ALGEBRAIC TOPOLOGY II SUMMARY

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Abstract. This is a rough summary of the course *Algebraic Topology II* held at *ETH Zurich* by *Prof. Dr. William J. Merry* in spring 2018. The main focus of this summary is to give a neat preparation for the oral exam.

Contents

Homology of Product Spaces

The Universal Coefficient and the Künneth Theorem.

Proposition 1.1. *Let* $A \in \mathsf{Ab}$. *Then* $(-) \otimes A : \mathsf{Ab} \to \mathsf{Ab}$ *and* $A \otimes (-) : \mathsf{Ab} \to \mathsf{Ab}$ *are both right exact.*

Example 1.1. $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n = \mathbb{Z}_{\gcd(m,n)}$.

Definition 1.1 (Tor). *Let* $A \in \mathsf{Ab}$ *and*

$$0 \longrightarrow K \xrightarrow{f} F \longrightarrow A \longrightarrow 0$$

a short free resolution of A. *Given any* $B \in \mathsf{Ab}$, *set*

$$\operatorname{Tor}(A, B) := \ker(f \otimes \mathrm{id}_B).$$

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Example 1.2. If either A or B are torsion free, then $\operatorname{Tor}(A, B) = 0$.

Example 1.3. $\operatorname{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) = \mathbb{Z}_{\gcd(m,n)}$.

Theorem 1.1 (Universal Coefficient Theorem). *Let $(C_\bullet, \partial_\bullet)$ be a free chain complex and $A \in \mathsf{Ab}$. Then for any $n \in \omega$ there is a split exact sequence*

$$0 \longrightarrow H_n(C_\bullet) \otimes A \longrightarrow H_n(C_\bullet \otimes A) \longrightarrow \operatorname{Tor}(H_{n-1}(C_\bullet), A) \longrightarrow 0.$$

Theorem 1.2 (Künneth Theorem). *Let $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ be two non-negative free chain complexes. Then there exists a split exact sequence*

$$0 \to \bigoplus_{i+j=n} H_i(C_\bullet) \otimes H_j(C'_\bullet) \to H_n(C_\bullet \otimes C'_\bullet) \to \bigoplus_{k+l=n-1} \operatorname{Tor}\big(H_k(C_\bullet), H_l(C'_\bullet)\big) \to 0.$$

The Eilenberg-Zilber Theorem and the Künneth Formula.

Theorem 1.3 (The Augmented Acyclic Models Theorem). *A Let $\mathcal{C}$ be a category with family of models $\mathcal{M}$. Consider*

$$S, T : \mathcal{C} \to \operatorname{AugCh}(\mathsf{Ab})$$

such that:

- *S_n is free with basis contained in $\mathcal{M}$ for any $n \in \omega$.*
- *Any $M \in \mathcal{M}$ is totally T-acyclic, i.e. $H_n(S(M)) = 0$ for all $n \geq 1$ and $H_0(S(M)) = \mathbb{Z}$.*

Then there exists a natural augmentation preserving chain map

$$\theta : S \Rightarrow T$$

Moreover, any two such natural augmenation preserving chain maps are naturally chain homotopic.
If additionally T_n is free with basis contained in $\mathcal{M}$ and each model $M \in \mathcal{M}$ is totally S-acyclic, then every such natural augmentation preserving chain map is a natural chain equivalence.

Theorem 1.4 (Eilenberg-Zilber). *Let $X, Y \in \mathsf{Top}$. Then there exists a chain equivalence*

$$\Omega : C_\bullet(X \times Y) \to C_\bullet(X) \otimes C_\bullet(Y)$$

unique up to chain homotopy. Any such map Ω is called an ***Eilenberg-Zilber morphism****.*

Proof. We make use of the augmented acyclic models theorem 1.3. In $\mathsf{Top} \times \mathsf{Top}$ define a family of models $\mathcal{M}$ by

$$\mathcal{M} := \{(\Delta^i, \Delta^j) : i, j \in \omega\}.$$

Moreover, define $S, T : \mathsf{Top} \times \mathsf{Top} \to \operatorname{AugCh}(\mathsf{Ab})$ by

$$S(X, Y) := C_\bullet(X \times Y) \qquad \text{and} \qquad T(X, Y) := C_\bullet(X) \otimes C_\bullet(Y).$$

