

$$[u] = \left[\prod_{k=1}^{n} (\gamma_{k-1} * u_k * \overline{\gamma}_k)\right]$$

holds. Thus

$$\begin{aligned}
\Psi\,[u] &= \Psi\left[\prod_{k=1}^{n} (\gamma_{k-1} * u_k * \overline{\gamma}_k)\right] \\
&= \prod_{k=1} \Psi\left[\gamma_{k-1} * u_k * \overline{\gamma}_k\right] \\
&= \prod_{k=1}^{n} \varphi_\bullet\left[\gamma_{k-1} * u_k * \overline{\gamma}_k\right] \\
&= \Phi\,[u]\,.
\end{aligned}$$

Exercise E.43. In the proof of the Seifert-Van Kampen theorem, show that $u_y * u'_k = u_k \circ \varphi$, where $\varphi \in \mathsf{Top}(I, I)$ is given by

$$\varphi(s) := \begin{cases} 2s(y - x_{k-1})/(x_k - x_{k-1}) & 0 \le s \le \frac{1}{2}, \\ 2(1-s)(y - x_{k-1})/(x_k - x_{k-1}) + 2s - 1 & \frac{1}{2} \le s \le 1. \end{cases}$$

E.1 Singular Simplices

Definition E.44 (Affinely Independent). Let $n, k \in \omega$. A family $(v_0, \dots, v_k)$ in $\mathbb{R}^n$ is said to be ***affinely independent***, iff the following condition is satisfied: Given $\lambda_0, \dots, \lambda_k \in \mathbb{R}$ such that

$$\sum_{i=0}^{k} \lambda_i = 0 \qquad \text{and} \qquad \sum_{i=0}^{k} \lambda_i v_i = 0$$

implies $c_0 = \cdots = c_k = 0$.

Lemma E.45. *Let $n, k \in \omega$. Then a family $(v_0, \dots, v_k)$ in $\mathbb{R}^n$ is affinely independent if and only if $(v_1 - v_0, \dots, v_k - v_0)$ is linearly independent in $\mathbb{R}^n$.*

Exercise E.46. Prove lemma E.45.

Definition E.47 (Simplex). Let $n, k \in \omega$ and $(v_0, \dots, v_k)$ affinely independent in $\mathbb{R}^n$. Define the ***simplex spanned by $(v_0, \dots, v_k)$***, written $[v_0, \dots, v_k]$, to be the topological subspace

$$[v_0, \dots, v_k] := \left\{ \sum_{i=0}^{k} \lambda_i v_i : \lambda_i \in \mathbb{R}_{\ge 0} \text{ and } \sum_{i=0}^{k} \lambda_i = 1 \right\} \subseteq \mathbb{R}^n.$$

Moreover, each of the v_i's, $i = 0, \dots, k$, is called a ***vertex*** of the simplex $[v_0, \dots, v_k]$.

Remark E.48. Let $\sigma := [v_0, \dots, v_k]$ be a simplex spanned by $(v_0, \dots, v_k)$. Then we will also simply call σ a k-simplex in $\mathbb{R}^n$.

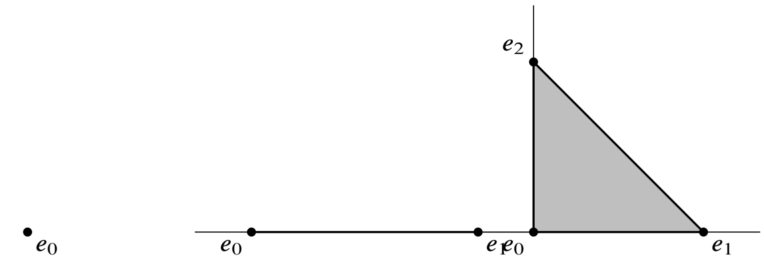
Example E.49 (Standard Simplex). Let $n \in \omega$. Then the family $(e_0, \dots, e_n)$ in $\mathbb{R}^n$, where $e_0 := 0$ and $(e_1, \dots, e_n)$ is the standard oriented basis of $\mathbb{R}^n$, is affinely independent by exercise E.46. The n-simplex spanned by this family is called the ***standard n-simplex*** and is denoted by Δ^n.

Lemma E.50. *Let $n, k \in \omega$ and $[v_0, \dots, v_k]$ a k-simplex in V. Then any $x \in [v_0, \dots, v_k]$ admits a unique representation $x = \sum_{i=0}^{k} \lambda_i v_i$.*

Exercise E.51. Prove lemma E.50.

Definition E.52 (Affinely Linear Mapping). Let $n, m \in \omega$. A mapping $A : \mathbb{R}^n \to \mathbb{R}^m$ is said to be ***affinely linear***, iff there exists an $\mathbb{R}$-linear vector space morphism $L : \mathbb{R}^n \to \mathbb{R}^m$ and $y \in \mathbb{R}^m$, such that

$$A(x) = L(x) + y$$



(a) Δ^0. (b) Δ^1. (c) Δ^2.

Fig. E.3: Standard n-simplices.

holds for all $x \in \mathbb{R}^n$.

Exercise E.53. Show that any affinely linear mapping is continuous with respect to the standard Euclidean topologies.

Exercise E.54. Show that the composition of affinely linear mappings is again affinely linear.

Proposition E.55 (Affine Map induced by Vertex Map). *Let $n, k, m \in \omega$ and $\sigma := [v_0, \dots, v_n]$ a k-simplex in $\mathbb{R}^n$. Given a function $f : \{v_0, \dots, v_k\} \to \mathbb{R}^m$, there exists a unique extension $\widetilde{f} : \sigma \to \mathbb{R}^m$, which is the restriction of an affinely linear map.*

Proof. We show first existence and then uniqueness.

Step 1: Existence. By exercise E.46, $(v_1 - v_0, \dots, v_k - v_0)$ is linearly independent in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is finite dimensional, we may complete this linearly independent subset to a basis of $\mathbb{R}^n$. Hence there exists a unique vector space morphism $L : \mathbb{R}^n \to \mathbb{R}^m$, mapping

$$v_i - v_0 \mapsto f(v_i) - f(v_0),$$

for $i = 1, \dots, k$ and to the zero vector else. Now $A : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$A := L - L(v_0) + f(v_0)$$

is the map we are looking for.
Step 2: Uniqueness. Given another such extension $\widetilde{g} : \sigma \to \mathbb{R}^m$ of f, say $\widetilde{g} = \widetilde{L} + y$, we have that $\widetilde{L}(v_i) = f(v_i) - y$ for all $i = 0, \dots, k$. Thus we compute

$$\widetilde{g}\left(\sum_{i=0}^{k} \lambda_i v_i\right) = \sum_{i=0}^{k} \lambda_i \widetilde{L}(v_i) + y = \sum_{i=0}^{k} \lambda_i f(v_i) - \sum_{i=0}^{k} \lambda_i y + y = \sum_{i=0}^{k} \lambda_i f(v_i).$$

Definition E.56 (Singular Simplex). Let $n \in \omega$ and $X \in \mathsf{Top}$. An element of $\mathsf{Top}(\Delta^n, X)$ is called a ***singular n-simplex in X***.

Example E.57 (Affine Singular Simplex). Let $n, m \in \omega$ and let Δ^n denote the standard n-simplex of example E.49. Given any $v_0, \dots, v_n \in \mathbb{R}^m$, define $A(v_0, \dots, v_n) : \Delta^n \to \mathbb{R}^m$ by the vertex map $e_i \mapsto v_i$ for $i = 0, \dots, n$ (see proposition E.55). By exercise E.53, $A(v_0, \dots, v_n)$ is continuous and thus a singular n-simplex, called an ***affine singular n-simplex***.

Example E.58 (Face Map). Let $n \in \omega$, $n \geq 1$, and let Δ^n denote the standard n-simplex of example E.49. For $k \in \omega$, $0 \leq k \leq n$, define a singular simplex $\varphi_k^n : \Delta^{n-1} \to \Delta^n$, called the ***k-th face map in dimension n***, by

$$\varphi_k^n := A(e_0, \dots, \widehat{e_k}, \dots, e_n).$$

This map is indeed well-defined as the uniqueness part of the proof of proposition E.55 shows.

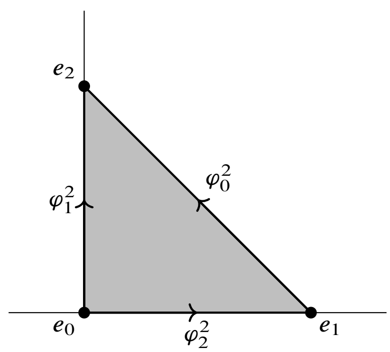


Fig. E.4: Face maps for $n = 2$.

E.2 The Singular Chain Complex

Proposition E.59. *Let $R \in$ Ring. Then the forgetful functor $U : {}_R\mathsf{Mod} \to \mathsf{Set}$ admits a left adjoint.*

Proof. Consider the ***free module functor*** $F : \mathsf{Set} \to {}_R\mathsf{Mod}$ defined as follows:

Step 1: Definition on objects. Let $S \in \mathsf{Set}$ and define

$$F(S) := \left\{ f \in R^S : \operatorname{supp} f \text{ is finite} \right\}.$$

Equipped with pointwise defined addition and multiplication, $F(S)$ is a left R-module. Moreover, there is an inclusion $\iota : S \hookrightarrow U(F(S))$ sending $x \in S$ to the function taking the value one at x and zero else. It is easy to check that $F(S)$ is free on S.

Step 2: Definition on morphisms. Let $f : S \to S'$ in Set, define $F(f) : F(S) \to F(S')$ by setting

$$F(f)\left(\sum_{x \in S} r_x x\right) := \sum_{x \in S} r_x f(x).$$

Step 3: $F \dashv U$. Let $M \in {}_R\mathsf{Mod}$ and $\varphi \in {}_R\mathsf{Mod}(F(S), M)$. Define $\overline{\varphi} \in {}_R\mathsf{Mod}(S, U(M))$ to be the restriction to S of the underlying map of sets. Conversly, if $f \in \mathsf{Set}(S, U(G))$, ***extending by linearity*** yields $\overline{f} \in {}_R\mathsf{Mod}(F(S), M)$ given by

$$\overline{f}\left(\sum_{x \in S} r_x x\right) := \sum_{x \in S} r_x f(x).$$

It is now easy to check that $\overline{\overline{\varphi}} = \varphi$ and $\overline{\overline{f}} = f$ holds. ☐

Exercise E.60. In the proof of proposition F.150, check functoriality of F and naturality of the bijection ${}_R\mathsf{Mod}(F(S), M) \cong \mathsf{Set}(S, U(M))$.

Theorem E.61 (Singular Chain Complex Functor). *Let* $R \in \mathsf{Ring}$. *Then there exists a functor*

$$C_\bullet : \mathsf{Top} \to \mathrm{Ch}_{\geq 0}({}_R\mathsf{Mod}).$$

Proof. The proof is divided into two steps.

Step 1: Definition on objects. Let $X \in \mathsf{Top}$. Then define

$$C_\bullet(X)_n := F(\mathsf{Top}(\Delta^n, X))$$

for all $n \in \omega$, where $F : \mathsf{Set} \to {}_R\mathsf{Mod}$ denotes the free module functor from proposition F.150 and Δ^n denotes the n-th standard simplex from example E.49. Let $\sigma \in \mathsf{Top}(\Delta^n, X)$, $n \geq 1$. Define

$$\partial_n \sigma := \sum_{k=0}^{n} (-1)^k \sigma \circ \varphi_k^n, \tag{E.5}$$

where φ_k^n denotes the k-th face map in dimension n from example E.58. Extending by linearity yields a morphism of R-modules $\partial_n : C_\bullet(X)_n \to C_\bullet(X)_{n-1}$. For any $\sigma \in \mathsf{Top}(\Delta^{n+1}, X)$ we compute

$$
\begin{aligned}
(\partial_n \circ \partial_{n+1})(\sigma) &= \partial_n \left(\sum_{k=0}^{n+1} (-1)^k \sigma \circ \varphi_k^{n+1} \right) \\
&= \sum_{k=0}^{n+1} (-1)^k \partial_n \left(\sigma \circ \varphi_k^{n+1} \right) \\
&= \sum_{k=0}^{n+1} \sum_{j=0}^{n} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\
&= \sum_{0 \le k \le j \le n} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n + \sum_{0 \le j < k \le n+1} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\
&= \sum_{0 \le j \le k \le n} (-1)^{k+j} \sigma \circ \varphi_j^{n+1} \circ \varphi_k^n + \sum_{0 \le j < k \le n+1} (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \\
&= \sum_{0 \le j < k \le n+1} \left((-1)^{k+j-1} \sigma \circ \varphi_j^{n+1} \circ \varphi_{k-1}^n + (-1)^{k+j} \sigma \circ \varphi_k^{n+1} \circ \varphi_j^n \right)
\end{aligned}
$$

Since $\varphi_j^{n+1} \circ \varphi_{k-1}^n$ and $\varphi_k^{n+1} \circ \varphi_j^n$ are both equal to $A(e_0, \ldots, \widehat{e_j}, \ldots, \widehat{e_k}, \ldots, e_{n+1})$, it follows that

$$\partial_n \circ \partial_{n+1} = 0.$$

Indeed, consider the following chart of vertex maps:

$$
\begin{array}{ccccc}
 & \varphi_{k-1}^n & & \varphi_j^{n+1} & \\
e_0 & \mapsto & e_0 & \mapsto & e_0 \\
\vdots & & \vdots & & \vdots \\
e_j & \mapsto & e_j & \mapsto & e_{j+1} \\
\vdots & & \vdots & & \vdots \\
e_{k-1} & \mapsto & e_k & \mapsto & e_{k+1} \\
\vdots & & \vdots & & \vdots \\
e_{n-1} & \mapsto & e_n & \mapsto & e_{n+1}
\end{array}
\qquad
\begin{array}{ccccc}
 & \varphi_j^n & & \varphi_k^{n+1} & \\
e_0 & \mapsto & e_0 & \mapsto & e_0 \\
\vdots & & \vdots & & \vdots \\
e_j & \mapsto & e_{j+1} & \mapsto & e_{j+1} \\
\vdots & & \vdots & & \vdots \\
e_{k-1} & \mapsto & e_k & \mapsto & e_{k+1} \\
\vdots & & \vdots & & \vdots \\
e_{n-1} & \mapsto & e_n & \mapsto & e_{n+1}
\end{array}.
$$

Step 2: Definition on morphisms. Let $f : X \to Y$ be a morphism in Top. For $n \in \omega$, define $C_\bullet(f)_n : C_\bullet(X)_n \to C_\bullet(Y)_n$ by

$$C_\bullet(f)_n(\sigma) := f \circ \sigma,$$

for any $\sigma \in \mathsf{Top}(\Delta^n, X)$. We compute

$$(\partial_n \circ C_\bullet(f)_n)(\sigma) = \sum_{k=0}^{n} (-1)^k f \circ \sigma \circ \varphi_k^n = (C_\bullet(f)_{n-1} \circ \partial_n)(\sigma).$$

Thus $C_\bullet(f)$ is a morphism in $\mathrm{Ch}_{\ge 0}({}_R\mathsf{Mod})$.

Checking functoriality is left as an exercise. □

Exercise E.62. Check that $C_\bullet : \mathsf{Top} \to \mathrm{Ch}_{\geq 0}({}_R\mathsf{Mod})$ defined in theorem E.61 is indeed a functor.

Theorem E.63 (Relative Singular Chain Complex Functor). *Let $R \in \mathsf{Ring}$. Then there exists a functor*

$$C_\bullet : \mathsf{Top}^2 \to \mathrm{Ch}_{\geq 0}({}_R\mathsf{Mod}).$$

Proof.

E.3 Homology of Product Spaces

E.3.1 The Universal Coefficient and the Künneth Theorem

Proposition E.64. *Let $A \in \mathsf{Ab}$. Then $(-) \otimes A : \mathsf{Ab} \to \mathsf{Ab}$ and $A \otimes (-) : \mathsf{Ab} \to \mathsf{Ab}$ are both right exact.*

Example E.65. $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n = \mathbb{Z}_{\gcd(m,n)}$.

Definition E.66 (Tor). Let $A \in \mathsf{Ab}$ and

$$0 \longrightarrow K \xrightarrow{f} F \longrightarrow A \longrightarrow 0$$

a short free resolution of A. Given any $B \in \mathsf{Ab}$, set

$$\mathrm{Tor}(A, B) := \ker(f \otimes \mathrm{id}_B).$$

Example E.67. If either A or B are torsion free, then $\mathrm{Tor}(A, B) = 0$.

Example E.68. $\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_{\gcd(m,n)}$.

Theorem E.69 (Universal Coefficient Theorem). *Let $(C_\bullet, \partial_\bullet)$ be a free chain complex and $A \in \mathsf{Ab}$. Then for any $n \in \omega$ there is a split exact sequence*

$$0 \longrightarrow H_n(C_\bullet) \otimes A \longrightarrow H_n(C_\bullet \otimes A) \longrightarrow \mathrm{Tor}(H_{n-1}(C_\bullet), A) \longrightarrow 0.$$

Theorem E.70 (Künneth Theorem). *Let $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ be two non-negative free chain complexes. Then there exists a split exact sequence*

$$0 \to \bigoplus_{i+j=n} H_i(C_\bullet) \otimes H_j(C'_\bullet) \to H_n(C_\bullet \otimes C'_\bullet) \to \bigoplus_{k+l=n-1} \mathrm{Tor}\big(H_k(C_\bullet), H_l(C'_\bullet)\big) \to 0.$$

E.3.2 The Eilenberg-Zilber Theorem and the Künneth Formula

Theorem E.71 (The Augmented Acyclic Models Theorem). *A Let $\mathcal{C}$ be a category with family of models $\mathcal{M}$. Consider*

$$S, T : \mathcal{C} \to \mathrm{AugCh}(\mathsf{Ab})$$

such that:

- S_n *is free with basis contained in* $\mathcal{M}$ *for any* $n \in \omega$.
- *Any* $M \in \mathcal{M}$ *is totally* T*-acyclic, i.e.* $H_n(S(M)) = 0$ *for all* $n \geq 1$ *and* $H_0(S(M)) = \mathbb{Z}$.

Then there exists a natural augmentation preserving chain map

$$\theta : S \Rightarrow T$$

Moreover, any two such natural augmenation preserving chain maps are naturally chain homotopic.
If additionally T_n *is free with basis contained in* $\mathcal{M}$ *and each model* $M \in \mathcal{M}$ *is totally* S*-acyclic, then every such natural augmentation preserving chain map is a natural chain equivalence.*

Theorem E.72 (Eilenberg-Zilber). *Let* $X, Y \in \mathsf{Top}$. *Then there exists a chain equivalence*

$$\Omega : C_\bullet(X \times Y) \to C_\bullet(X) \otimes C_\bullet(Y)$$

unique up to chain homotopy. Any such map Ω *is called an* ***Eilenberg-Zilber morphism***.

Proof. We make use of the augmented acyclic models theorem E.71. In $\mathsf{Top} \times \mathsf{Top}$ define a family of models $\mathcal{M}$ by

$$\mathcal{M} := \{(\Delta^i, \Delta^j) : i, j \in \omega\}.$$

Moreover, define $S, T : \mathsf{Top} \times \mathsf{Top} \to \mathrm{AugCh}(\mathsf{Ab})$ by

$$S(X, Y) := C_\bullet(X \times Y) \qquad \text{and} \qquad T(X, Y) := C_\bullet(X) \otimes C_\bullet(Y).$$

Since $\Delta^i \times \Delta^j$ is convex, we get that each model $M := (\Delta^i, \Delta^j)$ is totally S-acyclic. Moreover, the Künneth theorem E.70 implies that each model M is totally T-acyclic. That S_n is free with basis contained in $\mathcal{M}$ can be seen by choosing the diagonal map $d_n : \Delta^n \to \Delta^n \times \Delta^n$ for any $n \in \omega$. Finally, T_n is also free with basis contained in $\mathcal{M}$, since we can choose the model basis

$$\{(\Delta^i, \Delta^j) : i + j = n\}$$

for fixed $n \in \omega$ and $\iota_i \otimes \iota_j \in (C_\bullet(\Delta^i) \otimes C_\bullet(\Delta^j))_n$, where $\iota_k : \Delta^k \to \Delta^k$ denotes the identity map. $\square$

Corollary E.73 (Künneth Formula). *Let* $X, Y \in \mathsf{Top}$. *Then there is a split exact sequence*

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \longrightarrow H_n(X \times Y) \longrightarrow \bigoplus_{k+l=n-1} \mathrm{Tor}\big(H_k(X), H_l(Y)\big) \longrightarrow 0.$$

Example E.74. Let $n \in \omega$, $n \geq 1$. Define the ***n-torus*** $\mathbb{T}^n$ by

$$\mathbb{T}^n := \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n}.$$

Using induction and the KÃijnneth theorem E.73, one can show that

$$H_k(\mathbb{T}^n) = \mathbb{Z}^{\binom{n}{k}}.$$

E.4 Singular Cohomology

Proposition E.75. *Let* $A \in \mathsf{Ab}$. *Then* $\mathrm{Hom}(-, A) : \mathsf{Ab} \to \mathsf{Ab}$ *and* $\mathrm{Hom}(A, -) : \mathsf{Ab} \to \mathsf{Ab}$ *are both left exact.*

Corollary E.76. *Let* $X \in \mathsf{Top}$ *be of finite type, i.e.* $H_n(X)$ *is finitely generated for any* $n \in \mathbb{Z}$. *Then*

$$H^n(X) \cong H_n(X)/T_n(X) \oplus T_{n-1}(X)$$

where $T_n(X)$ *denotes the torsion subgroup of* $H_n(X)$, *i.e. the subgroup consisting of all elements of finite order.*

Theorem E.77 (Universal Coefficient Theorem for Cohomology). *Let* $X \in \mathsf{Top}$ *of finite type and* $A \in \mathsf{Ab}$. *Then there is a split exact sequence*

$$0 \longrightarrow H^n(X) \otimes A \longrightarrow H^n(X; A) \longrightarrow \mathrm{Tor}(H^{n+1}(X), A) \longrightarrow 0.$$

E.4.1 The Cohomology Ring

Proposition E.78. *Let* $X \in \mathsf{Top}$ *and* $R \in \mathsf{Ring}$. *Then there exists a contravariant functor*

$$C(-; R) : \mathsf{Top} \to \mathsf{GRing}.$$

Proof. We proceed in two (uncomplete) steps.

Step 1: Definition on objects. Let $X \in \mathsf{Top}$. For $\alpha \in C^n(X; R)$ and $\beta \in C^m(X; R)$ define

$$(\alpha \cup \beta)(\sigma) := \alpha(\sigma \circ A(e_0, \ldots, e_n))\beta(\sigma \circ A(e_n, \ldots, e_{n+m})),$$

for all singular $n + m$-simplices σ in X. Hence extending by linearity yields a map

$$\cup : C^n(X; R) \times C^m(X; R) \to C^{n+m}(X; R).$$

Moreover, if

$$C(X;R) := \bigoplus_{n\in\omega} C^n(X;R),$$

we define $\cup : C(X;R) \times C(X;R) \to C(X;R)$ by

$$\sum_i \alpha_i \cup \sum_j \beta_j := \sum_{i,j} \alpha_i \cup \beta_j .$$

This is called the ***cup product on*** $\boldsymbol{C(X;R)}$. It is easily verified that $(C(X;R),\cup) \in \mathsf{GRing}$.

