

LIE ALGEBRA COHOMOLOGY

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Abstract. Aim of this talk is to give a short overview of the *cohomology of Lie algebras with coefficients in modules*. We follow the original construction of Chevalley-Eilenberg via complexes. We then state two results concerning *semisimple* Lie algebras, known as the *first and second Whitehead lemma*, and calculate an example.

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Introduction

The archetypal example of a *cohomology theory* arises in differential topology: The *de Rham cohomology*. Given a smooth manifold M , we define $\Omega^n(M) := \Gamma(\Lambda^n T^*M)$ for each $n \in \omega$, the space of *smooth differential n -forms on M* . Moreover, is a sequence of mappings $(d^n : \Omega^n(M) \rightarrow \Omega^{n+1}(M))_{n \in \omega}$, called *exterior differentiation operators*, which roughly speaking generalize the notion of a differential of a function. They do satisfy the relation $d^n \circ d^{n-1} = 0$ and thus we can define the *n -th de Rham cohomology group* to be the quotient space

$$H_{\text{dR}}^n(M) := \ker d^n / \text{im } d^{n-1}.$$

The Chevalley-Eilenberg Complex

The definition of the n -th de Rham cohomology group $H_{\text{dR}}^n(M)$ can actually be thought of a two-stage process: First we go from Diff to an intermediate category and then we apply a *homology functor*, which is a purely algebraic construct, to go from this intermediate

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category to $\mathbb{R}\text{-Vect}$. Aim of this section is to give the definition of this intermediate category and then to define such a functor explicitly for the case of Lie algebras.

Definition 1.1 (Chain Complex). Let \mathcal{A} be an abelian category. A \mathbb{Z} -graded chain complex in \mathcal{A} is defined to be a tuple $((C_n)_{n \in \mathbb{Z}}, (\partial_n)_{n \in \mathbb{Z}})$, consisting of a sequence $(C_n)_{n \in \mathbb{Z}}$ of objects in \mathcal{A} and a sequence $(\partial_n)_{n \in \mathbb{Z}}$ of morphisms in \mathcal{A} , such that

$$\partial_n \in \text{Hom}_{\mathcal{A}}(C_n, C_{n-1}) \quad \text{and} \quad \partial_n \circ \partial_{n+1} = 0$$

for all $n \in \mathbb{Z}$.

Dually, a \mathbb{Z} -graded cochain complex in \mathcal{A} is a \mathbb{Z} -graded chain complex in \mathcal{A}^{op} .

Remark 1.1. For notational simplicity, we will write $(C_\bullet, \partial_\bullet)$ for a chain complex in \mathcal{A} .

Remark 1.2. For each abelian category \mathcal{A} , there is an abelian category $\text{Ch}(\mathcal{A})$ of chain complexes in \mathcal{A} (see [Wei94, p. 7]).

Definition 1.2 (Non-Negative Chain Complex). Let \mathcal{A} be an abelian category. A chain complex $(C_\bullet, \partial_\bullet) \in \text{Ch}(\mathcal{A})$ is said to be **non-negative**, iff $C_n = 0$ for all $n < 0$. We denote by $\text{Ch}_{\geq 0}(\mathcal{A})$ the full subcategory of $\text{Ch}(\mathcal{A})$ of non-negative chain complexes.

Definition 1.3 (Chevalley-Eilenberg Complex). Let $R \in \text{CRing}$ and $\mathfrak{g} \in {}_R\text{LieAlg}$ which is free as an R -module. Denote by $U\mathfrak{g}$ the universal enveloping algebra of \mathfrak{g} . Define a non-negative chain complex $(C_\bullet, \partial_\bullet) \in \text{Ch}_{\geq 0}(U\mathfrak{g}\text{Mod})$ by

$$C_n := U\mathfrak{g} \otimes_R \Lambda^n \mathfrak{g}$$

for all $n \in \omega$ and

$$\partial_n(u \otimes x_1 \wedge \cdots \wedge x_n) := \begin{cases} ux_1 & n = 1, \\ \theta_1 + \theta_2 & n > 1, \end{cases}$$

where

$$\theta_1 := \sum_{i=0}^n (-1)^{i+1} ux_i \otimes x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n,$$

and

$$\theta_2 := \sum_{1 \leq i < j \leq n} (-1)^{i+j} u \otimes [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n.$$

