THE WHITEHEAD PRODUCT

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Abstract. Aim of this paper is to give a short overview of the definition and the basic properties of the non-generalized *Whitehead product*.

Contents

1. Introduction

In the category of *compactly generated spaces*, suppose G is an H-group, i.e. a space satisfying the group axioms up to homotopy, then $[X, G]$ is a group for any space X. This group need not be abelian. Thus a natural question is, if $[X, G]$ is *nilpotent*. As the notion of nilpotence is based on the behaviour of *commutators*, it is natural to consider certain related products: First of all the *commutator product* or *Samelson product* defined as follows: If $[\alpha] \in [X, G]$ and $[\beta] \in [Y, G]$, define $\gamma : X \times Y \to G$ by

$$\gamma(x, y) := \alpha(x)\beta(y)\left(\alpha(x)\right)^{-1}\left(\beta(y)\right)^{-1}.$$

Then $\gamma|_{X\vee Y}$ is nullhomotopic and thus yields a map $\gamma : X \wedge Y \to G$, whose homotopy class is defined to be the product of $[\alpha]$ and $[\beta]$. When $X = \mathbb{S}^n$, $Y = \mathbb{S}^m$ and $G = \Omega X$, then $[\mathbb{S}^n, G]$ is identified with $\pi_n(G)$ since the π_1 action is trivial for H-spaces, and the Samelson product

$$\pi_n(G) \otimes \pi_m(G) \to \pi_{n+m}(G)$$

translates to a pairing

$$\pi_{n+1}(X) \otimes \pi_{m+1}(X) \to \pi_{n+m+1}(X),$$

the *Whitehead product*, since $\pi_n(G) \cong \pi_{n+1}(X)$ (see [Whi78, pp. 456–457]).

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2. Definition of the Whitehead Product

Notice, that for any $(X, x_0), (Y, y_0) \in \mathsf{Top}_*$, their coproduct is given by

$$X \coprod Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subseteq X \times Y,$$

with basepoint (x_0, y_0).

Lemma 2.1. *Let $n, m \in \omega$, $n, m \geq 1$. The space $\mathbb{S}^n \times \mathbb{S}^m$ can be obtained from $\mathbb{S}^n \vee \mathbb{S}^m$ by attaching an $n + m$-cell.*

Proof. Observe, that $\mathbb{D}^{n+m} \cong \mathbb{D}^n \times \mathbb{D}^m$ and hence

$$\mathbb{S}^{n+m-1} = \partial\mathbb{D}^{n+m} \cong (\partial\mathbb{D}^n \times \mathbb{D}^m) \cup (\mathbb{D}^n \times \partial\mathbb{D}^m) = (\mathbb{S}^{n-1} \times \mathbb{D}^m) \cup (\mathbb{D}^n \times \mathbb{S}^{m-1}).$$

Let

$$f_1 : \mathbb{S}^{n-1} \times \mathbb{D}^m \to (\mathbb{S}^{n-1} \times \mathbb{D}^m)/(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \cong * \times \mathbb{S}^m$$

and

$$f_2 : \mathbb{D}^n \times \mathbb{S}^{m-1} \to (\mathbb{D}^n \times \mathbb{S}^{m-1})/(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \cong \mathbb{S}^n \times *$$

be the quotient maps. An application of the gluing lemma thus yields a map

$$f : \mathbb{S}^{n+m-1} \to \mathbb{S}^n \vee \mathbb{S}^m.$$

Moreover, define

$$q : \mathbb{D}^n \times \mathbb{D}^m \to \mathbb{D}^n/\mathbb{S}^{n-1} \times \mathbb{D}^m/\mathbb{S}^{m-1} \cong \mathbb{S}^n \times \mathbb{S}^m$$

to be the product of the two quotient maps

$$\mathbb{D}^n \to \mathbb{D}^n/\mathbb{S}^{n-1} \qquad \text{and} \qquad \mathbb{D}^m \to \mathbb{D}^m/\mathbb{S}^{m-1}.$$

Hence by [Mun00, p. 186], q itself is a quotient map, and we get a commutative diagram

$$\begin{tikzcd}
\mathbb{S}^{n+m-1} \arrow[r, "f"] \arrow[d, hook] & \mathbb{S}^n \vee \mathbb{S}^m \arrow[d, hook] \\
\mathbb{D}^{n+m} \arrow[r, "q"'] & \mathbb{S}^n \times \mathbb{S}^m
\end{tikzcd}$$

