FUNCTIONAL ANALYSIS II SUMMARY

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Abstract. This is a rough summary of the course *Functional Analysis II* held at *ETH Zurich* by *Prof. Dr. Alessandro Carlotto* in spring 2018. The main focus of this summary is to give a neat preparation for the oral exam.

Contents

Sobolev Space Theory

In what follows, let $n \in \omega$, $n \geq 1$, and $1 \leq p \leq \infty$.

Definition 1.1 (Distributional and Weak Derivative). *Let $\Omega \subseteq \mathbb{R}^n$ open and $u \in L^1_{\mathrm{loc}}(\Omega)$. For any multiindex α, the **distributional derivative of order α of u**, written $D^\alpha u$, is defined to be the mapping $D^\alpha u : C_c^\infty(\Omega) \to \mathbb{R}$ defined by*

$$\varphi \mapsto (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi dx.$$

*Moreover, a function $D^\alpha u \in L^p(\Omega)$ is called **weak derivative of order α of u with exponent p**, iff*

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega D^\alpha u \varphi dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi dx.$$

Theorem 1.1 (Fundamental Lemma of Variational Calculus). *Let $\Omega \subseteq \mathbb{R}^n$ open and $f \in L^1_{\mathrm{loc}}(\Omega)$. If*

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega f \varphi dx = 0,$$

then $f = 0$ a.e.

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Remark 1.1. Let $\Omega \subseteq \mathbb{R}^n$ open. Then $L^p(\Omega) \subseteq L^1_{\text{loc}}(\Omega)$.

Remark 1.2. From the fundamental lemma of variational calculus 1.1 it follows that *weak derivatives, if they exist, are unique*.

Examples 1.1.

(a) Consider $\Omega := (-1, 1)$ and $u := |x|$. Then $u' = \chi_{[0,1)} - \chi_{(-1,0)}$.

(b) Consider $\Omega := \mathbb{R}$ and $u := \chi_{(0,\infty)}$. Then the weak derivative u' does not exist. Indeed, the *Dirac distribution* is not representable as one may see by considering the smooth family $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ for $\varepsilon > 0$ defined by

$$\varphi_\varepsilon(x) := \begin{cases} e^{\varepsilon^2/(x^2-\varepsilon^2)} & |x| < \varepsilon, \\ 0 & |x| \geq \varepsilon. \end{cases}$$

Definition 1.2 (Sobolev Space). *Let $\Omega \subseteq \mathbb{R}^n$ open. For any $k \in \omega$, the* ***Sobolev space of index (k, p)****, written $W^{k,p}(\Omega)$, is defined to be the space*

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ exists for all } |\alpha| \leq k\},$$

with norm

$$\|-\|_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha -\|_{L^p(\Omega)}.$$

Moreover, define

$$W_0^{k,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|-\|_{W^{k,p}(\Omega)}},$$

and $H^k(\Omega) := W^{k,2}(\Omega)$ as well as $H_0^k(\Omega) := W_0^{k,2}(\Omega)$.

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^n$ open. Then $W^{k,p}(\Omega)$ is*

(*a*) *a Banach space for all $1 \leq p \leq \infty$.*

(*b*) *separable for all $1 \leq p < \infty$.*

(*c*) *reflexive for all $1 < p < \infty$.*

Proof.

(a) This follows from the fact that $L^p(\Omega)$ is a Banach space for all $1 \leq p \leq \infty$. Let $(f_i)_{i\in\omega}$ be a Cauchy sequence in $W^{k,p}$. By definition of the $W^{k,p}$-norm, $(D^\alpha f_i)_{i\in\omega}$ is a Cauchy sequence in L^p. Thus we get $D^\alpha f_i \to f_\alpha$ in L^p, in particular, $f_i \to f$ in L^p. Using Hölder's inequality we compute

$$\int_\Omega f_\alpha \varphi dx = \lim_{i\to\infty} \int_\Omega D^\alpha f_i \varphi dx = (-1)^{|\alpha|} \lim_{i\to\infty} \int_\Omega f_i D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi dx$$

for $\varphi \in C_c^\infty(\Omega)$.

(b) For simplicity, we consider $k = 1$ only. Consider $\iota : W^{1,p} \hookrightarrow (L^p)^{n+1}$ defined in the obvious way. Then ι is an isometry and the statement follows.

(c) Same argument as in part (b).

□

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Elliptic Operators in Divergence Form

Lemma 1.1 (Poincaré Inequality).

