

# FUNCTIONAL ANALYSIS II SUMMARY

YANNIS BÄHNI

**Abstract.** This is a rough summary of the course *Functional Analysis II* held at *ETH Zurich* by *Prof. Dr. Alessandro Carlotto* in spring 2018. The main focus of this summary is to give a neat preparation for the oral exam.

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(Yannis Bähni) UNIVERSITY OF ZURICH, RÄMISTRASSE 71, 8006 ZURICH  
E-mail address: [yannis.baehni@uzh.ch](mailto:yannis.baehni@uzh.ch).

## Introduction

This serves as a summary of useful facts from *measure theory* which are used throughout the text.

**Theorem 1.1 (Transformation Formula).** Let  $n \in \omega$ ,  $n \geq 0$ ,  $U, V \subseteq \mathbb{R}^n$  open and  $\varphi : U \rightarrow V$  a  $C^1$ -diffeomorphism. A function  $f : V \rightarrow \mathbb{R}$  is in  $\mathcal{L}^1(V)$  if and only if  $(f \circ \varphi) |\det(D\varphi)|$  is in  $\mathcal{L}^1(U)$ . Then

$$\int_V f = \int_U (f \circ \varphi) |\det(D\varphi)|.$$

**Theorem 1.2.** Let  $\Omega \subseteq \mathbb{R}^n$  be open and  $1 \leq p < \infty$ . Then  $C_c^\infty(\Omega)$  is dense in  $L^p(\Omega)$ .

**Proposition 1.1.** If  $|\Omega| < \infty$  and  $0 < p < q \leq \infty$ . Then  $L^q(\Omega) \subseteq L^p(\Omega)$ .

**Proposition 1.2 (Jensen's Inequality).** Let  $\Omega \subseteq \mathbb{R}^n$  bounded and  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  convex. Then

$$\varphi\left(\frac{1}{|\Omega|} \int_\Omega f\right) \leq \frac{1}{|\Omega|} \int_\Omega \varphi \circ f$$

for any  $f \in L^1(\Omega)$ .

**Proposition 1.3 (Dual of  $L^p(\Omega)$ ).** Let  $\Omega \subseteq \mathbb{R}^n$  and  $1 \leq p < \infty$ . Then the mapping  $T : L^q(\Omega) \rightarrow (L^p(\Omega))^*$  defined by

$$T(f)(g) := \int_\Omega fg$$

is an isometric isomorphism.

**Proposition 1.4 (Integration by Parts).** Let  $(M, g)$  be a compact Riemannian manifold with boundary. Then

$$\int_M \langle \text{grad } f, X \rangle_g dV_g = \int_{\partial M} f \langle X, N \rangle dV_{\tilde{g}} - \int_M (f \operatorname{div} X) dV_g$$

for  $f \in C^\infty(M)$  and  $X \in \mathfrak{X}(M)$ . Moreover, **Green's identities** hold:

$$\int_M u \Delta v dV_g = \int_M \langle \text{grad } u, \text{grad } v \rangle_g dV_g - \int_{\partial M} u N v dV_{\tilde{g}}$$

and

$$\int_M (u \Delta v - v \Delta u) dV_g = \int_{\partial M} (v N u - u N v) dV_{\tilde{g}}$$

for  $u, v \in C^\infty(M)$ .

## Sobolev Space Theory

**The Spaces  $W^{k,p}(\Omega)$ .** In what follows, let  $n \in \omega$ ,  $n \geq 1$ , and  $1 \leq p \leq \infty$ .

**Definition 1.1 (Distributional and Weak Derivative).** Let  $\Omega \subseteq \mathbb{R}^n$  open and  $u \in L^1_{\text{loc}}(\Omega)$ . For any multiindex  $\alpha$ , the **distributional derivative of order  $\alpha$  of  $u$** , written  $D^\alpha u$ , is defined to be the mapping  $D^\alpha u : C_c^\infty(\Omega) \rightarrow \mathbb{R}$  defined by

$$\varphi \mapsto (-1)^{|\alpha|} \int_{\Omega} u D^\alpha \varphi dx.$$

Moreover, a function  $D^\alpha u \in L^p(\Omega)$  is called **weak derivative of order  $\alpha$  of  $u$  with exponent  $p$** , iff

$$\forall \varphi \in C_c^\infty(\Omega) : \int_{\Omega} D^\alpha u \varphi dx = (-1)^{|\alpha|} \int_{\Omega} u D^\alpha \varphi dx.$$

**Theorem 1.3 (Fundamental Lemma of Variational Calculus).** Let  $\Omega \subseteq \mathbb{R}^n$  open and  $f \in L^1_{\text{loc}}(\Omega)$ . If

$$\forall \varphi \in C_c^\infty(\Omega) : \int_{\Omega} f \varphi dx = 0,$$

then  $f = 0$  a.e.

**Remark 1.1.** Let  $\Omega \subseteq \mathbb{R}^n$  open. Then  $L^p(\Omega) \subseteq L^1_{\text{loc}}(\Omega)$ .

**Remark 1.2.** From the fundamental lemma of variational calculus 1.3 it follows that *weak derivatives, if they exist, are unique*.

**Examples 1.1 (Weak Derivatives).**

(a) Suppose  $u$  is classically differentiable. Then  $u$  is weakly differentiable using integration by parts 1.4.

(b) Consider  $\Omega := (-1, 1)$  and  $u := |x|$ . Then  $u' = \chi_{[0,1)} - \chi_{(-1,0]}$ .

(c) Consider  $\Omega := \mathbb{R}$  and  $u := \chi_{(0,\infty)}$ . Then the weak derivative  $u'$  does not exist. Indeed, the *Dirac distribution* is not representable as one may see by considering the smooth family  $\varphi_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$  for  $\varepsilon > 0$  defined by

$$\varphi_\varepsilon(x) := \begin{cases} e^{\varepsilon^2/(x^2-\varepsilon^2)} & |x| < \varepsilon, \\ 0 & |x| \geq \varepsilon. \end{cases}$$

(d) Let  $\Omega := (0, 1)$  and consider the *Cantor function*  $u : \Omega \rightarrow \Omega$ . Then  $u' = 0$  classically a.e. but the distributional derivative of  $u$  does not vanish.

(e) Let  $f \in L^p(\Omega)$ . Then the computation performed in the proof of lemma 1.3 shows, that the function  $u : I \rightarrow \mathbb{R}$  defined by

$$u(x) := \int_{x_0}^x f(t) dt$$

for  $x_0 \in I$ , admits the weak derivative  $f$ .

**Definition 1.2 (Sobolev Space).** Let  $\Omega \subseteq \mathbb{R}^n$  open. For any  $k \in \omega$ , the **Sobolev space of index  $(k, p)$** , written  $W^{k,p}(\Omega)$ , is defined to be the space

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ exists for all } |\alpha| \leq k\},$$

with norm

$$\|-\|_{W^{k,p}(\Omega)} := \sum_{|\alpha| \leq k} \|D^\alpha -\|_{L^p(\Omega)}.$$

Moreover, define

$$W_0^{k,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|-\|_{W^{k,p}(\Omega)}},$$

and  $H^k(\Omega) := W^{k,2}(\Omega)$  as well as  $H_0^k(\Omega) := W_0^{k,2}(\Omega)$ .

**Examples 1.2 (Sobolev Functions).** A main tool in constructing Sobolev functions for  $n \geq 2$  is using that the origin in  $\mathbb{R}^n$  has vanishing  $W^{1,p}$ -capacity.