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Since $\Delta^i \times \Delta^j$ is convex, we get that each model $M := (\Delta^i, \Delta^j)$ is totally S-acyclic. Moreover, the Künneth theorem 1.2 implies that each model M is totally T-acyclic.
That S_n is free with basis contained in $\mathcal{M}$ can be seen by choosing the diagonal map $d_n : \Delta^n \to \Delta^n \times \Delta^n$ for any $n \in \omega$. Finally, T_n is also free with basis contained in $\mathcal{M}$, since we can choose the model basis

$$\{(\Delta^i, \Delta^j) : i + j = n\}$$

for fixed $n \in \omega$ and $\iota_i \otimes \iota_j \in (C_\bullet(\Delta^i) \otimes C_\bullet(\Delta^j))_n$, where $\iota_k : \Delta^k \to \Delta^k$ denotes the identity map. □

Corollary 1.1 (Künneth Formula). *Let $X, Y \in \mathsf{Top}$. Then there is a split exact sequence*

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \longrightarrow H_n(X \times Y) \longrightarrow \bigoplus_{k+l=n-1} \operatorname{Tor}\big(H_k(X), H_l(Y)\big) \longrightarrow 0.$$

Example 1.4. Let $n \in \omega$, $n \geq 1$. Define the ***n-torus*** $\mathbb{T}^n$ by

$$\mathbb{T}^n := \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n}.$$

Using induction and the Künneth theorem 1.1, one can show that

$$H_k(\mathbb{T}^n) = \mathbb{Z}^{\binom{n}{k}}.$$

Cohomology

Proposition 1.2. *Let $A \in \mathsf{Ab}$. Then* $\operatorname{Hom}(-, A) : \mathsf{Ab} \to \mathsf{Ab}$ *and* $\operatorname{Hom}(A, -) : \mathsf{Ab} \to \mathsf{Ab}$ *are both left exact.*

Corollary 1.2. *Let $X \in \mathsf{Top}$ be of finite type, i.e. $H_n(X)$ is finitely generated for any $n \in \mathbb{Z}$. Then*

$$H^n(X) \cong H_n(X)/T_n(X) \oplus T_{n-1}(X)$$

where $T_n(X)$ denotes the torsion subgroup of $H_n(X)$, i.e. the subgroup consisting of all elements of finite order.

Theorem 1.5 (Universal Coefficient Theorem for Cohomology). *Let $X \in \mathsf{Top}$ of finite type and $A \in \mathsf{Ab}$. Then there is a split exact sequence*

$$0 \longrightarrow H^n(X) \otimes A \longrightarrow H^n(X; A) \longrightarrow \operatorname{Tor}(H^{n+1}(X), A) \longrightarrow 0.$$

The Cohomology Ring.

Proposition 1.3. *Let $X \in \mathsf{Top}$ and $R \in \mathsf{Ring}$. Then there exists a contravariant functor*

$$C(-; R) : \mathsf{Top} \to \mathsf{GRing}.$$

Proof. We proceed in two (uncomplete) steps.

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Step 1: Definition on objects. Let $X \in \mathsf{Top}$. For $\alpha \in C^n(X;R)$ and $\beta \in C^m(X;R)$ define

$$(\alpha \cup \beta)(\sigma) := \alpha(\sigma \circ A(e_0, \dots, e_n))\beta(\sigma \circ A(e_n, \dots, e_{n+m})),$$

for all singular $n+m$-simplices σ in X. Hence extending by linearity yields a map

$$\cup : C^n(X;R) \times C^m(X;R) \to C^{n+m}(X;R).$$

Moreover, if

$$C(X;R) := \bigoplus_{n \in \omega} C^n(X;R),$$

we define $\cup : C(X;R) \times C(X;R) \to C(X;R)$ by

$$\sum_i \alpha_i \cup \sum_j \beta_j := \sum_{i,j} \alpha_i \cup \beta_j.$$

This is called the ***cup product on $C(X;R)$***. It is easily verified that $(C(X;R), \cup) \in \mathsf{GRing}$.
Step 2: Definition on morphisms. Let $n \in \omega$ and $f \in \mathsf{Top}(X,Y)$. For $\alpha \in C^n(Y;R)$ define

$$C(f;R)(\alpha) := C^n(f;R)(\alpha) \in C^n(X;R),$$

and extend by linearity.

