LIE ALGEBRA COHOMOLOGY

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Abstract. Aim of this talk is to give a short overview of the *cohomology of Lie algebras with coefficients in modules*. We follow the original construction of Chevalley-Eilenberg via complexes. We then state two results concerning *semisimple* Lie algebras, known as the *first and second Whitehead lemma*, and calculate an example.

Contents

Introduction

The archetypal example of a *cohomology theory* arises in differential topology: The *de Rham cohomology*. Given a smooth manifold M, we define $\Omega^n(M) := \Gamma(\Lambda^n T^\vee M)$ for each $n \in \omega$, the space of *smooth differential n-forms on M*. Moreover, is a sequence of mappings $\left(d^n : \Omega^n(M) \to \Omega^{n+1}(M)\right)_{n\in\omega}$, called *exterior differentiation operators*, which roughly speaking generalize the notion of a differential of a function. They do satisfy the relation $d^n \circ d^{n-1} = 0$ and thus we can define the *n-th de Rham cohomology group* to be the quotient space

$$H^n_{\mathrm{dR}}(M) := \ker d^n / \operatorname{im} d^{n-1}.$$

The Chevalley-Eilenberg Complex

The definition of the n-th de Rham cohomology group $H^n_{\mathrm{dR}}(M)$ can actually be thought of a two-stage process: First we go from Diff to an intermediate category and then we apply a *homology functor*, which is a purely algebraic construct, to go from this intermediate

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category to ${}_{\mathbb{R}}\mathsf{Vect}$. Aim of this section is to give the definition of this intermediate category and then to define such a functor explicitely for the case of Lie algebras.

Definition 1.1 (Chain Complex in ${}_R\mathsf{Mod}$). *Let $R \in \mathsf{Ring}$. A **$\mathbb{Z}$-graded chain complex in ${}_R\mathsf{Mod}$** is defined to be a tuple $(C_\bullet, \partial_\bullet)$, consisting of an infinite sequence*

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots$$

in ${}_R\mathsf{Mod}$ such that $\partial_n \circ \partial_{n+1} = 0$ holds for all $n \in \mathbb{Z}$.
*Dually, a **$\mathbb{Z}$-graded cochain complex in ${}_R\mathsf{Mod}$** is simply a $\mathbb{Z}$-graded chain complex in ${}_R\mathsf{Mod}^{\mathrm{op}}$, i.e. a tuple $(C^\bullet, d^\bullet)$ consisting of an infinite sequence*

$$\dots \longrightarrow C^{n-1} \xrightarrow{d^n} C^n \xrightarrow{d^{n+1}} C^{n+1} \longrightarrow \dots$$

such that $d^{n+1} \circ d^n = 0$ holds for all $n \in \mathbb{Z}$.

In a more general setting, this translates as follows.

Definition 1.2 (Chain Complex). *Let $\mathcal{A}$ be an abelian category. A **$\mathbb{Z}$-graded chain complex in $\mathcal{A}$** is defined to be a tuple $((C_n)_{n\in\mathbb{Z}}, (\partial_n)_{n\in\mathbb{Z}})$, consisting of a sequence $(C_n)_{n\in\mathbb{Z}}$ of objects in $\mathcal{A}$ and a sequence $(\partial_n)_{n\in\mathbb{Z}}$ of morphisms in $\mathcal{A}$, such that*

$$\partial_n \in \mathrm{Hom}_{\mathcal{A}}(C_n, C_{n-1}) \qquad \text{and} \qquad \partial_n \circ \partial_{n+1} = 0$$

for all $n \in \mathbb{Z}$.
*Dually, a **$\mathbb{Z}$-graded cochain complex in $\mathcal{A}$** is a $\mathbb{Z}$-graded chain complex in $\mathcal{A}^{\mathrm{op}}$.*

Remark 1.1. For notational simplicity, we will write $(C_\bullet, \partial_\bullet)$ for a chain complex in $\mathcal{A}$.

Remark 1.2. The notions of chain and cochain complexes are actually equivalent. Indeed, changing from one to the other amounts simply to a reversing of the $\mathbb{Z}$-grading.

Remark 1.3. For each abelian category $\mathcal{A}$, there is an abelian category $\mathrm{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$ (see [Wei94, p. 7]).