- (a) Let  $\Omega := \mathbb{R}$ ,  $A$  Lebesgue-measurable and  $u := \chi_A$ . Then  $u \notin W^{1,p}(\Omega)$ , since by theorem 1.7  $u$  must admit a continuous representant, which it obviously does not.
- (b) Let  $\Omega := B_1(0) \subseteq \mathbb{R}^n$  for  $n \geq 2$ . Then  $u : \Omega \rightarrow \bar{\mathbb{R}}$  defined by  $u(x) := \log|x|$  belongs to  $L^p(\Omega)$  for any  $1 \leq p < \infty$  and moreover,  $u \in W^{1,p}(\Omega)$  for any  $p < n$ .
- (c) Let  $\Omega := B_{1/e}(0) \subseteq \mathbb{R}^n$  for  $n \geq 2$ . Then  $u : \Omega \rightarrow \bar{\mathbb{R}}$  defined by  $u(x) := \log \log \frac{1}{|x|}$  belongs to  $W^{1,n}(\Omega)$ .
- (d) Let  $\Omega := B_{1/2}(0) \subseteq \mathbb{R}^n$ . For  $\alpha \in \mathbb{R}$  define  $u_\alpha : \Omega \rightarrow \bar{\mathbb{R}}$  by  $u_\alpha(x) := |\log|x||^\alpha$ . Then  $u_\alpha \in H^1(\Omega)$  for  $n = 1$  if and only if  $\alpha = 0$ , for  $n = 2$  if and only if  $\alpha \in (-\infty, 1/2)$  and for  $n \geq 3$  if and only if  $\alpha \in \mathbb{R}$ .

**Remark 1.3.** Using proposition 1.1, we immediately get

$$W^{1,q}(\Omega) \hookrightarrow W^{1,p}(\Omega)$$

for all  $1 \leq p \leq q \leq \infty$  whenever  $\Omega \subseteq \mathbb{R}^n$ .

**Theorem 1.4.** Let  $\Omega \subseteq \mathbb{R}^n$  open. Then  $W^{k,p}(\Omega)$  is

- (a) a Banach space for all  $1 \leq p \leq \infty$ .
- (b) separable for all  $1 \leq p < \infty$ .
- (c) reflexive for all  $1 < p < \infty$ .

*Proof.* The proof basically boils down to using the corresponding properties of the Lebesgue spaces  $L^p(\Omega)$ .

(a) This follows from the fact that  $L^p(\Omega)$  is a Banach space for all  $1 \leq p \leq \infty$ . Let  $(f_i)_{i \in \omega}$  be a Cauchy sequence in  $W^{k,p}$ . By definition of the  $W^{k,p}$ -norm,  $(D^\alpha f_i)_{i \in \omega}$  is a Cauchy sequence in  $L^p$ . Thus we get  $D^\alpha f_i \rightarrow f_\alpha$  in  $L^p$ , in particular,  $f_i \rightarrow f$  in  $L^p$ . Using Hölder's inequality we compute

$$\int_\Omega f_\alpha \varphi dx = \lim_{i \rightarrow \infty} \int_\Omega D^\alpha f_i \varphi dx = (-1)^{|\alpha|} \lim_{i \rightarrow \infty} \int_\Omega f_i D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi dx$$

for  $\varphi \in C_c^\infty(\Omega)$ .

(b) For simplicity, we consider  $k = 1$  only. Consider  $\iota : W^{1,p} \hookrightarrow (L^p)^{n+1}$  defined in the obvious way. Then  $\iota$  is an isometry and the statement follows.

(c) Same argument as in part (b).

□

### Elliptic Operators in Divergence Form.

**Lemma 1.1 (Poincaré Inequality).** *Let  $\Omega \subseteq \mathbb{R}^n$  and  $1 \leq p < \infty$ . Then for any  $u \in C_c^\infty(\Omega)$  we have that*

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p}.$$

*Proof.* Let  $n = 1$ . Since  $\Omega$  is bounded, we get that  $\Omega \subseteq (a, b)$  and we may extend  $u$  on  $[a, b] =: I$  to be zero. Hence an application of Jensen's inequality 1.2 yields

$$|u(x)|^p = \left| \int_a^x u'(t) dt \right|^p \leq (x-a)^{p-1} \int_a^x |u'(t)|^p dt \leq (b-a)^{p-1} \|u'\|_{L^p(I)}^p.$$

Thus

$$\|u\|_{L^p(\Omega)}^p = \|u\|_{L^p(I)}^p \leq (b-a)^p \|u'\|_{L^p(I)}^p = (b-a)^p \|u'\|_{L^p(\Omega)}^p$$

where the last equality follows due to the fact that  $u$  and thus  $u'$  is compactly supported in  $\Omega$ . If  $n > 1$ , we have  $\Omega \subseteq (a, b) \times \mathbb{R}^{n-1}$ . Hence for fixed  $y \in \mathbb{R}^{n-1}$ , above computation yields

$$|u(x, y)|^p \leq (b-a)^{p-1} \|\partial_x u(-, y)\|_{L^p(I)}^p$$

for any  $x \in I$ . Hence

$$\begin{aligned} \|u\|_{L^p(\Omega)}^p &= \|u\|_{L^p((a,b) \times \mathbb{R}^{n-1})}^p \\ &\leq (b-a)^p \int_{\mathbb{R}^{n-1}} \|\partial_x u(-, y)\|_{L^p(I)}^p dy \\ &\leq (b-a)^p \|\nabla u\|_{L^p((a,b) \times \mathbb{R}^{n-1})}^p \\ &= (b-a)^p \|\nabla u\|_{L^p(\Omega)}^p. \end{aligned}$$

□

**Theorem 1.5 (Riesz Representation Theorem).** *Let  $H$  be a real Hilbert space. Then the mapping  $J : H \rightarrow H^*$  defined by  $J(x) := \langle x, - \rangle$  is an isometric isomorphism.*

**Theorem 1.6.** *Let  $\Omega \subseteq \mathbb{R}^n$  and consider the elliptic operator*

$$A_0 := -\frac{\partial}{\partial x^i} \left( a^{ij} \frac{\partial}{\partial x^j} \right),$$

for  $a^{ij} \in L^\infty(\Omega)$  symmetric. Then: Given  $f \in L^2(\Omega)$ , the homogenous Dirichlet problem

$$\begin{cases} A_0 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

admits a unique weak solution  $u \in H_0^1(\Omega)$ .

*Proof.* The proof is divided into two steps.

*Step 1: Derivation of Weak Formulation.* Suppose  $u \in C^2(\bar{\Omega})$  is a solution of (1). Let  $\varphi \in C_c^\infty(\Omega)$ . Then integration by parts (see [Lee13, p. 436]) yields

$$-\int_{\Omega} A_0 u \varphi = -\sum_{j=1}^n \int_{\Omega} \operatorname{div}(X_j) \varphi = \int_{\Omega} a^{ij} \frac{\partial u}{\partial x^j} \frac{\partial \varphi}{\partial x^i} = \int_{\Omega} a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial \varphi}{\partial x^j},$$

where  $X_j := \left(a^{ij} \frac{\partial}{\partial x^j}\right)_i$ . Thus we get the weak formulation:

$$\forall \varphi \in C_c^\infty(\Omega) : \int_{\Omega} a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial \varphi}{\partial x^j} = \int_{\Omega} f \varphi. \quad (2)$$

*Step 2: Existence and Uniqueness of Weak Solutions.* Since  $A_0$  is uniformly elliptic, there exists  $\lambda > 0$  such that

$$\xi^t(a^{ij}(x))\xi = a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

holds for any  $x \in \Omega$  and  $\xi \in \mathbb{R}^n$ . Moreover, since  $a^{ij} \in L^\infty(\Omega)$ , we get that  $A_0$  is uniformly bounded, i.e. there exists  $\Lambda > 0$  such that

$$a^{ij}(x)\xi_i\eta_j \leq \Lambda|\xi||\eta|$$

for

$$\Lambda = \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(\Omega)}$$

holds for almost all  $x \in \Omega$  and  $\xi, \eta \in \mathbb{R}^n$ . Now define a bilinear form

$$\langle \cdot, \cdot \rangle_a : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$$

by

$$\langle u, v \rangle_a := \int_{\Omega} a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j} \quad (3)$$