Suppose (X, g, h) is another cocone in Top for the pushout diagram:

$$\begin{tikzcd}
\mathbb{S}^{n+m-1} \arrow[r, "f"] \arrow[d, hook] & \mathbb{S}^n \vee \mathbb{S}^m \arrow[d, hook] \arrow[ddr, bend left, "h"] & \\
\mathbb{D}^{n+m} \arrow[r, "q"'] \arrow[drr, bend right, "g"'] & \mathbb{S}^n \times \mathbb{S}^m & \\
& & X.
\end{tikzcd}$$

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Then g is constant on the fibers of q. Indeed, we have that $q(x, y) = q(x', y)$ for all $x, x' \in \mathbb{S}^{n-1}$ and $y \in \mathbb{D}^m$, as well as $q(x, y) = q(x, y')$ for all $x \in \mathbb{D}^n$ and $y, y' \in \mathbb{S}^{m-1}$. We compute

$$g(x, y) = (h \circ f)(x, y) = (h \circ f)(x', y) = g(x', y),$$

and similarly for the other case. Thus g passes to the quotient by [Lee11, p. 72] to yield a unique map

$$\widetilde{g} : \mathbb{S}^n \times \mathbb{S}^m \to X,$$

such that $g = \widetilde{g} \circ q$. Finally, it is easy to check that

$$\begin{tikzcd}
\mathbb{S}^{n+m-1} \arrow[r, "f"] \arrow[d, hook] & \mathbb{S}^n \vee \mathbb{S}^m \arrow[d, hook] \arrow[ddr, bend left, "h"] & \\
\mathbb{D}^{n+m} \arrow[r, "q"'] \arrow[drr, bend right, "g"'] & \mathbb{S}^n \times \mathbb{S}^m \arrow[dr, "\widetilde{g}"] & \\
 & & X.
\end{tikzcd}$$

commutes. □

For $n, m \in \omega$, $n, m \geq 1$, consider the map f from lemma 2.1 and note that this is actually a pointed map. Let $(X, p) \in \mathsf{Top}_*$. If $[\alpha] \in \pi_n(X, p)$ and $[\beta] \in \pi_m(X, p)$, we get two pointed maps

$$\alpha : (\mathbb{S}^n, *) \to (X, p) \qquad \text{and} \qquad \beta : (\mathbb{S}^m, *) \to (X, p).$$

Forming their wedge $\alpha \vee \beta : (\mathbb{S}^n \vee \mathbb{S}^m, (*, *)) \to (X, p)$, defined by

$$(\alpha \vee \beta)(x, y) := \begin{cases} \alpha(x) & y = *, \\ \beta(y) & x = *, \end{cases}$$

and precomposing with f, yields a pointed map

$$(\alpha \vee \beta) \circ f : (\mathbb{S}^{n+m-1}, *) \to (X, p).$$

Explicitely, if we consider

$$\alpha : (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (X, p) \qquad \text{and} \qquad \beta : (\mathbb{D}^m, \mathbb{S}^{m-1}) \to (X, p),$$

we get that

$$\big((\alpha \vee \beta) \circ f\big)(x, y) = \begin{cases} \alpha(x) & x \in \mathbb{D}^n, y \in \mathbb{S}^{m-1}, \\ \beta(y) & x \in \mathbb{S}^{n-1}, y \in \mathbb{D}^m. \end{cases} \tag{1}$$

Hence if $F : \alpha \simeq_{\mathbb{S}^{n-1}} \alpha'$ and $F' : \beta \simeq_{\mathbb{S}^{m-1}} \beta'$, we get that

$$H : ((\alpha \vee \beta) \circ f) \simeq_* ((\alpha' \vee \beta') \circ f),$$

where $H : \mathbb{S}^{n+m-1} \times I \to X$ is defined by

$$H(x, y, t) := \begin{cases} F(x, t) & x \in \mathbb{D}^n, y \in \mathbb{S}^{m-1}, \\ F'(y, t) & x \in \mathbb{S}^{n-1}, y \in \mathbb{D}^m. \end{cases}$$

