LIE ALGEBRA COHOMOLOGY

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Abstract.

Contents

Left $\mathfrak{g}$-Modules

Definition 1.1 (Category of Left $\mathfrak{g}$-Modules). *Let $R \in \mathsf{CRing}$ and $\mathfrak{g} \in {}_R\mathsf{LieAlg}$. The **category of left $\mathfrak{g}$-modules**, written ${}_{\mathfrak{g}}\mathsf{Mod}$, is defined to be the category with objects **left $\mathfrak{g}$-modules**, i.e. modules $M \in {}_R\mathsf{Mod}$ equipped with a R-bilinear product $\mathfrak{g} \times M \to M$, $(x, m) \mapsto xm$, such that*

$$[x, y]m = x(ym) - y(xm)$$

*holds for all $x, y \in \mathfrak{g}$ and $m \in M$, and **left $\mathfrak{g}$-module homomorphisms** as morphisms, i.e. morphisms $f \in {}_R\mathsf{Mod}(M, N)$ such that*

$$f(xm) = xf(m)$$

holds for all $x \in \mathfrak{g}$ and $m \in M$.

Proposition 1.1. *Let $R \in \mathsf{CRing}$ and $\mathfrak{g} \in {}_R\mathsf{LieAlg}$. Then ${}_{\mathfrak{g}}\mathsf{Mod}$ is an abelian category.*

Proof. See [Wei94, p. 220]. □

Proposition 1.2 (Invariant Submodule Functor). *Let $R \in \mathsf{CRing}$ and $\mathfrak{g} \in {}_R\mathsf{LieAlg}$. Then there exists a left exact functor*

$$(-)^{\mathfrak{g}} : {}_{\mathfrak{g}}\mathsf{Mod} \to {}_R\mathsf{Mod}.$$

Proof. The proof is divided into three steps.

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Step 1: *Definition on objects.* Let $M \in {}_{\mathfrak{g}}\mathsf{Mod}$. Define

$$M^{\mathfrak{g}} := \{m \in M : \forall x \in \mathfrak{g}(xm = 0)\}.$$

Then $M^{\mathfrak{g}} \leq M$ in ${}_R\mathsf{Mod}$. Indeed, $M^{\mathfrak{g}} \neq \varnothing$ since $0 \in M^{\mathfrak{g}}$ and if $m, n \in M^{\mathfrak{g}}$ and $r \in R$, we have that

$$x(m-n) = xm - xn = 0 \qquad \text{and} \qquad x(rm) = r(xm) = 0,$$

for all $x \in \mathfrak{g}$ by bilinearity of the product. The R-submodule $M^{\mathfrak{g}}$ is called the ***invariant submodule of*** $\boldsymbol{M}$.
Step 2: *Definition on morphisms.* Let $f \in {}_{\mathfrak{g}}\mathsf{Mod}(M, N)$. Then simply let $f^{\mathfrak{g}} := f|_{M^{\mathfrak{g}}}$. This is well defined. Indeed, if $m \in M^{\mathfrak{g}}$, then for any $x \in \mathfrak{g}$ we have that

$$xf|_{M^{\mathfrak{g}}}(m) = xf(m) = f(xm) = f(0) = 0.$$

Step 3: *Left exactness.* Consider the functor $\varepsilon : {}_R\mathsf{Mod} \to {}_{\mathfrak{g}}\mathsf{Mod}$ defined by sending M to the ***trivial*** $\boldsymbol{\mathfrak{g}}$***-module***, i.e. $xm := 0$ for all $x \in \mathfrak{g}$ and $m \in M$. Then $\varepsilon \dashv (-)^{\mathfrak{g}}$. Thus $(-)^{\mathfrak{g}}$ preserves limits by [Lei16, p. 159] and hence since a left exact functor equivalently preserves kernels (see [Fre64, p. 65]), we have that $(-)^{\mathfrak{g}}$ is left exact.

$\square$

Universal Envelopping Algebras and Injectives in $\mathfrak{g}$-Mod

There is an intrinsic connection between $\mathfrak{g}$-modules and the notion of universal enveloping algebras for Lie algebras. Recall, that for $R \in \mathsf{CRing}$ and $\mathfrak{g} \in {}_R\mathsf{LieAlg}$, the ***universal envelopping algebra*** $\boldsymbol{U\mathfrak{g}}$ ***of*** $\boldsymbol{\mathfrak{g}}$ is defined to be the quotient of the ***tensor algebra*** $\boldsymbol{T\mathfrak{g}}$

$$T\mathfrak{g} := \bigoplus_{n \in \omega} \mathfrak{g}^{\otimes n}$$

by the 2-sided ideal generated by the relations

$$\iota[x, y] = \iota(x)\iota(y) - \iota(y)\iota(x),$$

for all $x, y \in \mathfrak{g}$, where $\iota : \mathfrak{g} \hookrightarrow T\mathfrak{g}$ denotes inclusion.

Theorem 1.1. *Let* $R \in \mathsf{CRing}$ *and* $\mathfrak{g} \in {}_R\mathsf{LieAlg}$. *Then*

$${}_{\mathfrak{g}}\mathsf{Mod} \cong {}_{U\mathfrak{g}}\mathsf{Mod}$$

naturally.

