FUNCTIONAL ANALYSIS II SUMMARY

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Abstract. This is a rough summary of the course *Functional Analysis II* held at *ETH Zurich* by *Prof. Dr. Alessandro Carlotto* in spring 2018. The main focus of this summary is to give a neat preparation for the oral exam.

Contents

Sobolev Space Theory

The Spaces $W^{k,p}(\Omega)$. In what follows, let $n \in \omega$, $n \geq 1$, and $1 \leq p \leq \infty$.

Definition 1.1 (Distributional and Weak Derivative). *Let $\Omega \subseteq \mathbb{R}^n$ open and $u \in L^1_{\text{loc}}(\Omega)$. For any multiindex α, the* ***distributional derivative of order α of u****, written $D^\alpha u$, is defined to be the mapping $D^\alpha u : C_c^\infty(\Omega) \to \mathbb{R}$ defined by*

$$\varphi \mapsto (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi dx.$$

Moreover, a function $D^\alpha u \in L^p(\Omega)$ is called ***weak derivative of order α of u with exponent p****, iff*

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega D^\alpha u \varphi dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi dx.$$

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Theorem 1.1 (Fundamental Lemma of Variational Calculus). *Let $\Omega \subseteq \mathbb{R}^n$ open and $f \in L^1_{\mathrm{loc}}(\Omega)$. If*

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega f \varphi dx = 0,$$

then $f = 0$ a.e.

Remark 1.1. Let $\Omega \subseteq \mathbb{R}^n$ open. Then $L^p(\Omega) \subseteq L^1_{\mathrm{loc}}(\Omega)$.

Remark 1.2. From the fundamental lemma of variational calculus 1.1 it follows that *weak derivatives, if they exist, are unique.*

Examples 1.1.

(a) Consider $\Omega := (-1, 1)$ and $u := |x|$. Then $u' = \chi_{[0,1)} - \chi_{(-1,0)}$.

(b) Consider $\Omega := \mathbb{R}$ and $u := \chi_{(0,\infty)}$. Then the weak derivative u' does not exist. Indeed, the *Dirac distribution* is not representable as one may see by considering the smooth family $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ for $\varepsilon > 0$ defined by

$$\varphi_\varepsilon(x) := \begin{cases} e^{\varepsilon^2/(x^2-\varepsilon^2)} & |x| < \varepsilon, \\ 0 & |x| \geq \varepsilon. \end{cases}$$

Definition 1.2 (Sobolev Space). *Let $\Omega \subseteq \mathbb{R}^n$ open. For any $k \in \omega$, the* ***Sobolev space of index (k, p)****, written $W^{k,p}(\Omega)$, is defined to be the space*

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ exists for all } |\alpha| \leq k\},$$

with norm

$$\|-\|_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha -\|_{L^p(\Omega)}.$$

Moreover, define

$$W_0^{k,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|-\|_{W^{k,p}(\Omega)}},$$

and $H^k(\Omega) := W^{k,2}(\Omega)$ as well as $H_0^k(\Omega) := W_0^{k,2}(\Omega)$.

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^n$ open. Then $W^{k,p}(\Omega)$ is*

(a) a Banach space for all $1 \leq p \leq \infty$.

(b) separable for all $1 \leq p < \infty$.

(c) reflexive for all $1 < p < \infty$.

Proof.

(a) This follows from the fact that $L^p(\Omega)$ is a Banach space for all $1 \leq p \leq \infty$. Let $(f_i)_{i\in\omega}$ be a Cauchy sequence in $W^{k,p}$. By definition of the $W^{k,p}$-norm, $(D^\alpha f_i)_{i\in\omega}$ is a Cauchy sequence in L^p. Thus we get $D^\alpha f_i \to f_\alpha$ in L^p, in particular, $f_i \to f$ in L^p. Using Hölder's inequality we compute

$$\int_\Omega f_\alpha \varphi dx = \lim_{i\to\infty} \int_\Omega D^\alpha f_i \varphi dx = (-1)^{|\alpha|} \lim_{i\to\infty} \int_\Omega f_i D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi dx$$

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for $\varphi \in C_c^\infty(\Omega)$.
(b) For simplicity, we consider $k = 1$ only. Consider $\iota : W^{1,p} \hookrightarrow (L^p)^{n+1}$ defined in the obvious way. Then ι is an isometry and the statement follows.
(c) Same argument as in part (b).

$\square$

Elliptic Operators.

