

FUNCTIONAL ANALYSIS II SUMMARY

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Abstract. This is a rough summary of the course *Functional Analysis II* held at *ETH Zurich* by *Prof. Dr. Alessandro Carlotto* in spring 2018. The main focus of this summary is to give a neat preparation for the oral exam.

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Introduction

Theorem 1.1 (Transformation Formula). Let $n \in \omega$, $n > 0$, $U, V \subseteq \mathbb{R}^n$ open and $\varphi : U \rightarrow V$ a C^1 -diffeomorphism. A function $f : V \rightarrow \bar{\mathbb{R}}$ is in $\mathcal{L}^1(V)$ if and only if $(f \circ \varphi) |\det(D\varphi)|$ is in $\mathcal{L}^1(U)$. Then

$$\int_V f = \int_U (f \circ \varphi) |\det(D\varphi)|.$$

Theorem 1.2. Let $\Omega \subseteq \mathbb{R}^n$ be open and $1 \leq p < \infty$. Then $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.

Proposition 1.1. If $\mu(X) < \infty$ and $0 < p < q \leq \infty$. Then $L^q(\mu) \subseteq L^p(\mu)$.

Proposition 1.2 (Integration by Parts). Let (M, g) be a compact Riemannian manifold with boundary. Then

$$\int_M \langle \text{grad } f, X \rangle_g dV_g = \int_{\partial M} f \langle X, N \rangle dV_{\tilde{g}} - \int_M (f \operatorname{div} X) dV_g$$

for $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$. Moreover, **Green's identities** hold:

$$\int_M u \Delta v dV_g = \int_M \langle \text{grad } u, \text{grad } v \rangle_g dV_g - \int_{\partial M} u N v dV_{\tilde{g}}$$

and

$$\int_M (u \Delta v - v \Delta u) dV_g = \int_{\partial M} (v N u - u N v) dV_{\tilde{g}}$$

for $u, v \in C^\infty(M)$.

Suppose $u \in \mathcal{A}$ is a minimizer of E_p and $\varphi \in C^2(\bar{\Omega})$ with $\varphi|_{\partial\Omega} = 0$. We compute

$$\begin{aligned} \frac{d}{dt} E_p(u + t\varphi) &= \frac{d}{dt} \int_{\Omega} |\nabla u + t \nabla \varphi|^p \\ &= \frac{d}{dt} \int_{\Omega} \langle \nabla u + t \nabla \varphi, \nabla u + t \nabla \varphi \rangle^{p/2} \\ &= p \int_{\Omega} |\nabla u + t \nabla \varphi|^{p-2} \langle \nabla \varphi, \nabla u + t \nabla \varphi \rangle. \end{aligned}$$

In particular

$$\left. \frac{d}{dt} \right|_{t=0} E_p(u + t\varphi) = p \int_{\Omega} |\nabla u|^{p-2} \langle \nabla \varphi, \nabla u \rangle = - \int_{\Omega} \operatorname{div}(|\nabla u|^{p-2} \nabla u) \varphi.$$

Sobolev Space Theory

The Spaces $W^{k,p}(\Omega)$. In what follows, let $n \in \omega$, $n \geq 1$, and $1 \leq p \leq \infty$.

Definition 1.1 (Distributional and Weak Derivative). Let $\Omega \subseteq \mathbb{R}^n$ open and $u \in L^1_{\text{loc}}(\Omega)$. For any multiindex α , the **distributional derivative of order α of u** , written $D^\alpha u$, is defined to be the mapping $D^\alpha u : C_c^\infty(\Omega) \rightarrow \mathbb{R}$ defined by

$$\varphi \mapsto (-1)^{|\alpha|} \int_{\Omega} u D^\alpha \varphi dx.$$

Moreover, a function $D^\alpha u \in L^p(\Omega)$ is called **weak derivative of order α of u with exponent p** , iff

$$\forall \varphi \in C_c^\infty(\Omega) : \int_{\Omega} D^\alpha u \varphi dx = (-1)^{|\alpha|} \int_{\Omega} u D^\alpha \varphi dx.$$

Theorem 1.3 (Fundamental Lemma of Variational Calculus). Let $\Omega \subseteq \mathbb{R}^n$ open and $f \in L^1_{\text{loc}}(\Omega)$. If

$$\forall \varphi \in C_c^\infty(\Omega) : \int_{\Omega} f \varphi dx = 0,$$

then $f = 0$ a.e.

Remark 1.1. Let $\Omega \subseteq \mathbb{R}^n$ open. Then $L^p(\Omega) \subseteq L^1_{\text{loc}}(\Omega)$.

Remark 1.2. From the fundamental lemma of variational calculus 1.3 it follows that *weak derivatives, if they exist, are unique*.

Examples 1.1.

