# Lecture 5: Noether's Theorem

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May 2, 2021

# Poisson Actions

Let $\theta \in C^\infty(G \times M, M)$ be a left action of a Lie group $G$ on a smooth manifold $M$ and denote by $\mathfrak{g} := \operatorname{Lie}(G)$ the corresponding Lie algebra. Each element $\xi \in \mathfrak{g}$ determines a smooth global flow on $M$ by

$$(t, x) \mapsto \theta_{\exp(-t\xi)}(x).$$

Define $\widehat{\xi} \in \mathfrak{X}(M)$ to be the infinitesimal generator of this flow, that is,

$$\widehat{\xi}_x = \frac{d}{dt}\bigg|_{t=0} \theta_{\exp(-t\xi)}(x).$$

The map $\xi \mapsto \widehat{\xi}$ is a Lie algebra homomorphism.

## Definition (Weakly Hamiltonian Action)

A left action of a Lie group $G$ on a symplectic manifold $(M, \omega)$ by symplectomorphisms is said to be ***weakly Hamiltonian***, iff for each $\xi \in \mathfrak{g}$ there exists $H_\xi \in C^\infty(M)$ such that

$$X_{H_\xi} = \widehat{\xi}.$$

## Definition (Comoment Map)

Given a weakly Hamiltonian action of a Lie group $G$ on a symplectic manifold $(M, \omega)$, define its ***comoment map*** by

$$\mu^* \colon \mathfrak{g} \to C^\infty(M), \qquad \mu^*(\xi) := H_\xi.$$

$$\begin{tikzcd}
C^\infty(M) \arrow[rr, "f \mapsto X_f"] & & \mathfrak{X}(M, \omega) \\
& \mathfrak{g} \arrow[ul, "\mu^*"] \arrow[ur, "\xi \mapsto \widehat{\xi}"'] &
\end{tikzcd}$$

## Definition (Poisson Action)

A left action of a Lie group $G$ on a symplectic manifold $(M, \omega)$ by symplectomorphisms is said to be ***Poisson***, iff it is a weakly Hamiltonian action such that the corresponding comoment map

$$\mu^*\colon (\mathfrak{g}, [\cdot, \cdot]) \to (C^\infty(M), \{\cdot, \cdot\})$$

is a Lie algebra homomorphism.

# The Momentum Lemma

## Lemma (Momentum Lemma)

*Let* $\theta \in C^\infty(G \times M, M)$ *be a Lie group action on an exact symplectic manifold* $(M, d\lambda)$ *such that* $\theta_g^*\lambda = \lambda$ *for all* $g \in G$ *holds. Then the action* $\theta$ *is Poisson with*

$$\mu^*(\xi) = i_{\hat{\xi}}(\lambda), \qquad \forall \xi \in \mathfrak{g}.$$

## Corollary

*Let $\theta \in C^\infty(G \times M, M)$ be a Lie group action on a smooth manifold $M$. Then the lifted action $g \mapsto D\theta_g^\dagger$ on $(T^*M, d\lambda)$ is Poisson with*

$$\mu^*(\xi)(q, p) = p(\widehat{\xi}).$$

**Definition (Symmetry Group)**

A Lie group $G$ is said to be a ***symmetry group of a Hamiltonian system*** $\boldsymbol{(M, \omega, H)}$, iff there exists a weakly Hamiltonian action $\theta$ of $G$ on $(M, \omega)$, such that $\theta_g^* H = H$ for all $g \in G$.

## Theorem (Noether's Theorem)

*Let $G$ be a symmetry group of a Hamiltonian system $(M, \omega, H)$. Then $\mu^*(\xi)$ is an integral of motion for all $\xi \in \mathfrak{g}$.*

Let $\mathfrak{g}$ be a Lie algebra. Define

$$C^k := \Lambda^k \mathfrak{g}^*$$

and $d\colon C^k \to C^{k+1}$ by

$$\begin{aligned} df(X_0, \ldots, X_k) \\ &:= \sum_{0 \le i < j \le k} (-1)^{i+j} f([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k). \end{aligned}$$

Then one checks that $d \circ d = 0$. The resulting nonnegative chain complex is called the ***Chevalley–Eilenberg cochain complex***.

**Definition (Lie Algebra Cohomology)**

Let $\mathfrak{g}$ be a Lie algebra. Then the ***k-th cohomology group of*** $\mathfrak{g}$ is defined by

$$H^k(\mathfrak{g};\mathbb{R}) := \frac{\ker d\colon C^k \to C^{k+1}}{\operatorname{im} d\colon C^{k-1} \to C^k}.$$

## Theorem

*Let $\theta \in C^\infty(G \times M, M)$ be a weakly Hamiltonian action on a connected symplectic manifold $(M, \omega)$. If $H^2(\mathfrak{g}; \mathbb{R}) = 0$, then the action is Poisson.*

## Theorem

*Let* $\theta \in C^\infty(G \times M, M)$ *be a Poisson action on a connected symplectic manifold* $(M, \omega)$ *with comoment maps* $\mu_1^*$ *and* $\mu_2^*$*. If* $H^1(\mathfrak{g}; \mathbb{R}) = 0$*, then*

$$\mu_1^* = \mu_2^*.$$

# Whitehead Lemmas

**Lemma (Whitehead's First Lemma)**

*Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $H^1(\mathfrak{g};\mathbb{R}) = 0$.*

**Lemma (Whitehead's Second Lemma)**

*Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $H^2(\mathfrak{g};\mathbb{R}) = 0$.*