FUNCTIONAL ANALYSIS II SUMMARY

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Abstract. This is a rough summary of the course *Functional Analysis II* held at *ETH Zurich* by *Prof. Dr. Alessandro Carlotto* in spring 2018. The main focus of this summary is to give a neat preparation for the oral exam.

Contents

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Introduction

This serves as a summary of useful facts from *measure theory* which are used throughout the text.

Theorem 1.1 (Transformation Formula). *Let $n \in \omega$, $n > 0$, $U, V \subseteq \mathbb{R}^n$ open and $\varphi : U \to V$ a C^1-diffeomorphism. A function $f : V \to \overline{\mathbb{R}}$ is in $\mathcal{L}^1(V)$ if and only if $(f \circ \varphi)\,|\det(D\varphi)|$ is in $\mathcal{L}^1(U)$. Then*

$$\int_V f = \int_U (f \circ \varphi)\,|\det(D\varphi)|\,.$$

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^n$ be open and $1 \leq p < \infty$. Then $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.*

Proposition 1.1. *If $|\Omega| < \infty$ and $0 < p < q \leq \infty$. Then $L^q(\Omega) \subseteq L^p(\Omega)$.*

Proposition 1.2 (Jensen's Inequality). *Let $\Omega \subseteq \mathbb{R}^n$ bounded and $\varphi : \mathbb{R} \to \mathbb{R}$ convex. Then*

$$\varphi\left(\frac{1}{|\Omega|}\int_\Omega f\right) \leq \frac{1}{|\Omega|}\int_\Omega \varphi \circ f$$

for any $f \in L^1(\Omega)$.

Proposition 1.3 (Dual of $L^p(\Omega)$). *Let $\Omega \subseteq \mathbb{R}^n$ and $1 \leq p < \infty$. Then the mapping $T : L^q(\Omega) \to (L^p(\Omega))^*$ defined by*

$$T(f)(g) := \int_\Omega fg$$

is an isometric isomorphism.

Proposition 1.4 (Integration by Parts). *Let (M, g) be a compact Riemannian manifold with boundary. Then*

$$\int_M \langle \operatorname{grad} f, X\rangle_g dV_g = \int_{\partial M} f\langle X, N\rangle dV_{\tilde{g}} - \int_M (f \operatorname{div} X) dV_g$$

for $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$. Moreover, ***Green's identities*** *hold:*

$$\int_M u\Delta v dV_g = \int_M \langle \operatorname{grad} u, \operatorname{grad} v\rangle_g dV_g - \int_{\partial M} uNv dV_{\tilde{g}}$$

and

$$\int_M (u\Delta v - v\Delta u) dV_g = \int_{\partial M} (vNu - uNv) dV_{\tilde{g}}$$

for $u, v \in C^\infty(M)$.

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Sobolev Space Theory

The Spaces $W^{k,p}(\Omega)$. In what follows, let $n \in \omega$, $n \geq 1$, and $1 \leq p \leq \infty$.

Definition 1.1 (Distributional and Weak Derivative). *Let $\Omega \subseteq \mathbb{R}^n$ open and $u \in L^1_{\text{loc}}(\Omega)$. For any multiindex α, the **distributional derivative of order α of u**, written $D^\alpha u$, is defined to be the mapping $D^\alpha u : C_c^\infty(\Omega) \to \mathbb{R}$ defined by*

$$\varphi \mapsto (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi dx.$$

*Moreover, a function $D^\alpha u \in L^p(\Omega)$ is called **weak derivative of order α of u with exponent p**, iff*

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega D^\alpha u \varphi dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \varphi dx.$$

Theorem 1.3 (Fundamental Lemma of Variational Calculus). *Let $\Omega \subseteq \mathbb{R}^n$ open and $f \in L^1_{\text{loc}}(\Omega)$. If*

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega f \varphi dx = 0,$$

then $f = 0$ a.e.

Remark 1.1. Let $\Omega \subseteq \mathbb{R}^n$ open. Then $L^p(\Omega) \subseteq L^1_{\text{loc}}(\Omega)$.

Remark 1.2. From the fundamental lemma of variational calculus 1.3 it follows that *weak derivatives, if they exist, are unique*.

Examples 1.1 (Weak Derivatives).

(a) Suppose u is classically differentiable. Then u is weakly differentiable using integration by parts 1.4.
(b) Consider $\Omega := (-1, 1)$ and $u := |x|$. Then $u' = \chi_{[0,1)} - \chi_{(-1,0)}$.
(c) Consider $\Omega := \mathbb{R}$ and $u := \chi_{(0,\infty)}$. Then the weak derivative u' does not exist. Indeed, the *Dirac distribution* is not representable as one may see by considering the smooth family $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ for $\varepsilon > 0$ defined by

$$\varphi_\varepsilon(x) := \begin{cases} e^{\varepsilon^2/(x^2-\varepsilon^2)} & |x| < \varepsilon, \\ 0 & |x| \geq \varepsilon. \end{cases}$$

(d) Let $\Omega := (0, 1)$ and consider the *Cantor function* $u : \Omega \to \Omega$. Then $u' = 0$ classically a.e. but the distributional derivative of u does not vanish.
(e) Let $f \in L^p(\Omega)$. Then the computation performed in the proof of lemma 1.3 shows, that the function $u : I \to \mathbb{R}$ defined by

$$u(x) := \int_{x_0}^x f(t) dt$$

for $x_0 \in I$, admits the weak derivative f.

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Definition 1.2 (Sobolev Space). *Let $\Omega \subseteq \mathbb{R}^n$ open. For any $k \in \omega$, the **Sobolev space of index (k, p)**, written $W^{k,p}(\Omega)$, is defined to be the space*

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ exists for all } |\alpha| \leq k\},$$

with norm

$$\|-\|_{W^{k,p}(\Omega)} := \sum_{|\alpha|\leq k} \|D^\alpha -\|_{L^p(\Omega)}.$$

Moreover, define

$$W_0^{k,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{\|-\|_{W^{k,p}(\Omega)}},$$

and $H^k(\Omega) := W^{k,2}(\Omega)$ as well as $H_0^k(\Omega) := W_0^{k,2}(\Omega)$.

Examples 1.2 (Sobolev Functions). A main tool in constructing Sobolev functions for $n \geq 2$ is using that the origin in $\mathbb{R}^n$ has vanishing $W^{1,p}$-capacity.

(a) Let $\Omega := \mathbb{R}$, A Lebesgue-measurable and $u := \chi_A$. Then $u \notin W^{1,p}(\Omega)$, since by theorem 1.7 u must admit a continuous representant, which it obviously does not.

(b) Let $\Omega := B_1(0) \subseteq \mathbb{R}^n$ for $n \geq 2$. Then $u : \Omega \to \overline{\mathbb{R}}$ defined by $u(x) := \log|x|$ belongs to $L^p(\Omega)$ for any $1 \leq p < \infty$ and moreover, $u \in W^{1,p}(\Omega)$ for any $p < n$.

(c) Let $\Omega := B_{1/e}(0) \subseteq \mathbb{R}^n$ for $n \geq 2$. Then $u : \Omega \to \overline{\mathbb{R}}$ defined by $u(x) := \log\log\frac{1}{|x|}$ belongs to $W^{1,n}(\Omega)$.