- (a) Consider $\Omega := (-1, 1)$ and $u := |x|$. Then $u' = \chi_{[0,1)} - \chi_{(-1,0)}$.
- (b) Consider $\Omega := \mathbb{R}$ and $u := \chi_{(0,\infty)}$. Then the weak derivative u' does not exist. Indeed, the *Dirac distribution* is not representable as one may see by considering the smooth family $\varphi_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ for $\varepsilon > 0$ defined by

$$\varphi_\varepsilon(x) := \begin{cases} e^{\varepsilon^2/(x^2-\varepsilon^2)} & |x| < \varepsilon, \\ 0 & |x| \geq \varepsilon. \end{cases}$$

Definition 1.2 (Sobolev Space). Let $\Omega \subseteq \mathbb{R}^n$ open. For any $k \in \omega$, the **Sobolev space of index (k, p)** , written $W^{k,p}(\Omega)$, is defined to be the space

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ exists for all } |\alpha| \leq k\},$$

with norm

$$\|-\|_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha -\|_{L^p(\Omega)}.$$

Moreover, define

$$W_0^{k,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|-\|_{W^{k,p}(\Omega)}},$$

and $H^k(\Omega) := W^{k,2}(\Omega)$ as well as $H_0^k(\Omega) := W_0^{k,2}(\Omega)$.

Theorem 1.4. Let $\Omega \subseteq \mathbb{R}^n$ open. Then $W^{k,p}(\Omega)$ is

- (a) a Banach space for all $1 \leq p \leq \infty$.
 (b) separable for all $1 \leq p < \infty$.
 (c) reflexive for all $1 < p < \infty$.
Proof.

(a) This follows from the fact that $L^p(\Omega)$ is a Banach space for all $1 \leq p \leq \infty$. Let $(f_i)_{i \in \omega}$ be a Cauchy sequence in $W^{k,p}$. By definition of the $W^{k,p}$ -norm, $(D^\alpha f_i)_{i \in \omega}$ is a Cauchy sequence in L^p . Thus we get $D^\alpha f_i \rightarrow f_\alpha$ in L^p , in particular, $f_i \rightarrow f$ in L^p . Using Hölder's inequality we compute

$$\int_{\Omega} f_{\alpha} \varphi dx = \lim_{i \rightarrow \infty} \int_{\Omega} D^{\alpha} f_i \varphi dx = (-1)^{|\alpha|} \lim_{i \rightarrow \infty} \int_{\Omega} f_i D^{\alpha} \varphi dx = (-1)^{|\alpha|} \int_{\Omega} f D^{\alpha} \varphi dx$$

for $\varphi \in C_c^{\infty}(\Omega)$.

- (b) For simplicity, we consider $k = 1$ only. Consider $\iota : W^{1,p} \hookrightarrow (L^p)^{n+1}$ defined in the obvious way. Then ι is an isometry and the statement follows.
 (c) Same argument as in part (b). □

Elliptic Operators in Divergence Form.

Lemma 1.1 (Poincaré Inequality). *Let $\Omega \subseteq \mathbb{R}^n$ open and bounded. Then for any $u \in C_c^{\infty}(\Omega)$ we have that*

$$\|u\|_{L^2} \leq C \|\nabla u\|_{L^2}.$$

Proof. Let $n = 1$. Since Ω is bounded, we get that $\Omega \subseteq [a, b]$ and we may extend u on $[a, b] =: I$ to be zero. Hence an application of Jensen's inequality (or Cauchy-Schwarz) yields

$$|u(x)|^2 = |u(x) - u(a)|^2 = \left| \int_a^x u'(t) dt \right|^2 \leq (x-a) \int_a^x |u'(t)|^2 dt \leq (b-a) \|u'\|_{L^2(I)}^2.$$

Thus

$$\|u\|_{L^2(\Omega)}^2 \leq \|u\|_{L^2(I)}^2 \leq (b-a)^2 \|u'\|_{L^2(I)}^2 = (b-a)^2 \|u'\|_{L^2(\Omega)}^2$$

where the last equality follows due to the fact that u and thus u' is compactly supported in Ω . If $n > 1$, we have $\Omega \subseteq [a, b] \times \mathbb{R}^{n-1}$ and thus the claim follows by reduction to the previous case. □

Theorem 1.5 (Riesz Representation Theorem). *Let H be a real Hilbert space. Then the mapping $J : H \rightarrow H^*$ defined by $J(x) := \langle x, - \rangle$ is an isometric isomorphism.*

Theorem 1.6. *Let $\Omega \subseteq \mathbb{R}^n$ and consider the elliptic operator*

$$L := \frac{\partial}{\partial x^i} \left(a^{ij} \frac{\partial}{\partial x^j} \right),$$

for $a^{ij} \in L^\infty(\Omega)$ symmetric. Then: Given $f \in L^2(\Omega)$, the homogenous Dirichlet problem

$$\begin{cases} -Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

admits a unique weak solution $u \in H_0^1(\Omega)$.

Proof.