Definition 1.3 (Non-Negative Chain Complex). *Let $\mathcal{A}$ be an abelian category. A chain complex $(C_\bullet, \partial_\bullet) \in \mathrm{Ch}(\mathcal{A})$ is said to be **non-negative**, iff $C_n = 0$ for all $n < 0$. We denote by $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ the full subcategory of $\mathrm{Ch}(\mathcal{A})$ of non-negative chain complexes.*

In what follows, suppose that all rings admit an identity element and all modules are unital. Recall, that for $R \in \mathsf{CRing}$ and $\mathfrak{g} \in {}_R\mathsf{LieAlg}$, the ***universal envelopping algebra $U\mathfrak{g}$ of $\mathfrak{g}$*** is defined to be the quotient of the ***tensor algebra $T\mathfrak{g}$***

$$T\mathfrak{g} := \bigoplus_{n\in\omega} \mathfrak{g}^{\otimes n}$$

by the 2-sided ideal generated by the relations

$$\iota[x, y] = \iota(x)\iota(y) - \iota(y)\iota(x),$$

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for all $x, y \in \mathfrak{g}$, where $\iota : \mathfrak{g} \hookrightarrow T\mathfrak{g}$ denotes inclusion. Moreover, U is a functor from ${}_R\mathsf{LieAlg}$ to associative R-algebras. Hence we can define a $U\mathfrak{g}$-action on $U\mathfrak{g} \otimes_R \Lambda^n\mathfrak{g}$ simply by

$$u(v \otimes x_1 \wedge \cdots \wedge x_n) := uv \otimes x_1 \wedge \cdots \wedge x_n,$$

for all $n \in \omega$.

Definition 1.4 (Chevalley-Eilenberg Complex). *Let $R \in \mathsf{CRing}$ and $\mathfrak{g} \in {}_R\mathsf{LieAlg}$ which is free as an R-module. Denote by $U\mathfrak{g}$ the universal enveloping algebra of $\mathfrak{g}$. Define a non-negative chain complex $(C_\bullet, \partial_\bullet) \in \mathsf{Ch}_{\geq 0}({}_{U\mathfrak{g}}\mathsf{Mod})$ by*

$$C_n := U\mathfrak{g} \otimes_R \Lambda^n\mathfrak{g}$$

for all $n \in \omega$ and

$$\partial_n(u \otimes x_1 \wedge \cdots \wedge x_n) := \begin{cases} ux_1 & n = 1, \\ \theta_1 + \theta_2 & n > 1, \end{cases}$$

where

$$\theta_1 := \sum_{i=0}^{n} (-1)^{i+1} ux_i \otimes x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n,$$

and

$$\theta_2 := \sum_{1 \leq i < j \leq n} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n.$$

Remark 1.4. It is by no means obvious, that $\partial_n \circ \partial_{n+1} = 0$ holds for the Chevalley-Eilenberg complex 1.4. However, it is a tedious computation, and we will only demonstrate the case $n = 1$. In this case

$$\begin{aligned}(\partial_1 \circ \partial_2)(u \otimes x \wedge y) &= \partial_1(ux \otimes y - uy \otimes x - u \otimes [x, y]) \\ &= u(xy - yx) - u[x, y] \\ &= 0,\end{aligned}$$

for all $u \in U\mathfrak{g}$ and $x, y \in \mathfrak{g}$.

Remark 1.5. The definition of the boundary map ∂_n in the Chevalley-Eilenberg complex 1.4 is not as arbitrary at it might seem at first sight. Given $\alpha \in \Omega^n(M)$ for a smooth manifold M, then we have that $d\alpha(X_1, \ldots, X_{n+1})$ is of the same form for any $X_1, \ldots, X_{n+1} \in \mathfrak{X}(M)$. Actually, this formula can be used to give an invariant definition of the exterior derivative d in the de Rham theory (see [Lee13, pp. 370–372]).

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Left $\mathfrak{g}$-Modules and the Cohomology of Lie Algebras

Definition 1.5 (Category of Left $\mathfrak{g}$-Modules). *Let $R \in \mathsf{CRing}$ and $\mathfrak{g} \in {}_R\mathsf{LieAlg}$. The **category of left $\mathfrak{g}$-modules**, written ${}_{\mathfrak{g}}\mathsf{Mod}$, is defined to be the category with objects **left $\mathfrak{g}$-modules**, i.e. modules $M \in {}_R\mathsf{Mod}$ equipped with an R-bilinear product $\mathfrak{g} \times M \to M$, $(x, m) \mapsto xm$, such that*

$$[x, y]m = x(ym) - y(xm)$$

*holds for all $x, y \in \mathfrak{g}$ and $m \in M$, and **left $\mathfrak{g}$-module homomorphisms** as morphisms, i.e. morphisms $f \in \mathrm{Hom}_R(M, N)$ such that*

$$f(xm) = xf(m)$$

holds for all $x \in \mathfrak{g}$ and $m \in M$.

