Chapter 2
Classical Mechanics

2.1 The Hamiltonian Formalism

Definition 2.1 (Hamiltonian System). A ***Hamiltonian system*** is defined to be a tuple $((M, \omega), H)$, where (M, ω) is a finite-dimensional symplectic manifold, called the ***phase space***, and $H \in C^\infty(M)$ is a smooth function, called the ***Hamiltonian function***.

Definition 2.2 (Hamiltonian Vector Field). Let (M, ω, H) be a Hamiltonian system. The corresponding ***Hamiltonian vector field*** is the vector field $X_H \in \mathfrak{X}(M)$ given implicitly by

$$i_{X_H}\omega = -dH.$$

Proposition 2.3. *Let (M, ω) and $(\widetilde{M}, \widetilde{\omega})$ be two symplectic manifolds and suppose that $\varphi \in C^\infty(M, \widetilde{M})$ is a symplectomorphism. Then*

$$\varphi^* X_f = X_{\varphi^* f}, \qquad \forall f \in C^\infty(\widetilde{M}).$$

Proof. We compute

$$i_{X_{\varphi^* f}}\omega = -d\varphi^* f = -\varphi^* df = \varphi^*(i_{X_f}\widetilde{\omega}) = i_{\varphi^* X_f}(\varphi^*\widetilde{\omega}) = i_{\varphi^* X_f}\omega.$$

□

Proposition 2.4. *Let (M, ω) and $(\widetilde{M}, \widetilde{\omega})$ be two symplectic manifolds and suppose that $\varphi \in C^\infty(M, \widetilde{M})$ is a symplectomorphism. Then*

$$\theta_t^{X_{\varphi^* f}} = \varphi^{-1} \circ \theta_t^{X_f} \circ \varphi, \qquad \forall f \in C^\infty(\widetilde{M})$$

whenever either side is defined.

Proof. Using proposition 2.3 we compute

$$\begin{aligned}\frac{d}{dt}\varphi^{-1}\circ\theta_t^{X_f}\circ\varphi &= D\varphi^{-1}\circ\frac{d}{dt}\theta_t^{X_f}\circ\varphi\\ &= D\varphi^{-1}\circ X_f\circ\theta_t^{X_f}\circ\varphi\\ &= D\varphi^{-1}\circ X_f\circ\varphi\circ\varphi^{-1}\circ\theta_t^{X_f}\circ\varphi\\ &= \varphi^* X_f\circ\varphi^{-1}\circ\theta_t^{X_f}\circ\varphi\\ &= X_{\varphi^* f}\circ\varphi^{-1}\circ\theta_t^{X_f}\circ\varphi\end{aligned}$$

and the result follows by the uniqueness of integral curves. □

Definition 2.5 (Poisson Bracket). Let (M,ω) be a symplectic manifold. Define a mapping

$$\{\cdot,\cdot\}: C^\infty(M)\times C^\infty(M)\to C^\infty(M)$$

by

$$\{f,g\} := \omega(X_f, X_g)$$

where X_f and X_g are Hamiltonian vector fields associated to the Hamiltonian systems (M,ω,f) and (M,ω,g), respectively. The mapping $\{\cdot,\cdot\}$ is called the ***Poisson bracket on the algebra of observables*** $\boldsymbol{C^\infty(M)}$.

Let (M,ω) be a symplectic manifold. We show that the algebra of observables $C^\infty(M)$ together with the Poisson bracket is a Poisson algebra.

Definition 2.6 (Poisson Algebra). A ***Poisson algebra*** is defined to be a real commutative algebra $\mathfrak{p}$ together with a Lie bracket $\{\cdot,\cdot\}$ on $\mathfrak{p}$ satisfying the ***Leibniz rule***

$$\{f,gh\} = h\{f,g\} + g\{f,h\} \qquad \forall f,g,h\in\mathfrak{p}.$$

Proposition 2.7. *Let (M,ω) be a symplectic manifold. Then*

$$X_{\{f,g\}} = [X_f, X_g] \qquad \forall f,g\in C^\infty(M).$$

Proof. We compute

$$\begin{aligned}i_{[X_f,X_g]}\omega &= \mathcal{L}_{X_f} i_{X_g}\omega - i_{X_g}\mathcal{L}_{X_f}\omega\\ &= -\mathcal{L}_{X_f} dg - i_{X_g}(i_{X_f} d\omega + d i_{X_f}\omega)\\ &= -(i_{X_f} ddg + d i_{X_f} dg) + i_{X_g}(ddf)\\ &= -d i_{X_f} dg\\ &= d i_{X_f} i_{X_g}\omega\\ &= -d\{f,g\}.\end{aligned}$$

□

Proposition 2.8. *Let (M,ω) be a symplectic manifold. Then $(C^\infty(M),\{\cdot,\cdot\})$ is a Poisson algebra.*

Proof. The bilinearity and antisymmetry of the Poisson bracket is immediate from the definition. Moreover, the Leibniz rule follows from the computation

$$\begin{aligned}\{f, gh\} &= \omega(X_f, X_{gh}) \\ &= \omega(X_f, hX_g + gX_h) \\ &= h\omega(X_f, X_g) + g\omega(X_f, X_h) \\ &= h\{f, g\} + g\{f, h\}\end{aligned}$$

for all $f, g, h \in C^\infty(M)$. Finally, the Jacobi identity follows using proposition 2.7 from the computation

$$\begin{aligned}\{f, \{g, h\}\} &= \omega(X_f, X_{\{g,h\}}) \\ &= \omega(X_f, [X_g, X_h]) \\ &= i_{X_f}\omega[X_g, X_h] \\ &= -df[X_g, X_h] \\ &= -[X_g, X_h]f \\ &= -X_g X_h f + X_h X_g f \\ &= -X_g\{h, f\} + X_h\{g, f\} \\ &= -\{g, \{h, f\}\} + \{h, \{g, f\}\} \\ &= -\{g, \{h, f\}\} - \{h, \{f, g\}\}.\end{aligned}$$

□

Corollary 2.9. *Let* (M, ω) *be a symplectic manifold. Then*

$$(C^\infty(M), \{f, g\}) \to (\mathfrak{X}(M), [\cdot, \cdot]), \qquad f \mapsto X_f$$

is a Lie algebra homomorphism.

