Lecture 2: Symplectic Manifolds

Yannis Bähni

University of Augsburg

yannis.baehni@math.uni-augsburg.de

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Linear Symplectic Geometry

$(V, \langle \cdot, \cdot \rangle)$ inner product space

$\langle \cdot, \cdot \rangle$ bilinear symmetric form which is pos. definite.

$\Downarrow$

nondegeneracy

Definition (Symplectic Vector Space)

A ***symplectic vector space*** is defined to be a tuple (V, ω), where V is a finite-dimensional real vector space and $\omega \colon V \times V \to \mathbb{R}$ is a nondegenerate skew-symmetric bilinear form.

- $\omega(v, u) = -\omega(u, v) \quad \forall u, v \in V$
- $\begin{aligned} V &\to V^* \\ v &\mapsto i_v \omega \end{aligned}$ is an isomorphism

$\Leftrightarrow \forall v \in V \setminus \{0\} \; \exists u \in V$ s.t. $\omega(v, u) \neq 0$

"Every nonzero vector has a friend."

symplectic version of Gram-Schmidt

Theorem

Let (V, ω) be a symplectic vector space. Then there exists a basis (a_i, b_i) of V such that

$$\omega(b_i, a_j) = \delta_{ij} \qquad \textit{and} \qquad \omega(a_i, a_j) = \omega(b_i, b_j) = 0$$

for all i, j. Any such basis is called a ***symplectic basis for V****.*

$$[\omega] = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} \Rightarrow \text{Every symplectic vector space is of even dimension!}$$

Proof by induction. $\dim V = 0$ is obvious.
So assume $\dim V \geq 1$. This means, there exists $b_1 \in V \setminus \{0\}$. By nondegeneracy, there exists a friend a_1 of b_1, i.e.

$$\omega(b_1, a_1) = 1.$$

Claim. $\{b_1, a_1\}$ is linearly independent.

Proof of Claim. If not, then $\exists \lambda \in \mathbb{R}$ s.t.

$$a_1 = \lambda b_1.$$

If dim of S is 1 then $S \subseteq S^\omega$.

But then by skew-symmetry:

$$\omega(b_1, a_1) = \lambda \omega(b_1, b_1) = 0. \quad \left(= \lambda \|b_1\|^2 \neq 0\right) \quad \text{↯} \quad \square$$

Now decompose

$$V = \underbrace{S}_{=\operatorname{span}\{b_1, a_1\}} \oplus S^\omega \overset{(+)}{:=} \{v \in V : \omega(v,u) = 0 \; \forall u \in S\}$$

$\omega|_S : S \times S \to \mathbb{R}$ symplectic complement to S.

- $\dim S + \dim S^\omega = \dim V \quad [(S^\omega)^\omega = S]$
- S is symplectic if and only if S^ω is symplectic.

Thus the induction hypothesis applies to the symplectic vector space S^ω. $\square$

Corollary

Every symplectic vector space (V, ω) is of even dimension $2n$ and

$$\omega = \sum_{i=1}^{n} \beta^i \wedge \alpha^i$$

$$i_{X_H}\omega = -dH$$

canonical / standard form

in the dual basis (α^i, β^i) of a symplectic basis (a_i, b_i) for V.

$$\omega^n = \left(\sum_{i=1}^{n} \beta^i \wedge \alpha^i\right)^n = n!\left(\beta^1 \wedge \alpha^1 \wedge \dots \wedge \beta^n \wedge \alpha^n\right) \neq 0$$

Corollary

A skew-symmetric bilinear form ω on a real vector space V^{2n} is nondegenerate if and only if $\omega^n \neq 0$.

$$\omega^n = \underbrace{\omega \wedge \dots \wedge \omega}_{n}$$

Manifold setting:

Let M be a smooth manifold. A form $\omega \in \Omega^2(M)$ is said to be nondegenerate, iff

$$(T_xM, \omega_x)$$

is a symplectic vector space $\forall x \in M$.

