WHITEHEAD PRODUCT

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Abstract. Aim of this paper is to give a short overview of the definition and the basic properties of the non-generalized *Whitehead product*.

Contents

1. Definition of the Whitehead Product

Notice, that for any $(X, x_0), (Y, y_0) \in \mathsf{Top}_*$, their coproduct is given by

$$X \coprod Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subseteq X \times Y,$$

with basepoint (x_0, y_0).

Lemma 1.1. *Let $n, m \in \omega$, $n, m \geq 1$. The space $\mathbb{S}^n \times \mathbb{S}^m$ can be obtained from $\mathbb{S}^n \vee \mathbb{S}^m$ by attaching an $n + m$-cell.*

Proof. Observe, that $\mathbb{D}^{n+m} \cong \mathbb{D}^n \times \mathbb{D}^m$ and hence

$$\mathbb{S}^{n+m-1} = \partial\mathbb{D}^{n+m} \cong (\partial\mathbb{D}^n \times \mathbb{D}^m) \cup (\mathbb{D}^n \times \partial\mathbb{D}^m) = (\mathbb{S}^{n-1} \times \mathbb{D}^m) \cup (\mathbb{D}^n \times \mathbb{S}^{m-1}).$$

Let

$$f_1 : \mathbb{S}^{n-1} \times \mathbb{D}^m \to (\mathbb{S}^{n-1} \times \mathbb{D}^m)/(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \cong * \times \mathbb{S}^m$$

and

$$f_2 : \mathbb{D}^n \times \mathbb{S}^{m-1} \to (\mathbb{D}^n \times \mathbb{S}^{m-1})/(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \cong \mathbb{S}^n \times *$$

be the quotient maps. An application of the gluing lemma thus yields a map

$$f : \mathbb{S}^{n+m-1} \to \mathbb{S}^n \vee \mathbb{S}^m.$$

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Moreover, define

$$q : \mathbb{D}^n \times \mathbb{D}^m \to \mathbb{D}^n/\mathbb{S}^{n-1} \times \mathbb{D}^m/\mathbb{S}^{m-1} \cong \mathbb{S}^n \times \mathbb{S}^m$$

to be the product of quotient maps. Thus we get a commutative diagram

$$\begin{tikzcd}
\mathbb{S}^{n+m-1} \arrow[r, "f"] \arrow[d, hook] & \mathbb{S}^n \vee \mathbb{S}^m \arrow[d, hook] \\
\mathbb{D}^{n+m} \arrow[r, "q"'] & \mathbb{S}^n \times \mathbb{S}^m
\end{tikzcd}$$

Suppose (X, g, h) is another cocone for the pushout diagram:

$$\begin{tikzcd}
\mathbb{S}^{n+m-1} \arrow[r, "f"] \arrow[d, hook] & \mathbb{S}^n \vee \mathbb{S}^m \arrow[d, hook] \arrow[ddr, bend left, "h"] & \\
\mathbb{D}^{n+m} \arrow[r, "q"'] \arrow[drr, bend right, "g"'] & \mathbb{S}^n \times \mathbb{S}^m & \\
& & X.
\end{tikzcd}$$

By [Mun00, p. 186], q is a quotient map. Moreover, for $(x, y) \in \mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$, we have that

$$g(x, y) = (h \circ f)(x, y) = h(*, *).$$

Thus g passes to the quotient by [Lee11, p. 72] to yield a unique map

$$\tilde{g} : \mathbb{S}^n \times \mathbb{S}^m \to X,$$

such that $g = \tilde{g} \circ q$. Finally, it is easy to check that

$$\begin{tikzcd}
\mathbb{S}^{n+m-1} \arrow[r, "f"] \arrow[d, hook] & \mathbb{S}^n \vee \mathbb{S}^m \arrow[d, hook] \arrow[ddr, bend left, "h"] & \\
\mathbb{D}^{n+m} \arrow[r, "q"'] \arrow[drr, bend right, "g"'] & \mathbb{S}^n \times \mathbb{S}^m \arrow[dr, "\tilde{g}"] & \\
& & X.
\end{tikzcd}$$

commutes. ∎

For $n, m \in \omega$, $n, m \geq 1$, consider the map f from lemma 1.1. Let $(X, p) \in \mathsf{Top}_*$. If $[\alpha] \in \pi_n(X, p)$ and $[\beta] \in \pi_m(X, p)$, we get two pointed maps

$$\alpha : \mathbb{S}^n \to X \qquad \text{and} \qquad \beta : \mathbb{S}^m \to X.$$