Definition 2.10 (Evolution Operator). Let (M, ω, H) be a complete Hamiltonian system. Define the ***evolution operator***

$$U_t\colon C^\infty(M) \to C^\infty(M), \qquad U_t(f) := f \circ \theta_t^{X_H}$$

for all $t \in \mathbb{R}$.

Proposition 2.11. *Let* (M, ω, H) *be a complete Hamiltonian system. Then*

$$\frac{d}{dt}U_t(f) = U_t\{H, f\} \qquad \forall f \in C^\infty, t \in \mathbb{R}.$$

Proof. Using Fisherman's formula we compute

$$\frac{d}{dt}U_t(f) = \frac{d}{dt}\left(\theta_t^{X_H}\right)^* f = \left(\theta_t^{X_H}\right)^* \mathfrak{L}_{X_H} f = \left(\theta_t^{X_H}\right)^* \{H, f\} = U_t\{H, f\}.$$

□

More generally, we have the following fundamental property of an autonomous Hamiltonian function on a symplectic manifold.

Corollary 2.12 (Preservation of Energy). *Let (M, ω, H) be a Hamiltonian system and denote by $\theta_t^{X_H} : \mathcal{D} \to M$ the flow of the Hamiltonian vector field X_H. Then*

$$H\left(\theta_t^{X_H}(x)\right) = H(x) \qquad \forall (t, x) \in \mathcal{D}.$$

Proof. This follows immediately from a local version of Proposition 2.11 since

$$\frac{d}{dt} U_t(H) = U_t\{H, H\} = 0.$$

□

Motivtated by proposition 2.11 we give the following definition of a preserved quantity in a Hamiltonian system.

Definition 2.13 (Integral of Motion). An ***integral of motion*** for a Hamiltonian system (M, ω, H) is defined to be a smooth function $I \in C^\infty(M)$ such that $\{H, I\} = 0$.

Remark 2.14. It is easy to check that the integrals of motion form a Lie subalgebra of $(C^\infty(M), \{\cdot, \cdot\})$.

Let $\theta : G \times M \to M$ be a smooth left action of a Lie group G on a smooth manifold M and denote by $\mathfrak{g} := T_e G \cong \mathfrak{X}_L(G)$ the corresponding Lie algebra. Every $\xi \in \mathfrak{g}$ determines a smooth global flow on M by $(t, x) \mapsto \theta_{\exp(-t\xi)}(x)$, where

$$\exp : \mathfrak{g} \to G, \qquad \exp(\xi) := \gamma_\xi(1)$$

denotes the exponential map and γ_ξ is the integral curve of the left-invariant vector field X_ξ starting at e with $\dot{\gamma}_\xi(0) = \xi$. Note that if there exists a bi-invariant Riemannian metric on G, then this exponential map coincides with the exponential map of the associated Levi–Civita connection at the identity. Define $\widehat{\xi} \in \mathfrak{X}(M)$ to be the infinitesimal generator of this flow, that is,

$$\widehat{\xi}_x = \left.\frac{d}{dt}\right|_{t=0} \theta_{\exp(-t\xi)}(x) \qquad \forall x \in M.$$

Then

$$(\mathfrak{g}, [\cdot, \cdot]) \to (\mathfrak{X}(M), [\cdot, \cdot]), \qquad \xi \mapsto \widehat{\xi}$$

is a Lie algebra homomorphism.

Definition 2.15 (Weakly Hamiltonian Action). Let (M, ω) be a symplectic manifold. A smooth action $\theta : G \times M \to M$ of a Lie group G is said to be ***weakly Hamiltonian***, iff $\theta_g^* \omega = \omega$ for all $g \in G$ and there exists a linear map

$$\mu : \mathfrak{g} \to C^\infty(M),$$

called a ***momentum map***, such that the diagram

$$\begin{tikzcd} C^\infty(M) \arrow[rr, "f\mapsto X_f"] & & \mathfrak{X}(M,\omega) \\ & \mathfrak{g} \arrow[ul, "\mu"] \arrow[ur, "\xi\mapsto\widehat{\xi}"'] & \end{tikzcd}$$

commutes.

Definition 2.16. A weakly Hamiltonian action of a Lie group G on a symplectic manifold (M,ω) is called

- ***Hamiltonian***, iff the momentum map $\mu\colon \mathfrak{g} \to C^\infty(M)$ is G-equivariant with respect to the adjoint action of G on its associated Lie algebra $\mathfrak{g}$ and the induced action of G on $C^\infty(M)$, that is

$$\mu(\mathrm{Ad}_{g^{-1}}(\xi)) = \mu(\xi)\circ\theta_g \qquad \forall g\in G, \xi\in\mathfrak{g},$$

  where

$$\mathrm{Ad}_{g^{-1}}(\xi) := \frac{d}{dt}\bigg|_{t=0} g^{-1}\exp(t\xi)g.$$

- ***Poisson***, iff the associated momentum map

$$\mu\colon (\mathfrak{g},[\cdot,\cdot]) \to (C^\infty(M),\{\cdot,\cdot\})$$

  is a Lie algebra homomorphism.

For showing existence and uniqueness results for Poisson actions, we recall the basic notions of Lie algebra cohomology. Let $\mathfrak{g}$ be a Lie algebra. Define

$$C^k := \Lambda^k\mathfrak{g}^*$$

and $d: C^k \to C^{k+1}$ by

$$d\tau(\xi_0,\dots,\xi_k) := \sum_{0\le i<j\le k} (-1)^{i+j}\tau([\xi_i,\xi_j],\xi_0,\dots,\overline{\xi_i},\dots,\overline{\xi_j},\dots,\xi_k).$$

Then one checks that $d\circ d = 0$. The resulting nonnegative chain complex is called the ***Chevalley–Eilenberg cochain complex***. Then the ***k-th cohomology group of*** $\mathfrak{g}$ is defined by

$$H^k(\mathfrak{g};\mathbb{R}) := \frac{\ker d: C^k\to C^{k+1}}{\operatorname{im} d: C^{k-1}\to C^k}.$$