(d) Let $\Omega := B_{1/2}(0) \subseteq \mathbb{R}^n$. For $\alpha \in \mathbb{R}$ define $u_\alpha : \Omega \to \overline{\mathbb{R}}$ by $u_\alpha(x) := |\log|x||^\alpha$. Then $u_\alpha \in H^1(\Omega)$ for $n = 1$ if and only if $\alpha = 0$, for $n = 2$ if and only if $\alpha \in (-\infty, 1/2)$ and for $n \geq 3$ if and only if $\alpha \in \mathbb{R}$.

Remark 1.3. Using proposition 1.1, we immediately get

$$W^{1,q}(\Omega) \hookrightarrow W^{1,p}(\Omega)$$

for all $1 \leq p \leq q \leq \infty$ whenever $\Omega \subseteq\subseteq \mathbb{R}^n$.

Theorem 1.4. *Let $\Omega \subseteq \mathbb{R}^n$ open. Then $W^{k,p}(\Omega)$ is*

(a) a Banach space for all $1 \leq p \leq \infty$.

(b) separable for all $1 \leq p < \infty$.

(c) reflexive for all $1 < p < \infty$.

Proof. The proof basically boils down to using the correponding properties of the Lebesgue spaces $L^p(\Omega)$.

(a) This follows from the fact that $L^p(\Omega)$ is a Banach space for all $1 \leq p \leq \infty$. Let $(f_i)_{i\in\omega}$ be a Cauchy sequence in $W^{k,p}$. By definition of the $W^{k,p}$-norm, $(D^\alpha f_i)_{i\in\omega}$ is a Cauchy sequence in L^p. Thus we get $D^\alpha f_i \to f_\alpha$ in L^p, in particular, $f_i \to f$ in L^p. Using Hölder's inequality we compute

$$\int_\Omega f_\alpha \varphi dx = \lim_{i\to\infty} \int_\Omega D^\alpha f_i \varphi dx = (-1)^{|\alpha|} \lim_{i\to\infty} \int_\Omega f_i D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega f D^\alpha \varphi dx$$

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for $\varphi \in C_c^\infty(\Omega)$.
(b) For simplicity, we consider $k = 1$ only. Consider $\iota : W^{1,p} \hookrightarrow (L^p)^{n+1}$ defined in the obvious way. Then ι is an isometry and the statement follows.
(c) Same argument as in part (b).

□

Elliptic Operators in Divergence Form.

Lemma 1.1 (Poincaré Inequality). *Let $\Omega \subseteq\subseteq \mathbb{R}^n$ and $1 \leq p < \infty$. Then for any $u \in C_c^\infty(\Omega)$ we have that*

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p}.$$

Proof. Let $n = 1$. Since Ω is bounded, we get that $\Omega \subseteq (a, b)$ and we may extend u on $[a, b] =: I$ to be zero. Hence an application of Jensen's inequality 1.2 yields

$$|u(x)|^p = \left| \int_a^x u'(t)dt \right|^p \leq (x-a) \int_a^x |u'(t)|^p \, dt \leq (b-a)^{p-1} \|u'\|_{L^p(I)}^p.$$

Thus

$$\|u\|_{L^p(\Omega)}^p = \|u\|_{L^p(I)}^p \leq (b-a)^p \|u'\|_{L^p(I)}^p = (b-a)^p \|u'\|_{L^p(\Omega)}^p$$

where the last equality follows due to the fact that u and thus u' is compactly supported in Ω. If $n > 1$, we have $\Omega \subseteq (a, b) \times \mathbb{R}^{n-1}$. Hence for fixed $y \in \mathbb{R}^{n-1}$, above computation yields

$$|u(x, y)|^p \leq (b-a)^{p-1} \|\partial_x u(-, y)\|_{L^p(I)}^p$$

for any $x \in I$. Hence

$$\begin{aligned}
\|u\|_{L^p(\Omega)}^p &= \|u\|_{L^p((a,b)\times\mathbb{R}^{n-1})} \\
&\leq (b-a)^p \int_{\mathbb{R}^{n-1}} \|\partial_x u(-, y)\|_{L^p(I)}^p \, dy \\
&\leq (b-a)^p \|\nabla u\|_{L^p((a,b)\times\mathbb{R}^{n-1})}^p \\
&= (b-a)^p \|\nabla u\|_{L^p(\Omega)}^p.
\end{aligned}$$

□

Theorem 1.5 (Riesz Representation Theorem). *Let H be a real Hilbert space. Then the mapping $J : H \to H^*$ defined by $J(x) := \langle x, - \rangle$ is an isometric isomorphism.*

Theorem 1.6. *Let $\Omega \subseteq\subseteq \mathbb{R}^n$ and consider the elliptic operator*

$$A_0 := -\frac{\partial}{\partial x^i}\left(a^{ij} \frac{\partial}{\partial x^j}\right),$$

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for $a^{ij} \in L^\infty(\Omega)$ symmetric. Then: Given $f \in L^2(\Omega)$, the homogenous Dirichlet problem

$$\begin{cases} A_0 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

admits a unique weak solution $u \in H_0^1(\Omega)$.

Proof. The proof is divided into two steps.
Step 1: Derivation of Weak Formulation. Suppose $u \in C^2(\overline{\Omega})$ is a solution of (1). Let $\varphi \in C_c^\infty(\Omega)$. Then integration by parts (see [Lee13, p. 436]) yields

$$-\int_\Omega A_0 u\varphi = -\sum_{j=1}^n \int_\Omega \operatorname{div}(X_j)\varphi = \int_\Omega a^{ij}\frac{\partial u}{\partial x^j}\frac{\partial \varphi}{\partial x^i} = \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial \varphi}{\partial x^j},$$

where $X_j := \left(a^{ij}\frac{\partial}{\partial x^j}\right)_i$. Thus we get the weak formulation:

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial \varphi}{\partial x^j} = \int_\Omega f\varphi. \tag{2}$$

Step 2: Existence and Uniqueness of Weak Solutions. Since A_0 is uniformly elliptic, there exists $\lambda > 0$ such that

$$\xi^t(a^{ij}(x))\xi = a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$$

holds for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Moreover, since $a^{ij} \in L^\infty(\Omega)$, we get that A_0 is uniformly bounded, i.e. there exists $\Lambda > 0$ such that

$$a^{ij}(x)\xi_i\eta_j \leq \Lambda|\xi||\eta|$$

for

$$\Lambda = \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(\Omega)}$$

holds for almost all $x \in \Omega$ and $\xi, \eta \in \mathbb{R}^n$. Now define a bilinear form

$$\langle\cdot,\cdot\rangle_a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

by

$$\langle u, v\rangle_a := \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^j} \tag{3}$$

Then it is easy to see, that $\langle\cdot,\cdot\rangle_a$ is symmetric. Also, $\langle\cdot,\cdot\rangle_a$ is positive definite since

$$\langle u, u\rangle_a = \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j} \geq \lambda\int_\Omega |\nabla u|^2 \geq C\lambda\int_\Omega |u|^2$$

using ellipticity and Poincaré's inequality. Moreover by Poincaré's inequality we have that

$$\lambda\|u\|^2_{H_0^1(\Omega)} \leq \|u\|_a^2 \leq \Lambda\|u\|^2_{H_0^1(\Omega)}$$

