INTRO TO QUANTUM COMPUTING

LECTURE #15

QUANTUM COMPUTATION PT. 3 :

QUANTUM CIRCUIT MATHEMATICS

FRANCISCA VASCONCELOS

2/28/2021

ANNOUNCEMENTS

QUANTUM COMPUTATION LECTURE SERIES

Lecture 1 – The Quantum Circuit Model *How can we perform computation with quantum systems?*	CONCEPTS
Lecture 2 – Qiskit Tutorial *How can we program quantum circuits?*	PROGRAMMING
Lecture 3 – Quantum Circuit Mathematics *How can we represent quantum circuits mathematically?*	MATH
Lectures 4-6 – Introductory Quantum Protocols and Algorithms *How can we leverage quantum for cryptography, teleportation, and algorithms?*	APPLICATION

TODAY'S LECTURE

1. Quantum Circuit Model Review
2. Single-Qubit Circuit Math
   a) Single-Qubit States Math
   b) Single-Qubit Gates Math
   c) Single-Qubit Measurement Math
   d) Putting it All Together!
3. Multi-Qubit Circuit Math
   a) The Tensor Product
   b) Multi-Qubit States Math
   c) Multi-Qubit Gates Math
   d) Multi-Qubit Measurement Math
   e) Putting it All Together!

QUANTUM CIRCUIT MODEL REVIEW

THE QUANTUM CIRCUIT MODEL

The quantum circuit is a theoretical model for quantum computation. It has three key components:

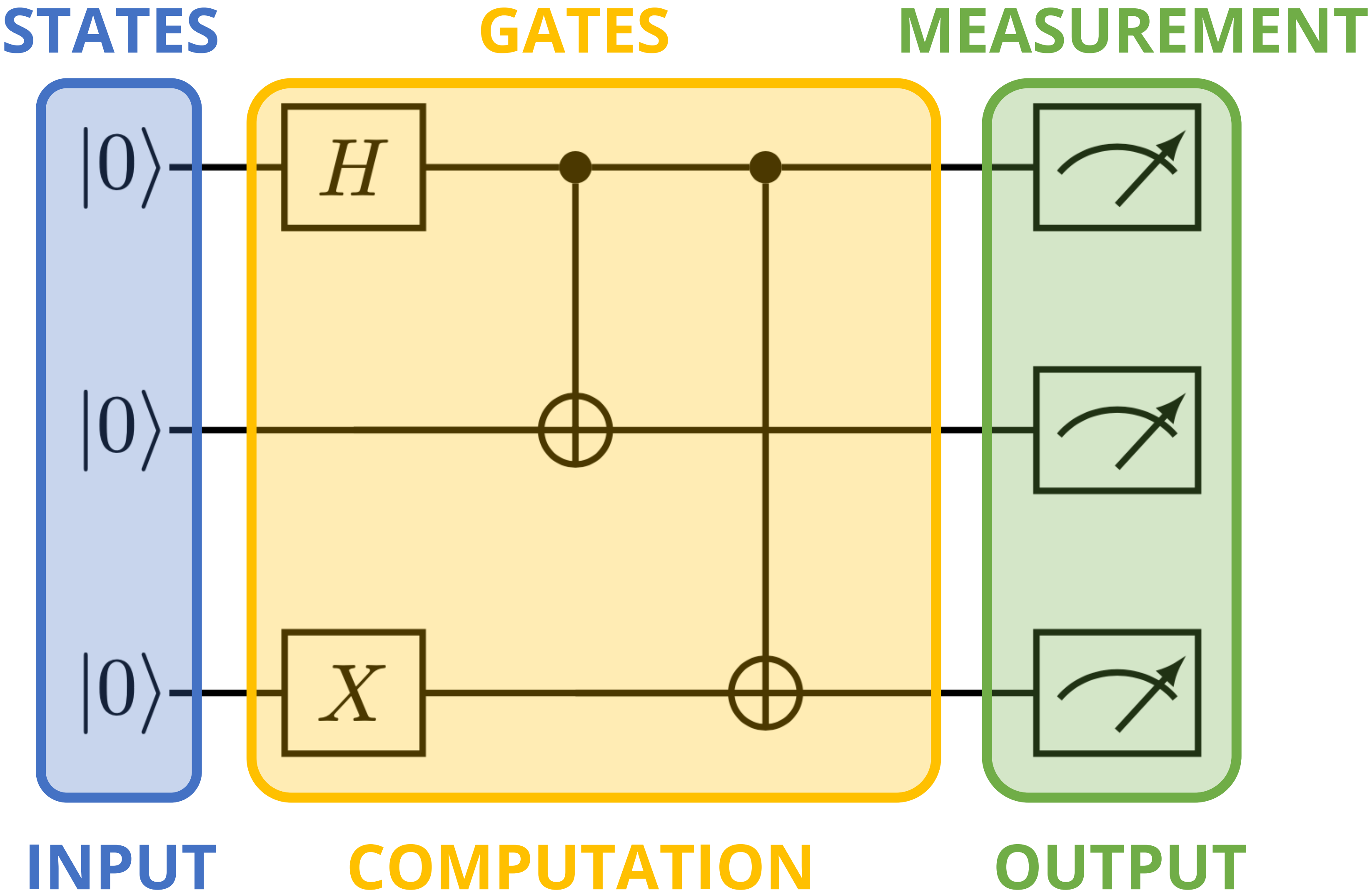
STATES,

GATES,

& MEASUREMENT

Now, how do we put these all together to create a quantum circuit?

THE QUANTUM CIRCUIT MODEL



Last class, you learned how to program these circuits in Qiskit!

Construct Bell state

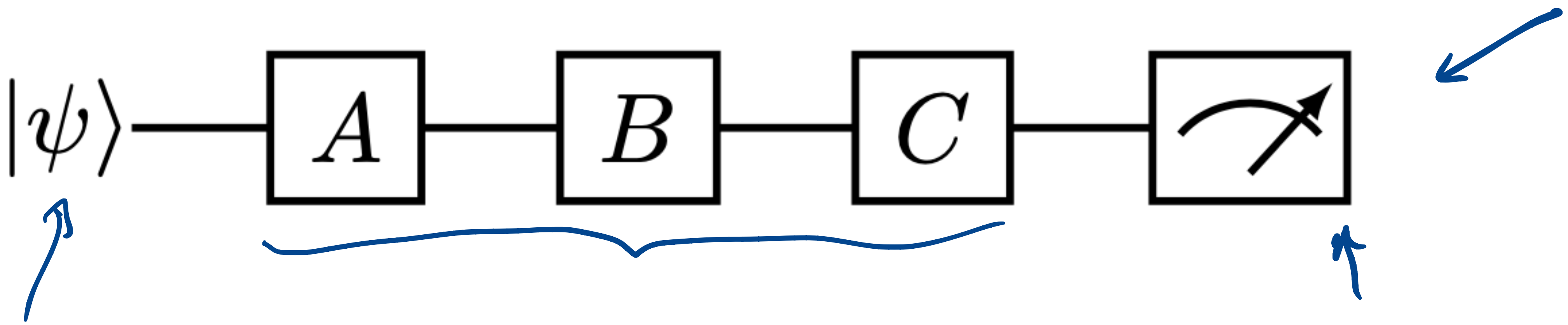
In [17]:
```
bell = QuantumCircuit(2, 2)
bell.h(1)
bell.cx(1, 0)
bell.measure(range(2), range(2))
bell.draw()
```

Out[17]:

What math are we learning today and why do we need it ???

SINGLE-QUBIT CIRCUIT MATH

THE SINGLE-QUBIT CIRCUIT

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qc_z = QuantumCircuit(1, 1)
qc_z.h(0)
qc_z.z(0)
qc_z.h(0)
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```

REMEMBER, MATHEMATICALLY...

Quantum states are complex vectors!

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\alpha|0\rangle + \beta|1\rangle$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \qquad \psi_1, \psi_2 \in \mathbb{C}$$

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}$$

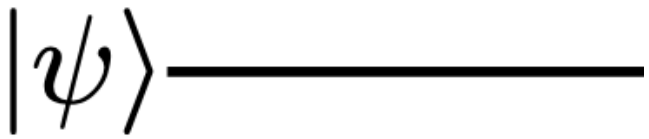
Quantum gates are complex (unitary) matrices!

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad a_{..} \in \mathbb{C}$$

$$A = \begin{pmatrix} \cos(\theta/2) & -e^{i\lambda}\sin(\theta/2) \\ e^{i\phi}\sin(\theta/2) & e^{i\lambda+i\phi}\cos(\theta/2) \end{pmatrix}$$

REMEMBER, MATHEMATICALLY…

Quantum states are complex vectors!



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REVIEW OF SINGLE QUBIT STATES

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

REVIEW OF SINGLE QUBIT GATES

$X =$ $\rightarrow |0\rangle\langle 1| + |1\rangle\langle 0|$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$Z =$ $|0\rangle\langle 0| - |1\rangle\langle 1|$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

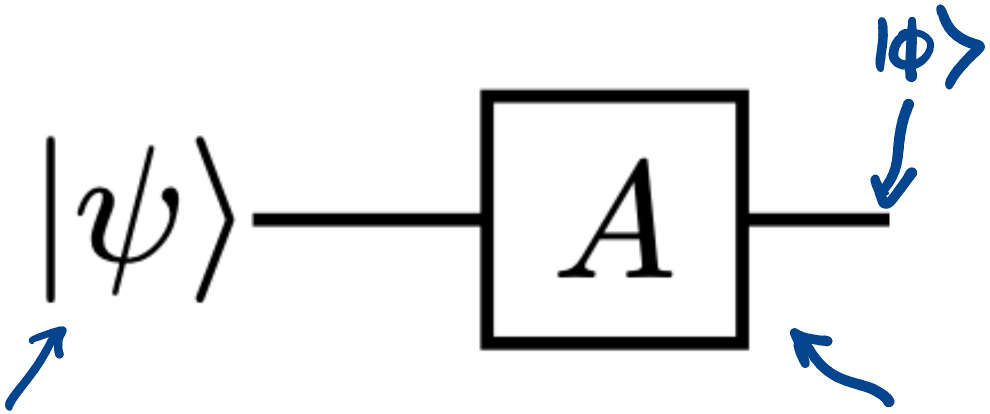
$Y =$ $-i|0\rangle\langle 1| + i|1\rangle\langle 0|$

$= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

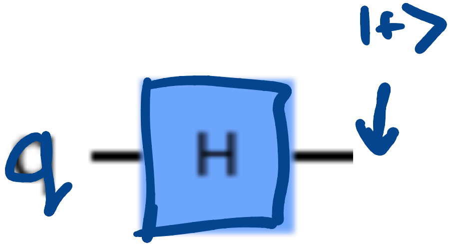
$H|0\rangle = |+\rangle$

$H|1\rangle = |-\rangle$




```python
qc_z = QuantumCircuit(1, 1)
qc_z.h(0)
```

$|q\rangle = |0\rangle$



$$|\phi\rangle = A|\psi\rangle$$

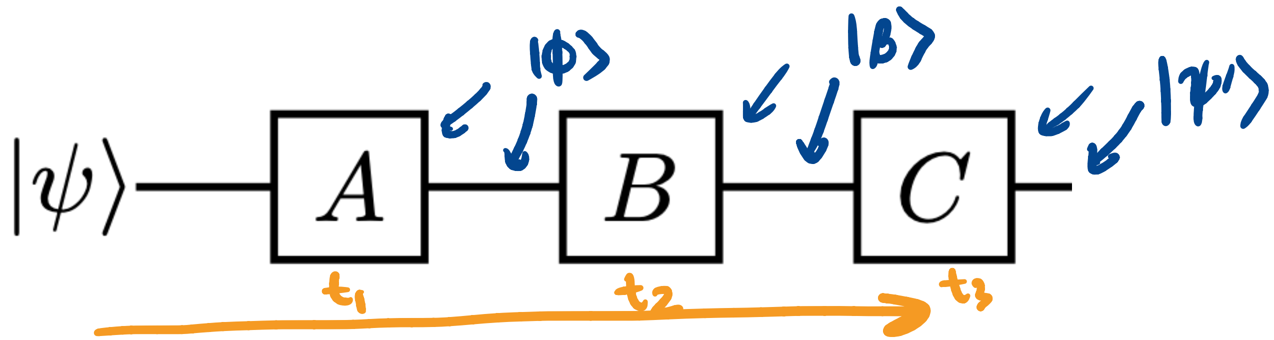
$$|\phi\rangle = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$H|q\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

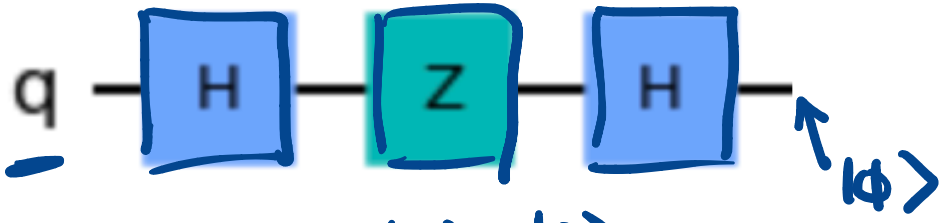
$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = |+\rangle$$

$$H|0\rangle = |+\rangle$$

AND IN A LARGER CIRCUIT...




```
qc_z = QuantumCircuit(1, 1)
qc_z.h(0)
qc_z.z(0)
qc_z.h(0)
```



$$C(\underbrace{B(\underbrace{A|\psi\rangle}_{|\phi\rangle})}_{|\beta\rangle})$$

$$|\psi'\rangle = CBA|\psi\rangle$$

$$HZH|q\rangle \quad |q\rangle = |0\rangle$$

$$= \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|\phi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

ORDER MATTERS!

$|\psi\rangle - \boxed{A} - \boxed{B} - \boxed{C} -$ $\neq$ $|\psi\rangle - \boxed{C} - \boxed{A} - \boxed{B} -$

generally

$CBA|\psi\rangle$ $\qquad$ $BAC|\psi\rangle$

$CBA \neq BAC$

generally

Heisenberg Uncertainty Prin.

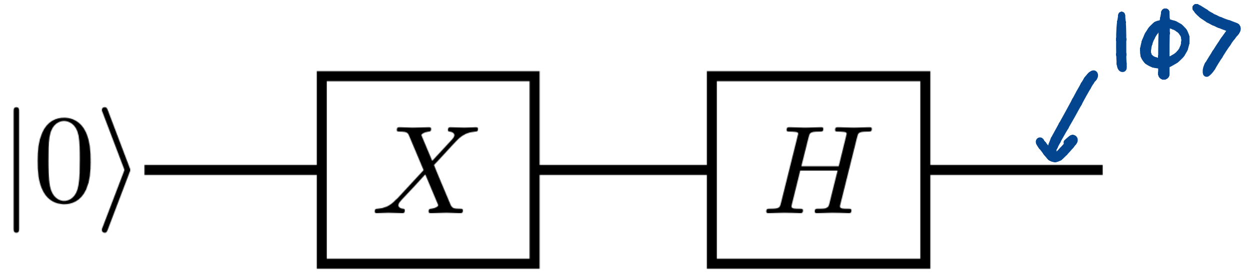
Commutators

$5 \times 4 \times 1$

$= 1 \times 4 \times 5$

$= 4 \times 1 \times 5$

QUANTUM PRACTICE TIME!

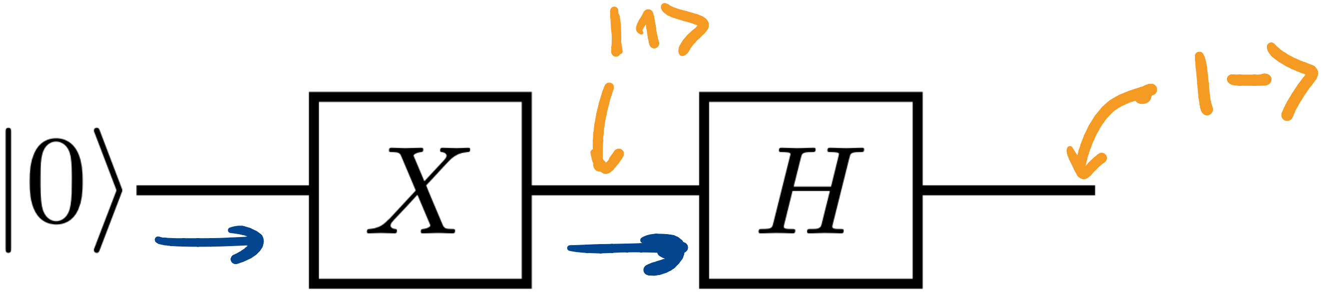


$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

QUANTUM PRACTICE SOLUTION!

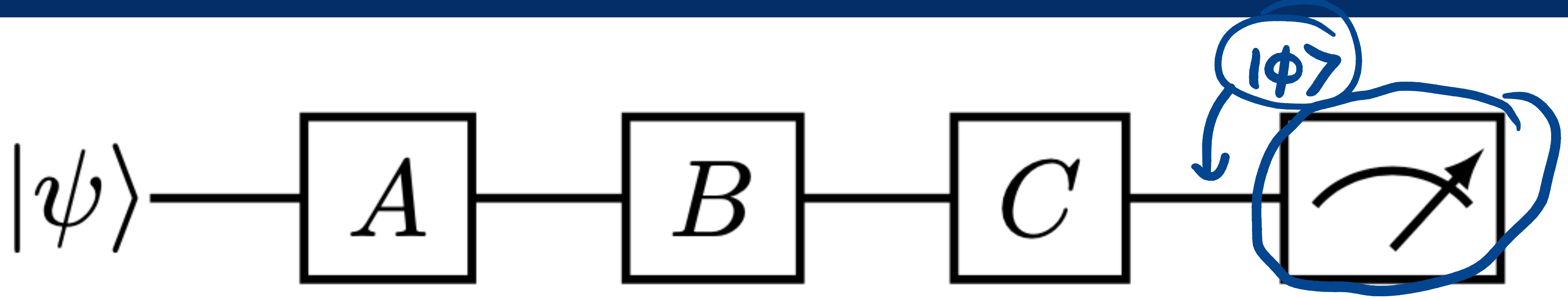


$$H\,X|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle$$

SO, WHAT IS MEASUREMENT?



Assume we are measuring in the Z-basis!

```
qc_z = QuantumCircuit(1, 1)
qc_z.h(0)
qc_z.z(0)
qc_z.h(0)
qc_z.measure(0, 0)
```

$|\phi\rangle = CBA|\psi\rangle$

$|\phi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

α → prob. amp. of measuring the state $|0\rangle$

β → prob. amp. of measuring the state $|1\rangle$

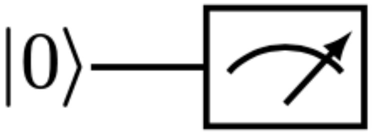
$\mathbb{P}(\text{meas. } |0\rangle) = |\alpha|^2$

$\mathbb{P}(\text{meas. } |1\rangle) = |\beta|^2$

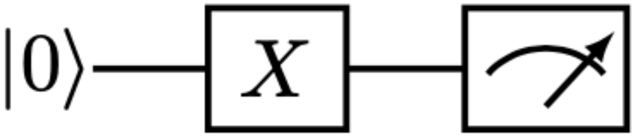
QUANTUM PRACTICE TIME!

Solve for the likelihoods of measuring $|0\rangle$ and $|1\rangle$ for each of the following quantum circuits…

(1)



(2)



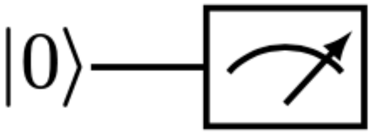
(3)

(4)

QUANTUM PRACTICE SOLUTIONS!

Solve for the likelihoods of measuring $|0\rangle$ and $|1\rangle$ for each of the following quantum circuits…

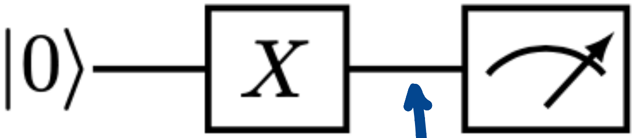
(1)



$\rightarrow |0\rangle$ 100%

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{array}{l} \rightarrow \mathbb{P}(\text{meas. } |0\rangle) = |1|^2 = 1 \rightarrow 100\% \\ \rightarrow \mathbb{P}(\text{meas. } |1\rangle) = |0|^2 = 0 \rightarrow 0\% \end{array}$$

(2)



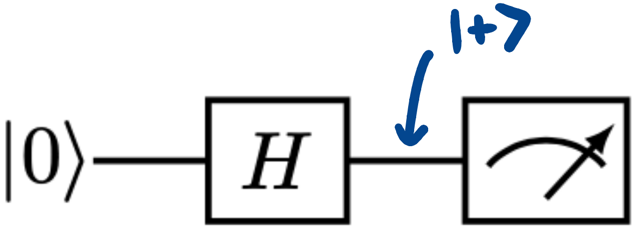
$|1\rangle$

$$\rightarrow |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \mathbb{P}(\text{meas. } |1\rangle) = 1 \; (100\%)$$

$$\mathbb{P}(\text{meas. } |0\rangle) = 0 \; (0\%)$$

Solve for the likelihoods of measuring $|0\rangle$ and $|1\rangle$ for each of the following quantum circuits…

(3)



$H|0\rangle = |+\rangle$

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

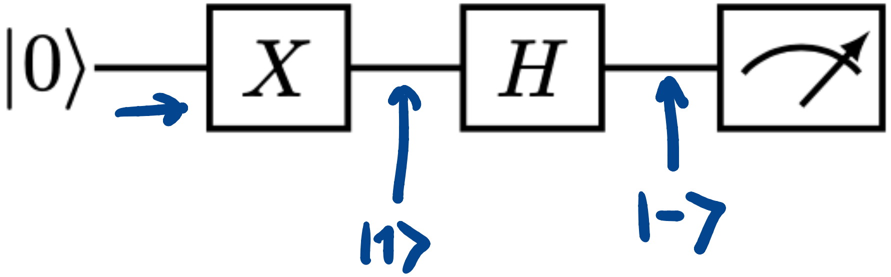
$\rightarrow \mathbb{P}(\text{meas. } |0\rangle) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$ 50%

$\rightarrow \mathbb{P}(\text{meas } |1\rangle) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$ 50%

↳ 50%

50%

(4)

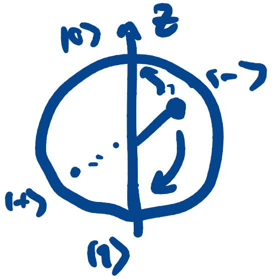


$H|1\rangle = |-\rangle$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\rightarrow \mathbb{P}(\text{meas } |0\rangle) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$

$\rightarrow \mathbb{P}(\text{meas } |1\rangle) = \left|-\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$



↳ 50%

50%

- Since matrix multiplication results in a new matrix, a series of gates can be represented by a single matrix, which is just the product of all the gate matrices…

- The "wire" without a gate is equivalent to applying the identity matrix/operation!