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Forming their wedge $\alpha \vee \beta : \mathbb{S}^n \vee \mathbb{S}^m \to X$, defined by

$$(\alpha \vee \beta)(x, y) := \begin{cases} \alpha(x) & y = *, \\ \beta(y) & x = *, \end{cases}$$

and precomposing with f, yields a pointed map

$$(\alpha \vee \beta) \circ f : \mathbb{S}^{n+m-1} \to X.$$

Explicitely, if we consider

$$\alpha : (\mathbb{D}^n, \mathbb{S}^{n-1}) \to (X, p) \qquad \text{and} \qquad \beta : (\mathbb{D}^m, \mathbb{S}^{m-1}) \to (X, p),$$

we get that

$$\big((\alpha \vee \beta) \circ f\big)(x, y) = \begin{cases} \alpha(x) & x \in \mathbb{D}^n, y \in \mathbb{S}^{m-1}, \\ \beta(y) & x \in \mathbb{S}^{n-1}, y \in \mathbb{D}^m. \end{cases}$$

Hence if $F : \alpha \simeq_{\mathbb{S}^{n-1}} \alpha'$ and $F' : \beta \simeq_{\mathbb{S}^{m-1}} \beta'$, we get that

$$H : ((\alpha \vee \beta) \circ f) \simeq_* ((\alpha' \vee \beta') \circ f),$$

where $H : \mathbb{S}^{n+m-1} \times I \to X$ is defined by

$$H(x, y, t) := \begin{cases} F(x, t) & x \in \mathbb{D}^n, y \in \mathbb{S}^{m-1}, \\ F'(y, t) & x \in \mathbb{S}^{n-1}, y \in \mathbb{D}^m. \end{cases}$$

Thus we get a well defined map $[-, -] : \pi_n(X) \times \pi_m(X) \to \pi_{n+m-1}(X)$, defined by

$$[\alpha, \beta] := [(\alpha \vee \beta) \circ f].$$

Definition 1.1 (Whitehead Product). *Let $n, m \in \omega$, $n, m \geq 1$, and $(X, p) \in \mathsf{Top}_*$. The product*

$$[-, -] : \pi_n(X, p) \times \pi_m(X, p) \to \pi_{n+m-1}(X, p)$$

defined by

$$[\alpha, \beta] := [(\alpha \vee \beta) \circ f],$$

*is called the **Whitehead product** and $[-, -]$ is called the **Whitehead bracket**.*

2. The Whitehead Product and the Conjugation Action

In this section, we want to have a closer look at $[-, -] : \pi_1(X) \times \pi_n(X) \to \pi_n(X)$. From figure 1a we immediately deduce that

$$[\alpha, \beta] = [\alpha\beta\alpha^{-1}\beta^{-1}].$$

Thus $[\alpha, \beta]$ coincides with the notation of a commutator.
Let $n > 1$. Then it follows from figure 1b, that

$$[\alpha, \beta] = [\alpha \cdot \beta] - [\beta],$$

where $\alpha \cdot \beta$ denotes the *conjugation action*, i.e. the action of $\pi_1(X)$ on $\pi_n(X)$.