Step 1: Derivation of Weak Formulation. Suppose $u \in C^2(\bar{\Omega})$ is a solution of (1). Let $\varphi \in C_c^\infty(\Omega)$. Then integration by parts (see [Lee13, p. 436]) yields

$$-\int_{\Omega} L(u)\varphi = -\sum_{j=1}^n \int_{\Omega} \operatorname{div}(X_j)\varphi = \int_{\Omega} a^{ij} \frac{\partial u}{\partial x^j} \frac{\partial \varphi}{\partial x^i} = \int_{\Omega} a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial \varphi}{\partial x^j},$$

where $X_j := \left(a^{ij} \frac{\partial}{\partial x^j}\right)_i$. Thus we get the weak formulation:

$$\forall \varphi \in C_c^\infty(\Omega) : \int_{\Omega} a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial \varphi}{\partial x^j} = \int_{\Omega} f \varphi. \quad (2)$$

Step 2: Existence and Uniqueness of Weak Solutions. Since L is uniformly elliptic, there exists $\lambda > 0$ such that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

holds for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Moreover, since $a^{ij} \in L^\infty(\Omega)$, we get that L is uniformly bounded, i.e. there exists $\Lambda > 0$ such that

$$a^{ij}(x)\xi_i\eta_j \leq \Lambda|\xi||\eta|$$

for

$$\Lambda = \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(\Omega)}$$

holds for almost all $x \in \Omega$ and $\xi, \eta \in \mathbb{R}^n$. Now define a bilinear form $\langle \cdot, \cdot \rangle_a : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ by

$$\langle u, v \rangle_a := \int_{\Omega} a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} \quad (3)$$

Then it is easy to see, that $\langle \cdot, \cdot \rangle_a$ is symmetric. Also, $\langle \cdot, \cdot \rangle_a$ is positive definite since

$$\langle u, u \rangle_a = \int_{\Omega} a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \geq \lambda \int_{\Omega} |\nabla u|^2 \geq C\lambda \int_{\Omega} |u|^2$$

using ellipticity and Poincaré's inequality. Moreover by Poincaré's inequality we have that

$$\lambda \|u\|_{H_0^1(\Omega)}^2 \leq \|u\|_a^2 \leq \Lambda \|u\|_{H_0^1(\Omega)}^2$$

for the induced norm $\|\cdot\|_a$. Hence the induced norm is equivalent to the standard norm on $H_0^1(\Omega)$ and thus $(H_0^1(\Omega), \|\cdot\|_a)$ is a Hilbert space. Thus an application of Riesz representation theorem 1.5 yields the existence of a unique $u \in H_0^1(\Omega)$, such that

$$\langle u, \varphi \rangle_a = l(\varphi) := \int_{\Omega} f \varphi$$

holds for all $\varphi \in H_0^1(\Omega)$, since $l \in (H_0^1(\Omega))^*$. □

Sobolev Spaces on an Interval. In what follows, let $-\infty \leq a < b \leq \infty$ and $I := (a, b)$.

Lemma 1.2 (Du Bois-Reymond). *Let $f \in L_{\text{loc}}^1(I)$ such that*

$$\forall \varphi \in C_c^\infty(I) : \int_I f \varphi' dx = 0.$$

Then f is almost everywhere constant.

Proof. Let $v := w - c_0 \psi$ for $w, \psi \in C_c^\infty(I)$ such that $\int_I \psi = 1$ and $\int_I v = 0$. This implies $c_0 = \int_I w$. By the fundamental theorem of calculus, the function $\varphi : I \rightarrow \mathbb{R}$ defined by

$$\varphi(x) := \int_I v(t) dt$$

belongs to $C_c^\infty(I)$ with $\varphi' = v$. Thus we compute

$$0 = \int_I f \varphi' = \int_I f v = \int_I f w - c_0 \int_I f \psi = \int_I f w - \int_I w \int_I f \psi = \int_I (f - c) w,$$

where $c := \int_I f \psi$. Since w was arbitrary, we conclude by the fundamental lemma of variational calculus 1.3. □

Lemma 1.3. *Let $f \in L_{\text{loc}}^1(I)$ and $x_0 \in I$. Then $u : I \rightarrow \mathbb{R}$ defined by*

$$u(x) := \int_{x_0}^x f(t) dt$$

is absolutely continuous and belongs to $W_{\text{loc}}^{1,1}(I)$ with $u' = f$ a.e.

Proof. Absolute continuity follows from real analysis. Let $\varphi \in C_c^\infty(I)$. Then Fubini yields

$$\begin{aligned} \int_I u \varphi' &= \int_a^{x_0} \int_{x_0}^x f(t) \varphi'(x) dt dx + \int_{x_0}^b \int_{x_0}^x f(t) \varphi'(x) dt dx \\ &= - \int_a^{x_0} \int_x^{x_0} f(t) \varphi'(x) dt dx + \int_{x_0}^b \int_{x_0}^x f(t) \varphi'(x) dt dx \end{aligned}$$

$$\begin{aligned}
&= - \int_a^{x_0} \int_a^t f(t) \varphi'(x) dx dt + \int_{x_0}^b \int_t^b f(t) \varphi'(x) dx dt \\
&= - \int_a^{x_0} f(t) \varphi(t) dt - \int_{x_0}^b f(t) \varphi(t) dt \\
&= - \int_I f \varphi.
\end{aligned}$$

□

Theorem 1.7. Let $u \in W^{1,p}(I)$. Then there exists an absolutely continuous representant \tilde{u} of u on \bar{I} , such that

$$\tilde{u}(x) = \tilde{u}(x_0) + \int_{x_0}^x u'(t) dt$$

holds for all $x, x_0 \in I$.

