Lecture 5: Noether's Theorem

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Lie Group Actions

The most important applications of Lie groups to smooth manifold theory involve actions by Lie groups on other manifolds.

Definition (Lie Group Action)

A ***left action of a Lie group G on a smooth manifold M*** is defined to be a smooth map

$$\theta \colon G \times M \to M$$

identity in G group law

such that

$$\theta_e = \mathrm{id}_M \qquad \text{and} \qquad \theta_{gh} = \theta_g \circ \theta_h$$

for all $g, h \in G$ where $\theta_g := \theta(g, \cdot) \in \mathrm{Diff}(M)$.

This follows from

$$\mathrm{id}_M = \theta_e = \theta_{gg^{-1}} = \theta_g \circ \theta_{g^{-1}}$$

$$\Rightarrow \theta_g^{-1} = \theta_{g^{-1}}$$

Lemma

Let M be a smooth manifold and $X \in \mathfrak{X}(M)$ complete. Then the flow of X is a left $(\mathbb{R}, +)$-action on M.

the flow exists for all $t \in \mathbb{R}$

Proof. $\theta^X : \mathbb{R} \times M \longrightarrow M$

$$\theta^X_e = \mathrm{id}_M$$

$$\theta^X_{s+t} = \theta_s \circ \theta_t$$

"Equivalently"

A left G-action can be thought of a Lie group homomorphism

$$\theta : G \to \mathrm{Diff}(M), \quad g \mapsto \theta_g$$

smooth?

Poisson Actions

Let $\theta: G \times M \to M$ be a smooth left action of a Lie group G on a smooth manifold M and denote by $\mathfrak{g} := T_e G \cong \mathfrak{X}_L(G)$ the corresponding Lie algebra. Each element $\xi \in \mathfrak{g}$ determines a smooth global flow on M by

$$(t, x) \mapsto \theta_{\exp(-t\xi)}(x).$$

$\exp(s)\cdot\exp(t) = \exp(s+t)\quad \forall s,t \in \mathbb{R}$

uniform time

lemma

Define $\widehat{\xi} \in \mathfrak{X}(M)$ to be the infinitesimal generator of this flow, that is,

$$\boxed{\widehat{\xi}_x = \frac{d}{dt}\Big|_{t=0} \theta_{\exp(-t\xi)}(x) \qquad \forall x \in M.}$$

If $\exists \varepsilon > 0$ s.t. $\forall x \in M$ $\gamma_x : (-\varepsilon_x, \varepsilon_x) \to M$ exists, the flow complete.

The map $\xi \mapsto \widehat{\xi}$ is a Lie algebra homomorphism.

Suppose $\theta: \mathbb{R} \times M \to M$ is a left $(\mathbb{R}, +)$-action, then

$$X := \frac{d}{dt}\Big|_{t=0} \theta_t \in \mathfrak{X}(M). \qquad \theta^X = \theta.$$

infinitesimal generator

Definition (Weakly Hamiltonian Action)

Let (M,ω) be a symplectic manifold. A smooth action $\theta\colon G \times M \to M$ of a Lie group G is said to be ***weakly Hamiltonian***, iff $\theta_g^*\omega = \omega$ for all $g \in G$ and there exists a linear map $\Leftrightarrow \theta_g \in \mathrm{Symp}(M,\omega)$

$$\mu\colon \mathfrak{g} \to C^\infty(M),$$

called a ***momentum map***, such that the diagram

$$\begin{tikzcd} C^\infty(M) \arrow[rr, "f \mapsto X_f"] & & \mathfrak{X}(M,\omega) \\ & \mathfrak{g} \arrow[ul, "\mu"] \arrow[ur, "\xi \mapsto \hat{\xi}"'] & \end{tikzcd}$$

locally constant fun.

symplectic vector fields on (M,ω)

$X_{\mu(\xi)} = \hat{\xi}$ $\forall \xi \in \mathfrak{g}$.

commutes.

"Comomentum map"

$\mu^*: M \to \mathfrak{g}^* \rightsquigarrow \mu(\xi)(x) := \mu^*(x)(\xi)$

Definition (Poisson Action)

A weakly Hamiltonian G-action on a symplectic manifold (M, ω) is said to be ***Poisson***, iff the corresponding momentum map

$$\mu\colon (\mathfrak{g}, [\cdot,\cdot]) \to (C^\infty(M), \{\cdot,\cdot\})$$

is a Lie algebra homomorphism.

Poisson bracket

Lie bracket
of left-invariant
vector fields

Physical case: $M = T^*N$, N smooth manifold.

