

The anti-Ramsey number of $K_4^{(3)-}$

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Graph coloring

Theorem 1.1 (Appel, Haken [AH89])

Every planar graph is four-colorable.

- ◀ Vertex-coloring: chromatic polynomial $\chi_G(k)$, chromatic number $\chi(G)$ (Birkhoff).
- ◀ Edge-coloring: chromatic index $\chi'(G)$.

Anti-Ramsey number

Definition 1.2 (Hypergraph)

A hypergraph \mathcal{H} is a pair $\mathcal{H} = (V, E)$, where elements in V are called vertices, and elements in $E \subseteq \mathcal{P}(V)$ are called (hyper)edges. If $E \subseteq \binom{V}{r}$, then \mathcal{H} is called an r -graph.

Definition 1.3 (Erdős, Simonovits, and Sós, [ESS75])

For r -graph G and r -graph H , the anti-Ramsey number of H in G is defined by

$$ar(G, r, H) = \max_k \{k : \text{there is a } k\text{-coloring of } G \text{ with no rainbow copy of } H\}.$$

Graph anti-Ramsey theorem

Theorem 1.4 (Erdős et al. [ESS75])

For all $n \geq 3$, $ar(n, 2, K_3) = n - 1$.

Theorem 1.5 (Schiermeyer [Sch04])

For all $n \geq k \geq 4$, $ar(n, 2, K_k) = ex(n, 2, K_{k-1}) + 1$.

Theorem 1.6 (Montellano-Ballesteros, Neumann-Lara [MN05])

For all $n \geq k \geq 3$,

$$ar(n, 2, C_k) = \left(\frac{k-2}{2} + \frac{1}{k-1} \right) n + O(1).$$

Hypergraph anti-Ramsey theorem

Theorem 1.7 (Erdős et al., [ESS75])

For every r -graph with $r \geq 2$,

$$ex(n, r, H_-) + 1 \leq ar(n, r, H) \leq ex(n, r, H_-) + o(n^r).$$

Theorem 1.8 (M. Guo, H. Lu, and Z. Peng, [GLP23])

For sufficiently large n , the following holds,

$$ar(n, 3, M_s) = \begin{cases} ex(n, 3, M_{s-1}) + 1, & \text{if } 3s < n < 5s - 2; \\ ex(n, 3, M_{s-1}) + 4, & \text{if } n = 3s. \end{cases}$$

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Determine $ar(n, 3, K_4^{(3)-})$

Conjecture 1 (Lu, 2025)

For $n \equiv 3 \pmod{6}$, $ar(n, 3, K_4^{(3)-}) = \binom{n}{2}/3 + 1$.

Determine $ar(n, 3, K_4^{(3)-})$

Conjecture 1 (Lu, 2025)

For $n \equiv 3 \pmod{6}$, $ar(n, 3, K_4^{(3)-}) = \binom{n}{2}/3 + 1$.

Lemma 1 (Lu, 2025)

For $n \equiv 1$ or $3 \pmod{6}$, $ar(n, 3, K_4^{(3)-}) \geq \binom{n}{2}/3 + 1$.

Definition 2.1 (Steiner triple system)

A Steiner triple system of order n is a pair (X, B) , where X is the set of n elements, and $B \subseteq \binom{X}{3}$ such that each pair of elements occurs in exactly one triple of B .

The upper bound

Our goal

Prove that any $((\binom{n}{2})/3 + 2)$ -coloring of $K_n^{(3)}$ contains a rainbow copy of $K_4^{(3)-}$.

Remark 1

There is a trivial upper bound given by $ar(n, 2, K_3) = n - 1$,

$$ar(n, 3, K_4^{(3)-}) \leq \frac{n(n-2)}{3}.$$

An important substructure

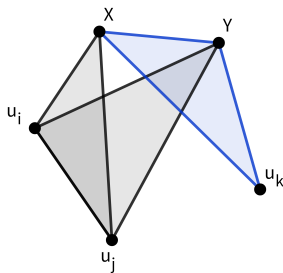
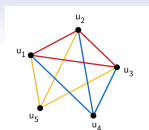


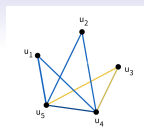
Figure 1: A substructure

Lemma 2

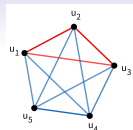
$$ar(5, 3, K_4^{(3)-}) = 3.$$



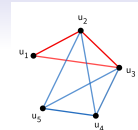
(a) (1,4,5)-1



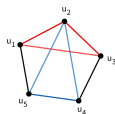
(b) (1,4,5)-2



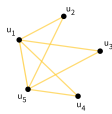
(c) (1,6,3)



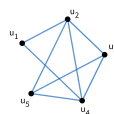
(d) (1,7,2)



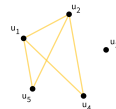
(e) (1,8,1)



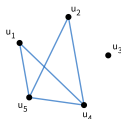
(f) (4,3,3)-1



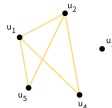
(g) (4,3,3)-2



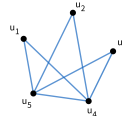
(h) (6,2,2)-1



(i) (6,2,2)-2



(j) (5,3,2)-1



(k) (5,3,2)-2

Figure 2: Extremal colorings of $K_5^{(3)}$

The next step

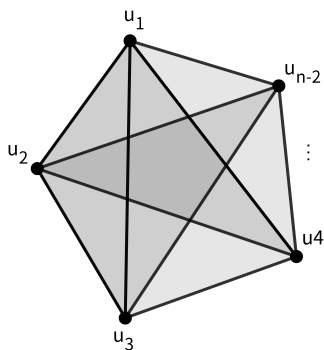


Figure 3: The remaining $K_{n-2}^{(3)}$

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Q&A

Thank you!