

# LIE ALGEBRA COHOMOLOGY

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**Abstract.** Aim of this talk is to give a short overview of the *cohomology of Lie algebras with coefficients in modules*. We follow the original construction of Chevalley-Eilenberg via complexes. We then state two results concerning *semisimple* Lie algebras, known as the *first and second Whitehead lemma*, and calculate an example.

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## Introduction

The archetypal example of a *cohomology theory* arises in differential topology: The *de Rham cohomology*. Given a smooth manifold  $M$ , we define  $\Omega^n(M) := \Gamma(\Lambda^n T^*M)$  for each  $n \in \omega$ , the space of *smooth differential  $n$ -forms on  $M$* . Moreover, is a sequence of mappings  $(d^n : \Omega^n(M) \rightarrow \Omega^{n+1}(M))_{n \in \omega}$ , called *exterior differentiation operators*, which roughly speaking generalize the notion of a differential of a function. They do satisfy the relation  $d^n \circ d^{n-1} = 0$  and thus we can define the  *$n$ -th de Rham cohomology group* to be the quotient space

$$H_{\text{dR}}^n(M) := \ker d^n / \text{im } d^{n-1}.$$

## The Chevalley-Eilenberg Complex

The definition of the  $n$ -th de Rham cohomology group  $H_{\text{dR}}^n(M)$  can actually be thought of a two-stage process: First we go from Diff to an intermediate category and then we apply a *homology functor*, which is a purely algebraic construct, to go from this intermediate

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category to  $\mathbb{R}\text{-Vect}$ . Aim of this section is to give the definition of this intermediate category and then to define such a functor explicitly for the case of Lie algebras.

**Definition 1.1 (Chain Complex).** Let  $\mathcal{A}$  be an abelian category. A  $\mathbb{Z}$ -graded chain complex in  $\mathcal{A}$  is defined to be a tuple  $((C_n)_{n \in \mathbb{Z}}, (\partial_n)_{n \in \mathbb{Z}})$ , consisting of a sequence  $(C_n)_{n \in \mathbb{Z}}$  of objects in  $\mathcal{A}$  and a sequence  $(\partial_n)_{n \in \mathbb{Z}}$  of morphisms in  $\mathcal{A}$ , such that

$$\partial_n \in \text{Hom}_{\mathcal{A}}(C_n, C_{n-1}) \quad \text{and} \quad \partial_n \circ \partial_{n+1} = 0$$

for all  $n \in \mathbb{Z}$ .

Dually, a  $\mathbb{Z}$ -graded cochain complex in  $\mathcal{A}$  is a  $\mathbb{Z}$ -graded chain complex in  $\mathcal{A}^{\text{op}}$ .

**Remark 1.1.** For notational simplicity, we will write  $(C_\bullet, \partial_\bullet)$  for a chain complex in  $\mathcal{A}$ .

**Remark 1.2.** For each abelian category  $\mathcal{A}$ , there is an abelian category  $\text{Ch}(\mathcal{A})$  of chain complexes in  $\mathcal{A}$  (see [Wei94, p. 7]).

**Definition 1.2 (Non-Negative Chain Complex).** Let  $\mathcal{A}$  be an abelian category. A chain complex  $(C_\bullet, \partial_\bullet) \in \text{Ch}(\mathcal{A})$  is said to be **non-negative**, iff  $C_n = 0$  for all  $n < 0$ . We denote by  $\text{Ch}_{\geq 0}(\mathcal{A})$  the full subcategory of  $\text{Ch}(\mathcal{A})$  of non-negative chain complexes.

In what follows, suppose that all rings admit an identity element and all modules are unital. Recall, that for  $R \in \text{CRing}$  and  $\mathfrak{g} \in {}_R\text{LieAlg}$ , the **universal enveloping algebra**  $U\mathfrak{g}$  of  $\mathfrak{g}$  is defined to be the quotient of the **tensor algebra**  $T\mathfrak{g}$

$$T\mathfrak{g} := \bigoplus_{n \in \omega} \mathfrak{g}^{\otimes n}$$

by the 2-sided ideal generated by the relations

$$\iota[x, y] = \iota(x)\iota(y) - \iota(y)\iota(x),$$

for all  $x, y \in \mathfrak{g}$ , where  $\iota : \mathfrak{g} \hookrightarrow T\mathfrak{g}$  denotes inclusion. Moreover,  $U$  is a functor from  ${}_R\text{LieAlg}$  to associative  $R$ -algebras. Hence we can define a  $U\mathfrak{g}$ -action on  $U\mathfrak{g} \otimes_R \Lambda^n \mathfrak{g}$  simply by

$$u(v \otimes x_1 \wedge \cdots \wedge x_n) := uv \otimes x_1 \wedge \cdots \wedge x_n,$$

for all  $n \in \omega$ .

