# The anti-Ramsey number of $K_4^{(3)-}$

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# Table of contents

# Table of contents

# Graph coloring

**Theorem 1.1 (Appel, Haken [AH89])**

*Every planar graph is four-colorable.*

- Vertex-coloring: chromatic polynomial $\chi_G(k)$, chromatic number $\chi(G)$ (Birkhoff).
- Edge-coloring: chromatic index $\chi'(G)$.

# Anti-Ramsey number

## Definition 1.2 (Hypergraph)

A hypergraph $\mathcal{H}$ is a pair $\mathcal{H} = (V, E)$, where elements in $V$ are called vertices, and elements in $E \subseteq \mathcal{P}(V)$ are called (hyper)edges. If $E \subseteq \binom{V}{r}$, then $\mathcal{H}$ is called an $r$-graph.

## Definition 1.3 (Erdös, Simonovits, and Sós, [ESS75])

For $r$-graph $G$ and $r$-graph $H$, the anti-Ramsey number of $H$ in $G$ is defined by

$$ar(G, r, H) = \max_k \{k : \text{there is a } k\text{-coloring of } G \text{ with no rainbow copy of } H\}.$$

# Graph anti-Ramsey theorem

**Theorem 1.4 (Erdös et al. [ESS75])**

*For all* $n \geq 3$, $ar(n, 2, K_3) = n - 1$.

**Theorem 1.5 (Schiermeyer [Sch04])**

*For all* $n \geq k \geq 4$, $ar(n, 2, K_k) = ex(n, 2, K_{k-1}) + 1$.

**Theorem 1.6 (Montellano-Ballesteros, Neumann-Lara [MN05])**

*For all* $n \geq k \geq 3$,

$$ar(n, 2, C_k) = \left(\frac{k-2}{2} + \frac{1}{k-1}\right) n + O(1).$$

# Hypergraph anti-Ramsey theorem

**Theorem 1.7 (Erdös et al. [ESS75])**

*For every $r$-graph with $r \geq 2$,*

$$ex(n, r, H_-) + 1 \leq ar(n, r, H) \leq ex(n, r, H_-) + o(n^r).$$

**Theorem 1.8 ( M. Guo, H. Lu, and Z. Peng, [GLP23])**

*For sufficiently large $n$, the following holds,*

$$ar(n, 3, M_s) = \begin{cases} ex(n, 3, M_{s-1}) + 1, & \textit{if } 3s < n < 5s - 2; \\ ex(n, 3, M_{s-1}) + 4, & \textit{if } n = 3s. \end{cases}$$

# Table of contents

# Determine $ar(n, 3, K_4^{(3)-})$

## Conjecture 1

For $n \equiv 3 \pmod 6$, $ar(n, 3, K_4^{(3)-}) = \binom{n}{2}/3 + 1$.

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## Lemma 1

For $n \equiv 1$ or $3 \pmod 6$, $ar(n, 3, K_4^{(3)-}) \geq \binom{n}{2}/3 + 1$.

## Definition 2.1 (Steiner triple system)

A Steiner triple system of order $n$ is a pair $(X, B)$, where $X$ is the set of $n$ elements, and $B \subseteq \binom{X}{3}$ such that each pair of elements occurs in exactly one triple of $B$.

# The upper bound

Goal: prove that any $(\binom{n}{2}/3+2)$-coloring of $K_n^{(3)}$ contains a rainbow copy of $K_4^{(3)-}$.
A trivial upper bound given by $ar(n,2,K_3)=n-1$:
$ar(n,3,K_4^{(3)-})\leq n\times(n-2)/3$.

# An important substructure

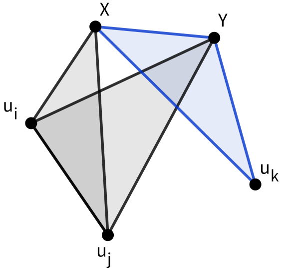
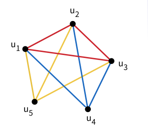


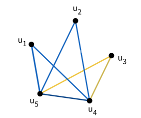

Figure 1: A substructure

## Lemma 2

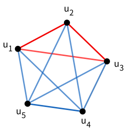
$$ar(5, 3, K_4^{(3)-}) = 3.$$



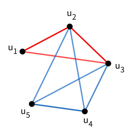

(a) (1,4,5)-1



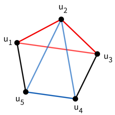

(b) (1,4,5)-2



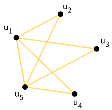

(c) (1,6,3)



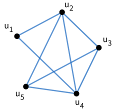

(d) (1,7,2)



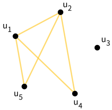

(e) (1,8,1)



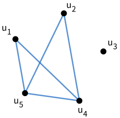

(f) (4,3,3)-1



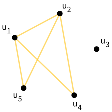

(g) (4,3,3)-2



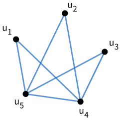

(h) (6,2,2)-1




(i) (6,2,2)-2




(j) (5,3,2)-1




(k) (5,3,2)-2

Figure 2: Extremal colorings of $K_5^{(3)}$

# The next step

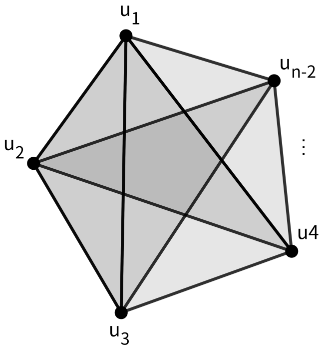



Figure 3: The remaining $K_{n-2}^{(3)}$

# References

[AH89] Kenneth Appel and Wolfgang Haken. *Every Planar Map Is Four Colorable*. Contemporary Mathematics 98. American Mathematical Society, 1989, pp. 236–240.

[ESS75] P. Erdős, M. Simonovits, and V. T. Sós. "Anti-Ramsey theorems". In: *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II*. Colloq. Math. Soc. János Bolyai, Vol. 10. Amsterdam: North-Holland, 1975, pp. 633–643.

[Sch04] Ingo Schiermeyer. "Rainbow numbers for matchings and complete graphs". In: *Discrete Mathematics* 286 (Sept. 2004), pp. 157–162. DOI: 10.1016/j.disc.2003.11.057.

[MN05] J. J. Montellano-Ballesteros and V. Neumann-Lara. "An Anti-Ramsey Theorem on Cycles". In: *Graph. Comb.* 21.3 (Sept. 2005), pp. 343–354. ISSN: 0911-0119. DOI: 10.1007/s00373-005-0619-y. URL: https://doi.org/10.1007/s00373-005-0619-y.

[GLP23] Mingyang Guo, Hongliang Lu, and Xing Peng. *Anti-Ramsey number of matchings in 3-uniform hypergraphs*. 2023. arXiv: 2305.15631 [math.CO]. URL: https://arxiv.org/abs/2305.15631.

# Q&A

Thank you!