Step 2: Definition on morphisms. Let $n \in \omega$ and $f \in \mathsf{Top}(X,Y)$. For $\alpha \in C^n(Y;R)$ define

$$C(f;R)(\alpha) := C^n(f;R)(\alpha) \in C^n(X;R),$$

and extend by linearity. □

Lemma E.79. *Let* $R \in \mathsf{GRing}$ *and* I *be a two-sided homogeneous ideal in* R*. Then also* $R/I \in \mathsf{GRing}$ *with*

$$R/I = \bigoplus_{n\in\omega} R^n/(R^n \cap I).$$

Theorem E.80. *Let* $R \in \mathsf{Ring}$*. Then there is a contravariant functor*

$$H(-;R) : \mathsf{hTop} \to \mathsf{GRing}.$$

Proof. Set

$$Z := \bigoplus_{n\in\omega} Z^n(X;R) \qquad \text{and} \qquad B := \bigoplus_{n\in\omega} B^n(X;R).$$

Then Z is a homogeneous subring of $C(X;R)$ by using the fact that

$$d(\alpha \cup \beta) = d\alpha \cup \beta + (-1)^n \alpha \cup d\beta$$

for any $\alpha \in C^n(X;R)$ and $\beta \in C^m(X;R)$ holds. Moreover, B is a homogeneous two-sided ideal in Z. Therefore by lemma E.79, we have

$$H(X;R) = \bigoplus_{n\in\omega} Z^n(X;R)/B^n(X;R) = \bigoplus_{n\in\omega} H^n(X;R).$$

Example E.81. Let $n \in \omega$, $n \geq 1$. Then using the fact that $\widetilde{H}_k(\mathbb{S}^n) = \mathbb{Z}$ if $k = n$ and zero otherwise, corollary E.76 implies that

$$H^0(\mathbb{S}^n) = \mathbb{Z} \qquad \text{and} \qquad H^n(\mathbb{S}^n) = \mathbb{Z}$$

and zero otherwise. Thus

$$H(\mathbb{S}^n;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}.$$

Denote the generator of the first summand by 1 and the second by X, we get that $X \cup X \in H^{2n}(\mathbb{S}^n) = 0$ and thus

$$H(\mathbb{S}^n; \mathbb{Z}) \cong \mathbb{Z}[X]/(X^2).$$

Actually, if $R \in \mathsf{CRing}$, then $H(-; R)$ attains values in CGRing.

Definition E.82 (Diagonal Approximation). A ***diagonal approximation*** is defined to be a natural chain map

$$C_\bullet(-) \to C_\bullet(-) \otimes C_\bullet(-)$$

such that $D_0(x) = x \otimes x$ holds for any $x \in X$, $X \in \mathsf{Top}$.

Theorem E.83 (Alexander-Whitney Formula). *An Eilenberg Zilber morphism*

$$\Omega : C_\bullet(X \times Y) \to C_\bullet(X) \otimes C_\bullet(Y)$$

is given by the ***Alexander-Whitney formula***

$$\Omega(\sigma) := \sum_{i=0}^{n} (\pi_1 \circ \sigma \circ A(e_0, \ldots, e_i)) \otimes (\pi_2 \circ \sigma \circ A(e_i, \ldots, e_n)) \tag{E.6}$$

for any $\sigma : \Delta^n \to X \times Y$.

Proposition E.84. *For the Alexander-Whitney choice of an Eilenberg-Zilber morphism Ω, the composition*

$$C^\bullet\delta \circ \mathrm{Hom}(\Omega, R) \circ \mu$$

where $\mu : C^\bullet(X; R) \otimes C^\bullet(X; R) \to \mathrm{Hom}(C_\bullet(X) \otimes C_\bullet(X), R)$ is defined by

$$\mu(\alpha \otimes \beta)\left(\sum_{k=0}^{n+m} \sigma_k \otimes \sigma'_{n+m-k}\right) := \alpha(\sigma_n)\beta(\sigma'_m)$$

coincides with the cup product.

Proof. Let $\alpha \in C^n(X; R)$, $\beta \in C^m(X; R)$ and $\sigma \in C^{n+m}(X)$. We compute

$$\begin{aligned}(C^\bullet\delta \circ \mathrm{Hom}(\Omega, R) \circ \mu)(\alpha \otimes \beta)(\sigma) &= \mathrm{Hom}(\Omega \circ \delta, R)(\mu(\alpha \otimes \beta))(\sigma)\\ &= \mu(\alpha \otimes \beta) \circ \Omega \circ C_\bullet\delta(\sigma)\\ &= \mu(\alpha \otimes \beta)(\Omega(\delta \circ \sigma))\\ &= (\alpha \cup \beta)(\sigma).\end{aligned}$$

Theorem E.85. *Let $R \in \mathsf{CRing}$ and $X \in \mathsf{Top}$. Then*

$$\langle\alpha\rangle \cup \langle\beta\rangle = (-1)^{nm}\langle\beta\rangle \cup \langle\alpha\rangle$$

for any $\langle\alpha\rangle \in H^n(X; R)$ *and* $\langle\beta\rangle \in H^m(X; R)$.

Proof. Since $\Omega \circ C_\bullet\delta$ and twist $\circ\Omega \circ C_\bullet\delta$ are both diagonal approximations, hence naturally chain homotopic. Now just evaluate both compositions. □

Corollary E.86. *Let* $X, Y \in \mathsf{Top}$ *of finite type and suppose that* $H_n(Y)$ *is free abelian for any* $n \in \mathbb{Z}$*. Then the cross product*

$$H(X) \otimes H(Y) \xrightarrow{\times} H(X \times Y)$$

is an isomorphism of graded rings.

Example E.87. Suppose $\mathbb{T}^n$ is the n-torus from example E.74. We claim that

$$H(\mathbb{T}^n; \mathbb{Z}) \cong \mathbb{Z}[X_1, \dots, X_n]/(X_k^2).$$

Indeed, example E.81, implies the base case for an induction over n. Suppose the claim holds for some $n \in \omega$, $n \geq 1$. Then using corollary E.86 implies that

$$\begin{aligned} H(\mathbb{T}^{n+1}) &= H(\mathbb{T}^n \times \mathbb{S}^1) \\ &= H(\mathbb{T}^n) \otimes H(\mathbb{S}^1) \\ &= \mathbb{Z}[X_1, \dots, X_n]/(X_k^2) \otimes \mathbb{Z}[X_{n+1}]/(X_{n+1}^2) \\ &= \mathbb{Z}[X_1, \dots, X_{n+1}]/(X_k^2). \end{aligned}$$

Appendix F
Review of Differential Topology

We follow the treatment as provided by *Will J. Merry* in the year course *Differential Geometry I and II* at the *ETH Zurich* in the autumn semester 2018 and spring semester 2019, respectively. The course notes are available at

https://www.merry.io/differential-geometry/.

Additionally, we rely on [6] as well as [16].

F.1 The Category of Smooth Manifolds

Definition F.1 (Topological Manifold). Let $n \in \mathbb{N}$. A topological space M is said to be a ***topological manifold of dimension n***, iff

(i) M is locally Euclidean of dimension n, that is, for every $x \in M$ there exist an open subset $U \subseteq M$ and a function $\varphi : U \to \mathbb{R}^n$ such that $\varphi(U) \subseteq \mathbb{R}^n$ is open and $\varphi : U \to \varphi(U)$ is a homeomorphism. Every such pair (U, φ) is called a ***chart on M about x***.
(ii) M is Hausdorff and has at most countably many connected components.
(iii) M is paracompact, that is, every open cover of M admits a locally finite open refinement.

Example F.2 (The Empty Manifold). Let $n \in \mathbb{N}$. Then the empty set $\varnothing$ endowed with the trivial topology is a topological manifold of dimension n.

Instead of requiring a topological manifold to be paracompact and to admit only countably many connected components, many authors instead use that any manifold is second countable. This is due to the following point-set topological result:

Theorem F.3 ([5, 126]). *Every Hausdorff locally Euclidean paracompact topological space is second countable if and only if it admits countably many connected components.*

Definition F.4 (Lindelöf Space). A topological space is said to be a ***Lindelöf space***, iff every open cover admits a countable subcover.

Theorem F.5. *Every second countable space is a Lindelöf space.*

Corollary F.6. *Every topological manifold is a Lindelöf space.*

Definition F.7 (Smooth Atlas). A ***smooth atlas for a topological manifold M*** is a collection $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ of charts on M such that

(i) $(U_\alpha)_{\alpha \in A}$ is an open cover for M.
(ii) For all $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$, the function

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$$

is smooth. The function $\varphi_\alpha \circ \varphi_\beta^{-1}$ is called a ***transition function***.

Let $\mathcal{A}$ and $\mathcal{A}'$ be two smooth atlases on a topological manifold M. Define a relation on the set of all smooth atlases on M (this is a subset of the power set 2^M) by

$$\mathcal{A} \sim \mathcal{A}' \quad :\Leftrightarrow \quad \mathcal{A} \cup \mathcal{A}' \text{ is an atlas for } M.$$

Exercise F.8. Show that above relation is actually an equivalence relation on the set of all smooth atlases on a topological manifold M.

Definition F.9 (Smooth Structure). A ***smooth structure on a topological manifold M*** is an equivalence class $[\mathcal{A}]$ where $\mathcal{A}$ is a smooth atlas for M.

Definition F.10 (Maximal Smooth Atlas). Let $[\mathcal{A}]$ be a smooth structure on a topological manifold M. Define the ***maximal smooth atlas on M*** by $\bigcup_{\mathcal{A}' \in [\mathcal{A}]} \mathcal{A}'$.

Definition F.11 (Smooth Manifold). Let $n \in \mathbb{N}$. A ***smooth manifold of dimension n*** is defined to be a pair $(M, \mathcal{A})$, where M is a topological manifold of dimension n and $\mathcal{A}$ is a maximal smooth atlas on M.

Example F.12 (Open Subsets). Let M be a smooth manifold and $U \subseteq M$ open. Then U inherits a smooth manifold structure from M.

Example F.13 (Vector Spaces). Let V be a finite-dimensional real vector space. Then $V \cong \mathbb{R}^{\dim V}$ and V inherits a norm from the standard norm on $\mathbb{R}^{\dim V}$. In fact, by a standard result in functional analysis the choice of norm does not matter, since any two norms on a finite-dimensional vector space are equivalent. Define the ***standard smooth structure on V*** to be the maximal atlas containg the smooth atlas consisting of the single chart induced by the coordinate isomorphism.

Example F.14 (n-Spheres). Let $n \in \mathbb{N}$. If $n = 0$, we have that $\mathbb{S}^0 = \{\pm 1\}$. It is easily seen that $\mathbb{S}^0$ is a smooth manifold of dimension 0. Let $n \geq 1$. Define $N := e_{n+1}$ and $S := -e_{n+1}$, where e_{n+1} denotes the $n+1$-th standard basis vector of $\mathbb{R}^{n+1}$. Moreover, set

$$U_+ := \mathbb{S}^n \setminus S \qquad \text{and} \qquad U_- := \mathbb{S}^n \setminus N.$$

Then U_+ and U_- are open subsets of $\mathbb{S}^n$, the upper and lower hemisphere, respectively. Then the functions $\varphi_\pm : U_\pm \to \mathbb{R}^n$ defined by

$$\varphi_\pm(x) := \frac{1}{1 \pm x_{n+1}}(x_1, \ldots, x_n),$$

are homeomorphisms. Indeed, one can check that $\psi_\pm : \mathbb{R}^n \to U_\pm$ defined by

$$\psi_\pm(x) := \left(\frac{2x}{1 + |x|^2}, \frac{\pm(1 - |x|^2)}{1 + |x|^2} \right)$$

is a continuous inverse for φ_+ and φ_-, respectively. We claim that $\{(U_\pm, \varphi_\pm)\}$ is a smooth atlas for $\mathbb{S}^n$. Clearly, $\mathbb{S}^n$ is covered by the two charts. Next we have to calculate the transition functions $\varphi_\mp \circ \varphi_\pm^{-1} = \varphi_\mp \circ \psi_\pm : \varphi_\pm(U_+ \cap U_-) \to \varphi_\mp(U_+ \cap U_-)$. It is easy to see that $\varphi_\pm(U_+ \cap U_-) = \mathbb{R}^n \setminus \{0\}$ and that

$$\varphi_\mp \circ \psi_\pm = \frac{x}{|x|^2},$$

which is smooth. Since $\mathbb{S}^n$ is Hausdorff as a metric space and as a subspace of a second countable space, itself second countable, $\mathbb{S}^n$ equipped with the smooth structure induced by the smooth atlas constructed above, is a smooth manifold of dimension n.

Example F.15 (Real Projective Spaces). Let $n \in \mathbb{N}$ and define an equivalence relation on $\mathbb{R}^{n+1} \setminus \{0\}$ by

$$x \sim y \quad :\Leftrightarrow \quad \exists \lambda \in \mathbb{R}^\times : x = \lambda y.$$

Define the ***real projective space of dimension n***, written $\mathbb{RP}^n$, to be the quotient space of the above equivalence relation. Then $\mathbb{RP}^n$ admits a smooth structure by defining a smooth atlas via the charts $(U_i, \varphi_i)_{i=1,\ldots,n+1}$, where

$$U_i := \left\{[x] : x^i \neq 0\right\},$$

and $\varphi_i : U_i \to \mathbb{R}^n$ is defined by

$$\varphi_i\,[x] := \frac{1}{x^i}\left(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{n+1}\right).$$

That each (U_i, φ_i) is indeed a chart, can be seen by using the fact that an explicit inverse of φ_i is given by $\psi_i : \mathbb{R}^n \to U_i$ defined by

$$\psi_i(x) := \left[x^1, \ldots, x^{i-1}, 1, x^{i+1}, \ldots, x^n\right].$$

Exercise F.16. Check that the relation defined in example F.15 is indeed an equivalence relation.

Proposition F.17 (Smooth Manifold Chart Lemma). *Let M be a set and suppose $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ is a family of subsets $U_\alpha \subseteq M$ and maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, for some fixed $n \in \mathbb{N}$, such that:*

(i) *For all $\alpha \in A$, $\varphi_\alpha(U_\alpha)$ is open and $\varphi : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a bijection.*
(ii) *For all $\alpha, \beta \in A$, $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$.*
(iii) *If $U_\alpha \cap U_\beta \neq \varnothing$, then $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is smooth.*
(iv) *Countably many of the sets U_α cover M.*
(v) *If $x, y \in M$ such that $x \neq y$, there either exists some $\alpha \in A$ such that $x, y \in U_\alpha$ or there exists $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta = \varnothing$, $x \in U_\alpha$ and $y \in U_\beta$.*

Then M admits a unique smooth structure containing the atlas $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$.

Definition F.18 (Smooth Map). Let M and N be smooth manifolds and $F : M \to N$ a map. We say that F is ***smooth***, iff for all $x \in M$, there exists a chart (U, φ) on M about x and a chart (V, ψ) on N about $F(x)$ such that

(i) $U \cap F^{-1}(V)$ is open in M.
(ii) $\psi \circ F \circ \psi^{-1} : \varphi\left(U \cap F^{-1}(V)\right) \to \psi(V)$ is smooth.

The ***set of all smooth maps from M to N*** is denoted by $C^\infty(M, N)$ and the ***set of all smooth functions on M*** is denoted by $C^\infty(M)$.

Exercise F.19. Let M be a smooth manifold. Show that $C^\infty(M)$ is an $\mathbb{R}$-algebra under pointwise defined operations.

Example F.20. Coordinate Functions Let M^n be a smooth manifold and (U, φ) be a chart about some $x \in M$. Let $\pi^i : \mathbb{R}^n \to \mathbb{R}$ be defined by $\pi^i(x^1, \ldots, x^n) := x^i$ for $i = 1, \ldots, n$. Define $x^i : U \to \mathbb{R}$ by $x^i := \pi^i \circ \varphi$. Then $x^i \in C^\infty(U)$ and we call x^i a ***coordinate function***. Moreover, we may denote the chart (U, φ) by $\left(U, (x^i)\right)$ and say that (x^i) are ***local coordinates about x***.

F.2 Tangent Spaces and the Differential

Let M be a smooth manifold and let $x \in M$. Define a binary relation on the set

$$X := \left\{(U, f) : U \subseteq M \text{ neighbourhood of } x,\ f \in C^\infty(U)\right\}$$

by

$$(U, f) \sim (V, g) \quad :\Leftrightarrow \quad \exists W \subseteq U \cap V \text{ neighbourhood of } x \text{, such that } f|_W = g|_W.$$

Exercise F.21. Show that the above relation is actually an equivalence relation.

Definition F.22 (Germ). Let M be a smooth manifold and let $x \in M$. The set of ***germs at p***, written $C_x^\infty(M)$ is defined to be $C_x^\infty(M) := X/\sim$.

Exercise F.23. Show that $C_x^\infty(M)$ is an $\mathbb{R}$-algebra under the obvious operations.

Remark F.24. Note that if $f \in C^\infty(M)$, then $[(M, f)] \sim [(U, f|_U)]$ for any neighbourhood U of x. Thus any germ at p contains a representant which is defined on the whole manifold and we thus may simply write $[f]$ for a germ at p.

Remark F.25. Let $[f]$ be a germ at $x \in M$. Then $f(x)$ is well-defined. Indeed, if $f|_U = g|_U$ on some neighbourhood of x, then in particular $f(x) = g(x)$.

Definition F.26 (Tangent Space). Let M be a smooth manifold and let $x \in M$. The ***tangent space of M at x***, written $T_x M$, is defined to be the vector space $\left(C_x^\infty(M)\right)^*$ such that

$$v([f]\,[g]) = v\,[f]\,g(x) + f(x)v\,[g]$$

holds.

Lemma F.27. *Let M be a smooth manifold and $x \in M$. Suppose $\lambda \in C^\infty(M)$ is a constant function. Then $v\,[\lambda] = 0$ for all $v \in T_x M$.*

Proof. This immediately follows from

$$v\,[\lambda] = v\,[\lambda \cdot 1] = \lambda v\,[1] = \lambda v\,[1 \cdot 1] = 2\lambda v\,[1] = 2v\,[\lambda]\,.$$

Definition F.28 (Derivation). Let M be a smooth manifold, $x \in M$ and U a neighbourhood of x. The ***space of derivations of $C^\infty(U)$ at x***, written $\mathcal{D}_x(U)$, is defined to be the vector space $\left(C^\infty(U)\right)^*$ such that

$$v(fg) = v(f)g(x) + f(x)v(g)$$

holds.

Proposition F.29. *Let M be a smooth manifold, $x \in M$ and U be a neighbourhood of x. Then*

$$T_x M \cong \mathcal{D}_x(U).$$

Proof. Let $\Phi : T_x M \to \mathcal{D}_x(U)$ be defined by

$$\Phi(v)(f) := v\,[f]$$

for all $f \in C^\infty(U)$. Clearly Φ is well-defined and linear. We want to construct an inverse $\Psi : \mathcal{D}_x(U) \to T_x M$ for Φ. This implies, that we should define

$$\Psi(v)\,[f] = v\left(\widetilde{f}\right)$$

where $\widetilde{f} \in C^\infty(U)$ such that $\left[\widetilde{f}\right] = [f]$.

Step 1: Existence of $\widetilde{f}$. Let (V, f) be a representant of $[f]$. As in the proof of the smoothness criteria for tensor fields 2.33, we find a neighbourhood W about x such that $\overline{W} \subseteq U \cap V$. Then there exists a smooth bump function $\psi \in C^\infty(U \cap V)$ such that $\psi|_W = 1$ and supp $\psi \subseteq U \cap V$. Let $\widetilde{f} := \psi f$ extended to be zero on U. Then clearly $\left[\widetilde{f}\right] = [f]$ since $\widetilde{f} = f$ on W.

Step 2: Ψ is well-defined. Suppose that $[f] = [g]$ in $C_x^\infty(M)$. Then $f = g$ on some neighbourhood V of x. We claim that $v(f) = v(g)$ on $U \cap V$. Indeed, let ψ be a smooth bump function for $\{x\}$ supported in $U \cap V$. Then $\psi(f - g) = 0$ on U and we compute

$$0 = v\left(\psi(f - g)\right) = v(\psi)(f - g)(x) + \psi(x)v(f - g) = v(f - g).$$

Lemma F.30. *Let M be a smooth manifold and U a neighbourhood of $x \in M$. Suppose $\lambda \in C^\infty(U)$ is a constant function. Then $v(\lambda) = 0$ for all $v \in \mathcal{D}_x(U)$.*

Proof. Using the notation of the proof of proposition F.29, lemma F.27 yields

$$v(\lambda) = (\Phi \circ \Psi)(v)(\lambda) = \Psi(v)\,[\lambda] = 0.$$

Example F.31. Coordinate Derivation Let M^n be a smooth manifold and (U, φ) be a chart on M. For every $x \in U$ and every $i = 1, \ldots, n$ define

$$\left.\frac{\partial}{\partial x^i}\right|_x : C^\infty(U) \to \mathbb{R}$$

by

$$\left.\frac{\partial}{\partial x^i}\right|_x (f) := D_i(f \circ \varphi^{-1})\left(\varphi(x)\right).$$

Then clearly $\left.\frac{\partial}{\partial x^i}\right|_x$ is a derivation of $C^\infty(U)$ at x. Thus by proposition F.29, $\left.\frac{\partial}{\partial x^i}\right|_x \in T_x M$.

One of the profound features of tangent spaces to a smooth manifold are that they are finite dimensional. In fact, they admit the same dimension as the manifold.

Lemma F.32. *Let $\Omega \subseteq \mathbb{R}^n$ be open and star-shaped about $x_0 \in \Omega$. Suppose $f \in C^\infty(\Omega)$. Then there exists $\varphi_1, \ldots, \varphi_n \in C^\infty(\Omega)$ such that $\varphi_i(x_0) = D_i f(x_0)$ and*

$$f(x) = f(x_0) + \pi^i(x - x_0)\varphi_i(x)$$

holds for all $x \in \Omega$

Proof. For $x \in \Omega$ define $\gamma_x : [0, 1] \to \Omega$ by $\gamma_x(t) := x_0 + t(x - x_0)$ (note that this is only possible since Ω is assumed to be star-shaped with centre x_0). Then

$$\begin{aligned}
f(x) - f(x_0) &= \int_0^1 (f \circ \gamma_x)'(t)dt \\
&= \int_0^1 D_i f\left(\gamma_x(t)\right) \dot{\gamma}_x^i(t)dt \\
&= \int_0^1 D_i f\left(\gamma_x(t)\right) \pi^i(x - x_0)dt \\
&= \pi^i(x - x_0)\varphi_i(x)
\end{aligned}$$

where

$$\varphi_i(x) := \int_0^1 D_i f\left(\gamma_x(t)\right) dt.$$

Proposition F.33 (Basis for the Tangent Space). *Let M^n be a smooth manifold and (U, φ) a chart on M. Then*

$$\left\{ \left.\frac{\partial}{\partial x^i}\right|_x : i = 1, \ldots, n \right\}$$

is a basis for $T_x M$ for all $x \in U$, where $x^i := \pi^i \circ \varphi$. In particular, $\dim T_x M = \dim M = n$.