**Definition 1.3 (Chevalley-Eilenberg Complex).** Let  $R \in \text{CRing}$  and  $\mathfrak{g} \in {}_R\text{LieAlg}$  which is free as an  $R$ -module. Denote by  $U\mathfrak{g}$  the universal enveloping algebra of  $\mathfrak{g}$ . Define a non-negative chain complex  $(C_\bullet, \partial_\bullet) \in \text{Ch}_{\geq 0}(U\mathfrak{g}\text{Mod})$  by

$$C_n := U\mathfrak{g} \otimes_R \Lambda^n \mathfrak{g}$$

for all  $n \in \omega$  and

$$\partial_n(u \otimes x_1 \wedge \cdots \wedge x_n) := \begin{cases} ux_1 & n = 1, \\ \theta_1 + \theta_2 & n > 1, \end{cases}$$

where

$$\theta_1 := \sum_{i=0}^n (-1)^{i+1} u x_i \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n,$$

and

$$\theta_2 := \sum_{1 \leq i < j \leq n} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n.$$

**Remark 1.3.** It is by no means obvious, that  $\partial_n \circ \partial_{n+1} = 0$  holds for the Chevalley-Eilenberg complex 1.3. However, it is a tedious computation, and we will only demonstrate the case  $n = 1$ . In this case

$$\begin{aligned} (\partial_1 \circ \partial_2)(u \otimes x \wedge y) &= \partial_1(ux \otimes y - uy \otimes x - u \otimes [x, y]) \\ &= u(xy - yx) - u[x, y] \\ &= 0, \end{aligned}$$

for all  $u \in U\mathfrak{g}$  and  $x, y \in \mathfrak{g}$ .

**Remark 1.4.** The definition of the boundary map  $\partial_n$  in the Chevalley-Eilenberg complex 1.3 is not as arbitrary as it might seem at first sight. Given  $\alpha \in \Omega^n(M)$  for a smooth manifold  $M$ , then we have that  $d\alpha(X_1, \dots, X_{n+1})$  is of the same form for any  $X_1, \dots, X_{n+1} \in \mathfrak{X}(M)$ . Actually, this formula can be used to give an invariant definition of the exterior derivative  $d$  in the de Rham theory (see [Lee13, pp. 370–372]).

### Left $\mathfrak{g}$ -Modules and the Cohomology of Lie Algebras

**Definition 1.4 (Category of Left  $\mathfrak{g}$ -Modules).** Let  $R \in \mathbf{CRing}$  and  $\mathfrak{g} \in {}_R\mathbf{LieAlg}$ . The *category of left  $\mathfrak{g}$ -modules*, written  ${}_g\mathbf{Mod}$ , is defined to be the category with objects *left  $\mathfrak{g}$ -modules*, i.e. modules  $M \in {}_R\mathbf{Mod}$  equipped with an  $R$ -bilinear product  $\mathfrak{g} \times M \rightarrow M$ ,  $(x, m) \mapsto xm$ , such that

$$[x, y]m = x(ym) - y(xm)$$

holds for all  $x, y \in \mathfrak{g}$  and  $m \in M$ , and *left  $\mathfrak{g}$ -module homomorphisms* as morphisms, i.e. morphisms  $f \in \mathbf{Hom}_R(M, N)$  such that

$$f(xm) = xf(m)$$

holds for all  $x \in \mathfrak{g}$  and  $m \in M$ .

**Proposition 1.1.** Let  $R \in \mathbf{CRing}$  and  $\mathfrak{g} \in {}_R\mathbf{LieAlg}$ . Then  ${}_g\mathbf{Mod}$  is an abelian category.

*Proof.* See [Wei94, p. 220]. □

We follow [KS06, p. 178].

**Proposition 1.2.** *Let  $\mathcal{A}$  be an abelian category and  $(C_\bullet, \partial_\bullet) \in \text{Ch}(\mathcal{A})$ . Then for every  $n \in \mathbb{Z}$ , there exists a unique monic*

$$\text{im } \partial_{n+1} \rightarrow \ker \partial_n,$$

where  $\text{im } \partial_{n+1} := \ker(\text{coker } \partial_{n+1})$ .

**Exercise 1.1.** Prove proposition 1.2. *Hint:* Use that  $\text{im } \partial_{n+1} \rightarrow C_n$  is monic by [Lan78, p. 199].