Thus we get a well defined map $[-, -] : \pi_n(X) \times \pi_m(X) \to \pi_{n+m-1}(X)$, defined by

$$[\alpha, \beta] := [(\alpha \vee \beta) \circ f].$$

Definition 2.1 (Whitehead Product). *Let $n, m \in \omega$, $n, m \geq 1$, and $(X, p) \in \mathsf{Top}_*$. The product*

$$[-, -] : \pi_n(X, p) \times \pi_m(X, p) \to \pi_{n+m-1}(X, p)$$

defined by

$$[\alpha, \beta] := [(\alpha \vee \beta) \circ f],$$

is called the ***Whitehead product*** *and $[-, -]$ is called the* ***Whitehead bracket****.*

3. The Whitehead Product and the Conjugation Action

In this section, we want to have a closer look at $[-, -] : \pi_1(X) \times \pi_n(X) \to \pi_n(X)$. If $n = 1$, the definition of the Whitehead product in equation (1) results in figure 1a and using that $\mathbb{S}^1$ is parametrized by $\theta \mapsto e^{i\theta}$, i.e. oriented counter clockwise, we get that

$$[\alpha, \beta] = [\alpha][\beta][\alpha]^{-1}[\beta]^{-1},$$

since any reparametrization of a path is homotopic relative to ∂I to the original path (a reparametrization of a path f in X is just a path $f \circ \varphi$, where $\varphi : I \to I$ is continuous and $\varphi|_{\partial I} = \mathrm{id}_{\partial I}$). Thus $[\alpha, \beta]$ coincides with the notation of a commutator in $\pi_1(X)$.
Let $n > 1$. Let us briefly introduce the notion of *cellular homology*. If X is a cell complex with skeleton filtration $\mathscr{F} : \varnothing =: X^{-1} \subseteq X^0 \subseteq \cdots \subseteq X$, we have for all $n \in \omega$ that $H_n(X^n, X^{n-1})$ is free abelian with a basis in one-to-one correspondence with the n-cells if X. Define a chain complex $C_\bullet(X, \mathscr{F}) \in \mathsf{Ch}_{\geq 0}({}_{\mathbb{Z}}\mathsf{Mod})$, the ***cellular chain complex of X***, by

$$C_n(X, \mathscr{F}) := H_n(X^n, X^{n-1}),$$

for all $n \in \omega$. Moreover, for $n \geq 1$, define $\partial_n : C_n(X, \mathscr{F}) \to C_{n-1}(X, \mathscr{F})$ to be the composition

$$H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2}).$$

Following [Hat01, p. 269], for $n \geq 1$, define an ***orientation of $\mathbb{D}^n$*** to be a choice of a generator of $H_n(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong \mathbb{Z}$ ($\mathbb{D}^n$ can be obtained from $\mathbb{S}^{n-1}$ by attaching a single n-cell). In what follows, we fix an orientation of $\mathbb{D}^n$, say e^n. Moreover, we can also fix an orientation of $\mathbb{D}^0 = \{*\}$, since there is a rather canonical choice: Just take it to be the single 0-cell $*$.
By [Hat01, p. 269], the boundary map in the cellular chain complex $C_\bullet(X \times Y, \mathscr{F}_{X \times Y})$,

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for another cell complex Y, is determined by the boundary maps in the cellular chain complexes $C_\bullet(X, \mathcal{F}_X)$ and $C_\bullet(Y, \mathcal{F}_Y)$ via the formula

$$\partial(e^n \times e^m) = \partial e^n \times e^m + (-1)^n e^n \times e^m. \tag{2}$$

If e^1 denotes the single 1-cell in I and e^n the single n-cell in $\mathbb{D}^n$, we thus obtain

$$\partial(e^1 \times e^n) = 1 \times e^n - 0 \times e^n - e^1 \times \partial e^n.$$

from formula (2). Now the definition of the Whitehead product in equation (1) results in figure 1b. Thus using lemma 3.1 below yields

$$[\alpha, \beta] = [\alpha \cdot \beta] - [\beta],$$

where $\alpha \cdot \beta$ denotes the *conjugation action*, i.e. the action of $\pi_1(X)$ on $\pi_n(X)$, since the boundary of the cylinder $I \times \mathbb{D}^n$ is oriented coherently with $1 \times \mathbb{D}^n$ and discoherently with $0 \times \mathbb{D}^n$, as above calculation suggests.