Proof. Since $\varphi(U) \subseteq \mathbb{R}^n$ is open, there exists $\varepsilon > 0$ such that $B_\varepsilon\left(\varphi(x)\right) \subseteq \varphi(U)$. Set $V := \varphi^{-1}\left(B_\varepsilon\left(\varphi(x)\right)\right)$. Then V is a neighbourhood of x in M and thus by proposition F.29, we have that $T_x M \cong \mathcal{D}_x(V)$. Let $f \in C^\infty(V)$. An application of lemma F.32 to $f \circ \varphi^{-1} \in C^\infty\left(B_\varepsilon\left(\varphi(x)\right)\right)$ yields

$$\begin{aligned}
(f \circ \varphi^{-1})(y) &= f(x) + \pi^i\left(y - \varphi(x)\right)\varphi_i(y) \\
&= f(x) + \left(\pi^i(y) - x^i(x)\right)\varphi_i(y) \\
&= f(x) + \left((\pi^i \circ \varphi)\left(\varphi^{-1}(y)\right) - x^i(x)\right)(\varphi_i \circ \varphi)\left(\varphi^{-1}(y)\right).
\end{aligned}$$

Thus

$$f = f(x) + \left(x^i - x^i(x)\right)(\varphi_i \circ \varphi)$$

on V. Using lemma F.30 we compute

$$v(f) = v\left(\left(x^i - x^i(x)\right)(\varphi_i \circ \varphi)\right) = v(x^i)\varphi_i\left(\varphi(x)\right) = v(x^i)\left.\frac{\partial}{\partial x^i}\right|_x (f). \quad \text{(F.1)}$$

Suppose that $\lambda^i \left.\frac{\partial}{\partial x^i}\right|_x = 0$. Then using example C.5 and proposition C.12 we compute

$$0 = \lambda^i \left.\frac{\partial}{\partial x^i}\right|_x (x^j) = \lambda^i D_i \pi^j\left(\varphi(x)\right) = \lambda^i \pi^j(e_i) = \lambda^i \delta_i^j = \lambda^j. \quad \text{(F.2)}$$

Proposition F.34. *Let M^n be a smooth manifold and $x \in M$. Suppose (e_i) is a basis for T_xM. Then there exists a chart $\left(U, x^i\right)$ centred about x such that*

$$\left.\frac{\partial}{\partial x^i}\right|_x = e_i \qquad \forall i = 1, \ldots, n.$$

Proof. Let $\left(U, (\tilde{x}^i)\right)$ be a chart about $x \in M$. Since (e_i) and $\left.\frac{\partial}{\partial \tilde{x}^i}\right|_x$ are bases for T_xM, we find an invertible matrix (A^i_j) such that

$$\left.\frac{\partial}{\partial \tilde{x}^j}\right|_x = A^i_j e_i.$$

Define new coordinates $x^i : U \to \mathbb{R}^n$ by

$$x^i := A^i_j \left(\tilde{x}^j - \tilde{x}^j(x)\right).$$

Then $x^i(x) = 0$ and using (F.1) we compute

$$\begin{aligned}
\left.\frac{\partial}{\partial x^i}\right|_x &= \left.\frac{\partial}{\partial x^i}\right|_x \left(\tilde{x}^j\right) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_x \\
&= \left.\frac{\partial}{\partial x^i}\right|_x \left(\left(A^{-1}\right)^j_k x^k + \tilde{x}^j(x)\right) \left.\frac{\partial}{\partial \tilde{x}^j}\right|_x \\
&= \left(A^{-1}\right)^j_k \delta^k_i \left.\frac{\partial}{\partial \tilde{x}^j}\right|_x \\
&= \left(A^{-1}\right)^j_i \left.\frac{\partial}{\partial \tilde{x}^j}\right|_x \\
&= \left(A^{-1}\right)^j_i A^k_j e_k \\
&= \delta^k_i e_k \\
&= e_i.
\end{aligned}$$

□

Definition F.35 (Derivative). Let M and N be smooth manifolds and $F \in C^\infty(M, N)$. For $x \in M$, define a map $DF_x : T_xM \to T_{F(x)}N$ by

$$DF_x(v)(f) := v\left(\tilde{f} \circ F\right)$$

for all $f \in C^\infty(V)$, where $V \subseteq N$ open and $\tilde{f}$ is any extension of f in some neighbourhood of $F(x)$. This map is called the ***derivative of F at x***.

Proposition F.36. *Let M and N be smooth manifolds. Then for any $(x, y) \in M \times N$ there is a canonical isomorphism*

$$T_{(x,y)}(M \times N) \cong T_xM \times T_yN. \tag{F.3}$$

Proof. Observe that $\Phi : T_{(x,y)}(M \times N) \to T_x M \times T_y N$ defined by

$$\Phi(v) := \left(D\pi^1_{(x,y)}(v), D\pi^2_{(x,y)}(v)\right)$$

and $\Psi : T_x M \times T_y N \to T_{(x,y)}(M \times N)$ defined by

$$\Psi(v, w) := D(\iota_y)_x(v) + D(\iota_x)_y(w)$$

are linear and inverse to each other. □

Definition F.37 (Velocity of a Curve). Let $J \subseteq \mathbb{R}$ be an open interval and $\gamma \in C^\infty(J, M)$ be a curve in a smooth manifold M. For every $t \in J$, define the ***velocity vector of γ at t***, written $\gamma'(t)$, by

$$\gamma'(t) := D\gamma_t\left(\left.\frac{d}{dt}\right|_t\right) \in T_{\gamma(t)}M.$$

It is immediate from the definition of the velocity vector of a curve F.37, that

$$\gamma'(t)(f) = D\gamma_t\left(\left.\frac{d}{dt}\right|_t\right)(f) = \left.\frac{d}{dt}\right|_t (f \circ \gamma) = (f \circ \gamma)'(t)$$

for all $f \in C^\infty(M)$. Moreover, if (U, φ) is a chart on M, then equation F.1 yields

$$\gamma'(t) = \gamma'(t)(x^i)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} = (x^i \circ \gamma)'(t)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} = \dot{\gamma}^i(t)\left.\frac{\partial}{\partial x^i}\right|_{\gamma(t)} \tag{F.4}$$

at least sufficiently close to t.

Proposition F.38 (The Velocity of a Composite Curve). *Let $F \in C^\infty(M, N)$ and $\gamma \in C^\infty(J, M)$ for some interval $J \subseteq \mathbb{R}$. Then*

$$(F \circ \gamma)'(t) = DF\left(\gamma'(t)\right)$$

for all $t \in J$.

Proof. This is immediate by

$$(F \circ \gamma)'(t) = D(F \circ \gamma)\left(\left.\frac{d}{dt}\right|_t\right) = DF \circ D\gamma\left(\left.\frac{d}{dt}\right|_t\right) = DF\left(\gamma'(t)\right).$$

Lemma F.39. *Let V be a finite-dimensional real vector space and $x \in V$. Define $\Phi_x : V \to T_x V$ by $\Phi_x(v) := \gamma'(0)$, where $\gamma : \mathbb{R} \to V$ is defined by $\gamma(t) := x + tv$. Then Φ_x is an isomorphism.*

Proof. By (F.4) we have that

$$\gamma'(0) = \dot{\gamma}^i \left.\frac{\partial}{\partial x^i}\right|_x = v^i \left.\frac{\partial}{\partial x^i}\right|_x.$$

Thus Φ_x maps bases to bases. $\square$

Using lemma F.39 we can relate the two notions of a derivative on Eudlidean spaces.

Proposition F.40. *Let $U \subseteq \mathbb{R}^n$ open and $F \in C^\infty(U, \mathbb{R}^m)$. Let $x_0 \in U$. Since F is differentiable at x_0, there exists a map $\varphi : U \to \mathrm{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that φ is continuous at x_0 and for all $x \in U$*

$$F(x) - F(x_0) = \varphi(x)(x - x_0)$$

holds. Then the following diagram commutes:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\varphi(x_0)"] \arrow[d, "\Phi_{x_0}"'] & \mathbb{R}^m \arrow[d, "\Phi_{Fx_0}"] \\
T_{x_0}\mathbb{R}^n \arrow[r, "DF_{x_0}"'] & T_{F(x_0)}\mathbb{R}^m .
\end{tikzcd}$$

Proof. Problem F.255. $\square$

Velocity vectors to a curve give yet another way to think about the tangent space $T_x M$ to a point $x \in M$ of a smooth maifold M. Consider the set

$$X := \left\{\gamma \in C^\infty(J, M) : J \subseteq \mathbb{R} \text{ open interval with } 0 \in J, \gamma(0) = x\right\}.$$

Define a binary relation on X as follows:

$$\gamma_1 \sim \gamma_2 \quad :\Leftrightarrow \quad \exists \text{ chart } (U, \varphi) \text{ about } x \text{ such that } (\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0).$$

Exercise F.41. Show that the above relation is an equivalence relation.

Let $\mathcal{V}_x M := X/\!\sim$.

Proposition F.42. *Let M be a smooth manifold and $x \in M$. Then $T_x M \cong \mathcal{V}_x M$ as sets.*

Proof. Define $\Phi : \mathcal{V}_x M \to T_x M$ by $\Phi[\gamma] := \gamma'(0)$. This map is well-defined. Indeed, if $[\gamma_1] = [\gamma_2]$, there exists a chart (U, φ) about x such that $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$. This immediately implies that $\dot{\gamma}_1^i(0) = \dot{\gamma}_2^i(0)$ for all $i = 1, \dots, n$. Thus (F.4) yields $\gamma_1'(0) = \gamma_2'(0)$. From this also follows that Φ is injective. Indeed, if $\gamma_1'(0) = \gamma_2'(0)$, then $\dot{\gamma}_1^i(0) = \dot{\gamma}_2^i(0)$ for all $i = 1, \dots, n$ by (F.4) and proposition F.33. Let $v \in T_x M$. Then in any chart (U, φ) centered about x we have that $v = v^i \frac{\partial}{\partial x^i}\big|_x$. Hence for $\varepsilon > 0$ sufficiently small we can define $\gamma_v : (-\varepsilon, \varepsilon) \to M^n$ by

$$\gamma_v(t) := \varphi^{-1}\left(t v^i, \dots, t v^n\right).$$

Thus Φ is surjective. $\square$

We can equip $\mathcal{V}_x M$ with the structure of a vector space by means of the following lemma.

Lemma F.43. *Let V be a a finite-dimensional real vector space and S be a set. If there exists a bijection $\varphi : S \to V$, we can equip V with a structure of a real vector space such that φ is an isomorphism.*

Proof. Just define
$$\lambda x + y := \varphi^{-1}\left(\lambda\varphi(x) + \varphi(y)\right)$$
for all $x, y \in S$ and $\lambda \in \mathbb{R}$. □

Definition F.44 (Cotangent Space). Let M be a smooth manifold. For $x \in M$, define the ***cotangent space of M at x***, written $T_x^* M$, to be
$$T_x^* M := (T_x M)^*.$$

Definition F.45 (Differential). Let M be a smooth manifold, U a neighbourhood of $x \in M$ and $f \in C^\infty(U)$. Define the ***differential of f at x***, written df_x, to be the element $df_x \in T_x^* M$ given by
$$df_x(v) := v(f).$$

Lemma F.46 (Basis for the Cotangent Space). *Let M^n be a smooth manifold and (U, φ) a chart on M. Then*
$$\left\{dx^i|_x : i = 1, \ldots, n\right\}$$
is a basis for $T_x^* M$ for all $x \in U$, where $x^i := \pi^i \circ \varphi$.

Proof. We only need to note that this is the dual basis of the tangent space basis F.33. This follows from (F.2) since
$$dx^i|_x \left(\left.\frac{\partial}{\partial x^j}\right|_x\right) = \left.\frac{\partial}{\partial x^j}\right|_x (x^i) = \delta^i_j.$$

F.3 Submanifolds

Proposition F.47. *Let M^n and N^n be smooth manifolds, $F \in C^\infty(M, N)$ and $x \in M$. If DF_x is invertible then there exists a neighbourhood U of x in M such that $F : U \to F(U)$ is a diffeomorphism.*

Proof. Let (V, φ) be a chart about x and (W, ψ) be a chart about $F(x)$. Then
$$\psi \circ F \circ \varphi^{-1} : \varphi\left(V \cap F^{-1}(W)\right) \to \mathbb{R}^n$$

and using the chain rule yields

$$D\left(\psi \circ F \circ \varphi^{-1}\right)_{\varphi(x)} = D\psi_{F(x)} \circ DF_x \circ D\left(\varphi^{-1}\right)_{\varphi(x)}$$

and thus $D\left(\psi \circ F \circ \varphi^{-1}\right)_{\varphi(x)}$ is invertible. An application of the inverse function theorem C.13 yields a neighbourhood $\widetilde{U}$ in $\varphi\left(V \cap F^{-1}(W)\right)$ about $\varphi(x)$ such that the restriction $\psi \circ F \circ \varphi^{-1}|_{\widetilde{U}}$ is a diffeomorhism. Set $U := \varphi^{-1}(\widetilde{U})$. □

Proposition F.48. *Let $U \subseteq \mathbb{R}^n$ be a neighbourhood about 0 and $f : U \to \mathbb{R}^k$ smooth such that $f(0) = 0$. Then:*

(a) *If $n \leq k$ and the matrix Df_0 has maximal rank, then there exists a chart ψ about 0 on $\mathbb{R}^k$ such that $\psi \circ f = \iota$, where $\iota : \mathbb{R}^n \hookrightarrow \mathbb{R}^k$ denotes the inclusion.*
(b) *If $n \geq k$ and the matrix Df_0 has maximal rank, then there exists a chart φ about 0 on $\mathbb{R}^n$ such that $f \circ \varphi = \pi$, where $\pi : \mathbb{R}^n \to \mathbb{R}^k$ denotes the projection.*

Definition F.49 (Immersion). A smooth map $F : M \to N$ is said to be an ***immersion***, iff DF_x is injective for all $x \in M$.

Definition F.50 (Embedding). A smooth map $F : M \to N$ is said to be an ***embedding***, iff F is an injective immersion and $F : M \to F(M)$ is a homeomorphism, where $F(M)$ is endowed with the subspace topology.

Every immersion is a local embedding.

Proposition F.51. *Suppose $F : M^n \to N^k$ is an immersion. Then for any $x \in M$, there exists a chart U of x and a chart (V, ψ) about $F(x)$ such that*

(a) *If $y^i := \pi^i \circ \psi$, then*

$$F(U) \cap V = \left\{y \in V : y^{n+1}(y) = \cdots = y^k(y) = 0\right\}.$$

(b) *$F|_U$ is an embedding.*

Corollary F.52. *Suppose $F : M^n \to N^k$ is an embedding. Then for any $x \in M$, there exists a chart U of x and a chart (V, ψ) about $F(x)$ such that if $y^i := \pi^i \circ \psi$, then*

$$F(M) \cap V = \left\{y \in V : y^{n+1}(y) = \cdots = y^k(y) = 0\right\}.$$

*If F is simply inclusion of M into N, we call the above choice of coordinates a **slice chart for M in N**.*

Proof. Since F is a homeomorphism onto $F(M)$, we have that $F(U)$ is open in $F(M)$. By definition of the subspace toology, $F(U) = F(M) \cap W$, where W is open in N. But then

$$F(M) \cap (W \cap V) = \left\{y \in V : y^{n+1}(y) = \cdots = y^k(y) = 0\right\}$$

by proposition F.51. □

Definition F.53 (Immersed Submanifold). Let M and N be smooth manifolds and $M \subseteq N$ as sets. We say that M is an ***immersed submanifold of N***, iff the inclusion $M \hookrightarrow N$ is an immersion.

Definition F.54 (Embedded Submanifold). Let M and N be smooth manifolds and $M \subseteq N$ as sets. We say that M is a ***embedded submanifold of N***, iff the inclusion $M \hookrightarrow N$ is an embedding.

By corollary F.52 every embedded submanifold admits an atlas consisting of slice charts. In fact, the converse is also true.

Proposition F.55. *Let N be a smooth manifold and $M \subseteq N$ a subset, such that for every $x \in M$ there exists a slice chart for M in N. If M is endowed with the subspace topology, then M admits a smooth structure making it into an embedded submanifold of N.*

Definition F.56 (Regular and Critical Point). Let $F : M \to N$ be smooth. A point $x \in M$ is said to be a ***regular point***, iff $\operatorname{rank} DF_x = \dim N$. A point $x \in M$ is said to be a ***critical point***, iff x is not a regular point.

Definition F.57 (Regular and Critical Value). Let $F : M \to N$ be smooth. A point $y \in N$ is said to be a ***regular value***, iff $F^{-1}(y)$ consist only of regular points. A point $y \in N$ is said to be a ***critical value***, iff y is not a regular value.

Theorem F.58 (The Implicit Function Theorem for Manifolds). *Let $F : M^n \to N^k$ be smooth and suppose that $y \in N$ is a regular value of F such that $F^{-1}(y) \neq \varnothing$. Then $F^{-1}(y)$ is a topological manifold of dimension $n-k$. Moreover, there exists a smooth structure on $F^{-1}(y)$ making it into an embedded submanifold of M.*

Proposition F.59. *Let $F : M \to N$ be smooth and $y \in N$ a regular value of F such that $F^{-1}(y) \neq \varnothing$. Then*

$$D\iota_x \left(T_x F^{-1}(y)\right) = \ker DF_x$$

holds for all $x \in F^{-1}(y)$ where $\iota : F^{-1}(y) \hookrightarrow M$ denotes the inclusion.

Proof. Observe, that both sides are subspaces of dimension $n-k$ of $T_x M$. Thus it suffices to show that $D\iota_x \left(T_x F^{-1}(y)\right) \subseteq \ker DF_x$. Let $v \in T_x F^{-1}(y)$ and $f \in C^\infty(N)$. Using the chain rule and lemma F.30 we compute

$$(DF_x \circ D\iota_x)(v) f = D(F \circ \iota)_x(v) f = v(f \circ F \circ \iota) = v\left(f(y)\right) = 0.$$

Example F.60 (*n*-Spheres). Let $n \in \mathbb{N}$ with $n \geq 1$. Then we can define

$$F : \mathbb{R}^{n+1} \to \mathbb{R} \qquad \text{by} \qquad F(x) := |x|^2 .$$

Then $\mathbb{S}^n = F^{-1}(\{1\})$ and it is easy to see that for any $x \in \mathbb{R}^{n+1}$ we have that

$$DF_x(v) = 2\langle x, v\rangle$$

for any $v \in \mathbb{R}^{n+1}$. Hence any $x \in \mathbb{R} \setminus \{0\}$ is a regular point of F under the identification given by proposition F.40 (Φ_x and $\Phi_{F(x)}$ are isomorphisms). In particular, 1 is a regular value of F and thus $\mathbb{S}^n$ is an embedded submanifold of $\mathbb{R}^{n+1}$ by the implicit function theorem for manifolds F.58. Moreover, using proposition F.59, we have that

$$D\iota_x\left(T_x\mathbb{S}^n\right) = \ker DF_x = \left\{v \in \mathbb{R}^{n+1} : \langle x, v\rangle = 0\right\} = x^\perp \tag{F.5}$$

for all $x \in \mathbb{S}^n$, again under the identification given by proposition F.40 (see figure F.1).

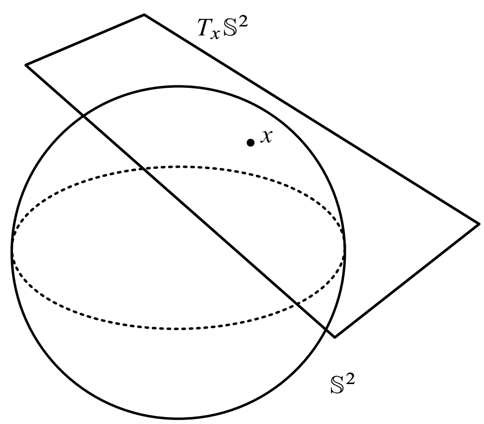


Fig. F.1: Tangent space $T_x\mathbb{S}^2$ at $x \in \mathbb{S}^2$.

Definition F.61 (Submersion). A smooth map $F : M \to N$ is said to be a ***submersion***, iff every point of M is a regular value.

The next theorem is the main reason why we require smooth manifolds to admit only countably many connected components.

Theorem F.62 (Sard's Theorem for Manifolds). *Let $F : M^n \to N^k$ be smooth. The set of critical values of F has measure zero in N and the set of regular values is dense in N. In particular, if $n < k$, every point is critical, and thus $N \setminus F(M)$ is dense in N.*

Theorem F.63 (The Strong Whitney Embedding Theorem). *Let M^n be a smooth manifold. Then there exists a proper embedding $M \to \mathbb{R}^{2n}$.*

Theorem F.64 (The Weak Whitney Embedding Theorem). *Let M^n be a smooth manifold. Then there exists a proper embedding $M \to \mathbb{R}^{2n+1}$.*

Proof. We will only give a sketch of the proof if M is compact. We proceed in two steps.

Step 1: *M^n embeds into some Euclidean space.* Let $(V_\alpha)_{\alpha \in A}$ be a finite open cover for M such that $\overline{V}_\alpha \subseteq U_\alpha$ for charts $(U_\alpha, \varphi_\alpha)$ and $|A| = k$. Moreover, let ψ_α be cutoff-functions for $\overline{V}_\alpha$ supported in U_α and set $f_\alpha := \psi_\alpha \varphi_\alpha$. Define $F : M \to \mathbb{R}^{kn+k}$ by

$$F(x) := \big(f_1(x), \dots, f_k(x), \psi_1(x), \dots, \psi_k(x)\big).$$

Then F is an injective immersion and hence an embedding.
Step 2: *Inductively reducing the dimension obtained in step* 1. Replacing M by $F(M)$, we can assume $M \subseteq \mathbb{R}^N$. Suppose $N > 2n + 1$, otherwise there is nothing to prove. We are looking for unit vectors $v \in \mathbb{R}^N \setminus \mathbb{R}^{N-1}$ such that the projection P_v parallel to v induces an embedding

$$P_v|_M : M \to \mathbb{R}^{N-1}.$$

It can be shown that P_v is an injective immersion if and only if

$$v \neq \frac{x - y}{|x - y|} \qquad \text{and} \qquad v \neq \frac{w}{|w|}$$

for all $x, y \in M$ and $w \in T_x M$. Using Sard's theorem for manifolds F.62, one can show the existence of such a v. $\square$

Using the Whitney embedding theorems we can prove a foundational result in the de Rham cohomology.

Proposition F.65. *Let M be a smooth manifold and $g \in C(M, \mathbb{R}^k)$. For any positive $\delta \in C(M)$, there exists $f \in C^\infty(M, \mathbb{R}^k)$ such that*

$$|f(x) - g(x)| < \delta(x)$$

holds for all $x \in M$.

Next we want to improve above proposition to the case where the codomain itself is an arbitrary manifold. For this we need the notion of a *tubular neighbourhood*.

Theorem F.66 (The Tubular Neighbourhood Theorem, Euclidean Case). *Every embedded submanifold $M \subseteq \mathbb{R}^k$ admits a tubular neighbourhood.*

Theorem F.67 (The Whitney Approximation Theorem). *Let $F : M \to N$ be a continuous map between two smooth manifolds M and N. Then F is homotopic to a smooth map.*

Proof. Use the Whitney embedding theorems together with the existence of tubular neighbourhoods F.66 and proposition F.65. $\square$

F.4 Vector Fields

Definition F.68 (Vector Field). Let M be a smooth manifold and $U \subseteq M$ open and non-empty. A ***vector field on U*** is defined to be a section of the projection $\pi : TU \to U$. The set of all vector fields on U is denoted by $\mathfrak{X}(U)$.

Example F.69. Coordinate Vector Fields Let M be a smooth manifold and $\big(U, (x^i)\big)$ be a chart on M. Define $\frac{\partial}{\partial x^i} : U \to TM$ by

$$\frac{\partial}{\partial x^i}(x) := \left.\frac{\partial}{\partial x^i}\right|_x .$$

It immediately follows from the smoothness criteria for tensor fields 2.33 that $\frac{\partial}{\partial x^i} \in \mathfrak{X}(U)$.