Remark 1.3. It is by no means obvious, that $\partial_n \circ \partial_{n+1} = 0$ holds for the Chevalley-Eilenberg complex 1.3. However, it is a tedious computation, and we will only demonstrate the case $n = 1$. In this case

$$\begin{aligned} (\partial_1 \circ \partial_2)(u \otimes x \wedge y) &= \partial_1(ux \otimes y - uy \otimes x - u \otimes [x, y]) \\ &= u(xy - yx) - u[x, y] \\ &= 0, \end{aligned}$$

for all $u \in U\mathfrak{g}$ and $x, y \in \mathfrak{g}$ since the relation

$$\iota[x, y] = \iota(x)\iota(y) - \iota(y)\iota(x)$$

for the inclusion $\iota : \mathfrak{g} \hookrightarrow U\mathfrak{g}$ holds by definition of $U\mathfrak{g}$.

Remark 1.4. The definition of the boundary map ∂_n in the Chevalley-Eilenberg complex 1.3 is not as arbitrary as it might seem at first sight. Given $\alpha \in \Omega^n(M)$ for a smooth manifold M , then we have that $d\alpha(X_1, \dots, X_{n+1})$ is of the same form for any $X_1, \dots, X_{n+1} \in \mathfrak{X}(M)$. Actually, this formula can be used to give an invariant definition of the exterior derivative d in the de Rham theory (see [Lee13, pp. 370–372]).

Left \mathfrak{g} -Modules and the Cohomology of Lie Algebras

Definition 1.4 (Category of Left \mathfrak{g} -Modules). Let $R \in \mathbf{CRing}$ and $\mathfrak{g} \in {}_R\mathbf{LieAlg}$. The category of left \mathfrak{g} -modules, written ${}_{{}_\mathfrak{g}}\mathbf{Mod}$, is defined to be the category with objects **left \mathfrak{g} -modules**, i.e. modules $M \in {}_R\mathbf{Mod}$ equipped with an R -bilinear product $\mathfrak{g} \times M \rightarrow M$, $(x, m) \mapsto xm$, such that

$$[x, y]m = x(ym) - y(xm)$$

holds for all $x, y \in \mathfrak{g}$ and $m \in M$, and **left \mathfrak{g} -module homomorphisms** as morphisms, i.e. morphisms $f \in \mathbf{Hom}_{{}_{{}_\mathfrak{g}}\mathbf{Mod}}(M, N)$ such that

$$f(xm) = xf(m)$$

holds for all $x \in \mathfrak{g}$ and $m \in M$.

Proposition 1.1. Let $R \in \mathbf{CRing}$ and $\mathfrak{g} \in {}_R\mathbf{LieAlg}$. Then ${}_{{}_\mathfrak{g}}\mathbf{Mod}$ is an abelian category.

Proof. See [Wei94, p. 220]. □

We follow [KS06, p. 178].

Proposition 1.2. Let \mathcal{A} be an abelian category and $(C_\bullet, \partial_\bullet) \in \mathbf{Ch}(\mathcal{A})$. Then for every $n \in \mathbb{Z}$, there exists a unique monic

$$\mathrm{im} \partial_{n+1} \rightarrow \ker \partial_n,$$

where $\mathrm{im} \partial_{n+1} := \ker(\mathrm{coker} \partial_{n+1})$.

Exercise 1.1. Prove proposition 1.2. *Hint:* Use that $\mathrm{im} \partial_{n+1} \rightarrow C_n$ is monic by [Lan78, p. 199].

