

WHITEHEAD PRODUCT

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Abstract. Aim of this paper is to give a short overview of the definition and the basic properties of the non-generalized *Whitehead product*.

Contents

1	Introduction	1
2	Definition of the Whitehead Product	2
3	The Whitehead Product and the Conjugation Action	4
4	Grading	6
	References	7

1. Introduction

In the category of *compactly generated spaces*, suppose G is an H -group, i.e. a space satisfying the group axioms up to homotopy, then $[X, G]$ is a group for any space X . This group need not be abelian. Thus a natural question is, if $[X, G]$ is *nilpotent*. As the notion of nilpotence is based on the behaviour of *commutators*, it is natural to consider certain related products: First of all the *commutator product* or *Samelson product* defined as follows: If $[\alpha] \in [X, G]$ and $[\beta] \in [Y, G]$, define $\gamma : X \times Y \rightarrow G$ by

$$\gamma(x, y) := \alpha(x)\beta(y) (\alpha(x))^{-1} (\beta(y))^{-1}.$$

Then $\gamma|_{X \vee Y}$ is nullhomotopic and thus yields a map $\gamma : X \wedge Y \rightarrow G$, whose homotopy class is defined to be the product of $[\alpha]$ and $[\beta]$. When $G = \Omega X$, then $\pi_n(G) \cong \pi_{n+1}(X)$ and the Samelson product

$$\pi_n(G) \otimes \pi_m(G) \rightarrow \pi_{n+m}(G)$$

translates to a pairing

$$\pi_{n+1}(X) \otimes \pi_{m+1}(X) \rightarrow \pi_{n+m+1}(X),$$

the *Whitehead product* (see [Whi78, pp. 456–457]).

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2. Definition of the Whitehead Product

Notice, that for any $(X, x_0), (Y, y_0) \in \text{Top}_*$, their coproduct is given by

$$X \coprod Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subseteq X \times Y,$$

with basepoint (x_0, y_0) .

Lemma 2.1. *Let $n, m \in \omega, n, m \geq 1$. The space $S^n \times S^m$ can be obtained from $S^n \vee S^m$ by attaching an $n + m$ -cell.*

Proof. Observe, that $\mathbb{D}^{n+m} \cong \mathbb{D}^n \times \mathbb{D}^m$ and hence

$$S^{n+m-1} = \partial \mathbb{D}^{n+m} \cong (\partial \mathbb{D}^n \times \mathbb{D}^m) \cup (\mathbb{D}^n \times \partial \mathbb{D}^m) = (S^{n-1} \times \mathbb{D}^m) \cup (\mathbb{D}^n \times S^{m-1}).$$

Let

$$f_1 : S^{n-1} \times \mathbb{D}^m \rightarrow (S^{n-1} \times \mathbb{D}^m) / (S^{n-1} \times S^{m-1}) \cong * \times S^m$$

and

$$f_2 : \mathbb{D}^n \times S^{m-1} \rightarrow (\mathbb{D}^n \times S^{m-1}) / (S^{n-1} \times S^{m-1}) \cong S^n \times *$$

be the quotient maps. An application of the gluing lemma thus yields a map

$$f : S^{n+m-1} \rightarrow S^n \vee S^m.$$

Moreover, define

$$q : \mathbb{D}^n \times \mathbb{D}^m \rightarrow \mathbb{D}^n / S^{n-1} \times \mathbb{D}^m / S^{m-1} \cong S^n \times S^m$$

to be the product of quotient maps. Thus we get a commutative diagram

$$\begin{array}{ccc} S^{n+m-1} & \xrightarrow{f} & S^n \vee S^m \\ \downarrow & & \downarrow \\ \mathbb{D}^{n+m} & \xrightarrow{q} & S^n \times S^m \end{array}$$

Suppose (X, g, h) is another cocone in Top for the pushout diagram:

$$\begin{array}{ccc} S^{n+m-1} & \xrightarrow{f} & S^n \vee S^m \\ \downarrow & & \downarrow \\ \mathbb{D}^{n+m} & \xrightarrow{q} & S^n \times S^m \end{array} \quad \begin{array}{c} \searrow h \\ \nearrow g \\ \downarrow \\ X \end{array}$$