Exercise F.70. Show that $\mathfrak{X}(U)$ is a $C^\infty(U)$-module. *Hint:* Use the smoothness criteria for tensor fields 2.33.

In contrast to arbitrary tensor fields, vector fields can act on smooth functions.

Proposition F.71. *Let M be a smooth manifold and $U \subseteq M$ open. Then $X \in \mathfrak{X}(U)$ if and only if the function $Xf : U \to \mathbb{R}$ defined by $Xf(x) := X(x)f$ is smooth for all $f \in C^\infty(V)$, where $V \subseteq U$ is open.*

Proof. Using the smoothness criteria for tensor fields 2.33, we locally write $X = X^i \frac{\partial}{\partial x^i}$, where X^i are smooth functions. Hence

$$Xf = X^i \frac{\partial f}{\partial x^i}$$

which is smooth.
Conversely, suppose that Xf is smooth for any $f \in C^\infty(V)$. Then in particular

$$X(x^j) = X^i \frac{\partial x^j}{\partial x^i} = X^j$$

is smooth. □

We adopt the terminology from [20, 218].

Definition F.72 (Derivation). Let M be a smooth manifold and $U \subseteq M$ open. A ***derivation of $C^\infty(U)$*** is a linear map $D : C^\infty(U) \to C^\infty(U)$ such that

$$D(fg) = D(f)g + fD(g)$$

holds for all $f, g \in C^\infty(U)$. Denote the set of all derivations of $C^\infty(U)$ by $\mathrm{Der}(U)$.

Exercise F.73. Show that $\mathrm{Der}(U)$ is a $C^\infty(M)$-module.

Proposition F.74. *Let M be a smooth manifold and $U \subseteq M$ be open and non-empty. Then $\mathfrak{X}(U) \cong \mathrm{Der}(U)$ as modules over $C^\infty(U)$.*

Proof. Define $\Phi : \mathfrak{X}(U) \to \mathrm{Der}(U)$ by $\Phi(X)(f) := Xf$ using proposition F.71. Moreover, define $\Psi : \mathrm{Der}(U) \to \mathfrak{X}(U)$ by $\Psi(D)(x)(f) := Df$ for all $f \in C^\infty(U)$ again using proposition F.71. □

Remark F.75. From now on we will identify vector fields in $\mathfrak{X}(U)$ with derivations $\mathrm{Der}(U)$ by means of proposition F.74.

Proposition F.74 yields a new tool for constructing vector fields.

Exercise F.76. Let M be a smooth manifold and $U \subseteq M$ a non-empty open subset. Show that

$$[X, Y] := X \circ Y - Y \circ X \in \mathfrak{X}(U)$$

for any $X, Y \in \mathfrak{X}(U)$.

Definition F.77. Let M and N be smooth manifold and $F \in C^\infty(M, N)$. Two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are said to be ***F-related***, iff

$$DF_x(X|_x) = Y_{F(x)}$$

holds for all $x \in M$.

Proposition F.78. *Let M and N be smooth manifolds and $F \in C^\infty(M, N)$. Then $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F-related if and only if*

$$X(f \circ F) = (Yf) \circ F$$

holds for all $f \in C^\infty(V)$, where $V \subseteq M$ is open.

Proposition F.79. *Let M and N be smooth manifolds and $F \in C^\infty(M, N)$. Suppose $X_1, X_2 \in \mathfrak{X}(M)$ and $Y_1, Y_2 \in \mathfrak{X}(N)$ such that X_1 is F-related to Y_1 and X_2 is F-related to Y_2. Then $[X_1, X_2]$ is F-related to $[Y_1, Y_2]$.*

F.5 Flows

Definition F.80 (Integral Curve). Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. A curve $\gamma \in C^\infty(J, M)$, where $J \subseteq \mathbb{R}$ is an interval, is said to be an ***integral curve of X***, iff

$$\gamma'(t) = X_{\gamma(t)}$$

holds for all $t \in J$.

Proposition F.81 (Fundamental Theorem for Autonomous ODEs). *Let $U \subseteq \mathbb{R}^n$ open and $X \in C^\infty(U, \mathbb{R}^n)$. Consider the initial value problem*

$$\begin{cases} \gamma'(t) = X\left(\gamma^1(t), \dots, \gamma^n(t)\right) \\ \gamma(t_0) = c, \end{cases} \tag{F.6}$$

for $t_0 \in \mathbb{R}$ and $c \in U$. Then:

(a) *For any $t_0 \in \mathbb{R}$ and $x_0 \in U$ there exists an open interval J_0 containing t_0 and an open subset $U_0 \subseteq U$ containing x_0 such that for each $c \in U_0$ there is a map $\gamma \in C^1(J_0, U)$ that solves (F.6).*
(b) *Any two solutions to (F.6) agree on their common domain.*
(c) *Define*
$$\theta : J_0 \times U_0 \to U$$
by $\theta(t, x) := \gamma(t)$, where $\gamma(t)$ is the unique solution of (F.6) such that $\gamma(t_0) = x$. Then θ is smooth.

Theorem F.82 (Local Flow). *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. For every $x \in M$ there exists a neighbourhood U of x, $\varepsilon > 0$ and a smooth map*

$$\theta : (-\varepsilon, \varepsilon) \times U \to M$$

such that: $\theta(\cdot, x)$ is the unique integral curve of X passing through x.

Definition F.83 (Maximal Integral Curve). Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. Given $x \in M$, denote by $\left(t^-(x), t^+(x)\right)$ the maximal interval around 0 such that the integral curve γ_x of X starting at x is defined. This integral curve is called the ***maximal integral curve of X starting at x.***

Theorem F.84 (Fundamental Theorem of Flows). *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. Then there exists a unique open set $\mathcal{D} \subseteq \mathbb{R} \times M$ and a unique smooth map $\theta : \mathcal{D} \to M$, called the **maximal flow associated to X**, such that*

(a) *For all $x \in M$ we have that*
$$\mathcal{D} \cap (\mathbb{R} \times \{x\}) = \left(t^-(x), t^+(x)\right) \times \{x\}.$$
(b) *$\theta(t, x) = \gamma_x(t)$ for all $(t, x) \in \mathcal{D}$.*

Proof. Observe that (a) and (b) determine $\mathcal{D}$ and θ uniquely. Therefore it suffices to show that $\mathcal{D}$ is open and θ is smooth. In order to show this, it is enough to show that for every $x \in M$ the set A_x consisting of all $t \in \left(t^-(x), t^+(x)\right)$ such that there exists a neighboruhood of (t, x) in $\mathcal{D}$ such that θ is smooth, is closed and non-empty. □

Definition F.85 (Complete Vector Field). A vector field $X \in \mathfrak{X}(M)$ on a smooth manifold M is said to be ***complete***, iff its flow θ admits the domain $\mathbb{R} \times M$.

A sufficient condition for completeness is given in the following lemma.

Lemma F.86. *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. Suppose that there exists $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq \left(t^-(x), t^+(x)\right)$ for all $x \in M$. Then X is complete.*

Proposition F.87. *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$ with compact support. Then X is complete.*

Corollary F.88. *Every vector field on a compact smooth manifold is complete.*

F.6 Lie groups and Lie algebras

Definition F.89 (Lie Group). A ***Lie group*** is defined to be a group object in Man.

Examples F.90 (Lie Groups). The following are examples of Lie groups.

(a) $(\mathrm{GL}(V), \circ)$.
(b) Consider $\mathbb{S}^1 \subseteq \mathbb{C}$. Then $\mathbb{S}^1$ is an abelian Lie group under complex multiplication (see problem F.254).
(c) Let $G_1, \ldots, G_n$ be Lie groups. Then $G_1 \times \cdots \times G_n$ is a Lie group. If $G_1, \ldots, G_n$ are abelian Lie groups, then so is $G_1 \times \cdots \times G_n$.
(d) The torus
$$\mathbb{T}^n := \underbrace{\mathbb{S}^n \times \cdots \times \mathbb{S}^n}_{n}$$
is an abelian Lie group from part (b) and (c).

Definition F.91 (Lie Group Homomorphism). A map $F \in C^\infty(G, H)$ between two Lie groups G and H is said to be a ***Lie group homomorphism***, iff $F : G \to H$ is a homomorphism.

The group structure of a Lie group induces canonical maps.

Definition F.92 (Translation). Let G be a Lie group and $g \in G$. Define morphisms $L_g, R_g \in \mathrm{Diff}(G)$ by

$$L_g(h) := gh \qquad \text{and} \qquad R_g(h) := hg.$$

These maps are called ***left translation by g*** and ***right translation by g***, respectively.

Proposition F.93. *Every Lie group homomorphism has constant rank.*

Definition F.94 (Lie Subgroup). A ***Lie subgroup*** of a Lie group G is defined to be a subgroup of G, which is itself a Lie group and an immersed submanifold of G.

Proposition F.95. *Let G be a Lie group and H be a subgroup of G such that H is an embedded submanifold of G. Then H is a Lie subgroup of G.*

To every Lie group we can associate an algebraic object.

Definition F.96 (Lie Algebra). A ***Lie algebra*** is defined to be a real vector space $\mathfrak{g}$, such that there exists a bilinear mapping

$$[\,\cdot\,,\,\cdot\,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

called the ***Lie bracket on*** $\mathfrak{g}$, such that:

(i) (***Antisymmetry***) $[x, y] = -[y, x]$,
(ii) (***Jacobi's Identity***) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$,

holds for all $x, y, z \in \mathfrak{g}$.

Example F.97. Vector Fields Let M be a smooth manifold and $U \subseteq M$ open. Then $\mathfrak{X}(U)$ together with $[\,\cdot\,,\,\cdot\,]$ defined in exercise F.76 is a Lie algebra, called the ***Lie algebra of vector fields on*** $\boldsymbol{U}$.

Definition F.98 (Left-Invariance). Let G be a Lie group. A vector field on G is said to be ***left-invariant***, iff it is L_g-related to itself for all $g \in G$. The vector space of left-invariant vector fields on G is denoted by $\mathfrak{X}_{\mathrm{L}}(G)$.

Proposition F.99. *Let G be a Lie group. Then $\mathfrak{X}_{\mathrm{L}}(G)$ is a Lie-subalgebra of $\mathfrak{X}(G)$.*

Proof. By definition, X and Y are L_g-related to themselves for all $g \in G$. Hence by proposition F.79 we have that $[X, Y]$ is L_g-related to itself and so $[X, Y] \in \mathfrak{X}_{\mathrm{L}}(G)$.$\square$

Theorem F.100. *Let G be a Lie group. Then $T_eG \cong \mathfrak{X}_{\mathrm{L}}(G)$ as real vector spaces.*

Proof. Consider the map $\mathrm{ev}\colon \mathfrak{X}_{\mathrm{L}}(G) \to \mathfrak{g}$ defined by $\mathrm{ev}(X) := X_e$. Then ev is linear and injective by left-invariance. So we need to show that ev is surjective. Let $v \in \mathfrak{g}$. Define $X_v\colon G \to TG$ by

$$X_v|_g := D(L_g)_e(v). \tag{F.7}$$

Then $X_v \in \mathfrak{X}_{\mathrm{L}}(G)$. Indeed, by F.71 it is enough to show that $X_v f$ is smooth for every $f \in C^\infty(G)$. Moreover, by proposition F.42 we find a smooth path $\gamma : (-\varepsilon, \varepsilon) \to G$ such that $\gamma(0) = e$ and $\gamma'(0) = v$. Hence

$$\begin{aligned}
(X_v f)(g) &= X_v|_g(f) \\
&= D(L_g)_e(v)(f) \\
&= v(f \circ L_g) \\
&= \gamma'(0)(f \circ L_g) \\
&= (f \circ L_g \circ \gamma)'(0).
\end{aligned}$$

Also X_v is left-invariant by the chain rule. $\square$

Definition F.101 (Lie Algebra associated to a Lie Group). Let G be a Lie group. The Lie algebra $\mathfrak{g} := T_eG$ is called the ***Lie algebra associated to the Lie group*** $\boldsymbol{G}$.

Definition F.102 (Lie Algebra Homomorphism). Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. A ***Lie algebra homomorphism between $\mathfrak{g}$ and $\mathfrak{h}$*** is defined to be a homomorphism $L \in \mathrm{L}(\mathfrak{g}, \mathfrak{h})$ such that

$$L[x, y] = [Lx, Ly]$$

holds for all $x, y \in \mathfrak{g}$.

Proposition F.103. *Let G and H be Lie groups and $F : G \to H$ a Lie group homomorphism. Then*

$$DF_e : \mathfrak{g} \to \mathfrak{h}$$

is a Lie algebra homomorphism.

Proposition F.104. *Every left-invariant vector field is complete.*

Definition F.105 (One-Parameter Subgroup). Let G be a Lie group. A ***one-parameter subgroup of G*** is defined to be a Lie group homomorphism $(\mathbb{R}, +) \to G$.

Proposition F.106 (Characterisation of One-Parameter Subgroups). *The one-parameter subgroups of a Lie group are in one-to-one correspondence with the maximal integral curves of left-invariant vector fields starting at the identity.*

Proof. Suppose γ is an integral curve of some left-invariant vector field $X \in \mathfrak{X}_{\mathrm{L}}(G)$. By proposition F.104, $\gamma : \mathbb{R} \to G$. Let $s \in \mathbb{R}$ and consider the path $\widetilde{\gamma} \in C^{\infty}(\mathbb{R}, G)$ defined by

$$\widetilde{\gamma}(t) := \gamma(s)^{-1}\gamma(s + t) = L_{\gamma(s)^{-1}} \left(\gamma(s + t)\right).$$

Then $\widetilde{\gamma}(0) = e$ and

$$\begin{aligned} \widetilde{\gamma}'(t) &= D\left(L_{\gamma(s)^{-1}}\right)_{\gamma(s+t)} \left(\gamma'(s + t)\right) \\ &= D\left(L_{\gamma(s)^{-1}}\right)_{\gamma(s+t)} \left(X_{\gamma(s+t)}\right) \\ &= X_{\gamma(s)^{-1}\gamma(s+t)} \\ &= X_{\widetilde{\gamma}(t)}. \end{aligned}$$

Thus by uniqueness $\widetilde{\gamma} = \gamma$. But this implies

$$\gamma(s + t) = \gamma(s)\gamma(t)$$

for all $s, t \in \mathbb{R}$.

Conversly, suppose that $\gamma : \mathbb{R} \to G$ is a one-parameter subgroup of G. Then $\gamma'(0) \in T_eG$ and thus by theorem F.100, we can associate to $\gamma'(0)$ a left-invariant vector field X. Then γ is an integral curve of X. Indeed, we compute

$$\begin{aligned} X_{\gamma(t)} &= D\left(L_{\gamma(t)}\right)_e \left(\gamma'(0)\right) \\ &= \left.\frac{d}{ds}\right|_{s=0} \left(L_{\gamma(t)} \circ \gamma\right) \\ &= \left.\frac{d}{ds}\right|_{s=0} \gamma(s+t) \\ &= \gamma'(t). \end{aligned}$$

Definition F.107 (Exponential Map). Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Then the map $\exp\colon \mathfrak{g} \to G$, defined by $v \mapsto \gamma(1)$, where γ is the unique one-parameter subgroup of G associated to v, is called the ***exponential map***.

Theorem F.108. *The exponential map is smooth and* $D\exp_0 = \mathrm{id}_{\mathfrak{g}}$.

Proof. Consider the map $\widetilde{X}$ on $G \times \mathfrak{g}$ defined by

$$\widetilde{X}|_{(g,v)} := \left(X_v|_g, 0\right) \in T_g G \times T_v \mathfrak{g} \cong T_{(g,v)}(G \times \mathfrak{g})$$

by proposition F.36. Then $\widetilde{X}$ is a vector field on $G\times\mathfrak{g}$. Indeed, for any $f \in C^\infty(G\times\mathfrak{g})$ we compute

$$\left(\widetilde{X} f\right)(g,v) = \widetilde{X}|_{(g,v)} f = D(\iota_v)(X_v|_g) f = X_v|_g (f \circ \iota_v) = \left(X_v(f \circ \iota_v)\right)(g)$$

using proposition F.36. The latter function is smooth and so is $\widetilde{X} f$. Now the flow $\theta : \mathbb{R} \times G \times \mathfrak{g} \to G \times \mathfrak{g}$ of $\widetilde{X}$ is given by

$$\theta(t,g,v) = \left(g \exp(tv), v\right).$$

Thus $\theta(1,e,\cdot)$ is smooth, but this is simply the map $v \mapsto \left(\exp(v), v\right)$. So exp is smooth since $\exp = \pi^1 \circ \theta(1,e,\cdot)$. ☐

Proposition F.109. *Let G and H be Lie groups with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. If $F : G \to H$ is a Lie algebra homomorphism, then the diagram*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "DF_e"] \arrow[d, "\exp"'] & \mathfrak{h} \arrow[d, "\exp"] \\ G \arrow[r, "F"'] & H \end{tikzcd}$$

commutes.

A prominent feature of Lie groups is their action on smooth manifolds.

Definition F.110 (Left Action). Let G be a Lie group and M be a smooth manifold. A ***left action of G on M*** is defined to be a smooth map $\theta \in C^\infty(G \times M, M)$ such that

$$\theta\left(g, \theta(h, x)\right) = \theta(gh, x) \qquad \text{and} \qquad \theta(e, x) = x$$

holds for all $g, h \in G$ and $x \in M$.

Example F.111. The Conjugation Action Let G be a Lie group. Then G acts on itself via the ***conjugation action***, written $C : G \times G \to G$, defined by

$$C(g, h) := ghg^{-1} = L_g\left(R_{g^{-1}}(h)\right).$$

Definition F.112 (Transitive Action). A left Lie group action of a Lie group G on a smooth manifold M is said to be ***transitive***, iff for all $x, y \in M$ there exists $g \in G$ such that $g \cdot x = y$.

Definition F.113 (Free Action). A left Lie group action of a Lie group G on a smooth manifold M is said to be ***free***, iff $g \cdot x = x$ implies $g = e$ for all $g \in G$ and $x \in M$.

Definition F.114 (Effective Action). A left action $\theta : G \times M \to M$ of a Lie group G on a smooth manifold M is said to be ***effective***, iff $\theta_g = \mathrm{id}_M$ if and only if $g = e$.

It immediately follows from the definitions, that every free action is effective. Recall, that a morphism $f : X \to Y$ in Top is said to be *proper*, iff $f^{-1}(K)$ is compact for every $K \subseteq Y$ compact.

Definition F.115 (Proper Action). A left Lie group action of a Lie group G on a smooth manifold M is said to be ***proper***, iff the map $G \times M \to G \times M$ defined by $(g, x) \mapsto (g, gx)$ is proper.

Finally, we want to give a short introduction in a subject called *representation theory*, which has many applications.

Definition F.116 (Representation). Let G be a Lie group. A ***representation of G*** is defined to be a tuple (V, ρ) consisting of a finite-dimensional real vector space and a Lie group homomorphism $\rho : G \to \mathrm{GL}(V)$.

Definition F.117 (Linear Action). Let V be a finite-dimensional real vector space and G a Lie group. A left action $\theta : G \times V \to V$ is said to be a ***linear action***, iff $\theta_g \in \mathrm{GL}(V)$ for all $g \in G$.

Definition F.118 (Fixed Point). Let θ be a left action of a Lie group G on a smooth manifold M. A ***fixed point of θ*** is defined to be a point $x \in M$, such that $\theta(g, x) = x$ holds for all $g \in G$.

Proposition F.119. *Let θ be a left action of a Lie group G on a smooth manifold M. Suppose $x \in M$ is a fixed point of θ. Then $\rho : G \to \mathrm{GL}(T_x M)$ defined by*

$$\rho(g)(v) := D(\theta_g)_x(v)$$

is a representation of G.

Definition F.120 (The Adjoint Representation). Let G be a Lie group. The representation induced by proposition F.119 at the identity of G from the conjugation action is called the ***adjoint representation of G***, written $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$.

We can go one step further and differentiate the adjoint representation Ad. We write

$$\mathrm{ad} := D(\mathrm{Ad})_e : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}).$$

There is an easy description of this representation.

Proposition F.121. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then* $\mathrm{ad}_v(w) = [v, w]$.

F.7 Distributions

Definition F.122 (Distribution). Let M^n be a smooth manifold and $k \le n$. A ***distribution Δ on M of dimension k*** is defined to be a choice of a k-dimensional subspace $\Delta_x \subseteq T_x M$ for every $x \in M$ such that the following smoothness condition is satisfied: For every $x_0 \in M$ there exists a neighbourhood U of x_0 and k vector fields $X_1, \ldots, X_k \in \mathfrak{X}(U)$ such that

$$\Delta_x = \mathrm{span}_{\mathbb{R}} \left\{ X_1|_x, \ldots, X_k|_x \right\}$$

holds for all $x \in U$.

Example F.123. Nowhere-Vanishing Vector Field Let M be a smooth manifold and $X \in \mathfrak{X}(M)$ nowhere-vanishing, that is, $X_x \neq 0$ for all $x \in M$. Then

$$\Delta_x := \mathrm{span}_{\mathbb{R}} X_x$$

defines a one-dimensional distribution on M.

Definition F.124 (Integral Manifold). Let Δ be a k-dimensional distribution on a smooth manifold M. An immersed submanifold $L \subseteq M$ is said to be an ***integral manifold of Δ***, iff

$$D\iota_x(T_x L) = \Delta_x$$

holds for all $x \in L$.

Definition F.125 (Integrable Distribution). A distribution Δ on a smooth manifold M is said to be ***integrable***, iff the following condition is satisfied: If $X, Y \in \mathfrak{X}(M)$ such that $X_x, Y_x \in \Delta_x$ for all $x \in M$, then also $[X, Y]_x \in \Delta_x$ for all $x \in M$.

Theorem F.126 (The Local Frobenius Theorem). *Let M^n be a smooth manifold and Δ an integrable k-dimensional distribution on M. Then for every $x \in M$ there exists a chart $\varphi : U \to (-1, 1)^n$ centered at x and such that for every $c \in (-1, 1)^{n-k}$ the slice*

$$\left\{ x \in U : x^{k+1}(x) = c^1, \ldots, x^n = c^{n-k} \right\}$$

is an integral manifold of Δ*. Moreover, every connected integral manifold of* Δ *contained in* U *is of this form.*

The following proposition is crucial in the proof of the local Frobenius theorem F.126.

Proposition F.127. *Let* M^n *be a smooth manifold and* $W \subseteq M$ *a non-empty open subset. Suppose* $X_1, \ldots, X_k \in \mathfrak{X}(W)$ *are such that*

(i) *There exists* $x_0 \in W$ *such that* $\left(X_i|_{x_0}\right)$ *is linearly independent.*
(ii) $[X_i, X_j] = 0$ *for all* i *and* j.

Then there exists a chart (U, φ) *contained in* W *about* x_0 *such that* $X_i|_U = \frac{\partial}{\partial x^i}$.

Definition F.128 (Foliation). Let M^n be a smooth manifold. A ***k-dimensional foliation*** $\mathscr{F}$ ***of M*** is a partition of M into k-dimensional connected immersed submanifolds, called the ***leaves***, such that:

(i) The collection of tangent spaces of the leaves defines a distribution on M.
(ii) Any connected integral manifold of this distribution is contained in a leaf.

Example F.129. Let $F : M^n \to N^k$ be a surjective submersion. Then $\left(F^{-1}(y)\right)_{y \in N}$ is an $(n-k)$-dimensional foliation of M.