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for the induced norm $\|\cdot\|_a$. Hence the induced norm is equivalent to the standard norm on $H_0^1(\Omega)$ and thus $(H_0^1(\Omega), \|\cdot\|_a)$ is a Hilbert space. Thus an application of Riesz representation theorem 1.5 yields the existence of a unique $u \in H_0^1(\Omega)$, such that

$$\langle u, \varphi\rangle_a = l(\varphi) := \int_\Omega f\varphi$$

holds for all $\varphi \in H_0^1(\Omega)$, since $l \in (H_0^1(\Omega))^*$ by

$$|l(\varphi)| \leq \|f\|_{L^2}\|\varphi\|_{L^2} \leq \|f\|_{L^2}\|\varphi\|_{H^1}.$$

$\square$

Examples 1.3 (Elliptic Operators in Divergence Form).
(a) Set $a^{ij}(x) := \delta^{ij}$ for all $x \in \Omega$. Then $A_0 = -\Delta$. Moroever A_0 is uniformly elliptic, since $\delta^{ij}\xi_i\xi_j = |\xi|^2$ for all $\xi \in \mathbb{R}^n$.
(b) For $\Omega \subseteq \mathbb{R}^2$ consider

$$(a^{ij}(x,y)) := \begin{pmatrix} 2 & xy/|xy| \\ xy/|xy| & 2 \end{pmatrix}.$$

Then A_0 is elliptic. Indeed, a^{ij} admits the eigenvalues 1 and 3, thus by the *Min-Max theorem* we get that

$$1 \leq R_{A(x,y)}(z) \leq 3$$

for all $(x, y) \in \Omega$ and where $R_A(z)$ denotes the *Rayleigh-Ritz quotient* defined by

$$R_A(z) := \frac{\langle Az, z\rangle}{\|z\|^2}$$

for $z \in \mathbb{C}^2$.
(c) A non-example would be $a^{ij}(x) := 0$.
(d) Another non-example is given by

$$(a^{ij}(x,y)) := \begin{pmatrix} x^2+y^2 & x+y \\ x+y & 1 \end{pmatrix}$$

for any $\Omega \subseteq \mathbb{R}^2$ containing the origin. Indeed, we get $\det(a^{ij}(0,0)) = 0$.

Sobolev Spaces on an Interval. In what follows, let $-\infty \leq a < b \leq \infty$ and $I := (a, b)$.

Lemma 1.2 (Du Bois-Reymond). *Let $f \in L^1_{\text{loc}}(I)$ such that*

$$\forall \varphi \in C_c^\infty(I): \int_I f\varphi' dx = 0.$$

Then f is almost everywhere constant.

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Proof. Let $v := w - c_0\psi$ for $w, \psi \in C_c^\infty(I)$ such that $\int_I \psi = 1$ and $\int_I v = 0$. This implies $c_0 = \int_I w$. By the fundamental theorem of calculus, the function $\varphi : I \to \mathbb{R}$ defined by

$$\varphi(x) := \int_I v(t)dt$$

belongs to $C_c^\infty(I)$ with $\varphi' = v$. Thus we compute

$$0 = \int_I f\varphi' = \int_I fv = \int_I fw - c_0 \int_I f\psi = \int_I fw - \int_I w \int_I f\psi = \int_I (f - c)w,$$

where $c := \int_I f\psi$. Since w was arbitrary, we conclude by the fundamental lemma of variational calculus 1.3. □

Lemma 1.3. *Let $f \in L^1_{\text{loc}}(I)$ and $x_0 \in I$. Then $u : I \to \mathbb{R}$ defined by*

$$u(x) := \int_{x_0}^x f(t)dt$$

is absolutely continuous and belongs to $W^{1,1}_{\text{loc}}(I)$ with $u' = f$ a.e.

Proof. Absolute continuity follows from real analysis. Let $\varphi \in C_c^\infty(I)$. Then Fubini yields

$$\begin{aligned}
\int_I u\varphi' &= \int_a^{x_0} \int_{x_0}^x f(t)\varphi'(x)dtdx + \int_{x_0}^b \int_{x_0}^x f(t)\varphi'(x)dtdx \\
&= -\int_a^{x_0} \int_x^{x_0} f(t)\varphi'(x)dtdx + \int_{x_0}^b \int_{x_0}^x f(t)\varphi'(x)dtdx \\
&= -\int_a^{x_0} \int_a^t f(t)\varphi'(x)dxdt + \int_{x_0}^b \int_t^b f(t)\varphi'(x)dxdt \\
&= -\int_a^{x_0} f(t)\varphi(t)dt - \int_{x_0}^b f(t)\varphi(t)dt \\
&= -\int_I f\varphi.
\end{aligned}$$

□

Theorem 1.7. *Let $u \in W^{1,p}(I)$. Then there exists an absolutely continuous representant $\widetilde{u}$ of u on $\overline{I}$, such that*

$$\widetilde{u}(x) = \widetilde{u}(x_0) + \int_{x_0}^x u'(t)dt$$

holds for all $x, x_0 \in I$. In particular, $\widetilde{u}$ is classically differentiable a.e. and $\widetilde{u}' = u'$.

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Proof. By lemma 1.3, the function $v(x) := \int_{x_0}^x u'(t)dt$ is in $W^{1,1}_{\mathrm{loc}}(I)$ with weak derivative u'. Moreover, for any $\varphi \in C_c^\infty(I)$ we compute

$$\int_I (u-v)\varphi' = \int_I u\varphi' - \int_I v\varphi' = -\int_I u'\varphi + \int_I u'\varphi = 0.$$

Thus lemma 1.2 yields $u = c + v$, for some $c \in \mathbb{R}$. Set

$$\widetilde{u}(x) := c + \int_{x_0}^x u'(t)dt.$$

Then $\widetilde{u}(x_0) = c$ and thus the statement follows. □

Theorem 1.8 (Characterization of $W^{1,p}(I)$). *Let $1 < p \leq \infty$ and $u \in L^p(I)$. Then the following statements are equivalent:*

(a) $u \in W^{1,p}(I)$.

(b) *There exists $C \geq 0$ such that*

$$\forall \varphi \in C_c^\infty(I) : \left| \int_I u\varphi' \right| \leq C \|\varphi\|_{L^q}.$$

(c) *There exists $C \geq 0$ such that for all $I' \subseteq\subseteq I$ and $|h| < \operatorname{dist}(I', \partial I)$ holds*

$$\|\tau_h u - u\|_{L^p(I')} \leq C|h|,$$

where $\tau_h u(x) := u(x+h)$.

Proof. The implication $(a) \Rightarrow (b)$ follows immediately from Hölder's inequality. To prove $(b) \Rightarrow (a)$, we observe that $l : C_c^\infty(I) \to \mathbb{R}$ defined by

$$L(\varphi) := \int_I u\varphi'$$

is continuous. Since $C_c^\infty(I)$ is dense in $L^q(I)$, we get that $l \in (L^q(I))^*$. Hence we find $g \in L^p$, such that $\int_I g\varphi = l(\varphi)$ and so $u' = -g$.