- $\dim M = \dim T_xM \; \forall x \in M \Rightarrow M$ is even dimensional!!
- M^{2n} is always orientable! Just observe that $\omega^n \in \Omega^{2n}(M)$ is a volume form on M.

The Category of Symplectic Manifolds

Definition (Symplectic Manifold)

A ***symplectic manifold*** is defined to be a tuple (M, ω) where M is a smooth finite-dimensional manifold and $\omega \in \Omega^2(M)$ is closed and nondegenerate.

Analytical condition

Definition (Symplectomorphism)

Let (M, ω) and $(\widetilde{M}, \widetilde{\omega})$ be two symplectic manifolds. A ***symplectomorphism*** is a diffeomorphism $\varphi\colon M \to \widetilde{M}$ such that $\varphi^*\widetilde{\omega} = \omega$.

if $\varphi \in C^\infty(M, \widetilde{M})$ satisfies $\varphi^*\widetilde{\omega} = \omega$, then φ is an immersion.

- Let Σ be an orientable surface. Then (Σ, ω) is a symplectic manifold for any volume form ω on Σ.

bijective smooth with smooth inverse

Lemma

Let (M,ω) be a compact symplectic manifold. Then $H^{2p}_{\mathrm{dR}}(M) \neq 0$. $p \in \mathbb{N}$

Proof. Exercise. Hint: Stokes's theorem.

$(T^*M, d\lambda)$ ⇒ Every exact symplectic manifold without boundary is non-compact.

exact symplectic form

Assume $(M^{2n}, d\lambda)$ is exact symplectic manifold, $\omega = d\lambda$. Then

Liouville domain $(M, d\lambda)$ compact. $\Rightarrow \partial M \neq \emptyset$.

$$0 < \int_M \omega^n = \int_M d(\lambda \wedge \omega^{n-1})$$

$$\overset{\text{Stokes}}{=} \int_{\partial M} \lambda \wedge \omega^{n-1} = 0. \quad ↯$$

$\partial M = \emptyset$

symplectic reduction

Oberseminar 10.5 twisted RFH.

Corollary

No even sphere $\mathbb{S}^{2n}$ admits a symplectic form for $n \geq 2$.

The only sphere admitting a symplectic form is $\mathbb{S}^2$.

Proof.

$$H^k_{dR}(\mathbb{S}^{2n}) = \begin{cases} \mathbb{R} & k = 0, 2n \\ 0 & k \neq 0, 2n \end{cases}$$

$\square$

Lemma

Let M be a smooth manifold. Then $(T^*M, d\lambda)$ is a symplectic manifold with the ***Liouville form***

$$\lambda_{(x,\xi)}(v) := \xi(D\pi(v)) \qquad \forall (x,\xi) \in T^*M, v \in T_{(x,\xi)}T^*M,$$

where $\pi\colon T^*M \to M$ denotes the cotangent bundle projection. The symplectic form $d\lambda$ on T^*M is called the ***canonical symplectic form****.*

$\Rightarrow$ Every cotangent bundle is orientable!

Let (x^i) be coordinates on M. Then (x^i, ξ_i) are coordinates on T^*M.

just one element in $T^*_x M$ in basis dx^i.

- $\lambda = \xi_i dx^i$.
- $d\lambda = \underline{d\xi_i \wedge dx^i}$.

(x^i, ξ_i) are Darboux coordinates!

Lemma

*Let $\varphi\colon M \to \widetilde{M}$ be a diffeomorphism between smooth manifolds M and $\widetilde{M}$. Then the **cotangent lift***

$$D\varphi^\dagger\colon T^*M \to T^*\widetilde{M}, \qquad D\varphi^\dagger_{(x,\xi)}(\varphi(x), v) := \xi(D\varphi^{-1}(v))$$

is an exact symplectomorphism.