By [Mun00, p. 186], q is a quotient map. Moreover, for $(x, y) \in S^{n-1} \times S^{m-1}$, we have that

$$g(x, y) = (h \circ f)(x, y) = h(*, *).$$

Thus g passes to the quotient by [Lee11, p. 72] to yield a unique map

$$\tilde{g} : \mathbb{S}^n \times \mathbb{S}^m \rightarrow X,$$

such that $g = \tilde{g} \circ q$. Finally, it is easy to check that

$$\begin{array}{ccccc} \mathbb{S}^{n+m-1} & \xrightarrow{f} & \mathbb{S}^n \vee \mathbb{S}^m & & \\ \downarrow & & \downarrow & \searrow h & \\ \mathbb{D}^{n+m} & \xrightarrow{q} & \mathbb{S}^n \times \mathbb{S}^m & \xrightarrow{\tilde{g}} & X. \\ & \searrow g & & & \end{array}$$

commutes. □

For $n, m \in \omega$, $n, m \geq 1$, consider the map f from lemma 2.1. Let $(X, p) \in \text{Top}_*$. If $[\alpha] \in \pi_n(X, p)$ and $[\beta] \in \pi_m(X, p)$, we get two pointed maps

$$\alpha : \mathbb{S}^n \rightarrow X \quad \text{and} \quad \beta : \mathbb{S}^m \rightarrow X.$$

Forming their wedge $\alpha \vee \beta : \mathbb{S}^n \vee \mathbb{S}^m \rightarrow X$, defined by

$$(\alpha \vee \beta)(x, y) := \begin{cases} \alpha(x) & y = *, \\ \beta(y) & x = *, \end{cases}$$

and precomposing with f , yields a pointed map

$$(\alpha \vee \beta) \circ f : \mathbb{S}^{n+m-1} \rightarrow X.$$

Explicitly, if we consider

$$\alpha : (\mathbb{D}^n, \mathbb{S}^{n-1}) \rightarrow (X, p) \quad \text{and} \quad \beta : (\mathbb{D}^m, \mathbb{S}^{m-1}) \rightarrow (X, p),$$

we get that

$$((\alpha \vee \beta) \circ f)(x, y) = \begin{cases} \alpha(x) & x \in \mathbb{D}^n, y \in \mathbb{S}^{m-1}, \\ \beta(y) & x \in \mathbb{S}^{n-1}, y \in \mathbb{D}^m. \end{cases}$$

Hence if $F : \alpha \simeq_{\mathbb{S}^{n-1}} \alpha'$ and $F' : \beta \simeq_{\mathbb{S}^{m-1}} \beta'$, we get that

$$H : ((\alpha \vee \beta) \circ f) \simeq_* ((\alpha' \vee \beta') \circ f),$$

where $H : \mathbb{S}^{n+m-1} \times I \rightarrow X$ is defined by

$$H(x, y, t) := \begin{cases} F(x, t) & x \in \mathbb{D}^n, y \in \mathbb{S}^{m-1}, \\ F'(y, t) & x \in \mathbb{S}^{n-1}, y \in \mathbb{D}^m. \end{cases}$$

Thus we get a well defined map $[-, -] : \pi_n(X) \times \pi_m(X) \rightarrow \pi_{n+m-1}(X)$, defined by

$$[\alpha, \beta] := [(\alpha \vee \beta) \circ f].$$

Definition 2.1 (Whitehead Product). Let $n, m \in \omega$, $n, m \geq 1$, and $(X, p) \in \text{Top}_*$. The product

$$[-, -] : \pi_n(X, p) \times \pi_m(X, p) \rightarrow \pi_{n+m-1}(X, p)$$

defined by

$$[\alpha, \beta] := [(\alpha \vee \beta) \circ f],$$

is called the **Whitehead product** and $[-, -]$ is called the **Whitehead bracket**.

3. The Whitehead Product and the Conjugation Action

In this section, we want to have a closer look at $[-, -] : \pi_1(X) \times \pi_n(X) \rightarrow \pi_n(X)$. If $n = 1$, we immediately deduce from figure 1a that

$$[\alpha, \beta] = [\alpha][\beta][\alpha]^{-1}[\beta]^{-1}.$$

Thus $[\alpha, \beta]$ coincides with the notation of a commutator in $\pi_1(X)$. Let $n > 1$. An **orientation of \mathbb{D}^n** is simply a choice of a generator of $H_n(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong \mathbb{Z}$ (follows from the long exact sequence axiom).