Examples 1.1.

(a) A ***trivial $\mathfrak{g}$-module*** is an R-module M where $xm := 0$ for all $x \in \mathfrak{g}$ and $m \in M$.

(b) The Lie bracket makes $\mathfrak{g}$ itself into a left $\mathfrak{g}$-module by the Jacobi identity. This module is usually called the ***adjoint representation of $\mathfrak{g}$***.

Proposition 1.1. *Let $R \in \mathsf{CRing}$ and $\mathfrak{g} \in {}_R\mathsf{LieAlg}$. Then ${}_{\mathfrak{g}}\mathsf{Mod}$ is an abelian category.*

Proof. See [Wei94, p. 220]. □

We follow [KS06, p. 178].

Proposition 1.2. *Let $\mathcal{A}$ be an abelian category and $(C_\bullet, \partial_\bullet) \in \mathrm{Ch}(\mathcal{A})$. Then for every $n \in \mathbb{Z}$, there exists a unique monic*

$$\mathrm{im}\,\partial_{n+1} \to \ker \partial_n,$$

where $\mathrm{im}\,\partial_{n+1} := \ker(\mathrm{coker}\,\partial_{n+1})$.

Exercise 1.1. Prove proposition 1.2. *Hint:* Use that $\mathrm{im}\,\partial_{n+1} \to C_n$ is monic by [Lan78, p. 199].

Definition 1.6 (Homology of a Chain Complex). *Let $\mathcal{A}$ be an abelian category and $(C_\bullet, \partial_\bullet) \in \mathrm{Ch}(\mathcal{A})$. Moreover, let $n \in \mathbb{Z}$ and $\mathrm{im}\,\partial_{n+1} \to \ker \partial_n$ be the unique morphism assured by lemma 1.2. Then we define the **n-th homology object**, written $H_n(C_\bullet, \partial_\bullet)$, by*

$$H_n(C_\bullet, \partial_\bullet) := \mathrm{coker}(\mathrm{im}\,\partial_{n+1} \to \ker \partial_n) \in \mathrm{ob}(\mathcal{A}).$$

Explicitely, in our setting this translates as follows.

Definition 1.7 (Homology of a Chain Complex in ${}_R\mathsf{Mod}$). *Let $R \in \mathsf{Ring}$ and $n \in \mathbb{Z}$. The **n-th homology module** of a chain complex $(C_\bullet, \partial_\bullet)$, written $H_n(C_\bullet, \partial_\bullet)$, is defined to be*

$$H_n(C_\bullet, \partial_\bullet) := \ker \partial_n / \mathrm{im}\,\partial_{n+1}.$$

*Dually, the **n-th cohomology module** of a cochain complex $(C^\bullet, d^\bullet)$ in ${}_R\mathsf{Mod}$, written $H^n(C^\bullet, d^\bullet)$, is defined to be*

$$H^n(C^\bullet, d^\bullet) := \ker d^{n+1} / \mathrm{im}\, d^n.$$

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Observe, that for each $n \in \omega$, we have that $C_n \in {}_{\mathfrak{g}}\mathsf{Mod}$ via $\iota : \mathfrak{g} \hookrightarrow U\mathfrak{g}$.

Definition 1.8 (Cohomology of Lie Algebras). *Let $R \in \mathsf{CRing}$ and $\mathfrak{g} \in {}_R\mathsf{LieAlg}$ which is free as an R-module. Moreover, let $M \in {}_{\mathfrak{g}}\mathsf{Mod}$ and $(C_\bullet, \partial_\bullet)$ denote the Chevalley-Eilenberg complex 1.4. For $n \in \omega$, define the **n-th cohomology group of $\mathfrak{g}$ with coefficients in M**, written $H^n(\mathfrak{g}, M)$, to be the n-th cohomology module of the cochain complex* $\operatorname{Hom}_{\mathfrak{g}}((C_\bullet, \partial_\bullet), M)$.