Theorem 2.17 (Uniqueness of Momentum Maps for Poisson Actions). *Let μ and $\widetilde{\mu}$ be two momentum maps for a Poisson G-action on a connected symplectic manifold. If $H^1(\mathfrak{g};\mathbb{R}) = 0$, then $\mu = \widetilde{\mu}$.*

Proof. By assumption there exists $\sigma\in\mathfrak{g}^*$ such that

$$\mu(\xi) - \widetilde{\mu}(\xi) = \sigma(\xi) \qquad \forall \xi \in \mathfrak{g}.$$

Since both μ and $\widetilde{\mu}$ are Lie algebra homomorphisms, we have that $d\sigma = 0$. Indeed, for $\xi, \eta \in \mathfrak{g}$ we compute

$$d\sigma(\xi, \eta) = \sigma([\eta, \xi]) = \mu([\eta, \xi]) - \widetilde{\mu}([\eta, \xi]) = \{\mu(\eta), \mu(\xi)\} - \{\widetilde{\mu}(\eta), \widetilde{\mu}(\xi)\} = 0.$$

Thus $\sigma \in H^1(\mathfrak{g}; \mathbb{R}) = 0$, implying $\sigma = 0$ and the statement follows. □

Theorem 2.18 (Existence of Poisson Actions). *Suppose we are given a weakly Hamiltonian G-action on a connected symplectic manifold (M, ω). If $H^2(\mathfrak{g}; \mathbb{R}) = 0$, then the action is Poisson.*

Proof. For $\xi, \eta \in \mathfrak{g}$ we compute

$$X_{\mu([\xi,\eta])} = \widehat{[\xi, \eta]} = [\widehat{\xi}, \widehat{\eta}] = [X_{\mu(\xi)}, X_{\mu(\eta)}] = X_{\{\mu(\xi),\mu(\eta)\}}$$

using Proposition 2.7. Thus by connectedness of M there exists $\tau \in \Lambda^2\mathfrak{g}^*$ such that

$$\{\mu(\xi), \mu(\eta)\} - \mu([\xi, \eta]) = \tau(\xi, \eta) \qquad \forall \xi, \eta \in \mathfrak{g}.$$

Invoking the Jacobi identity for the Lie as well as the Poisson bracket, yields $d\tau = 0$ and so $\tau \in H^2(\mathfrak{g}; \mathbb{R}) = 0$. Hence there exists $\sigma \in H^1(\mathfrak{g}; \mathbb{R})$ such that $\tau = d\sigma$. The momentum map

$$\mathfrak{g} \to C^\infty(M), \qquad \xi \mapsto \mu(\xi) - \sigma(\xi)$$

is a Lie algebra homomorphism. □

Recall, that a Lie algebra $\mathfrak{g}$ is said to be semisimple iff $\mathfrak{g}$ does not admit any nontrivial abelian ideals.

Corollary 2.19. *Let G be a Lie group with semisimple Lie algebra. Then every weakly Hamiltonian G-action on a connected symplectic manifold is Poisson and admits a unique momentum map.*

Proof. The statement immediately follows from Theorem 2.17 and 2.18 as the two Whitehead lemmas imply $H^1(\mathfrak{g}; \mathbb{R}) = H^2(\mathfrak{g}; \mathbb{R}) = 0$. □

We give now a profoundly deep example of an action that is simultaniously a Hamiltonian action and a Poisson action.

Lemma 2.20 (Momentum Lemma). *Let $\theta\colon G \times M \to M$ be a smooth Lie group action on an exact symplectic manifold $(M, d\lambda)$ such that $\theta_g^*\lambda = \lambda$ for all $g \in G$ holds. Then the action is Hamiltonian and Poisson with momentum map*

$$\mu(\xi) = i_{\widehat{\xi}}(\lambda), \qquad \forall \xi \in \mathfrak{g}.$$

Proof. We show the result in four steps. Obviously, $\theta_g^* d\lambda = d\lambda$ for all $g \in G$.

Step 1: θ is a weakly Hamiltonian action. Let $\xi \in \mathfrak{g}$. We compute

$$\begin{aligned}
i_{\widehat{\xi}} d\lambda &= \mathcal{L}_{\widehat{\xi}}\lambda - d i_{\widehat{\xi}}\lambda \\
&= \frac{d}{dt}\bigg|_{t=0} \theta^*_{\exp(-t\xi)}\lambda - d\mu(\xi) \\
&= \frac{d}{dt}\bigg|_{t=0} \lambda - d\mu(\xi) \\
&= -d\mu(\xi).
\end{aligned}$$

Step 2: $\theta_g^*\widehat{\xi} = \widehat{\mathrm{Ad}_{g^{-1}}(\xi)}$ for all $g \in G$ and $\xi \in \mathfrak{g}$. We have a commutative diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\mathrm{Ad}_{g^{-1}}"] \arrow[d, "\exp"'] & \mathfrak{g} \arrow[d, "\exp"] \\
G \arrow[r, "C_{g^{-1}}"'] & G,
\end{tikzcd}$$

where $C_{g^{-1}}(h) = g^{-1}hg$ denotes the conjugation action on G. Let $g \in G$. Then we compute

$$\begin{aligned}
\theta_g^*\widehat{\xi} &= D\theta_{g^{-1}} \circ \widehat{\xi} \circ \theta_g \\
&= D\theta_{g^{-1}} \circ \frac{d}{dt}\bigg|_{t=0} \theta_{\exp(-t\xi)} \circ \theta_g \\
&= \frac{d}{dt}\bigg|_{t=0} \theta_{g^{-1}} \circ \theta_{\exp(-t\xi)} \circ \theta_g \\
&= \frac{d}{dt}\bigg|_{t=0} \theta_{g^{-1}\exp(-t\xi)g} \\
&= \frac{d}{dt}\bigg|_{t=0} \theta_{\exp(-t\,\mathrm{Ad}_{g^{-1}}(\xi))} \\
&= \widehat{\mathrm{Ad}_{g^{-1}}(\xi)}.
\end{aligned}$$

Step 3: θ is a Hamiltonian action. Using step 2, we compute

$$\begin{aligned}
\mu(\mathrm{Ad}_{g^{-1}}(\xi)) &= i_{\widehat{\mathrm{Ad}_{g^{-1}}(\xi)}}\lambda \\
&= i_{\theta_g^*\widehat{\xi}}\lambda \\
&= \lambda(D\theta_{g^{-1}} \circ \widehat{\xi} \circ \theta_g) \\
&= \theta_{-g}^*\lambda(\widehat{\xi} \circ \theta_g) \\
&= i_{\widehat{\xi}}\lambda \circ \theta_g \\
&= \mu(\xi) \circ \theta_g
\end{aligned}$$

for all $g \in G$ and $\xi \in \mathfrak{g}$.

