WHITEHEAD PRODUCT

YANNIS BÄHNI

Abstract.

Contents

1. Definition of the Whitehead Product

Notice, that for any $(X, x_0), (Y, y_0) \in \mathsf{Top}_*$, their coproduct is given by

$$X \coprod Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subseteq X \times Y,$$

with basepoint (x_0, y_0) .

Lemma 1.1. Let $n, m \in \omega$, $n, m \ge 1$. The space $\mathbb{S}^n \times \mathbb{S}^m$ can be obtained from $\mathbb{S}^n \vee \mathbb{S}^m$ by attaching an n + m-cell.

Proof. Observe, that $\mathbb{D}^{n+m} \cong \mathbb{D}^n \times \mathbb{D}^m$ and hence

$$\mathbb{S}^{n+m-1} = \partial \mathbb{D}^{n+m} \cong (\partial \mathbb{D}^n \times \mathbb{D}^m) \cup (\mathbb{D}^n \times \partial \mathbb{D}^m) = (\mathbb{S}^{n-1} \times \mathbb{D}^m) \cup (\mathbb{D}^n \times \mathbb{S}^{m-1}).$$

Let

$$f_1: \mathbb{S}^{n-1} \times \mathbb{D}^m \to (\mathbb{S}^{n-1} \times \mathbb{D}^m)/(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \cong * \times \mathbb{S}^m$$

and

$$f_2: \mathbb{D}^n \times \mathbb{S}^{m-1} \to (\mathbb{D}^n \times \mathbb{S}^{m-1})/(\mathbb{S}^{n-1} \times \mathbb{S}^{m-1}) \cong \mathbb{S}^n \times *$$

be the quotient maps. An application of the gluing lemma thus yields a map

$$f: \mathbb{S}^{n+m-1} \to \mathbb{S}^n \vee \mathbb{S}^m$$
.

Moreover, define

$$q: \mathbb{D}^n \times \mathbb{D}^m \to \mathbb{D}^n / \mathbb{S}^{n-1} \times \mathbb{D}^m / \mathbb{S}^{m-1} \cong \mathbb{S}^n \times \mathbb{S}^m$$

(Yannis Bähni) University of Zurich, Rämistrasse 71, 8006 Zurich *E-mail address*: yannis.baehni@uzh.ch.

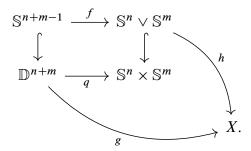
to be the product of quotient maps. Thus we get a commutative diagram

$$\mathbb{S}^{n+m-1} \xrightarrow{f} \mathbb{S}^n \vee \mathbb{S}^m$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathbb{D}^{n+m} \xrightarrow{a} \mathbb{S}^n \times \mathbb{S}^m$$

Suppose (X, g, h) is another cocone for the pushout diagram:



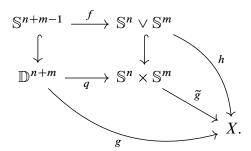
By [Mun00, p. 186], q is a quotient map. Moreover, for $(x, y) \in \mathbb{S}^{n-1} \times \mathbb{S}^{m-1}$, we have that

$$g(x, y) = (h \circ f)(x, y) = h(*, *).$$

Thus g passes to the quotient by [Lee11, p. 72] to yield a unique map

$$\tilde{g}: \mathbb{S}^n \times \mathbb{S}^m \to X$$
,

such that $g = \tilde{g} \circ q$. Finally, it is easy to check that



commutes.

For $n, m \in \omega$, $n, m \ge 1$, consider the map f from lemma 1.1. Let $(X, p) \in \mathsf{Top}_*$. If $[\alpha] \in \pi_n(X, p)$ and $[\beta] \in \pi_m(X, p)$, we get two pointed maps

$$\alpha: \mathbb{S}^n \to X$$
 and $\beta: \mathbb{S}^m \to X$.

Forming their wedge $\alpha \vee \beta : \mathbb{S}^n \vee \mathbb{S}^m \to X$, defined by

$$(\alpha \vee \beta)(x, y) := \begin{cases} \alpha(x) & y = *, \\ \beta(y) & x = *, \end{cases}$$

and precomposing with f, yields a pointed map

$$(\alpha \vee \beta) \circ f : \mathbb{S}^{n+m-1} \to X.$$

Moreover, it is easy to check that above map is well behaved under pointed homotopies, hence gives raise to a class $[(\alpha \lor \beta) \circ f]$.

Definition 1.1 (Whitehead Product). Let $n, m \in \omega$, $n, m \geq 1$, and $(X, p) \in \mathsf{Top}_*$. The product

$$\pi_n(X, p) \times \pi_m(X, p) \to \pi_{n+m-1}(X, p)$$

defined by

$$([\alpha], [\beta]) \mapsto [(\alpha \vee \beta) \circ f]$$

is called the Whitehead product.