Remark 1.6. Actually, for $n \in \omega$ we have that

$$\operatorname{Hom}_{\mathfrak{g}}(C_n, M) \cong \operatorname{Hom}_R(\Lambda^n \mathfrak{g}, M),$$

in ${}_R\mathsf{Mod}$. Indeed, if $n \geq 1$ and $\varphi \in \operatorname{Hom}_{\mathfrak{g}}(C_n, M)$, define $\overline{\varphi} \in \operatorname{Hom}_R(\Lambda^n \mathfrak{g}, M)$ by

$$\overline{\varphi}(x_1 \wedge \cdots \wedge x_n) := \varphi(1 \otimes x_1 \wedge \cdots \wedge x_n),$$

and conversely, if $\varphi \in \operatorname{Hom}_R(\Lambda^n \mathfrak{g}, M)$, define $\overline{\varphi} \in \operatorname{Hom}_{\mathfrak{g}}(C_n, M)$ by

$$\overline{\varphi}(u \otimes x_1 \wedge \cdots \wedge x_n) := u\varphi(x_1 \wedge \cdots \wedge x_n).$$

This is possible, since every left $\mathfrak{g}$-module is naturally a left $U\mathfrak{g}$-module (see [Wei94, pp. 224–225]). If $n = 0$, we define $\overline{\varphi} \in \operatorname{Hom}_R(R, M)$ by

$$\overline{\varphi}(r) := r\varphi(1),$$

for $\varphi \in \operatorname{Hom}_{\mathfrak{g}}(U\mathfrak{g} \otimes_R R, M) \cong \operatorname{Hom}_{\mathfrak{g}}(U\mathfrak{g}, M)$ and for $\varphi \in \operatorname{Hom}_R(R, M)$ define similarly $\overline{\varphi} \in \operatorname{Hom}_{\mathfrak{g}}(U\mathfrak{g}, M)$ by

$$\overline{\varphi}(u) := u\varphi(1).$$

Hence we get an induced morphism

$$\begin{tikzcd}
\operatorname{Hom}_{\mathfrak{g}}(C_{n-1}, M) \arrow[r, "d^n"] \arrow[d, "\overline{\cdot}"'] & \operatorname{Hom}_{\mathfrak{g}}(C_n, M) \arrow[d, "\overline{\cdot}"] \\
\operatorname{Hom}_R(\Lambda^{n-1}\mathfrak{g}, M) \arrow[r, dashed] & \operatorname{Hom}_R(\Lambda^n \mathfrak{g}, M).
\end{tikzcd}$$

Explicitely

$$\begin{aligned}
d^n f(x_1, \ldots, x_n) = &\sum_{i=1}^n (-1)^{i+1} x_i f(x_1, \ldots, \hat{x}_i, \ldots, x_n) \\
&+ \sum_{1 \leq i < j \leq n} (-1)^{i+j} f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_n),
\end{aligned}$$

for $n > 1$ and

$$d^1 f(x_1) = x_1 f(1).$$

Remark 1.7. There is a more general approach to the definition of the cohomology of Lie algebras via the notion of *right derived functors* which does not use the intermediate step of the Chevalley-Eilenberg complex.

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Example 1.1 ($H^3(\mathfrak{sl}_2, k)$). Let k be a field with characteristic not equal to two and consider the *special linear Lie algebra over* k, i.e. $A \in M_2(k)$ such that $\operatorname{tr} A = 0$. This is a three dimensional Lie algebra with ordered basis

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad e_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Hence the crucial portion of the Eilenberg-Chevalley cochain complex is given by

$$\ldots \longrightarrow \operatorname{Hom}_k(\Lambda^2\mathfrak{sl}_2, k) \xrightarrow{d} \operatorname{Hom}_k(\Lambda^3\mathfrak{sl}_2, k) \longrightarrow 0.$$

We compute

$$\begin{aligned} df(e_1, e_2, e_3) =& e_1 f(e_2, e_3) - e_2 f(e_1, e_3) + e_3 f(e_1, e_2) \\ & - f([e_1, e_2], e_3) + f([e_1, e_3], e_2) - f([e_2, e_3], e_1) \\ =& -2f(e_2, e_3) - 2f(e_3, e_2) - f(e_1, e_1) \\ =& -2f(e_2, e_3) + 2f(e_2, e_3) \\ =& 0, \end{aligned}$$

since k is interpreted as a trivial $\mathfrak{sl}_2$-module and by the alternating k-multilinear properties of f. Hence

$$H^3(\mathfrak{sl}_2, k) \cong \operatorname{Hom}_k(\Lambda^3\mathfrak{sl}_2, k) \cong k,$$

since $\dim \operatorname{Hom}_k(\Lambda^3\mathfrak{sl}_2, k) = 1$.