Then it is easy to see, that  $\langle \cdot, \cdot \rangle_a$  is symmetric. Also,  $\langle \cdot, \cdot \rangle_a$  is positive definite since

$$\langle u, u \rangle_a = \int_{\Omega} a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \geq \lambda \int_{\Omega} |\nabla u|^2 \geq C\lambda \int_{\Omega} |u|^2$$

using ellipticity and Poincaré's inequality. Moreover by Poincaré's inequality we have that

$$\lambda \|u\|_{H_0^1(\Omega)}^2 \leq \|u\|_a^2 \leq \Lambda \|u\|_{H_0^1(\Omega)}^2$$

for the induced norm  $\|\cdot\|_a$ . Hence the induced norm is equivalent to the standard norm on  $H_0^1(\Omega)$  and thus  $(H_0^1(\Omega), \|\cdot\|_a)$  is a Hilbert space. Thus an application of Riesz representation theorem 1.5 yields the existence of a unique  $u \in H_0^1(\Omega)$ , such that

$$\langle u, \varphi \rangle_a = l(\varphi) := \int_{\Omega} f \varphi$$

holds for all  $\varphi \in H_0^1(\Omega)$ , since  $l \in (H_0^1(\Omega))^*$  by

$$|l(\varphi)| \leq \|f\|_{L^2} \|\varphi\|_{L^2} \leq \|f\|_{L^2} \|\varphi\|_{H^1}.$$

□

### Examples 1.3 (Elliptic Operators in Divergence Form).

(a) Set  $a^{ij}(x) := \delta^{ij}$  for all  $x \in \Omega$ . Then  $A_0 = -\Delta$ . Moreover  $A_0$  is uniformly elliptic, since  $\delta^{ij} \xi_i \xi_j = |\xi|^2$  for all  $\xi \in \mathbb{R}^n$ .

(b) For  $\Omega \subseteq \mathbb{R}^2$  consider

$$(a^{ij}(x, y)) := \begin{pmatrix} 2 & xy/|xy| \\ xy/|xy| & 2 \end{pmatrix}.$$

Then  $A_0$  is elliptic. Indeed,  $a^{ij}$  admits the eigenvalues 1 and 3, thus by the *Min-Max theorem* we get that

$$1 \leq R_{A(x,y)}(z) \leq 3$$

for all  $(x, y) \in \Omega$  and where  $R_A(z)$  denotes the *Rayleigh-Ritz quotient* defined by

$$R_A(z) := \frac{\langle Az, z \rangle}{\|z\|^2}$$

for  $z \in \mathbb{C}^2$ .

(c) A non-example would be  $a^{ij}(x) := 0$ .

(d) Another non-example is given by

$$(a^{ij}(x, y)) := \begin{pmatrix} x^2 + y^2 & x + y \\ x + y & 1 \end{pmatrix}$$

for any  $\Omega \subseteq \mathbb{R}^2$  containing the origin. Indeed, we get  $\det(a^{ij}(0, 0)) = 0$ .

**Sobolev Spaces on an Interval.** In what follows, let  $-\infty \leq a < b \leq \infty$  and  $I := (a, b)$ .

**Lemma 1.2 (Du Bois-Reymond).** Let  $f \in L_{\text{loc}}^1(I)$  such that

$$\forall \varphi \in C_c^\infty(I) : \int_I f \varphi' dx = 0.$$

Then  $f$  is almost everywhere constant.

*Proof.* Let  $v := w - c_0\psi$  for  $w, \psi \in C_c^\infty(I)$  such that  $\int_I \psi = 1$  and  $\int_I v = 0$ . This implies  $c_0 = \int_I w$ . By the fundamental theorem of calculus, the function  $\varphi : I \rightarrow \mathbb{R}$  defined by

$$\varphi(x) := \int_I v(t)dt$$

belongs to  $C_c^\infty(I)$  with  $\varphi' = v$ . Thus we compute

$$0 = \int_I f\varphi' = \int_I f v = \int_I f w - c_0 \int_I f \psi = \int_I f w - \int_I w \int_I f \psi = \int_I (f - c)w,$$

where  $c := \int_I f \psi$ . Since  $w$  was arbitrary, we conclude by the fundamental lemma of variational calculus 1.3.  $\square$

**Lemma 1.3.** Let  $f \in L^1_{\text{loc}}(I)$  and  $x_0 \in I$ . Then  $u : I \rightarrow \mathbb{R}$  defined by

$$u(x) := \int_{x_0}^x f(t)dt$$

is absolutely continuous and belongs to  $W^{1,1}_{\text{loc}}(I)$  with  $u' = f$  a.e.

*Proof.* Absolute continuity follows from real analysis. Let  $\varphi \in C_c^\infty(I)$ . Then Fubini yields

$$\begin{aligned} \int_I u\varphi' &= \int_a^{x_0} \int_{x_0}^x f(t)\varphi'(x)dt dx + \int_{x_0}^b \int_{x_0}^x f(t)\varphi'(x)dt dx \\ &= - \int_a^{x_0} \int_x^{x_0} f(t)\varphi'(x)dt dx + \int_{x_0}^b \int_{x_0}^x f(t)\varphi'(x)dt dx \\ &= - \int_a^{x_0} \int_a^t f(t)\varphi'(x)dx dt + \int_{x_0}^b \int_t^b f(t)\varphi'(x)dx dt \\ &= - \int_a^{x_0} f(t)\varphi(t)dt - \int_{x_0}^b f(t)\varphi(t)dt \\ &= - \int_I f\varphi. \end{aligned}$$

$\square$

**Theorem 1.7.** Let  $u \in W^{1,p}(I)$ . Then there exists an absolutely continuous representant  $\tilde{u}$  of  $u$  on  $\bar{I}$ , such that

$$\tilde{u}(x) = \tilde{u}(x_0) + \int_{x_0}^x u'(t)dt$$

holds for all  $x, x_0 \in I$ . In particular,  $\tilde{u}$  is classically differentiable a.e. and  $\tilde{u}' = u'$ .



*Proof.* By lemma 1.3, the function  $v(x) := \int_{x_0}^x u'(t)dt$  is in  $W_{\text{loc}}^{1,1}(I)$  with weak derivative  $u'$ . Moreover, for any  $\varphi \in C_c^\infty(I)$  we compute

$$\int_I (u - v)\varphi' = \int_I u\varphi' - \int_I v\varphi' = - \int_I u'\varphi + \int_I u'\varphi = 0.$$

Thus lemma 1.2 yields  $u = c + v$ , for some  $c \in \mathbb{R}$ . Set

$$\tilde{u}(x) := c + \int_{x_0}^x u'(t)dt.$$

Then  $\tilde{u}(x_0) = c$  and thus the statement follows.  $\square$

**Theorem 1.8 (Characterization of  $W^{1,p}(I)$ ).** *Let  $1 < p \leq \infty$  and  $u \in L^p(I)$ . Then the following statements are equivalent:*

- (a)  $u \in W^{1,p}(I)$ .
- (b) There exists  $C \geq 0$  such that

$$\forall \varphi \in C_c^\infty(I) : \left| \int_I u\varphi' \right| \leq C \|\varphi\|_{L^q}.$$

- (c) There exists  $C \geq 0$  such that for all  $I' \subseteq\subseteq I$  and  $|h| < \text{dist}(I', \partial I)$  holds

$$\|\tau_h u - u\|_{L^p(I')} \leq C|h|,$$

where  $\tau_h u(x) := u(x + h)$ .