Lemma 3.1. Let $n \in \omega$, $n > 1$, $[\alpha] \in \pi_n(X)$ and $h : (\mathbb{D}^n, \mathbb{S}^{n-1}) \rightarrow (\mathbb{D}^n, \mathbb{S}^{n-1})$ an orientation reversing homeomorphism, i.e. h is a homeomorphism and $H_n(h)\langle e \rangle = -\langle e \rangle$ for a generator $\langle e \rangle$ of $H_n(\mathbb{D}^n, \mathbb{S}^{n-1})$. Then

$$[\alpha \circ h] = -[\alpha].$$

Proof. Let $\rho : \pi_n(Y, A) \rightarrow H_n(Y, A)$ denote the **Hurewicz homomorphism** defined by

$$\rho[f] := H_n(f)\langle e \rangle,$$

for a generator $\langle e \rangle$ of $H_n(\mathbb{D}^n, \mathbb{S}^{n-1})$ (see [Whi78, p. 166]). Using that

$$\rho : \pi_n(\mathbb{D}^n, \mathbb{S}^{n-1}) \rightarrow H_n(\mathbb{D}^n, \mathbb{S}^{n-1})$$

is an isomorphism for $n > 1$ (see [Whi78, p. 168]), we compute

$$\begin{aligned} [\alpha \circ h] &= \pi_n(\alpha)[h] \\ &= \pi_n(\alpha)\rho^{-1}\rho[h] \\ &= \pi_n(\alpha)\rho^{-1}(H_n(h)\langle e \rangle) \\ &= -\pi_n(\alpha)\rho^{-1}\langle e \rangle \\ &= -\pi_n(\alpha)[\text{id}_{\mathbb{D}^n}] \\ &= -[\alpha]. \end{aligned}$$

□

Using lemma 3.1, it follows from figure 1b, that

$$[\alpha, \beta] = [\alpha \cdot \beta] - [\beta],$$

where $\alpha \cdot \beta$ denotes the *conjugation action*, i.e. the action of $\pi_1(X)$ on $\pi_n(X)$, since the boundary of the cylinder $I \times \mathbb{D}^n$ is oriented coherently with $1 \times \mathbb{D}^n$ and therefore discoherently with $0 \times \mathbb{D}^n$. Indeed, this can be seen as follows: If e^n and e^m are orientations of \mathbb{D}^n and \mathbb{D}^m , respectively, then their *cross product* $e^n \times e^m$ is an orientation of $\mathbb{D}^n \times \mathbb{D}^m$ (see [Whi78, p. 64]). Moreover, we have that

$$\partial(e^n \times e^m) = \partial e^n \times e^m + (-1)^n e^n \times \partial e^m.$$

Therefore

$$\partial(e^1 \times e^n) = 1 \times e^n - 0 \times e^n - e^1 \times \partial e^n.$$

And we conclude by observing that the sum $[\alpha] + [\beta]$ in $\pi_n(X)$ is given by the pointed homotopy class of the composition

$$\mathbb{S}^n \xrightarrow{c} \mathbb{S}^n \vee \mathbb{S}^n \xrightarrow{\alpha \vee \beta} X,$$

where $c : \mathbb{S}^n \rightarrow \mathbb{S}^n \vee \mathbb{S}^n$ denotes the mapping which collapses the equatorial \mathbb{S}^{n-1} in \mathbb{S}^n to a point, depicted in figure 2 (see [Hat01, p. 341]).

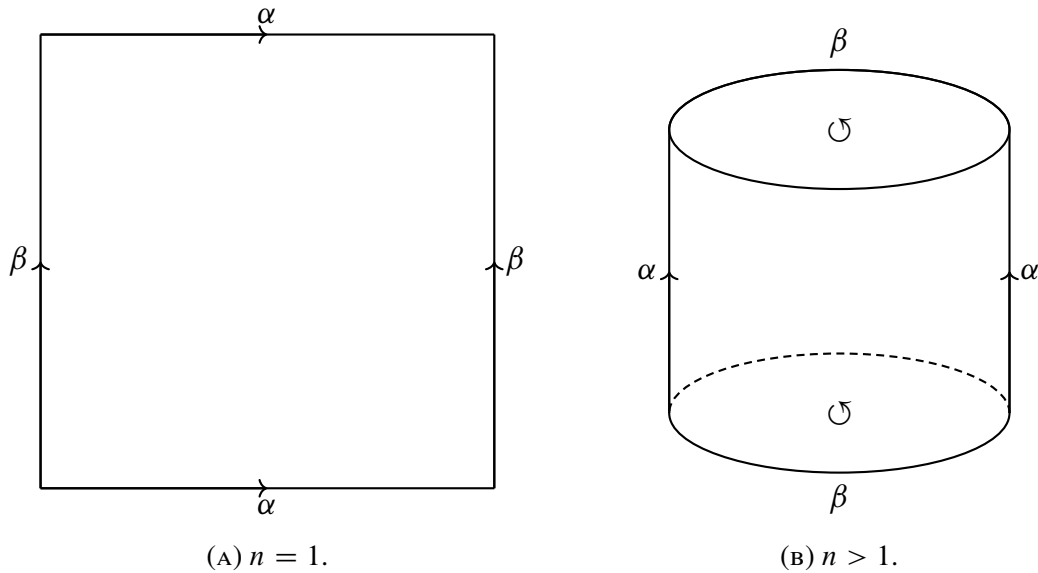


FIGURE 1. Whitehead bracket and the conjugation action.