**Definition 1.5 (Homology of a Chain Complex).** *Let  $\mathcal{A}$  be an abelian category and  $(C_\bullet, \partial_\bullet) \in \text{Ch}(\mathcal{A})$ . Moreover, let  $n \in \mathbb{Z}$  and  $\text{im } \partial_{n+1} \rightarrow \ker \partial_n$  be the unique morphism assured by lemma 1.2. Then we define the  **$n$ -th homology object**, written  $H_n(C_\bullet, \partial_\bullet)$ , by*

$$H_n(C_\bullet, \partial_\bullet) := \text{coker}(\text{im } \partial_{n+1} \rightarrow \ker \partial_n) \in \text{ob}(\mathcal{A}).$$

Observe, that for each  $n \in \omega$ , we have that  $C_n \in {}_{\mathfrak{g}}\text{Mod}$  via  $\iota : \mathfrak{g} \hookrightarrow U\mathfrak{g}$ .

**Definition 1.6 (Cohomology of Lie Algebras).** *Let  $R \in \text{CRing}$  and  $\mathfrak{g} \in {}_R\text{LieAlg}$  which is free as an  $R$ -module. Moreover, let  $M \in {}_{\mathfrak{g}}\text{Mod}$  and  $(C_\bullet, \partial_\bullet)$  denote the Chevalley-Eilenberg complex 1.3. For  $n \in \omega$ , define the  **$n$ -th cohomology group of  $\mathfrak{g}$  with coefficients in  $M$** , written  $H^n(\mathfrak{g}, M)$ , to be the  $n$ -th homology object of the cochain complex*

$$\text{Hom}_{\mathfrak{g}}((C_\bullet, \partial_\bullet), M).$$

**Remark 1.5.** Actually, for  $n \in \omega$  we have that

$$\text{Hom}_{\mathfrak{g}}(C_n, M) \cong \text{Hom}_R(\Lambda^n \mathfrak{g}, M),$$

in  ${}_{\mathbb{Z}}\text{Mod}$ . Indeed, if  $n \geq 1$  and  $\varphi \in \text{Hom}_{\mathfrak{g}}(C_n, M)$ , define  $\bar{\varphi} \in \text{Hom}_R(\Lambda^n \mathfrak{g}, M)$  by

$$\bar{\varphi}(x_1 \wedge \cdots \wedge x_n) := \varphi(1 \otimes x_1 \wedge \cdots \wedge x_n),$$

and conversly, if  $\varphi \in \text{Hom}_R(\Lambda^n \mathfrak{g}, M)$ , define  $\bar{\varphi} \in \text{Hom}_{\mathfrak{g}}(C_n, M)$  by

$$\bar{\varphi}(u \otimes x_1 \wedge \cdots \wedge x_n) := u\varphi(x_1 \wedge \cdots \wedge x_n).$$

This is possible, since every left  $\mathfrak{g}$ -module is naturally a left  $U\mathfrak{g}$ -module (see [Wei94, pp. 224–225]). If  $n = 0$ , we define  $\bar{\varphi} \in \text{Hom}_R(R, M)$  by

$$\bar{\varphi}(r) := r\varphi(1),$$

for  $\varphi \in \text{Hom}_{\mathfrak{g}}(U\mathfrak{g} \otimes_R R, M) \cong \text{Hom}_{\mathfrak{g}}(U\mathfrak{g}, M)$  and for  $\varphi \in \text{Hom}_R(R, M)$  define similarly  $\bar{\varphi} \in \text{Hom}_{\mathfrak{g}}(U\mathfrak{g}, M)$  by

$$\bar{\varphi}(u) := u\varphi(1).$$

Hence we get an induced morphism

$$\begin{array}{ccc} \text{Hom}_{\mathfrak{g}}(C_{n-1}, M) & \xrightarrow{d^n} & \text{Hom}_{\mathfrak{g}}(C_n, M) \\ \downarrow \bar{\cdot} & & \downarrow \bar{\cdot} \\ \text{Hom}_R(\Lambda^{n-1} \mathfrak{g}, M) & \dashrightarrow & \text{Hom}_R(\Lambda^n \mathfrak{g}, M). \end{array}$$

Explicitly

$$\begin{aligned} d^n f(x_1, \dots, x_n) &= \sum_{i=1}^n (-1)^{i+1} x_i f(x_1, \dots, \hat{x}_i, \dots, x_n) \\ &\quad + \sum_{1 \leq i < j \leq n} (-1)^{i+j} f([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n), \end{aligned}$$

for  $n > 1$  and

$$d^1 f(x_1) = x_1 f(1).$$

**Remark 1.6.** There is a more general approach to the definition of the cohomology of Lie algebras via the notion of *right derived functors* which does not use the intermediate step of the Chevalley-Eilenberg complex.