*Proof.* The implication (a)  $\Rightarrow$  (b) follows immediately from Hölder's inequality. To prove (b)  $\Rightarrow$  (a), we observe that  $l : C_c^\infty(I) \rightarrow \mathbb{R}$  defined by

$$L(\varphi) := \int_I u\varphi'$$

is continuous. Since  $C_c^\infty(I)$  is dense in  $L^q(I)$ , we get that  $l \in (L^q(I))^*$ . Hence we find  $g \in L^p$ , such that  $\int_I g\varphi = l(\varphi)$  and so  $u' = -g$ .

Next we show (a)  $\Rightarrow$  (c). By theorem 1.7, we find an absolutely continuous representant  $\tilde{u}$  of  $u$ . Thus

$$\tilde{u}(x + h) - \tilde{u}(x) = h \int_0^1 u'(x + th)dt$$

Hence Jensen's inequality yields

$$\|\tau_h u - u\|_{L^p(I')} \leq |h| \int_0^1 \|u'(\cdot + th)\|_{L^p(I')} dt \leq |h| \|u'\|_{L^p(I)}.$$

Lastly, we prove (c)  $\Rightarrow$  (b). Let  $\varphi \in C_c^\infty(I)$ . Then we may find  $I' \subseteq\subseteq I$  such that  $\text{supp } \varphi \subseteq I'$ . Hence we compute

$$\left| \int_I u\varphi' \right| = \lim_{h \rightarrow 0} \frac{1}{|h|} \left| \int_I u(x) (\varphi(x + h) - \varphi(x)) dx \right|$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{1}{|h|} \left| \int_I (u(x-h) - u(x)) \varphi(x) dx \right| \\
&= \lim_{h \rightarrow 0} \frac{1}{|h|} \left| \int_I (\tau_{-h} u - u) \varphi \right| \\
&\leq \lim_{h \rightarrow 0} \frac{1}{|h|} \|\tau_{-h} u - u\|_{L^p(I')} \|\varphi\|_{L^q(I)} \\
&\leq C \|\varphi\|_{L^q(I)}.
\end{aligned}$$

□

**Theorem 1.9 (Extension Theorem).** *There exists a continuous linear operator*

$$E : W^{1,p}(I) \rightarrow W^{1,p}(\mathbb{R})$$

*such that:*

- (i)  $Eu|_I = u$ .
- (ii)  $\|Eu\|_{L^p(\mathbb{R})} \leq C \|u\|_{L^p(I)}$ .
- (iii)  $\|(Eu)'\|_{L^p(\mathbb{R})} \leq C \|u\|_{W^{1,p}(I)}$ .

*Proof.* First we consider the case  $I = (0, \infty)$ . We extend  $u$  by continuity to 0 and then we extend  $u$  by means of *even symmetry*. If  $I$  is bounded we can without loss of generality assume that  $I = (0, 1)$ . Now use a cut-off function. □

**Theorem 1.10 (Approximation Theorem).** *Let  $1 \leq p < \infty$  and  $u \in W^{1,p}(I)$ . Then there exists a sequence  $(u_i)_{i \in \omega}$  in  $C_c^\infty(\mathbb{R})$  such that*

$$\|u_i|_I - u\|_{W^{1,p}(I)} \rightarrow 0.$$

*Proof.* The main idea of the proof is to use convolutions. Moreover, it is enough to consider the case  $I = \mathbb{R}$  only, due to the extension theorem 1.9. □

**Theorem 1.11 (Sobolev Embedding).** *There is a continuous embedding*

$$W^{1,p}(I) \hookrightarrow L^\infty(I).$$

*Proof.* First consider  $I$  bounded. By theorem 1.7 we get that

$$\|u\|_{L^\infty} = \sup_{x \in I} |u(x)| \leq |u(y)| + \sup_{x \in I} \left| \int_y^x u'(t) dt \right| \leq |u(y)| + \|u'\|_{L^1},$$

for any  $y \in I$ . Hence

$$\|u\|_{L^\infty} \leq \inf_{y \in I} |u(y)| + \|u'\|_{L^1} \leq \frac{1}{|I|} \int_I |u(y)| + \|u'\|_{L^1} \leq C \|u\|_{W^{1,1}} \leq C \|u\|_{W^{1,p}}.$$

Assume now that  $I$  is unbounded. Then we find  $I' \subseteq \subseteq I$  such that

$$\|u\|_{L^\infty(I')} \geq \frac{1}{2} \|u\|_{L^\infty(I)}$$

and thus the claim follows by the previous computation. Indeed, note that by theorem 1.7, we have that

$$|u(x)| \leq |u(y)| + \|u'\|_{L^1(I)}$$

for all  $x \in I$  and fixed  $y \in I$ , and thus  $u \in L^\infty(I)$ . Moreover, there exists  $x_0 \in I$  such that  $|u(x_0)| > \frac{1}{2}\|u\|_{L^\infty(I)}$ , if not, this would contradict the definition of the supremum norm. Since  $u$  is continuous by theorem 1.7, we find  $\delta > 0$  such that

$$|u(x) - u(x_0)| \leq |u(x_0)| - \frac{1}{2}\|u\|_{L^\infty(I)}$$

for all  $x \in I$  such that  $|x - x_0| < \delta$ . Hence the reversed triangle inequality yields

$$\frac{1}{2}\|u\|_{L^\infty(I)} - |u(x_0)| \leq |u(x)| - |u(x_0)| \leq |u(x_0)| - \frac{1}{2}\|u\|_{L^\infty(I)}$$

and so

$$\frac{1}{2}\|u\|_{L^\infty(I)} \leq |u(x)|$$

for all  $x \in I \cap (x_0 - \delta, x_0 + \delta) =: I'$ . □

**Corollary 1.1.** *Let  $I$  be unbounded and  $u \in W^{1,p}(I)$  for  $1 \leq p < \infty$ . Then  $u \rightarrow 0$  as  $|x| \rightarrow \infty$ .*

**Dirichlet and Neumann Boundary Problems on an Interval.** In what follows, let us consider  $-\infty < a < b < \infty$  and  $I := (a, b)$ .

**Proposition 1.5.** *Let  $f \in C^0(\bar{I})$ . Then the weak solution  $u$  of the homogenous Dirichlet problem*

$$\begin{cases} -u'' = f & \text{in } I, \\ u(a) = 0 = u(b), \end{cases}$$

*is a classical solution, i.e.  $u \in C^2(\bar{I})$ .*

**Proposition 1.6.** *Let  $f \in C^0(\bar{I})$ . Then the weak solution  $u$  of the homogenous Neumann problem*

$$\begin{cases} -u'' + u = f & \text{in } I, \\ u'(a) = 0 = u'(b), \end{cases}$$

*is a classical solution, i.e.  $u \in C^2(\bar{I})$ .*

### Sobolev Spaces on a Domain.

**Theorem 1.12 (Meyers-Serrin).** *Let  $\Omega \subseteq \mathbb{R}^n$  be open. Then  $C^\infty(\Omega) \cap W^{1,p}(\Omega)$  is dense in  $W^{1,p}(\Omega)$  for every  $1 \leq p < \infty$ .*

*Proof.* Convolutions and a partition of unity argument.  $\square$

**Proposition 1.7 (Product Rule).** *Let  $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ . Then  $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$  and*

$$\partial_\alpha(uv) = (\partial_\alpha u)v + u(\partial_\alpha v).$$

*Proof.* First consider the case  $p < \infty$ . Then

$$\|uv\|_{L^p} \leq \|u\|_{L^\infty} \|v\|_{L^p}$$

and

$$\|(\partial_\alpha u)v + u(\partial_\alpha v)\|_{L^p} \leq \|\partial_\alpha u\|_{L^p} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|\partial_\alpha v\|_{L^p}.$$

