WHITEHEAD PRODUCT

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Abstract.

Contents

1. Definition of the Whitehead Product

Notice, that for any $(X, x_0), (Y, y_0) \in \mathsf{Top}_*$, their coproduct is given by

$$X \coprod Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subseteq X \times Y,$$

with basepoint (x_0, y_0).

Lemma 1.1. *Let $n, m \in \omega$, $n, m \geq 1$. The space $\mathbb{S}^n \times \mathbb{S}^m$ can be obtained from $\mathbb{S}^n \vee \mathbb{S}^m$ by attaching an $n + m$-cell.*

Proof. Observe, that $\mathbb{D}^{n+m} \cong \mathbb{D}^n \times \mathbb{D}^m$ and hence

$$\mathbb{S}^{n+m-1} = \partial\mathbb{D}^{n+m} \cong (\partial\mathbb{D}^n \times \mathbb{D}^m) \cup (\mathbb{D}^n \times \partial\mathbb{D}^m) = (\mathbb{S}^{n-1} \times \mathbb{D}^m) \cup (\mathbb{D}^n \times \mathbb{S}^{m-1}).$$

Let

$$f_1 : \mathbb{S}^{n-1} \times \mathbb{D}^m \to (\mathbb{S}^{n-1} \times \mathbb{D}^m)/(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \cong * \times \mathbb{S}^m$$

and

$$f_2 : \mathbb{D}^n \times \mathbb{S}^{m-1} \to (\mathbb{D}^n \times \mathbb{S}^{m-1})/(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \cong \mathbb{S}^n \times *$$

be the quotient maps. An application of the gluing lemma thus yields a map

$$f : \mathbb{S}^{n+m-1} \to \mathbb{S}^n \vee \mathbb{S}^m.$$

Moreover, define

$$q : \mathbb{D}^n \times \mathbb{D}^m \to \mathbb{D}^n/\mathbb{S}^{n-1} \times \mathbb{D}^m/\mathbb{S}^{m-1} \cong \mathbb{S}^n \times \mathbb{S}^m$$

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to be the product of quotient maps. Thus we get a commutative diagram

$$\begin{tikzcd}
\mathbb{S}^{n+m-1} \arrow[r, "f"] \arrow[d, hook] & \mathbb{S}^n \vee \mathbb{S}^m \arrow[d, hook] \\
\mathbb{D}^{n+m} \arrow[r, "q"'] & \mathbb{S}^n \times \mathbb{S}^m
\end{tikzcd}$$

Suppose (X, g, h) is another cocone for the pushout diagram:

$$\begin{tikzcd}
\mathbb{S}^{n+m-1} \arrow[r, "f"] \arrow[d, hook] & \mathbb{S}^n \vee \mathbb{S}^m \arrow[d, hook] \arrow[ddr, bend left, "h"] & \\
\mathbb{D}^{n+m} \arrow[r, "q"'] \arrow[drr, bend right, "g"'] & \mathbb{S}^n \times \mathbb{S}^m & \\
& & X.
\end{tikzcd}$$

By [Mun00, p. 186], q is a quotient map. Moreover, for $(x, y) \in \mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$, we have that

$$g(x, y) = (h \circ f)(x, y) = h(*, *).$$

Thus g passes to the quotient by [Lee11, p. 72] to yield a unique map

$$\widetilde{g} : \mathbb{S}^n \times \mathbb{S}^m \to X,$$

such that $g = \widetilde{g} \circ q$. Finally, it is easy to check that

$$\begin{tikzcd}
\mathbb{S}^{n+m-1} \arrow[r, "f"] \arrow[d, hook] & \mathbb{S}^n \vee \mathbb{S}^m \arrow[d, hook] \arrow[ddr, bend left, "h"] & \\
\mathbb{D}^{n+m} \arrow[r, "q"'] \arrow[drr, bend right, "g"'] & \mathbb{S}^n \times \mathbb{S}^m \arrow[dr, "\widetilde{g}"] & \\
& & X.
\end{tikzcd}$$

commutes. $\square$

For $n, m \in \omega$, $n, m \geq 1$, consider the map f from lemma 1.1. Let $(X, p) \in \mathsf{Top}_*$. If $[\alpha] \in \pi_n(X, p)$ and $[\beta] \in \pi_m(X, p)$, we get two pointed maps

$$\alpha : \mathbb{S}^n \to X \qquad \text{and} \qquad \beta : \mathbb{S}^m \to X.$$

Forming their wedge $\alpha \vee \beta : \mathbb{S}^n \vee \mathbb{S}^m \to X$, defined by

$$(\alpha \vee \beta)(x, y) := \begin{cases} \alpha(x) & y = *, \\ \beta(y) & x = *, \end{cases}$$

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and precomposing with f, yields a pointed map

$$(\alpha \vee \beta) \circ f : \mathbb{S}^{n+m-1} \to X.$$

Moreover, it is easy to check that above map is well behaved under pointed homotopies, hence gives raise to a class $[(\alpha \vee \beta) \circ f]$.

Definition 1.1 (Whitehead Product). *Let $n, m \in \omega$, $n, m \geq 1$, and $(X, p) \in \mathsf{Top}_*$. The product*

$$\pi_n(X, p) \times \pi_m(X, p) \to \pi_{n+m-1}(X, p)$$

defined by

$$([\alpha], [\beta]) \mapsto [(\alpha \vee \beta) \circ f]$$

*is called the **Whitehead product**.*