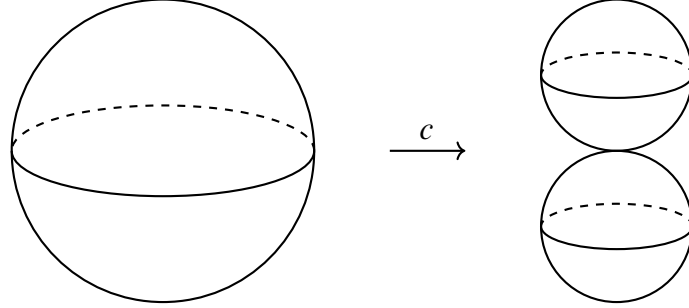


FIGURE 2. The collapsing map $c : \mathbb{S}^n \rightarrow \mathbb{S}^n \vee \mathbb{S}^n$.

4. Grading

Let $(X, p) \in \text{Top}_*$. For $n \in \omega$ let $L^n := \pi_{n+1}(X, p)$ and define

$$L := \bigoplus_{n \in \omega} L^n.$$

Moreover, define $[-, -] : L \times L \rightarrow L$ by

$$\left[\sum_i \alpha_i, \sum_j \beta_j \right] := \sum_{i,j} [\alpha_i, \beta_j].$$

Then clearly $L^n L^m \subseteq L^{n+m}$ holds. It also turns out, that we have a Lie algebra-like structure on L , i.e. the bracket is bilinear, alternating and there is a Jacobi identity (for more details see [Whi78, pp. 474–478]).

Proposition 4.1. *Let $n, m \in \omega$, $n \geq 1$, $[\alpha_1], [\alpha_2] \in \pi_{n+1}(X)$ and $[\beta] \in \pi_{m+1}(X)$. Then*

$$[\alpha_1 + \alpha_2, \beta] = [\alpha_1, \beta] + [\alpha_2, \beta] \quad \text{and} \quad [\beta, \alpha_1 + \alpha_2] = [\beta, \alpha_1] + [\beta, \alpha_2].$$

Recall, that for $n \geq 1$ we have that $H_n(\mathbb{S}^n) \cong \mathbb{Z}$. Thus if we are given any continuous map $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$, the induced map $H_n(f)$ is simply a multiplication by a unique integer. This integer is defined to be the **degree of f** , written $\deg f$.

Proposition 4.2. *Let $n, m \in \omega$, $[\alpha] \in \pi_{n+1}(X)$ and $[\beta] \in \pi_{m+1}(X)$. Then*

$$[\beta, \alpha] = (-1)^{(n+1)(m+1)} [\alpha, \beta].$$

Proof. Consider the **permutation map** $\sigma : \mathbb{S}^{m+n+1} \rightarrow \mathbb{S}^{n+m+1}$ defined by

$$(y_1, \dots, y_{m+1}, x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_{n+1}, y_1, \dots, y_{m+1}).$$

Then clearly $\deg \sigma = (-1)^{(n+1)(m+1)}$, since σ is the composition of permutations and hence orthogonal transformations. Using lemma 3.1, we compute

$$[\beta, \alpha] = [(\beta \vee \alpha) \circ f] = [(\alpha \vee \beta) \circ f \circ \sigma] = (-1)^{(n+1)(m+1)} [\alpha, \beta].$$

□

Proposition 4.3. *Let $n, m, r \in \omega$, $n, m, r \geq 1$, $[\alpha] \in \pi_{n+1}(X)$, $[\beta] \in \pi_{m+1}(X)$ and $[\gamma] \in \pi_{r+1}(X)$. Then*

$$(-1)^{r(n+1)}[\alpha, [\beta, \gamma]] + (-1)^{n(m+1)}[\beta, [\gamma, \alpha]] + (-1)^{m(r+1)}[\gamma, [\alpha, \beta]] = 0$$

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