Theorem 1.3 (Riesz Representation Theorem). *Let H be a real Hilbert space. Then the mapping $J : H \to H^*$ defined by $J(x) := \langle x, - \rangle$ is an isometric isomorphism.*

Theorem 1.4. *Let $\Omega \subseteq\subseteq \mathbb{R}^n$, $k \in \omega$ and consider the elliptic operator*

$$L := \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(a_{ij} \frac{\partial}{\partial x^j} \right),$$

for $a_{ij} \in C^{k+1}(\bar{\Omega})$ symmetric. Then:

(a) Given $f \in L^2(\Omega)$, the homogenous Dirichlet problem

$$\begin{cases} -L(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

admits a unique weak solution $u \in H_0^1(\Omega)$.

(b) If $f \in H^k(\Omega)$ for some $k \in \omega$, then we have $u \in H_{\mathrm{loc}}^{k+2}(\Omega)$ for the unique weak solution of part (a) and moreover, for any $\Omega' \subseteq\subseteq \Omega$ we have the estimate

$$\|u\|_{H^{k+2}(\Omega')} \leq C \left(\|f\|_{H^k(\Omega)} + \|u\|_{H^1(\Omega)} \right).$$

Proof.

Step 1: Derivation of Weak Formulation. Suppose $u \in C^2(\bar{\Omega})$ is a solution of (1). Let $\varphi \in C_c^\infty(\Omega)$. Then integration by parts (see [Lee13, p. 436]) yields

$$-\int_\Omega L(u)\varphi = -\sum_{j=1}^{n} \int_\Omega \operatorname{div}(X_j)\varphi = \sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial u}{\partial x^j} \frac{\partial \varphi}{\partial x^i} = \sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial u}{\partial x^i} \frac{\partial \varphi}{\partial x^j},$$

where $X_j := \left(a_{ij} \frac{\partial}{\partial x^j} \right)_i$. Thus we get the weak formulation:

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial u}{\partial x^i} \frac{\partial \varphi}{\partial x^j} = \int_\Omega f\varphi \qquad \forall \varphi \in C_c^\infty(\Omega). \tag{2}$$

Step 2: Existence and Uniqueness of Weak Solutions. Since L is uniformly elliptic, there exists $\lambda > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

holds for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Moreover, since $a_{ij} \in C^0(\overline{\Omega})$, we get that L is uniformly bounded, i.e. there exists $\Lambda > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

holds for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Now define a bilinear form $\langle\cdot,\cdot\rangle_a : H_0^1(\Omega)\times H_0^1(\Omega) \to \mathbb{R}$ by

$$\langle u, v\rangle_a := \sum_{i,j=1}^n \int_\Omega a_{ij}\frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^j} \tag{3}$$

Then it is easy to see, that $\langle\cdot,\cdot\rangle_a$ is symmetric. Also, $\langle\cdot,\cdot\rangle_a$ is positive definite since

$$\langle u, u\rangle_a = \sum_{i,j=1}^n \int_\Omega a_{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j} \geq \lambda \int_\Omega |\nabla u|^2 \geq C\lambda \int_\Omega |u|^2$$

using ellipticity and Poincaré's inequality. Moreover by Poincaré's inequality we have that

$$C\lambda\|u\|^2_{H_0^1(\Omega)} \leq \|u\|_a \leq \Lambda\|u\|^2_{H_0^1(\Omega)}$$

for the induced norm $\|\cdot\|_a$. Hence the induced norm is equivalent to the standard norm on $H_0^1(\Omega)$ and thus $(H_0^1(\Omega), \|\cdot\|_a)$ is a Hilbert space. Thus an application of Riesz representation theorem 1.3 yields the existence of a unique $u \in H_0^1(\Omega)$, such that

$$\langle u, \varphi\rangle_a = l(\varphi) := \int_\Omega f\varphi$$

holds for all $\varphi \in H_0^1(\Omega)$, since $l \in (H_0^1(\Omega))^*$ This proves part (a).
Step 3: H^1-Estimate. The main idea in proving part (b) is an induction on $k \in \omega$. So let us assume that $k = 0$. Let $u \in H_0^1(\Omega)$ denote the unique solution of part (a).

Lemma 1.2.

$$\|\nabla u\|_{L^2(\Omega')}$$

□

References

[Lee13] John M. Lee. *Introduction to Smooth Manifolds*. Second Edition. Graduate Texts in Mathematics. Springer, 2013.