

# WHITEHEAD PRODUCT

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**Abstract.** Aim of this paper is to give a short overview of the definition and the basic properties of the non-generalized *Whitehead product*.

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## 1. Introduction

In the category of *compactly generated spaces*, suppose  $G$  is an  $H$ -group, i.e. a space satisfying the group axioms up to homotopy, then  $[X, G]$  is a group for any space  $X$ . This group need not be abelian. Thus a natural question is, if  $[X, G]$  is *nilpotent*. As the notion of nilpotence is based on the behaviour of *commutators*, it is natural to consider certain related products: First of all the *commutator product* or *Samelson product* defined as follows: If  $[\alpha] \in [X, G]$  and  $[\beta] \in [Y, G]$ , define  $\gamma : X \times Y \rightarrow G$  by

$$\gamma(x, y) := \alpha(x)\beta(y) (\alpha(x))^{-1} (\beta(y))^{-1}.$$

Then  $\gamma|_{X \vee Y}$  is nullhomotopic and thus yields a map  $\gamma : X \wedge Y \rightarrow G$ , whose homotopy class is defined to be the product of  $[\alpha]$  and  $[\beta]$ . When  $X = \mathbb{S}^n$ ,  $Y = \mathbb{S}^m$  and  $G = \Omega X$ , then  $[\mathbb{S}^n, G]$  is identified with  $\pi_n(G)$  since the  $\pi_1$  action is trivial for  $H$ -spaces, and the Samelson product

$$\pi_n(G) \otimes \pi_m(G) \rightarrow \pi_{n+m}(G)$$

translates to a pairing

$$\pi_{n+1}(X) \otimes \pi_{m+1}(X) \rightarrow \pi_{n+m+1}(X),$$

the *Whitehead product*, since  $\pi_n(G) \cong \pi_{n+1}(X)$  (see [Whi78, pp. 456–457]).

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## 2. Definition of the Whitehead Product

Notice, that for any  $(X, x_0), (Y, y_0) \in \text{Top}_*$ , their coproduct is given by

$$X \coprod Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subseteq X \times Y,$$

with basepoint  $(x_0, y_0)$ .

**Lemma 2.1.** *Let  $n, m \in \omega, n, m \geq 1$ . The space  $S^n \times S^m$  can be obtained from  $S^n \vee S^m$  by attaching an  $n + m$ -cell.*

*Proof.* Observe, that  $\mathbb{D}^{n+m} \cong \mathbb{D}^n \times \mathbb{D}^m$  and hence

$$S^{n+m-1} = \partial \mathbb{D}^{n+m} \cong (\partial \mathbb{D}^n \times \mathbb{D}^m) \cup (\mathbb{D}^n \times \partial \mathbb{D}^m) = (S^{n-1} \times \mathbb{D}^m) \cup (\mathbb{D}^n \times S^{m-1}).$$

Let

$$f_1 : S^{n-1} \times \mathbb{D}^m \rightarrow (S^{n-1} \times \mathbb{D}^m) / (S^{n-1} \times S^{m-1}) \cong * \times S^m$$

and

$$f_2 : \mathbb{D}^n \times S^{m-1} \rightarrow (\mathbb{D}^n \times S^{m-1}) / (S^{n-1} \times S^{m-1}) \cong S^n \times *$$

be the quotient maps. An application of the gluing lemma thus yields a map

$$f : S^{n+m-1} \rightarrow S^n \vee S^m.$$

Moreover, define

$$q : \mathbb{D}^n \times \mathbb{D}^m \rightarrow \mathbb{D}^n / S^{n-1} \times \mathbb{D}^m / S^{m-1} \cong S^n \times S^m$$

to be the product of the two quotient maps

$$\mathbb{D}^n \rightarrow \mathbb{D}^n / S^{n-1} \quad \text{and} \quad \mathbb{D}^m \rightarrow \mathbb{D}^m / S^{m-1}.$$

