Lecture 1: Review of Differential Topology

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Sheaves

Definition (Presheaf)

Let $\mathscr{C}$ be a category and X a topological space. A ***presheaf on X with values in $\mathscr{C}$*** is a contravariant functor $\mathscr{O}(X) \to \mathscr{C}$, where $\mathscr{O}(X)$ denotes the poset category of open subsets of X.

Definition (Sheaf)

Let X be a topological space. A ***sheaf on X*** is defined to be a presheaf

$$F\colon \mathcal{O}(X) \to \mathsf{Vect}$$

satisfying the following ***gluing condition*** for all $U \in \mathcal{O}(X)$.
Given any open cover $(U_\alpha)_{\alpha \in A}$ of U together with $f_\alpha \in F(U_\alpha)$ for all $\alpha \in A$ such that

$$f_\alpha|_{U_\alpha \cap U_\beta} = f_\beta|_{U_\alpha \cap U_\beta} \qquad \forall \alpha, \beta \in A,$$

then there exists a unique $f \in F(U)$ such that $f|_{U_\alpha} = f_\alpha$ for all $\alpha \in A$.

Let $\pi\colon E \to M^n$ be a smooth vector bundle. Then

$$\Gamma_E\colon \mathcal{O}(M) \to \mathsf{Vect}, \qquad \Gamma_E(U) := \Gamma(U, E)$$

is a sheaf on M. In this talk we will focus on the sheaves

$$\mathcal{T}_M := \bigoplus_{k,l\geq 0} \mathcal{T}_M^{k,l} \qquad \text{and} \qquad \Omega_M := \bigoplus_{0\leq k\leq n} \Omega_M^k$$

on M, the ***total sheaf of tensor fields on M*** and the ***total sheaf of differential forms on M***, respectively, where

$$\mathcal{T}_M^{k,l} := \Gamma_{T^{(k,l)}TM} \qquad \text{and} \qquad \Omega_M^k := \Gamma_{\Omega^k(M)}.$$

More concretely, for every $U \in \mathcal{O}(M)$ there is a canonical identification between $\Omega^k_M(U)$ and alternating $C^\infty(U)$-multilinear maps

$$\underbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}_{k} \to C^\infty(U)$$

and likewise between $\mathcal{T}^{k,l}_M(U)$ and $C^\infty(U)$-multilinear maps

$$\underbrace{\Omega^1(U) \times \cdots \times \Omega^1(U)}_{k} \times \underbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}_{l} \to C^\infty(U).$$

The Lie Derivative

The Lie derivative generalises the directional derivative of a function to arbitrary tensor fields on a smooth manifold. Let M be a smooth manifold and $A \in \mathcal{T}^{k,l}(M)$. Then for any $X \in \mathfrak{X}(M)$ we define the ***Lie derivative of A with respect to X*** to be the tensor field $\mathcal{L}_X A \in \mathcal{T}^{k,l}(M)$ given by

$$\mathcal{L}_X A := \left.\frac{d}{dt}\right|_{t=0} \theta_t^* A.$$

Definition (Tensor Derivation)

A ***tensor derivation on a smooth manifold M*** is defined to be a sheaf morphism $\mathcal{D} : \mathcal{T}_M \to \mathcal{T}_M$ that preserves type and satisfies:

- For all $U \in \mathcal{O}(M)$, $\mathcal{D}_U$ commutes with all contractions of $\mathcal{T}_M(U)$.
- For all $U \in \mathcal{O}(M)$, $\mathcal{D}_U$ is a derivation, that is

$$\mathcal{D}_U(A \otimes B) = \mathcal{D}_U A \otimes B + A \otimes \mathcal{D}_U B$$

  holds for all $A, B \in \mathcal{T}(U)$.

Lemma (Contraction Lemma)

Let $\mathcal{D}$ be a tensor derivation, $U \in \mathcal{O}(M)$ and $A \in \mathcal{T}^{k,l}(U)$. Then for all $\omega^1, \ldots, \omega^k \in \Omega^1(U)$ and $X_1, \ldots, X_l \in \mathfrak{X}(U)$ we have that

$$
\begin{aligned}
\mathcal{D}_U(A)\left(\omega^1, \ldots, \omega^k, X_1, \ldots, X_l\right) = {} & \mathcal{D}_U\left(A\left(\omega^1, \ldots, \omega^k, X_1, \ldots, X_l\right)\right) \\
& - \sum_{i=1}^{k} A\left(\omega^1, \ldots, \mathcal{D}_U\left(\omega^i\right), \ldots, \omega^k, X_1, \ldots, X_l\right) \\
& - \sum_{i=1}^{l} A\left(\omega^1, \ldots, \omega^k, X_1, \ldots, \mathcal{D}_U(X_i), \ldots, X_l\right).
\end{aligned}
$$

Theorem

Let $\mathfrak{D}$ and $\mathfrak{D}'$ be two tensor derivations on a smooth manifold which agree on functions and vector fields. Then $\mathfrak{D} = \mathfrak{D}'$.