Proof. By lemma 1.3, the function $v(x) := \int_{x_0}^x u'(t) dt$ is in $W_{\text{loc}}^{1,1}(I)$ with weak derivative u' . Moreover, for any $\varphi \in C_c^\infty(I)$ we compute

$$\int_I (u - v) \varphi' = \int_I u \varphi' - \int_I v \varphi' = - \int_I u' \varphi + \int_I u' \varphi = 0.$$

Thus lemma 1.2 yields $u = c + v$, for some $c \in \mathbb{R}$. But $c = u(x_0)$ and we conclude by setting

$$\tilde{u}(x) := u(x_0) + \int_{x_0}^x u'(t) dt.$$

□

Theorem 1.8 (Characterization of $W^{1,p}(I)$). Let $1 < p \leq \infty$ and $u \in L^p(I)$. Then the following statements are equivalent:

- (a) $u \in W^{1,p}(I)$.
- (b) There exists $C \geq 0$ such that

$$\forall \varphi \in C_c^\infty(I) : \left| \int_I u \varphi' \right| \leq C \|\varphi\|_{L^q}.$$

- (c) There exists $C \geq 0$ such that for all $I' \subseteq\subseteq I$ and $|h| < \text{dist}(I', \partial I)$ holds

$$\|\tau_h u - u\|_{L^p(I')} \leq C |h|,$$

where $\tau_h u(x) := u(x + h)$.

Proof. The implication (a) \Rightarrow (b) follows immediately from Hölder's inequality. To prove (b) \Rightarrow (a), we observe that $l : C_c^\infty(I) \rightarrow \mathbb{R}$ defined by

$$L(\varphi) := \int_I u \varphi'$$

is continuous. Since $C_c^\infty(I)$ is dense in $L^q(I)$, we get that $l \in (L^q(I))^*$. Hence we find $g \in L^p$, such that $\int_I g\varphi = l(\varphi)$ and so $u' = -g$.

Next we show (a) \Rightarrow (c). By theorem 1.7, we find an absolutely continuous representant \tilde{u} of u . Thus

$$\tilde{u}(x+h) - \tilde{u}(x) = h \int_0^1 u'(x+th) dt$$

Hence Jensen's inequality yields

$$\|\tau_h u - u\|_{L^p(I')} \leq |h| \int_0^1 \|u'(\cdot + th)\|_{L^p(I')} dt \leq |h| \|u'\|_{L^p(I)}.$$

Lastly, we prove (c) \Rightarrow (b). Let $\varphi \in C_c^\infty(I)$. Then we may find $I' \subseteq\subseteq I$ such that $\text{supp } \varphi \subseteq I'$. Hence we compute

$$\begin{aligned} \left| \int_I u\varphi' \right| &= \lim_{h \rightarrow 0} \frac{1}{|h|} \left| \int_I u(x) (\varphi(x+h) - \varphi(x)) dx \right| \\ &= \lim_{h \rightarrow 0} \frac{1}{|h|} \left| \int_I (u(x-h) - u(x)) \varphi(x) dx \right| \\ &= \lim_{h \rightarrow 0} \frac{1}{|h|} \left| \int_I (\tau_{-h} u - u) \varphi \right| \\ &\leq \lim_{h \rightarrow 0} \frac{1}{|h|} \|\tau_{-h} u - u\|_{L^p(I')} \|\varphi\|_{L^q(I)} \\ &\leq C \|\varphi\|_{L^q(I)}. \end{aligned}$$

□

Theorem 1.9 (Extension Theorem). *There exists a continuous linear operator*

$$E : W^{1,p}(I) \rightarrow W^{1,p}(\mathbb{R})$$

such that:

- (i) $Eu|_I = u$.
- (ii) $\|Eu\|_{L^p(\mathbb{R})} \leq C \|u\|_{L^p(I)}$.
- (iii) $\|(Eu)'\|_{L^p(\mathbb{R})} \leq C \|u\|_{W^{1,p}(I)}$.

Proof. First we consider the case $I = (0, \infty)$. We extend u by continuity to 0 and then we extend u by means of even symmetry. If I is bounded we can without loss of generality assume that $I = (0, 1)$. Now use a cut-off function. □

Theorem 1.10 (Approximation Theorem). *Let $1 \leq p < \infty$ and $u \in W^{1,p}(I)$. Then there exists a sequence $(u_i)_{i \in \omega}$ in $C_c^\infty(\mathbb{R})$ such that*

$$\|u_i|_I - u\|_{W^{1,p}(I)} \rightarrow 0.$$

Proof. The main idea of the proof is to use convolutions. Moreover, it is enough to consider the case $I = \mathbb{R}$ only, due to the extension theorem 1.9. \square