Next we show $(a) \Rightarrow (c)$. By theorem 1.7, we find an absolutely continuous representant $\widetilde{u}$ of u. Thus

$$\widetilde{u}(x+h) - \widetilde{u}(x) = h\int_0^1 u'(x+th)dt$$

Hence Jensen's inequality yields

$$\|\tau_h u - u\|_{L^p(I')} \leq |h| \int_0^1 \|u'(\cdot + th)\|_{L^p(I')}\, dt \leq |h|\, \|u'\|_{L^p(I)}.$$

Lastly, we prove $(c) \Rightarrow (b)$. Let $\varphi \in C_c^\infty(I)$. Then we may find $I' \subseteq\subseteq I$ such that $\operatorname{supp}\varphi \subseteq I'$. Hence we compute

$$\left|\int_I u\varphi'\right| = \lim_{h\to 0} \frac{1}{|h|} \left|\int_I u(x)\left(\varphi(x+h) - \varphi(x)\right)dx\right|$$

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$$
\begin{aligned}
&= \lim_{h\to 0}\frac{1}{|h|}\left|\int_I \big(u(x-h)-u(x)\big)\,\varphi(x)dx\right| \\
&= \lim_{h\to 0}\frac{1}{|h|}\left|\int_I (\tau_{-h}u-u)\,\varphi\right| \\
&\le \lim_{h\to 0}\frac{1}{|h|}\,\|\tau_{-h}u-u\|_{L^p(I')}\|\varphi\|_{L^q(I)} \\
&\le C\|\varphi\|_{L^q(I)}.
\end{aligned}
$$

□

Theorem 1.9 (Extension Theorem). *There exists a continuous linear operator*

$$E : W^{1,p}(I) \to W^{1,p}(\mathbb{R})$$

such that:

(i) $Eu|_I = u$.
(ii) $\|Eu\|_{L^p(\mathbb{R})} \le C\|u\|_{L^p(I)}$.
(iii) $\|(Eu)'\|_{L^p(\mathbb{R})} \le C\|u\|_{W^{1,p}(I)}$.

Proof. First we consider the case $I = (0,\infty)$. We extend u by continuity to 0 and then we extend u by means of *even symmetry*. If I is bounded we can without loss of generality assume that $I = (0,1)$. Now use a cut-off function. □

Theorem 1.10 (Approximation Theorem). *Let* $1 \le p < \infty$ *and* $u \in W^{1,p}(I)$. *Then there exists a sequence* $(u_i)_{i\in\omega}$ *in* $C_c^\infty(\mathbb{R})$ *such that*

$$\|u_i|_I - u\|_{W^{1,p}(I)} \to 0.$$

Proof. The main idea of the proof is to use convolutions. Moreover, it is enough to consider the case $I = \mathbb{R}$ only, due to the extension theorem 1.9. □

Theorem 1.11 (Sobolev Embedding). *There is a continuous embedding*

$$W^{1,p}(I) \hookrightarrow L^\infty(I).$$

Proof. First consider I bounded. By theorem 1.7 we get that

$$\|u\|_{L^\infty} = \sup_{x\in I}|u(x)| \le |u(y)| + \sup_{x\in I}\left|\int_y^x u'(t)dt\right| \le |u(y)| + \|u'\|_{L^1},$$

for any $y \in I$. Hence

$$\|u\|_{L^\infty} \le \inf_{y\in I}|u(y)| + \|u'\|_{L^1} \le \frac{1}{|I|}\int_I |u(y)| + \|u'\|_{L^1} \le C\|u\|_{W^{1,1}} \le C\|u\|_{W^{1,p}}.$$

Assume now that I is unbounded. Then we find $I' \subseteq\subseteq I$ such that

$$\|u\|_{L^\infty(I')} \ge \frac{1}{2}\|u\|_{L^\infty(I)}$$

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and thus the claim follows by the previous computation. Indeed, note that by theorem 1.7, we have that

$$|u(x)| \leq |u(y)| + \|u'\|_{L^1(I)}$$

for all $x \in I$ and fixed $y \in I$, and thus $u \in L^\infty(I)$. Moreover, there exists $x_0 \in I$ such that $|u(x_0)| > \frac{1}{2}\|u\|_{L^\infty(I)}$, if not, this would contradict the definition of the supremum norm. Since u is continuous by theorem 1.7, we find $\delta > 0$ such that

$$|u(x) - u(x_0)| \leq |u(x_0)| - \frac{1}{2}\|u\|_{L^\infty(I)}$$

for all $x \in I$ such that $|x - x_0| < \delta$. Hence the reversed triangle inequality yields

$$\frac{1}{2}\|u\|_{L^\infty(I)} - |u(x_0)| \leq |u(x)| - |u(x_0)| \leq |u(x_0)| - \frac{1}{2}\|u\|_{L^\infty(I)}$$

and so

$$\frac{1}{2}\|u\|_{L^\infty(I)} \leq |u(x)|$$

for all $x \in I \cap (x_0 - \delta, x_0 + \delta) =: I'$. $\square$

Corollary 1.1. *Let I be unbounded and $u \in W^{1,p}(I)$ for $1 \leq p < \infty$. Then $u \to 0$ as $|x| \to \infty$.*

Dirichlet and Neumann Boundary Problems on I. In what follows, let us consider $-\infty < a < b < \infty$ and $I := (a, b)$.

Proposition 1.5. *Let $f \in C^0(\overline{I})$. Then the weak solution u of the homogenous Dirichlet problem*

$$\begin{cases} -u'' = f & \text{in } I, \\ u(a) = 0 = u(b). \end{cases}$$

is a classical solution, i.e. $u \in C^2(\overline{I})$.

Proposition 1.6. *Let $f \in C^0(\overline{I})$. Then the weak solution u of the homogenous Neumann problem*

$$\begin{cases} -u'' + u = f & \text{in } I, \\ u'(a) = 0 = u'(b). \end{cases}$$

is a classical solution, i.e. $u \in C^2(\overline{I})$.

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Sobolev Spaces on a Domain.