Meyers-Serrin 1.12 yields the existence of sequences  $u_k$  and  $v_k$  in  $C^\infty(\Omega) \cap W^{1,p}(\Omega)$  such that  $u_k \rightarrow u$  and  $v_k \rightarrow v$  in  $W^{1,p}(\Omega)$ . For any  $\varphi \in C_c^\infty(\Omega)$ , we compute

$$\begin{aligned} \int_\Omega uv \partial_\alpha \varphi &= \lim_{k \rightarrow \infty} \int_\Omega u_k v_k \partial_\alpha \varphi \\ &= - \lim_{k \rightarrow \infty} \int_\Omega ((\partial_\alpha u_k)v_k + u_k(\partial_\alpha v_k))\varphi \\ &= - \int_\Omega ((\partial_\alpha u)v + u(\partial_\alpha v))\varphi. \end{aligned}$$

Now consider the case  $p = \infty$ . We have  $uv \in L^\infty(\Omega)$  as well as  $(\partial_\alpha u_k)v_k + u_k(\partial_\alpha v_k) \in L^\infty(\Omega)$ . Let  $\varphi \in C_c^\infty(\Omega)$ . Hence we find  $\Omega' \subseteq \subseteq \Omega$  with  $\text{supp } \varphi \subseteq \Omega'$ . But then the above calculation holds on  $\Omega'$ .  $\square$

**Theorem 1.13 (Characterization of  $W^{1,p}(\Omega)$ ).** *Let  $1 < p \leq \infty$  and  $u \in L^p(\Omega)$ . Then the following statements are equivalent:*

- (a)  $u \in W^{1,p}(\Omega)$ .
- (b) There exists  $C \geq 0$  such that

$$\forall |\alpha| \leq 1 \forall \varphi \in C_c^\infty(\Omega) : \left| \int_I u D^\alpha \varphi \right| \leq C \|\varphi\|_{L^q}.$$

- (c) There exists  $C \geq 0$  such that for all  $\Omega' \subseteq \subseteq \Omega$  and  $|h| < \text{dist}(I', \partial I)$  holds

$$\|\tau_h u - u\|_{L^p(\Omega')} \leq C|h|,$$

where  $\tau_h u(x) := u(x + h)$ .

*Proof.* The proof (c)  $\Rightarrow$  (b)  $\Rightarrow$  (a) is almost the same as the one given in the characterization theorem for  $\Omega$  an interval. For proving (a)  $\Rightarrow$  (c), use Meyers-Serrin.  $\square$

**Corollary 1.2.** *Let  $u \in L^\infty(\Omega)$ . Then  $u \in W^{1,\infty}(\Omega)$  if and only if  $u$  admits a locally Lipschitz continuous representant. Moreover, if  $\Omega$  is convex, then  $u \in W^{1,\infty}(\Omega)$  if and only if  $u$  admits a Lipschitz continuous representant.*

**Extension and Trace Operator.** We start off with *local theory*. In what follows, define

$$Q := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < 1, |x_n| < 1\}.$$

Moreover

$$Q_+ := \{(x', x_n) \in Q : x_n > 0\} \quad \text{and} \quad Q_0 := \{(x', x_n) \in Q : x_n = 0\}.$$

**Lemma 1.4.** *Let  $u \in W^{1,p}(Q_+)$ . Set*

$$u^*(x', x_n) := \begin{cases} u(x', x_n) & x_n > 0, \\ u(x', -x_n) & x_n < 0. \end{cases}$$

*Then  $u^* \in W^{1,p}(Q)$  and  $\|u^*\|_{W^{1,p}(Q)} \leq C \|u\|_{W^{1,p}(Q_+)}$ .*

Now to the *global theory*.

**Theorem 1.14 (Extension).** *Let  $\Omega \subseteq \subseteq \mathbb{R}^n$  of class  $C^1$ . Then there exists a continuous linear operator*

$$E : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^n)$$

*such that:*

- (i)  $Eu|_\Omega = u$ .
- (ii)  $\|Eu\|_{L^p(\mathbb{R}^n)} \leq C \|u\|_{L^p(\Omega)}$ .
- (iii)  $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\Omega)}$ .

**Corollary 1.3.** *Let  $\Omega \subseteq \subseteq \mathbb{R}^n$  of class  $C^1$  and  $1 \leq p < \infty$ . Then  $C^\infty(\bar{\Omega})$  is dense in  $W^{1,p}(\Omega)$ .*

Again, we tackle first the *local theory*.

**Lemma 1.5.** *Let  $u \in W^{1,p}(Q_+)$ . Then  $u|_{Q_0} \in L^p(Q_0)$  is well defined and the induced trace operator  $W^{1,p}(Q_+) \rightarrow L^p(Q_0)$  is linear and continuous.*

*Proof.* We consider the case  $1 \leq p < \infty$ . The main idea is to show this for  $u \in C^\infty(Q)$ , then for  $u \in W^{1,p}(Q)$  and then finally for  $u \in W^{1,p}(Q_+)$  by extension.

Consider now  $p = \infty$ . Since  $Q_+$  is convex,  $u \in W^{1,\infty}(Q_+)$  admits a Lipschitz continuous representant and the result follows by extending via continuity.  $\square$

**Theorem 1.15 (Characterization of  $H^1(\Omega)$ ).** *Let  $\Omega \subseteq \subseteq \mathbb{R}^n$ . Then*

$$H^1(\Omega) = H_0^1(\Omega) \oplus \{u \in H^1(\Omega) : \Delta u = 0\}.$$

*Proof.* Let  $u \in H^1(\Omega)$  and let  $u_0 \in H_0^1(\Omega)$  denote the unique solution of

$$\forall \varphi \in C_c^\infty(\Omega) : \int_{\Omega} \nabla u_0 \nabla \varphi = \int_{\Omega} \nabla u \nabla \varphi.$$

Set  $u_1 := u - u_0$ . Then for any  $\varphi \in C_c^\infty(\Omega)$  we compute

$$-\int_{\Omega} u_1 \Delta \varphi = \int_{\Omega} \nabla u_1 \nabla \varphi = \int_{\Omega} \nabla u \nabla \varphi - \int_{\Omega} \nabla u_0 \nabla \varphi = 0.$$

Thus  $u = u_0 + u_1$  is of the desired form. Moreover, we have

$$\|\nabla u\|_{L^2(\Omega)}^2 = \|\nabla u_0\|_{L^2(\Omega)}^2 + \|\nabla u_1\|_{L^2(\Omega)}^2 + 2 \int_{\Omega} \nabla u_0 \nabla u_1.$$

Since  $u_0 \in H_0^1(\Omega)$ , we find a sequence  $\varphi_k$  in  $C_c^\infty(\Omega)$  such that  $\varphi_k \rightarrow u$  in  $H^1(\Omega)$ . But then

$$\int_{\Omega} \nabla u_0 \nabla u_1 = \lim_{k \rightarrow \infty} \int_{\Omega} \nabla \varphi_k \nabla u_1 = 0.$$

Hence

$$\|\nabla u\|_{L^2(\Omega)}^2 = \|\nabla u_0\|_{L^2(\Omega)}^2 + \|\nabla u_1\|_{L^2(\Omega)}^2$$

which implies that the decomposition is direct. Indeed, suppose  $u \in H_0^1(\Omega)$  such that  $\Delta u = 0$ . Then  $u = u/2 + u/2$  which yields  $u = 0$  by the above computation.  $\square$