Step 4: θ is a Poisson action. For $\xi, \eta \in \mathfrak{g}$ we compute

$$\begin{aligned}
\mu[\xi, \eta] &= i_{\widehat{[\xi,\eta]}}\lambda \\
&= i_{[\widehat{\xi},\widehat{\eta}]}\lambda \\
&= \mathcal{L}_{\widehat{\xi}} i_{\widehat{\eta}}\lambda - i_{\widehat{\eta}}\mathcal{L}_{\widehat{\xi}}\lambda \\
&= \mathcal{L}_{\widehat{\xi}} i_{\widehat{\eta}}\lambda \\
&= \mathcal{L}_{\widehat{\xi}}\mu(\eta) \\
&= \widehat{\xi}\mu(\eta) \\
&= X_{\mu(\xi)}\mu(\eta) \\
&= \{\mu(\xi), \mu(\eta)\}.
\end{aligned}$$

□

Definition 2.21 (Symmetry Group). A Lie group G is said to be a ***symmetry group of a Hamiltonian system*** (M, ω, H), iff there exists a weakly Hamiltonian action θ of G on (M, ω), such that $H \circ \theta_g = H$ for all $g \in G$.

Theorem 2.22 (Noether's Theorem). *Let G be a symmetry group of a Hamiltonian system (M, ω, H). Then $\mu(\xi)$ is an integral of motion for all $\xi \in \mathfrak{g}$.*

Proof. For $\xi \in \mathfrak{g}$ we compute

$$\{\mu(\xi), H\} = X_{\mu(\xi)}H = \widehat{\xi}H = \left.\frac{d}{dt}\right|_{t=0} H \circ \theta_{\exp(-t\xi)} = \left.\frac{d}{dt}\right|_{t=0} H = 0.$$

□

2.2 The Lagrangian Formalism

Definition 2.23 (Lagrangian System). A ***Lagrangian system*** is a tuple (M, L), where M is a finite-dimensional smooth manifold, called the ***configuration space***, and $L \in C^\infty(TM)$ is a smooth function, called the ***Lagrangian function***. Moreover, the tangent bundle TM of the configuration space M is called the ***state space***.

Remark 2.24. Infinite-dimensional configuration spaces are treated in classical field theory (see [10]).

Remark 2.25. Let $\pi \colon E \to M$ be a vector bundle, $\mathcal{H}$ a connection on E and $\kappa \colon TE \to E$ be the associated connection map. Then

$$(D\pi, \kappa) \colon TE \to TM \oplus E$$

is a vector bundle isomorphism along π. In particular

$$TTM \cong TM \oplus TM$$

as vector bundles for every smooth manifold M. Note that this isomorphism depends on the choice of a connection and is therefore not canonical.

Definition 2.26 (Mechanical Lagrangian Function). Let (M, m) be a pseudo-Riemannian manifold and $V \in C^\infty(M)$. A ***mechanical Lagrangian function*** is defined to be the function $L_V \in C^\infty(TM)$

$$L_V(q, v) := \frac{1}{2}|v|_m^2 - V(q).$$

If $F \in C^\infty(M, N)$, then the derivative of F can be interpreted as a vector bundle homomorphism $DF : TM \to F^*TN$. Indeed, define

$$DF(q, v) := (q, (F(q), DF_q(v)))$$

for any $(q, v) \in TM$. If $\pi : E \to M$ is a fibre bundle, we can define

$$VE := \coprod_{p \in E} \ker D\pi_p.$$

Then VE with the usual footpoint projection is a vector bundle over E, called the ***vertical bundle of E***. Moreover, one can show that VE is isomorphic to π^*E. Explicitely, the isomorphism $\Phi : \pi^*E \to VE$ is given by

$$\Phi(v, u) := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (v + \varepsilon u). \tag{2.1}$$

Definition 2.27 (Associated Form). Let (M, L) be a Lagrangian system. Define the ***associated form***, written $\lambda_L \in \Omega^1(TM)$, by

$$\lambda_L(v) := dL((\Phi \circ D\pi_{TM})v) \qquad \forall v \in TTM. \tag{2.2}$$

Definition 2.28 (Legendre Transform). A ***Legendre transform of a Lagrangian system (M, L)*** is defined to be a mapping $\tau_L \in C^\infty(TM, T^*M)$ such that

$$\pi_{T^*M} \circ \tau_L = \pi_{TM} \qquad \text{and} \qquad \lambda_L = \tau_L^*\lambda.$$

Proposition 2.29. *Let (M, L) be a Lagrangian system. Then $\tau_L \in C^\infty(TM, T^*M)$ is a Legendre transform if and only if*

$$\tau_L(v)(u) = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L_q(v + \varepsilon u) \qquad \forall v, u \in T_qM, q \in M.$$

Proof. Suppose $\tau_L \in C^\infty(TM, T^*M)$ is a Legendre transform. For $\zeta \in D\pi_{TM}^{-1}(u)$ we compute on one hand

$$\begin{aligned}(\tau_L^*\lambda)_{(x,v)}(\xi) &= \lambda_{(x,\tau_L(v))}(D\tau_L(\zeta)) \\ &= \tau_L(v)\left((D\pi_{T^*M} \circ D\tau_L)(\zeta)\right) \\ &= \tau_L(v)\left(D(\pi_{T^*M} \circ \tau_L)(\zeta)\right) \\ &= \tau_L(v)\left(D\pi_{TM}(\zeta)\right) \\ &= \tau_L(v)(u),\end{aligned}$$

and on the other

$$\begin{aligned}\lambda_L|_{(x,v)}(\xi) &= dL((\Phi \circ D\pi_{TM})(\zeta)) \\ &= dL\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0}\left(v + \varepsilon D\pi_{TM}(\zeta)\right)\right) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L_q(v + \varepsilon D\pi_{TM}(\zeta)) \\ &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L_q(v + \varepsilon u).\end{aligned}$$

□

Definition 2.30 (Energy). The ***energy of a Lagrangian system*** (M, L) is defined to be the function $E_L \in C^\infty(TM)$ given by

$$E_L(q, v) := \tau_L(v)(v) - L(x, v)$$

for $(q, v) \in TM$.