Lemma 1.1 (Poincaré Inequality). *Let $\Omega \subseteq \mathbb{R}^n$ open and bounded. Then for any $u \in C_c^\infty(\Omega)$ we have that*

$$\|u\|_{L^2} \leq C \|\nabla u\|_{L^2}.$$

Proof. Let $n = 1$. Since Ω is bounded, we get that $\Omega \subseteq [a,b]$ and we may extend u on $[a,b] =: I$ to be zero. Hence an application of Jensen's inequality (or Cauchy-Schwarz) yields

$$|u(x)|^2 = |u(x) - u(a)|^2 = \left| \int_a^x u'(t)dt \right|^2 \leq (x-a) \int_a^x |u'(t)|^2 \, dt \leq (b-a) \|u'\|_{L^2(I)}^2.$$

Thus

$$\|u\|_{L^2(\Omega)}^2 \leq \|u\|_{L^2(I)}^2 \leq (b-a)^2 \|u'\|_{L^2(I)}^2 = (b-a)^2 \|u'\|_{L^2(\Omega)}^2$$

where the last equality follows due to the fact that u and thus u' is compactly supported in Ω. If $n > 1$, we have $\Omega \subseteq [a,b] \times \mathbb{R}^{n-1}$ and thus the claim follows by reduction to the previous case. $\square$

Theorem 1.3 (Riesz Representation Theorem). *Let H be a real Hilbert space. Then the mapping $J : H \to H^*$ defined by $J(x) := \langle x, - \rangle$ is an isometric isomorphism.*

Theorem 1.4. *Let $\Omega \subseteq\subseteq \mathbb{R}^n$ and consider the elliptic operator*

$$L := \frac{\partial}{\partial x^i}\left(a^{ij}\frac{\partial}{\partial x^j}\right),$$

for $a^{ij} \in L^\infty(\Omega)$ symmetric. Then: Given $f \in L^2(\Omega)$, the homogenous Dirichlet problem

$$\begin{cases} -Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

admits a unique weak solution $u \in H_0^1(\Omega)$.

Proof.
Step 1: Derivation of Weak Formulation. Suppose $u \in C^2(\overline{\Omega})$ is a solution of (1). Let $\varphi \in C_c^\infty(\Omega)$. Then integration by parts (see [Lee13, p. 436]) yields

$$-\int_\Omega L(u)\varphi = -\sum_{j=1}^n \int_\Omega \operatorname{div}(X_j)\varphi = \int_\Omega a^{ij}\frac{\partial u}{\partial x^j}\frac{\partial \varphi}{\partial x^i} = \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial \varphi}{\partial x^j},$$

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where $X_j := \left(a^{ij}\frac{\partial}{\partial x^j}\right)_i$. Thus we get the weak formulation:

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial \varphi}{\partial x^j} = \int_\Omega f\varphi. \tag{2}$$

Step 2: Existence and Uniqueness of Weak Solutions. Since L is uniformly elliptic, there exists $\lambda > 0$ such that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

holds for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Moreover, since $a^{ij} \in L^\infty(\Omega)$, we get that L is uniformly bounded, i.e. there exists $\Lambda > 0$ such that

$$a^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

holds for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Now define a bilinear form $\langle\cdot,\cdot\rangle_a : H_0^1(\Omega)\times H_0^1(\Omega) \to \mathbb{R}$ by

$$\langle u, v\rangle_a := \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^j} \tag{3}$$

Then it is easy to see, that $\langle\cdot,\cdot\rangle_a$ is symmetric. Also, $\langle\cdot,\cdot\rangle_a$ is positive definite since

$$\langle u, u\rangle_a = \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j} \geq \lambda\int_\Omega |\nabla u|^2 \geq C\lambda\int_\Omega |u|^2$$

using ellipticity and Poincaré's inequality. Moreover by Poincaré's inequality we have that

$$\lambda\|u\|^2_{H_0^1(\Omega)} \leq \|u\|_a^2 \leq \Lambda\|u\|^2_{H_0^1(\Omega)}$$

for the induced norm $\|\cdot\|_a$. Hence the induced norm is equivalent to the standard norm on $H_0^1(\Omega)$ and thus $(H_0^1(\Omega), \|\cdot\|_a)$ is a Hilbert space. Thus an application of Riesz representation theorem 1.3 yields the existence of a unique $u \in H_0^1(\Omega)$, such that

$$\langle u, \varphi\rangle_a = l(\varphi) := \int_\Omega f\varphi$$

holds for all $\varphi \in H_0^1(\Omega)$, since $l \in (H_0^1(\Omega))^*$.

□

Sobolev Spaces on an Interval. In what follows, let $-\infty \leq a < b \leq \infty$ and $I := (a, b)$.

Lemma 1.2 (Du Bois-Reymond). *Let $f \in L^1_{\mathrm{loc}}(I)$ such that*

$$\forall \varphi \in C_c^\infty(I) : \int_I f\varphi' dx = 0.$$

Then f is almost everywehere constant.