Thus we get a commutative diagram

$$\begin{array}{ccc} S^{n+m-1} & \xrightarrow{f} & S^n \vee S^m \\ \downarrow & & \downarrow \\ \mathbb{D}^{n+m} & \xrightarrow{q} & S^n \times S^m \end{array}$$

Suppose  $(X, g, h)$  is another cocone in  $\text{Top}$  for the pushout diagram:

$$\begin{array}{ccc} S^{n+m-1} & \xrightarrow{f} & S^n \vee S^m \\ \downarrow & & \downarrow \\ \mathbb{D}^{n+m} & \xrightarrow{q} & S^n \times S^m \end{array} \quad \begin{array}{c} \searrow h \\ \downarrow \\ \searrow g \end{array} \quad \begin{array}{c} \\ \\ X \end{array}$$

By [Mun00, p. 186],  $q$  is a quotient map. Moreover, for  $(x, y) \in \mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$ , we have that

$$g(x, y) = (h \circ f)(x, y) = h(*, *).$$

Thus  $g$  passes to the quotient by [Lee11, p. 72] to yield a unique map

$$\tilde{g} : \mathbb{S}^n \times \mathbb{S}^m \rightarrow X,$$

such that  $g = \tilde{g} \circ q$ . Finally, it is easy to check that

$$\begin{array}{ccc} \mathbb{S}^{n+m-1} & \xrightarrow{f} & \mathbb{S}^n \vee \mathbb{S}^m \\ \downarrow & & \downarrow \\ \mathbb{D}^{n+m} & \xrightarrow{q} & \mathbb{S}^n \times \mathbb{S}^m \end{array} \quad \begin{array}{c} \searrow h \\ \nearrow \tilde{g} \\ \searrow g \end{array} \quad \begin{array}{c} \\ \\ X \end{array}$$

commutes. □

For  $n, m \in \omega$ ,  $n, m \geq 1$ , consider the map  $f$  from lemma 2.1. Let  $(X, p) \in \text{Top}_*$ . If  $[\alpha] \in \pi_n(X, p)$  and  $[\beta] \in \pi_m(X, p)$ , we get two pointed maps

$$\alpha : \mathbb{S}^n \rightarrow X \quad \text{and} \quad \beta : \mathbb{S}^m \rightarrow X.$$

Forming their wedge  $\alpha \vee \beta : \mathbb{S}^n \vee \mathbb{S}^m \rightarrow X$ , defined by

$$(\alpha \vee \beta)(x, y) := \begin{cases} \alpha(x) & y = *, \\ \beta(y) & x = *, \end{cases}$$

and precomposing with  $f$ , yields a pointed map

$$(\alpha \vee \beta) \circ f : \mathbb{S}^{n+m-1} \rightarrow X.$$

Explicitly, if we consider

$$\alpha : (\mathbb{D}^n, \mathbb{S}^{n-1}) \rightarrow (X, p) \quad \text{and} \quad \beta : (\mathbb{D}^m, \mathbb{S}^{m-1}) \rightarrow (X, p),$$

we get that

$$((\alpha \vee \beta) \circ f)(x, y) = \begin{cases} \alpha(x) & x \in \mathbb{D}^n, y \in \mathbb{S}^{m-1}, \\ \beta(y) & x \in \mathbb{S}^{n-1}, y \in \mathbb{D}^m. \end{cases} \quad (1)$$

Hence if  $F : \alpha \simeq_{\mathbb{S}^{n-1}} \alpha'$  and  $F' : \beta \simeq_{\mathbb{S}^{m-1}} \beta'$ , we get that

$$H : ((\alpha \vee \beta) \circ f) \simeq_* ((\alpha' \vee \beta') \circ f),$$

where  $H : \mathbb{S}^{n+m-1} \times I \rightarrow X$  is defined by

$$H(x, y, t) := \begin{cases} F(x, t) & x \in \mathbb{D}^n, y \in \mathbb{S}^{m-1}, \\ F'(y, t) & x \in \mathbb{S}^{n-1}, y \in \mathbb{D}^m. \end{cases}$$

