Betti tables forcing failure of the Weak Lefschetz Property

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The Weak Lefschetz Property

Definition. Let $R = \mathbb{K}[x_1, \dots, x_n]$ and let $A = R/I$ be a standard graded Artinian algebra, where $I \subseteq R$ is a homogeneous ideal. Then A has the *Weak Lefschetz Property (WLP)* if multiplication by a general linear form

$$A_i \xrightarrow{\cdot l} A_{i+1}$$

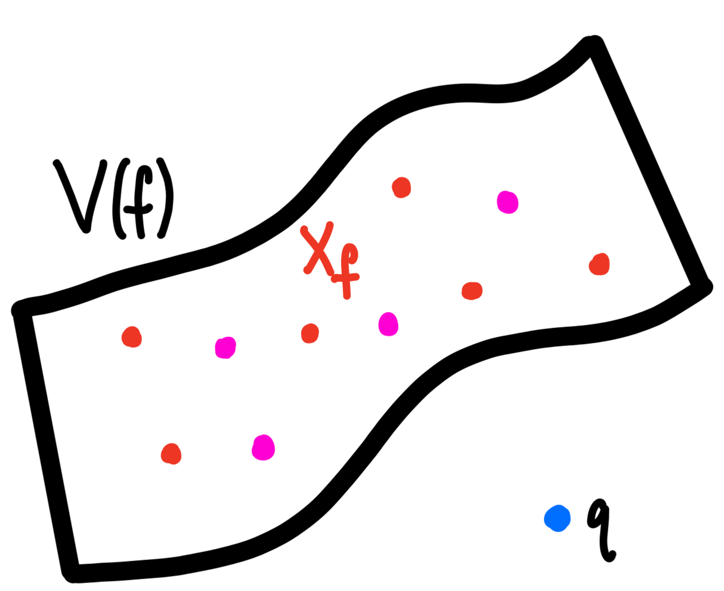
has full rank for all $i \geq 0$, so it is either injective or surjective.

Motivating Questions

- Is there ever a set of points $Y \subset X$ (and perhaps conditions on $|Y|$ relative to $|X|$) that guarantee failure or success of WLP?
- Given a configuration of points X, if WLP fails, can we predict the degree in which WLP fails in terms of the geometry of X?
- If a set of points has multiple "subconfigurations" of points–for example, two susbets of points lying on different hypersurfaces $V(f)$ and $V(g)$–then how does this structure influence WLP? What if there are conditions on f and g?

Points on a hypersurface

Theorem. Assume X_f is a finite set of points lying on a unique hypersurface $V(f)$ with $\deg(f) = d$, and q is a point not on $V(f)$. Set $X \coloneqq X_f \cup \{q\}$. Then the Artinian reduction A of X does not have the WLP. In particular, if l is a general linear form, the map $A_d \xrightarrow{\cdot l} A_{d+1}$ does not have full rank.



How many points?

To get a *unique* hyperfurface $V(f)$ with $\deg(f) = d$, we need $|X_f| = \binom{n+d}{d} - 1$ (if chosen generically). This ensures that

$$\dim_{\mathbb{K}}(I_{X_f})_d = 1 \quad \text{and} \quad \dim_{\mathbb{K}}(I_{X_f})_{<d} = 0.$$

Solving the equation

$$\dim(R_{d+1}) - (|\tilde{X}| + m) = n$$
$$\downarrow$$
$$\binom{n+d+1}{d+1} - \left(\binom{n+d}{d} + m\right) = n$$

We can then choose $m \coloneqq \binom{n+d}{d+1} - n$ distinct generic points on $V(f) \setminus X_f$ to ensure that we have enough points on $V(f)$.

Hypersurface example

Consider the points $X \subset \mathbb{P}^3$ given as the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

with X_f given as the first seven columns, and q the last column. The Hilbert functions of the Artinian reductions are

$$h(A_{X_f}) = \{1,2,3,1\} \quad \text{and} \quad h(A_X) = \{1,3,3,1\}.$$

All of the points X_f lie on the plane $x_3 = 0$ and q lies off of it. In this way,

$$(I_{X_f})_1 = (x_3) \quad \text{and} \quad I_{\{q\}} = (x_0, x_1, x_2),$$

and so,

$$(I_X)_2 = (I_{X_f})_1 \cap (I_{\{q\}})_1 = (x_3) \cap (x_0, x_1, x_2) = (x_0x_3, x_1x_3, x_2x_3).$$

By the previous theorem, an Artinian reduction of X will not have WLP.