Definition 2.31 (Hamiltonian Function). Let (M, L) be a Lagrangian system such that the Legendre transform τ_L is a diffeomorphism. The function $H_L \in C^\infty(T^*M)$ defined by

$$H_L := E_L \circ \tau_L^{-1}$$

is called the ***Hamiltonian function associated to the Lagrangian function*** L.

Remark 2.32 (Tonelli Lagrangians). Let (M, L) be a Lagrangian system and fix a Riemannian metric m on M. The Lagrangian function L is said to be ***Tonelli***, iff the following conditions are satisfied:

(T1) The fibrewise Hessian of L is positive-definite, that is,

$$\frac{\partial^2 L}{\partial v^i \partial v^j}(q, v)u^i u^j > 0$$

for all $(q, v) \in TM$ and $u := u^i \partial_i \in T_x M$ such that $u \neq 0$.

(T2) The Lagrangian function L is fibrewise supercoersive, that is,

$$\lim_{|v|_m \to \infty} \frac{L(q, v)}{|v|_m} = +\infty$$

for all $x \in M$.

By [23, Proposition 1.2.1], for a fibrewise convex Lagrangian function L, the associated Legendre transformation $\tau_L\colon TM \to T^*M$ is a diffeomorphism, if and only if L is Tonelli.

Definition 2.33 (Symmetry Group). A Lie group G is called a ***symmetry group of a Lagrangian system*** (M, L), iff there exists a left action of G on M with

$$L \circ D\theta_g = L \qquad \forall g \in G.$$

Theorem 2.34. *Let (M, L) be a Lagrangian system with symmetry group G and such that the Legendre transform τ_L is a diffeomorphism. Then G is a symmetry group of the corresponding Hamiltonian system $(T^*M, d\lambda, H_L)$ with*

$$\mu(\xi)(q, p) = p\left(\left.\frac{d}{dt}\right|_{t=0} \theta_{\exp(-t\xi)}(q)\right) \qquad \forall \xi \in \mathfrak{g}, (q, p) \in T^*M,$$

where θ denotes the smooth left G-action on the configuration space M. Moreover, the induced action on the phase space T^*M is Hamiltonian and Poisson.

Proof. Define a smooth left G-action Θ on the phase space T^*M by

$$\Theta_g := \tau_L \circ D\theta_g \circ \tau_L^{-1} \qquad \forall g \in G.$$

Applying the Momentum Lemma 2.20 to this action yields the Theorem. We proceed in five steps.

Step 1: $D\theta_g^* \lambda_L = \lambda_L$ for all $g \in G$. We compute

$$\begin{aligned}
((D\theta_g)^* \lambda_L)(\zeta) &= dL((\Phi \circ D\pi_{TM} \circ DD\theta_g)(\zeta)) \\
&= dL(\Phi(D\theta_g(v), D\pi_{TM} \circ DD\theta_g(\zeta))) \\
&= dL(\Phi(D\theta_g(v), D\theta_g(D\pi_{TM}(\zeta)))) \\
&= dL\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} D\theta_g(v + \varepsilon D\pi_{TM}(\zeta))\right) \\
&= dL \circ D\theta_g\left(\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} (v + \varepsilon D\pi_{TM}(\zeta))\right) \\
&= dL((\Phi \circ D\pi_{TM})(\zeta)) \\
&= \lambda_L(\zeta)
\end{aligned}$$

for all $\zeta \in T_{(q,v)}TM$.

Step 2: The induced action Θ preserves the Liouville form. For $g \in G$ we compute

$$\begin{aligned}
\Theta_g^* \lambda &= \left(\tau_L \circ D\theta_g \circ \tau_L^{-1}\right)^* \lambda \\
&= \left(\tau_L^{-1}\right)^* (D\theta_g)^* \tau_L^* \lambda \\
&= \left(\tau_L^{-1}\right)^* (D\theta_g)^* \lambda_L
\end{aligned}$$

$$
\begin{aligned}
&= \left(\tau_L^{-1}\right)^* \lambda_L \\
&= \lambda
\end{aligned}
$$

by Step 1.

Step 3: The momentum map is of the stated form. We compute

$$
\begin{aligned}
\mu(\xi)(q,p) &= i_{\widehat{\xi}}\lambda(q,p) \\
&= \lambda_{(q,p)}\left(\left.\frac{d}{dt}\right|_{t=0} \Theta_{\exp(-t\xi)}(q,p)\right) \\
&= p\left(\left.\frac{d}{dt}\right|_{t=0} \pi_{T^*M} \circ \Theta_{\exp(-t\xi)}(q,p)\right) \\
&= p\left(\left.\frac{d}{dt}\right|_{t=0} \pi_{TM} \circ D\theta_{\exp(-t\xi)} \circ \tau_L^{-1}(q,p)\right) \\
&= p\left(\left.\frac{d}{dt}\right|_{t=0} \theta_{\exp(-t\xi)} \circ \pi_{TM} \circ \tau_L^{-1}(q,p)\right) \\
&= p\left(\left.\frac{d}{dt}\right|_{t=0} \theta_{\exp(-t\xi)} \circ \pi_{T^*M}(q,p)\right) \\
&= p\left(\left.\frac{d}{dt}\right|_{t=0} \theta_{\exp(-t\xi)}(q)\right)
\end{aligned}
$$

for all $\xi \in \mathfrak{g}$ and $(q,p) \in T^*M$.

Step 4: $E_L \circ D\theta_g = E_L$ for all $g \in G$. We compute

$$
\begin{aligned}
E_L(\theta_g(q), D\theta_g(v)) &= \tau_L(D\theta_g(v))(D\theta_g(v)) - L \circ D\theta_g(q,v) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L_{\theta_g(q)} \circ D\theta_g((1+\varepsilon)v) - L(q,v) \\
&= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L_q((1+\varepsilon)v) - L(q,v) \\
&= E_L(q,v)
\end{aligned}
$$

for all $g \in G$ and $(q,v) \in TM$.