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Proof. Let $v := w - c_0\psi$ for $w, \psi \in C_c^\infty(I)$ such that $\int_I \psi = 1$ and $\int_I v = 0$. This implies $c_0 = \int_I w$. By the fundamental theorem of calculus, the function $\varphi : I \to \mathbb{R}$ defined by

$$\varphi(x) := \int_I v(t)dt$$

belongs to $C_c^\infty(I)$ with $\varphi' = v$. Thus we compute

$$0 = \int_I f\varphi' = \int_I fv = \int_I fw - c_0 \int_I f\psi = \int_I fw - \int_I w \int_I f\psi = \int_I (f - c)w,$$

where $c := \int_I f\psi$. Since w was arbitrary, we conclude by the fundamental lemma of variational calculus 1.1. □

Lemma 1.3. *Let $f \in L^1_{\text{loc}}(I)$ and $x_0 \in I$. Then $u : I \to \mathbb{R}$ defined by*

$$u(x) := \int_{x_0}^x f(t)dt$$

is absolutely continuous and belongs to $W^{1,1}_{\text{loc}}(I)$ with $u' = f$ a.e.

Proof. Absolute continuity follows from real analysis. Let $\varphi \in C_c^\infty(I)$. Then Fubini yields

$$\begin{aligned}
\int_I u\varphi' &= \int_a^{x_0} \int_{x_0}^x f(t)\varphi'(x)dtdx + \int_{x_0}^b \int_{x_0}^x f(t)\varphi'(x)dtdx \\
&= -\int_a^{x_0} \int_x^{x_0} f(t)\varphi'(x)dtdx + \int_{x_0}^b \int_{x_0}^x f(t)\varphi'(x)dtdx \\
&= -\int_a^{x_0} \int_a^t f(t)\varphi'(x)dxdt + \int_{x_0}^b \int_t^b f(t)\varphi'(x)dxdt \\
&= -\int_a^{x_0} f(t)\varphi(t)dt - \int_{x_0}^b f(t)\varphi(t)dt \\
&= -\int_I f\varphi.
\end{aligned}$$

□

Theorem 1.5. *Let $u \in W^{1,p}(I)$. Then there exists an absolutely continuous representant $\widetilde{u}$ of u on $\overline{I}$, such that*

$$\widetilde{u}(x) = \widetilde{u}(x_0) + \int_{x_0}^x u'(t)dt$$

holds for all $x, x_0 \in I$.

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Proof. By lemma 1.3, the function $v(x) := \int_{x_0}^{x} u'(t)dt$ is in $W^{1,1}_{\text{loc}}(I)$ with weak derivative u'. Moreover, for any $\varphi \in C_c^\infty(I)$ we compute

$$\int_I (u - v)\varphi' = \int_I u\varphi' - \int_I v\varphi' = -\int_I u'\varphi + \int_I u'\varphi = 0.$$

Thus lemma 1.2 yields $u = c + v$, for some $c \in \mathbb{R}$. But $c = u(x_0)$ and we conclude by setting

$$\widetilde{u}(x) := u(x_0) + \int_{x_0}^{x} u'(t)dt.$$

□

Theorem 1.6 (Characterization of $W^{1,p}(I)$). *Let $1 < p \leq \infty$ and $u \in L^p(I)$. Then the following statements are equivalent:*

(*a*) $u \in W^{1,p}(I)$.

(*b*) *There exists $C \geq 0$ such that*

$$\forall \varphi \in C_c^\infty(I) : \left| \int_I u\varphi' \right| \leq C \|\varphi\|_{L^q}.$$

(*c*) *There exists $C \geq 0$ such that for all $I' \subseteq\subseteq I$ and $|h| < \operatorname{dist}(I', \partial I)$ holds*

$$\|\tau_h u - u\|_{L^p(I')} \leq C|h|,$$

where $\tau_h u(x) := u(x + h)$.

Proof. The implication $(a) \Rightarrow (b)$ follows immediately from Hölder's inequality. To prove $(b) \Rightarrow (a)$, we observe that $l : C_c^\infty(I) \to \mathbb{R}$ defined by

$$L(\varphi) := \int_I u\varphi'$$

is continuous. Since $C_c^\infty(I)$ is dense in $L^q(I)$, we get that $l \in (L^q(I))^*$. Hence we find $g \in L^p$, such that $\int_I g\varphi = l(\varphi)$ and so $u' = -g$.