Theorem 1.11 (Sobolev Embedding). *There is a continuous embedding*

$$W^{1,p}(I) \hookrightarrow L^\infty(I).$$

Proof. Without loss of generality, consider $|I| \leq 1$. By theorem 1.7 we get that

$$\|u\|_{L^\infty} = \sup_{x \in I} |u(x)| \leq |u(y)| + \sup_{x \in I} \left| \int_y^x u'(t) dt \right| \leq |u(y)| + \|u'\|_{L^1},$$

for any $y \in I$. Hence

$$\|u\|_{L^\infty} \leq \inf_{y \in I} |u(y)| + \|u'\|_{L^1} \leq \frac{1}{|I|} \int_I |u(y)| + \|u'\|_{L^1} \leq C \|u\|_{W^{1,1}} \leq C \|u\|_{W^{1,p}}.$$

\square

Corollary 1.1. *Let I be unbounded and $u \in W^{1,p}(I)$ for $1 \leq p < \infty$. Then $u \rightarrow 0$ as $|x| \rightarrow \infty$.*

Dirichlet and Neumann Boundary Problems on I . In what follows, let us consider $-\infty < a < b < \infty$ and $I := (a, b)$.

Proposition 1.3. *Let $f \in C^0(\bar{I})$. Then the weak solution u of the homogenous Dirichlet problem*

$$\begin{cases} -u'' = f & \text{in } I, \\ u(a) = 0 = u(b). \end{cases}$$

is a classical solution, i.e. $u \in C^2(\bar{I})$.

Proof.

\square

Proposition 1.4. *Let $f \in C^0(\bar{I})$. Then the weak solution u of the homogenous Neumann problem*

$$\begin{cases} -u'' + u = f & \text{in } I, \\ u'(a) = 0 = u'(b). \end{cases}$$

is a classical solution, i.e. $u \in C^2(\bar{I})$.

Proof.

\square

Sobolev Spaces on a Domain.

Example 1.1 (Vanishing $W^{1,p}$ -Capacity). For $n \in \omega$, $n > 1$, and $1 \leq p \leq n$, the set $\{0\}$ has vanishing $W^{1,p}$ -capacity.

Theorem 1.12 (Meyers-Serrin). Let $\Omega \subseteq \mathbb{R}^n$ be open. Then $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for every $1 \leq p < \infty$.

Proof. Convolutions and a partition of unity argument. \square

Theorem 1.13 (Characterization of $W^{1,p}(\Omega)$). Let $1 < p \leq \infty$ and $u \in L^p(\Omega)$. Then the following statements are equivalent:

- (a) $u \in W^{1,p}(\Omega)$.
- (b) There exists $C \geq 0$ such that

$$\forall |\alpha| \leq 1 \forall \varphi \in C_c^\infty(\Omega) : \left| \int_I u D^\alpha \varphi \right| \leq C \|\varphi\|_{L^q}.$$

- (c) There exists $C \geq 0$ such that for all $\Omega' \subseteq \subseteq \Omega$ and $|h| < \text{dist}(I', \partial I)$ holds

$$\|\tau_h u - u\|_{L^p(\Omega')} \leq C |h|,$$

where $\tau_h u(x) := u(x + h)$.

Proof. The proof $(c) \Rightarrow (b) \Rightarrow (a)$ is almost the same as the one given in the characterization theorem for Ω an interval. For proving $(a) \Rightarrow (c)$, use Meyers-Serrin. \square

Corollary 1.2. Let $u \in L^\infty(\Omega)$. Then $u \in W^{1,\infty}(\Omega)$ if and only if u admits a locally Lipschitz continuous representant. Moreover, if Ω is convex, then $u \in W^{1,\infty}(\Omega)$ if and only if u admits a Lipschitz continuous representant.

Extension and Trace Operator. We start off with *local theory*. In what follows, define

$$Q := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < 1, |x_n| < 1\}.$$

Moreover

$$Q_+ := \{(x', x_n) \in Q : x_n > 0\} \quad \text{and} \quad Q_0 := \{(x', x_n) \in Q : x_n = 0\}.$$

Lemma 1.4. Let $u \in W^{1,p}(Q_+)$. Set

$$u^*(x', x_n) := \begin{cases} u(x', x_n) & x_n > 0, \\ u(x', -x_n) & x_n < 0. \end{cases}$$

Then $u^* \in W^{1,p}(Q)$ and $\|u^*\|_{W^{1,p}(Q)} \leq C \|u\|_{W^{1,p}(Q_+)}.$

Now to the *global theory*.

Theorem 1.14 (Extension). *Let $\Omega \subseteq \mathbb{R}^n$ of class C^1 . Then there exists a continuous linear operator*

$$E : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^n)$$

such that:

- (i) $Eu|_{\Omega} = u$.
- (ii) $\|Eu\|_{L^p(\mathbb{R}^n)} \leq C \|u\|_{L^p(\Omega)}$.
- (iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\Omega)}$.

Corollary 1.3. *Let $\Omega \subseteq \mathbb{R}^n$ of class C^1 and $1 \leq p < \infty$. Then $C^\infty(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$.*

Again, we tackle first the *local theory*.

