ADDITIONAL PRACTICE 5

DISCRETE PROBABILITY

This worksheet is meant to provide additional practice problems for the major concepts from lecture 5 and those in the homework problems. Quantum states are closely linked to probability distributions, so it is important to develop an understanding of the laws of probability. This worksheet is not graded, but should help students get a solid foundation in the mathematics we will use throughout the course. The solutions to the additional practice problems can be found at the end of worksheet.

Problem 1: Coins and Dice

Suppose you have a fair 6-sided die ($\frac{1}{6}$ probability of rolling each number), a fair 4-sided die ($\frac{1}{4}$ probability of rolling each number), and a fair coin ($\frac{1}{2}$ probability of flipping heads or tails).

Calculate the probability of each of the following events:

a) Flipping heads three times in a row

b) Flipping tails twice in a row

c) Flipping a head and then a tails

d) Rolling a 1 and then a 3 on a 6-sided die

e) Rolling a 1 and then a 3 on a 4-sided die

f) Flipping a heads twice and rolling an odd number twice

g) Assume that someone switches out the fair coin for a coin which has a 70% chance of flipping a head and 30% chance of flipping a tail. ($\frac{7}{10}$ probability of flipping a head and $\frac{3}{10}$ probability flipping a tail)

   i) What is the probability of flipping two heads?

   ii) What is the probability of flipping a head and then a tails?

Problem 2: Random Variables

Let X and Y be independent discrete random variables that can be one of the following values: $\{1, 4, 5, 6, 9\}$. However, the probability for X and Y to take each value is different. The following table shows the corresponding probability distributions for X and Y

v	P(X=v)	P(Y=v)
1	0.15	0.22
4	0.20	0.15
5	0.15	0.25
6	0.30	0.18
9	0.20	0.20

a) What is the expectation of X?

b) What is the expectation of Y?

c) What is the variance of X?

d) What is the variance of Y?

e) Evaluate the following probabilities:

i) P(X=1 and Y=1)

ii) P(X=1 and Y=4)

iii) P(X=9 and Y=1)

iv) P(X=4 and Y=4)

v) P(X=4 and Y=1)

vi) P(X=6 and Y=6)

vii) P(X=6 and Y=5)

viii) P(X=6 and Y=4)

ix) P(X=9 and Y=4)

x) P(X=9 and Y=9)

Problem 3: Law of Large Numbers

Remember the **Law of Large Numbers:** a large number of repetitions of a process results in the average value of measurements converging to the expected value of the distribution. In this problem we will analyze an aspect of the law of large numbers, which is that the more measurements we do, the less outlier measurements affect the average we record.

1. A fair die has been rolled 12 times and that the average value of the 12 die rolls is 3.5.

   a) What is the sum of values of the first 12 rolls?

   Now we roll the die one additional time and get a 6.

   b) What is the new average of the 13 die rolls?

   c) What is the change in the average due to the 13th roll, $\Delta_1 = \text{Avg}(13) - \text{Avg}(12)$?

2. Now suppose a fair die has been rolled 150 times and that the average value of the 150 die rolls is 3.5. We roll the die one additional time and get a 6.

   d) What is the new average of the 151 die rolls?

   e) What is the change in the average due to the 151st roll, $\Delta_2 = \text{Avg}(151) - \text{Avg}(150)$

   f) Calculate $\left|\frac{\Delta_2}{\Delta_1}\right|$, which is the ratio between the effect of the 13th roll of a die and the 151st roll of a die on the average value.

3. Now suppose a fair die has been rolled 1000 times and that the average value of the 1000 die rolls is 3.5. We roll the die one additional time and get a 6.

   g) What is the new average of the 1001 die rolls?

   h) What is the change in the average due to the 1001st roll, $\Delta_3 = \text{Avg}(1001) - \text{Avg}(1000)$

   i) Calculate $\left|\frac{\Delta_3}{\Delta_2}\right|$, which is the ratio between the effect of the 1001th roll of a die and the 151st roll of a die on the average value.

   j) Calculate $\left|\frac{\Delta_3}{\Delta_1}\right|$, which is the ratio between the effect of the 1001th roll of a die and the 151st roll of a die on the average value.