□

Lemma 1.1. *Let $R \in \mathsf{GRing}$ and I be a two-sided homogeneous ideal in R. Then also $R/I \in \mathsf{GRing}$ with*

$$R/I = \bigoplus_{n \in \omega} R^n/(R^n \cap I).$$

Theorem 1.6. *Let $R \in \mathsf{Ring}$. Then there is a contravariant functor*

$$H(-;R) : \mathsf{hTop} \to \mathsf{GRing}.$$

Proof. Set

$$Z := \bigoplus_{n \in \omega} Z^n(X;R) \qquad \text{and} \qquad B := \bigoplus_{n \in \omega} B^n(X;R).$$

Then Z is a homogeneous subring of $C(X;R)$ by using the fact that

$$d(\alpha \cup \beta) = d\alpha \cup \beta + (-1)^n \alpha \cup d\beta$$

for any $\alpha \in C^n(X;R)$ and $\beta \in C^m(X;R)$ holds. Moreover, B is a homogeneous two-sided ideal in Z. Therefore by lemma 1.1, we have

$$H(X;R) = \bigoplus_{n \in \omega} Z^n(X;R)/B^n(X;R) = \bigoplus_{n \in \omega} H^n(X;R).$$

□

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Example 1.5. Let $n \in \omega$, $n \geq 1$. Then using the fact that $\widetilde{H}_k(\mathbb{S}^n) = \mathbb{Z}$ if $k = n$ and zero otherwise, corollary 1.2 implies that

$$H^0(\mathbb{S}^n) = \mathbb{Z} \qquad \text{and} \qquad H^n(\mathbb{S}^n) = \mathbb{Z}$$

and zero otherwise. Thus

$$H(\mathbb{S}^n; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}.$$

Denote the generator of the first summand by 1 and the second by X, we get that $X \cup X \in H^{2n}(\mathbb{S}^n) = 0$ and thus

$$H(\mathbb{S}^n; \mathbb{Z}) \cong \mathbb{Z}[X]/(X^2).$$

Actually, if $R \in \mathsf{CRing}$, then $H(-; R)$ attains values in CGRing.

Definition 1.2 (Diagonal Approximation). *A* ***diagonal approximation*** *is defined to be a natural chain map*

$$C_\bullet(-) \to C_\bullet(-) \otimes C_\bullet(-)$$

such that $D_0(x) = x \otimes x$ holds for any $x \in X$, $X \in \mathsf{Top}$.

Theorem 1.7 (Alexander-Whitney Formula). *An Eilenberg Zilber morphism*

$$\Omega : C_\bullet(X \times Y) \to C_\bullet(X) \otimes C_\bullet(Y)$$

is given by the ***Alexander-Whitney formula***

$$\Omega(\sigma) := \sum_{i=0}^{n} (\pi_1 \circ \sigma \circ A(e_0, \ldots, e_i)) \otimes (\pi_2 \circ \sigma \circ A(e_i, \ldots, e_n)) \tag{1}$$

for any $\sigma : \Delta^n \to X \times Y$.

Proposition 1.4. *For the Alexander-Whitney choice of an Eilenberg-Zilber morphism Ω, the composition*

$$C^\bullet\delta \circ \mathrm{Hom}(\Omega, R) \circ \mu$$

where $\mu : C^\bullet(X; R) \otimes C^\bullet(X; R) \to \mathrm{Hom}(C_\bullet(X) \otimes C_\bullet(X), R)$ is defined by

$$\mu(\alpha \otimes \beta)\left(\sum_{k=0}^{n+m} \sigma_k \otimes \sigma'_{n+m-k}\right) := \alpha(\sigma_n)\beta(\sigma'_m)$$

coincides with the cup product.

Proof. Let $\alpha \in C^n(X; R)$, $\beta \in C^m(X; R)$ and $\sigma \in C^{n+m}(X)$. We compute

$$\begin{aligned}(C^\bullet\delta \circ \mathrm{Hom}(\Omega, R) \circ \mu)(\alpha \otimes \beta)(\sigma) &= \mathrm{Hom}(\Omega \circ \delta, R)(\mu(\alpha \otimes \beta))(\sigma) \\ &= \mu(\alpha \otimes \beta) \circ \Omega \circ C_\bullet\delta(\sigma) \\ &= \mu(\alpha \otimes \beta)(\Omega(\delta \circ \sigma)) \\ &= (\alpha \cup \beta)(\sigma).\end{aligned}$$

$\square$

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Theorem 1.8. *Let $R \in \mathsf{CRing}$ and $X \in \mathsf{Top}$. Then*

$$\langle\alpha\rangle \cup \langle\beta\rangle = (-1)^{nm}\langle\beta\rangle \cup \langle\alpha\rangle$$

for any $\langle\alpha\rangle \in H^n(X;R)$ and $\langle\beta\rangle \in H^m(X;R)$.