Lemma 1.5. *Let $u \in W^{1,p}(Q_+)$. Then $u|_{Q_0} \in L^p(Q_0)$ is well defined and the induced trace operator $W^{1,p}(Q_+) \rightarrow L^p(Q_0)$ is linear and continuous.*

Proof. We consider the case $1 \leq p < \infty$. The main idea is to show this for $u \in C^\infty(Q)$, then for $u \in W^{1,p}(Q)$ and then finally for $u \in W^{1,p}(Q_+)$ by extension. Consider now $p = \infty$. Since Q_+ is convex, $u \in W^{1,\infty}(Q_+)$ admits a Lipschitz continuous representant and the result follows by extending via continuity. \square

Theorem 1.15 (Characterization of $H^1(\Omega)$). *Let $\Omega \subseteq \mathbb{R}^n$ of class C^1 . Then*

$$H^1(\Omega) = H_0^1(\Omega) \oplus \{u \in H^1(\Omega) : \Delta u = 0\}.$$

Corollary 1.4 (Characterization of $H_0^1(\Omega)$). *Let $\Omega \subseteq \mathbb{R}^n$. Then*

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}.$$

Sobolev Embeddings.

$p < n$.

Theorem 1.16 (Sobolev-Gagliardo-Nirenberg). *Let $1 \leq p < n$ and let $p^* := \frac{np}{n-p}$. Then $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ with*

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p}.$$

Theorem 1.17 (Sobolev-Gagliardo-Nirenberg Compactness). *Let $\Omega \subseteq \mathbb{R}^n$ and $1 \leq p < n$. Then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $1 \leq q \leq p^*$ and the embedding is compact if $q < p^*$.*

$p > n$.

Theorem 1.18. *Let $p > n$. Then $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,\alpha}(\mathbb{R}^n)$ with $\alpha := 1 - \frac{n}{p}$ and*

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq \|u\|_{W^{1,p}(\Omega)}.$$

Remark 1.3. For $p = \infty$, the statement is trivially true, since any function in $W^{1,\infty}(\mathbb{R}^n)$ is Lipschitz continuous since \mathbb{R}^n is convex, and thus belongs to $C^{0,1}(\mathbb{R}^n)$.

The proof uses the notion of so-called *Morrey-Campanato spaces*.

Theorem 1.19. *Let $\Omega \subseteq \mathbb{R}^n$ of type A for some $A > 0$ and $1 \leq p < \infty$, $\lambda > n$, $\alpha := \frac{\lambda-n}{p}$. Then*

$$\mathcal{L}^{p,\lambda}(\Omega) \cong C^{0,\alpha}(\bar{\Omega}).$$

Proof. The inclusion $\mathcal{L}^{p,\lambda}(\Omega) \cong C^{0,\alpha}(\bar{\Omega})$ follows from the Campanato-theorem and does also hold for general $\Omega \subseteq \mathbb{R}^n$ open. \square

Theorem 1.20 (Poincaré-Wirtinger Inequality). *Let $u \in W^{1,p}(\mathbb{R}^n)$, $1 \leq p < \infty$. Then for all $x_0 \in \mathbb{R}^n$ and $r > 0$ we have that*

$$\|u - u_{x_0,r}\|_{L^p(B_r(x_0))}^p \leq C r^p \|\nabla u\|_{L^p(B_r(x_0))}^p.$$

Now the proof of the Sobolev embedding theorem for $p > n$ is immediaty by considering

$$W^{1,p}(\mathbb{R}^n) \xrightarrow{\text{P.W.}} \mathcal{L}^{p,p}(\mathbb{R}^n) \xrightarrow{\text{Campanato}} C^{0,\alpha}(\mathbb{R}^n)$$

and observing that \mathbb{R}^n is of type $\frac{\pi^{n/2}}{\Gamma(n/2+1)} > 0$.

Theorem 1.21. *Let $\Omega \subseteq \mathbb{R}^n$ connected and of class C^1 and $1 \leq p < \infty$. Then*

$$\|u - \bar{u}\|_{L^p(\Omega)}^p \leq C \|\nabla u\|_{L^p(\Omega)}^p.$$

$p = n$.

Theorem 1.22. *It holds that $W^{1,n}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ for $n \leq p < \infty$.*

Regularity Theory

Goal of this section is to prove the following regularity result.

Theorem 1.23 (Global H^2 -Estimate). *Let $\Omega \subseteq \mathbb{R}^n$ of class C^{k+2} and $f \in H^k(\Omega)$ for some $k \in \omega$. Moreover, let $u \in H_0^1(\Omega)$ be the unique solution of the homogenous Dirichlet boundary value problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $u \in H^{k+2}(\Omega)$ and

$$\|u\|_{H^{k+2}(\Omega)} \leq C \|f\|_{H^k(\Omega)}.$$

Interior Regularity.

Theorem 1.24. *Let $\Omega \subseteq \mathbb{R}^n$ of class C^1 and L an elliptic operator in divergence form satisfying $a^{ij} \in C^{k+1}(\bar{\Omega})$. If $f \in H^k(\Omega)$, the unique weak solution $u \in H_0^1(\Omega)$ of the homogenous Dirichlet boundary value problem*

$$\begin{cases} -Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

belongs to $H_{\text{loc}}^{k+2}(\Omega)$ and for all $\Omega' \subseteq \subseteq \Omega$ we have the estimate

$$\|u\|_{H^{k+2}(\Omega')} \leq C(\|f\|_{H^k(\Omega)} + \|u\|_{H^1(\Omega)}).$$

Proof. Step 1: $k = 0$.

