MATHEMATICAL METHODS OF QUANTUM MECHANICS SUMMARY

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Abstract.

Contents

Postulates of Quantum Mechanics

Quantum mechanical system	Hilbert space $\mathcal{H}$
State	$\psi \in \mathcal{H}, \|\psi\| = 1$
Observables	Self-adjoint operators on $\mathcal{H}$
Expected Value of an observable A in the state ψ	$\langle \psi, A\psi \rangle$
Variance of an observable A in the state ψ	$\Delta A_\psi := \langle \psi, A^2\psi \rangle - \langle \psi, A\psi \rangle^2$

Lemma 1.1 (Heisenberg Uncertainity Principle). *Let A and B two self-adjoint operators on a Hilbert space $\mathcal{H}$. Then for any state ψ*

$$\Delta A_\psi \Delta B_\psi \geq \frac{1}{4} \left| \langle \psi, [A, B]\psi \rangle \right|^2 .$$

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Unbounded Operators

Definition 1.1 (Linear Operator). *Let $\mathcal{H}$ be a Hilbert space. A* ***(linear) operator on $\mathcal{H}$*** *is simply a linear map $A : D(A) \to \mathcal{H}$, where $D(A)$ is a linear subspace of $\mathcal{H}$.*

Definition 1.2 (Closed Operator). *An operator A on $\mathcal{H}$ is said to be* ***closed****, iff Γ_A is closed in $\mathcal{H} \times \mathcal{H}$.*

Definition 1.3 (Closable Operator). *An operator A on $\mathcal{H}$ is said to be* ***closable****, iff $\overline{\Gamma}_A$ is a linear graph, i.e. $(0, y) \in \overline{\Gamma}_A$ implies $y = 0$. The corresponding operator associated to $\overline{\Gamma}_A$ is denoted by $\overline{A}$ and called the* ***closure*** *of A. Clearly $A \subseteq \overline{A}$.*

Definition 1.4 (Adjoint). *Let A be a densly defined operator on $\mathcal{H}$. Set*

$$D(A^*) := \{\psi \in \mathcal{H} : \exists \eta \in \mathcal{H} \text{ s.t. } \forall \varphi \in D(A)\ \langle A\varphi, \psi\rangle = \langle \varphi, \eta\rangle\},$$

and $A^*\psi := \eta$. The operator A^* is called the ***adjoint*** *of A.*

Theorem 1.1. *Let A be a densly defined operator on $\mathcal{H}$. Then:*
(a) A^* is closed.
(b) A is closable if and only if $D(A^*)$ is dense.
(c) If A is closable, then $(\overline{A})^* = A^*$.

Definition 1.5 (Symmetric Operator). *A densly defined operator A is said to be* ***symmetric****, iff $A \subseteq A^*$.*

Definition 1.6 (Self-adjoint Operator). *A densly defined operator is said to be* ***self-adjoint****, iff $A = A^*$.*

Definition 1.7 (Essentially Self-adjoint Operator). *A A symmetric operator A is said to be* ***essentially self-adjoint****, iff $\overline{A}$ is self-adjoint.*

The Spectral Theorem

Projection Valued Measures.

Definition 1.8 (Projection Valued Measure). *Let $\mathcal{H}$ be a Hilbert space. A function $P : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ is said to be a* ***projection valued measure****, iff*
(i) For all $\Omega \in \mathcal{B}(\mathbb{R})$, $P(\Omega)$ is an orthogonal projection, i.e.

$$P(\Omega)^2 = P(\Omega) = P(\Omega)^*.$$

(ii) $P(\mathbb{R}) = \mathrm{id}_{\mathcal{H}}$.
(iii) If $(\Omega_n)_{n\in\omega}$ is a sequence of pairwise disjoint elements of $\mathcal{B}(\mathbb{R})$, then

$$P(\Omega)\psi = \sum_{n\in\omega} P(\Omega_n)\psi,$$

for all $\psi \in \mathcal{H}$.

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Definition 1.9 (Resolution of the Identity). *Let $\mathcal{H}$ be a Hilbert space and $P : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ a projection valued measure. The function $p : \mathbb{R} \to \mathcal{L}(\mathcal{H})$ defined by*

$$p(\lambda) := P\left(-\infty, \lambda\right],$$

is called the ***resolution of the identity associated to a projection valued measure****.*

The Spectral Theorem.

Theorem 1.2 (Spectral Theorem). *Let A be a self-adjoint operator. Then there exists a unique projection valued measure P^A such that $D(A) = \{\psi \in \mathcal{H} : \int |\lambda|^2 \, d\mu_\psi^A(\lambda)\}$ and*

$$A = \int \lambda dp^A(\lambda).$$

The Schrödinger Equation.

Theorem 1.3. *Let $\mathcal{H}$ be a Hilbert space and $H : D(H) \to \mathcal{H}$ be self adjoint. Moreover, set $U(t) := \exp(-iHt)$ for $t \in \mathbb{R}$. Then:*

(*a*) *$U(t)$ is a strongly continuous one parameter unitary group.*

(*b*) *The limit*

$$\lim_{t \to 0} \frac{U(t) - 1}{t} \psi$$

exists if and only if $\psi \in D(H)$. Then

$$\lim_{t \to 0} \frac{U(t) - 1}{t} \psi = -iH\psi.$$

(*c*) *$U(t)D(H) = D(H)$ and $[U(t), H] = 0$ on $D(H)$.*

(*d*) *Let $\psi_0 \in D(H)$. Then $U(t)\psi_0$ uniquely solves the initial value problem*

$$\begin{cases} i\partial_t \psi(t) = H\psi(t) \\ \psi(0) = \psi_0, \end{cases} \tag{1}$$

called the ***Schrödinger equation****.*