Step 5: $H_L \circ \Theta_g = H_L$ for all $g \in G$. Using Step 4 we conclude

$$
H_L \circ \Theta_g = E_L \circ D\theta_g \circ \tau_L^{-1} = E_L \circ \tau_L^{-1} = H_L
$$

for all $g \in G$. □

2.3 Periodic Orbits on Regular Energy Surfaces

Let (M, ω, H) be a Hamiltonian system. Then ***Hamilton's description of classical dynamics*** is given by

$$\frac{d\mu}{dt} = 0 \qquad \text{and} \qquad \frac{df}{dt} = \{H, f\},$$

for all states $\mu \in \mathcal{M}(M)$ and observables $f \in C^\infty(M)$, where $\mathcal{M}(M)$ denotes the convex space of all Borel probability measures on M. Moreover, a ***process of measurement*** is the map

$$C^\infty(M) \times \mathcal{M}(M) \to \mathcal{M}(\mathbb{R}), \qquad (f, \mu) \mapsto f_*\mu,$$

where $f_*\mu \in \mathcal{M}(\mathbb{R})$ denotes the pushforward measure of μ by f.

Lemma 2.35. *Let (M, Ω) be a compact oriented smooth manifold of positive dimension and suppose that $\varphi \in \mathrm{Diff}(M)$ such that $\varphi^*\Omega = \Omega$. Then there exists a unique regular φ-invariant Borel probability measure μ_Ω such that*

$$\int_M f\Omega = \int_M f d\mu_\Omega \qquad \forall f \in C(M).$$

The measure μ_Ω is called the ***measure associated with the volume form Ω****.*

Proof. Define a nonnegative normalised continuous linear functional I_Ω on $C^\infty(M)$ by

$$I_\Omega(f) := \int_M f\bar{\Omega}, \qquad \text{where} \qquad \bar{\Omega} := \Omega / \textstyle\int_M \Omega.$$

Because $C^\infty(M)$ is dense in $C(M)$ by the Stone–Weierstrass Theorem [9, p. 392], we can uniquely extend I_Ω to a nonnegative normalised continuous linear functional on $C(M)$. Thus by the Riesz Representation Theorem [9, Theorem 7.2.8] there exists a unique regular Borel probability measure μ_Ω such that

$$I_\Omega(f) = \int_M f d\mu_\Omega \qquad \forall f \in C(M).$$

We compute

$$I_\Omega(f) = \int_M f\bar{\Omega} = \int_M (f \circ \varphi)\varphi^*\bar{\Omega} = \int_M (f \circ \varphi)\bar{\Omega} = I_\Omega(f \circ \varphi)$$

for all $f \in C^\infty(M)$ and by density also for all $f \in C(M)$. In particular

$$\int_M f d\mu_\Omega = \int_M (f \circ \varphi) d\mu_\Omega = \int_M f d(\varphi_*\mu_\Omega)$$

for all $f \in C(M)$ and consequently, $\varphi_*\mu_\Omega = \mu_\Omega$ by the uniqueness part of the Riesz Representation Theorem. □

Lemma 2.36. *Let M be a smooth manifold. Suppose that $\eta \in \Omega^1(M)$ is nowhere vanishing and $\xi \in \Omega^k(M)$. Then $\eta \wedge \xi = 0$ if and only if there exists $\zeta \in \Omega^{k-1}(M)$ such that $\xi = \eta \wedge \zeta$.*

Proof. Choose a Riemannian metric m on M and define $x \in \mathfrak{X}(M)$ by

$$X := \widehat{m}^{-1}(\eta)/\left\|\widehat{m}^{-1}(\eta)\right\|_m^2,$$

where $\widehat{m}\colon \mathfrak{X}(M) \to \Omega^1(M)$ denotes the tangent-cotangent bundle isomorphism. Set $\zeta := i_X(\xi)$ and assume that $\eta \wedge \xi = 0$. Then we compute

$$\eta \wedge \zeta = \eta \wedge i_X(\xi) = i_X(\eta) \wedge \xi - i_X(\eta \wedge \xi) = \eta(X)\xi = \xi.$$

The other direction is immediate. □

Definition 2.37 (Regular Energy Surface). A ***regular energy surface in a Hamiltonian system*** (M, ω, H) is defined to be an embedded hypersurface $\Sigma = H^{-1}(0)$ such that $\mathrm{Crit}(H) \cap \Sigma = \varnothing$.

Lemma 2.38. *Let Σ be a compact regular energy surface in a Hamiltonian system (M^{2n}, ω, H). Denote by θ the flow of X_H on Σ. Then there exists a unique regular θ-invariant probability measure μ_Σ on Σ, that is, we have that*

$$(\theta_t)_*\mu_\Sigma = \mu_\Sigma, \qquad \forall t \in \mathbb{R}.$$

Proof. By Lemma 2.35, it is enough to construct a θ-invariant volume form on Σ^{2n-1}. We proceed similar to the proof of Lemma 2.36. Since $dH \neq 0$ on Σ, we find a neighbourhood U of Σ in M such that $dH \neq 0$ on U. We claim that there exists $\eta \in \Omega^{2n-1}(U)$ such that

$$\omega^n = dH \wedge \eta. \tag{2.3}$$

Indeed, let m be any Riemannian metric on U. Define $X \in \mathfrak{X}(U)$ by

$$X := \mathrm{grad}_m H / \left\|\mathrm{grad}_m H\right\|_m^2.$$

Set $\eta := i_X(\omega^n)$ and compute

$$dH \wedge \eta = dH \wedge i_X(\omega^n) = i_X(dH) \wedge \omega^n - i_X(dH \wedge \omega^n) = dH(X)\omega^n = \omega^n.$$