Thus we get a well defined map  $[-, -] : \pi_n(X) \times \pi_m(X) \rightarrow \pi_{n+m-1}(X)$ , defined by

$$[\alpha, \beta] := [(\alpha \vee \beta) \circ f].$$

**Definition 2.1 (Whitehead Product).** Let  $n, m \in \omega$ ,  $n, m \geq 1$ , and  $(X, p) \in \text{Top}_*$ . The product

$$[-, -] : \pi_n(X, p) \times \pi_m(X, p) \rightarrow \pi_{n+m-1}(X, p)$$

defined by

$$[\alpha, \beta] := [(\alpha \vee \beta) \circ f],$$

is called the **Whitehead product** and  $[-, -]$  is called the **Whitehead bracket**.

### 3. The Whitehead Product and the Conjugation Action

In this section, we want to have a closer look at  $[-, -] : \pi_1(X) \times \pi_n(X) \rightarrow \pi_n(X)$ . If  $n = 1$ , the definition of the Whitehead product in equation (1) results in figure 1a and using that  $\mathbb{S}^1$  is parametrized by  $\theta \mapsto e^{i\theta}$ , i.e. oriented counter clockwise, we get that

$$[\alpha, \beta] = [\alpha][\beta][\alpha]^{-1}[\beta]^{-1},$$

since any reparametrization of a path is homotopic relative to  $\partial I$  to the original path (a reparametrization of a path  $f$  in  $X$  is just a path  $f \circ \varphi$ , where  $\varphi : I \rightarrow I$  is continuous and  $\varphi|_{\partial I} = \text{id}_{\partial I}$ ). Thus  $[\alpha, \beta]$  coincides with the notation of a commutator in  $\pi_1(X)$ .

Let  $n > 1$ . In the previous case we implicitly used an explicit parametrization of the unit sphere  $\mathbb{S}^1$ , which resulted in a geometric notion called *orientation*. This can also be done in higher dimensions, however the geometric picture there is not that clear. We define an **orientation of  $\mathbb{D}^n$**  simply to be a choice of a generator of  $H_n(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong \mathbb{Z}$  (follows from the long exact sequence axiom).

**Lemma 3.1.** Let  $n \in \omega$ ,  $n > 1$ ,  $[\alpha] \in \pi_n(X)$  and  $h : (\mathbb{D}^n, \mathbb{S}^{n-1}) \rightarrow (\mathbb{D}^n, \mathbb{S}^{n-1})$  an orientation reversing homeomorphism, i.e.  $h$  is a homeomorphism and  $H_n(h)\langle e \rangle = -\langle e \rangle$  for a generator  $\langle e \rangle$  of  $H_n(\mathbb{D}^n, \mathbb{S}^{n-1})$ . Then

$$[\alpha \circ h] = -[\alpha].$$

*Proof.* Let  $\rho : \pi_n(Y, A) \rightarrow H_n(Y, A)$  denote the **Hurewicz homomorphism** defined by

$$\rho[f] := H_n(f)\langle e \rangle,$$

for a generator  $\langle e \rangle$  of  $H_n(\mathbb{D}^n, \mathbb{S}^{n-1})$  (see [Whi78, p. 166]). Using that

$$\rho : \pi_n(\mathbb{D}^n, \mathbb{S}^{n-1}) \rightarrow H_n(\mathbb{D}^n, \mathbb{S}^{n-1})$$

is an isomorphism for  $n > 1$  (see [Whi78, p. 168]), we compute

$$\begin{aligned} [\alpha \circ h] &= \pi_n(\alpha)[h] \\ &= \pi_n(\alpha)\rho^{-1}\rho[h] \\ &= \pi_n(\alpha)\rho^{-1}(H_n(h)\langle e \rangle) \\ &= -\pi_n(\alpha)\rho^{-1}\langle e \rangle \\ &= -\pi_n(\alpha)[\text{id}_{\mathbb{D}^n}] \\ &= -[\alpha]. \end{aligned}$$