Koszul tails

Definition. A Betti table B has an (n,d)-Koszul tail if it has an upper-left principal block of the form

$$\begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ \hline
0 & 1 & . & . & . & \dots & . & . & . \\
1 & . & . & . & . & \dots & . & . & . \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
d-1 & . & . & . & . & \dots & . & . & . \\
d & . & n & \binom{n}{2} & \binom{n}{3} & \dots & \binom{n}{n-2} & n & 1
\end{array}.$$

If B has an (n,d)-Koszul tail and is the Betti table for an Artinian ring $\mathbb{K}[x_1, \dots, x_n]/I$, then we say B has a *maximal* (n,d)-Koszul tail.

Koszul tail example

The Betti table for the Artinian reduction A_X of the pointset X above is

$$\mathrm{Betti}(A_X) = \begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\ \hline
0 & 1 & . & . & . \\
1 & . & 3 & 3 & 1 \\
2 & . & 3 & 4 & 1 \\
3 & . & . & 1 & 1
\end{array}.$$

So $\mathrm{Betti}(A_X)$ has a $(3,1)$-Koszul tail; moreover, this is a *maximal* $(3,1)$-Koszul tail.

Maximal Koszul tails force failure of the WLP

Theorem. An Artinian algebra $A = \mathbb{K}[x_1, \dots, x_n]/I$ whose Betti table has a maximal (n,d)-Koszul tail does not have the WLP; the map from $A_d \xrightarrow{\cdot l} A_{d+1}$ is not injective.

Corollary. If $T = \mathbb{K}[x_1, \dots, x_n]/I$ is Cohen-Macaulay of dimension m, and $\mathrm{Betti}(T)$ has a maximal $(n-m,d)$-Koszul tail, then the Artinian reduction of T does not have the WLP.

Corollary. If $A = \mathbb{K}[x_1, \dots, x_{n+m}]/I$ is Artinian with an (n,d)-Koszul tail, and there exists a sequence of linearly independent linear forms $\{l_1, \dots, l_m\}$ such that A/I_L has the same top row Betti table as A, then A/I_L does not have the WLP.

Another Koszul tail example

Consider the pointset $X_f \subset \mathbb{P}^4$ lying on a unique hypersurface $\mathbf{V}(f)$ with $\deg(f) = 3$. Take 5 points X_Q lying off of $\mathbf{V}(f)$, but on a codimension 3 linear space. Let $X \coloneqq X_f \cup X_Q$.

$$\mathrm{Betti}(A) = \begin{array}{c|ccccc}
 & 0 & 1 & 2 & 3 & 4 \\ \hline
0 & 1 & . & . & . & . \\
1 & . & . & . & . & . \\
2 & . & . & . & . & . \\
3 & . & 3 & 3 & 1 & . \\
4 & . & 44 & 111 & 90 & 20 \\
5 & . & . & . & . & 3
\end{array}.$$

So the Betti table of A has a $(3,3)$-Koszul tail, and in this case, A has WLP. However, if we form A^* from A by quotienting by yet another generic linear form, i.e. $A^* = A/(L')$ with $L' \in A_1$, then

$$\mathrm{Betti}(A^*) = \begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\ \hline
0 & 1 & . & . & . \\
1 & . & . & . & . \\
2 & . & . & . & . \\
3 & . & 3 & 3 & 1 \\
4 & . & 15 & 27 & 12
\end{array}.$$

The Betti table of A^* has a maximal $(3,3)$-Koszul tail, and A^* fails WLP from degree 3 to degree 4.

Important note

We are assuming $n \geq 3$, i.e. we assume we are working with pointsets in $\mathbb{P}^n$ with $n \geq 3$. Our technique does not apply to $(2,d)$-Koszul tails.

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