**Corollary 1.4 (Characterization of  $H_0^1(\Omega)$ ).** Let  $\Omega \subseteq \mathbb{R}^n$  of class  $C^1$ . Then

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}.$$

*Proof.* Suppose  $u \in H_0^1(\Omega)$ . Then  $\varphi_k \rightarrow u$  in  $H^1(\Omega)$  for some sequence  $\varphi_k$  in  $C_c^\infty(\Omega)$ . Hence

$$u|_{\partial\Omega} = \lim_{k \rightarrow \infty} \varphi_k|_{\partial\Omega} = 0$$

by continuity of the trace operator. Conversely, suppose  $u \in H^1(\Omega)$  with  $u|_{\partial\Omega} = 0$ . Using the characterization of  $H^1(\Omega)$  1.15, we get a unique decomposition  $u = u_0 + u_1$  for  $u_0 \in H_0^1(\Omega)$  and  $u_1 \in H^1(\Omega)$  with  $\Delta u_1 = 0$ . Moreover, observe that

$$0 = u|_{\partial\Omega} = u_0|_{\partial\Omega} + u_1|_{\partial\Omega} = u_1|_{\partial\Omega}.$$

Since harmonic extensions are unique, we conclude  $u_1 = 0$ .  $\square$

## Sobolev Embeddings.

**Theorem 1.16 (Sobolev Embedding Theorem).** Let  $\Omega \subseteq \mathbb{R}^n$  of class  $C^1$  and  $k \in \omega$ ,  $k \geq 1$ . Then:

- (a) If  $kp < n$ , then  $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$  for all  $1 \leq q \leq p^* := \frac{np}{n-pk}$  and the embedding is compact for  $q < p^*$ .
- (b) If  $kp = n$ , then  $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$  for all  $1 \leq q < \infty$  and those embeddings are compact.

- (c) If  $kp > n$  and  $k - \frac{n}{p} \notin \omega$ , then  $W^{k,p}(\Omega) \hookrightarrow C^{l,\alpha}(\Omega)$  for  $l := \lfloor k - \frac{n}{p} \rfloor$  and  $0 \leq \alpha \leq \alpha^* := k - l - \frac{n}{p}$  and those embeddings are compact for  $\alpha < \alpha^*$ .  
(d) If  $kp > n$  and  $k - \frac{n}{p} = l + 1 \in \omega$ , then  $W^{k,p}(\Omega) \hookrightarrow C^{l,\alpha}(\Omega)$  for  $0 \leq \alpha < 1$  and those embeddings are compact.

**Corollary 1.5.** Let  $\Omega \subseteq \mathbb{R}^n$  of class  $C^1$  and  $u \in H^1(\Omega)$ . Moreover, assume that  $u \in H^k(\Omega)$  for some  $k > \frac{n}{2} + 2$ . Then  $u \in C^2(\Omega)$ .

$p < n$ .

**Theorem 1.17 (Sobolev-Gagliardo-Nirenberg).** Let  $1 \leq p < n$  and let  $p^* := \frac{np}{n-p}$ . Then  $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$  with

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p}.$$

**Theorem 1.18 (Sobolev-Gagliardo-Nirenberg Compactness).** Let  $\Omega \subseteq \mathbb{R}^n$  and  $1 \leq p < n$ . Then  $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$  for  $1 \leq q \leq p^*$  and the embedding is compact if  $q < p^*$ .

$p > n$ .

**Theorem 1.19.** Let  $p > n$ . Then  $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,\alpha}(\mathbb{R}^n)$  with  $\alpha := 1 - \frac{n}{p}$  and

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq \|u\|_{W^{1,p}(\Omega)}.$$

**Remark 1.4.** For  $p = \infty$ , the statement is trivially true, since any function in  $W^{1,\infty}(\mathbb{R}^n)$  is Lipschitz continuous since  $\mathbb{R}^n$  is convex, and thus belongs to  $C^{0,1}(\mathbb{R}^n)$ .

The proof uses the notion of so-called *Campanato spaces*.

**Theorem 1.20.** Let  $\Omega \subseteq \mathbb{R}^n$  of type  $A$  for some  $A > 0$  and  $1 \leq p < \infty$ ,  $\lambda > n$ ,  $\alpha := \frac{\lambda-n}{p}$ . Then

$$\mathcal{L}^{p,\lambda}(\Omega) \cong C^{0,\alpha}(\bar{\Omega}).$$

*Proof.* The inclusion  $\mathcal{L}^{p,\lambda}(\Omega) \cong C^{0,\alpha}(\bar{\Omega})$  follows from the Campanato-theorem and does also hold for general  $\Omega \subseteq \mathbb{R}^n$  open.  $\square$

**Lemma 1.6.** Let  $u \in W^{1,p}(\mathbb{R}^n)$ ,  $1 \leq p < \infty$ . Then for all  $x_0 \in \mathbb{R}^n$  and  $r > 0$  we have that

$$\|u - u_{x_0,r}\|_{L^p(B_r(x_0))}^p \leq C r^p \|\nabla u\|_{L^p(B_r(x_0))}^p.$$

*Proof.* This is an application of the Poincaré-Wirtinger inequality 1.21 since without loss of generality, we may assume  $x_0 = 0$  and  $r = 1$ .  $\square$

Now the proof of the Sobolev embedding theorem for  $p > n$  is immediaty by considering

$$W^{1,p}(\mathbb{R}^n) \xrightarrow{\text{P.W.}} \mathcal{L}^{p,p}(\mathbb{R}^n) \xrightarrow{\text{Campanato}} C^{0,\alpha}(\mathbb{R}^n)$$

and observing that  $\mathbb{R}^n$  is of type  $\frac{\pi^{n/2}}{\Gamma(n/2+1)} > 0$ .

**Theorem 1.21 (Poincaré-Wirtinger Inequality).** *Let  $\Omega \subseteq \mathbb{R}^n$  connected and of class  $C^1$  and  $1 \leq p < \infty$ . Then there exists  $C \geq 0$  such that*

$$\|u - \bar{u}\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}$$

*holds for all  $u \in W^{1,p}(\Omega)$ .*

*Proof.* Towards a contradiction, assume that for any  $C \geq 0$  there exists  $u \in W^{1,p}(\Omega)$  such that

$$\|u - \bar{u}\|_{L^p(\Omega)} > C \|\nabla u\|_{L^p(\Omega)}.$$

In particular, there exists a sequence  $u_k$  in  $W^{1,p}(\Omega)$ , such that

$$\|u_k - \bar{u}_k\|_{L^p(\Omega)} > k \|\nabla u_k\|_{L^p(\Omega)}$$

holds for each  $k \in \omega$ ,  $k \geq 1$ . Defining  $v_k := u_k - \bar{u}_k$  and normalizing, i.e. setting  $w_k := v_k / \|v_k\|_{L^p(\Omega)}$  (this is valid since  $\|v_k\|_{L^p(\Omega)} > 0$ ), yields a sequence  $w_k$  in  $W^{1,p}(\Omega)$  such that

$$\bar{w}_k = 0, \quad \|w_k\|_{L^p(\Omega)} = 1 \quad \text{and} \quad \|\nabla w_k\|_{L^p(\Omega)} \rightarrow 0$$

for any  $k \in \omega$ ,  $k \geq 1$ . Using the Sobolev embedding theorem 1.16, we get

$$W^{1,p}(\Omega) \hookrightarrow W^{1,n}(\Omega) \hookrightarrow L^p(\Omega) \quad \text{and} \quad W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$

if  $p \geq n$  and  $p < n$ , respectively. Moreover, those are compact embeddings. Thus since  $w_k$  is bounded in  $W^{1,p}(\Omega)$ , we have that  $w_{k_i} \rightarrow w$  in  $L^p(\Omega)$  for a subsequence  $w_{k_i}$  of  $w_k$ . Moreover,  $\nabla w = 0$ . Indeed, for any  $\varphi \in C_c^\infty(\Omega)$  we compute

$$\int_{\Omega} w \nabla \varphi = \lim_{i \rightarrow \infty} \int_{\Omega} w_{k_i} \nabla \varphi = - \lim_{i \rightarrow \infty} \int_{\Omega} \nabla w_{k_i} \varphi = 0.$$

By the constancy lemma we therefore conclude that  $w = c \in \mathbb{R}$  a.e. But

$$\bar{w} = \frac{1}{|\Omega|} \int_{\Omega} w = \frac{1}{|\Omega|} \lim_{i \rightarrow \infty} \int_{\Omega} w_{k_i} = 0$$

implies  $w = 0$  a.e. contradicting

$$\|w\|_{L^p(\Omega)} = \lim_{i \rightarrow \infty} \|w_{k_i}\|_{L^p(\Omega)} = 1.$$

□

**$p = n$ .**

**Theorem 1.22.** *It holds that  $W^{1,n}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$  for  $n \leq p < \infty$ .*



## Regularity Theory

Goal of this section is to prove the following regularity result.