**Example 1.1** ( $H^3(\mathfrak{sl}_2, k)$ ). Let  $k$  be a field with characteristic not equal to two and consider the *special linear Lie algebra over  $k$* , i.e.  $A \in M_2(k)$  such that  $\text{tr } A = 0$ . This is a three dimensional Lie algebra with ordered basis

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad e_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Hence the crucial portion of the Eilenberg-Chevalley cochain complex is given by

$$\dots \longrightarrow \text{Hom}_k(\Lambda^2 \mathfrak{sl}_2, k) \xrightarrow{d} \text{Hom}_k(\Lambda^3 \mathfrak{sl}_2, k) \longrightarrow 0.$$

We compute

$$\begin{aligned} df(e_1, e_2, e_3) &= e_1 f(e_2, e_3) - e_2 f(e_1, e_3) + e_3 f(e_1, e_2) \\ &\quad - f([e_1, e_2], e_3) + f([e_1, e_3], e_2) - f([e_2, e_3], e_1) \\ &= -2f(e_2, e_3) - 2f(e_3, e_2) - f(e_1, e_1) \\ &= -2f(e_2, e_3) + 2f(e_2, e_3) \\ &= 0, \end{aligned}$$

since  $k$  is interpreted as a trivial  $\mathfrak{sl}_2$ -module and by the alternating  $k$ -multilinear properties of  $f$ . Hence

$$H^3(\mathfrak{sl}_2, k) \cong \text{Hom}_k(\Lambda^3 \mathfrak{sl}_2, k) \cong k,$$

since  $\dim \text{Hom}_k(\Lambda^3 \mathfrak{sl}_2, k) = 1$ .

### The Whitehead Lemmas

Let  $M \in {}_{\mathfrak{g}}\text{Mod}$  and denote by  $\text{Der}(\mathfrak{g}, M)$  the *derivations from  $\mathfrak{g}$  to  $M$* , i.e. mappings  $D \in {}_R\text{Mod}$ , such that  $D[x, y] = x(Dy) - y(Dx)$  holds for all  $x, y \in \mathfrak{g}$ . If  $m \in M$ , define  $D_m(x) := xm$  for  $x \in \mathfrak{g}$ . This is called an *inner derivation*. The  $R$ -module of

all inner derivations from  $\mathfrak{g}$  to  $M$  is denoted by  $\text{Der}_{\text{Inn}}(\mathfrak{g}, M)$ . Clearly,  $\text{Der}_{\text{Inn}}(\mathfrak{g}, M)$  is an  $R$ -submodule of  $\text{Der}(\mathfrak{g}, M)$ . By [Wei94, p. 230] we have that

$$H^1(\mathfrak{g}, M) \cong \text{Der}(\mathfrak{g}, M) / \text{Der}_{\text{Inn}}(\mathfrak{g}, M).$$

Let  $M \in {}_R\text{LieAlg}$  be abelian. An *extension of  $\mathfrak{g}$  by  $M$*  is a short exact sequence

$$0 \longrightarrow M \longrightarrow \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

in  ${}_R\text{LieAlg}$ . There is a notion of equivalence classes of extensions of  $\mathfrak{g}$  by  $M$ . Denote by  $\text{Ext}(\mathfrak{g}, M)$  the set of all such equivalence classes. Then there is a one-to-one correspondence

$$\text{Ext}(\mathfrak{g}, M) \xleftarrow{1:1} H^2(\mathfrak{g}, M)$$

if  $M \in {}_{\mathfrak{g}}\text{Mod}$  (see [Wei94, p. 235]).

However, the situation gets much more easier if we make some assumptions.

**Theorem 1.2 (Whitehead's First Lemma).** *Let  $k$  be a field of characteristic zero and  $\mathfrak{g} \in {}_k\text{LieAlg}$  semisimple. Then for any finite-dimensional  $M \in {}_{\mathfrak{g}}\text{Mod}$  we have that*

$$H^1(\mathfrak{g}, M) = 0.$$

**Theorem 1.3 (Whitehead's Second Lemma).** *Let  $k$  be a field of characteristic zero and  $\mathfrak{g} \in {}_k\text{LieAlg}$  semisimple. Then for any finite-dimensional  $M \in {}_{\mathfrak{g}}\text{Mod}$  we have that*

$$H^2(\mathfrak{g}, M) = 0.$$

**Remark 1.7.** There cannot be a third Whitehead lemma, since

$$H^3(\mathfrak{sl}_2, k) \cong k,$$

by exercise 1.1.

There is a quite strong result obtained via the second Whitehead lemma, which gives rise to the *structure theory of Lie algebras*.

**Theorem 1.4 (Levi's Theorem).** *Let  $k$  be a field of characteristic zero and  $\mathfrak{g} \in {}_k\text{LieAlg}$  finite dimensional. Then there exists a semisimple Lie subalgebra  $\mathcal{L} \subseteq \mathfrak{g}$  (called a **Levi factor**), such that*

$$\mathfrak{g} \cong \mathcal{L} \ltimes \text{rad}(\mathfrak{g}).$$

*Proof.* See [Wei94, p. 247]. □

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