MATHEMATICAL METHODS OF QUANTUM MECHANICS SUMMARY

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Abstract.

Contents

The Spectral Theorem

Projection Valued Measures.

Definition 1.1 (Projection Valued Measure). *Let $\mathscr{H}$ be a Hilbert space. A function $P : \mathscr{B}(\mathbb{R}) \to \mathscr{L}(\mathscr{H})$ is said to be a **projection valued measure**, iff*

(*i*) *For all $\Omega \in \mathscr{B}(\mathbb{R})$, $P(\Omega)$ is an orthogonal projection, i.e.*

$$P(\Omega)^2 = P(\Omega) = P(\Omega)^*.$$

(*ii*) $P(\mathbb{R}) = \mathrm{id}_{\mathscr{H}}$.

(*iii*) *If $(\Omega_n)_{n\in\omega}$ is a sequence of pairwise disjoint elements of $\mathscr{B}(\mathbb{R})$, then*

$$P(\Omega)\psi = \sum_{n\in\omega} P(\Omega_n)\psi,$$

for all $\psi \in \mathscr{H}$.

Definition 1.2 (Resolution of the Identity). *Let $\mathscr{H}$ be a Hilbert space and $P : \mathscr{B}(\mathbb{R}) \to \mathscr{L}(\mathscr{H})$ a projection valued measure. The function $p : \mathbb{R} \to \mathscr{L}(\mathscr{H})$ defined by*

$$p(\lambda) := P\left(-\infty, \lambda\right],$$

*is called the **resolution of the identity associated to a projection valued measure**.*

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The Spectral Theorem.

Theorem 1.1 (Spectral Theorem). *Let A be a self-adjoint operator. Then there exists a unique projection valued measure P^A such that $D(A) = \{\psi \in \mathscr{H} : \int |\lambda|^2 \, d\mu_\psi^A(\lambda)\}$ and*

$$A = \int \lambda dp^A(\lambda).$$

The Schrödinger Equation.

Theorem 1.2. *Let $\mathscr{H}$ be a Hilbert space and $H : D(H) \to \mathscr{H}$ be self adjoint. Moreover, set $U(t) := \exp(-iHt)$ for $t \in \mathbb{R}$. Then:*

(a) $U(t)$ is a strongly continuous one parameter unitary group.

(b) The limit

$$\lim_{t \to 0} \frac{U(t) - 1}{t} \psi$$

exists if and only if $\psi \in D(H)$. Then

$$\lim_{t \to 0} \frac{U(t) - 1}{t} \psi = -iH\psi.$$

(c) $U(t)D(H) = D(H)$ and $[U(t), H] = 0$ on $D(H)$.

(d) Let $\psi_0 \in D(H)$. Then $U(t)\psi_0$ uniquely solves the initial value problem

$$\begin{cases} i\partial_t \psi(t) = H\psi(t) \\ \psi(0) = \psi_0, \end{cases} \tag{1}$$

called the ***Schrödinger equation****.*