Every diffeomorphism can be lifted to a symplectomorphism w.r.t. the canonical symplectic forms on the cotangent bundles.

$$\begin{tikzcd} M \arrow[r, "\varphi"] & \widetilde{M} \\ T^*M \arrow[u] \arrow[r, "D\varphi^\dagger"'] \arrow[ur, phantom, "\circlearrowright"] & T^*\widetilde{M} \arrow[u] \end{tikzcd}$$

Good exercise to get used to tangent bundles of cotangent bundles!
i.e. compute $(D\varphi^\dagger)^*\tilde{\lambda} = \lambda$.

Definition (Lagrangian Submanifold)

Let (M, ω) be a symplectic manifold. An embedded submanifold L of M is said to be ***Lagrangian***, iff $\dim L = \frac{1}{2} \dim M$ and $\iota_L^* \omega = 0$.

Lemma

Let $\varphi \colon M \to \tilde{M}$ *be a diffeomorphism between symplectic manifolds* (M, ω) *and* $(\tilde{M}, \tilde{\omega})$. *Then* φ *is a symplectomorphism if and only if its graph*

$$\Gamma_\varphi \subseteq (M \times \tilde{M}, -\omega \oplus \tilde{\omega})$$

is a Lagrangian submanifold. $\dim \Gamma_\varphi = \frac{1}{2} \dim (M \times \tilde{M})$

Proof. Exercise.

The Darboux Theorem and Moser's Trick

Nonlinear analogue of the canonical form Thm.

Theorem (Darboux Theorem) (Weinstein)

Let M^{2n} be a smooth manifold and $\omega \in \Omega^2(M)$ nondegenerate. Then ω is closed if and only if for every $x \in M$ there exists a chart $(U, (x^i, y^i))$ about x such that

$$\omega|_U = \sum_{i=1}^{n} dy^i \wedge dx^i.$$ (Darboux Coordinates)

Symplectic manifolds are locally indistinguishable!

Proof. Let $x \in M$. By the canonical form theorem there exists a chart such that

$$\omega_x = \sum_{i=1}^{n} dy^i|_x \wedge dx^i|_x$$

only at one point!

$$\omega_1 := \sum_{i=1}^{n} dy^i \wedge dx^i \qquad \omega_0 := \omega|_U$$

Moser's Trick the theorem is proven, if we find a chart (U, φ) about x s.t. $\varphi: U \to \varphi(U) \subseteq \mathbb{C}^n$

$$\varphi^* \omega_1 = \omega_0.$$

Show existence of such a φ by letting it be the time-1 map of a time-dependent vector field,

$$\varphi = \psi_1,$$

- $\frac{d}{dt}\psi_t = X_t \circ \psi_t$,
- $\psi_0 = id$.

$\omega_t|_x = \omega_x$ by construction.

- ω_t is closed.
- ω_t is nondegenerate for some compact interval around $[0,1]$ in some neighborhood

For $t \in \mathbb{R}$, set

$$\omega_t := (1-t)\omega_0 + t\,\omega_1$$

We compute improved Fisherman's formula

$$\frac{d}{dt}\varphi_t^*\omega_t = \varphi_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t\right) = \varphi_t^*(\mathcal{L}_{X_t}\omega_t + \underbrace{\omega_1 - \omega_0}_{-d\eta \text{ by Poincaré}})$$

$$= \varphi_t^*\left(\cancel{i_{X_t} d\omega_t} + d\, i_{X_t}\omega_t - d\eta\right) = d\varphi_t^*\left(\underline{i_{X_t}\omega_t - \eta}\right)$$

Cartan's magic formula

Need $\frac{d}{dt}\varphi_t^*\omega_t = 0$, because then

$$\varphi^*\omega_1 = \varphi_1^*\omega_1 = \varphi_0^*\omega_0 = \omega_0 .$$

Why does such an X_t even exist?

$$i_{X_t}\omega_t = \eta \Rightarrow X_t := \hat{\omega}_t^{-1}(\eta)$$

due to nondegeneracy of ω_t $\forall t \in J \supseteq [0,1]$.

"Symplectic version of ε - δ proof!"

$\square$