Definition 1.5 (Homology of a Chain Complex). Let \mathcal{A} be an abelian category and $(C_\bullet, \partial_\bullet) \in \mathbf{Ch}(\mathcal{A})$. Moreover, let $n \in \mathbb{Z}$ and $\mathrm{im} \partial_{n+1} \rightarrow \ker \partial_n$ be the unique morphism assured by lemma 1.2. Then we define the **n -th homology object**, written $H_n(C_\bullet, \partial_\bullet)$, by

$$H_n(C_\bullet, \partial_\bullet) := \mathrm{coker}(\mathrm{im} \partial_{n+1} \rightarrow \ker \partial_n) \in \mathbf{ob}(\mathcal{A}).$$

Definition 1.6 (Cohomology of Lie Algebras). Let $R \in \mathbf{CRing}$ and $\mathfrak{g} \in {}_R\mathbf{LieAlg}$ which is free as an R -module. Moreover, let $M \in {}_{\mathfrak{g}}\mathbf{Mod}$ and $(C_{\bullet}, \partial_{\bullet})$ denote the Chevalley-Eilenberg complex 1.3. For $n \in \omega$, define the **n -th cohomology group of \mathfrak{g} with coefficients in M** , written $H^n(\mathfrak{g}, M)$, to be the n -th homology object of the cochain complex

$$\mathrm{Hom}_{{}_{\mathfrak{g}}\mathbf{Mod}}((C_{\bullet}, \partial_{\bullet}), M).$$

Remark 1.5. Actually, we have that

$$\mathrm{Hom}_{{}_{\mathfrak{g}}\mathbf{Mod}}(C_n, M) \cong \mathrm{Hom}_{{}_R\mathbf{Mod}}(\Lambda^n \mathfrak{g}, M),$$

and thus

$$\begin{aligned} d^n f(x_1, \dots, x_n) &= \sum_{i=1}^n (-1)^{i+1} x_i f(x_1, \dots, \hat{x}_i, \dots, x_n) \\ &\quad + \sum_{1 \leq i < j \leq n} (-1)^{i+j} f([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_n), \end{aligned}$$

for all $n \in \omega$.

Remark 1.6. There is a more general approach to the definition of the cohomology of Lie algebras via the notion of *right derived functors* which does not use the intermediate step of the Chevalley-Eilenberg complex.

Example 1.1 ($H^3(\mathfrak{sl}_2, k)$). Let k be a field with characteristic not equal to two and consider the *special linear Lie algebra over k* , i.e. $A \in M_2(k)$ such that $\mathrm{tr} A = 0$. This is a three dimensional Lie algebra with ordered basis

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad e_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad e_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Hence the crucial portion of the Eilenberg-Chevalley cochain complex is given by

$$\dots \longrightarrow \mathrm{Hom}_k(\Lambda^2, k) \xrightarrow{d} \mathrm{Hom}_k(\Lambda^3, k) \longrightarrow 0.$$

We compute

$$\begin{aligned} df(e_1, e_2, e_3) &= e_1 f(e_2, e_3) - e_2 f(e_1, e_3) + e_3 f(e_1, e_2) \\ &\quad - f([e_1, e_2], e_3) + f([e_1, e_3], e_2) - f([e_2, e_3], e_1) \\ &= -2f(e_2, e_3) - 2f(e_3, e_2) - f(e_1, e_1) \\ &= -2f(e_2, e_3) + 2f(e_2, e_3) \\ &= 0, \end{aligned}$$

since k is interpreted as a trivial \mathfrak{sl}_2 -module and by the alternating k -multilinear properties of f . Hence

$$H^3(\mathfrak{sl}_2, k) \cong \mathrm{Hom}_k(\Lambda^3 \mathfrak{g}, k) \cong k,$$

since $\dim \mathrm{Hom}_k(\Lambda^3 \mathfrak{sl}_2, k) = 1$.

The Whitehead Lemmas

Theorem 1.2 (Whitehead's First Lemma). *Let k be a field of characteristic zero and $\mathfrak{g} \in {}_k\text{LieAlg}$ semisimple. Then for any finite-dimensional $M \in {}_{\mathfrak{g}}\text{Mod}$ we have that*

$$H^1(\mathfrak{g}, M) = 0.$$

Theorem 1.3 (Whitehead's Second Lemma). *Let k be a field of characteristic zero and $\mathfrak{g} \in {}_k\text{LieAlg}$ semisimple. Then for any finite-dimensional $M \in {}_{\mathfrak{g}}\text{Mod}$ we have that*

$$H^2(\mathfrak{g}, M) = 0.$$

Remark 1.7. There cannot be a third Whitehead lemma, since

$$H^3(\mathfrak{sl}_2, k) \cong k,$$

by exercise 1.1.

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