Proof. Since $\Omega \circ C_\bullet\delta$ and twist $\circ\Omega\circ C_\bullet\delta$ are both diagonal approximations, hence naturally chain homotopic. Now just evaluate both compositions. □

Corollary 1.3. *Let $X, Y \in \mathsf{Top}$ of finite type and suppose that $H_n(Y)$ is free abelian for any $n \in \mathbb{Z}$. Then the cross product*

$$H(X) \otimes H(Y) \xrightarrow{\times} H(X \times Y)$$

is an isomorphism of graded rings.

Example 1.6. Suppose $\mathbb{T}^n$ is the n-torus from example 1.4. We claim that

$$H(\mathbb{T}^n;\mathbb{Z}) \cong \mathbb{Z}[X_1,\dots,X_n]/(X_k^2).$$

Indeed, example 1.5, implies the base case for an induction over n. Suppose the claim holds for some $n \in \omega$, $n \geq 1$. Then using corollary 1.3 implies that

$$\begin{aligned}
H(\mathbb{T}^{n+1}) &= H(\mathbb{T}^n \times \mathbb{S}^1)\\
&= H(\mathbb{T}^n) \otimes H(\mathbb{S}^1)\\
&= \mathbb{Z}[X_1,\dots,X_n]/(X_k^2) \otimes \mathbb{Z}[X_{n+1}]/(X_{n+1}^2)\\
&= \mathbb{Z}[X_1,\dots,X_{n+1}]/(X_k^2).
\end{aligned}$$

Fibre Bundles

Definition 1.3 (Fibre Bundle). *Let $p \in \mathsf{Top}(E,X)$ surjective and $F \in \mathsf{Top}$ non-empty. We say that **p is a fibre bundle over X with fibre F** iff for any $x \in X$ there exists a neighbourhood U of x in X and a homeomorphism $h : p^{-1}(U) \to U \times F$ such that*

$$\begin{tikzcd}
p^{-1}(U) \arrow[r, "h"] \arrow[dr, "p"'] & U \times F \arrow[d, "\pi"] \\
& U
\end{tikzcd}$$

commutes.

Theorem 1.9 (Leray-Hirsch). *Let $F \to E \xrightarrow{p} X$ be a fibre bundle and $R \in \mathsf{CRing}$ such that $H^n(F;R)$ is a finitely generated free R-module for any $n \in \mathbb{Z}$ and that a cohomology extension ξ of the fibre F exists. Then the mapping*

$$L : H(X;R) \otimes_R H(F;R) \to H(E;R)$$

defined by

$$L(\langle\alpha\rangle \otimes \langle\beta\rangle) := H(p)\langle\alpha\rangle \cup \xi\langle\beta\rangle$$

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is an isomorphism.

Proof. Step 1: X paracompact and pointed contractible. Then the fibre bundle is trivial, i.e. $E \cong X \times F$ and there exists $x_0 \in X$, such that $\iota_{x_0} : E_{x_0} \hookrightarrow E$ is a homotopy equivalence. Thus $H^\bullet(\iota_{x_0})$ is an isomorphism and hence is ξ. Using $H(X; R) \cong R$, we get

$$H(X; R) \otimes_R H(F; R) \cong H(F; R) \cong H(E; R).$$

Step 2: X finite dimensional cell complex. We perform an induction over $\dim X$. The base case follows immediately from step 1. Suppose $\dim X = n$. Define U to be the union of n-cells, where we remove from each n-cell a single point. Moreover, let V be the union of all n-cells. Then $X^n = U \cup V$ and the dual version of Mayer-Vietoris yields

$$\begin{tikzcd}
H(X^n; R) \otimes_R M \arrow[r, "L_{X^n}"] \arrow[d] & H(E_{X^n}; R) \arrow[d] \\
(H(U; R) \otimes_R M) \oplus (H(V; R) \otimes_R M) \arrow[r, "(L_U, L_V)"] \arrow[d] & H(E_U; R) \oplus H(E_V; R) \arrow[d] \\
H(U \cap V; R) \otimes_R M \arrow[r, "L_{U\cap V}"] & H(E_{U\cap V}; R)
\end{tikzcd}$$

where $M := H(F; R)$. Since M is free, the left hand side is exact.
Step 3: L_U, L_V and $L_{U\cap V}$ are isomorphisms. First of all X^{n-1} is a strong deformation retract of U and thus L_U is an isomorphism by induction hypothesis (since $L_{X^{n-1}}$ is an isomorphism).