Theorem F.130 (The Frobenius Theorem). *Let* Δ *be an integrable distribution on* M*. Then* Δ *is induced by a foliation.*

The Frobenius theorem has many applications.

Theorem F.131. *Let* G *be a Lie group with associated Lie algebra* $\mathfrak{g}$*. If* $\mathfrak{h} \subseteq \mathfrak{g}$ *is a Lie subalgebra, then there exists a unique connected Lie subgroup* $H \subseteq G$*, whose associated Lie algebra is* $\mathfrak{h}$*.*

Proof. Consider the distribution Δ on G defined by

$$\Delta_g := \{X_v|_g : v \in \mathfrak{h}\}$$

and apply the Frobenius theorem F.7. □

One particularly important application of the Frobenius theorem F.7 is the next theorem.

Theorem F.132. *Let* G *be a Lie group and* $H \subseteq G$ *a closed subgroup. If* G/H *denotes the set of left cosets of* H *in* G*, then* G/H *is a topological manifold of dimension*

$$\dim(G/H) = \dim G - \dim H$$

endowed with the quotient topology. Moreover, there exists a smooth structure on G/H *making* $\pi : G \to G/H$ *into a smooth submersion.*

Definition F.133 (Homogeneous Space). A ***homogeneous space*** is defined to be a smooth manifold M, such that there exists a Lie group G and a closed subgroup $H \subseteq G$, such that $M \cong G/H$ in Man, where G/H is endowed with the smooth structure of theorem F.132.

Theorem F.134. *Let θ be a transitive left action of a Lie group G on a smooth manifold M. Fix $x \in M$ and let*

$$H := \{g \in G : \theta_g(x) = x\}\,.$$

Then M is a homogeneous space with $M \cong G/H$, where an explicit diffeomorphism is given by $F : G/H \to M$ defined by $F\big(\pi(g)\big) := \theta_g(x)$.

F.8 Vector Bundles

Definition F.135 (Fibre Bundle). A ***fibre bundle*** is defined to be a tuple (E, M, π, F) consisting of smooth manifolds E, M and F together with a surjective map $\pi \in C^\infty(E, M)$ such that there exists an open cover $(U_\alpha)_{\alpha \in A}$ of M and maps $\varphi_\alpha \in C^\infty\big(\pi^{-1}(U_\alpha), F\big)$ for all $\alpha \in A$ such that $(\pi, \varphi_\alpha) : \pi^{-1}(U_\alpha) \to U_\alpha \times F$ is a diffeomorphism. If (E, M, π, F) is a fibre bundle, we call M the ***base space***, E the ***total space*** and F the ***fibre***. Moreover, the family $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ is called a ***bundle atlas*** for (E, M, π, F).

Example F.136 (Trivial Bundle). Let M and F be smooth manifolds. Then

$$\pi : M \times F \to M$$

is a fibre bundle.

The fibre F of a fibre bundle (E, M, π, F) is completely determined by $\pi : E \to M$.

Proposition F.137. *Let (E, M, π, F) be a fibre bundle. Then π is a submersion, $E_x := \pi^{-1}(x)$ is an embedded submanifold of E for all $x \in M$ and $E_x \cong F$ in* Man.

Proof. Let $x \in M$. Then there exists a neighbourhood U_α of x such that $\pi = \pi^1 \circ (\pi, \varphi_\alpha)$. But then π is a submersion as a composition of submersions. Thus an application of the implicit function theorem for manifolds F.58 yields that E_x is an embedded submanifold of E. Now $E_x \cong \{x\} \times F$ by φ_α, but $\{x\} \times F \cong F$ in Man. $\square$

Exercise F.138. Let M and N be smooth manifolds and $G \in C^\infty(M, N)$. Moreover, suppose that (E, N, π) is a fibre bundle. Define

$$G^*E := \big\{(x, p) \in M \times E : G(x) = \pi(p)\big\}\,.$$

Show that (G^*E, M, π^1, F) is a fibre bundle. This fibre bundle is called the ***pullback bundle***.

One particularly important notion concerning vector bundles are sections.

Definition F.139 (Local Section). Let $\pi : E \to M$ be a fibre bundle. A ***local section of E*** is defined to be a section of the fibre bundle $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ for some $U \in \mathcal{O}(M)$. The set of all local sections on U is denoted by $\Gamma(U, E)$.

Definition F.140 (Compatibility). Let (E, M, π) be a fibre bundle and $\theta : G \times F \to F$ an effective Lie group action. Let $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$. We say that $\varphi_\alpha : \pi^{-1}(U_\alpha) \to F$ and $\varphi_\beta : \pi^{-1}(U_\beta) \to F$ are ***(G, θ) - compatible***, iff there exists $\widetilde{\rho}_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that

$$\rho_{\alpha\beta}(x)(y) = \widetilde{\rho}_{\alpha\beta}(x) \cdot y$$

holds for all $x \in U_\alpha \cap U_\beta$ and $y \in F$, where $\rho_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$ is defined by

$$\rho_{\alpha\beta}(x) := \varphi_\alpha|_{E_x} \circ \varphi_\beta|_{E_x}^{-1}.$$

Definition F.141 (Structure Group). A ***structure group*** of a fibre bundle (E, M, π) is a Lie group G such that there exists an effective Lie group action on F and a bundle atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ which is G-compatible.

Definition F.142 (Vector Bundle). Let $k \in \mathbb{N}$. A ***vector bundle of rank k*** is defined to be a fibre bundle $(E, M, \pi, \mathbb{R}^k)$ admitting a matrix Lie subgroup of $\mathrm{GL}(k)$ as a structure group.

As aestetically pleasing the definition of a vector bundle F.142 may be, in practice, it is not that useful. Hence we give an alternative definition.

Definition F.143 (Vector Bundle). Let $\pi : E \to M$ be a fibre bundle with fibre $\mathbb{R}^k$. We say that (E, M, π) is a ***vector bundle of rank k***, iff

(i) For all $x \in M$, the fibre E_x admits the structure of a k-dimensional real vector space.
(ii) For all $x \in M$, $\varphi_\alpha|_{E_x} : E_x \to \mathbb{R}^k$ is an isomorphism of vector spaces.

Example F.144 (The Tangent Bundle). Let M^n be a smooth manifold. Define

$$TM := \coprod_{x \in M} T_x M$$

and $\pi : TM \to M$ by $\pi(x, v) := x$. Then π is certainly surjective. If $(U_\alpha, \psi_\alpha)_{\alpha \in A}$ is a countable atlas of M (this is possible since every smooth manifold is Lindelöf by corollary F.6), define $TU_\alpha := \coprod_{x \in U_\alpha} T_x M$ and

$$\widetilde{\varphi}_\alpha : TU_\alpha \to \mathbb{R}^n$$

by setting

$$\widetilde{\varphi}_\alpha(x, v) := dx^i_\alpha|_x(v) e_i.$$

Then $\left(\pi, \widetilde{\varphi}_\alpha\right) : TU_\alpha \to U_\alpha \times \mathbb{R}^n$ is a bijection since the explicit inverse is given by

$$\left(\pi, \widetilde{\varphi}_\alpha\right)^{-1}(x, v) := \left(x, v^i \left.\frac{\partial}{\partial x_\alpha^i}\right|_x\right).$$

Moreover, if $U_\alpha \cap U_\beta \neq \varnothing$, the transition function

$$\left(\varphi_\alpha \circ \pi, \widetilde{\varphi}_\alpha\right) \circ \left(\varphi_\beta \circ \pi, \widetilde{\varphi}_\beta\right)^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}^n \to \varphi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is given by

$$\left(\varphi_\alpha \circ \pi, \widetilde{\varphi}_\alpha\right) \circ \left(\varphi_\beta \circ \pi, \widetilde{\varphi}_\beta\right)^{-1} \left(\varphi_\beta(x), v\right) = \left(\varphi_\alpha(x), v^i \frac{\partial x_\alpha^j}{\partial x_\beta^i}(x) e_j\right).$$

Hence the transition functions are smooth and by the smooth manifold chart lemma F.17, TM admits a smooth structure that makes it into a smooth manifold of dimension $2n$ and moreover, $\pi : TM \to M$ is a vector bundle of rank n, called the ***tangent bundle***.

Definition F.145 (Vector Bundle Morphism). Let (E, M, π) and (E', M', π') be two vector bundles and $f \in C^\infty(M, M')$. A ***vector bundle morphism along f*** is defined to be a map $F \in C^\infty(E, E')$ such that

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[f]{} & M' \end{array}$$

commutes and $F|_{E_x} : E_x \to E'_{f(x)}$ is linear for all $x \in M$.

Example F.146. The derivative as a Vector Bundle Morphism Let $F \in C^\infty(M, N)$. Then DF is a vectro bundle morphism along F.

Definition F.147 (Vector Bundle Homomorphism). Let (E, M, π) and (E', M, π') be two vector bundles over the same base space. A ***vector bundle homomorphism*** is a vector bundle morphism along id_M.

Definition F.148. A functor

$$\mathscr{F} : \underbrace{\mathsf{Vect} \times \cdots \times \mathsf{Vect}}_{k} \to \mathsf{Vect}$$

is said to be ***smooth***, iff for all $V_1, \ldots, V_k, W_1, \ldots, W_k \in \mathsf{Vect}$, the map

$$\bigoplus_{i=1}^{k} \widetilde{\mathrm{L}}(V_i, W_i) \to \mathrm{L}\left(\mathscr{F}(V_1, \ldots, V_k), \mathscr{F}(W_1, \ldots, W_k)\right)$$

where

$$\widetilde{\mathrm{L}}(V_i, W_i) := \begin{cases} \mathrm{L}(V_i, W_i) & \mathcal{F} \text{ covariant in the } i\text{-th argument,} \\ \mathrm{L}(W_i, V_i) & \mathcal{F} \text{ contravariant in the } i\text{-th argument} \end{cases}$$

given by

$$(T_1, \ldots, T_k) \mapsto \mathcal{F}(T_1, \ldots, T_k)$$

is smooth.

Theorem F.149. *Let*

$$\mathcal{F} : \underbrace{\mathsf{Vect} \times \cdots \times \mathsf{Vect}}_{k} \to \mathsf{Vect}$$

be a smooth functor of mixed variance and $\pi_i : E_i \to M$ vector bundles for $i = 1, \ldots, k$. Then

$$\pi : \coprod_{x \in M} \mathcal{F}(E_1|_x, \ldots, E_k|_x) \to M$$

is a vector bundle.

There are two particularly important constructions from linear algebra in differential topology, namely the tensor product and the exterior product.

Proposition F.150 (The Free Module Functor). *Let $R \in \mathsf{Ring}$. Then the forgetful functor $U : {}_R\mathsf{Mod} \to \mathsf{Set}$ admits a left adjoint.*

Proof. Consider the ***free module functor*** $F : \mathsf{Set} \to {}_R\mathsf{Mod}$ defined as follows:

Step 1: Definition on objects. Let $S \in \mathsf{Set}$ and define

$$F(S) := \left\{ f \in R^S : \operatorname{supp} f \text{ is finite} \right\}.$$

Equipped with pointwise defined addition and multiplication, $F(S)$ is a left R-module. Moreover, there is an inclusion $\iota : S \hookrightarrow U(F(S))$ sending $x \in S$ to the function taking the value one at x and zero else. It is easy to check that $F(S)$ is free on S.

Step 2: Definition on morphisms. Let $f : S \to S'$ in Set, define $F(f) : F(S) \to F(S')$ by setting

$$F(f)\left(\sum_{x \in S} r_x x\right) := \sum_{x \in S} r_x f(x).$$

Step 3: $F \dashv U$. Let $M \in {}_R\mathsf{Mod}$ and $\varphi \in {}_R\mathsf{Mod}(F(S), M)$. Define $\overline{\varphi} \in {}_R\mathsf{Mod}(S, U(M))$ to be the restriction to S of the underlying map of sets. Conversely, if $f \in \mathsf{Set}(S, U(G))$, ***extending by linearity*** yields $\overline{f} \in {}_R\mathsf{Mod}(F(S), M)$ given by

$$\overline{f}\left(\sum_{x \in S} r_x x\right) := \sum_{x \in S} r_x f(x).$$

It is now easy to check that $\overline{\overline{\varphi}} = \varphi$ and $\overline{\overline{f}} = f$ holds. $\square$

Exercise F.151. In the proof of proposition F.150, check functoriality of F and naturality of the bijection ${}_R\mathsf{Mod}(F(S), M) \cong \mathsf{Set}(S, U(M))$.

Definition F.152 (Universal Property of the Tensor Product). Let $V, W \in \mathsf{Vect}$. The ***tensor product of V and W*** is defined to be a tuple $(V \otimes W, \otimes)$, where $V \otimes W \in \mathsf{Vect}$ and $\otimes : V \times W \to V \otimes W$ is a bilinear mapping such that the following universal property in Vect is satisfied:

$$\begin{tikzcd} V \times W \arrow[rr, "\otimes"] \arrow[dr, "\forall f \text{ bilinear}"'] & & V \otimes W \arrow[dl, "\exists! \tilde{f}"] \\ & \forall U & \end{tikzcd}$$

Lemma F.153. *Let $V, W \in \mathsf{Vect}$. The $V^* \otimes W \cong \mathrm{Hom}(V, W)$.*

Proof. Just apply the universal property of the tensor product F.152 to the map $f : V^* \times W \to \mathrm{Hom}(V, W)$ defined by

$$f(\omega, w)(v) := \omega(v)w.$$

□

Definition F.154 (Pairing). Let $V, W \in \mathsf{Vect}$. A bilinear form β is said to be a ***noon-degenerate pairing***, iff $\beta(v, \cdot) = 0$ if and only if $v = 0$, and $\beta(\cdot, w) = 0$ if and only if $w = 0$.

Proposition F.155. *Let $k, l \in \mathbb{N}$ and $V \in \mathsf{Vect}$. Then*

$$\underbrace{V \otimes \cdots \otimes V}_{k} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l} \cong \mathrm{L}(\underbrace{V^*, \ldots, V^*}_{k}, \underbrace{V, \ldots, V}_{l}; \mathbb{R}).$$

Lemma F.156 (Permutation Lemma). *Let $V \in \mathsf{Vect}$, $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$. Then*

$$\begin{aligned}(\omega \wedge \eta)\left(v_1, \ldots, v_{k+l}\right) &= \\ &\frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\omega\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right)\eta\left(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}\right)\end{aligned} \tag{F.8}$$

for all $v_1, \ldots, v_{k+l} \in V$.

One particular advantage of studying vector bundles instead of mere fibre bundles is that the set of sections admits an additional structure.

Lemma F.157. *Let (E, M, π) be a vector bundle. Then for any $U \subseteq M$ open and non-empty, the set $\Gamma(U, E)$ is a vector space and a $C^\infty(U)$-module.*

Proof. Let $\varphi : \pi^{-1}(U) \to \mathbb{R}^k$ be a vector bundle chart and (V, ψ) be a chart on M such that $U \cap V \neq \varnothing$. Then

$$(\psi \circ \pi, \varphi) : \pi^{-1}(U \cap V) \to \psi(U \cap V) \times \mathbb{R}^k$$

is a chart on E compatible with its smooth structure. Since σ is a section, we have that

$$(\psi \circ \pi, \varphi) \circ \sigma \circ \psi^{-1} : \psi(U \cap V) \to \psi(U \cap V) \times \mathbb{R}^k$$

Hence the coordinate representation of σ is of the form $\big(\mathrm{id}, \widetilde{\sigma}\big)$. Hence σ is smooth if and only if $\widetilde{\sigma}$ is smooth for all charts. This readily implies the statement. □

Definition F.158 (Local Frame). Let $\pi : E \to M$ be a vector bundle of rank k and $U \in \mathcal{O}(M)$. A ***local frame for E over U*** is defined to be a family $(e_1, \dots, e_k)$ of sections in $\Gamma(U, E)$ such that $(e_1|_x, \dots, e_k|_x)$ is a basis for E_x for all $x \in U$.

Lemma F.159. *Let $\pi : E \to M$ be vector bundle. Then for every $x \in M$ a local frame exists.*

Proof. Let $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ a vector bundle atlas and assume that the vector bundle is of rank k. Let $(e_1, \dots, e_k)$ denote the standard basis of $\mathbb{R}^k$. For $\alpha \in A$ define $e_i : U_\alpha \to \pi^{-1}(U_\alpha)$ by

$$e_i(x) := \varphi_\alpha|_{E_x}^{-1}(e_i).$$

Then $e_i \in \Gamma(U, E)$ by the argument in the proof of lemma F.157 and $(e_1, \dots, e_k)$ forms a local frame since $\varphi_\alpha|_{E_x}$ is an isomorphism for all $x \in U_\alpha$. □

Theorem F.160 (The Hom $-\Gamma$-Theorem). *Let $\pi : E \to M$ and $\pi' : E' \to M$ be two vector bundles. Then there is a one-to-one correspodence between vector bundle homomorphisms from E to E' and $C^\infty(M)$-linear maps from $\Gamma(E)$ to $\Gamma(E')$. Explicitely, if $\Phi : E \to E'$ is a vector bundle homomorphism, then the induced map $\chi : \Gamma(E) \to \Gamma(E')$ is given by*

$$\chi(\sigma) = \Phi \circ \sigma.$$

If M is a smooth manifold, so is U for any open subset $U \subseteq M$. Most of the constructions we performed so far also work for this induced smooth structure on U. However, it is tedious to explicitely mention this all the time. So we introduce now a foundational notion of a mathematical field called *Algebraic Geometry*.

Let $(X, \mathcal{T}) \in \mathsf{Top}$. Then denote by $\mathcal{O}(X)$ the category of open subsets of X, that is the category associated to the poset $(\mathcal{T}, \subseteq)$ (see [8, 24]). Recall, that for any two categories $\mathcal{C}$ and $\mathcal{D}$, there exists the functor category $\mathcal{D}^{\mathcal{C}}$ from $\mathcal{C}$ to $\mathcal{D}$ (see [8, 30]).

Definition F.161 (Presheaf). Let $X \in \mathsf{Top}$ and $\mathcal{C}$ be a category. A ***presheaf of $\mathcal{C}$ on X*** is defined to be a contravariant functor $\mathcal{O}(X) \to \mathcal{C}$. The ***category of presheaves of $\mathcal{C}$ on X*** is denoted by $\mathsf{PSh}(X; \mathcal{C})$.

Remark F.162. Let $F : \mathcal{O}(X) \to \mathcal{C}$, where $\mathcal{C}$ is the category of a mathematical structure, that is $\mathsf{Grp}, \mathsf{Ring}, \mathsf{Vect}, \dots$, be a presheaf of $\mathcal{C}$ on X. Then if $U \subseteq V$ for $U, V \in \mathcal{O}(X)$, we simply write $f|_U$ for $F(U \hookrightarrow V)(f)$, where $f \in F(V)$.

Example F.163 (Presheaf of Sections of a Vector Bundle). Let (E, M, π) be a vector bundle. Define $\mathcal{E}_E : \mathcal{O}(X) \to \mathsf{Vect}$ on objects $U \in \mathcal{O}(X)$ by $\mathcal{E}_E(U) := \Gamma(U, E)$ and on morphisms by restriction.

Definition F.164 (Local Operator). Let $\pi : E \to M$ and $\pi' : E' \to M$ be two vector bundles. An $\mathbb{R}$-linear operator $\chi : \Gamma(E) \to \Gamma(E')$ is said to be a ***local operator***, iff the following condition is satisfied: if $\sigma \in \Gamma(E)$ such that $\sigma|_U = 0$ for some $U \in \mathcal{O}(M)$, then also $\chi(\sigma)|_U = 0$.

Proposition F.165. *Let $\pi : E \to M$ and $\pi' : E' \to M$ be two vector bundles. Every local operator $\chi : \Gamma(E) \to \Gamma(E')$ uniquely induces a morphism of presheaves $\chi : \mathcal{E}_E \to \mathcal{E}_{E'}$.*

Proof. Let $U \in \mathcal{O}(M)$. Define a morphism $\chi_U : \mathcal{E}_E(U) \to \mathcal{E}_{E'}(U)$ by

$$\chi_U(\sigma)(x) := \chi\left(\widetilde{\sigma}\right)(x)$$

for all $x \in U$ where $\widetilde{\sigma} \in \Gamma(E)$ is any extension of σ in a neighbourhood of x. Since χ is a local operator, this is well defined. It is easy to check that $(\chi_U)_{U \in \mathcal{O}(M)}$ is a natural transformation. □

Proposition F.166. *Let $\pi : E \to M$ and $\pi' : E' \to M$ be two vector bundles. Every $C^\infty(M)$-linear operator $\chi : \Gamma(E) \to \Gamma(E')$ is a local operator.*

Proof. The usual argument via bump functions. □

Definition F.167 (Point Operator). Let $\pi : E \to M$ and $\pi' : E' \to M$ be two vector bundles. An $\mathbb{R}$-linear operator $\chi : \Gamma(E) \to \Gamma(E')$ is said to be a ***point operator***, iff the following condition is satisfied: if $\sigma \in \Gamma(E)$ such that $\sigma_x = 0$ for some $x \in M$, then also $\chi(\sigma)_x = 0$.

Proposition F.168. *Let $\pi : E \to M$ and $\pi' : E' \to M$ be two vector bundles. Every $C^\infty(M)$-linear map $\Gamma(E) \to \Gamma(E')$ is a point operator.*

Proof. Let $\sigma \in \Gamma(E)$ and suppose that $\sigma_x = 0$. By lemma F.159, there exists a local frame (e_i) on a neighbourhood U about x. Then $\sigma|_U = f^i e_i$ for some $f^i \in C^\infty(U)$ and $f^i(x) = 0$ for all i. By proposition F.166, χ is a local operator, and thus using proposition F.165 we compute

$$\chi(\sigma)(x) = \chi_U(\sigma|_U)(x) = f^i(x)\chi_U(e_i)(x) = 0$$

since it is easy to show via a bump function argument that χ_U is $C^\infty(U)$-linear. □

Proof (of theorem F.160). Let $\chi : \Gamma(E) \to \Gamma(E')$ be $C^\infty(M)$-linear. Define $\Phi : E \to E'$ as follows. If $x \in M$, defined

$$\Phi(p) := \chi(\sigma)(x)$$

for $p \in E_x$, where $\sigma \in \Gamma(E)$ such that $\sigma_x = p$. This is well-defined, since χ is a point opertaor by proposition F.168. Moreover, Φ is fibre-preserving and linear on the fibres. Also one can show that Φ is smooth, hence a vector bundle homomorphism. □

F.9 Sheaves

Definition F.169 (Sheaf). Let $X \in \mathsf{Top}$ and F a presheaf of Set (Grp, Ring, Vect, …) on X. We say that F is a ***sheaf on X***, iff for all $U \in \mathcal{O}(X)$ the following ***gluing condition*** is satisfied: Given any open cover $(U_\alpha)_{\alpha\in A}$ for U and $f_\alpha \in F(U_\alpha)$ for all $\alpha \in A$ such that

$$f_\alpha|_{U_\alpha\cap U_\beta} = f_\beta|_{U_\alpha\cap U_\beta}$$

for all $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \varnothing$, then there exists a unique element $f \in F(U)$ with $f|_{U_\alpha} = f_\alpha$ for all $\alpha \in A$. A morphism of sheaves is simply defined to be a morphism of presheaves.

From example F.163 we already know that $\mathcal{E}_E$ is a presheaf. In fact, more is true.