Theorem 1.12 (Meyers-Serrin). *Let $\Omega \subseteq \mathbb{R}^n$ be open. Then $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$ for every $1 \leq p < \infty$.*

Proof. Convolutions and a partition of unity argument. □

Proposition 1.7 (Product Rule). *Let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and*

$$\partial_\alpha(uv) = (\partial_\alpha u)v + u(\partial_\alpha v).$$

Proof. First consider the case $\underline{p < \infty}$. Then

$$\|uv\|_{L^p} \leq \|u\|_{L^\infty} \|v\|_{L^p}$$

and

$$\|(\partial_\alpha u)v + u(\partial_\alpha)v\|_{L^p} \leq \|\partial_\alpha u\|_{L^p} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|\partial_\alpha v\|_{L^p}.$$

Meyers-Serrin 1.12 yields the existence of sequences u_k and v_k in $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that $u_k \to u$ and $v_k \to v$ in $W^{1,p}(\Omega)$. For any $\varphi \in C_c^\infty(\Omega)$, we compute

$$\begin{aligned}
\int_\Omega uv\partial_\alpha \varphi &= \lim_{k\to\infty} \int_\Omega u_k v_k \partial_\alpha \varphi \\
&= -\lim_{k\to\infty} \int_\Omega ((\partial_\alpha u_k)v_k + u_k(\partial_\alpha v_k))\varphi \\
&= -\int_\Omega ((\partial_\alpha u)v + u(\partial_\alpha v))\varphi.
\end{aligned}$$

Now consider the case $\underline{p = \infty}$. We have $uv \in L^\infty(\Omega)$ as well as $(\partial_\alpha u_k)v_k + u_k(\partial_\alpha v_k) \in L^\infty(\Omega)$. Let $\varphi \in C_c^\infty(\Omega)$. Hence we find $\Omega' \subseteq\subseteq \Omega$ with $\operatorname{supp} \varphi \subseteq \Omega'$. But then the above calculation holds on Ω'. □

Theorem 1.13 (Characterization of $W^{1,p}(\Omega)$). *Let $1 < p \leq \infty$ and $u \in L^p(\Omega)$. Then the following statements are equivalent:*

(a) $u \in W^{1,p}(\Omega)$.

(b) There exists $C \geq 0$ such that

$$\forall |\alpha| \leq 1 \forall \varphi \in C_c^\infty(\Omega) : \left| \int_I u D^\alpha \varphi \right| \leq C \|\varphi\|_{L^q}.$$

(c) There exists $C \geq 0$ such that for all $\Omega' \subseteq\subseteq \Omega$ and $|h| < \operatorname{dist}(I', \partial I)$ holds

$$\|\tau_h u - u\|_{L^p(\Omega')} \leq C|h|,$$

where $\tau_h u(x) := u(x+h)$.

Proof. The proof $(c) \Rightarrow (b) \Rightarrow (a)$ is almost the same as the one given in the characterization theorem for Ω an interval. For proving $(a) \Rightarrow (c)$, use Meyers-Serrin. □

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Corollary 1.2. *Let $u \in L^\infty(\Omega)$. Then $u \in W^{1,\infty}(\Omega)$ if and only if u admits a locally Lipschitz continuous representant. Moreover, if Ω is convex, then $u \in W^{1,\infty}(\Omega)$ if and only if u admits a Lipschitz continuous representant.*

Extension and Trace Operator. We start off with *local theory*. In what follows, define

$$Q := \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'| < 1, |x_n| < 1\}.$$

Moreover

$$Q_+ := \left\{(x', x_n) \in Q : x_n > 0\right\} \qquad \text{and} \qquad Q_0 := \left\{(x', x_n) \in Q : x_n = 0\right\}.$$

Lemma 1.4. *Let $u \in W^{1,p}(Q_+)$. Set*

$$u^*(x', x_n) := \begin{cases} u(x', x_n) & x_n > 0, \\ u(x', -x_n) & x_n < 0. \end{cases}$$

Then $u^* \in W^{1,p}(Q)$ and $\|u^*\|_{W^{1,p}(Q)} \leq C\|u\|_{W^{1,p}(Q_+)}$.

Now to the *global theory*.

Theorem 1.14 (Extension). *Let $\Omega \subseteq\subseteq \mathbb{R}^n$ of class C^1. Then there exists a continuous linear operator*

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$$

such that:

(i) $Eu|_\Omega = u$.
(ii) $\|Eu\|_{L^p(\mathbb{R}^n)} \leq C\|u\|_{L^p(\Omega)}$.
(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}$.

Corollary 1.3. *Let $\Omega \subseteq\subseteq \mathbb{R}^n$ of class C^1 and $1 \leq p < \infty$. Then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$.*

Again, we tackle first the *local theory*.

Lemma 1.5. *Let $u \in W^{1,p}(Q_+)$. Then $u|_{Q_0} \in L^p(Q_0)$ is well defined and the induced trace operator $W^{1,p}(Q_+) \to L^p(Q_0)$ is linear and continuous.*

Proof. We consider the case $\underline{1 \leq p < \infty}$. The main idea is to show this for $u \in C^\infty(Q)$, then for $u \in W^{1,p}(Q)$ and then finally for $u \in W^{1,p}(Q_+)$ by extension.
Consider now $\underline{p = \infty}$. Since Q_+ is convex, $u \in W^{1,\infty}(Q_+)$ admits a Lipschitz continuous representant and the result follows by extending via continuity. □

Theorem 1.15 (Characterization of $H^1(\Omega)$). *Let $\Omega \subseteq\subseteq \mathbb{R}^n$. Then*

$$H^1(\Omega) = H^1_0(\Omega) \oplus \{u \in H^1(\Omega) : \Delta u = 0\}.$$

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Proof. Let $u \in H^1(\Omega)$ and let $u_0 \in H_0^1(\Omega)$ denote the unique solution of

$$\forall \varphi \in C_c^\infty(\Omega) : \int_\Omega \nabla u_0 \nabla \varphi = \int_\Omega \nabla u \nabla \varphi.$$

Set $u_1 := u - u_0$. Then for any $\varphi \in C_c^\infty(\Omega)$ we compute

$$-\int_\Omega u_1 \Delta \varphi = \int_\Omega \nabla u_1 \nabla \varphi = \int_\Omega \nabla u \nabla \varphi - \int_\Omega \nabla u_0 \nabla \varphi = 0.$$

Thus $u = u_0 + u_1$ is of the desired form. Moreover, we have

$$\|\nabla u\|^2_{L^2(\Omega)} = \|\nabla u_0\|^2_{L^2(\Omega)} + \|\nabla u_1\|^2_{L^2(\Omega)} + 2\int_\Omega \nabla u_0 \nabla u_1.$$

Since $u_0 \in H_0^1(\Omega)$, we find a sequence φ_k in $C_c^\infty(\Omega)$ such that $\varphi_k \to u$ in $H^1(\Omega)$. But then

$$\int_\Omega \nabla u_0 \nabla u_1 = \lim_{k\to\infty} \int_\Omega \nabla \varphi_k \nabla u_1 = 0.$$

Hence

$$\|\nabla u\|^2_{L^2(\Omega)} = \|\nabla u_0\|^2_{L^2(\Omega)} + \|\nabla u_1\|^2_{L^2(\Omega)}$$

which implies that the decomposition is direct. Indeed, suppose $u \in H_0^1(\Omega)$ such that $\Delta u = 0$. Then $u = u/2 + u/2$ which yields $u = 0$ by the above computation. □

Corollary 1.4 (Characterization of $H_0^1(\Omega)$). *Let $\Omega \subseteq\subseteq \mathbb{R}^n$ of class C^1. Then*

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\}.$$

Sobolev Embeddings.