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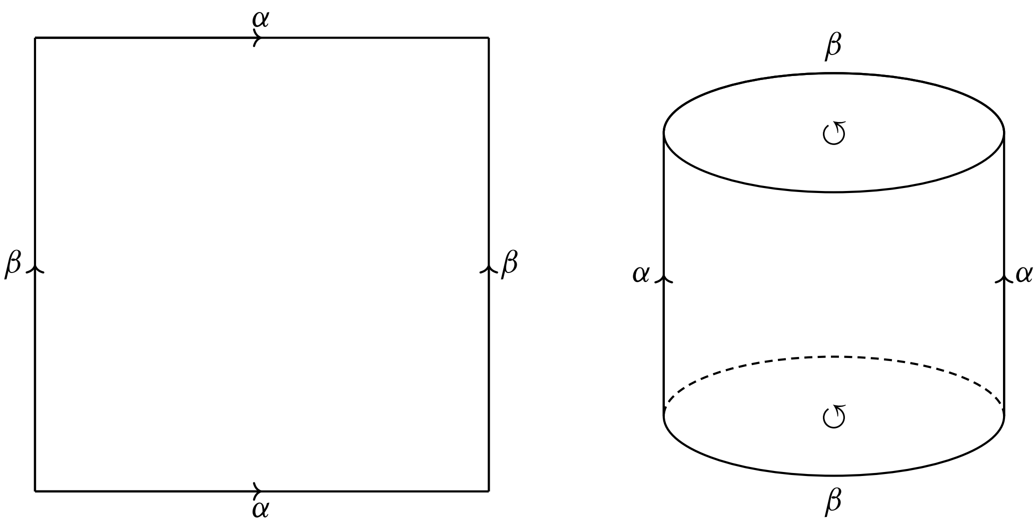

(A) $n = 1$. (B) $n > 1$.

FIGURE 1. Whitehead bracket and the conjugation action.

3. Grading

Let $(X, p) \in \mathsf{Top}_*$. For $n \in \omega$ let $L^n := \pi_{n+1}(X, p)$ and define

$$L := \bigoplus_{n\in\omega} L^n.$$

Moreover, define $[-,-] : L \times L \to L$ by

$$\left[\sum_i \alpha_i, \sum_j \beta_j\right] := \sum_{i,j} [\alpha_i, \beta_j].$$

Then clearly $L^n L^m \subseteq L^{n+m}$ holds. It also turns out, that we have a Lie algebra-like structure on L, i.e. the bracket is bilinear, alternating and there is a Jacobi identity (for more details see [Whi78, pp. 474–478]).

Proposition 3.1. *Let $n, m \in \omega$, $n \geq 1$, $[\alpha_1], [\alpha_2] \in \pi_{n+1}(X)$ and $[\beta] \in \pi_{m+1}(X)$. Then*

$$[\alpha_1 + \alpha_2, \beta] = [\alpha_1, \beta] + [\alpha_2, \beta] \qquad \text{and} \qquad [\beta, \alpha_1 + \alpha_2] = [\beta, \alpha_1] + [\beta, \alpha_2].$$

Proposition 3.2. *Let $n, m \in \omega$, $[\alpha] \in \pi_{n+1}(X)$ and $[\beta] \in \pi_{m+1}(X)$. Then*

$$[\beta, \alpha] = (-1)^{(n+1)(m+1)}[\alpha, \beta].$$

Proof. □

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Proposition 3.3. *Let* $n, m, r \in \omega$, $n, m, r \geq 1$, $[\alpha] \in \pi_{n+1}(X)$, $[\beta] \in \pi_{m+1}(X)$ *and* $[\gamma] \in \pi_{r+1}(X)$. *Then*

$$(-1)^{r(n+1)}[\alpha, [\beta, \gamma]] + (-1)^{n(m+1)}[\beta, [\gamma, \alpha]] + (-1)^{m(r+1)}[\gamma, [\alpha, \beta]] = 0$$

References

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[Mun00] James R. Munkres. *Topology*. Second edition. Prentice Hall, 2000.

[Whi78] George W. Whitehead. *Elements of Homotopy Theory*. Graduate Texts in Mathematics. Springer-Verlag, 1978.