LIE ALGEBRA COHOMOLOGY

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Abstract. Aim of this talk is to give a short overview of the *cohomology of Lie algebras with coefficients in modules*. We follow the original construction of Chevalley-Eilenberg via complexes. We then state two results concerning *semisimple* Lie algebras, known as the *first and second Whitehead lemma*, and calculate an example.

Contents

Introduction

The archetypal example of a *cohomology theory* arises in differential topology: The *de Rham cohomology*. Given a smooth manifold M, we define $\Omega^n(M) := \Gamma(\Lambda^n T^\vee M)$ for each $n \in \omega$, the space of *smooth differential n-forms on M*. Moreover, is a sequence of mappings $\left(d^n : \Omega^n(M) \to \Omega^{n+1}(M)\right)_{n\in\omega}$, called *exterior differentiation operators*, which roughly speaking generalize the notion of a differential of a function. They do satisfy the relation $d^n \circ d^{n-1} = 0$ and thus we can define the *n-th de Rham cohomology group* to be the quotient space

$$H^n_{\mathrm{dR}}(M) := \ker d^n / \operatorname{im} d^{n-1}.$$

The Chevalley-Eilenberg Complex

The definition of the n-th de Rham cohomology group $H^n_{\mathrm{dR}}(M)$ can actually be thought of a two-stage process: First we go from Diff to an intermediate category and then we apply a *homology functor*, which is a purely algebraic construct, to go from this intermediate

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category to ${}_{\mathbb{R}}\mathsf{Vect}$. Aim of this section is to give the definition of this intermediate category and then to define such a functor explicitely for the case of Lie algebras.

Definition 1.1 (Chain Complex). *Let $\mathcal{A}$ be an abelian category. A **$\mathbb{Z}$-graded chain complex in $\mathcal{A}$** is defined to be a tuple $((C_n)_{n\in\mathbb{Z}}, (\partial_n)_{n\in\mathbb{Z}})$, consisting of a sequence $(C_n)_{n\in\mathbb{Z}}$ of objects in $\mathcal{A}$ and a sequence $(\partial_n)_{n\in\mathbb{Z}}$ of morphisms in $\mathcal{A}$, such that*

$$\partial_n \in \mathrm{Hom}_{\mathcal{A}}(C_n, C_{n-1}) \qquad \textit{and} \qquad \partial_n \circ \partial_{n+1} = 0$$

for all $n \in \mathbb{Z}$.
*Dually, a **$\mathbb{Z}$-graded cochain complex in $\mathcal{A}$** is a $\mathbb{Z}$-graded chain complex in $\mathcal{A}^{\mathrm{op}}$.*

Remark 1.1. For notational simplicity, we will write $(C_\bullet, \partial_\bullet)$ for a chain complex in $\mathcal{A}$.

Remark 1.2. For each abelian category $\mathcal{A}$, there is an abelian category $\mathrm{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$ (see [Wei94, p. 7]).

Definition 1.2 (Non-Negative Chain Complex). *Let $\mathcal{A}$ be an abelian category. A chain complex $(C_\bullet, \partial_\bullet) \in \mathrm{Ch}(\mathcal{A})$ is said to be **non-negative**, iff $C_n = 0$ for all $n < 0$. We denote by $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ the full subcategory of $\mathrm{Ch}(\mathcal{A})$ of non-negative chain complexes.*

Recall, that for $R \in \mathsf{CRing}$ and $\mathfrak{g} \in {}_R\mathsf{LieAlg}$, the ***universal envelopping algebra $U\mathfrak{g}$ of $\mathfrak{g}$*** is defined to be the quotient of the ***tensor algebra $T\mathfrak{g}$***

$$T\mathfrak{g} := \bigoplus_{n\in\omega} \mathfrak{g}^{\otimes n}$$

by the 2-sided ideal generated by the relations

$$\iota[x, y] = \iota(x)\iota(y) - \iota(y)\iota(x),$$

for all $x, y \in \mathfrak{g}$, where $\iota : \mathfrak{g} \hookrightarrow T\mathfrak{g}$ denotes inclusion. Moreover, U is a functor from ${}_R\mathsf{LieAlg}$ to associative R-algebras. Hence we can define a $U\mathfrak{g}$-action on $U\mathfrak{g} \otimes_R \Lambda^n\mathfrak{g}$ simply by

$$u(v \otimes x_1 \wedge \cdots \wedge x_n) := uv \otimes x_1 \wedge \cdots \wedge x_n,$$

for all $n \in \omega$.