□

The definition of the Whitehead product in equation (1) results in 1b. Thus using lemma 3.1 yields

$$[\alpha, \beta] = [\alpha \cdot \beta] - [\beta],$$

where  $\alpha \cdot \beta$  denotes the *conjugation action*, i.e. the action of  $\pi_1(X)$  on  $\pi_n(X)$ , since the boundary of the cylinder  $I \times \mathbb{D}^n$  is oriented coherently with  $1 \times \mathbb{D}^n$  and therefore discoherently with  $0 \times \mathbb{D}^n$ . Indeed, this can be seen as follows: If  $e^n$  and  $e^m$  are orientations of  $\mathbb{D}^n$  and  $\mathbb{D}^m$ , respectively, then their *cross product*  $e^n \times e^m$  is an orientation of  $\mathbb{D}^n \times \mathbb{D}^m$  (see [Whi78, p. 64] and [Hat01, pp. 268–278] for a more involved treatment). Moreover, we have that

$$\partial(e^n \times e^m) = \partial e^n \times e^m + (-1)^n e^n \times \partial e^m.$$

Therefore

$$\partial(e^1 \times e^n) = 1 \times e^n - 0 \times e^n - e^1 \times \partial e^n.$$

And we conclude by observing that the sum  $[\alpha] + [\beta]$  in  $\pi_n(X)$  is given by the pointed homotopy class of the composition

$$\mathbb{S}^n \xrightarrow{c} \mathbb{S}^n \vee \mathbb{S}^n \xrightarrow{\alpha \vee \beta} X,$$

where  $c : \mathbb{S}^n \rightarrow \mathbb{S}^n \vee \mathbb{S}^n$  denotes the mapping which collapses the equatorial  $\mathbb{S}^{n-1}$  in  $\mathbb{S}^n$  to a point, depicted in figure 2 (see [Hat01, p. 341]).

#### 4. Grading

Let  $(X, p) \in \text{Top}_*$ . For  $n \in \omega$  let  $L^n := \pi_{n+1}(X, p)$  and define

$$L := \bigoplus_{n \in \omega} L^n.$$

Moreover, define  $[-, -] : L \times L \rightarrow L$  by

$$\left[ \sum_i \alpha_i, \sum_j \beta_j \right] := \sum_{i,j} [\alpha_i, \beta_j].$$

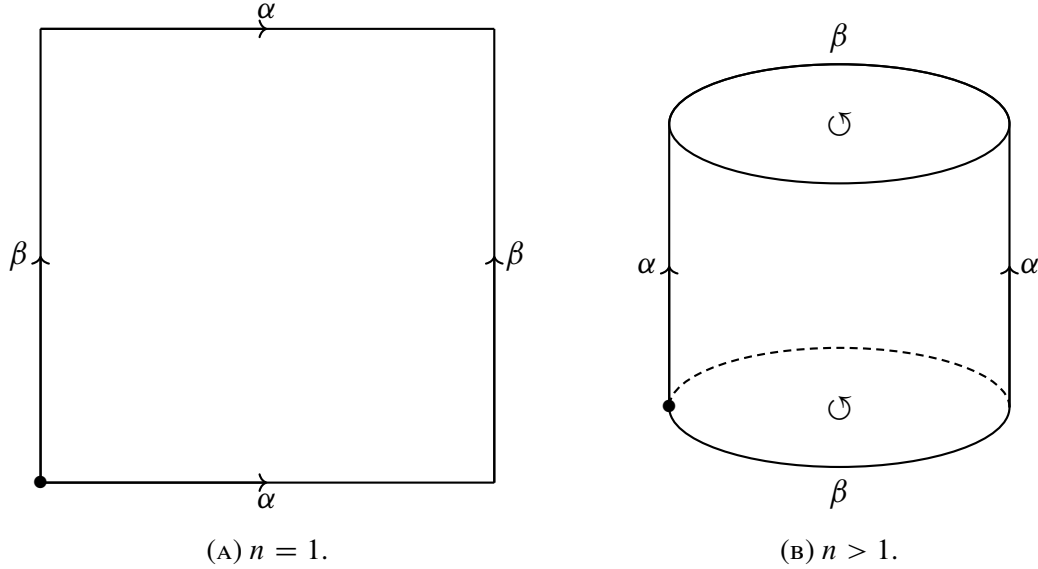


FIGURE 1. Whitehead bracket and the conjugation action.