Theorem 1.16 (Sobolev Embedding Theorem). *Let $\Omega \subseteq\subseteq \mathbb{R}^n$ of class C^1 and $k \in \omega$, $k \geq 1$. Then:*

(a) If $kp < n$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq q \leq p^* := \frac{np}{n-pk}$ and the embedding is compact for $q < p^*$.
(b) If $kp = n$, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq q < \infty$ and those embeddings are compact.
(c) If $kp > n$ and $k - \frac{n}{p} \notin \omega$, then $W^{k,p}(\Omega) \hookrightarrow C^{l,\alpha}(\Omega)$ for $l := \left[k - \frac{n}{p}\right]$ and $0 \leq \alpha \leq \alpha^* := k - l - \frac{n}{p}$ and those embeddings are compact for $\alpha < \alpha^*$.
(d) If $kp > n$ and $k - \frac{n}{p} = l + 1 \in \omega$, then $W^{k,p}(\Omega) \hookrightarrow C^{l,\alpha}(\Omega)$ for $0 \leq \alpha < 1$ and those embeddings are compact.

Corollary 1.5. *Let $\Omega \subseteq\subseteq \mathbb{R}^n$ of class C^1 and $u \in H^1(\Omega)$. Moreover, assume that $u \in H^k(\Omega)$ for some $k > \frac{n}{2} + 2$. Then $u \in C^2(\Omega)$.*

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p < n.

Theorem 1.17 (Sobolev-Gagliardo-Nirenberg). *Let* $1 \leq p < n$ *and let* $p^* := \frac{np}{n-p}$*. Then* $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ *with*

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p}.$$

Theorem 1.18 (Sobolev-Gagliardo-Nirenberg Compactness). *Let* $\Omega \subseteq\subseteq \mathbb{R}^n$ *and* $1 \leq p < n$*. Then* $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ *for* $1 \leq q \leq p^*$ *and the embedding is compact if* $q < p^*$*.*

p > n.

Theorem 1.19. *Let* $p > n$*. Then* $W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,\alpha}(\mathbb{R}^n)$ *with* $\alpha := 1 - \frac{n}{p}$ *and*

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq \|u\|_{W^{1,p}(\Omega)}.$$

Remark 1.4. For $p = \infty$, the statement is trivially true, since any function in $W^{1,\infty}(\mathbb{R}^n)$ is Lipschitz continuous since $\mathbb{R}^n$ is convex, and thus belongs to $C^{0,1}(\mathbb{R}^n)$.

The proof uses the notion of so-called *Campanato spaces*.

Theorem 1.20. *Let* $\Omega \subseteq\subseteq \mathbb{R}^n$ *of type* A *for some* $A > 0$ *and* $1 \leq p < \infty$, $\lambda > n$, $\alpha := \frac{\lambda - n}{p}$*. Then*

$$\mathcal{L}^{p,\lambda}(\Omega) \cong C^{0,\alpha}(\bar{\Omega}).$$

Proof. The inclusion $\mathcal{L}^{p,\lambda}(\Omega) \cong C^{0,\alpha}(\bar{\Omega})$ follows from the Campanato-theorem and does also hold for general $\Omega \subseteq \mathbb{R}^n$ open. □

Lemma 1.6. *Let* $u \in W^{1,p}(\mathbb{R}^n)$, $1 \leq p < \infty$*. Then for all* $x_0 \in \mathbb{R}^n$ *and* $r > 0$ *we have that*

$$\|u - u_{x_0,r}\|_{L^p(B_r(x_0))}^p \leq C r^p \|\nabla u\|_{L^p(B_r(x_0))}^p.$$

Proof. This is an application of the Poincaré-Wirtinger inequality 1.21 since without loss of generality, we may assume $x_0 = 0$ and $r = 1$. □

Now the proof of the Sobolev embedding theorem for $p > n$ is immediaty by considering

$$W^{1,p}(\mathbb{R}^n) \xhookrightarrow{\text{P.W.}} \mathcal{L}^{p,p}(\mathbb{R}^n) \xhookrightarrow{\text{Campanato}} C^{0,\alpha}(\mathbb{R}^n)$$

and observing that $\mathbb{R}^n$ is of type $\frac{\pi^{n/2}}{\Gamma(n/2+1)} > 0$.

Theorem 1.21 (Poincaré-Wirtinger Inequality). *Let* $\Omega \subseteq\subseteq \mathbb{R}^n$ *connected and of class* C^1 *and* $1 \leq p < \infty$*. Then there exists* $C \geq 0$ *such that*

$$\|u - \bar{u}\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}$$

holds for all $u \in W^{1,p}(\Omega)$*.*

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Proof. Towards a contradiction, assume that for any $C \geq 0$ there exists $u \in W^{1,p}(\Omega)$ such that

$$\|u - \overline{u}\|_{L^p(\Omega)} > C \|\nabla u\|_{L^p(\Omega)} .$$

In particular, there exists a sequence u_k in $W^{1,p}(\Omega)$, such that

$$\|u_k - \overline{u_k}\|_{L^p(\Omega)} > k \|\nabla u_k\|_{L^p(\Omega)}$$

holds for each $k \in \omega$, $k \geq 1$. Defining $v_k := u_k - \overline{u_k}$ and normalizing, i.e. setting $w_k := v_k / \|v_k\|_{L^p(\Omega)}$ (this is valid since $\|v_k\|_{L^p(\Omega)} > 0$), yields a sequence w_k in $W^{1,p}(\Omega)$ such that

$$\overline{w_k} = 0, \qquad \|w_k\|_{L^p(\Omega)} = 1 \qquad \text{and} \qquad \|\nabla w_k\|_{L^p(\Omega)} \to 0$$

for any $k \in \omega$, $k \geq 1$. Using the Sobolev embedding theorem 1.16, we get

$$W^{1,p}(\Omega) \hookrightarrow W^{1,n}(\Omega) \hookrightarrow L^p(\Omega) \qquad \text{and} \qquad W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$$

if $p \geq n$ and $p < n$, respectively. Moreover, those are compact embeddings. Thus since w_k is bounded in $W^{1,p}(\Omega)$, we have that $w_{k_i} \to w$ in $L^p(\Omega)$ for a subsequence w_{k_i} of w_k. Moreover, $\nabla w = 0$. Indeed, for any $\varphi \in C_c^\infty(\Omega)$ we compute

$$\int_\Omega w \nabla \varphi = \lim_{i \to \infty} \int_\Omega w_{k_i} \nabla \varphi = - \lim_{i \to \infty} \int_\Omega \nabla w_{k_i} \varphi = 0.$$

By the constancy lemma we therefore conclude that $w = c \in \mathbb{R}$ a.e. But

$$\overline{w} = \frac{1}{|\Omega|} \int_\Omega w = \frac{1}{|\Omega|} \lim_{i \to \infty} \int_\Omega w_{k_i} = 0$$

implies $w = 0$ a.e. contradicting

$$\|w\|_{L^p(\Omega)} = \lim_{i \to \infty} \|w_{k_i}\|_{L^p(\Omega)} = 1.$$

□

$p = n$.

Theorem 1.22. *It holds that $W^{1,n}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ for $n \leq p < \infty$.*

Regularity Theory

Goal of this section is to prove the following regularity result.