Definition 1.3 (Chevalley-Eilenberg Complex). *Let $R \in \mathsf{CRing}$ and $\mathfrak{g} \in {}_R\mathsf{LieAlg}$ which is free as an R-module. Denote by $U\mathfrak{g}$ the universal envelopping algebra of $\mathfrak{g}$. Define a non-negative chain complex $(C_\bullet, \partial_\bullet) \in \mathrm{Ch}_{\geq 0}({}_{U\mathfrak{g}}\mathsf{Mod})$ by*

$$C_n := U\mathfrak{g} \otimes_R \Lambda^n\mathfrak{g}$$

for all $n \in \omega$ and

$$\partial_n(u \otimes x_1 \wedge \cdots \wedge x_n) := \begin{cases} ux_1 & n = 1, \\ \theta_1 + \theta_2 & n > 1, \end{cases}$$

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where

$$\theta_1 := \sum_{i=0}^{n} (-1)^{i+1} u x_i \otimes x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n,$$

and

$$\theta_2 := \sum_{1 \leq i < j \leq n} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n.$$

Remark 1.3. It is by no means obvious, that $\partial_n \circ \partial_{n+1} = 0$ holds for the Chevalley-Eilenberg complex 1.3. However, it is a tedious computation, and we will only demonstrate the case $n = 1$. In this case

$$\begin{aligned}(\partial_1 \circ \partial_2)(u \otimes x \wedge y) &= \partial_1(ux \otimes y - uy \otimes x - u \otimes [x, y]) \\ &= u(xy - yx) - u[x, y] \\ &= 0,\end{aligned}$$

for all $u \in U\mathfrak{g}$ and $x, y \in \mathfrak{g}$.

Remark 1.4. The definition of the boundary map ∂_n in the Chevalley-Eilenberg complex 1.3 is not as arbitrary at it might seem at first sight. Given $\alpha \in \Omega^n(M)$ for a smooth manifold M, then we have that $d\alpha(X_1, \ldots, X_{n+1})$ is of the same form for any $X_1, \ldots, X_{n+1} \in \mathfrak{X}(M)$. Actually, this formula can be used to give an invariant definition of the exterior derivative d in the de Rham theory (see [Lee13, pp. 370–372]).

Left $\mathfrak{g}$-Modules and the Cohomology of Lie Algebras

Definition 1.4 (Category of Left $\mathfrak{g}$-Modules). *Let $R \in \mathsf{CRing}$ and $\mathfrak{g} \in {}_R\mathsf{LieAlg}$. The **category of left $\mathfrak{g}$-modules**, written ${}_\mathfrak{g}\mathsf{Mod}$, is defined to be the category with objects **left $\mathfrak{g}$-modules**, i.e. modules $M \in {}_R\mathsf{Mod}$ equipped with an R-bilinear product $\mathfrak{g} \times M \to M$, $(x, m) \mapsto xm$, such that*

$$[x, y]m = x(ym) - y(xm)$$

*holds for all $x, y \in \mathfrak{g}$ and $m \in M$, and **left $\mathfrak{g}$-module homomorphisms** as morphisms, i.e. morphisms $f \in \mathrm{Hom}_{{}_R\mathsf{Mod}}(M, N)$ such that*

$$f(xm) = xf(m)$$

holds for all $x \in \mathfrak{g}$ and $m \in M$.