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

- Experimentally, circuits generally have to start in the $|0\rangle$ state. So, while you can start your circuit in any state theoretically, it is good to keep in mind that experimentally this will involve applying gates.

BREAK TIME!

3:05 PM EST

$$|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \rightarrow |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \leftarrow \text{SUPERPOSITION}$$

1st qubit   2nd qubit

$$\frac{|\underline{0}\underline{0}\rangle + |11\rangle}{\sqrt{2}} \quad \begin{matrix} \nearrow |0\rangle|0\rangle \\ \searrow |1\rangle|1\rangle \end{matrix} \leftarrow \text{ENTANGLEMENT}$$

$$\underline{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \underline{|0\rangle\langle 1| + |1\rangle\langle 0|}$$

$$\langle 0|0\rangle = 1 \rightarrow (1\ 0)\begin{pmatrix}1\\0\end{pmatrix} = 1(1) + 0(0) = 1$$

$$\langle 1|1\rangle = 1$$

$$\langle 0|1\rangle = 0 \rightarrow (1\ 0)\begin{pmatrix}0\\1\end{pmatrix} = 1(0) + 0(1) = 0$$

$$\langle 1|0\rangle = 0$$

$$\boxed{X|0\rangle}$$

$$= \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)|0\rangle$$

$$= |0\rangle\underbrace{\langle 1|0\rangle}_{0} + |1\rangle\underbrace{\langle 0|0\rangle}_{1}$$

$$= \cancel{0|0\rangle} + 1|1\rangle$$

$$= \boxed{|1\rangle}$$

$$|0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}(1\ 0) = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \qquad \underline{|0\rangle\langle 1|} = \begin{pmatrix}1\\0\end{pmatrix}(0\ 1) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

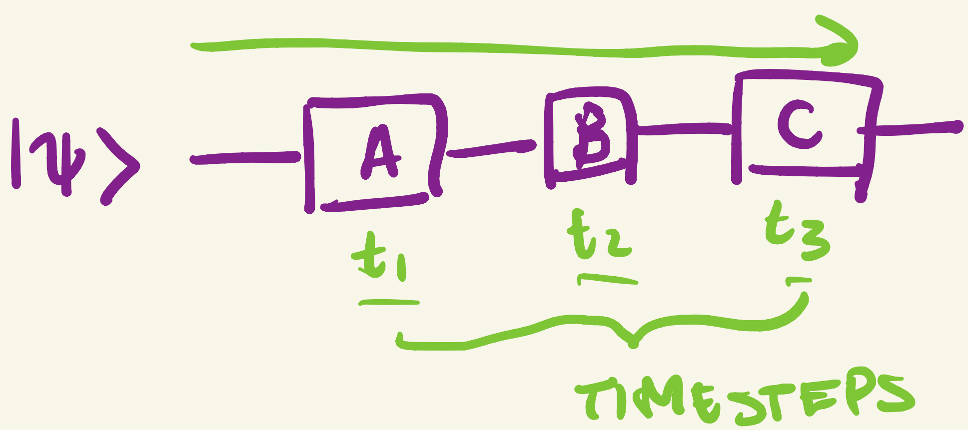
$$|1\rangle\langle 1| = \begin{pmatrix}0\\1\end{pmatrix}(0\ 1) = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \qquad \underline{|1\rangle\langle 0|} = \begin{pmatrix}0\\1\end{pmatrix}(1\ 0) = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

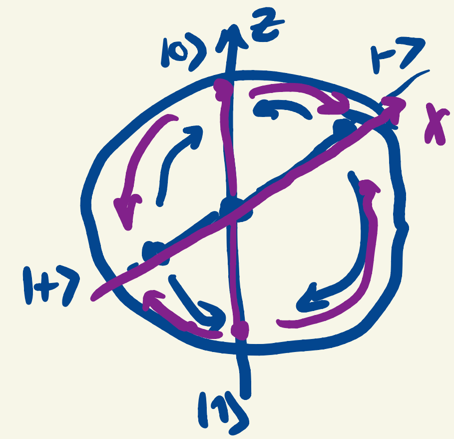
$$X\psi = (|1\rangle\langle 0| + |0\rangle\langle 1|)(\alpha|0\rangle + \beta|1\rangle)$$

$$= \alpha|1\rangle\underbrace{\langle 0|0\rangle}_{1} + \cancel{\beta|1\rangle\langle 0|1\rangle} + \cancel{\alpha|0\rangle\langle 1|0\rangle} + \beta|0\rangle\underbrace{\langle 1|1\rangle}_{1}$$

$$= \alpha|1\rangle + \beta|0\rangle$$

$$\psi = \alpha|0\rangle + \beta|1\rangle$$

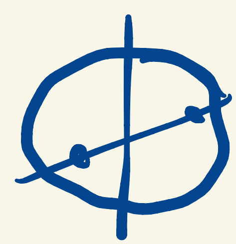


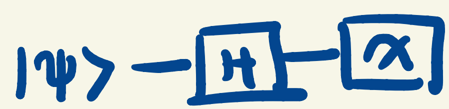


$$H|0\rangle = |+\rangle$$

$$H|+\rangle = |0\rangle$$

$$H|-\rangle = |1\rangle$$





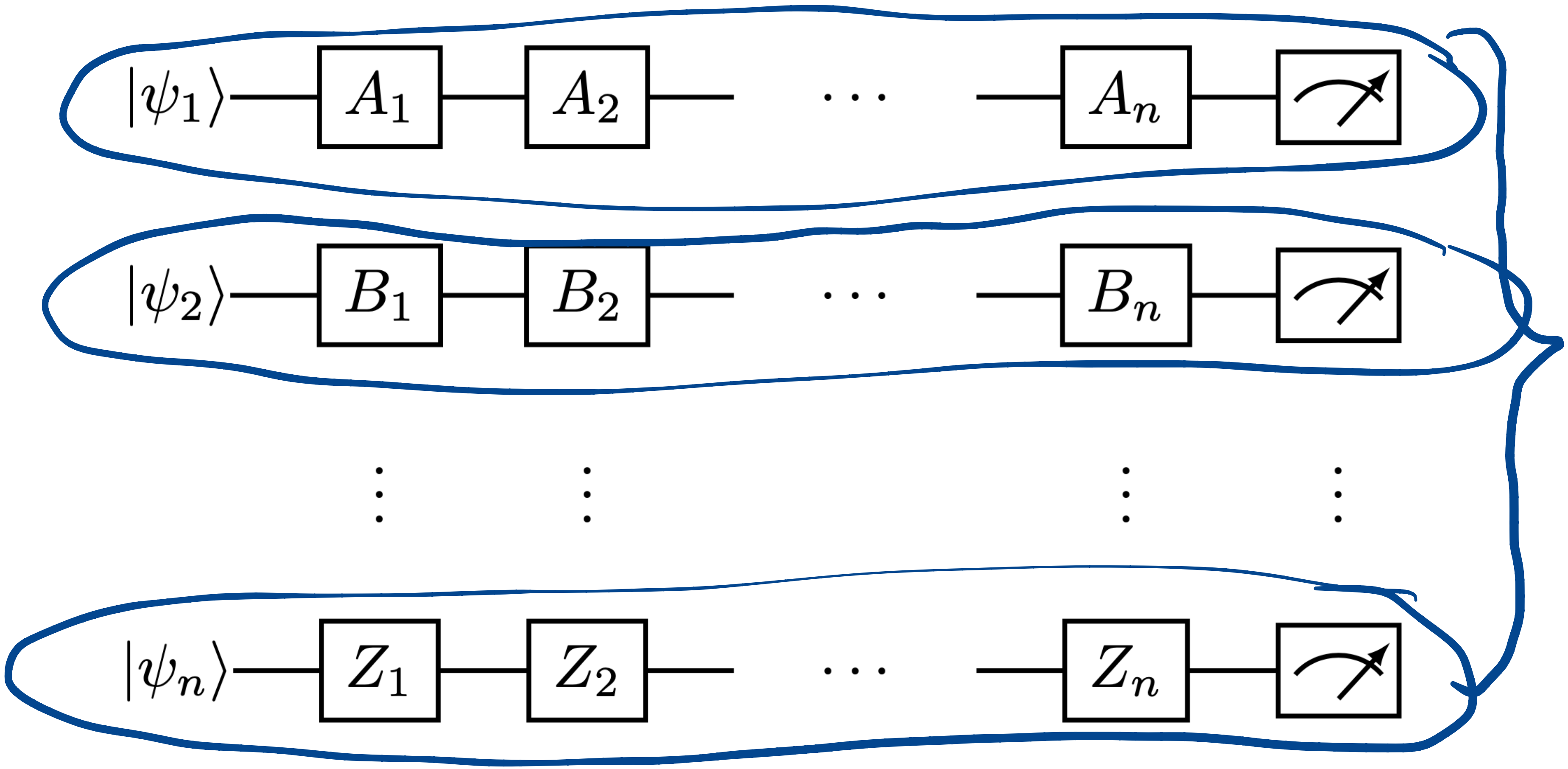
$$|+\rangle \rightarrow |0\rangle$$

$$|-\rangle \rightarrow |1\rangle$$

MULTI-QUBIT CIRCUIT MATH

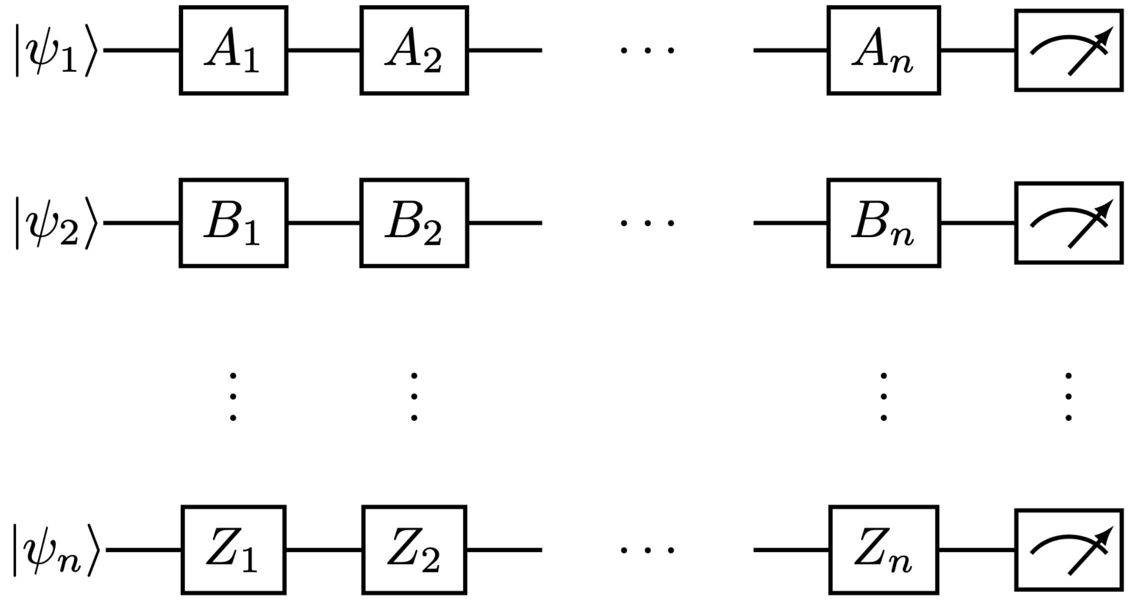
MANY SINGLE-QUBIT CIRCUITS

What if we have a collection of n single-qubit circuits?