**Theorem 1.23 (Global Regularity).** *Let  $\Omega \subseteq \mathbb{R}^n$  of class  $C^{k+2}$  and  $f \in H^k(\Omega)$  for some  $k \in \omega$ . Moreover, let  $u \in H_0^1(\Omega)$  be the unique solution of the homogenous Dirichlet boundary value problem*

$$\begin{cases} A_0 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where  $a^{ij} \in C^{k+1}(\bar{\Omega})$ . Then  $u \in H^{k+2}(\Omega)$  and

$$\|u\|_{H^{k+2}(\Omega)} \leq C \|f\|_{H^k(\Omega)}.$$

### Interior Regularity.

**Theorem 1.24.** *Let  $\Omega \subseteq \mathbb{R}^n$  of class  $C^1$  and  $L$  an elliptic operator in divergence form satisfying  $a^{ij} \in C^{k+1}(\bar{\Omega})$ . If  $f \in H^k(\Omega)$ , the unique weak solution  $u \in H_0^1(\Omega)$  of the homogenous Dirichlet boundary value problem*

$$\begin{cases} A_0 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

belongs to  $H_{\text{loc}}^{k+2}(\Omega)$  and for all  $\Omega' \subseteq \subseteq \Omega$  we have the estimate

$$\|u\|_{H^{k+2}(\Omega')} \leq C(\|f\|_{H^k(\Omega)} + \|u\|_{H^1(\Omega)}).$$

*Proof. Step 1:  $k = 0$ .*

(a) *A-priori Estimates.* First of all, we are assuming that  $u \in H_{\text{loc}}^2(\Omega)$ .

(i)  *$H^1$ -Estimate.* Choose a bump function  $\varphi \in C_c^\infty(\Omega)$  supported in  $\Omega'$ . Thus the weak formulation yields by plugging in the test function  $\varphi^2 u$

$$\int_{\Omega} \varphi^2 a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} + 2 \int_{\Omega} u \varphi a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial \varphi}{\partial x^j} = \int_{\Omega} f \varphi^2 u. \quad (4)$$

Rearranging formula (4) we compute

$$\begin{aligned} \int_{\Omega} \varphi^2 a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} &= \int_{\Omega} f \varphi^2 u - 2 \int_{\Omega} u \varphi a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial \varphi}{\partial x^j} \\ &\leq \|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 + 2\Lambda \int_{\Omega} (-u) \varphi |\nabla u| |\nabla \varphi| \\ &\leq \|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 + \Lambda \varepsilon \|\varphi \nabla u\|_{L^2(\Omega)}^2 + \frac{\Lambda}{\varepsilon} \|u \nabla \varphi\|_{L^2(\Omega)}^2 \end{aligned}$$

Noticing that

$$\int_{\Omega} \varphi^2 a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \geq \lambda \|\varphi \nabla u\|_{L^2(\Omega)}^2$$

yields

$$(\lambda - \Lambda\varepsilon) \|\varphi \nabla u\|_{L^2(\Omega)}^2 \leq \|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 + \frac{\Lambda}{\varepsilon} \|u \nabla \varphi\|_{L^2(\Omega)}^2$$

Picking  $\varepsilon > 0$  appropriately, yields

$$\|\varphi \nabla u\|_{L^2(\Omega)}^2 \leq C(\|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2)$$

and thus

$$\|\nabla u\|_{L^2(\Omega')}^2 \leq C(\|f\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2).$$

(ii)  $H^2$ -Estimate. Let  $1 \leq \mu \leq n$ . Then  $\partial_\mu u$  solves

$$\int_{\Omega} a^{ij} \partial_i \partial_\mu u \partial_j \varphi = - \int_{\Omega} f \partial_\mu \varphi - \int_{\Omega} \partial_\mu a^{ij} \partial_i u \partial_j \varphi$$

for all  $\varphi \in C_c^\infty(\Omega)$ . Now perform the  $H^1$ -estimate on  $\partial_\mu u$ .

(b) *Existence: The Nirenberg-Trick.* The trick is to use difference quotients

$$D_h u := \frac{\tau_h u - u}{|h|}$$

for  $h \in \mathbb{R}^n$  such that  $|h| < \text{dist}(\Omega', \partial\Omega)$ . The idea now is to find a PDE solved by  $D_h u$  in the weak sense and to use the characterization of the Sobolev space.

*Step 2: Induction Step.*

□

## Boundary Regularity.

**Proposition 1.8 (Minimality Property).** *Let  $\Omega \subseteq \mathbb{R}^n$ . Then  $u \in H_0^1(\Omega)$  solves (1) if and only if the **energy functional** satisfies*

$$E(u) := \frac{1}{2} \|u\|_a^2 - \int_{\Omega} f u = \inf_{v \in H_0^1(\Omega)} E(v).$$

*Proof.* Suppose  $u \in H_0^1(\Omega)$  solves (1) and let  $v \in H_0^1(\Omega)$ . Then  $v = u + \varphi$  for some  $\varphi \in H_0^1(\Omega)$  and we compute

$$E(v) = E(u + \varphi) = \frac{1}{2} \|u\|_a^2 + \langle u, \varphi \rangle_a + \frac{1}{2} \|\varphi\|_a^2 - \int_{\Omega} f(u + \varphi) = E(u) + \frac{1}{2} \|\varphi\|_a^2 \geq E(u).$$

Conversely, suppose  $u_0 \in H_0^1(\Omega)$  is a minimizer of the energy functional. Thus by elementary calculus

$$\left. \frac{d}{dt} \right|_{t=0} E(u_0 + tv) = 0$$

for all  $v \in H_0^1(\Omega)$ . But

$$\left. \frac{d}{dt} \right|_{t=0} E(u_0 + tv) = \langle u_0, v \rangle_a - \int_{\Omega} f v.$$

□

### Eigenfunctions of $-\Delta$ .

**Theorem 1.25.** *Let  $\Omega \subseteq \mathbb{R}^n$  of class  $C^2$ . Then there exists a Hilbert-space basis  $(\varphi_i)_{i \in \omega}$  of  $L^2(\Omega)$  consisting of eigenfunctions of the Laplace operator, i.e.*

$$\begin{cases} -\Delta \varphi_i = \lambda_i \varphi_i & \text{in } \Omega, \\ \varphi_i = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover  $0 < \lambda_i \rightarrow \infty$  are called **Dirichlet eigenvalues**.