Proposition F.170. *Let (E, M, π) be a vector bundle. Then $\mathcal{E}_E : \mathcal{O}(M) \to \mathsf{Vect}$ is a sheaf.*

Example F.171 (Tensor Sheaf). Let M be a smooth manifold. Then tensor fields of type (k, l) can be assembled in a sheaf by proposition F.170. Denote this sheaf by $\mathcal{T}_M^{k,l} := \mathcal{E}_{T^{(k,l)}TM}$. We can assemble these sheaves in a total sheaf $\mathcal{T}_M : \mathcal{O}(M) \to {}_{\mathbb{R}}\mathsf{GAlg}$ by setting

$$\mathcal{T}_M(U) := \bigoplus_{k,l\geq 0} \mathcal{T}_M^{k,l}(U).$$

We call $\mathcal{T}_M$ the ***tensor algebra sheaf on M***.

Proposition F.172 (The Tensor Characterisation Lemma). *Let M be a smooth manifold and $U \in \mathcal{O}(M)$ non-empty. Then there is a one-to-one correspondence between $\mathcal{T}^{k,l}(U)$ and $C^\infty(U)$-multilinear maps*

$$\underbrace{\Omega^1(U) \times \cdots \times \Omega^1(U)}_{k} \times \underbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}_{l} \to C^\infty(U).$$

Example F.173 (Sheaf of Differential Forms). Let M^n be a smooth manifold and let $0 \leq k \leq n$. Then by F.170, $\mathcal{E}_{\Lambda^k(T^*M)}$ is a sheaf. This sheaf is denoted by Ω_M^k and called the ***sheaf of differential k-forms***. As with tensor fields in example F.171,we can define a sheaf $\Omega_M : \mathcal{O}(M) \to {}_{\mathbb{R}}\mathsf{GSCAlg}$ by

$$\Omega_M(U) := \bigoplus_{0\leq k\leq n} \Omega_M^k(U).$$

Proposition F.174 (The Differential Form Characterisation Lemma). *Let M be a smooth manifold and $U \in \mathcal{O}(M)$ non-empty. Then there is a one-to-one correspondence between $\Omega^l(U)$ and alternating $C^\infty(U)$-multilinear maps*

$$\underbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}_{l} \to C^\infty(U).$$

F.10 The Lie Derivative

Definition F.175 (Pullback). Let $l \in \mathbb{N}$ and $F \in C^\infty(M, N)$. Define

$$F^* : \mathcal{T}^{0,l}(N) \to \mathcal{T}^{0,l}(M)$$

by

$$(F^* A)_x(v_1, \ldots, v_l) := A_{F(x)}\left(DF_x(v_1), \ldots, DF_x(v_l)\right)$$

for all $x \in M$ and $v_1, \ldots, v_l \in T_x M$, if $k \geq 1$ and by $F^* f := f \circ F$ if $k = 0$. We call $F^* A$ the ***pullback of A under F***.

To extend the notion of a pullback of a tensor field to arbitrary tensor fields, we must impose an additional condition on the map.

Definition F.176 (Cotangent Lift). Let $F \in C^\infty(M, N)$ be a diffeomorphism. Define a map $DF^\dagger : T^*M \to T^*N$ by

$$DF^\dagger(x, \xi)(v) := \xi\left((DF_x)^{-1}(v)\right)$$

for all $v \in T_{F(x)}N$. This map is called the ***cotangent lift of the diffeomorphism F***.

Definition F.177 (Pullback). Let $k, l \in \mathbb{N}$ and $f \in C^\infty(M, N)$ a diffeomorphism. Define

$$F^* : \mathcal{T}^{k,l}(N) \to \mathcal{T}^{k,l}(M)$$

by

$$A \mapsto (F^* A)_x\left(\xi^1, \ldots, \xi^k, v_1, \ldots, v_l\right)$$

for all $x \in M$, $\xi^1, \ldots, \xi^k \in T_x^* M$ and $v_1, \ldots, v_l \in T_x M$, if $k \geq 1$, where the latter is defined to be

$$A_{F(x)}\left(DF^\dagger(\xi^1), \ldots, DF^\dagger(\xi^k), DF_x(v_1), \ldots, DF_x(v_k)\right)$$

We call $F^* A$ the ***pullback of A under F***. Extending by linearity yields a morphism

$$F^* : \mathcal{T}(N) \to \mathcal{T}(M).$$

Proposition F.178. *Let F be a diffeomorphism. Then*

$$F^*(A \otimes B) = F^* A \otimes F^* B.$$

Definition F.179 (Pushforward). Let $F \in C^\infty(M, N)$ be a diffeomorphism. Define

$$F_* : \mathcal{T}(M) \to \mathcal{T}(N)$$

by

$$F_* A := \left(F^{-1}\right)^* A.$$

This morphism is called the ***pushforward by F***.

Proposition F.180. *Let $F \in C^\infty(M,N)$, $A \in \mathcal{T}^{0,k}(N)$ and let $X_1,\dots,X_k \in \mathfrak{X}(M)$, $Y_1,\dots,Y_k \in \mathfrak{X}(N)$ such that X_i is F-related to Y_i for $i=1,\dots,k$. Then*

$$(F^*A)(X_1,\dots,X_k) = A(Y_1,\dots,Y_k)\circ F.$$

Proposition F.181. *Let $F \in C^\infty(M,N)$ and $X \in \mathfrak{X}(N)$. Then F^*X is F-related to X.*

Definition F.182 (Trace). Let $v_1\otimes\cdots\otimes v_k\otimes\omega^1\otimes\cdots\otimes\omega^l \in T^{k,l}V$ for some vector space V such that $k,l \geq 1$. Define a ***trace of A*** to be the tensor $\operatorname{Tr} A \in T^{k-1,l-1}V$ defined by

$$\operatorname{Tr} A := \omega^j(v_i)v_1\otimes\cdots\otimes\widehat{v}_i\otimes\cdots\otimes v_k\otimes\omega^1\otimes\cdots\otimes\widehat{\omega}^j\otimes\cdots\otimes\omega^l$$

for some $1\leq i\leq k$ and $1\leq j\leq l$. Extend this map by linearity to $T^{k,l}V$ and then pointwise to a sheaf morphism $\operatorname{Tr}:\mathcal{T}^{k,l}_M \to \mathcal{T}^{k-1,l-1}_M$.

Proposition F.183 (Traces commute with Pullbacks). *Let $F \in C^\infty(M,N)$ and $A \in \mathcal{T}^{k,l}(N)$. Then*

$$\operatorname{Tr}(F^*A) = F^*(\operatorname{Tr} A)$$

for any trace Tr.

Definition F.184 (Tensor Derivation). A ***tensor derivation on a smooth manifold M*** is defined to be a sheaf morphism $\mathcal{D}:\mathcal{T}_M \to \mathcal{T}_M$ that preserves type and satisfies:

(i) For all $U \in \mathcal{O}(M)$, $\mathcal{D}_U$ commutes with all contractions of $\mathcal{T}_M(U)$.
(ii) For all $U \in \mathcal{O}(M)$, $\mathcal{D}_U$ is a derivation, that is

$$\mathcal{D}_U(A\otimes B) = \mathcal{D}_U A\otimes B + A\otimes\mathcal{D}_U B$$

holds for all $A,B \in \mathcal{T}(U)$.

Lemma F.185. *Let $\mathcal{D}$ be a tensor derivation, $U \in \mathcal{O}(M)$ and $A \in \mathcal{T}^{k,l}(U)$. Then for all $\omega^1,\dots,\omega^k \in \Omega^1(U)$ and $X_1,\dots,X_l \in \mathfrak{X}(U)$ we have that*

$$\begin{aligned}\mathcal{D}_U(A)\left(\omega^1,\dots,\omega^k,X_1,\dots,X_l\right) &= \mathcal{D}_U\left(A\left(\omega^1,\dots,\omega^k,X_1,\dots,X_l\right)\right)\\ &\quad-\sum_{i=1}^k A\left(\omega^1,\dots,\mathcal{D}_U\left(\omega^i\right),\dots,\omega^k,X_1,\dots,X_l\right)\\ &\quad-\sum_{i=1}^l A\left(\omega^1,\dots,\omega^k,X_1,\dots,\mathcal{D}_U(X_i),\dots,X_l\right).\end{aligned}$$

Proposition F.186. *Let $\mathcal{D}$ and $\mathcal{D}'$ be two tensor derivations on a smooth manifold which agree on functions and vector fields. Then $\mathcal{D}=\mathcal{D}'$.*

Proof. By the contraction lemma F.185 we have that tensor derivations are uniquely characterised by their action on functions, vector fields and covector fields. In fact, only on functions and vector fields. Indeed, if $\omega \in \Omega^1(U)$, then again by the contraction lemma F.185 we have that

$$\begin{aligned}\mathcal{D}_U(\omega)(X) &= \mathcal{D}_U\left(\omega(X)\right) - \omega\left(\mathcal{D}_U(X)\right)\\ &= \mathcal{D}'_U\left(\omega(X)\right) - \omega\left(\mathcal{D}'_U(X)\right)\\ &= \mathcal{D}'_U(\omega)(X)\end{aligned}$$

for all $X \in \mathfrak{X}(U)$. □

Proposition F.187. *Let $\mathcal{D}$ be a sheaf morphism on functions and vector fields. If*

$$\mathcal{D}_U(fg) = \mathcal{D}_U(f)g + f\mathcal{D}_U(g) \quad \text{and} \quad \mathcal{D}_U(fX) = \mathcal{D}_U(f)X + f\mathcal{D}_U(X)$$

holds for all $U \in \mathcal{O}(M)$, $f, g \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, then $\mathcal{D}$ extends uniquely to a tensor derivation on M.

Theorem F.188 (The Lie Derivative). *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. Then there exists a unique tensor derivation*

$$\mathcal{L}_X : \mathcal{T}_M \to \mathcal{T}_M$$

on M such that

$$\mathcal{L}_X f = Xf \qquad \text{and} \qquad \mathcal{L}_X Y = [X, Y]$$

*for all $U \in \mathcal{O}(M)$, $f \in C^\infty(U)$ and $Y \in \mathfrak{X}(U)$. This tensor derivation is called the **Lie derivative**.*

Proof. This immediately follows from proposition F.187 since

$$\begin{aligned}\left(\mathcal{L}_X(fY)\right)g &= [X, fY]\,g\\ &= X\left((fY)(g)\right) - fY\left(X(g)\right)\\ &= X\left(fY(g)\right) - fY\left(X(g)\right)\\ &= X(f)Y(g) + f\left(X\left(Y(g)\right)\right) - fY\left(X(g)\right)\\ &= X(f)Y(g) + f\,[X, Y]\,g\end{aligned}$$

implies

$$\mathcal{L}_X(fY) = \mathcal{L}_X(f)Y + f\mathcal{L}_X Y.$$

The next proposition shows why the name Lie derivative is appropriate.

Proposition F.189. *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$ with flow θ. Then*

$$\mathcal{L}_X A = \left.\frac{d}{dt}\right|_{t=0} \theta_t^*(A) =: \mathcal{L}'_X(A)$$

for any $A \in \mathcal{T}^{k,l}(U)$, $U \in \mathcal{O}(M)$.

Crucial in the proof of proposition F.189 are the following two results.

Lemma F.190. *Let $\varepsilon > 0$ and $f \in C^\infty\left((-\varepsilon,\varepsilon)\times U\right)$, where $U \in \mathcal{O}(M)$ for M a smooth manifold and $f(0,x) = 0$ for all $x \in U$. Then there exists a function $g \in C^\infty\left((-\varepsilon,\varepsilon)\times U\right)$ such that*

$$f(t,x) = tg(t,x) \qquad \text{and} \qquad \frac{\partial f}{\partial t}(0,x) = g(0,x)$$

holds for all $(t,x) \in (-\varepsilon,\varepsilon)\times U$.

Proof. Just set

$$g(t,x) := \int_0^1 \frac{\partial f}{\partial t}(st,x)ds.$$

□

Proposition F.191. *Let*

$$\mathcal{A} : \mathcal{T}_M^{k,l} \times \mathcal{T}_M^{k',l'} \to \mathcal{T}_M^{k'',l''}$$

be a $\mathcal{C}_M^\infty$-bilinear sheaf homomorphism. Moreover, suppose that for every local diffeomorphism $F \in C^\infty(U,V)$ for $U, V \in \mathcal{O}(M)$ we have that

$$F^*\left(\mathcal{A}_V(A,B)\right) = \mathcal{A}_U\left(F^*A, F^*B\right).$$

Then

$$\mathcal{L}'_X\left(\mathcal{A}(A,B)\right) = \mathcal{A}\left(\mathcal{L}'_X(A),B\right) + \mathcal{A}\left(A,\mathcal{L}'_X(B)\right).$$

Proof. We compute

$$\begin{aligned}
\mathcal{L}'_X\left(\mathcal{A}(A,B)\right) &= \left.\frac{d}{dt}\right|_{t=0} \theta_t^*\left(\mathcal{A}(A,B)\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \mathcal{A}\left(\theta_t^*A, \theta_t^*B\right) \\
&= \lim_{t\to 0} \frac{\mathcal{A}\left(\theta_t^*A, \theta_t^*B\right) - \mathcal{A}(A,B)}{t} \\
&= \lim_{t\to 0} \frac{\mathcal{A}\left(\theta_t^*A - A, \theta_t^*B\right) + \mathcal{A}(A, \theta_t^*B - B)}{t} \\
&= \mathcal{A}\left(\lim_{t\to 0}\frac{\theta_t^*A - A}{t}, B\right) + \mathcal{A}\left(A, \lim_{t\to 0}\frac{\theta_t^*B - B}{t}\right) \\
&= \mathcal{A}\left(\mathcal{L}'_X(A),B\right) + \mathcal{A}\left(A,\mathcal{L}'_X(B)\right).
\end{aligned}$$

□

Proof. We make use of F.186. Let $f \in C^\infty(M)$. Then for $x \in M$ we compute

$$\left.\frac{d}{dt}\right|_{t=0}\left(\theta_t^*f\right)_x = \left.\frac{d}{dt}\right|_{t=0}\left(f\circ\theta_t\right)(x) = \left(f\circ\theta^x\right)'(0) = (\theta^x)'(0)f = X|_x f.$$

Next, we show that $\mathcal{L}'_X Y = [X, Y]$. Let $x \in M$. Then there exists $\varepsilon > 0$ and neighbourhood U of x in M such that the flow θ of X is defined on $(-\varepsilon, \varepsilon) \times U$. Let $f \in C^\infty(U)$. Applying lemma F.190 to the function $f \circ \theta_t - f$ we compute

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} (\theta_t^* Y) f &= \left.\frac{d}{dt}\right|_{t=0} D(\theta_{-t}) \left(Y|_{\theta_t}\right) f \\
&= \left.\frac{d}{dt}\right|_{t=0} Y|_{\theta_t} \left(f \circ \theta_{-t}\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} Y|_{\theta_t} \left(f - t h_{-t}\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} Y|_{\theta_t} f - Y|_{\theta_0} h_0 \\
&= \left.\frac{d}{dt}\right|_{t=0} (Yf) \circ \theta_t - Y\left(\mathcal{L}_X f\right) \\
&= \mathcal{L}_X (Yf) - Y\left(\mathcal{L}_X f\right) \\
&= (XY - YX) f \\
&= [X, Y] f,
\end{aligned}$$

since by continuity of h, we have that

$$h_0 = \lim_{t \to 0} h_t = \left.\frac{d}{dt}\right|_{t=0} (f \circ \theta_t) = \mathcal{L}_X f.$$

$\square$

Proposition F.192. *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. If A is an arbitrary tensor field on M, we have that*

$$\left.\frac{d}{dt}\right|_{t=t_0} \theta_t^*(A) = \theta_{t_0}^* \left(\mathcal{L}_X A\right).$$

Lemma F.193. *Let $F \in C^\infty(M, N)$ for some smooth manifolds M and N, and $\omega, \eta \in \Omega(N)$. Then*

$$F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta.$$

Proof. Immediate from the permutation lemma F.156 and the definitions. $\square$

Proposition F.194. *Let $X \in \mathfrak{X}(M)$ for some smooth manifold M and $\omega, \eta \in \Omega(M)$. Then*

$$\mathcal{L}_X(\omega \wedge \eta) = \mathcal{L}_X \omega \wedge \eta + \omega \wedge \mathcal{L}_X \eta.$$

Proof. Using proposition F.189 together with lemma F.193 yields

$$
\begin{aligned}
\mathcal{L}_X(\omega \wedge \eta) &= \frac{d}{dt}\bigg|_{t=0} \theta_t^*(\omega \wedge \eta) \\
&= \frac{d}{dt}\bigg|_{t=0} \left(\theta_t^*\omega \wedge \theta_t^*\eta\right) \\
&= \frac{d}{dt}\bigg|_{t=0} \theta_t^*\omega \wedge \theta_0^*\eta + \theta_0^*\omega \wedge \frac{d}{dt}\bigg|_{t=0} \theta_t^*\eta \\
&= \frac{d}{dt}\bigg|_{t=0} \theta_t^*\omega \wedge \eta + \omega \wedge \frac{d}{dt}\bigg|_{t=0} \theta_t^*\eta \\
&= \mathcal{L}_X\omega \wedge \eta + \omega \wedge \mathcal{L}_X\eta
\end{aligned}
$$

since $\theta_0^* = \mathrm{id}_M^* = \mathrm{id}$. □

F.11 Differential Forms

Differential forms are a key technical tool in differential geometry. In contrast to mere tensor fields, they can be both differentiated and integrated.

Definition F.195. Let M be a smooth manifold and $l \in \mathbb{Z}$. A ***graded derivation of degree l on M*** is defined to be a sheaf morphism $\mathcal{D} : \Omega_M \to \Omega_M$ satisfying:

(i) If $\omega \in \Omega^k(U)$, then $\mathcal{D}_U(\omega) \in \Omega^{k+l}(U)$.
(ii) If $\omega \in \Omega^k(U)$ and $\eta \in \Omega(U)$, then

$$\mathcal{D}_U(\omega \wedge \eta) = \mathcal{D}_U(\omega) \wedge \eta + (-1)^{kl}\omega \wedge \mathcal{D}_U(\eta).$$

Note that by the contraction lemma F.185, the Lie derivative $\mathcal{L}_X$ can be seen as a sheaf morphism $\mathcal{L}_X : \Omega_M \to \Omega_M$ for any vector field $X \in \mathfrak{X}(M)$ on a smooth manifold M.

Example F.196 (The Lie Derivative). Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. Then the Lie derivative $\mathcal{L}_X$ is a graded derivation of degree 0 by proposition F.194.

Lemma F.197. *Let $\mathcal{D}$ and $\mathcal{D}'$ be two graded derivations of degree k and l, respectively. Then*

$$\mathcal{D} \circ \mathcal{D}' - (-1)^{kl}\mathcal{D}' \circ \mathcal{D}$$

is a graded derivation of degree $k + l$.

Exercise F.198. Prove lemma F.197.

Proposition F.199. *Let M be a smooth manifold and suppose that $\mathcal{D}$ and $\mathcal{D}'$ are two graded derivations on M of the same degree which coincide on functions and exact 1-forms, that is, forms $\omega \in \Omega^1(U)$ such that there exists $f \in C^\infty(U)$ with $\omega = df$, for $U \in \mathcal{O}(M)$. Then $\mathcal{D} = \mathcal{D}'$.*

Proof. Every graded derivation is entirely determined by what it does on a chart on expressions of the form

$$f dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Theorem F.200 (The Exterior Differential). *Let M be a smooth manifold. Then there exists a unique graded derivation $d : \Omega_M \to \Omega_M$ of degree 1 such that*

$$d_U(f) = df \qquad \textit{and} \qquad d \circ d = 0$$

*holds for all $f \in C^\infty(U)$. This graded derivation is called the **exterior differential**.*

Proof. It is enough to define $d_U : \Omega^k(U) \to \Omega^{k+1}(U)$ for some chart (U, φ). If $\omega = f_I dx^I$ in this chart, define

$$d_U(\omega) := df_I \wedge dx^I,$$

where I denotes an increasing multiindex. If $\omega \in \Omega^k(U)$, then for any $\eta \in \Omega(U)$, where $\eta = g_J dx^J$, we compute

$$\begin{aligned}
d_U(\omega \wedge \eta) &= d_U\left(f_I g_J dx^I \wedge dx^J\right) \\
&= d\left(f_I g_J\right) \wedge dx^I \wedge dx^J \\
&= \left((df_I)g_J + f_I(dg_J)\right) \wedge dx^I \wedge dx^J \\
&= g_J(df_I) \wedge dx^I \wedge dx^J + f_I dg_J \wedge dx^I \wedge dx^J \\
&= d_U(\omega) \wedge \eta + (-1)^k f_I dx^I \wedge dg_J \wedge dx^J \\
&= d_U(\omega) \wedge \eta + (-1)^k \omega \wedge d_U(\eta).
\end{aligned}$$

Moreover, for any $f \in C^\infty(U)$ we compute

$$\begin{aligned}
d_U(d_U f) &= d(df) \\
&= d\left(\frac{\partial f}{\partial x^j} dx^j\right) \\
&= d\left(\frac{\partial f}{\partial x^j}\right) \wedge dx^j \\
&= \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j \\
&= \sum_{i<j}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i}\right) dx^i \wedge dx^j \\
&= 0
\end{aligned}$$

by Schwarz and by the previous computation it follows that $d_U \circ d_U = 0$. Lastly, d_U is well-defined. Indeed, by proposition F.199 it is enough to check that if we have two charts (U, φ) and (V, ψ) with $U \cap V \neq \varnothing$, then the graded derivation $d_{U \cap V}$

on $\Omega(U \cap V)$ is the same with respect to both coordinates for smooth functions and exact 1-forms. But this is immediate by the previous computation. $\square$

Proposition F.201. *Let M and N be smooth manifolds and $F \in C^\infty(M, N)$. Then for $\omega \in \Omega(M)$ we have that*

$$F^*(d\omega) = d(F^*\omega).$$

Proof. First we prove this for functions $f \in C^\infty(N)$. Let $X \in \mathfrak{X}(M)$. Then we compute

$$\begin{aligned} F^*(df)(X) &= df\left(DF(X)\right) \\ &= DF(X)f \\ &= X(f \circ F) \\ &= d(f \circ F)(X) \\ &= d(F^* f)(X). \end{aligned}$$

Thus in a chart (U, φ) for $\omega = f_I\, dx^I$ we compute

$$\begin{aligned} F^*(d\omega) &= F^*\left(df_I \wedge dx^I\right) \\ &= F^*(df_I) \wedge F^*\left(dx^I\right) \\ &= d(F^* f_I) \wedge F^*\left(dx^I\right) \\ &= d\left((F^* f_I) F^*\left(dx^I\right)\right) \\ &= d(F^*\omega). \end{aligned}$$

Proposition F.202. *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. Then*

$$\mathcal{L}_X \circ d = d \circ \mathcal{L}_X.$$

Proof. By lemma F.197, $\mathcal{L}_X \circ d - d \circ \mathcal{L}_X$ is a graded derivation of degree 1. Thus by proposition F.199 it is enough to show that $\mathcal{L}_X \circ d - d \circ \mathcal{L}_X$ vanishes for smooth functions and exact 1-forms. Note that by the contraction lemma F.185, we have that

$$\mathcal{L}_X(\omega)(Y) = X\left(\omega(Y)\right) - \omega\left([X, Y]\right)$$

for all $Y \in \mathfrak{X}(M)$. Hence

$$\begin{aligned} \left(\mathcal{L}_X \circ d - d \circ \mathcal{L}_X\right) f(Y) &= \mathcal{L}_X(df)(Y) - d(Xf)(Y) \\ &= X\left(df(Y)\right) - \omega\left([X, Y]\right) - d(Xf)(Y) \\ &= XYf - \omega\left([X, Y]\right) - YXf \\ &= 0. \end{aligned}$$

for all $f \in C^\infty(M)$. Consider an exact form df. Then by the previous computation we have that

$$\left(\mathcal{L}_X \circ d - d \circ \mathcal{L}_X\right) df = -d\left(\mathcal{L}_X(df)\right) = -(d \circ d)\left(\mathcal{L}_X f\right) = 0.$$

Proposition F.203 (Interior Multiplication). *Let M be a smooth manifold and $x \in \mathfrak{X}(M)$. Then there exists a unique graded derivation $i_X : \Omega_M \to \Omega_M$ of degree -1 such that*

$$i_X(f) = 0 \qquad \text{and} \qquad i_X(\omega) = \omega(X)$$

for all smooth functions f and 1-forms ω.