There are 2 different ways we can approach solving for the output n - bit string.

Method #1: *Individually* solve each of the n single-qubit circuits.



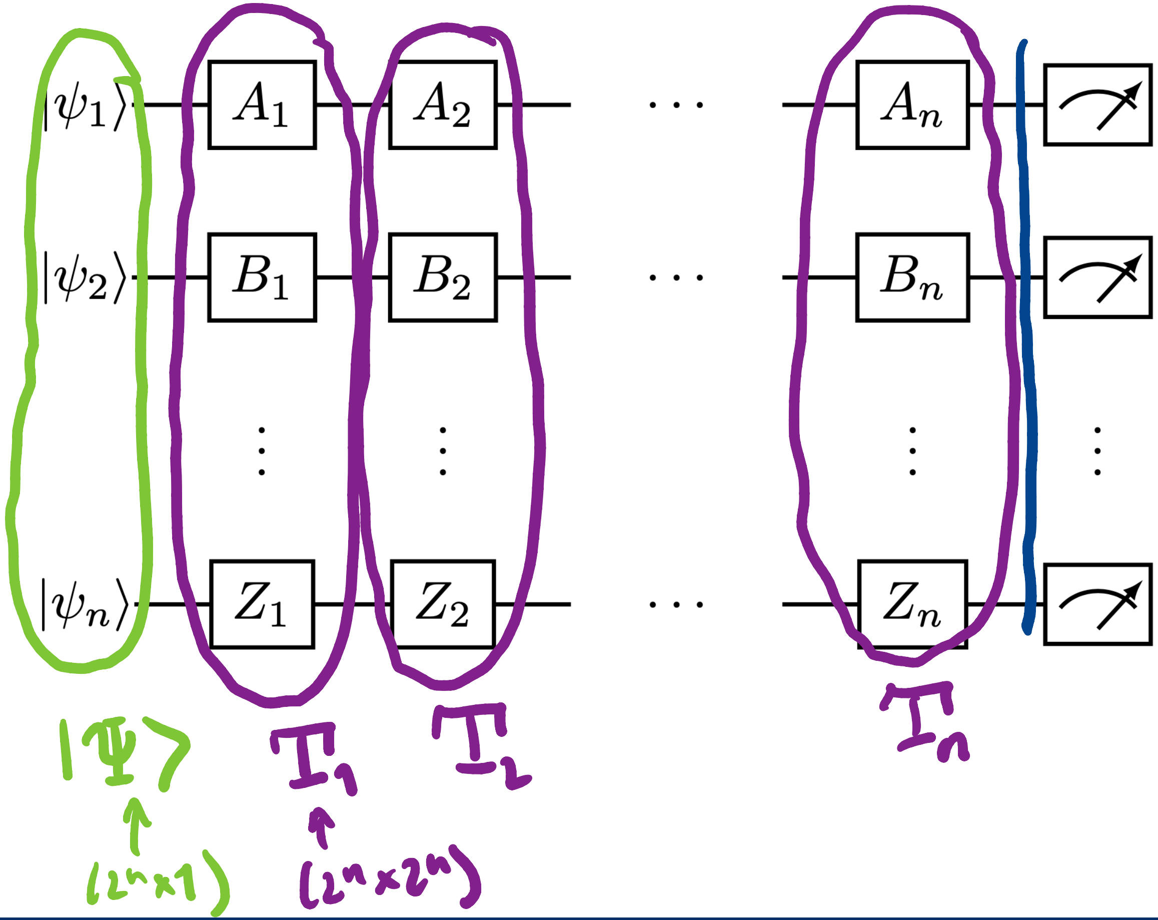
$$A_n \dots A_2 A_1 |\psi_1\rangle = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \begin{matrix} \to |\alpha_1|^2\% \ |0\rangle \\ \to |\beta_1|^2\% \ |1\rangle \end{matrix}$$

$$B_n \dots B_2 B_1 |\psi_2\rangle = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

$$Z_n \dots Z_2 Z_1 |\psi_n\rangle = \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}$$

$$|0\,1\,0\rangle = \mathbb{P}(|0\rangle)\,\mathbb{P}(|1\rangle)\,\mathbb{P}(|0\rangle) = |\alpha_1|^2 |\beta_2|^2 |\alpha_3|^2$$

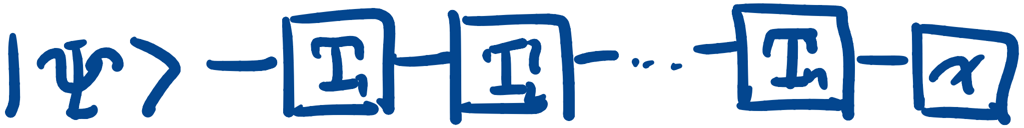
Method #2: *Simultaneously* solve all n single-qubit circuits.



$$T_n \cdots T_2 T_1 |\Psi\rangle = |\Psi'\rangle$$

$$\underbrace{(2^n \times 2^n) \cdots (2^n \times 2^n)}$$

$$\underbrace{(2^n \times 2^n)\ (2^n \times 1)}_{(2^n \times 1)}$$

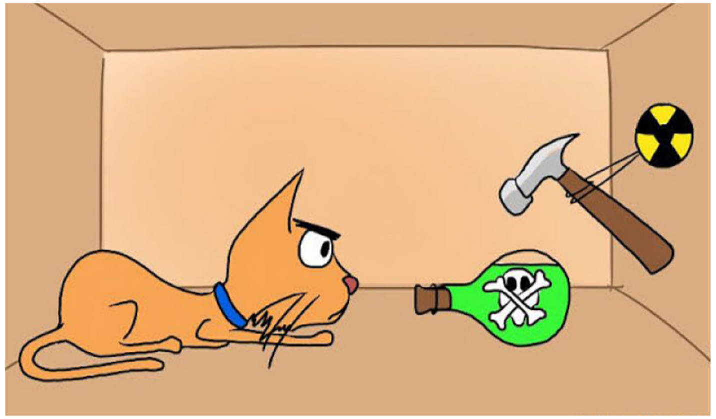


MULTI-QUBIT CIRCUITS

Are all multi-qubit circuits just collections of single-qubit circuits?

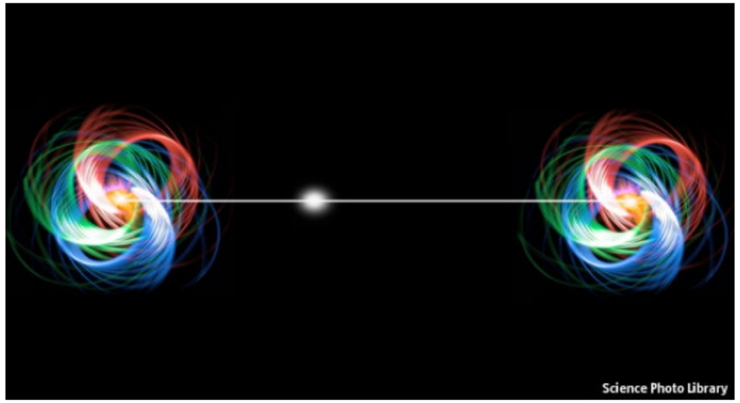
To answer this, let's think back to our 3 key quantum properties:

SUPERPOSITION



QUANTUM INTERFERENCE

ENTANGLEMENT



H!

$H|0\rangle = |+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$

$H|+\rangle = |0\rangle$

MULTI-QUBIT GATES

To achieve entanglement, our qubits need to interact, meaning we need 2-qubit (or larger) gates!

Construct Bell state

In [17]:
```
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bell.h(1)
bell.cx(1, 0)
bell.measure(range(2), range(2))
bell.draw()
```
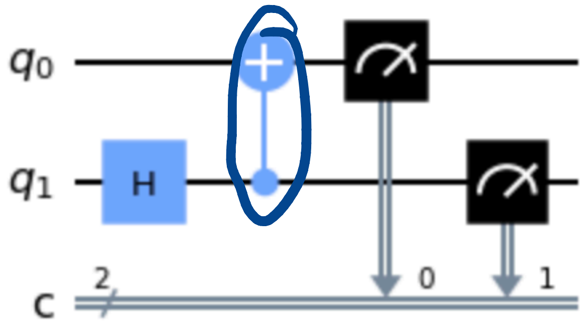
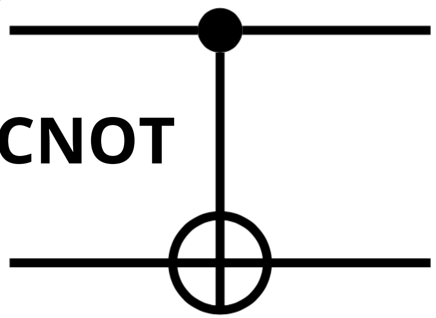

Out[17]:



