Lecture 4: The Poisson Algebra of Observables

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Lie Algebras and Poisson Algebras

Definition (Lie Algebra)

A ***real Lie algebra*** is defined to be a real vector space $\mathfrak{g}$ admitting a bilinear map

$$[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

called a ***Lie bracket***, satisfying the following conditions:

- $[\cdot,\cdot]$ is skew-symmetric.
- $[\cdot,\cdot]$ satisfies the ***Jacobi identity***, that is

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0 \qquad \forall X, Y, Z \in \mathfrak{g}.$$

Lemma

Let M be a smooth manifold. Then $(\mathfrak{X}(M), [\cdot,\cdot])$ is a Lie algebra.

Definition (Algebra of Classical Observables)

Let (M, ω) be a symplectic manifold. Then the commutative real algebra $C^\infty(M)$ of smooth functions on M is called the ***algebra of classical observables***.

Definition (Poisson Algebra)

A ***Poisson algebra*** is defined to be a real commutative algebra $\mathfrak{p}$ together with a Lie bracket $\{\cdot,\cdot\}$ on $\mathfrak{p}$ satisfying the ***Leibniz rule***

$$\{f, gh\} = h\{f, g\} + g\{f, h\} \qquad \forall\, f, g, h \in \mathfrak{p}.$$

Lemma

Let (M, ω) *be a symplectic manifold. Then* $(C^\infty(M), \{\cdot, \cdot\})$ *is a Poisson algebra, where*

$$\{f, g\} := \omega(X_f, X_g) \qquad \forall\, f, g \in C^\infty(M)$$

denotes the ***Poisson bracket of classical observables****.*

- $X_{\{f,g\}} = [X_f, X_g]$ for all $f, g \in C^\infty(M)$.

Lemma

Let (M, ω) *be a symplectic manifold and* $\varphi \in \operatorname{Symp}(M, \omega)$. *Then*

$$\varphi^*\{f, g\} = \{\varphi^* f, \varphi^* g\} \qquad \forall f, g \in C^\infty(M).$$

The Evolution Operator

Definition (Evolution Operator)

Let (M, ω, H) be a complete Hamiltonian system. Define the ***evolution operator***

$$U_t \colon C^\infty(M) \to C^\infty(M), \qquad U_t(f) := f \circ \theta_t^{X_H}$$

for all $t \in \mathbb{R}$.

Theorem

Let (M, ω, H) *be a complete Hamiltonian system. Then*

$$\frac{d}{dt} U_t(f) = U_t\{H, f\} \qquad \forall f \in C^\infty(M).$$

Preservation of Energy

Definition (Integral of Motion)

Let (M, ω, H) be a Hamiltonian system. An ***integral of motion*** is defined to be a function $I \in C^\infty(M)$ such that $\{H, I\} = 0$.

The Lie Algebra of a Lie Group

Definition (Lie Group)

A ***Lie group*** is defined to be a group object in the category of finite-dimensional smooth manifolds.

Given a Lie group G, the tangent space $\mathfrak{g} := T_e G$ to the identity element $e \in G$ is a Lie algebra. Indeed, there is a canonical isomorphism

$$T_e G \cong \mathfrak{X}_L(G),$$

where $\mathfrak{X}_L(G) \subseteq \mathfrak{X}(G)$ denotes the Lie subalgebra of all left-invariant vector fields on G.

As every left-invariant vector field is complete, we can define the ***exponential map***

$$\exp\colon \mathfrak{g} \to G, \qquad \xi \mapsto \gamma_\xi(1),$$

where $\gamma_\xi\colon \mathbb{R} \to G$ denotes the unique integral curve of $X_\xi \in \mathfrak{X}_L(G)$ with $X_\xi(e) = \xi$ such that $\gamma_\xi(0) = e$ and $\dot{\gamma}_\xi(0) = \xi$.