The volume form $\iota_\Sigma^*\eta$ is uniquely determined by the requirement (2.3). Indeed, suppose that there exists $\xi \in \Omega^{2n-1}(U)$ such that

$$\omega^n = dH \wedge \eta = dH \wedge \xi.$$

Then

$$dH \wedge (\eta - \xi) = 0,$$

and thus by lemma 2.36 there exists $\zeta \in \Omega^{2n-2}(U)$ such that

$$\eta - \xi = dH \wedge \zeta.$$

But then

$$\begin{aligned}\iota_\Sigma^*\eta &= \iota_\Sigma^*(dH)\wedge\iota_\Sigma^*\zeta+\iota_\Sigma^*\xi\\ &= d\iota_\Sigma^*H\wedge\iota_\Sigma^*\zeta+\iota_\Sigma^*\xi\\ &= d(H\circ\iota_\Sigma)\wedge\iota_\Sigma^*\zeta+\iota_\Sigma^*\xi\\ &= \iota_\Sigma^*\xi\end{aligned}$$

because $H\circ\iota_\Sigma$ is constant. Using preservation of energy 2.12 and Problem 2.2 (c) we compute

$$dH\wedge\theta_t^*\eta=\theta_t^*(dH\wedge\eta)=\theta_t^*\omega^n=\omega^n \qquad \forall t\in\mathbb{R}.$$

□

Corollary 2.39. *Let Σ be a compact regular energy surface in a Hamiltonian system (M,ω,H). Then $\theta_1\in\mathrm{Diff}(\Sigma)$ is a discrete reversible measure theoretical dynamical system on the probability space $(\Sigma,\mathcal{B}(\Sigma),\mu_\Sigma)$.*

Theorem 2.40 (Poincaré's Recurrence Theorem). *Let Σ be a compact regular energy surface in a Hamiltonian system. Then almost every $x\in\Sigma$ is a recurrent point with respect to the probability measure μ_Σ, that is, for μ_Σ-almost every point $x\in\Sigma$ there exists a sequence $(t_k)\subseteq\mathbb{R}$ such that*

$$t_k\to+\infty \qquad \text{and} \qquad \lim_{k\to\infty}\theta_t^{X_H}(x)=x.$$

Proof. Let $T:=\theta_1\in\mathrm{Diff}(\Sigma)$. Then a routine computation shows

$$\mu\left(A\cap\bigcap_{k\geq 0}\bigcup_{l\geq k}T^{-l}(A)\right)=\mu(A) \qquad \forall A\in\mathcal{B}(\Sigma). \tag{2.4}$$

Fix a Riemannian metric m on Σ. Then (Σ,d_m) is a metric space with metric topology coinciding with the manifold topology of Σ. Because Σ is compact, for every $n\in\mathbb{N}$ there exists a finite index set I_n such that $\left(B_{1/n}(x_{i,n})\right)_{i\in I_n}$ is an open cover for Σ. Define

$$N:=\bigcup_{n\in\mathbb{N}}\bigcup_{i\in I_n}\left(B_{1/n}(x_{i,n})\setminus\left(B_{1/n}(x_{i,n})\cap\bigcap_{k\geq 0}\bigcup_{l\geq k}T^{-l}\left(B_{1/n}(x_{i,n})\right)\right)\right).$$

Then $N\in\mathcal{B}(\Sigma)$ and $\mu(N)=0$. Indeed, we have that

$$\begin{aligned}\mu(N) &\leq \sum_{n\in\mathbb{N}}\sum_{i\in I_n}\mu\left(B_{1/n}(x_{i,n})\setminus\left(B_{1/n}(x_{i,n})\cap\bigcap_{k\geq 0}\bigcup_{l\geq k}T^{-l}\left(B_{1/n}(x_{i,n})\right)\right)\right)\\ &= \sum_{n\in\mathbb{N}}\sum_{i\in I_n}\mu\left(B_{1/n}(x_{i,n})\right)-\mu\left(B_{1/n}(x_{i,n})\cap\bigcap_{k\geq 0}\bigcup_{l\geq k}T^{-l}\left(B_{1/n}(x_{i,n})\right)\right)\\ &= 0,\end{aligned}$$

by (2.4). Moreover, every $x \in N^c$ is a recurrent point. Indeed, $x \in N^c$ means that for all $n \in \mathbb{N}$ and $i \in I_n$

$$x \in \left(B_{1/n}(x_{i,n})\right)^c \qquad \text{or} \qquad x \in \bigcap_{k \geq 0} \bigcup_{l \geq k} T^{-l}\left(B_{1/n}(x_{i,n})\right).$$

Since $\left(B_{1/n}(x_{i,n})\right)_{i \in I_n}$ is an open cover for S_c, we conclude that

$$x \in \bigcap_{k \geq 0} \bigcup_{l \geq k} T^{-l}\left(B_{1/n}(x_{i,n})\right) \tag{2.5}$$

for some $i \in I_n$. Consequently, for every $n \in \mathbb{N}$ there exists an index $i_n \in I_n$ such that (2.5) holds. □

2.4 Problems

2.1 (Cotangent Bundles). Let M be a smooth manifold. Show that $(T^*M, d\lambda)$ is a symplectic manifold, where $\lambda \in \Omega^1(T^*M)$ is the ***Liouville form*** defined by

$$\lambda_{(q,p)}(\zeta) := p(D\pi_{T^*M}(\zeta)) \qquad \forall (q,p) \in T^*M, \zeta \in T_{(q,p)}T^*M. \tag{2.6}$$

2.2 (Symplectic Vector Fields). Let (M, ω) be a symplectic manifold. Define the real vector space of ***symplectic vector fields*** by

$$\mathfrak{X}(M, \omega) := \{X \in \mathfrak{X}(M) \colon i_X\omega \text{ closed}\}.$$

(a) Show that $\mathfrak{X}(M, \omega) \subseteq \mathfrak{X}(M)$ is a Lie subalgebra.
(b) Let M be compact. Show that if $(\varphi_\sigma)_{\sigma \in I}$ is a smooth path in $\mathrm{Diff}(M)$ starting at id_M, then $(\varphi_\sigma)_{\sigma \in I}$ is a smooth path in $\mathrm{Symp}(M, \omega)$ if and only if $X_\sigma \in \mathfrak{X}(M, \omega)$, where

$$X_\sigma := \frac{d}{d\sigma}\varphi_\sigma \circ \varphi_\sigma^{-1},$$

for all $\sigma \in I$.
(c) Let $X \in \mathfrak{X}(M, \omega)$. Show that the flow θ^X of X is volume preserving.