The Whitehead Lemmas

Let $M \in {}_{\mathfrak{g}}\mathsf{Mod}$ and denote by $\operatorname{Der}(\mathfrak{g}, M)$ the *derivations from* $\mathfrak{g}$ *to* M, i.e. mappings $D \in {}_R\mathsf{Mod}$, such that $D[x, y] = x(Dy) - y(Dx)$ holds for all $x, y \in \mathfrak{g}$. If $m \in M$, define $D_m(x) := xm$ for $x \in \mathfrak{g}$. This is called an *inner derivation*. The R-module of all inner derivations from $\mathfrak{g}$ to M is denoted by $\operatorname{Der}_{\mathrm{Inn}}(\mathfrak{g}, M)$. Clearly, $\operatorname{Der}_{\mathrm{Inn}}(\mathfrak{g}, M)$ is an R-submodule of $\operatorname{Der}(\mathfrak{g}, M)$. By [Wei94, p. 230] we have that

$$H^1(\mathfrak{g}, M) \cong \operatorname{Der}(\mathfrak{g}, M)/\operatorname{Der}_{\mathrm{Inn}}(\mathfrak{g}, M).$$

Let $M \in {}_R\mathsf{LieAlg}$ be abelian. An *extension of* $\mathfrak{g}$ *by* M is a short exact sequence

$$0 \longrightarrow M \longrightarrow \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

in ${}_R\mathsf{LieAlg}$. There is a notion of equivalence classes of extensions of $\mathfrak{g}$ by M. Denote by $\operatorname{Ext}(\mathfrak{g}, M)$ the set of all such equivalence classes. Then there is a one-to-one correspondence

$$\operatorname{Ext}(\mathfrak{g}, M) \xleftrightarrow{1:1} H^2(\mathfrak{g}, M)$$

if $M \in {}_{\mathfrak{g}}\mathsf{Mod}$ (see [Wei94, p. 235]).
However, the situation gets much more easier if we make some assumptions.

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Theorem 1.2 (Whitehead's First Lemma). *Let k be a field of characteristic zero and $\mathfrak{g} \in {}_k\mathsf{LieAlg}$ semisimple. Then for any finite-dimensional $M \in {}_{\mathfrak{g}}\mathsf{Mod}$ we have that*

$$H^1(\mathfrak{g}, M) = 0.$$

That is, every derivation from $\mathfrak{g}$ into M is an inner derivation.

Theorem 1.3 (Whitehead's Second Lemma). *Let k be a field of characteristic zero and $\mathfrak{g} \in {}_k\mathsf{LieAlg}$ semisimple. Then for any finite-dimensional $M \in {}_{\mathfrak{g}}\mathsf{Mod}$ we have that*

$$H^2(\mathfrak{g}, M) = 0.$$

Remark 1.8. There cannot be a third Whitehead lemma, since

$$H^3(\mathfrak{sl}_2, k) \cong k,$$

by exercise 1.1.

There is a quite strong result obtained via the second Whitehead lemma, which gives rise to the *structure theory of Lie algebras*.

Theorem 1.4 (Levi's Theorem). *Let k be a field of characteristic zero and $\mathfrak{g} \in {}_k\mathsf{LieAlg}$ finite dimensional. Then there exists a semisimple Lie subalgebra $\mathscr{L} \subseteq \mathfrak{g}$ (called a **Levi factor**), such that*

$$\mathfrak{g} \cong \mathscr{L} \ltimes \mathrm{rad}(\mathfrak{g}).$$

Proof. We only provide a sketch. For full details see [Wei94, p. 247]. We know that $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple, so it suffices to show that the Lie algebra extension

$$0 \longrightarrow \mathrm{rad}(\mathfrak{g}) \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathrm{rad}(\mathfrak{g}) \longrightarrow 0$$

splits. If $\mathrm{rad}(\mathfrak{g})$ is abelian, we are done by Whitehead's second lemma 1.3. If $\mathrm{rad}(\mathfrak{g})$ is not abelian, proceed by induction on the derived length of $\mathrm{rad}(\mathfrak{g})$. □

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