Problem 4: Dirac Notation

Consider the following quantum states:

$$|\alpha\rangle = a\begin{pmatrix}0\\1\end{pmatrix} \qquad |\beta\rangle = b\begin{pmatrix}2\\-1\end{pmatrix} \qquad |\gamma\rangle = c\begin{pmatrix}4\\6e^{-i\frac{\pi}{2}}\end{pmatrix} \qquad |\varepsilon\rangle = d\begin{pmatrix}3\\4i\end{pmatrix}$$

a) Write the following in vector form:

i) $\langle\alpha|$

ii) $\langle\beta|$

iii) $\langle\gamma|$

iv) $\langle\varepsilon|$

b) One of the most important properties of a quantum state is that it is normalized. This means that for any quantum state $|\psi\rangle$, the following must be satisfied:

$$\langle\psi|\psi\rangle = 1$$

i) Using $\langle\alpha|\alpha\rangle = 1$, determine the value of a.

ii) Using $\langle\beta|\beta\rangle = 1$, determine the value of b.

iii) Using $\langle\gamma|\gamma\rangle = 1$, determine the value of c.

iv) Using $\langle\varepsilon|\varepsilon\rangle$=1, determine the value of d.

c) Calculate the following inner products:

i) $\langle\alpha|\gamma\rangle$

ii) $\langle\gamma|\beta\rangle$

iii) $\langle\beta|\gamma\rangle$

iv) $\langle\varepsilon|\alpha\rangle$

v) $\langle\gamma|\varepsilon\rangle$

Solutions to Problem 1

a) $\frac{1}{8}$

b) $\frac{1}{4}$

c) $\frac{1}{4}$

d) $\frac{1}{36}$

e) $\frac{1}{16}$

f) $\frac{1}{16}$

g) i) $\frac{49}{100}$
   ii) $\frac{21}{100}$

Solutions to Problem 2

a) $E[X] = 0.53$

b) $E[Y] = 0.495$

c) $\text{var}[X] = 6.01$

d) $\text{var}[Y] = 7.048$

e)
   i) 0.033
   ii) 0.0225
   iii) 0.044
   iv) 0.03
   v) 0.044
   vi) 0.054
   vii) 0.075
   viii) 0.045
   ix) 0.03
   x) 0.04

Solutions to Problem 3

a) 48

b) 3.69

c) 0.19

d) 3.5166

e) 0.0166

f) 0.087

g) 3.5025

h) 0.0025

i) 0.15.

j) 0.013

Solutions to Problem 4

Part a)

i) $\langle\alpha| = a\begin{pmatrix}0 & 1\end{pmatrix}$

ii) $\langle\beta| = b\begin{pmatrix}2 & -1\end{pmatrix}$

iii) $\langle\gamma| = c\begin{pmatrix}4 & 6e^{i\frac{\pi}{2}}\end{pmatrix}$

iv) $\langle\varepsilon| = d\begin{pmatrix}3 & -4i\end{pmatrix}$

Part b)

i) $a = 1$

ii) $b = \frac{1}{\sqrt{5}}$

iii) $c = \frac{1}{\sqrt{60}}$

iv) $\frac{1}{5}$

Part c)

i) $\langle\alpha|\gamma\rangle = \frac{6}{\sqrt{60}}e^{-i\frac{\pi}{2}}$

ii) $\langle\gamma|\beta\rangle = \frac{1}{\sqrt{300}}(8 - 6e^{i\frac{pi}{2}})$

iii) $\langle\beta|\gamma\rangle = \frac{1}{\sqrt{300}}(8 - 6e^{-i\frac{pi}{2}})$

iv) $\langle\varepsilon|\alpha\rangle = -\frac{4}{5}i$

v) $\langle\gamma|\varepsilon\rangle = \frac{1}{5\sqrt{60}}(12 + 24ie^{i\frac{\pi}{2}}) = -\frac{12}{5\sqrt{60}}$