Proposition 1.1. *Let $R \in \mathsf{CRing}$ and $\mathfrak{g} \in {}_R\mathsf{LieAlg}$. Then ${}_\mathfrak{g}\mathsf{Mod}$ is an abelian category.*

Proof. See [Wei94, p. 220]. □

We follow [KS06, p. 178].

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Proposition 1.2. *Let $\mathcal{A}$ be an abelian category and $(C_\bullet, \partial_\bullet) \in \mathsf{Ch}(\mathcal{A})$. Then for every $n \in \mathbb{Z}$, there exists a unique monic*

$$\operatorname{im}\partial_{n+1} \to \ker\partial_n,$$

where $\operatorname{im}\partial_{n+1} := \ker(\operatorname{coker}\partial_{n+1})$.

Exercise 1.1. Prove proposition 1.2. *Hint:* Use that $\operatorname{im}\partial_{n+1} \to C_n$ is monic by [Lan78, p. 199].

Definition 1.5 (Homology of a Chain Complex). *Let $\mathcal{A}$ be an abelian category and $(C_\bullet, \partial_\bullet) \in \mathsf{Ch}(\mathcal{A})$. Moreover, let $n \in \mathbb{Z}$ and $\operatorname{im}\partial_{n+1} \to \ker\partial_n$ be the unique morphism assured by lemma 1.2. Then we define the **n-th homology object**, written $H_n(C_\bullet, \partial_\bullet)$, by*

$$H_n(C_\bullet, \partial_\bullet) := \operatorname{coker}(\operatorname{im}\partial_{n+1} \to \ker\partial_n) \in \operatorname{ob}(\mathcal{A}).$$

Observe, that for each $n \in \omega$, we have that $C_n \in {}_{\mathfrak{g}}\mathsf{Mod}$ via $\iota : \mathfrak{g} \hookrightarrow U\mathfrak{g}$.

Definition 1.6 (Cohomology of Lie Algebras). *Let $R \in \mathsf{CRing}$ and $\mathfrak{g} \in {}_R\mathsf{LieAlg}$ which is free as an R-module. Moreover, let $M \in {}_{\mathfrak{g}}\mathsf{Mod}$ and $(C_\bullet, \partial_\bullet)$ denote the Chevalley-Eilenberg complex 1.3. For $n \in \omega$, define the **n-th cohomology group of $\mathfrak{g}$ with coefficients in M**, written $H^n(\mathfrak{g}, M)$, to be the n-th homology object of the cochain complex*

$$\operatorname{Hom}_{{}_{\mathfrak{g}}\mathsf{Mod}}((C_\bullet, \partial_\bullet), M).$$

Remark 1.5. Actually, we have that

$$\operatorname{Hom}_{{}_{\mathfrak{g}}\mathsf{Mod}}(C_n, M) \cong \operatorname{Hom}_{{}_R\mathsf{Mod}}(\Lambda^n\mathfrak{g}, M),$$

in ${}_{\mathbb{Z}}\mathsf{Mod}$. Indeed, if $\varphi \in \operatorname{Hom}_{{}_{\mathfrak{g}}\mathsf{Mod}}(C_n, M)$, define $\overline{\varphi} \in \operatorname{Hom}_{{}_R\mathsf{Mod}}(\Lambda^n\mathfrak{g}, M)$ by

$$\overline{\varphi}(x_1 \wedge \cdots \wedge x_n) := \varphi(1 \otimes x_1 \wedge \cdots \wedge x_n),$$

and conversly, if $\varphi \in \operatorname{Hom}_{{}_R\mathsf{Mod}}(\Lambda^n\mathfrak{g}, M)$, define $\overline{\varphi} \in \operatorname{Hom}_{{}_{\mathfrak{g}}\mathsf{Mod}}(C_n, M)$ by

$$\overline{\varphi}(u \otimes x_1 \wedge \cdots \wedge x_n) := u\varphi(x_1 \wedge \cdots \wedge x_n).$$