□

The Leray-Hirsch theorem 1.9 is useless unless a cohomology extension of the fibre exists. This is the content of the so-called *Thom-isomorphism theorem.*

Theorem 1.10 (Thom Isomorphism Theorem).

The Duality Theorem

Topological Manifolds.

Definition 1.4 ($\mathsf{Op}(X)$). *Let $X \in \mathsf{Top}$. Define $\mathsf{Op}(X)$ to be the category with objects all the open sets of X and* $\mathrm{Hom}(U, V)$ *to be the singleton $\iota_U^V : U \hookrightarrow V$ if $U \subseteq V$ and empty otherwise.*

Definition 1.5 (Cap Product). *Let $R \in \mathsf{CRing}$ and $X \in \mathsf{Top}$. The pairing*

$$\frown : C^n(X; R) \otimes C_{n+m}(X; R) \to C_m(X; R)$$

defined by

$$\alpha \frown (\sigma \otimes r) := r\alpha(\sigma \circ A(e_0, \dots, e_n)) \otimes (\sigma \circ A(e_n, \dots, e_{n+m}))$$

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*is called **cap product**.*

Cech Cohomology.

Definition 1.6. *Let $\mathcal{K} \subseteq \mathsf{Top}^2$ be the full subcategory with objects all pairs (L, K) with $K \subseteq L \subseteq X$ compact for some Euclidean neighbourhood retract X.*

Definition 1.7 (Cech Cohomology). *Let $K \subseteq L \subseteq X$ be compact subspaces of an Euclidean neighbourhood retract and let $A \in \mathsf{Ab}$. We define the **Cech cohomology of (L, K) with coefficients in A** to be the abelian group*

$$\check{H}^k(L, K; A) := \operatorname{colim} H^k(V, U; A).$$

Let M be an n-dimensional manifold. Then for any $R \in \mathsf{CRing}$ we can define

$$\mathcal{O}(M; R) := \coprod_{x \in M} H_n(M, M \setminus \{x\}; R).$$

Observe that $H_n(M, M \setminus \{x\}; R) \cong \mathbb{R}$ as R-modules by excision. Hence it makes sense to define an ***orientation of M*** to be a section $\sigma : M \to \mathcal{O}(M; A)$ of the projection

$$\pi : \mathcal{O}(M; A) \to M$$

in Top, i.e. $\pi \circ \sigma = \mathrm{id}_M$ such that $\sigma(x)$ is a generator for $H_n(M, M \setminus \{x\}); R$ for any $x \in M$.

Theorem 1.11 (Duality Theorem). *Let M be an n-dimensional oriented topological manifold. Then for any pair $K \subseteq L \subseteq M$ compact, the duality morphism*

$$D_{KL} : \check{H}^k(L, K) \to H_{n-k}(M \setminus K, M \setminus L)$$

is an isomorphism.

Corollary 1.4 (Poincaré Duality). *Let M be an n-dimensional oriented closed topological manifold with fundamental class $\langle o_M \rangle \in H_n(M)$ (i.e. $\langle o_M \rangle$ is a generator of $H_n(M)$). Then*

$$H^k(M) \to H_{n-k}(M) \qquad \langle c \rangle \mapsto \langle c \rangle \frown \langle o_M \rangle$$

is an isomorphism.

Homotopy Theory

Theorem 1.12 (Homotopy Sequence). *Let (X, X') be a pointed pair. Then there is a long exact sequence*

$$\ldots \longrightarrow \pi_n(X') \longrightarrow \pi_n(X) \longrightarrow \pi_n(X, X') \xrightarrow{\delta} \pi_{n-1}(X') \longrightarrow \ldots$$

Lemma 1.2. *Any trivial fibre bundle is a weak fibration.*

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Theorem 1.13 (Homotopy Sequence of a Fibration). *Let $p : E \to X$ be a fibration with fibre F. Then there exists a long exact sequence*

$$\ldots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(X) \longrightarrow \pi_{n-1}(F) \longrightarrow \ldots$$

Corollary 1.5. *Let $F \to E \overset{p}{\to} X$ be a fibre bundle. Then p is a weak fibration.*

Corollary 1.6 (Homotopy Sequence of a Weak Fibration). *Let $p : E \to X$ be a weak fibration. Choose basepoints $y_0 \in E$ and $x_0 := p(y_0) \in X$. Let $F := p^{-1}(x_0)$. Then there is a long exact sequence*

$$\ldots \longrightarrow \pi_n(F) \longrightarrow \pi_n(E) \longrightarrow \pi_n(X) \longrightarrow \pi_{n-1}(F) \longrightarrow \ldots$$