Lemma 3.1. *Let $n \in \omega$, $n > 1$, $[\alpha] \in \pi_n(X)$ and $h : (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (\mathbb{D}^n, \mathbb{S}^{n-1})$ an orientation reversing homeomorphism, i.e. h is a homeomorphism and $H_n(h)e^n = -e^n$. Then $[\alpha \circ h] = -[\alpha]$.*

Proof. Following [Whi78, p. 166], let $\rho : \pi_n(Y, A) \to H_n(Y, A)$ denote the ***Hurewicz homomorphism*** defined by

$$\rho[f] := H_n(f)e^n,$$

where e^n denotes an orientation of $\mathbb{D}^n$. Using that

$$\rho : \pi_n(\mathbb{D}^n, \mathbb{S}^{n-1}) \to H_n(\mathbb{D}^n, \mathbb{S}^{n-1})$$

is an isomorphism for $n > 1$ (see [Whi78, p. 168]), we compute

$$\begin{aligned}
[\alpha \circ h] &= \pi_n(\alpha)[h] \\
&= \pi_n(\alpha)\rho^{-1}\rho[h] \\
&= \pi_n(\alpha)\rho^{-1}(H_n(h)e^n) \\
&= -\pi_n(\alpha)\rho^{-1}e^n \\
&= -\pi_n(\alpha)[\mathrm{id}_{\mathbb{D}^n}] \\
&= -[\alpha].
\end{aligned}$$

□

In the above argument, We implicitely used the observation that the sum $[\alpha] + [\beta]$ in $\pi_n(X)$ is given by the pointed homotopy class of the composition

$$\mathbb{S}^n \xrightarrow{c} \mathbb{S}^n \vee \mathbb{S}^n \xrightarrow{\alpha \vee \beta} X,$$

where $c : \mathbb{S}^n \to \mathbb{S}^n \vee \mathbb{S}^n$ denotes the mapping which collapses the equatorial $\mathbb{S}^{n-1}$ in $\mathbb{S}^n$ to a point, depicted in figure 2 (see [Hat01, p. 341]).

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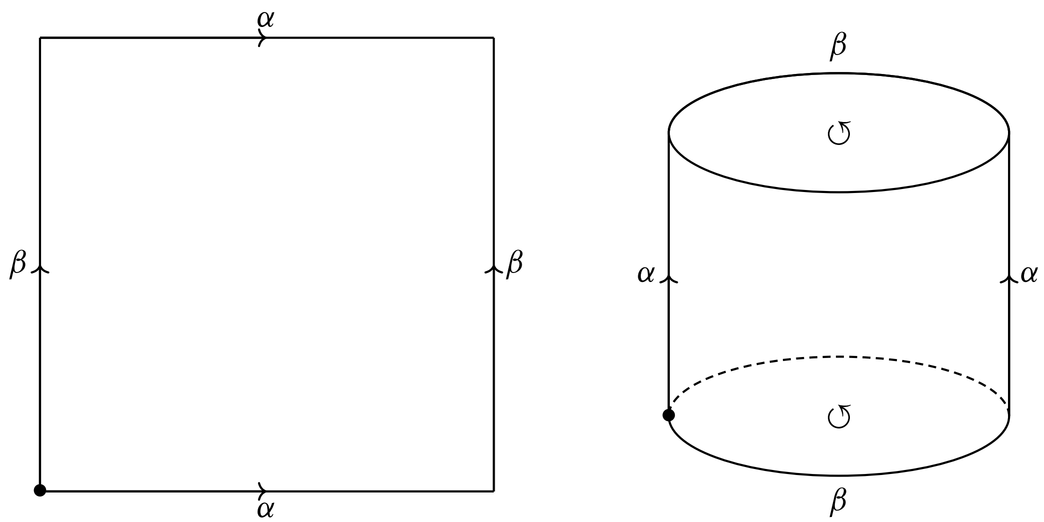

(A) $n = 1$.

(B) $n > 1$.

FIGURE 1. Whitehead bracket and the conjugation action.