*Proof.* Define  $K : L^2(\Omega) \rightarrow L^2(\Omega)$  by setting  $Kf$  to be the unique weak solution of the homogenous Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By the global regularity theorem,  $u \in H^2(\Omega)$  and thus we can write  $K$  as the composition

$$L^2(\Omega) \longrightarrow H^2(\Omega) \hookrightarrow L^2(\Omega).$$

Thus  $K$  is continuous as a composition of continuous mappings and moreover, since the embedding is compact by the Sobolev theorem, so is  $K$ . □

### Schauder Theory

#### Campanato-Estimates and Morrey-Spaces.

**Lemma 1.7 (Minimality of Mean-Value).** *Let  $\Omega \subseteq \mathbb{R}^n$  open,  $f \in L^2(\Omega)$ ,  $x_0 \in \Omega$  and  $r > 0$ . Then*

$$\|f - \bar{f}_{r,x_0}\|_{L^2(\Omega_r(x_0))}^2 = \min_{a \in \mathbb{R}} \|f - a\|_{L^2(\Omega_r(x_0))}^2.$$

#### Schauder Estimates.

**Theorem 1.26 (Global Schauder-Estimate).** *Let  $\Omega \subseteq \mathbb{R}^n$  of class  $C^{2,\alpha}$ ,  $0 < \alpha < 1$ . Moreover, let  $a^{ij} \in C^{1,\alpha}(\Omega)$  symmetric, uniformly elliptic and uniformly bounded,  $c \in C^\alpha(\Omega)$ ,  $u_0 \in C^{2,\alpha}(\bar{\Omega})$ ,  $f = (f^1, \dots, f^n) \in C^{1,\alpha}(\Omega)$  and  $h \in C^\alpha(\Omega)$ . Then any solution  $u \in C^{2,\alpha}(\Omega)$  of the Dirichlet boundary value problem*

$$\begin{cases} A_0 u + cu = -\frac{\partial}{\partial x^i} f^i + h & \text{in } \Omega, \\ u = u_0 & \text{on } \Omega \end{cases}$$

satisfies

$$\|u\|_{C^{2,\alpha}} \leq C(\|u\|_{H^1} + \|f\|_{C^{1,\alpha}} + \|h\|_{C^\alpha} + \|u_0\|_{C^{2,\alpha}})$$

where  $C$  does not depend on  $u$ .

## Existence Theorems.

**Proposition 1.9 (Method of Continuity).** *Let  $(X, \|\cdot\|_X)$  and  $(Y, \|\cdot\|_Y)$  be Banach spaces. Given  $A_0, A_1 \in \mathcal{L}(X, Y)$  define  $A_t := (1 - t)A_0 + tA_1$ ,  $t \in [0, 1]$ . Suppose that*

$$\exists C > 0 \forall t \in [0, 1] \forall x \in X : \|x\|_X \leq \|A_t x\|_Y.$$

*Then  $A_0$  is surjective if and only if  $A_1$  is surjective.*

Using the method of continuity 1.9, one can show existence results of solutions of Dirichlet boundary value problems. Define

$$A_0 := -\frac{\partial}{\partial x^i} \left( a^{ij} \frac{\partial}{\partial x^j} \right)$$

for  $a^{ij} \in C^{1,\alpha}$  symmetric, uniformly elliptic and uniformly bounded. Consider the problem

$$\begin{cases} A_0 u + cu = -\frac{\partial}{\partial x^j} f^j + h & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

for  $c \in C^\alpha$ ,  $f = (f^1, \dots, f^n) \in C^{1,\alpha}$  and  $h \in C^\alpha$ . If  $c \geq 0$ , one can show existence and uniqueness of  $C^{2,\alpha}$  solutions. First of all, suppose that solutions of

$$\begin{cases} A_0 u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

do exist. Let us define

$$X := \{u \in C^{2,\alpha} : u|_{\partial\Omega} = 0\} \quad \text{and} \quad Y := C^\alpha.$$

Then  $X$  and  $Y$  are Banach spaces, since  $X$  is a closed subset of a Banach space. Define now  $A_1 := A_0 + c$ . Then it is easy to show that  $A_0$  and  $A_1$  are continuous. Thus to apply the continuity method, we have to show the existence of a constant  $C > 0$ , such that for all  $t \in [0, 1]$  and  $u \in X$

$$\|x\|_{C^{2,\alpha}} \leq \|A_t x\|_{C^\alpha}$$

holds. But this looks like the Schauder-estimate 1.26. Indeed, since  $u \in C^{2,\alpha}$  solves  $A_t u = A_t u$ , we get

$$\|u\|_{C^{2,\alpha}} \leq C(\|u\|_{H^1} + \|A_t u\|_{C^\alpha}).$$

Using ellipticity, integration by parts (justified since any function in  $X$  vanishes on the boundary  $\partial\Omega$ ) and  $c \geq 0$ , we compute

$$\begin{aligned} \lambda \|u\|_{H^1}^2 &= \lambda \int_{\Omega} |\nabla u|^2 \\ &\leq \int_{\Omega} a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega} (A_0 u) u \\
&= \int_{\Omega} (A_0 u) u + ct u^2 - ct u^2 \\
&= \int_{\Omega} (A_t u) u - ct u^2 \\
&\leq \int_{\Omega} (A_t u) u \\
&\leq \|A_t u\|_{L^2} \|u\|_{L^2} \\
&\leq C \|A_t u\|_{C^\alpha} \|u\|_{H^1}.
\end{aligned}$$

### Maximum Principle

**Weak Maximum Principle.** Let  $\Omega \subseteq \mathbb{R}^n$ . In what follows, we consider the second order homogenous differential operator in non-divergence form

$$Lu := a^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + b^i \frac{\partial u}{\partial x^i} + cu$$

where  $a^{ij}, b^i, c \in C^0(\bar{\Omega})$  and  $L$  is uniformly elliptic, i.e. there exists  $\lambda > 0$  such that

$$a^{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2$$

holds for all  $\xi \in \mathbb{R}^n$  and  $x \in \Omega$ .

**Theorem 1.27 (Weak Maximum Principle).** Let  $\Omega \subseteq \mathbb{R}^n$  and  $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$  such that  $Lu \geq 0$ . Then:

- (a) If  $c \leq 0$  in  $\Omega$ , then  $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u_+$ .
- (b) If  $c = 0$  in  $\Omega$ , then  $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$ .

*Proof.* Consider the perturbation  $v_\varepsilon := u + \varepsilon e^{\gamma x_1}$  for  $\varepsilon, \gamma > 0$  and use the first and second derivative test.  $\square$

### Strong Maximum Principle.

**Lemma 1.8 (Boundary Point Lemma, E. Hopf).** Let  $B := B_\rho(y) \subseteq \mathbb{R}^n$  and  $u \in C^2(B) \cap C^0(\bar{B})$  such that  $Lu \geq 0$  in  $B$  with  $c \leq 0$ . Assume for some  $x_0 \in \partial B$  that  $u(x_0) \geq 0$  and  $u(x) < u(x_0)$  for every  $x \in B$ . Then

$$\limsup_{h \rightarrow 0} \frac{u(x_0 + h\eta) - u(x_0)}{h} < 0$$

for  $\eta$  the inward pointing unit normal at  $x_0$ .

*Proof.* Without loss of generality one can assume  $\rho = 1$  and  $y = 0$ . Then define  $w : \bar{B} \rightarrow \mathbb{R}$  by

$$w(x) := e^{-\alpha|x|^2} - e^{-\alpha}$$

for some  $\alpha > 0$  to be determined. We compute

$$Lw \geq e^{-\alpha|x|^2} (4\mu|x|^2 \alpha^2 - 2\alpha(\operatorname{tr} A + b^i x_i) + c).$$

Thus for some  $\alpha$  large enough, we get that  $Lw > 0$ . Set

$$v := u - u(x_0) + \varepsilon w$$

for some  $\varepsilon > 0$  on the annulus  $A := \bar{B}_1(0) \setminus B_{1/2}(0)$ . For  $\varepsilon > 0$  sufficiently small, we get that  $v \leq 0$  on  $\partial A$ . Since moreover

$$Lv = Lu - cu(x_0) + \varepsilon Lw > 0$$

the weak maximum principle implies  $v \leq 0$  on  $A$ . Hence  $D_\eta^+ v \leq 0$ , but

$$D_\eta^+ v = D_\eta^+ u + \varepsilon D_\eta^+ w$$

which yields the statement by observing that  $D_\eta^+ w > 0$ . □

### References

- [Lee13] John M. Lee. *Introduction to Smooth Manifolds*. Second Edition. Graduate Texts in Mathematics. Springer, 2013.