FUNCTIONAL ANALYSIS II SUMMARY

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Abstract.

Contents

Elliptic Operators in Divergence Form

Lemma 1.1 (Poincaré Inequality).

Theorem 1.1 (Riesz Representation Theorem).

Theorem 1.2. *Let $\Omega \subseteq\subseteq \mathbb{R}^n$, $k \in \omega$ and consider the elliptic operator*

$$L := \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(a_{ij} \frac{\partial}{\partial x^j} \right),$$

for $a_{ij} \in C^{k+1}(\bar{\Omega})$ symmetric. Then:

(a) *Given $f \in L^2(\Omega)$, the homogenous Dirichlet problem*

$$\begin{cases} -L(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

admits a unique weak solution $u \in H_0^1(\Omega)$.

(b) *If $f \in H^k(\Omega)$ for some $k \in \omega$, then we have $u \in H^{k+2}_{\mathrm{loc}}(\Omega)$ for the unique weak solution of part (a) and moreover, for any $\Omega' \subseteq\subseteq \Omega$ we have the estimate*

$$\|u\|_{H^{k+2}(\Omega')} \leq C \left(\|f\|_{H^k(\Omega)} + \|u\|_{H^1(\Omega)} \right).$$

Proof.

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Step 1: Derivation of Weak Formulation. Suppose $u \in C^2(\bar{\Omega})$ is a solution of (1). Let $\varphi \in C_c^\infty(\Omega)$. Then integration by parts (see [Lee13, p. 436]) yields

$$-\int_\Omega L(u)\varphi = -\sum_{j=1}^n \int_\Omega \operatorname{div}(X_j)\varphi = \sum_{i,j=1}^n \int_\Omega a_{ij} \frac{\partial u}{\partial x^j} \frac{\partial \varphi}{\partial x^i} = \sum_{i,j=1}^n \int_\Omega a_{ij} \frac{\partial u}{\partial x^i} \frac{\partial \varphi}{\partial x^j},$$

where $X_j := \left(a_{ij} \frac{\partial}{\partial x^j}\right)_i$. Thus we get the weak formulation:

$$\sum_{i,j=1}^n \int_\Omega a_{ij} \frac{\partial u}{\partial x^i} \frac{\partial \varphi}{\partial x^j} = \int_\Omega f\varphi \qquad \forall \varphi \in C_c^\infty(\Omega). \tag{2}$$

Step 2: Existence and Uniqueness of Weak Solutions. Since L is uniformly elliptic, there exists $\lambda > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

holds for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Moreover, since $a_{ij} \in C^0(\bar{\Omega})$, we get that L is uniformly bounded, i.e. there exists $\Lambda > 0$ such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2$$

holds for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Now define a bilinear form $\langle \cdot, \cdot \rangle_a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$\langle u, v \rangle_a := \sum_{i,j=1}^n \int_\Omega a_{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} \tag{3}$$

Then it is easy to see, that $\langle \cdot, \cdot \rangle_a$ is symmetric. Also, $\langle \cdot, \cdot \rangle_a$ is positive definite since

$$\langle u, u \rangle_a = \sum_{i,j=1}^n \int_\Omega a_{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \geq \lambda \int_\Omega |\nabla u|^2 \geq C\lambda \int_\Omega |u|^2$$

using ellipticity and Poincaré's inequality. Moreover by Poincaré's inequality we have that

$$C\lambda\|u\|_{H_0^1(\Omega)}^2 \leq \|u\|_a \leq \Lambda\|u\|_{H_0^1(\Omega)}^2$$

for the induced norm $\|\cdot\|_a$. Hence the induced norm is equivalent to the standard norm on $H_0^1(\Omega)$ and thus $(H_0^1(\Omega), \|\cdot\|_a)$ is a Hilbert space. Thus an application of Riesz representation theorem 1.1 yields the existence of a unique $u \in H_0^1(\Omega)$, such that

$$\langle u, \varphi \rangle_a = l(\varphi) := \int_\Omega f\varphi$$

holds for all $\varphi \in H_0^1(\Omega)$, since $l \in (H_0^1(\Omega))^*$ This proves part (a).
Step 3: H^1-Estimate. The main idea in proving part (b) is an induction on $k \in \omega$.

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□

References

[Lee13] John M. Lee. *Introduction to Smooth Manifolds*. Second Edition. Graduate Texts in Mathematics. Springer, 2013.