(i) *H^1 -Estimate.* Choose a bump function $\varphi \in C_c^\infty(\Omega)$ supported in Ω' . Thus the weak formulation yields by plugging in the test function $\varphi^2 u$

$$\int_{\Omega} \varphi^2 a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} + 2 \int_{\Omega} u \varphi a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial \varphi}{\partial x^j} = \int_{\Omega} f \varphi^2 u. \quad (4)$$

Rearranging formula (4) we compute

$$\begin{aligned} \int_{\Omega} \varphi^2 a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} &= \int_{\Omega} f \varphi^2 u - 2 \int_{\Omega} u \varphi a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial \varphi}{\partial x^j} \\ &\leq \|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 + 2\Lambda \int_{\Omega} (-u) \varphi |\nabla u| |\nabla \varphi| \\ &\leq \|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 + \Lambda \varepsilon \|\varphi \nabla u\|_{L^2(\Omega)}^2 + \frac{\Lambda}{\varepsilon} \|u \nabla \varphi\|_{L^2(\Omega)}^2 \end{aligned}$$

Noticing that

$$\int_{\Omega} \varphi^2 a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \geq \lambda \|\varphi \nabla u\|_{L^2(\Omega)}^2$$

yields

$$(\lambda - \Lambda \varepsilon) \|\varphi \nabla u\|_{L^2(\Omega)}^2 \leq \|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 + \frac{\Lambda}{\varepsilon} \|u \nabla \varphi\|_{L^2(\Omega)}^2$$

Picking $\varepsilon > 0$ appropriately, yields

$$\|\varphi \nabla u\|_{L^2(\Omega)}^2 \leq C(\|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2)$$

and thus

$$\|\nabla u\|_{L^2(\Omega')}^2 \leq C(\|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2).$$

(ii) *H^2 -Estimate.*

(iii) *Nirenberg-Trick.*

□

Boundary Regularity.

Eigenfunctions of $-\Delta$.

Theorem 1.25. *Let $\Omega \subseteq \mathbb{R}^n$ of class C^2 . Then there exists a Hilbert-space basis $(\varphi_i)_{i \in \omega}$ of $L^2(\Omega)$ consisting of eigenfunctions of the Laplace operator, i.e.*

$$\begin{cases} -\Delta \varphi_i = \lambda_i \varphi_i & \text{in } \Omega, \\ \varphi_i = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover $0 < \lambda_i \rightarrow \infty$.

Proof. Define $K : L^2(\Omega) \rightarrow L^2(\Omega)$ by setting Kf to be the unique weak solution of the homogenous Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By the global regularity theorem, $u \in H^2(\Omega)$ and thus we can write K as the composition

$$L^2(\Omega) \longrightarrow H^2(\Omega) \hookrightarrow L^2(\Omega).$$

Thus K is continuous as a composition of continuous mappings and moreover, since the embedding is compact by the Sobolev theorem, so is K . \square

Schauder Theory

Schauder Estimates.

Theorem 1.26 (Global $C^{2,\alpha}$ -Estimate). *Let $\Omega \subseteq \mathbb{R}^n$ of class $C^{2,\alpha}$, $0 < \alpha < 1$. Moreover, let $a^{ij} \in C^{1,\alpha}(\Omega)$ symmetric, uniformly elliptic and uniformly bounded, $c \in C^\alpha(\Omega)$, $u_0 \in C^{2,\alpha}(\bar{\Omega})$, $f = (f^1, \dots, f^n) \in C^{1,\alpha}(\Omega)$ and $h \in C^\alpha(\Omega)$. Then any solution $u \in C^{2,\alpha}(\Omega)$ of the Dirichlet boundary value problem*

$$\begin{cases} A_0 u + cu = -\frac{\partial}{\partial x^i} f^i + h & \text{in } \Omega, \\ u = u_0 & \text{on } \Omega \end{cases}$$

satisfies

$$\|u\|_{C^{2,\alpha}} \leq C(\|u\|_{H^1} + \|f\|_{C^{1,\alpha}} + \|h\|_{C^\alpha} + \|u_0\|_{C^{2,\alpha}})$$

where C does not depend on u .

Existence Theorems.

Proposition 1.5 (Method of Continuity). *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Given $A_0, A_1 \in \mathcal{L}(X, Y)$ define $A_t := (1-t)A_0 + tA_1$, $t \in [0, 1]$. Suppose that*

$$\exists C > 0 \forall t \in [0, 1] \forall x \in X : \|x\|_X \leq \|A_t x\|_Y.$$

Then A_0 is surjective if and only if A_1 is surjective.