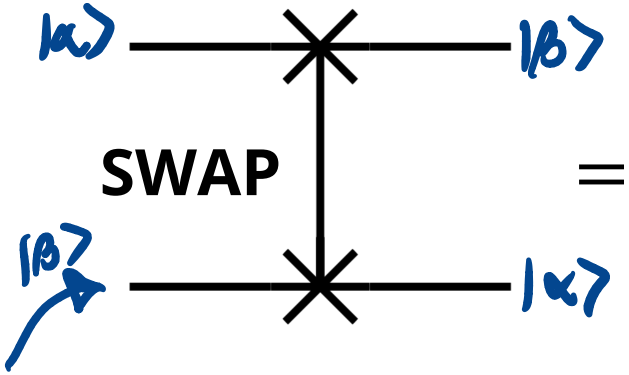
CNOT



$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

```
bell.cx(1, 0)
```

SWAP

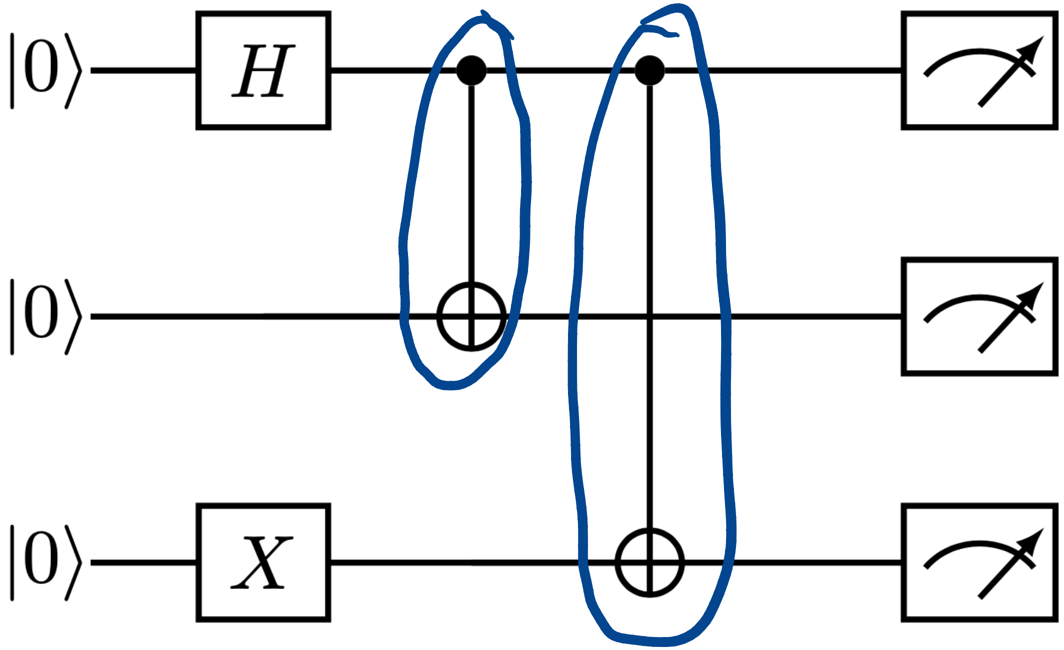


$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that these are $(2^2 \times 2^2) = (4 \times 4)$ matrices.

MULTI-QUBIT GATES

However, with 2-qubit gates, we can no longer use **Method #1** (individually calculating each single-qubit circuit) for calculating the output of the multi-qubit circuit…

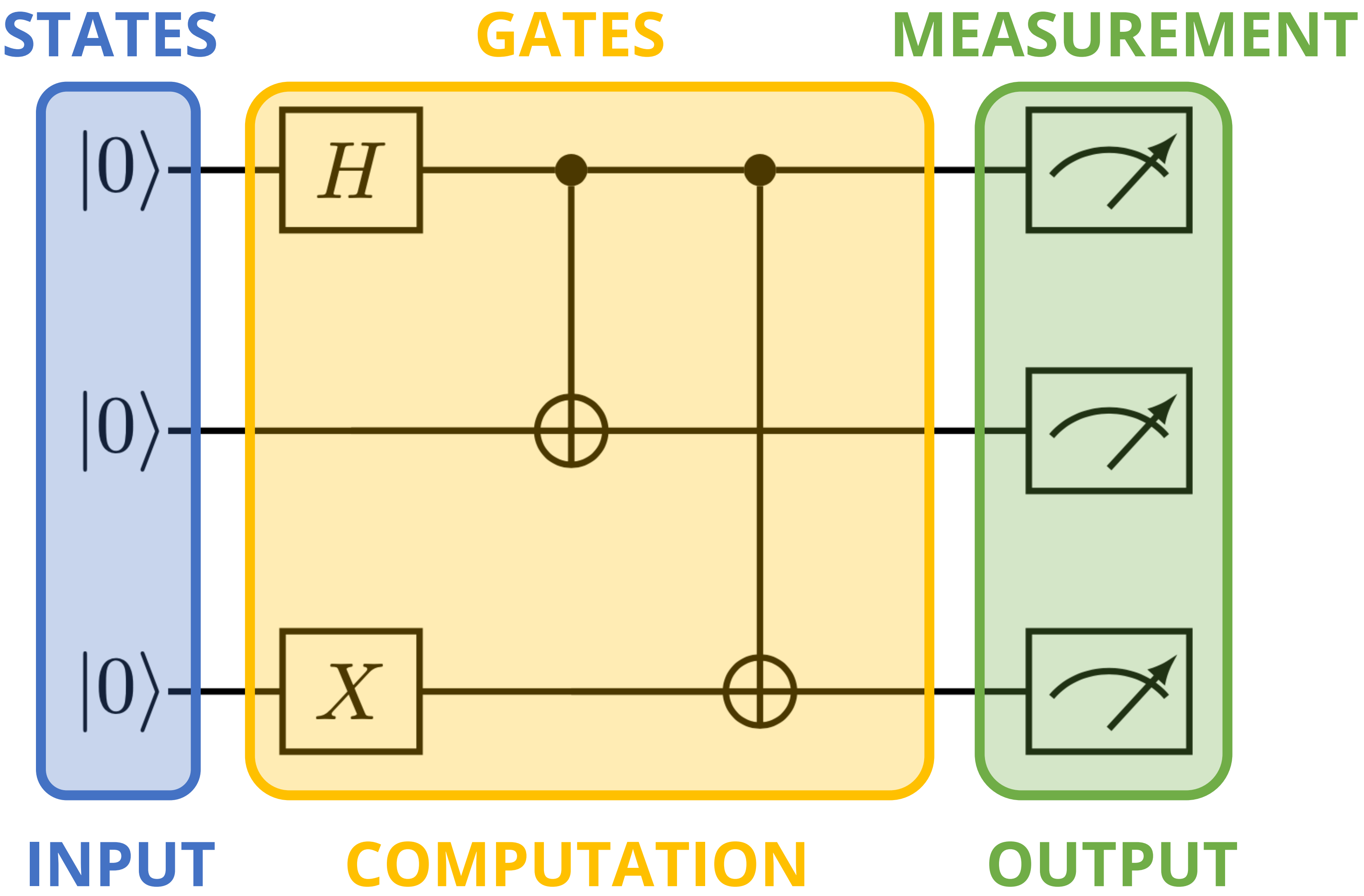


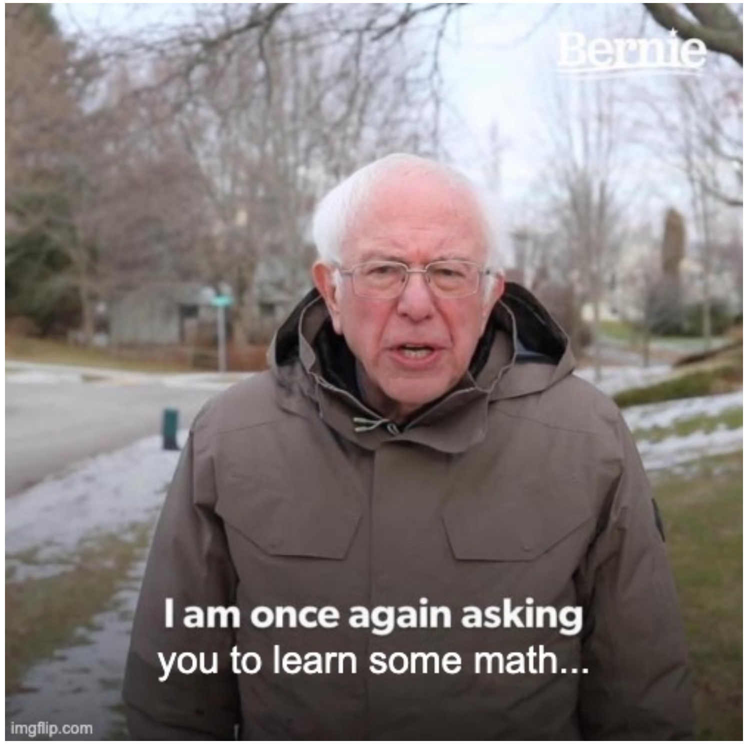
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ b \end{pmatrix}$$

$$\underbrace{(4 \times 4)\ (2 \times 2)}$$

So now let's learn **Method #2**!