Definition 1.2 (Chain Complex). *Let* $\mathcal{A}$ *be an abelian category. A* ***$\mathbb{Z}$-graded chain complex in*** $\boldsymbol{\mathcal{A}}$ *is a tuple* $((C_n)_{n\in\mathbb{Z}}, (\partial_n)_{n\in\mathbb{Z}})$*, consisting of a sequence* $(C_n)_{n\in\mathbb{Z}}$ *in* $\mathcal{A}$ *and a sequence* $(\partial_n)_{n\in\mathbb{Z}}$ *in* $\mathcal{A}$*, such that*

$$\partial_n \in \mathcal{A}(C_n, C_{n-1}) \qquad \text{and} \qquad \partial_n \circ \partial_{n+1} = 0$$

for all $n \in \mathbb{Z}$.
Dually, a ***$\mathbb{Z}$-graded cochain complex in*** $\boldsymbol{\mathcal{A}}$ *is a* $\mathbb{Z}$*-graded chain complex in* $\mathcal{A}^{\mathrm{op}}$.

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We follow [KS06, p. 178].

Proposition 1.3. *Let $\mathcal{A}$ be an abelian category and $(C_\bullet, \partial_\bullet) \in \mathrm{Ch}(\mathcal{A})$. Then for every $n \in \mathbb{Z}$, there exists a unique monic*

$$\operatorname{im} \partial_{n+1} \to \ker \partial_n,$$

where $\operatorname{im} \partial_{n+1} := \ker(\operatorname{coker} \partial_{n+1})$.

Exercise 1.2. Prove proposition 1.3. *Hint:* Use that $\operatorname{im} \partial_{n+1} \to C_n$ is monic by [Lan78, p. 199].

Definition 1.3 (Homology). *Let $\mathcal{A}$ be an abelian category and $(C_\bullet, \partial_\bullet) \in \mathrm{Ch}(\mathcal{A})$. For $n \in \mathbb{Z}$, we define the* ***n-th homology object****, written $H_n(C_\bullet, \partial_\bullet)$, by*

$$H_n(C_\bullet, \partial_\bullet) := \operatorname{coker}(\operatorname{im} \partial_{n+1} \to \ker \partial_n),$$

where $\operatorname{im} \partial_{n+1} \to \ker \partial_n$ is the unique morphism defined by lemma 1.3.

Definition 1.4 (Injective). *Let $\mathcal{A}$ be an abelian category. A object $I \in \mathcal{A}$ is said to be* ***injective****, iff it satisfies the following universal lifting property:*

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\forall"] \arrow[d, "\forall"'] & Y \arrow[dl, dashed, "\exists"] \\ & I & \end{tikzcd}$$

We say that ***$\mathcal{A}$ has enough injectives****, iff for all $X \in \mathcal{A}$ there exists an exact sequence*

$$0 \longrightarrow X \longrightarrow I,$$

with I injective.

Definition 1.5 (Injective Resolution). *Let $\mathcal{A}$ be an abelian category and $X \in \mathcal{A}$. An exact sequence*

$$0 \longrightarrow X \longrightarrow I_0 \xrightarrow{d_0} I_1 \xrightarrow{d_1} \dots$$

with I_n injective for all $n \in \omega$ is called a ***injective resolution of X****.*

Definition 1.6 (Right Derived Functor). *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categroies and $F : \mathcal{A} \to \mathcal{B}$ left exact. Moreover, assume that $\mathcal{A}$ has enough injectives. For $n \in \omega$, define the* ***right derived functors of F****, written $(\mathcal{R}^n F)_{n\in\omega}$, by*

$$\mathcal{R}^n F(X) := H_n(F(I_0 \to I_1 \to \dots)).$$

where $I_0 \to I_1 \to \dots$ is an injective resolution of X.

Remark 1.1. One can show that the definition 1.6 of right derived functors does not depend of the choice of an injective resolution (this invokes the so called *comparison theorem* of homological algebra).

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Definition 1.7 (Cohomology of Lie Algebras). *Let* $R \in \mathsf{CRing}$, $\mathfrak{g} \in {}_R\mathsf{LieAlg}$ *and* $M \in {}_{\mathfrak{g}}\mathsf{Mod}$. *For* $n \in \omega$, *define the* ***n-th cohomology group of*** $\mathfrak{g}$ ***with coefficients in*** M *by*

$$H^n(\mathfrak{g}, M) := \mathcal{R}^n(-)^{\mathfrak{g}}(M) \cong \operatorname{Ext}^n_{U\mathfrak{g}}(R, M) =: \mathcal{R}^n \operatorname{Hom}_{U\mathfrak{g}}(k, -)(M).$$

Remark 1.2. The definition of cohomology of Lie algebras 1.7 actually makes sense since by theorem 1.1, ${}_{\mathfrak{g}}\mathsf{Mod}$ has enough injectives and thus by [Wei94, p. 40], every object admits an injective resolution. Moreover, the isomorphism in definition 1.7 follows by

$$M^{\mathfrak{g}} \cong \operatorname{Hom}_{\mathfrak{g}}(R, M),$$

where we consider R as a trivial $\mathfrak{g}$-module.

The Whitehead Lemmas

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