Theorem 1.23 (Global Regularity). *Let $\Omega \subseteq\subseteq \mathbb{R}^n$ of class C^{k+2} and $f \in H^k(\Omega)$ for some $k \in \omega$. Moreover, let $u \in H_0^1(\Omega)$ be the unique solution of the homogenous Dirichlet boundary value problem*

$$\begin{cases} A_0 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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where $a^{ij} \in C^{k+1}(\bar{\Omega})$. Then $u \in H^{k+2}(\Omega)$ and

$$\|u\|_{H^{k+2}(\Omega)} \leq C \|f\|_{H^k(\Omega)}.$$

Interior Regularity.

Theorem 1.24. *Let $\Omega \subseteq\subseteq \mathbb{R}^n$ of class C^1 and L an elliptic operator in divergence form satisfying $a^{ij} \in C^{k+1}(\bar{\Omega})$. If $f \in H^k(\Omega)$, the unique weak solution $u \in H_0^1(\Omega)$ of the homogenous Dirichlet boundary value problem*

$$\begin{cases} A_0 u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

belongs to $H^{k+2}_{\text{loc}}(\Omega)$ and for all $\Omega' \subseteq\subseteq \Omega$ we have the estimate

$$\|u\|_{H^{k+2}(\Omega')} \leq C(\|f\|_{H^k(\Omega)} + \|u\|_{H^1(\Omega)}).$$

Proof. Step 1: $k = 0$.

(a) *A-priori Estimates.* First of all, we are assuming that $u \in H^2_{\text{loc}}(\Omega)$.

(i) *H^1-Estimate.* Choose a bump function $\varphi \in C_c^\infty(\Omega)$ supported in Ω'. Thus the weak formulation yields by plugging in the test function $\varphi^2 u$

$$\int_\Omega \varphi^2 a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} + 2 \int_\Omega u \varphi a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial \varphi}{\partial x^j} = \int_\Omega f \varphi^2 u. \tag{4}$$

Rearanging formula (4) we compute

$$\begin{aligned}
\int_\Omega \varphi^2 a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} &= \int_\Omega f \varphi^2 u - 2 \int_\Omega u \varphi a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial \varphi}{\partial x^j} \\
&\leq \|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + 2\Lambda \int_\Omega (-u)\varphi\, |\nabla u|\, |\nabla \varphi| \\
&\leq \|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + \Lambda\varepsilon \|\varphi \nabla u\|^2_{L^2(\Omega)} + \frac{\Lambda}{\varepsilon} \|u \nabla \varphi\|^2_{L^2(\Omega)}
\end{aligned}$$

Noticing that

$$\int_\Omega \varphi^2 a^{ij} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} \geq \lambda \|\varphi \nabla u\|^2_{L^2(\Omega)}$$

yields

$$(\lambda - \Lambda\varepsilon) \|\varphi \nabla u\|^2_{L^2(\Omega)} \leq \|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + \frac{\Lambda}{\varepsilon} \|u \nabla \varphi\|^2_{L^2(\Omega)}$$

Picking $\varepsilon > 0$ appropriately, yields

$$\|\varphi \nabla u\|^2_{L^2(\Omega)} \leq C(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)})$$

and thus

$$\|\nabla u\|^2_{L^2(\Omega')} \leq C(\|f\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}).$$

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(ii) H^2-*Estimate.* Let $1 \leq \mu \leq n$. Then $\partial_\mu u$ solves

$$\int_\Omega a^{ij} \partial_i \partial_\mu u \partial_j \varphi = -\int_\Omega f \partial_\mu \varphi - \int_\Omega \partial_\mu a^{ij} \partial_i u \partial_j \varphi$$

for all $\varphi \in C_c^\infty(\Omega)$. Now perform the H^1-estimate on $\partial_\mu u$.

(b) *Existence: The Nirenberg-Trick.* The trick is to use difference quotients

$$D_h u := \frac{\tau_h u - u}{|h|}$$

for $h \in \mathbb{R}^n$ such that $|h| < \operatorname{dist}(\Omega', \partial\Omega)$. The idea now is to find a PDE solved by $D_h u$ in the weak sense and to to use the characterization of the Sobolev space.
Step 2: Induction Step.

□

Boundary Regularity.

Proposition 1.8 (Minimality Property). *Let $\Omega \subseteq\subseteq \mathbb{R}^n$. Then $u \in H_0^1(\Omega)$ solves (1) if and only if the **energy functional** satisfies*

$$E(u) := \frac{1}{2}\|u\|_a^2 - \int_\Omega f u = \inf_{v \in H_0^1(\Omega)} E(v).$$

Proof. Suppose $u \in H_0^1(\Omega)$ solves (1) and let $v \in H_0^1(\Omega)$. Then $v = u + \varphi$ for some $\varphi \in H_0^1(\Omega)$ and we compute

$$E(v) = E(u+\varphi) = \frac{1}{2}\|u\|_a^2 + \langle u, \varphi\rangle_a + \frac{1}{2}\|\varphi\|_a^2 - \int_\Omega f(u+\varphi) = E(u) + \frac{1}{2}\|\varphi\|_a^2 \geq E(u).$$

Conversely, suppose $u_0 \in H_0^1(\Omega)$ is a minimizer of the energy functional. Thus by elementary calculus

$$\left.\frac{d}{dt}\right|_{t=0} E(u_0 + tv) = 0$$

for all $v \in H_0^1(\Omega)$. But

$$\left.\frac{d}{dt}\right|_{t=0} E(u_0 + tv) = \langle u_0, v\rangle_a - \int_\Omega f v.$$

□

Eigenfunctions of $-\Delta$.

Theorem 1.25. *Let $\Omega \subseteq\subseteq \mathbb{R}^n$ of class C^2. Then there exists a Hilbert-space basis $(\varphi_i)_{i\in\omega}$ of $L^2(\Omega)$ consisting of eigenfunctions of the Laplace opertator, i.e.*

$$\begin{cases} -\Delta\varphi_i = \lambda_i \varphi_i & \text{in } \Omega, \\ \varphi_i = 0 & \text{on } \partial\Omega. \end{cases}$$

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Moreover $0 < \lambda_i \to \infty$ *are called* ***Dirichlet eigenvalues***.

Proof. Define $K : L^2(\Omega) \to L^2(\Omega)$ by setting Kf to be the unique weak solution of the homogenous Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By the global regularity theorem, $u \in H^2(\Omega)$ and thus we can write K as the composition

$$L^2(\Omega) \longrightarrow H^2(\Omega) \hookrightarrow L^2(\Omega).$$

Thus K is continuous as a composition of continuous mappings an moreover, since the embedding is compact by the Sobolev theorem, so is K. $\square$

Schauder Theory

Campanato-Estimates and Morrey-Spaces.

Lemma 1.7 (Minimality of Mean-Value). *Let* $\Omega \subseteq \mathbb{R}^n$ *open,* $f \in L^2(\Omega)$*,* $x_0 \in \Omega$ *and* $r > 0$*. Then*

$$\left\| f - \bar{f}_{r,x_0} \right\|^2_{L^2(\Omega_r(x_0))} = \min_{a \in \mathbb{R}} \| f - a \|^2_{L^2(\Omega_r(x_0))}.$$

Schauder Estimates.