2.3 (The Euler–Lagrange Equations). A ***motion of a Lagrangian system*** (M, L) is defined to be a path

$$q \in \mathscr{P}_{x_0}^{x_1}M := \{q \in C^\infty(I, M) : q(0) = x_0 \text{ and } q(1) = x_1\}$$

for $x_0, x_1 \in M$ satisfying the ***Hamilton's principle of least action***

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathscr{E}_L(q_\varepsilon) = 0,$$

where

$$\mathcal{E}_L \colon \mathcal{P}_{x_0}^{x_1} M \to \mathbb{R}, \qquad \mathcal{E}_L(q) := \int_0^1 L(q(t), \dot{q}(t)) dt,$$

and q_ε is a variation of q with fixed ends, that is, the map

$$\Gamma \colon I \times (\varepsilon_0, \varepsilon_0) \to M, \qquad \Gamma(t, \varepsilon) := q_\varepsilon(t)$$

is smooth and satisfies $q_\varepsilon(0) = x_0, q_\varepsilon(1) = x_1$ for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ for some $\varepsilon_0 > 0$ such that $q_0 = q$.

(a) Show that q is a motion of (M, L) if and only if q satisfies

$$\frac{d}{dt}\frac{\partial L}{\partial v}(\dot{q}) = \frac{\partial L}{\partial q}(\dot{q})$$

locally in standard coordinates on the tangent bundle. This system of second order ordinary differential equations is referred to as the ***Euler–Lagrange equations of the Euler–Lagrange functional $\boldsymbol{\mathcal{E}_L}$***.

(b) Let (M, m) be a pseudo-Riemannian manifold and consider the Lagrangian

$$L \colon TM \to \mathbb{R}, \qquad L(x, v) := \frac{1}{2}\,|v|_m^2 .$$

Show that q is a motion of the Lagrangian system (M, L) if and only if q is a geodesic with respect to the induced Levi–Civita connection on TM.

(c) Let (M, L) be a Lagrangian system such that the Legendre transform τ_L is a diffeomorphism. Show that q is a motion of (M, L) if and only if ***Hamilton's equations***

$$\dot{q} = \frac{\partial H_L}{\partial p}(q, p) \qquad \text{and} \qquad \dot{p} = -\frac{\partial H_L}{\partial q}(q, p)$$

hold for $(q, p) := \tau_L(q, \dot{q})$ in standard coordinates on T^*M.

(d) Let (M, m) be a pseudo-Riemannian manifold. Compute Hamilton's equations for the mechanical Lagrangian

$$L \colon TM \to \mathbb{R}, \qquad L(q, v) := K(q, v) - V(q),$$

where $K \in C^\infty(TM)$ and $V \in C^\infty(M)$.

2.4 (Physical Transformation). Let $\varphi \colon M \to \widetilde{M}$ be a diffeomorphism between smooth manifolds M and $\widetilde{M}$. Define the ***cotangent lift of $\boldsymbol{\varphi}$***

$$D\varphi^\dagger \colon T^*M \to T^*\widetilde{M}, \qquad D\varphi^\dagger(q, p)(v) := p(D\varphi^{-1}(v))$$

for all $(q, p) \in T^*M$ and $v \in T_{\varphi(x)}\widetilde{M}$.

(a) Show that $(D\varphi^\dagger)^*\widetilde{\lambda} = \lambda$, where $\widetilde{\lambda}$ and λ are the Liouville forms on $\widetilde{M}$ and M, respectively.

(b) Deduce that for any action $\theta \in C^\infty(G \times M, M)$ of a Lie group G the lifted action $g \mapsto D\theta_g^\dagger$ is Poisson and show that the corresponding momentum map is given by

$$\mu\colon \mathfrak{g} \to C^\infty(T^*M), \qquad \mu(\xi)(q,p) = p(\widehat{\xi}).$$

2.5. Let G be a connected Lie group. Show that a weakly Hamiltonian G-action on a symplectic manifold is Hamiltonian if and only if it is Poisson.

2.6 (Complete Integrability and The Kepler Problem). A Hamiltonian system (M^{2n}, ω, H) is said to be ***completely integrable***, iff there exist Hamiltonian functions $H_1 = H, H_2, \dots, H_n \in C^\infty(M)$ such that $\{H_i, H_j\} = 0$ for all $i \neq j$. The ***Kepler problem*** is defined to be the Hamiltonian system $(T^*M^n, dp \wedge dq, H)$, where $M^n := \mathbb{R}^n \setminus \{0\}$ and the Hamiltonian function is given by

$$H(q,p) := \frac{1}{2}|p|^2 - \frac{1}{|q|}.$$

Show that the spatial Kepler Problem $(T^*M^3, dp \wedge dq, H)$ is completely integrable.

2.7 (Regular Energy Surfaces). Let Σ be an embedded hypersurface in a symplectic manifold (M, ω) and J and ω-compatible almost complex structure.

1. If Σ is compact, show that the following statements are equivalent.
   (a) Σ is orientable.
   (b) The normal bundle $N\Sigma \to \Sigma$ with respect to the Riemannian metric m_J is orientable.
   (c) The characteristic line bundle $\ker \omega|_\Sigma \to \Sigma$ is orientable.
   (d) There exists a parametrised family of hypersurfaces modelled on Σ, that is, there exists an embedding
   $$\psi\colon \Sigma \times (-\varepsilon, \varepsilon) \hookrightarrow M$$
   for some $\varepsilon > 0$ such that $\psi|_{\Sigma \times \{0\}} = \Sigma$.
   (e) There exists a local defining Hamiltonian function for Σ.
2. If Σ is orientable, show that there exists $H \in C^\infty(M)$ such that $\Sigma \subseteq H^{-1}(0)$, where 0 is a regular value of H.
3. If $H, \widetilde{H} \in C^\infty(M)$ are defining Hamiltonian functions for Σ, show that there exists a nowhere-vanishing function $f \in C^\infty(\Sigma)$ such that $X_{\widetilde{H}}|_\Sigma = f X_H|_\Sigma$.
4. If Σ is a connected regular energy surface in a connected Hamiltonian system (M, ω, H), then Σ separates M.