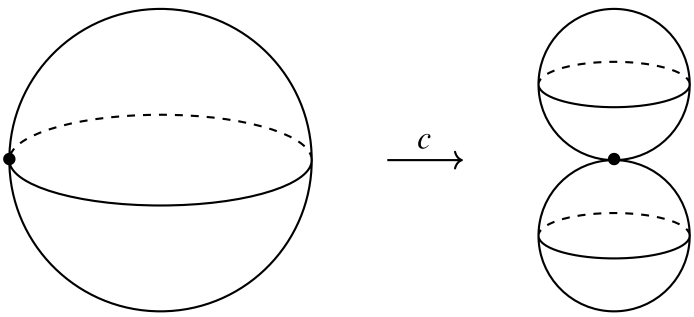

FIGURE 2. The collapsing map $c : \mathbb{S}^n \to \mathbb{S}^n \vee \mathbb{S}^n$.

4. Grading

Let $(X, p) \in \mathsf{Top}_*$. For $n \in \omega$ let $L^n := \pi_{n+1}(X, p)$ and define

$$L := \bigoplus_{n \in \omega} L^n.$$

Moreover, define $[-,-] : L \times L \to L$ by

$$\left[\sum_i \alpha_i, \sum_j \beta_j \right] := \sum_{i,j} [\alpha_i, \beta_j].$$

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Then clearly $L^n L^m \subseteq L^{n+m}$ holds. It also turns out, that we have a Lie algebra-like structure on L, i.e. the bracket is bilinear, alternating and there is a Jacobi identity (for more details see [Whi78, pp. 474–478]).

Proposition 4.1. *Let $n, m \in \omega$, $n \geq 1$, $[\alpha_1], [\alpha_2] \in \pi_{n+1}(X)$ and $[\beta] \in \pi_{m+1}(X)$. Then*

$$[\alpha_1 + \alpha_2, \beta] = [\alpha_1, \beta] + [\alpha_2, \beta] \qquad \text{and} \qquad [\beta, \alpha_1 + \alpha_2] = [\beta, \alpha_1] + [\beta, \alpha_2].$$

Recall, that for $n \geq 1$ we have that $H_n(\mathbb{S}^n) \cong \mathbb{Z}$. Thus if we are given any continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$, the induced map $H_n(f)$ is simply a multiplication by a unique integer. This integer is defined to be the ***degree of f***, written $\deg f$. Observe that lemma 3.1 stays true if we consider a homeomorphism $h : (\mathbb{S}^n, *) \to (\mathbb{S}^n, *)$ with $\deg h = -1$. Indeed, this basically follows from the relative homeomorphism theorem which states that

$$H_k(q) : H_k(\mathbb{D}^n, \mathbb{S}^{n-1}) \to H_k(\mathbb{S}^n, *) \cong \widetilde{H}_k(\mathbb{S}^k)$$

is an isomorphism for all $k \in \omega$.

Proposition 4.2. *Let $n, m \in \omega$, $[\alpha] \in \pi_{n+1}(X)$ and $[\beta] \in \pi_{m+1}(X)$. Then*

$$[\beta, \alpha] = (-1)^{(n+1)(m+1)}[\alpha, \beta].$$

Proof. Consider the ***permutation map*** $\sigma : \mathbb{S}^{n+m+1} \to \mathbb{S}^{n+m+1}$ defined by

$$(y_1, \dots, y_{m+1}, x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_{n+1}, y_1, \dots, y_{m+1}).$$

Then clearly $\deg \sigma = (-1)^{(n+1)(m+1)}$, since σ is the composition of permutations and hence orthogonal transformations. An application of lemma 3.1 yields

$$[\beta, \alpha] = [(\beta \vee \alpha) \circ f] = [(\alpha \vee \beta) \circ f \circ \sigma] = (-1)^{(n+1)(m+1)}[\alpha, \beta].$$

$\square$

Proposition 4.3. *Let $n, m, r \in \omega$, $n, m, r \geq 1$, $[\alpha] \in \pi_{n+1}(X)$, $[\beta] \in \pi_{m+1}(X)$ and $[\gamma] \in \pi_{r+1}(X)$. Then*

$$(-1)^{r(n+1)}[\alpha, [\beta, \gamma]] + (-1)^{n(m+1)}[\beta, [\gamma, \alpha]] + (-1)^{m(r+1)}[\gamma, [\alpha, \beta]] = 0$$

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