Theorem 1.26 (Global Schauder-Estimate). *Let* $\Omega \subseteq\subseteq \mathbb{R}^n$ *of class* $C^{2,\alpha}$, $0 < \alpha < 1$*. Moreover, let* $a^{ij} \in C^{1,\alpha}(\Omega)$ *symmetric, uniformly elliptic and uniformly bounded,* $c \in C^\alpha(\Omega)$*,* $u_0 \in C^{2,\alpha}(\overline{\Omega})$*,* $f = (f^1, \dots, f^n) \in C^{1,\alpha}(\Omega)$ *and* $h \in C^\alpha(\Omega)$*. Then any solution* $u \in C^{2,\alpha}(\Omega)$ *of the Dirichlet boundary value problem*

$$\begin{cases} A_0 u + cu = -\frac{\partial}{\partial x^i} f^i + h & \text{in } \Omega, \\ u = u_0 & \text{on } \Omega \end{cases}$$

satisfies

$$\|u\|_{C^{2,\alpha}} \leq C(\|u\|_{H^1} + \|f\|_{C^{1,\alpha}} + \|h\|_{C^\alpha} + \|u_0\|_{C^{2,\alpha}})$$

where C *does not depend on* u*.*

Existence Theorems.

Proposition 1.9 (Method of Continuity). *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be Banach spaces. Given* $A_0, A_1 \in \mathcal{L}(X, Y)$ *define* $A_t := (1-t)A_0 + tA_1$*,* $t \in [0, 1]$*. Suppose that*

$$\exists C > 0 \forall t \in [0, 1] \, \forall x \in X : \|x\|_X \leq \|A_t x\|_Y.$$

Then A_0 *is surjective if and only if* A_1 *is surjective.*

Using the method of continuity 1.9, one can show existence results of solutions of Dircihlet boundary value problems. Define

$$A_0 := -\frac{\partial}{\partial x^i}\left(a^{ij}\frac{\partial}{\partial x^j}\right)$$

for $a^{ij} \in C^{1,\alpha}$ symmetric, uniformly elliptic and uniformly bounded. Consider the problem

$$\begin{cases} A_0 u + cu = -\frac{\partial}{\partial x^j} f^i + h & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

for $c \in C^\alpha$, $f = (f^1, \dots, f^n) \in C^{1,\alpha}$ and $h \in C^\alpha$. If $c \geq 0$, one can show existence and uniqueness of $C^{2,\alpha}$ solutions. First of all, suppose that solutions of

$$\begin{cases} A_0 u = h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

do exist. Let us define

$$X := \{u \in C^{2,\alpha} : u|_{\partial\Omega} = 0\} \qquad \text{and} \qquad Y := C^\alpha.$$

Then X and Y are Banach spaces, since X is a closed subset of a Banach space. Define now $A_1 := A_0 + c$. Then it is easy to show that A_0 and A_1 are continuous. Thus to apply the continuity method, we have to show the existence of a constant $C > 0$, such that for all $t \in [0, 1]$ and $u \in X$

$$\|x\|_{C^{2,\alpha}} \leq \|A_t x\|_{C^\alpha}$$

holds. But this looks like the Schauder-estimate 1.26. Indeed, since $u \in C^{2,\alpha}$ solves $A_t u = A_t u$, we get

$$\|u\|_{C^{2,\alpha}} \leq C(\|u\|_{H^1} + \|A_t u\|_{C^\alpha}).$$

Using ellipticity, integration by parts (justified since any fucntion in X vanishes on the boundary $\partial\Omega$) and $c \geq 0$, we compute

$$\begin{aligned} \lambda\|u\|_{H^1}^2 &= \lambda\int_\Omega |\nabla u|^2 \\ &\leq \int_\Omega a^{ij}\frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j} \\ &= \int_\Omega (A_0 u)u \\ &= \int_\Omega (A_0 u)u + ctu^2 - ctu^2 \\ &= \int_\Omega (A_t u)u - ctu^2 \end{aligned}$$

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$$\begin{aligned}&\leq \int_\Omega (A_t u)u \\ &\leq \|A_t u\|_{L^2} \|u\|_{L^2} \\ &\leq C \|A_t u\|_{C^\alpha} \|u\|_{H^1}.\end{aligned}$$

Maximum Principle

Weak Maximum Principle. Let $\Omega \subseteq\subseteq \mathbb{R}^n$. In what follows, we consider the second order homogenous differential operator in non-divergence form

$$Lu := a^{ij} \frac{\partial^2 u}{\partial x^i \partial x^j} + b^i \frac{\partial u}{\partial x^i} + cu$$

where $a^{ij}, b^i, c \in C^0(\bar{\Omega})$ and L is uniformly elliptic, i.e. there exists $\lambda > 0$ such that

$$a^{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2$$

holds for all $\xi \in \mathbb{R}^n$ and $x \in \Omega$.

Theorem 1.27 (Weak Maximum Principle). *Let $\Omega \subseteq\subseteq \mathbb{R}^n$ and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that $Lu \geq 0$. Then:*

(a) If $c \leq 0$ in Ω, then $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u_+$.
(b) If $c = 0$ in Ω, then $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.

Proof. Consider the perturbation $v_\varepsilon := u + \varepsilon e^{\gamma x_1}$ for $\varepsilon, \gamma > 0$ and use the first and second derivative test. □

Strong Maximum Principle.

Lemma 1.8 (Boundary Point Lemma, E. Hopf). *Let $B := B_\rho(y) \subseteq \mathbb{R}^n$ and $u \in C^2(B) \cap C^0(\bar{B})$ such that $Lu \geq 0$ in B with $c \leq 0$. Assume for some $x_0 \in \partial B$ that $u(x_0) \geq 0$ and $u(x) < u(x_0)$ for every $x \in B$. Then*

$$\limsup_{h \to 0} \frac{u(x_0 + h\eta) - u(x_0)}{h} < 0$$

for η the invard pointing unit normal at x_0.

Proof. Without loss of generality one can assume $\rho = 1$ and $y = 0$. Then define $w : \bar{B} \to \mathbb{R}$ by

$$w(x) := e^{-\alpha|x|^2} - e^{-\alpha}$$

for some $\alpha > 0$ to be determined. We compute

$$Lw \geq e^{-\alpha|x|^2}(4\mu|x|^2\alpha^2 - 2\alpha(\operatorname{tr} A + b^i x_i) + c).$$

Thus for some α large enough, we get that $Lw > 0$. Set

$$v := u - u(x_0) + \varepsilon w$$

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for some $\varepsilon > 0$ on the anulus $A := \overline{B}_1(0) \setminus B_{1/2}(0)$. For $\varepsilon > 0$ sufficiently small, we get that $v \leq 0$ on ∂A. Since moreover

$$Lv = Lu - cu(x_0) + \varepsilon Lw > 0$$

the weak maximum principle implies $v \leq 0$ on A. Hence $D_\eta^+ v \leq 0$, but

$$D_\eta^+ v = D_\eta^+ u + \varepsilon D_\eta^+ w$$

which yields the statement by observing that $D_\eta^+ w > 0$. □

References

[Lee13] John M. Lee. *Introduction to Smooth Manifolds*. Second Edition. Graduate Texts in Mathematics. Springer, 2013.