Next we show $(a) \Rightarrow (c)$. By theorem 1.5, we find an absolutely continuous representant $\widetilde{u}$ of u. Thus

$$\widetilde{u}(x + h) - \widetilde{u}(x) = h \int_0^1 u'(x + th)dt$$

Hence Jensen's inequality yields

$$\|\tau_h u - u\|_{L^p(I')} \leq |h| \int_0^1 \|u'(\cdot + th)\|_{L^p(I')} dt \leq |h| \|u'\|_{L^p(I)}.$$

Lastly, we prove $(c) \Rightarrow (b)$. Let $\varphi \in C_c^\infty(I)$. Then we may find $I' \subseteq\subseteq I$ such that $\operatorname{supp}\varphi \subseteq I'$. Hence we compute

$$\left| \int_I u\varphi' \right| = \lim_{h \to 0} \frac{1}{|h|} \left| \int_I u(x) \big(\varphi(x + h) - \varphi(x)\big) dx \right|$$

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$$
\begin{aligned}
&= \lim_{h \to 0} \frac{1}{|h|} \left| \int_I \big(u(x-h) - u(x)\big)\, \varphi(x) dx \right| \\
&= \lim_{h \to 0} \frac{1}{|h|} \left| \int_I (\tau_{-h} u - u)\, \varphi \right| \\
&\leq \lim_{h \to 0} \frac{1}{|h|} \|\tau_{-h} u - u\|_{L^p(I')} \|\varphi\|_{L^q(I)} \\
&\leq C \|\varphi\|_{L^q(I)}.
\end{aligned}
$$

□

Theorem 1.7 (Extension Theorem). *There exists a continuous linear operator*

$$E : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$$

such that:

(i) $Eu|_I = u$.
(ii) $\|Eu\|_{L^p(\mathbb{R})} \leq C \|u\|_{L^p(I)}$.
(iii) $\|(Eu)'\|_{L^p(\mathbb{R})} \leq C \|u\|_{W^{1,p}(I)}$.

Proof. First we consider the case $I = (0, \infty)$. We extend u by continuity to 0 and then we extend u by means of even symmetry. If I is bounded we can without loss of generality assume that $I = (0, 1)$. Now use a cut-off function. □

Theorem 1.8 (Approximation Theorem). *Let* $1 \leq p < \infty$ *and* $u \in W^{1,p}(I)$. *Then there exists a sequence* $(u_i)_{i \in \omega}$ *in* $C_c^\infty(\mathbb{R})$ *such that*

$$\|u_i|_I - u\|_{W^{1,p}(I)} \to 0.$$

Proof. The main idea of the proof is to use convolutions. Moreover, it is enough to consider the case $I = \mathbb{R}$ only, due to the extension theorem 1.7. □

Theorem 1.9 (Sobolev Embedding). *There is a continuous embedding*

$$W^{1,p}(I) \hookrightarrow L^\infty(I).$$

Proof. Without loss of generality, consider $|I| \leq 1$. By theorem 1.5 we get that

$$\|u\|_{L^\infty} = \sup_{x \in I} |u(x)| \leq |u(y)| + \sup_{x \in I} \left| \int_y^x u'(t) dt \right| \leq |u(y)| + \|u'\|_{L^1},$$

for any $y \in I$. Hence

$$\|u\|_{L^\infty} \leq \inf_{y \in I} |u(y)| + \|u'\|_{L^1} \leq \frac{1}{|I|} \int_I |u(y)| + \|u'\|_{L^1} \leq C \|u\|_{W^{1,1}} \leq C \|u\|_{W^{1,p}}.$$

□

Corollary 1.1. *Let* I *be unbounded and* $u \in W^{1,p}(I)$ *for* $1 \leq p < \infty$. *Then* $u \to 0$ *as* $|x| \to \infty$.

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Dirichlet and Neumann Boundary Problems on I. In what follows, let us consider $-\infty < a < b < \infty$ and $I := (a,b)$.

Proposition 1.1. *Let $f \in C^0(\overline{I})$. Then the weak solution u of the homogenous Dirichlet problem*

$$\begin{cases} -u'' = f & \text{in } I, \\ u(a) = 0 = u(b). \end{cases}$$

is a classical solution, i.e. $u \in C^2(\overline{I})$.

Proof. □

Proposition 1.2. *Let $f \in C^0(\overline{I})$. Then the weak solution u of the homogenous Neumann problem*

$$\begin{cases} -u'' + u = f & \text{in } I, \\ u'(a) = 0 = u'(b). \end{cases}$$

is a classical solution, i.e. $u \in C^2(\overline{I})$.

Proof. □

References

[Lee13] John M. Lee. *Introduction to Smooth Manifolds*. Second Edition. Graduate Texts in Mathematics. Springer, 2013.