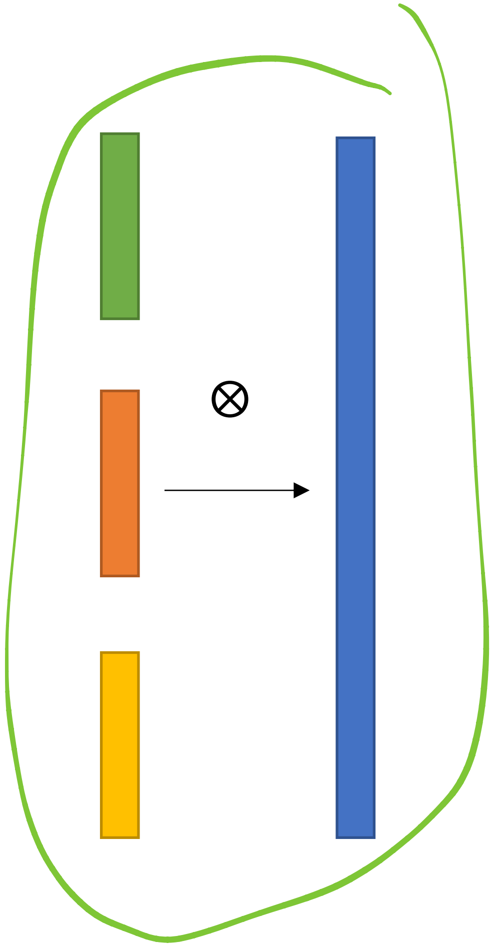
THE MULT-QUBIT CIRCUIT MODEL

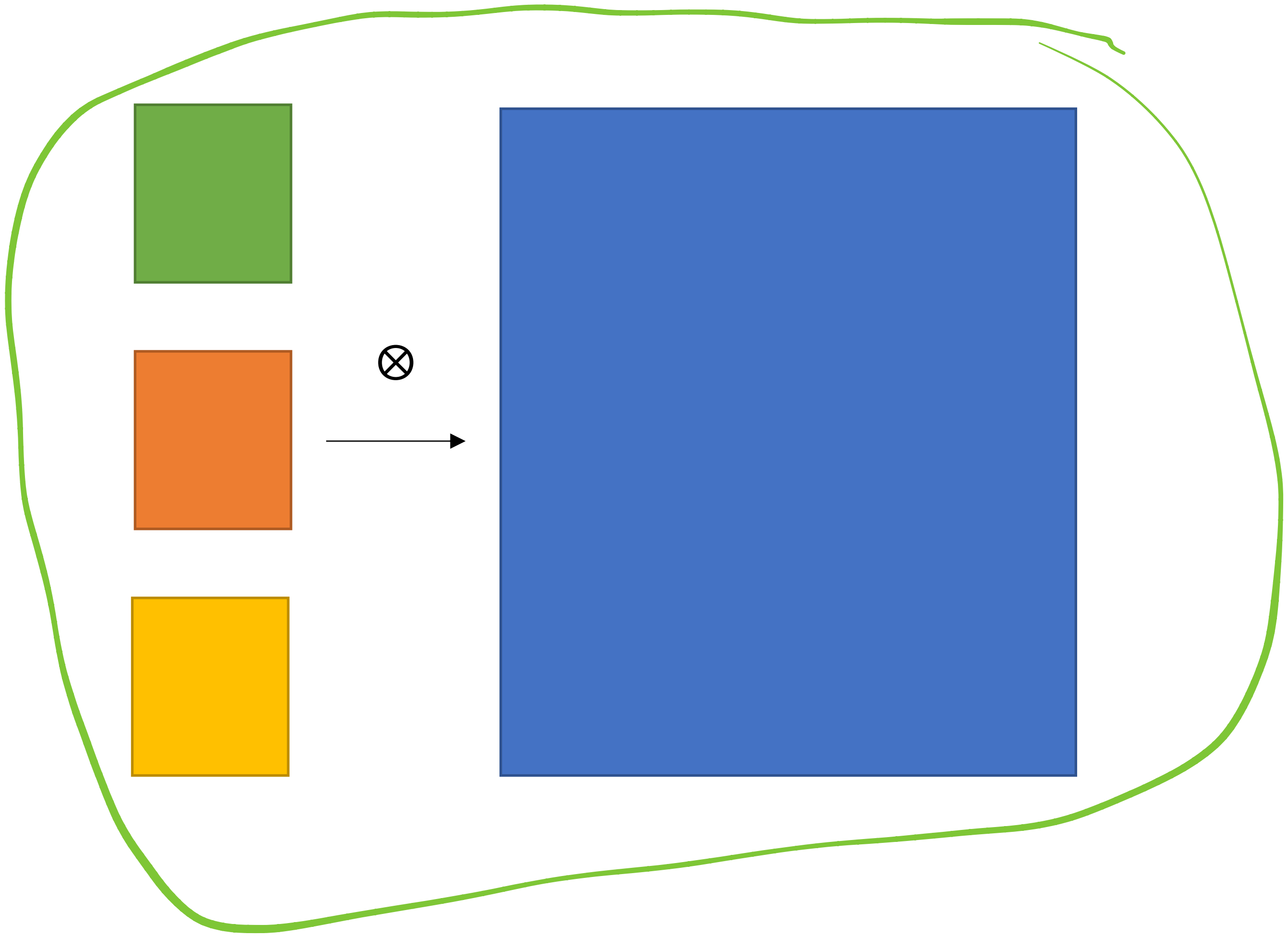




TENSOR PRODUCT

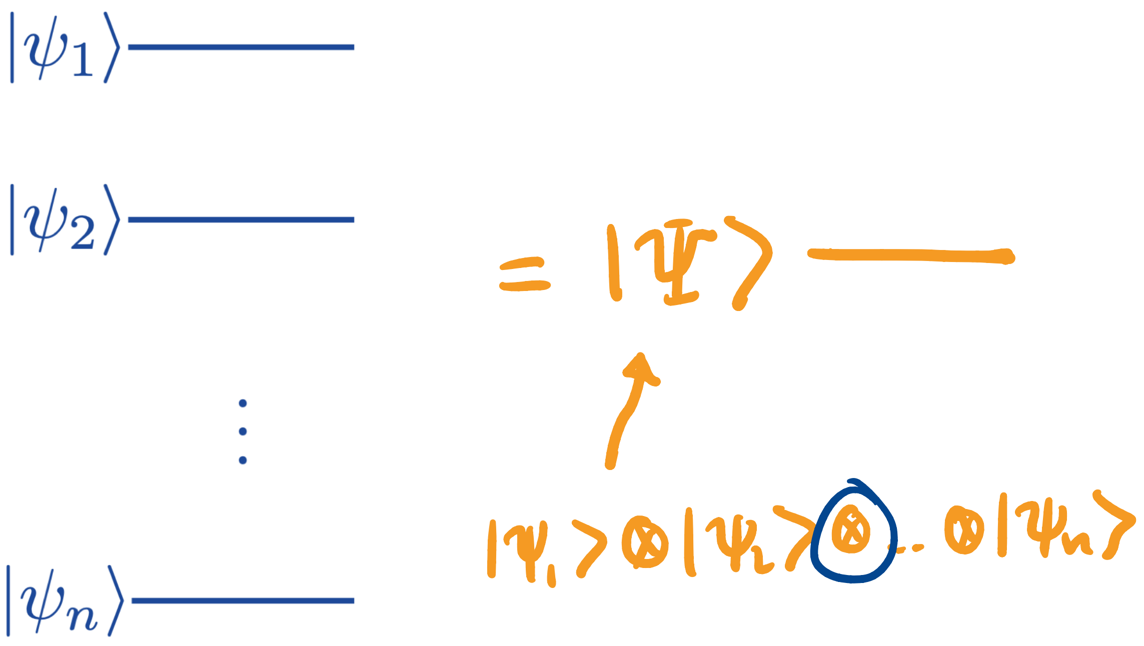
We use the ***tensor product*** to solve multi-qubit circuits!





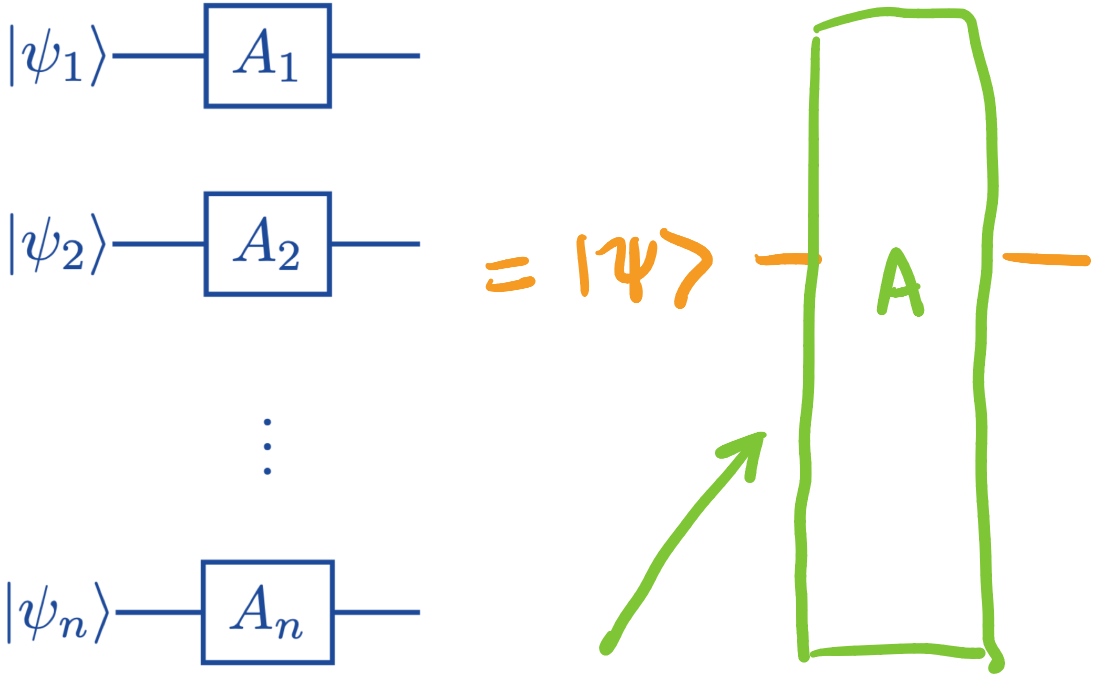
MULTI-QUBIT STATES & GATES

Mathematically, a quantum bit string is simply a ***tensor product*** of qubits!



$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes \dots \otimes |\psi_n\rangle$$

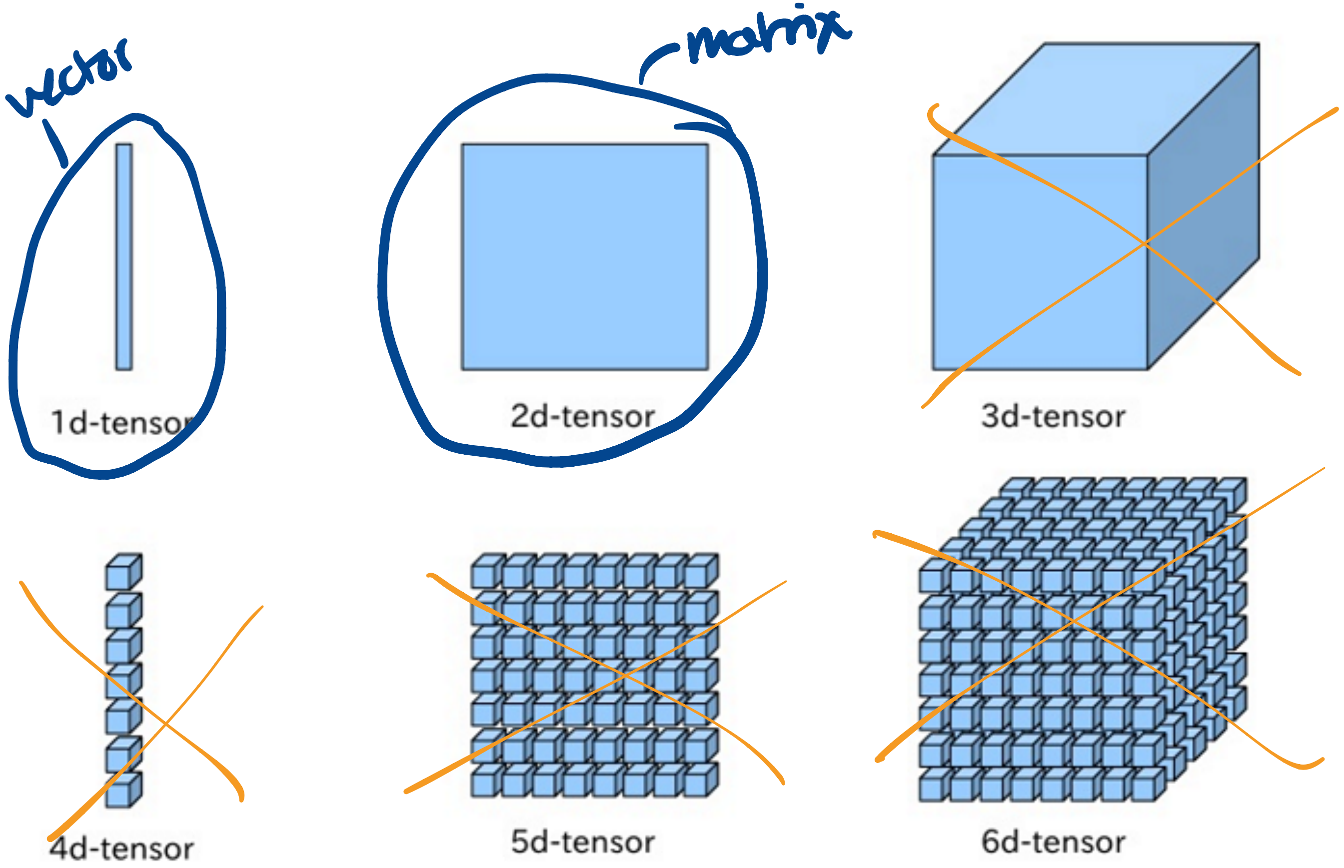
Mathematically, gates acting on different qubits are a ***tensor product*** of all the gates!