Proof. Let $U \in \mathcal{O}(M)$, $k \in \mathbb{N}$ and $X_1, \ldots, X_k \in \mathfrak{X}(U)$. For any $\omega \in \Omega^{k+1}(U)$ define

$$i_X(\omega)\left(X_1, \ldots, X_k\right) := \omega(X, X_1, \ldots, X_k). \tag{F.9}$$

Proposition F.204 (Cartan's Magic Formula). *Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. Then*

$$\mathcal{L}_X = d \circ i_X + i_X \circ d.$$

Proof. By lemma F.197, $d \circ i_X + i_X \circ d$ is a graded derivation of degree 0. Hence by proposition F.199, it is enough to check that $d \circ i_X + i_X \circ d$ and $\mathcal{L}_X$ coincide on smooth functions and exact 1-forms. Let $f \in C^\infty(M)$. Then we compute

$$\left(d \circ i_X + i_X \circ d\right) f = i_X(df) = df(X) = Xf = \mathcal{L}_X.$$

Moreover

$$\left(d \circ i_X + i_X \circ d\right) df = d\left(i_X(df)\right) = d\left(df(X)\right) = d(Xf)$$

which coincides with

$$\mathcal{L}_X(df) = d\left(\mathcal{L}_X f\right) = d(Xf)$$

by proposition F.202. □

Finally, using Cartan's magic formula **??**, we can give a coordinate free description of the exterior differential.

Proposition F.205. *Let M be a smooth manifold, $k \in \mathbb{N}$ and $\omega \in \Omega^k(M)$. Then for all $X_0, \ldots, X_k \in \mathfrak{X}(M)$ we have*

$$\begin{aligned} d\omega(X_0, \ldots, X_k) = &\sum_{i=0}^{k} (-1)^i X_i\left(\omega(X_0, \ldots, \widehat{X}_i, \ldots, X_k)\right) \\ &+ \sum_{i<j} (-1)^{i+j} \omega\left([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k\right). \end{aligned}$$

Exercise F.206. Prove proposition F.205.

F.12 Orientability and Orientations

Definition F.207 (Orientation). Let V be a real vector space. An ***orientation of V*** is defined to be a choice of one of the two connected components of $\Lambda^{\dim V}(V) \setminus \{0\}$.

Definition F.208 (Determinant Functor). Define a functor

$$\det : \mathsf{Vect}^{\geq 1} \to \mathsf{Vect}^1$$

on objects by $\det V := \Lambda^{\dim V} V$ and on morphisms $L : V \to W$ as follows: If $\dim V = \dim W = n$, then set

$$\det L(v_1 \wedge \cdots \wedge v_n) := Lv_1 \wedge \cdots \wedge Lv_n$$

and to be the zero-morphism otherwise.

Proposition F.209. *Let $\pi : E \to M$ be a vector bundle of rank k, $k \geq 1$. The following conditions are equivalent:*

(a) *There exists a nowhere-vanishing section $\sigma \in \Gamma(\det E^*)$.*
(b) *The structure group of E can be reduced to* $\mathrm{GL}^+(k)$.
(c) *The bundle* $\det E^* \to M$ *is trivial.*

Proof. That (i)⇔(iii) is trivial. To prove (i)⇒(ii), suppose that $\sigma \in \Gamma(\det E^*)$ is nowhere-vanishing. Suppose $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ is a vector bundle chart such that each U_α is connected. Moreover, let $(e_1^\alpha, \ldots, e_k^\alpha)_{\alpha \in A}$ be a family of corresponding local frames. Since σ is nowhere-vanishing, the function

$$\sigma\left(e_1^\alpha, \ldots, e_k^\alpha\right)$$

is either positive or negative. If it is negative, substitute the local frame $(e_1^\alpha, \ldots, e_k^\alpha)$ with the local frame $(-e_1^\alpha, \ldots, e_k^\alpha)$ and also the corresponding vector bundle chart. Thus $\sigma\left(e_1^\alpha, \ldots, e_k^\alpha\right)$ is positive for all $\alpha \in A$. Suppose now that $U_\alpha \cap U_\beta \neq \varnothing$. Then the transition matrix between the bases $\left(e_i^\alpha(x)\right)$ and $\left(e_i^\beta(x)\right)$ of E_x for $x \in U_\alpha \cap U_\beta$ is given by $\rho_{\alpha\beta}(x)$. But

$$\sigma\left(e_1^\beta, \ldots, e_k^\beta\right)(x) = \left(\det \rho_{\alpha\beta}(x)\right) \sigma\left(e_1^\alpha, \ldots, e_k^\alpha\right)(x)$$

and so $\det \rho_{\alpha\beta}(x)$.
Conversly, to prove (ii)⇔(i), suppose that $(U_\alpha, \varphi_\alpha)$ is a vector bundle atlas which is $\mathrm{GL}^+(k)$-compatible. Let $(\psi_\alpha)_{\alpha \in A}$ be a partition of unity subordinate to the open cover $(U_\alpha)_{\alpha \in A}$. Define $\sigma \in \Gamma(\det E^*)$ by

$$\sigma := \sum_{\alpha \in A} \psi_\alpha \varepsilon_\alpha^1 \wedge \cdots \wedge \varepsilon_\alpha^k,$$

where $\left(\varepsilon^i\right)$ is the dual frame corresponding to the local frame $\left(e_i\right)$. Then σ is nowhere-vanishing. Indeed, if $x \in M$, then $x \in U_\beta$ for some $\beta \in A$ and we compute

$$\sigma_x\left(e_1^\beta(x),\ldots,e_k^\beta(x)\right)=\sum_{\alpha\in A}\psi_\alpha(x)\det\rho_{\alpha\beta}(x)>0.$$

□

Definition F.210 (Orientability). A vector bundle $\pi : E \to M$ is said to be ***orientable***, iff one of the conditions of proposition F.209 is satisfied. A smooth manifold M is said to be ***orientable***, iff the tangent bundle $\pi : TM \to M$ is orientable.

Definition F.211 (Volume Form). Let M^n be a smooth manifold. A ***volume form on M*** is defined to be a nowhere-vanishing n-form.

Corollary F.212 (Orientability of Manifolds). *Let M be a smooth manifold. Then the following conditions are equivalent:*

(a) *M admits a volume form.*
(b) *There exists a smooth atlas $(U_\alpha,\varphi_\alpha)_{\alpha\in A}$ on M such that when $U_\alpha\cap U_\beta\neq\varnothing$*

$$\det D(\varphi_\alpha\circ\varphi_\beta^{-1})\left(\varphi_\beta(x)\right)>0$$

holds for all $x\in U_\alpha\cap U_\beta$.
(c) *The bundle* $\det T^*M\to M$ *is trivial.*

Example F.213 (Lie Groups are Orientable). Let G be a Lie group. Then from problem F.257 we know that $TG\cong G\times\mathfrak{g}$ in Man. In particular, TG is trivial and so is $\det(T^*G)$. Indeed, we have that

$$\det(T^*G)\cong\det(G\times\mathfrak{g}^*)\cong G\times\det(\mathfrak{g}^*).$$

Hence G is orientable.

Example F.214 (Spheres are Orientable). Let $n\geq 1$ and $\omega := dx^0\wedge\cdots\wedge dx^n$ be the standard volume form on $\mathbb{R}^{n+1}$. Moreover, define $X\in\mathfrak{X}(\mathbb{R}^{n+1}\setminus\{0\})$ by $X|_x := x$. View $\mathbb{S}^n\subseteq\mathbb{R}^{n+1}\setminus\{0\}$. Then

$$\iota^*(i_X\omega)\in\Omega^n(\mathbb{S}^n)$$

is a volume form. Indeed, let $v_1,\ldots,v_n\in T_x\mathbb{S}^n$ be a basis. Then $D\iota$ idenifies v_i with a vector in $x^\perp$. But then $(x,v_1,\ldots,v_n)$ is a basis for $\mathbb{R}^{n+1}$ and we have that

$$\iota_X\omega(v_1,\ldots,v_n)=\omega(x,v_1,\ldots,v_n)\neq 0.$$

Definition F.215 (Positively Oriented). Let $\pi : E\to M$ be an orientable vector bundle of rank k and denote by $\sigma\in\Gamma(\det E^*)$ a nowhere vanishing section. A basis $(v_1,\ldots,v_k)$ of E_x, $x\in M$, is said to be ***positively oriented***, iff $\sigma(v_1,\ldots,v_k)>0$.

Definition F.216 (Orientation). Let $\pi : E\to M$ be an orientable vector bundle. An ***orientation of E*** is defined to be an equivalence class $[\sigma]$ of a nowhere vanishing section $\sigma\in\Gamma(\det E^*)$ under the equivalence relation

$$\sigma \sim \sigma' \quad :\Leftrightarrow \quad \exists f \in C^\infty\left(M, \mathbb{R}^+\right) : \sigma = f\sigma'$$

on the set of all nowhere vanishing sections of the determinant bundle $\det E^*$. If an orientation $[\sigma]$ is fixed, we call $\pi : E \to M$ and ***oriented*** vector bundle.

Remark F.217. Let $\pi : E \to M$ be an oriented vector bundle with orientation $[\sigma]$. If M is connected, there are exactly two equivalence classes, $[\sigma]$ and $[-\sigma]$.

Exercise F.218. Prove the statement made in remark F.217.

F.13 Manifolds with Boundary

Definition F.219 (Half-Space). Let V be a vector space and $\rho \in V^*$. Define ***half-spaces associated to ρ*** by

$$V_\rho^+ := \left\{v \in V : \rho(v) \geq 0\right\} \qquad \text{and} \qquad V_\rho^- := \left\{v \in V : \rho(v) \leq 0\right\}.$$

Definition F.220 (Standard Half-Spaces). Let $n \in \mathbb{N}$. Then the half-spaces defined by

$$\mathbb{R}_+^n := \left(\mathbb{R}^n\right)_{\pi^1}^+ \qquad \text{and} \qquad \mathbb{R}_-^n := \left(\mathbb{R}^n\right)_{\pi^1}^-$$

are called the ***standard half-spaces***.

Definition F.221 (Topological Manifold with Boundary). Let $n \in \mathbb{N}$. A topological space M is said to be a ***topological manifold with boundary of dimension n***, iff

(i) M is locally Euclidean of dimension n with boundary, that is, for every $x \in M$ there exist an open subset $U \subseteq M$, $\rho \in \left(\mathbb{R}^n\right)^*$ and a function $\varphi : U \to \left(\mathbb{R}^n\right)_\rho^\pm$ such that $\varphi(U) \subseteq \left(\mathbb{R}^n\right)_\rho^\pm$ is open and $\varphi : U \to \varphi(U)$ is a homeomorphism. Every such pair (U, φ) is called a ***chart on M about x***.
(ii) M is Hausdorff and has at most countably many connected components.
(iii) M is paracompact.

Essentially, a smooth manifold with boundary is the same as an ordinary smooth manifold, but the each chart in the atlas is allowed to have its image in an open subset of some half-space.

Definition F.222 (Smooth Atlas). A ***smooth atlas for a topological manifold with boundary M^n*** is a collection $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ of charts on M such that

(i) $(U_\alpha)_{\alpha \in A}$ is an open cover for M.
(ii) For all $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \varnothing$, the function

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^n$$

is smooth in the sense that there exists a smooth extension. The function $\varphi_\alpha \circ \varphi_\beta^{-1}$ is called a ***transition function***.

Proposition F.223. *Let M^n be a smooth manifold with boundary. Then* $\operatorname{int} M$ *is a smooth manifold without boundary of dimension n and ∂M is a smooth manifold without boundary of dimension $n-1$.*

Lemma F.224. *Let $\rho \in \left(\mathbb{R}^n\right)^*$ such that $\rho \neq 0$. Then*

$$\left(\mathbb{R}^n\right)^{\pm}_{\rho} \cong \ker \rho \times \mathbb{R}^{\pm}$$

as manifolds with boundaries.

Remark F.225. By means of lemma F.224 we may assume always that a smooth manifold with boundary M^n admits an atlas where all the charts have image in some open subset of $\mathbb{R}^n_-$.

Definition F.226 (Outward Pointing). Let M be a smooth manifold with boundary. A tangent vector $v \in T_x M$ for $x \in \partial M$ is said to be an ***outward pointing vector***, iff

$$dx^1|_x(v) > 0$$

for some chart $\left(U, (x^i)\right)$ about x. Moreover, a section X of $TM|_{\partial M} \to \partial M$ is said to be an ***outward-pointing vector field***, iff X_x is an outward pointing vector for all $x \in \partial M$.

Lemma F.227. *Let M be a smooth manifold with nonempty boundary. Then there exists an outward pointing vector field.*

Proof. Let $U_\alpha, (x^i_\alpha)_{\alpha \in A}$ be an atlas for M and $(\psi_\alpha)_{\alpha \in A}$ a partition of unity subordinate to the atlas. Then

$$X := \sum_{\alpha \in A} \psi_\alpha \frac{\partial}{\partial x^1_\alpha}$$

is an outward-pointing vector field. Indeed, we have that

$$dx^1_\alpha|_x(X_x) = \sum_{\alpha \in A} \psi_\alpha(x) dx^1_\alpha|_x \left(\frac{\partial}{\partial x^1_\alpha}\right) = \sum_{\alpha \in A} \psi_\alpha(x) = 1$$

for all $x \in \partial M$. □

Definition F.228 (Induced Orientation). Let M be a smooth manifold with nonempty boundary and ω a volume form on M. Then the ***induced orientation on*** $\boldsymbol{\partial M}$ is defined to be the equivalence class $[i_X(\omega)]$, where X is an outward pointing vector field.

F.14 Integration on Manifolds

Definition F.229. Let M and N be smooth manifolds and $A \subseteq M$ a subset. A map $F : A \to N$ is said to be ***smooth on A***, iff for every $x \in A$ there exists a neighbourhood U of x and a map $\widetilde{F} \in C^\infty(U, N)$, such that $\widetilde{F}|_{U \cap A} = F$.

Definition F.230 (Singular Cube). Let $k \in \mathbb{N}$ and M be a smooth manifold. A ***singular k-cube in M*** is defined to be a morphism $\sigma \in C^\infty(I^k, M)$.

Definition F.231. Let $k \in \mathbb{N}$, $\omega \in \Omega^k(M)$ and $\sigma = \lambda^i \sigma_i$ be a singular k-chain in M. Then $\sigma_i^* \omega = f_i dx^1 \wedge \cdots \wedge dx^k$ for some $f_i \in C^\infty(I^k)$. Define the ***integral of ω over σ*** to be

$$\int_\sigma \omega := \lambda^i \int_{I^k} f_i \qquad \text{and} \qquad \int_\sigma f := \lambda^i f\left(\sigma_i(0)\right)$$

if $k \geq 1$ and $k = 0$, respectively.

Definition F.232 (Front and Back Face). Let σ be a singular k-cube. For $1 \leq i \leq k$ define the ***i-th front face of σ***, to be the singular $k-1$-cube $F_i\sigma$ defined by

$$F_i\sigma\left(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^k\right)$$

and the ***i-th back face of σ***, to be the singular $k-1$-cube $B_i\sigma$ defined by

$$B_i\sigma\left(x^1, \ldots, x^{i-1}, 1, x^{i+1}, \ldots, x^k\right).$$

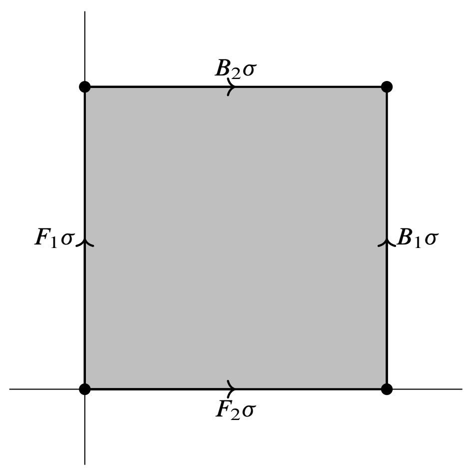


Fig. F.2: Face maps for $k = 2$ and σ the inclusion $I^2 \hookrightarrow \mathbb{R}^2$.

Definition F.233 (Boundary). Let σ be a singular k-cube, $k \geq 1$. Define the ***boundary of σ*** to be the singular k-chain $\partial\sigma$ defined by

$$\partial\sigma := \sum_{i=1}^{k} (-1)^k \left(F_i\sigma - B_i\sigma\right).$$

Moreover, define the boundary of a singular 0-cube to be 1.

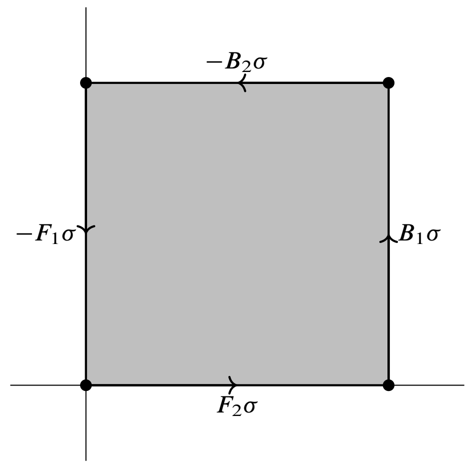


Fig. F.3: Boundary of the inclusion $I^2 \hookrightarrow \mathbb{R}^2$.

Proposition F.234 (Stoke's Theorem, Local Version). *Let M be a smooth manifold, $\sigma \in C_k(M)$ and $\omega \in \Omega^{k-1}(M)$. Then*

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega.$$

Definition F.235 (Orientation-Preserving). Let $\big(M^n, [\omega]\big)$ and $\big(N^n, [\eta]\big)$ be two oriented manifolds and $F \in C^\infty(M, N)$ a diffeomorphism. Then F is said to be ***orientation-preserving***, iff $f > 0$ where $f \in C^\infty(M)$ is defined by $F^*\eta = f\omega$.

Definition F.236. Let M^n be an oriented manifold. A singular n-cube is said to be ***orientation-preserving***, iff it admits an orientation-preserving extension which is also an embedding.

Definition F.237 (Special Singular Cube). Let M^n be an oriented smooth manifold with boundary. An orientation preserving singular n-cube σ is said to be ***special***, iff either $\operatorname{im}\sigma \subseteq \operatorname{int} M$ or $\partial M \cap \operatorname{im}\sigma = \operatorname{im}(F_1\sigma)$.

Definition F.238. Let M^n be an oriented manifold with boundary and $\omega \in \Omega_c^n(M)$. Then define

$$\int_M \omega := \sum_{\alpha\in A} \int_{\sigma_\alpha} \psi_\alpha \omega.$$

where $(\psi_\alpha)_{\alpha\in A}$ is a partition of unity subordinate to a cover $(U_\alpha)_{\alpha\in A}$ with the property that each U_α is conatined in the interior of the image of a special orientation preserving singular n-cube σ_α for all $\alpha \in A$.

Theorem F.239 (Stoke's Theorem, global Version). *Let M^n be an oriented smooth manifold with boundary and endow ∂M with the induced orientation. If $\omega \in \Omega_c^{n-1}(M)$, then*

$$\int_M d\omega = \int_{\partial M} \omega. \tag{F.10}$$

F.15 De Rham Cohomology

Definition F.240 (Closed and Exact Form). Let M be a smooth manifold, $U \in \mathcal{O}(M)$ non-empty and $k \in \mathbb{N}$, $k \geq 1$. A form $\omega \in \Omega^k(U)$ is said to be ***closed***, iff $d\omega = 0$, and ***exact***, iff there exists $\eta \in \Omega^{k-1}(U)$ with $\omega = d\eta$.

Definition F.241 (The de Rham Chain Complex). The contravariant functor

$$C^{\mathrm{dR}}_\bullet : \mathsf{Man} \to \mathsf{Ch}^{\geq 0,\mathrm{fin}}(\mathsf{Vect})$$

defined on objects $M \in \mathsf{Man}$ by

$$C^k_{\mathrm{dR}}(M) := \Omega^k(M) \qquad \text{and} \qquad d^k := d : \Omega^k(M) \to \Omega^{k+1}(M)$$

and on morphisms $F \in C^\infty(M, N)$ by

$$C^\bullet_{\mathrm{dR}}(F) := F^*$$

is called the ***de Rham chain complex functor***.

Using Stoke's theorem F.239 one can show the homotopy invariance of the de Rham cohomology.

Proposition F.242. *Let M be a smooth manifold. For $t \in I$ define $\iota_t : M \to M \times I$ by $\iota_t(x) := (x, t)$. Then there is a map*

$$h : \Omega^k(M \times I) \to \Omega^{k-1}(M)$$

such that

$$h(d\omega) + d\left(h(\omega)\right) = \iota_1^*(\omega) - \iota_0^*(\omega).$$

Proof. Define $h : \Omega^k(M \times I) \to \Omega^{k-1}(M)$ by

$$h(\omega)_x := \int_0^1 \iota_t^* \left(i_X \omega_{(x,t)}\right) dt$$

where $X_{(x,t)} := \left(0, \partial_t|_t\right)$. □

Theorem F.243 (The Poincaré Lemma). *Let M be a smooth manifold and $\omega \in \Omega^k(M)$ be closed. Then for every $x \in M$ there exists a neighbourhood U of x such that $\omega|_U$ is exact.*

F.16 Principal Bundles

Definition F.244 (Principal Bundle). A fibre bundle $\pi : P \to M$ with fibre a Lie group G is said to be a ***principal G-bundle***, iff

(i) There exists a fibre-preserving free right action of G on P.
(ii) There exists a bundle atlas such that each bundle chart $\varphi_\alpha : \pi^{-1}(U_\alpha) \to G$ is ***G-equivariant***, that is, we have that

$$\varphi_\alpha(p \cdot g) = \varphi_\alpha(p)g$$

for all $p \in \pi^{-1}(U_\alpha)$ and $g \in G$.

Proposition F.245. *The structure group of a principal G-bundle $\pi : P \to M$ is G, acting via left translations.*

A particular interesting example of a principal bundle is the following.

Proposition F.246 (The Frame Bundle). *Let $E \to M$ be a vector bundle of rank k. For all $x \in M$ define*

$$\mathrm{Fr}(E_x) := \{\text{isomorphisms } \mathbb{R}^k \to E_x\}\,.$$

Then

$$\pi : \mathrm{Fr}(E) := \coprod_{x \in M} \mathrm{Fr}(E_x) \to M$$

is a principal $\mathrm{GL}(k)$*-bundle. This bundle is called the* ***frame bundle***.