This is possible, since every left $\mathfrak{g}$-module is naturally a left $U\mathfrak{g}$-module (see [Wei94, pp. 224–225]). Hence we get an induced morphism

$$\begin{tikzcd}
\operatorname{Hom}_{{}_{\mathfrak{g}}\mathsf{Mod}}(C_{n-1}, M) \arrow[r, "d^n"] \arrow[d, "\overline{\cdot}"'] & \operatorname{Hom}_{{}_{\mathfrak{g}}\mathsf{Mod}}(C_n, M) \arrow[d, "\overline{\cdot}"] \\
\operatorname{Hom}_{{}_R\mathsf{Mod}}(\Lambda^{n-1}\mathfrak{g}, M) \arrow[r, dashed] & \operatorname{Hom}_{{}_R\mathsf{Mod}}(\Lambda^n\mathfrak{g}, M).
\end{tikzcd}$$

Explicitely

$$\begin{aligned}
d^n f(x_1, \ldots, x_n) = &\sum_{i=1}^n (-1)^{i+1} x_i f(x_1, \ldots, \widehat{x_i}, \ldots, x_n) \\
&+ \sum_{1 \le i < j \le n} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_n),
\end{aligned}$$

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for all $n \in \omega$.

Remark 1.6. There is a more general approach to the definition of the cohomology of Lie algebras via the notion of *right derived functors* which does not use the intermediate step of the Chevalley-Eilenberg complex.

Example 1.1 ($H^3(\mathfrak{sl}_2, k)$). Let k be a field with characteristic not equal to two and consider the *special linear Lie algebra over* k, i.e. $A \in M_2(k)$ such that $\operatorname{tr} A = 0$. This is a three dimensional Lie algebra with ordered basis

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad e_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Hence the crucial portion of the Eilenberg-Chevalley cochain complex is given by

$$\ldots \longrightarrow \operatorname{Hom}_k(\Lambda^2\mathfrak{sl}_2, k) \xrightarrow{d} \operatorname{Hom}_k(\Lambda^3\mathfrak{sl}_2, k) \longrightarrow 0.$$

We compute

$$\begin{aligned} df(e_1, e_2, e_3) =& e_1 f(e_2, e_3) - e_2 f(e_1, e_3) + e_3 f(e_1, e_2) \\ & - f([e_1, e_2], e_3) + f([e_1, e_3], e_2) - f([e_2, e_3], e_1) \\ =& -2f(e_2, e_3) - 2f(e_3, e_2) - f(e_1, e_1) \\ =& -2f(e_2, e_3) + 2f(e_2, e_3) \\ =& 0, \end{aligned}$$

since k is interpreted as a trivial $\mathfrak{sl}_2$-module and by the alternating k-multilinear properties of f. Hence

$$H^3(\mathfrak{sl}_2, k) \cong \operatorname{Hom}_k(\Lambda^3\mathfrak{sl}_2, k) \cong k,$$

since $\dim \operatorname{Hom}_k(\Lambda^3\mathfrak{sl}_2, k) = 1$.

The Whitehead Lemmas

Theorem 1.2 (Whitehead's First Lemma). *Let k be a field of characteristic zero and $\mathfrak{g} \in {}_k\mathsf{LieAlg}$ semisimple. Then for any finite-dimensional $M \in {}_{\mathfrak{g}}\mathsf{Mod}$ we have that*

$$H^1(\mathfrak{g}, M) = 0.$$

Theorem 1.3 (Whitehead's Second Lemma). *Let k be a field of characteristic zero and $\mathfrak{g} \in {}_k\mathsf{LieAlg}$ semisimple. Then for any finite-dimensional $M \in {}_{\mathfrak{g}}\mathsf{Mod}$ we have that*

$$H^2(\mathfrak{g}, M) = 0.$$

Remark 1.7. There cannot be a third Whitehead lemma, since

$$H^3(\mathfrak{sl}_2, k) \cong k,$$

by exercise 1.1.

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