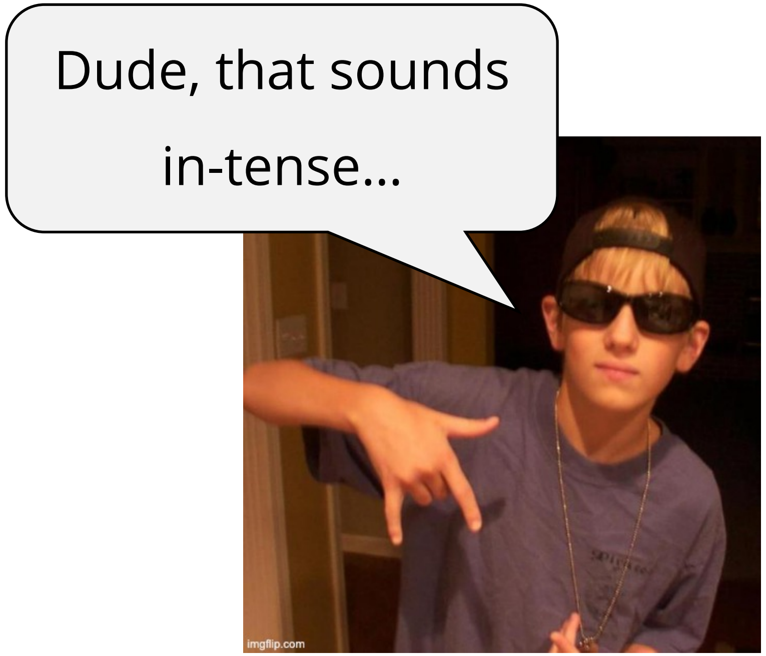


$$A = A_1 \otimes A_2 \otimes \dots \otimes A_n$$

TENSOR

A tensor is essentially an n-dimensional vector... In quantum computing, we care about vectors (states) and matrices (gates), which are 1-D and 2-D tensors!





THE TENSOR (KRONECKER) PRODUCT

If $\boldsymbol{A}$ is an (nxm) matrix and $\boldsymbol{B}$ is a (pxq) matrix, their tensor product is the (pn x mq) matrix:

$$A \otimes B = \begin{pmatrix} a_{11}\boldsymbol{B} & a_{12}\boldsymbol{B} & \cdots & a_{1m}\boldsymbol{B} \\ a_{21}\boldsymbol{B} & a_{22}\boldsymbol{B} & \cdots & a_{2m}\boldsymbol{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}\boldsymbol{B} & a_{n2}\boldsymbol{B} & \cdots & a_{nm}\boldsymbol{B} \end{pmatrix}$$

$$A = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad A \otimes B = \begin{pmatrix} a_1 B \\ a_2 B \end{pmatrix} = \begin{pmatrix} a_1 \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ a_2 \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_1 b_{11} & a_1 b_{12} \\ a_1 b_{21} & a_1 b_{22} \\ a_2 b_{11} & a_2 b_{12} \\ a_2 b_{21} & a_2 b_{22} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

QUANTUM PRACTICE TIME!

Write out the full matrix for the following tensor products…

(1) $\begin{pmatrix}1\\2\end{pmatrix} \otimes \begin{pmatrix}3\\4\end{pmatrix} =$

(3) $\begin{pmatrix}1\\-1\end{pmatrix} \otimes \begin{pmatrix}2 & 3\\4 & 5\end{pmatrix} =$

(2) $\begin{pmatrix}3\\4\end{pmatrix} \otimes \begin{pmatrix}1\\2\end{pmatrix} =$

(4) $\begin{pmatrix}2 & 3\\4 & 5\end{pmatrix} \otimes \begin{pmatrix}1\\-1\end{pmatrix} =$

QUANTUM PRACTICE SOLUTIONS!

Write out the full matrix for the following tensor products…

(1) $\begin{pmatrix}1\\2\end{pmatrix} \otimes \begin{pmatrix}3\\4\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}3\\4\end{pmatrix}\\ 2\begin{pmatrix}3\\4\end{pmatrix}\end{pmatrix} = \begin{pmatrix}3\\4\\6\\8\end{pmatrix}$

order matters!

(2) $\begin{pmatrix}3\\4\end{pmatrix} \otimes \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3\begin{pmatrix}1\\2\end{pmatrix}\\ 4\begin{pmatrix}1\\2\end{pmatrix}\end{pmatrix} = \begin{pmatrix}3\\6\\4\\8\end{pmatrix}$

$|\psi_1\rangle -\boxed{A}-$

$|\psi_2\rangle -\boxed{B}-$

$\neq$

$|\psi_1\rangle -\boxed{B}-$

$|\psi_2\rangle -\boxed{A}-$

(3) $\begin{pmatrix}1\\-1\end{pmatrix} \otimes \begin{pmatrix}2 & 3\\4 & 5\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}2 & 3\\4 & 5\end{pmatrix}\\ -1\begin{pmatrix}2 & 3\\4 & 5\end{pmatrix}\end{pmatrix} = \begin{pmatrix}2 & 3\\4 & 5\\-2 & -3\\-4 & -5\end{pmatrix}$

$\neq$

(4) $\begin{pmatrix}2 & 3\\4 & 5\end{pmatrix} \otimes \begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}2\begin{pmatrix}1\\-1\end{pmatrix} & 3\begin{pmatrix}1\\-1\end{pmatrix}\\ 4\begin{pmatrix}1\\-1\end{pmatrix} & 5\begin{pmatrix}1\\-1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}2 & 3\\-2 & -3\\4 & 5\\-4 & -5\end{pmatrix}$

MULTI-QUBIT STATES

Mathematically, a quantum bit string is simply a tensor product of qubits!

$|01\rangle = |0\rangle \otimes |1\rangle$

$|11\rangle = |1\rangle \otimes |1\rangle$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$|\psi_1\rangle$ ———

$|\psi_2\rangle$ ———

$\vdots$

$|\psi_n\rangle$ ———

$$\boxed{\underbrace{|\psi_1 \; \psi_2 \dots \psi_n\rangle}_{} = \underbrace{|\psi_1\rangle \cancel{\otimes} |\psi_2\rangle \otimes \dots \otimes |\psi_n\rangle}_{}}$$

n qubits are represented mathematically by a vector of length 2^n

$(2^n \times 1)$

```
bell = QuantumCircuit(2, 2)
```

WORKED EXAMPLES

(1)

$n=3$

$$|010\rangle = |0\rangle \otimes |1\rangle \otimes |0\rangle$$

$$= \begin{pmatrix}1\\0\end{pmatrix} \otimes \left(\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\begin{pmatrix}1\\0\end{pmatrix}\\1\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\\0\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\end{pmatrix} = \begin{array}{c} \begin{pmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix} \end{array} \begin{array}{l}|000\rangle\\|001\rangle\\|010\rangle\\|011\rangle\\|100\rangle\\|101\rangle\\|110\rangle\\|111\rangle\end{array}$$

$8 = 2^3$

(2)

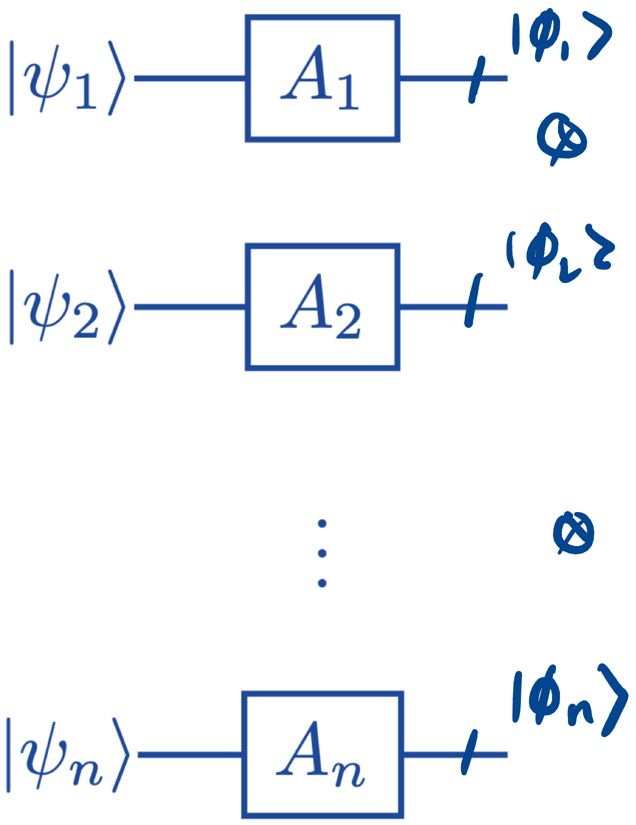
$$|+\,-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1\begin{pmatrix}1\\-1\end{pmatrix}\\1\begin{pmatrix}1\\-1\end{pmatrix}\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$$

MULTI-QUBIT GATES

If we want to calculate how a set of quantum gates will affect our multi-qubit state, we will need to take the tensor product of the gate matrices.



$$\underbrace{A_1\otimes A_2\otimes\ldots\otimes A_n}\underbrace{|\psi_1\ \psi_2\ldots\ \psi_n\rangle}$$

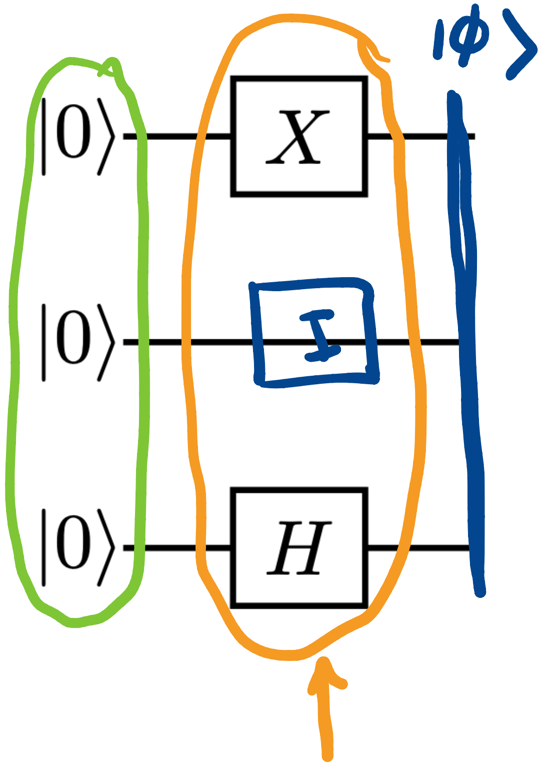
$$= \underbrace{A_1|\psi_1\rangle}_{|\phi_1\rangle}\otimes\underbrace{A_2|\psi_2\rangle}_{|\phi_2\rangle}\otimes\ldots\otimes\underbrace{A_n\,|\psi_n\rangle}_{|\phi_n\rangle}$$

n single qubit gates on different wires are represented mathematically by a matrix of dimension $(2^n{\times}2^n)$

```
bell = QuantumCircuit(2, 2)
bell.i(0)
bell.h(1)
```

Note that wherever there is not a gate, we must insert an identity matrix.