Proof. Let $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ be a vector bundle atlas for $E \to M$. For every $\alpha \in A$ define $\widetilde{\varphi}_\alpha : \pi^{-1}(U_\alpha) \to \mathrm{GL}(k)$ by

$$\widetilde{\varphi}_\alpha(x, A) := \varphi_\alpha|_{E_x} \circ A.$$

Then

$$\widetilde{\rho}_{\alpha\beta}(x)(A) = \widetilde{\varphi}_\alpha|_{\mathrm{Fr}(E_x)} \circ \widetilde{\varphi}_\beta|_{\mathrm{Fr}(E_x)}(A) = \rho_{\alpha\beta}(x) \circ A$$

for all $U_\alpha \cap U_\beta \neq \varnothing$. Hence $\mathrm{Fr}(E)$ can be given the structure of a smooth manifold. Define a right action

$$\mathrm{Fr}(E) \times \mathrm{GL}(k) \to \mathrm{Fr}(E)$$

by

$$\big((x, A), T\big) \mapsto (x, A \circ T).$$

This action is obviously free and we have that

$$\widetilde{\varphi}_\alpha\left((x, A) \cdot T\right) = \widetilde{\varphi}_\alpha(x, A \circ T) = \varphi_\alpha|_{E_x} \circ (A \circ T) = (\varphi_\alpha|_{E_x} \circ A) \circ T = \widetilde{\varphi}_\alpha(x, A) \circ T.$$

To each principal bundle one can associate a fibre bundle.

Definition F.247. Let $\pi : P \to M$ be a principal G-bundle and suppose that there is an effective action $\theta : G \times F \to F$ on some smooth manifold F. Define an equivalence relation $\sim$ on $P \times F$ via

$$(p \cdot g, v) \sim \big(p, \theta(g, v)\big)$$

for all $p \in P$, $g \in G$ and $v \in F$. Denote the quotient space by $P \times_{(G,\theta)} F$.

Theorem F.248. *Let $\pi : P \to M$ be a principal G-bundle nd suppose that there is an action $\theta : G \times F \to F$ on some smooth manifold F. Then:*

(a) $\widetilde{\pi} : P \times_{(G,\theta)} F \to M$ *is a fibre bundle with fibre F and structure group G, where*

$$\widetilde{\pi}[p, v] := \pi(p).$$

(b) *P is the principal bundle associated to $P \times_{(G,\theta)} F$.*

Lemma F.249. *Let $\pi : P \to M$ be a principal G-bundle. Then G acts transitively on the fibres.*

Proof. Let $x \in M$ and $p, q \in P_x$. Thus there exists $\varphi_\alpha : \pi^{-1}(U_\alpha) \to M$ such that $p, q \in \pi^{-1}(U_\alpha)$. Suppose that there exists $g \in G$ such that $p = q \cdot g$. Then equivariance yields

$$\varphi_\alpha(p) = \varphi_\alpha(q \cdot g) = \varphi_\alpha(q) g$$

which implies $g = \varphi_\alpha(q)^{-1} \varphi_\alpha(p) \in G$. □

Up to now we have never used the fact that the action is free. The next proposition however makes use of it.

Proposition F.250. *Let $\pi : P \to M$ and $\pi' : P' \to M'$ be two principal G-bundles. Suppose $\Phi : P \to P'$ is a principal bundle morphism along a diffeomorphism. Then Φ is a diffeomorphism.*

Proposition F.251. *Let $\pi : P \to M$ be a principal G-bundle. Then π admits a section if and only if P is trivial.*

Proof. Suppose P is trivial, that is $P \cong M \times G$. For any $g \in G$, $\sigma : M \to M \times G$ defined by $\sigma(x) := (x, g)$ is a section.
Conversly, suppose that π admits a section. Then for each $p \in P$, p and $\sigma\left(\pi(p)\right)$ belong to the same fibre. Since the action of G on the fibres is transitive by lemma F.249, we can define a map $\varphi : P \to G$ such that

$$p = \sigma\left(\pi(p)\right) \cdot \varphi(p).$$

Then (π, φ) is a principal G-bundle morphism along id_M, and thus a diffeomorphism by proposition F.251. □

F.17 Connections on Principal Bundles

Definition F.252 (Bundle Valued Differential Form). Let $\pi : E \to M$ be a vector bundle. A ***bundle-valued differential k-form*** is defined to be a section of the bundle

$$\Lambda^k(T^*M) \otimes E \to M.$$

The vector space of all such sections is denoted by $\Omega^k(M; E)$

Thus a bundle-valued differential k-form $\omega \in \Omega^k(M; E)$ is nothing more than an alternating map

$$\omega_x : \underbrace{T_x M \times \cdots \times T_x M}_{k} \to E_x$$

for all $x \in M$.

Proposition F.253 (The Bundle-Valued Differential Form Criterion). *There is a natural $C^\infty(M)$-module isomorphism between $\Omega^k(M; E)$ and alternating $C^\infty(M)$-multilinear maps*

$$\underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k} \to \Gamma(E).$$

Problems

F.254. Aim of this exercise is to show that $\mathbb{S}^1$ is a Lie group under complex multiplication.

(a) We can endow $\mathbb{S}^1$ with a different smooth atlas as follows: Construct two charts with range a bounded interval in $\mathbb{R}^2$. Those are called ***angle coordinates***.
(b) Show that complex multiplication in these coordinates is smooth.
(c) Show that complex inversion in these coordinates is smooth.

F.255. Prove proposition F.40.

F.256. Let M be a smooth manifold and $X, Y \in \mathfrak{X}(M)$. Show that

$$i_{[X,Y]} = \mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X.$$

F.257.

(a) Let M be a smooth manifold and suppose that there exist vector fields $X_1, \dots, X_n \in \mathfrak{X}(M)$ such that $(X_1|_x, \dots, X_n|_x)$ is a basis for $T_x M$ for every $x \in M$. Prove that the tangent bundle TM is trivial.
(b) Let G be a Lie group. Prove that $TG \cong G \times \mathfrak{g}$ in Man.

Appendix G
Review of Differential Geometry

An excellent introduction to the subject is given in [7].

G.1 Pseudo-Riemannian Manifolds

Definition G.1 (Pseudo-Riemannian Metric). Let M be a smooth manifold. A ***pseudo-Riemannian metric on M*** is defined to be a symmetric covariant 2-tensor field $g \in \mathcal{T}^{0,2}(M)$ which is ***nondegenerate*** at each point $x \in M$, that is, we have $g_x(v, w) = 0$ for all $w \in T_x M$ and some $v \in T_x M$ implies $v = 0$.

Definition G.2 (Pseudo-Riemannian Manifold). A ***Pseudo-Riemannian manifold*** is defined to be a tuple (M, g) consisting of a smooth manifold M and a pseudo-Riemannian metric g on M.

Remark G.3. The tangent-cotangent isomorphism from theorem 2.36 is also valid for a pseudo-Riemannian manifold since the proof is only based on nondegeneracy of a covariant 2-tensor field.

Proposition G.4 (Sylvester's Law of Inertia). *Let q be a nondegenerate symmetric bilinear form on a finite-dimensional real vector space V. Then there exists a basis $\left(\beta^i\right)$ for V^* such that*

$$q = \left(\beta^1\right)^2 + \cdots + \left(\beta^r\right)^2 - \left(\beta^{r+1}\right)^2 - \cdots - \left(\beta^{r+s}\right)^2 .$$

*Moreover, the natural numbers r and s are independent on the choice of basis. Thus the pair (r, s) is called the **signature of q**.*

Example G.5 (Riemannian Manifolds). ***Riemannian manifolds*** are pseudo-Riemannian manifolds (M, g) such that g_x has signature $(r, 0)$ for all $x \in M$.

Example G.6 (Lorentz Manifolds). ***Lorentz Manifolds*** are pseudo-Riemannian manifolds (M, g) such that g_x has signature $(1, s)$ for all $x \in M$.

G.2 Connections

Definition G.7 (Koszul Connection). Let $\pi : E \to M$ be a vector bundle. A ***connection in E*** is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$

written $(X, Y) \mapsto \nabla_X Y$ such that:

(i) ∇ is $C^\infty(M)$-linear in the first argument.
(ii) ∇ is $\mathbb{R}$-linear in the second argument.
(iii) The following ***Product rule*** holds:

$$\nabla_X(fY) = f\nabla_X Y + (Xf)Y$$

for all $f \in C^\infty(M)$.

Any section $\nabla_X Y$ is called the ***covariant derivative of Y in the direction X***.

Definition G.8 (Connection Coefficients). Let $\pi : E \to M$ be a vector bundle and ∇ be a connection in E. Let (e_i) be a local frame on $U \subseteq M$. Then

$$\nabla_{e_i} e_j = \Gamma_{ij}^k e_k$$

for some functions $\Gamma_{ij}^k \in C^\infty(U)$. The family $\left(\Gamma_{ij}^k\right)$ is called the ***connection coefficients of ∇ with respect to the local frame (e_i)***.

Remark G.9. There is an immediate transformation rule for the connection coefficients of a connection. Moreover, it can be seen that the connection coefficients do *not* transform like the component functions of a $(1, 2)$-tensor field due to an error term coming from the product rule.

Exercise G.10. Establish the transformation law for connection coefficients with respect to another local frame $\widetilde{e}_i = A_i^j e_j$ for some smooth functions A_i^j.

Proposition G.11. *Let M be a smooth manifold and ∇ a connection in TM. Then for every $X \in \mathfrak{X}(M)$, there exists a unique tensor derivation*

$$\widetilde{\nabla}_X : \mathcal{T}_M \to \mathcal{T}_M$$

such that

$$\widetilde{\nabla}_X f = Xf \qquad \text{and} \qquad \widetilde{\nabla}_X Y = \nabla_X Y$$

holds for all smooth functions f and vector fields Y.

Proof. Immediate by proposition F.187. ☐

Definition G.12 (Vector Field along a Curve). Let M be a smooth manifold and $\gamma \in C^\infty(J, M)$ a path in M where $J \subseteq M$ is an interval. A ***vector field along γ*** is defined to be a map $V \in C^\infty(J, TM)$ such that $V_t \in T_{\gamma(t)}M$ holds for all $t \in J$. The set of all vector fields along a curve γ is denoted by $\mathfrak{X}(\gamma)$.

Exercise G.13. Let M be a smooth manifold and $\gamma \in C^\infty(J, M)$ a curve in M. Show that $\mathfrak{X}(\gamma)$ is a module over $C^\infty(J)$.

Theorem G.14 (Covariant Derivative along a Curve). *Let M be a smooth manifold and ∇ a connection in TM. For each curve $\gamma \in C^\infty(J, M)$, there is a unique operator*

$$D_\gamma : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma),$$

*called the **covariant derivative along γ**, such that*

(i) *D_γ is $\mathbb{R}$-linear.*
(ii) *The following **Product rule** holds:*

$$D_\gamma(fV) = f'V + fD_\gamma V$$

for all $f \in C^\infty(J)$.
(iii) *If $V \in \mathfrak{X}(\gamma)$ is extendible, then for every extension $\widetilde{V}$ of V we have that*

$$D_\gamma V|_t = \nabla_{\gamma'(t)} \widetilde{V}.$$

for all $t \in J$.

Proof. Let $t_0 \in J$ and suppose $(U, (x^i))$ is a chart about $\gamma(t_0)$. Then for sufficiently $t \in J$ sufficiently close to t_0, we may write

$$V_t = V^j(t) \frac{\partial}{\partial x^j}\bigg|_{\gamma(t)}.$$

We compute

$$\begin{aligned}
D_\gamma V|_t &= \dot{V}^j(t) \frac{\partial}{\partial x^j}\bigg|_{\gamma(t)} + V^j(t) D_\gamma \left(\frac{\partial}{\partial x^j}\right)\bigg|_t \\
&= \dot{V}^j(t) \frac{\partial}{\partial x^j}\bigg|_{\gamma(t)} + V^j(t) \nabla_{\gamma'(t)} \left(\frac{\partial}{\partial x^j}\right)\bigg|_t \\
&= \dot{V}^j(t) \frac{\partial}{\partial x^j}\bigg|_{\gamma(t)} + V^j(t) \dot{\gamma}^i(t) \nabla_{\partial/\partial x^i|_{\gamma(t)}} \left(\frac{\partial}{\partial x^j}\right) \\
&= \dot{V}^j(t) \frac{\partial}{\partial x^j}\bigg|_{\gamma(t)} + V^j(t) \dot{\gamma}^i(t) \Gamma^k_{ij}\left(\gamma(t)\right) \frac{\partial}{\partial x^k}\bigg|_{\gamma(t)} \\
&= \left(\dot{V}^k(t) + V^j(t) \dot{\gamma}^i(t) \Gamma^k_{ij}\left(\gamma(t)\right)\right) \frac{\partial}{\partial x^k}\bigg|_{\gamma(t)}.
\end{aligned}$$

This shows existence and uniqueness. □

Definition G.15 (Acceleration of a Curve). Let M be a smooth manifold with a connection ∇ in TM. The ***acceleration of a curve*** $\gamma \in C^\infty(J, M)$ is defined to be the vector field $D_\gamma \gamma'$ along γ.

Definition G.16 (Geodesic). Let M be a smooth manifold with connection ∇ in TM. A curve is said to be a ***geodesic***, iff its acceleration vanishes.

More generally, we have the following definition.

Definition G.17 (Parallel Vector Field along a Curve). Let M be a smooth manifold with connection ∇ in TM. Suppose $\gamma \in C^\infty(J, M)$ is a curve. Then a vector field $V \in \mathfrak{X}(\gamma)$ is said to be ***parallel along*** $\boldsymbol{\gamma}$, iff $D_\gamma V = 0$.

Theorem G.18 (Existence and Uniqueness of Parallel Transport). *Let M be a smooth manifold with connection ∇. Given a curve $\gamma \in C^\infty(J, M)$, $t_0 \in J$ and $v \in T_{\gamma(t_0)}M$, there exists a unique parallel vector field along γ such that $V_{t_0} = v$.*

Definition G.19 (Parallel Transport). Let M be a smooth manifold and ∇ a connection on M. For every $\gamma \in C^\infty(J, M)$ and $t_0, t_1 \in J$ define a map

$$P^\gamma_{t_0 t_1} : T_{\gamma(t_0)}M \to T_{\gamma(t_1)}M$$

by $P^\gamma_{t_0 t_1}(v) := V_{t_1}$, where V is the unique parallel vector field along γ such that $V_{t_0} = v$ whose existence is guaranteed by theorem G.18.

Theorem G.20 (Parallel Transport Determines Covariant Derivative). *Let M be a smooth manifold with connection ∇. Suppose $\gamma \in C^\infty(J, M)$ is a path and $V \in \mathfrak{X}(\gamma)$. Then for each $t_0 \in J$ we have that*

$$D_\gamma V|_{t_0} = \lim_{t_1 \to t_0} \frac{P^\gamma_{t_1 t_0} V_{t_1} - V_{t_0}}{t_1 - t_0}.$$

Corollary G.21 (Parallel Transport Determines the Connection). *Let M be a smooth manifold with connection ∇. Then for all $X, Y \in \mathfrak{X}(M)$ and $x \in M$ we have that*

$$\nabla_X Y|_x = \lim_{h \to 0} \frac{P^\gamma_{h0} Y_{\gamma(h)} - Y_x}{h}$$

where $\gamma \in C^\infty(J, M)$ is a curve such that $\gamma(0) = x$ and $\gamma'(0) = X_x$.

Theorem G.22 (Fundamental Theorem of Riemannian Geometry). *Let (M, g) be a pseudo-Riemannian manifold. Then there exists a unique connection ∇ on M with:*

(i) Compatibility: $\nabla g = 0$.
(ii) Torsion Free: $\nabla_X Y - \nabla_Y X - [X, Y] = 0$ *for all* $X, Y \in \mathfrak{X}(M)$.

This connection is called the ***Levi-Civita connection***. *Explicitely, the connection coefficients in any chart are given by*

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right). \tag{G.1}$$

These connection coefficients are called the ***Christoffel symbols***.

Proposition G.23. *Let (M, g) be a pseudo-Riemannian metric. Then*

$$\mathcal{L}_X g(Y, Z) = g\left(\nabla_Y X, Z\right) + g\left(Y, \nabla_Z X\right)$$

for every $X, Y, Z \in \mathfrak{X}(M)$ where ∇ denotes the Levi-Civita connection on M.

Proof. Since ∇_X is a tensor derivation, compatibility implies

$$0 = (\nabla_X g)(Y, Z) = \nabla_X g(Y, Z) - g\left(\nabla_X Y, Z\right) - g\left(Y, \nabla_X Z\right).$$

Moreover, torsion-freeness implies

$$[X, Y] = \nabla_X Y - \nabla_Y X \qquad \text{and} \qquad [X, Z] = \nabla_X Z - \nabla_Z X.$$

Thus we compute

$$\begin{aligned}
(\mathcal{L}_X g)(Y, Z) =& \mathcal{L}_X g(Y, Z) - g\left(\mathcal{L}_X Y, Z\right) - g\left(Y, \mathcal{L}_X Z\right) \\
=& X\left(g(Y, Z)\right) - g\left([X, Y], Z\right) - g\left(Y, [X, Z]\right) \\
=& \nabla_X g(Y, Z) - g\left([X, Y], Z\right) - g\left(Y, [X, Z]\right) \\
=& g\left(\nabla_X Y, Z\right) + g\left(Y, \nabla_X Z\right) - g\left([X, Y], Z\right) - g\left(Y, [X, Z]\right) \\
=& g\left(\nabla_X Y, Z\right) + g\left(Y, \nabla_X Z\right) - g\left(\nabla_X Y, Z\right) + g\left(\nabla_Y X, Z\right) \\
& - g\left(Y, \nabla_X Z\right) + g\left(Y, \nabla_Z X\right) \\
=& g\left(\nabla_Y X, Z\right) + g\left(Y, \nabla_Z X\right).
\end{aligned}$$

□

Definition G.24 (The Exponential Map). Let (M, g) be a pseudo-Riemannian manifold. Define $\mathcal{E} \subseteq TM$ by

$$\mathcal{E} := \{v \in TM : \gamma_v \text{ is defined on an interval containing } I\}$$

and $\exp : \mathcal{E} \to M$ by

$$\exp v := \gamma_v(1).$$

This map is called the ***exponential map***.

Proposition G.25 (Properties of the Exponential Map). *Let (M, g) be a pseudo-Riemannian manifold.*

(a) *$\mathcal{E} \subseteq TM$ is open and contains the image of the zero section.*
(b) *For each $v \in TM$, the geodesic γ_v is given by*

$$\gamma_v(t) = \exp(tv)$$

for all t such that either side is defined.
(c) *The exponential map is smooth.*
(d) *For each $x \in M$, $D(\exp_x)_0 = \mathrm{id}_{T_x M}$.*

Definition G.26 (Normal Coordinates). Let (M^n, g) be a pseudo-Riemannian manifold. Let $x \in M$. Then there exists a star-shaped neighbourhood V of the origin in $T_x M$ and a neighbourhood U of x in M such that $\exp_x : V \to U$ is a diffeomorphism. Let (b_i) be an orthonormal basis of $T_x M$ with coordinate isomorphism B. Then

$$\varphi := B^{-1} \circ \left(\exp_x |_V\right)^{-1} : U \to \mathbb{R}^n$$

is called a ***normal coordinate chart about x***.

Proposition G.27 (Properties of Normal Coordinates). *Let (M^n, g) be a pseudo-Riemannian manifold and let $\left(U, (x^i)\right)$ be any normal coordinate chart about $x \in M$.*

(a) *The components of the metric are given by $g_{ij} = \pm\delta_{ij}$.*
(b) *For every $v = v^i \frac{\partial}{\partial x^i}|_x$, we have that*

$$\gamma_v(t) = \left(tv^1, \ldots, tv^n\right).$$

(c) *The Christoffel symbols vanish at x.*
(d) *All first partial derivatives of g_{ij} vanish at x.*

References

1. Grillet, P.A.: Abstract Algebra. Graduate Texts in Mathematics. Springer Science + Business Media, LLC (2007)
2. I. Arnol'd, V.: What is mathematical physics? Uspekhi Fizicheskikh Nauk - USP FIZ NAUK **174** (2004). DOI 10.3367/UFNr.0174.200412i.1381
3. Lane, S.M.: Categories for the Working Mathematician, second edition edn. No. 5 in Graduate Texts in Mathematics. Springer Science+Business Media New York (1978)
4. Lee, J.M.: Riemannian Manifolds - An Introduction to Curvature. Springer (1997)
5. Lee, J.M.: Introduction to Topological Manifolds, second edition edn. Springer Science+Business Media (2011)
6. Lee, J.M.: Introduction to Smooth Manifolds, second edition edn. Graduate Texts in Mathematics. Springer (2013)
7. Lee, J.M.: Introduction to Riemannian Manifolds, second edition edn. Graduate Texts in Mathematics. Springer International Pu (2018)
8. Leinster, T.: Basic category theory (2016). URL https://arxiv.org/abs/1612.09375
9. Mazzucchelli, M.: Critical Point Theory for Lagrangian Systems. No. 293 in Progress in Mathematics. Birkhäuser (2012)
10. McDuff, D., Salamon, D.: Introduction to Symplectic Topology, third edition edn. No. 27 in Oxford Graduate Texts in Mathematics. Oxford University Press (2017)
11. Munkres, J.: Topology, second edition edn. Pearson (2014)
12. Munkres, J.R.: Elements of Algebraic Topology. Perseus Publishing (1984)
13. Niculescu, C.P., Persson, L.E.: Convex Functions and Their Applications, second edition edn. CMS Books in Mathematics. Springer (2018)
14. Robbin, J.W., Salamon, D.A.: Introduction to differential topology (2018). URL https://people.math.ethz.ch/~salamon/PREPRINTS/difftop.pdf
15. da Silva, A.C.: Lectures on Symplectic Geometry, corrected 2nd printing edn. No. 1764 in Lecture Notes in Mathematics. Springer (2008)
16. Spivak, M.: A Comprehensive Introduction to Differential Geometry, vol. 1, third edition with corrections edn. Publish or Perish, Inc. (2005)
17. Struwe, M.: Variational Methods, fourth edition edn. No. 34 in Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer (2008)
18. Struwe, M.: Funktionalanalysis 1 und 2 (2014). URL https://people.math.ethz.ch/~struwe/Skripten/FA-I-II-11-9-2014.pdf
19. Takhtajan, L.A.: Quantum Mechanics for Mathematicians, *Graduate Studies in Mathematics*, vol. 95. American Mathematical Society (2008)
20. Weibel, C.A.: An introduction to homological algebra, *Cambridge studies in advanced mathematics*, vol. 38. Cambridge University Press (1994)

Index

ETH

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Declaration of originality

The signed declaration of originality is a component of every semester paper, Bachelor's thesis, Master's thesis and any other degree paper undertaken during the course of studies, including the respective electronic versions.

Lecturers may also require a declaration of originality for other written papers compiled for their courses.

I hereby confirm that I am the sole author of the written work here enclosed and that I have compiled it in my own words. Parts excepted are corrections of form and content by the supervisor.

Title of work (in block letters):

Mathematical Aspects of Classical Mechanics (cont.)

Authored by (in block letters):
For papers written by groups the names of all authors are required.

Name(s):	**First name(s):**
Bähni	Yannis

With my signature I confirm that
- I have committed none of the forms of plagiarism described in the 'Citation etiquette' information sheet.
- I have documented all methods, data and processes truthfully.
- I have not manipulated any data.
- I have mentioned all persons who were significant facilitators of the work.

I am aware that the work may be screened electronically for plagiarism.

Place, date
ETH Zürich, 23.4.19

Signature(s)

For papers written by groups the names of all authors are required. Their signatures collectively guarantee the entire content of the written paper.