Lemma (Momentum Lemma)

Let $\theta \in C^\infty(G \times M, M)$ be a Lie group action on an exact symplectic manifold $(M, d\lambda)$ such that $\theta_g^*\lambda = \lambda$ for all $g \in G$ holds. Then the action θ is Poisson with

$$\boxed{\mu(\xi) = i_{\hat{\xi}}(\lambda), \qquad \forall \xi \in \mathfrak{g}.}$$

Proof. ① $\theta_g^* d\lambda = d\theta_g^*\lambda = d\lambda \quad \forall g \in G.$

② I want $X_{i_{\hat\xi}\lambda} = \hat\xi \quad \forall \xi \in \mathfrak{g}.$

$$i_{\hat\xi} d\lambda = \mathcal{L}_{\hat\xi}\lambda - d i_{\hat\xi}\lambda = \frac{d}{dt}\Big|_{t=0} \theta^*_{\exp(-t\xi)}\lambda - d\mu(\xi)$$

Cartan's magic formula

$$= \frac{d}{dt}\Big|_{t=0} \lambda - d\mu(\xi)$$

$$= -d\mu(\xi).$$

[Weakly Hamiltonian]

③ $\mu[\xi,\eta] = i_{\widehat{[\xi,\eta]}}\lambda$

$= i_{[\hat\xi,\hat\eta]}\lambda$

$\xi \mapsto \hat\xi$ is a Lie algebra homomorphism

$= \mathcal{L}_{\hat\xi} i_{\hat\eta}\lambda - i_{\hat\eta}\underbrace{\mathcal{L}_{\hat\xi}\lambda}_{=0}$

$= \mathcal{L}_{\hat\xi} i_{\hat\eta}\lambda$

$= \mathcal{L}_{\hat\xi}\mu(\eta)$

$= \hat\xi\,\mu(\eta)$

weakly Hamiltonian $\to$ $= X_{\mu(\xi)}\mu(\eta)$

$= \{\mu(\xi),\mu(\eta)\}$ $\quad\forall \xi,\eta \in \mathfrak{g}$.

$\square$

Corollary

Let $\theta \in C^\infty(G \times M, M)$ be a Lie group action on a smooth manifold M. Then the lifted action $g \mapsto D\theta_g^\dagger$ on $(T^*M, d\lambda)$ is Poisson with

$$\mu(\xi)(x,p) = p(\widehat{\xi}) \qquad \forall \xi \in \mathfrak{g}, (x,p) \in T^*M.$$

cotangent lift

i.e. if $\varphi \in \mathrm{Diff}(M)$, then $D\varphi^\dagger : T^*M \to T^*M$

$$(x,p) \mapsto (\varphi(x), \underbrace{D\varphi^\dagger(x,p)}_{= p(D\varphi^{-1}\,\cdot)})$$

Momentum lemma

$$\mu(\xi)(x,p) \overset{\downarrow}{=} i_\xi \lambda = \lambda\left(\left.\frac{d}{dt}\right|_{t=0} D\theta^\dagger_{\exp(-t\xi)}(x,p)\right)$$

$$= p\left(D\pi_{T^*M} \cdot \left.\frac{d}{dt}\right|_{t=0} D\theta^\dagger_{\exp(-t\xi)}(x,p)\right)$$

Def. Liouville form

$$= p\left(\left.\frac{d}{dt}\right|_{t=0} \pi_{T^*M} \circ D\theta^\dagger_{\exp(-t\xi)}(x,p)\right)$$

chain rule

$$= p\left(\left.\frac{d}{dt}\right|_{t=0} \theta_{\exp(-t\xi)}(x)\right) = p(\widehat{\xi}) \qquad \square$$

Noether's Theorem

Definition (Symmetry Group)

A Lie group G is said to be a ***symmetry group of a Hamiltonian system*** $(\boldsymbol{M}, \boldsymbol{\omega}, \boldsymbol{H})$, iff there exists a weakly Hamiltonian action θ of G on (M, ω), such that $\theta_g^* H = H$ for all $g \in G$.

$\Updownarrow$

$H \circ \theta_g = H \quad \forall g \in G$

Theorem (Noether's Theorem)

Let G be a symmetry group of a Hamiltonian system (M, ω, H). Then $\mu(\xi)$ is an integral of motion for all $\xi \in \mathfrak{g}$.

Proof. Let $\xi \in \mathfrak{g}$. We compute

$$\{\mu(\xi), H\} = X_{\mu(\xi)} H \underset{\substack{\text{weakly}\\\text{Hamiltonian}\\\text{action}}}{=} \hat{\xi} H = \frac{d}{dt}\Big|_{t=0} H \circ \theta_{\exp(-t\xi)}$$

$$= \frac{d}{dt}\Big|_{t=0} H = 0.$$

$\square$

$$H(x,y) := \frac{1}{2}|y|^2 + V(|x|^2) \qquad (x,y) \in \mathbb{R}^{2n} \setminus \{0\}.$$

Pick $A \in \mathfrak{so}(n)$, then

$$H(Ax, Ay) = H(x,y).$$

[Kepler Problem]

Symmetries help you solving ODE's.