Using the method of continuity 1.5, one can show existence results of solutions of Dirichlet boundary value problems. Define

$$A_0 := -\frac{\partial}{\partial x^i} \left(a^{ij} \frac{\partial}{\partial x^j} \right)$$

for $a^{ij} \in C^{1,\alpha}$ symmetric, uniformly elliptic and uniformly bounded. Consider the problem

$$\begin{cases} A_0 u + cu = -\frac{\partial}{\partial x^j} f^j + h & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

for $c \in C^\alpha$, $f = (f^1, \dots, f^n) \in C^{1,\alpha}$ and $h \in C^\alpha$. If $c \geq 0$, one can show existence and uniqueness of $C^{2,\alpha}$ solutions. First of all, suppose that solutions of

$$\begin{cases} A_0 u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

do exist. Let us define

$$X := \{u \in C^{2,\alpha} : u|_{\partial\Omega} = 0\} \quad \text{and} \quad Y := C^\alpha.$$

Then X and Y are Banach spaces, since X is a closed subset of a Banach space. Define now $A_1 := A_0 + c$. Then it is easy to show that A_0 and A_1 are continuous. Thus to apply the continuity method, we have to show the existence of a constant $C > 0$, such that for all $t \in [0, 1]$ and $u \in X$

$$\|x\|_{C^{2,\alpha}} \leq \|A_t x\|_{C^\alpha}$$

holds. But this looks like the $C^{2,\alpha}$ -estimate 1.26. Indeed, since $u \in C^{2,\alpha}$ solves $A_t u = A_t u$, we get

$$\|u\|_{C^{2,\alpha}} \leq C(\|u\|_{H^1} + \|A_t u\|_{C^\alpha}).$$

Using ellipticity, integration by parts (justified since any function in X vanishes on the boundary $\partial\Omega$) and $c \geq 0$, we compute

$$\begin{aligned} \lambda \|u\|_{H^1}^2 &= \lambda \int_{\Omega} |\nabla u|^2 \\ &\leq \int_{\Omega} a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \\ &= \int_{\Omega} (A_0 u) u \\ &= \int_{\Omega} (A_0 u) u + ct u^2 - ct u^2 \\ &= \int_{\Omega} (A_t u) u - ct u^2 \end{aligned}$$

$$\begin{aligned} &\leq \int_{\Omega} (A_t u) u \\ &\leq \|A_t u\|_{L^2} \|u\|_{L^2} \\ &\leq C \|A_t u\|_{C^\alpha} \|u\|_{H^1}. \end{aligned}$$

Maximum Principle

Weak Maximum Principle. Let $\Omega \subseteq \mathbb{R}^n$. In what follows, we consider the second order homogenous differential operator in non-divergence form

$$Lu := a^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + b^i \frac{\partial u}{\partial x^i} + cu$$

where $a^{ij}, b^i, c \in C^0(\bar{\Omega})$ and L is uniformly elliptic, i.e. there exists $\lambda > 0$ such that

$$a^{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2$$

holds for all $\xi \in \mathbb{R}^n$ and $x \in \Omega$.

Theorem 1.27 (Weak Maximum Principle). Let $\Omega \subseteq \mathbb{R}^n$ and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that $Lu \geq 0$. Then:

- (a) If $c \leq 0$ in Ω , then $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u_+$.
- (b) If $c = 0$ in Ω , then $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.

Proof. Consider the perturbation $v_\varepsilon := u + \varepsilon e^{\gamma x_1}$ for $\varepsilon, \gamma > 0$ and use the first and second derivative test. \square

Strong Maximum Principle.

Lemma 1.6 (Boundary Point Lemma, E. Hopf). Let $B := B_\rho(y) \subseteq \mathbb{R}^n$ and $u \in C^2(B) \cap C^0(\bar{B})$ such that $Lu \geq 0$ in B with $c \leq 0$. Assume for some $x_0 \in \partial B$ that $u(x_0) \geq 0$ and $u(x) < u(x_0)$ for every $x \in B$. Then

$$\limsup_{h \rightarrow 0} \frac{u(x_0 + h\eta) - u(x_0)}{h} < 0$$

for η the inward pointing unit normal at x_0 .

Proof. Without loss of generality one can assume $\rho = 1$ and $y = 0$. Then define $w : \bar{B} \rightarrow \mathbb{R}$ by

$$w(x) := e^{-\alpha|x|^2} - e^{-\alpha}$$

for some $\alpha > 0$ to be determined. We compute

$$Lw \geq e^{-\alpha|x|^2} (4\mu|x|^2 \alpha^2 - 2\alpha(\text{tr } A + b^i x_i) + c).$$

Thus for some α large enough, we get that $Lw > 0$. Set

$$v := u - u(x_0) + \varepsilon w$$

for some $\varepsilon > 0$ on the annulus $A := \bar{B}_1(0) \setminus B_{1/2}(0)$. For $\varepsilon > 0$ sufficiently small, we get that $v \leq 0$ on ∂A . Since moreover

$$Lv = Lu - cu(x_0) + \varepsilon Lw > 0$$

the weak maximum principle implies $v \leq 0$ on A . Hence $D_\eta^+ v \leq 0$, but

$$D_\eta^+ v = D_\eta^+ u + \varepsilon D_\eta^+ w$$

which yields the statement by observing that $D_\eta^+ w > 0$. □

References

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