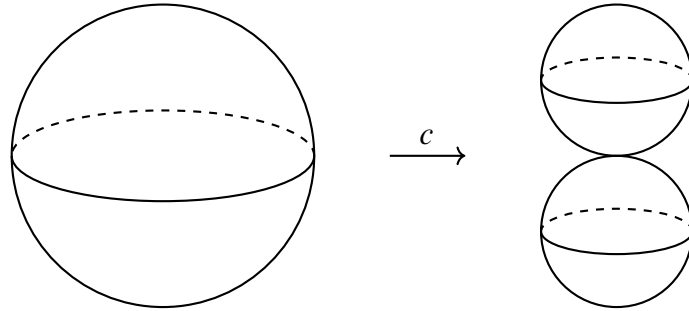


FIGURE 2. The collapsing map  $c : \mathbb{S}^n \rightarrow \mathbb{S}^n \vee \mathbb{S}^n$ .

Then clearly  $L^n L^m \subseteq L^{n+m}$  holds. It also turns out, that we have a Lie algebra-like structure on  $L$ , i.e. the bracket is bilinear, alternating and there is a Jacobi identity (for more details see [Whi78, pp. 474–478]).

**Proposition 4.1.** *Let  $n, m \in \omega$ ,  $n \geq 1$ ,  $[\alpha_1], [\alpha_2] \in \pi_{n+1}(X)$  and  $[\beta] \in \pi_{m+1}(X)$ . Then*

$$[\alpha_1 + \alpha_2, \beta] = [\alpha_1, \beta] + [\alpha_2, \beta] \quad \text{and} \quad [\beta, \alpha_1 + \alpha_2] = [\beta, \alpha_1] + [\beta, \alpha_2].$$

Recall, that for  $n \geq 1$  we have that  $H_n(\mathbb{S}^n) \cong \mathbb{Z}$ . Thus if we are given any continuous map  $f : \mathbb{S}^n \rightarrow \mathbb{S}^n$ , the induced map  $H_n(f)$  is simply a multiplication by a unique integer. This integer is defined to be the **degree of  $f$** , written  $\deg f$ .

**Proposition 4.2.** *Let  $n, m \in \omega$ ,  $[\alpha] \in \pi_{n+1}(X)$  and  $[\beta] \in \pi_{m+1}(X)$ . Then*

$$[\beta, \alpha] = (-1)^{(n+1)(m+1)}[\alpha, \beta].$$

*Proof.* Consider the **permutation map**  $\sigma : \mathbb{S}^{m+n+1} \rightarrow \mathbb{S}^{n+m+1}$  defined by

$$(y_1, \dots, y_{m+1}, x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_{n+1}, y_1, \dots, y_{m+1}).$$

Then clearly  $\deg \sigma = (-1)^{(n+1)(m+1)}$ , since  $\sigma$  is the composition of permutations and hence orthogonal transformations. Using lemma 3.1, we compute

$$[\beta, \alpha] = [(\beta \vee \alpha) \circ f] = [(\alpha \vee \beta) \circ f \circ \sigma] = (-1)^{(n+1)(m+1)}[\alpha, \beta].$$

□

**Proposition 4.3.** *Let  $n, m, r \in \omega$ ,  $n, m, r \geq 1$ ,  $[\alpha] \in \pi_{n+1}(X)$ ,  $[\beta] \in \pi_{m+1}(X)$  and  $[\gamma] \in \pi_{r+1}(X)$ . Then*

$$(-1)^{r(n+1)}[\alpha, [\beta, \gamma]] + (-1)^{n(m+1)}[\beta, [\gamma, \alpha]] + (-1)^{m(r+1)}[\gamma, [\alpha, \beta]] = 0$$

### References

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