Lie Algebra Cohomology

Let $\mathfrak{g}$ be a Lie algebra. Define

$\mu(\xi) + c_\xi \quad \forall \xi \in \mathfrak{g}$

$c : \mathfrak{g} \to \mathbb{R}$

$\Rightarrow c \in \mathfrak{g}^*$

$$C^k := \Lambda^k \mathfrak{g}^*$$

k-alternating forms on $\mathfrak{g}$

and $d : C^k \to C^{k+1}$ by

$$d\tau(\xi_0, \ldots, \xi_k) := \sum_{0 \le i < j \le k} (-1)^{i+j} \tau([\xi_i, \xi_j], \xi_0, \ldots, \bar{\xi}_i, \ldots, \bar{\xi}_j, \ldots, \xi_k).$$

This comes from exterior differential

omitted

Then one checks that $d \circ d = 0$. The resulting nonnegative chain complex is called the ***Chevalley–Eilenberg cochain complex***.

Definition (Lie Algebra Cohomology)

Let $\mathfrak{g}$ be a Lie algebra. Then the ***k-th cohomology group of*** $\mathfrak{g}$ is defined by

$$H^k(\mathfrak{g};\mathbb{R}) := \frac{\ker d\colon C^k \to C^{k+1}}{\operatorname{im} d\colon C^{k-1} \to C^k}.$$

Theorem (Uniqueness of Momentum Maps for Poisson Actions)

Suppose that μ and $\tilde{\mu}$ are two momentum maps for a Poisson G-action on a connected symplectic manifold. If $H^1(\mathfrak{g};\mathbb{R}) = 0$, then $\mu = \tilde{\mu}$.

Proof. We know that there exists $\sigma \in \mathfrak{g}^*$

$$\mu(\xi) - \tilde{\mu}(\xi) = \sigma(\xi) \quad \forall \xi \in \mathfrak{g}$$

Not nec. $\sigma \in H^1(\mathfrak{g};\mathbb{R})$.

$$d\sigma(\xi,\eta) = \sigma([\eta,\xi]) = \mu([\eta,\xi]) - \tilde{\mu}([\eta,\xi])$$

$$= \{\mu(\eta),\mu(\xi)\} - \{\tilde{\mu}(\eta),\tilde{\mu}(\xi)\}$$

$$= 0$$

$$\{\mu(\eta),\mu(\xi)\} = \omega(X_{\mu(\eta)}, X_{\mu(\xi)})$$
$$= \omega(\hat{\eta},\hat{\xi})$$
$$= \{\tilde{\mu}(\eta),\tilde{\mu}(\xi)\}$$

$\Rightarrow \sigma \in H^1(\mathfrak{g};\mathbb{R}) = 0$

$\Rightarrow \sigma = 0$

$\Rightarrow \mu = \tilde{\mu}$. $\square$

Theorem (Existence of Poisson Actions)

Suppose we are given a weakly Hamiltonian G-action on a connected symplectic manifold (M,ω). If $H^2(\mathfrak{g};\mathbb{R}) = 0$, then the action is Poisson.

Proof. Play the same game as in previous Theorem, i.e. $\tau \in \Lambda^2\mathfrak{g}^*$.

$$\{\mu(\xi),\mu(\eta)\} - \mu([\xi,\eta]) = \tau(\xi,\eta) \quad \forall \xi,\eta \in \mathfrak{g}.$$

- Show $d\tau = 0$. $\Rightarrow \tau \in H^2(\mathfrak{g};\mathbb{R}) = 0$

  $\Rightarrow \exists \sigma \in H^1(\mathfrak{g};\mathbb{R})$ s.t. $\tau = d\sigma$.

Now observe that

$$\tilde{\mu}(\xi) := \mu(\xi) - \sigma(\xi) \in C^\infty(M)$$

is a Lie algebra homomorphism.

$\square$

Whitehead Lemmas

$\mathfrak{g}$ semisimple :⇔ if there exist no nontrivial abelian ideals of $\mathfrak{g}$.

Lemma (Whitehead's First Lemma)

Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $H^1(\mathfrak{g};\mathbb{R}) = 0$.

Lemma (Whitehead's Second Lemma)

Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $H^2(\mathfrak{g};\mathbb{R}) = 0$.

Ref. Charles Weibel
"An introduction to homological algebra"