By theorem 5 the Lie derivative is the unique tensor derivation such that

$$\mathcal{L}_X f = Xf \qquad \text{and} \qquad \mathcal{L}_X Y = [X, Y]$$

for all $f \in C^\infty(M)$ and $X, Y \in \mathfrak{X}(M)$.

The Exterior Differential

Definition

Let M be a smooth manifold and $l \in \mathbb{Z}$. A ***graded derivation of degree l on M*** is defined to be a sheaf morphism $\mathcal{D} : \Omega_M \to \Omega_M$ satisfying:

- If $\omega \in \Omega^k(U)$, then $\mathcal{D}_U(\omega) \in \Omega^{k+l}(U)$.
- If $\omega \in \Omega^k(U)$ and $\eta \in \Omega(U)$, then

$$\mathcal{D}_U(\omega \wedge \eta) = \mathcal{D}_U(\omega) \wedge \eta + (-1)^{kl} \omega \wedge \mathcal{D}_U(\eta).$$

Lemma

Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. Then the Lie derivative $\mathcal{L}_X$ is a graded derivation of degree 0.

Theorem

Let M be a smooth manifold and suppose that $\mathfrak{D}$ and $\mathfrak{D}'$ are two graded derivations on M of the same degree which coincide on functions and exact 1*-forms. Then $\mathfrak{D} = \mathfrak{D}'$.*

Theorem (The Exterior Differential)

Let M be a smooth manifold. Then there exists a unique graded derivation $d : \Omega_M \to \Omega_M$ of degree 1 such that

$$d_U(f) = df \qquad \textit{and} \qquad d \circ d = 0$$

holds for all $f \in C^\infty(U)$. This graded derivation is called the ***exterior differential****.*

Cartan's Magic Formula

Theorem (Cartan's Magic Formula)

Let M be a smooth manifold and $X \in \mathfrak{X}(M)$. Then

$$\mathcal{L}_X = d \circ i_X + i_X \circ d.$$

Fisherman's Formula

Theorem (Fisherman's Formula)

Let M be a smooth manifold and suppose that $X\colon I \times M \to TM$ is a time-dependent vector field with time-dependent flow $\psi\colon \mathcal{D} \to M$. Then

$$\frac{d}{dt}\psi_t^*\omega = \psi_t^*\mathscr{L}_{X_t}\omega \qquad \forall \omega \in \Omega_M.$$

The Tangent-Cotangent Bundle Isomorphism

Definition (Nondegenerate Bilinear Form)

Let V be a finite-dimensional real vector space. A skew-symmetric bilinear form $\omega\colon V \times V \to \mathbb{R}$ is said to be ***nondegenerate***, iff the map $\widehat{\omega}\colon V \to V^*$ defined by $\widehat{\omega}(v) := i_v\omega$ is an isomorphism.

Lemma

Let V be a finite-dimensional real vector space and $\omega\colon V \times V \to \mathbb{R}$ skew-symmetric. Then the following statements are equivalent:

- *ω is symplectic.*
- *With respect to any basis for V, the matrix representing $\widehat{\omega}$ is invertible.*
- *If $\omega(v, u) = 0$ for all $u \in V$, then $v = 0$.*
- *If $v \neq 0$, then there exists some $u \in V$ such that $\omega(v, u) \neq 0$.*
- *The matrix representing ω in any basis of V is invertible.*

Theorem (Tangent-Cotangent Bundle Isomorphism)

Let M be a smooth manifold and $\omega \in \Omega^2(M)$ nondegenerate. Define

$$\widehat{\omega}: TM \to T^*M, \qquad \widehat{\omega}(v)(w) := \omega_x(v, w)$$

for all $x \in M$ and $v, w \in T_xM$. Then $\widehat{\omega}$ is a bundle isomorphism. The morphism $\widehat{\omega}$ is called the ***tangent-cotangent bundle isomorphism****.*

Lemma

Let M be a smooth manifold, $\omega \in \Omega^2(M)$ nondegenerate and $\lambda \in \Omega^1(M)$. Then there exists a unique vector field $X \in \mathfrak{X}(M)$ such that

$$i_X \omega = \lambda.$$