WORKED EXAMPLE



$$|0\rangle \otimes |0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\begin{pmatrix}1\\0\end{pmatrix}\\0\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$$

$$X \otimes I \otimes H = \underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix}} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$$

$$= \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0&1&1&0&0\\0&0&0&0&1&-1&0&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&-1\\1&1&0&0&0&0&0&0\\1&-1&0&0&0&0&0&0\\0&0&1&1&0&0&0&0\\0&0&1&-1&0&0&0&0\end{pmatrix}$$

$$|\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0&0&0&0&1&1&0&0\\0&0&0&0&1&-1&0&0\\0&0&0&0&0&0&1&1\\0&0&0&0&0&0&1&-1\\1&1&0&0&0&0&0&0\\1&-1&0&0&0&0&0&0\\0&0&1&1&0&0&0&0\\0&0&1&-1&0&0&0&0\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\0\\0\\1\\1\\0\\0\end{pmatrix} = \begin{array}{c}|000\rangle\\|001\rangle\\|010\rangle\\|011\rangle\\|100\rangle\\|101\rangle\\|110\rangle\\|111\rangle\end{array}\begin{pmatrix}0\\0\\0\\0\\1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{pmatrix}$$

$$|1/\sqrt{2}|^2 = 1/2 \sim 50\%$$

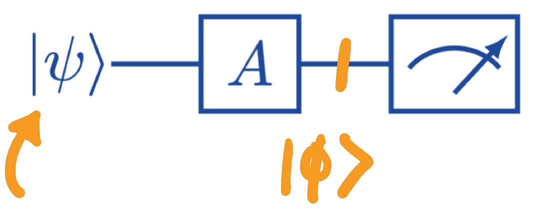
$$|1/\sqrt{2}|^2 = 1/2$$

MULTI-QUBIT MEASUREMENT

In single-qubit measurement, we were simply measuring 1 quantum state.

```
qc_z.measure(0, 0)
```

This state is in a superposition (linear combination) of $|0\rangle$ and $|1\rangle$:

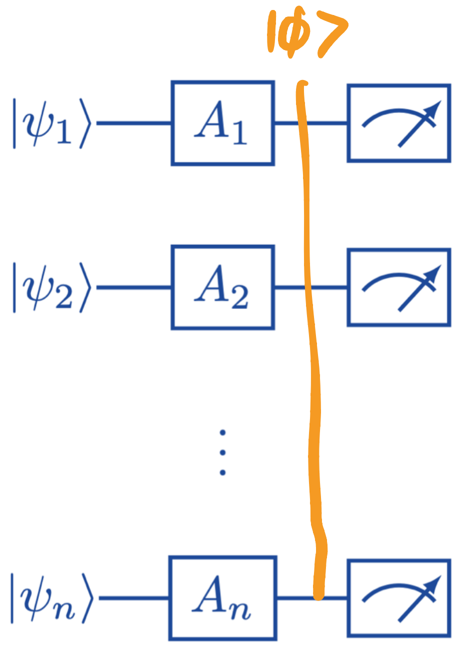


$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{matrix} \to |\alpha|^2 & \mathbb{P}(\text{meas } |0\rangle) \\ \to |\beta|^2 & \mathbb{P}(\text{meas } |1\rangle) \end{matrix}$$

In multi-qubit measurement, we are now measuring n states.

```
bell.measure(range(2), range(2))
```

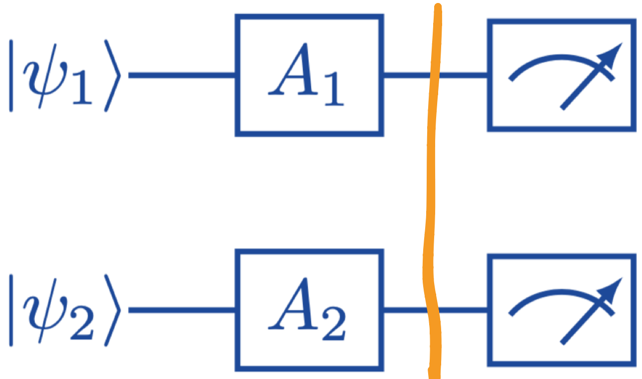
These n states are in a superposition of all 2^n possible bit strings! :



$$|\phi\rangle = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \vdots \\ \Gamma \end{pmatrix} \begin{matrix} \to |\alpha|^2 & \mathbb{P}(\text{meas } 0) \\ \to |\beta|^2 & \mathbb{P}(\text{meas } 1) \\ \to |\gamma|^2 & \mathbb{P}(\text{meas } 2) \\ & \\ \to |\Gamma|^2 & \mathbb{P}(\text{meas } n-1) \end{matrix}$$

MULTI-QUBIT MEASUREMENT

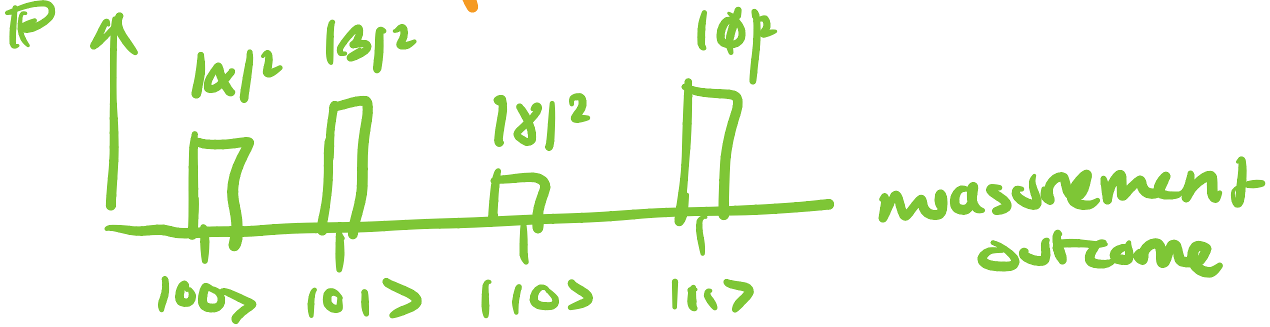
Example: 2-qubit circuit measurement



$$|0\rangle = |00\rangle \quad |1\rangle = |01\rangle \quad |2\rangle = |10\rangle \quad |3\rangle = |11\rangle$$

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \phi \end{pmatrix} \rightarrow \left.\begin{matrix} \mathbb{P}(|00\rangle) \\ \mathbb{P}(|01\rangle) \\ \mathbb{P}(|10\rangle) \\ \mathbb{P}(|11\rangle) \end{matrix}\right\}$$

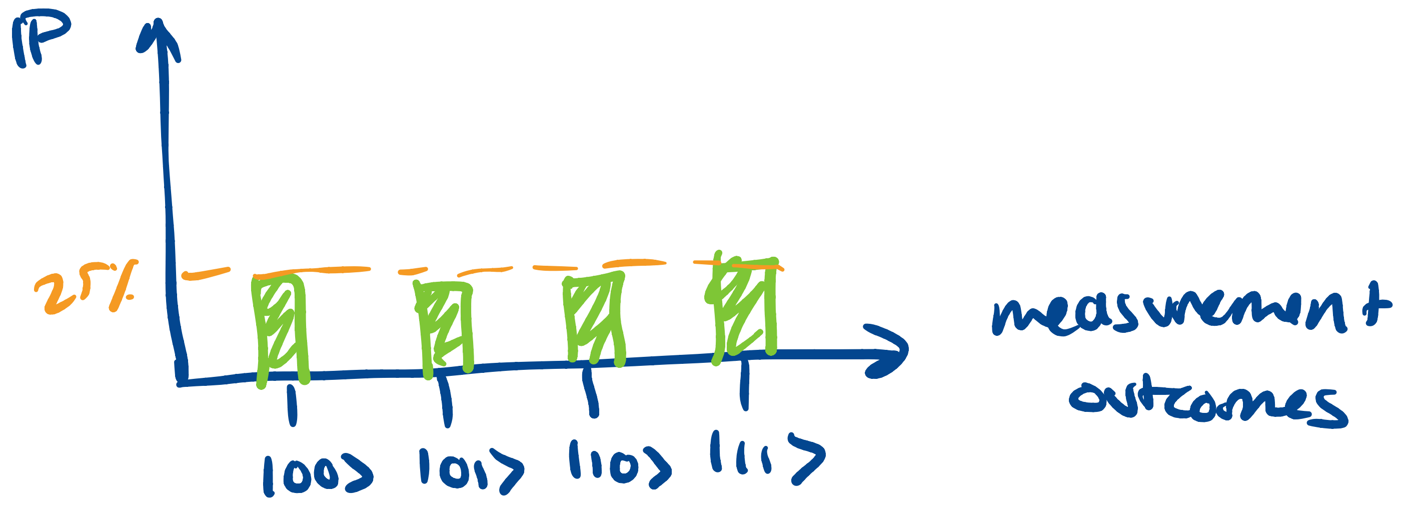
probabilities over all possible n-qubit strings!



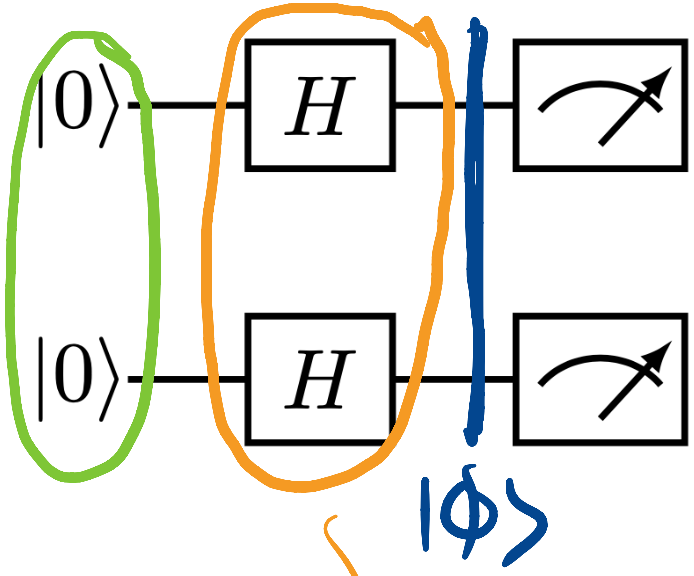
UNIFORM SUPERPOSITION?

Want to have equal likelihood of measuring every possible quantum bitstring.

ex) 2-qubit circuit : $\mathbb{P}(|00\rangle) = \mathbb{P}(|01\rangle) = \mathbb{P}(|10\rangle) = \mathbb{P}(|11\rangle)$



WORKED EXAMPLE



$$|0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\begin{pmatrix}1\\0\end{pmatrix}\\ 0\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$H \otimes H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} & 1\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\\ 1\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix} & -1\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\end{pmatrix}$$

$$|\phi\rangle = \frac{1}{2}\begin{pmatrix}1 & 1 & 1 & 1\\1 & -1 & 1 & -1\\1 & 1 & -1 & -1\\1 & -1 & -1 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}$$

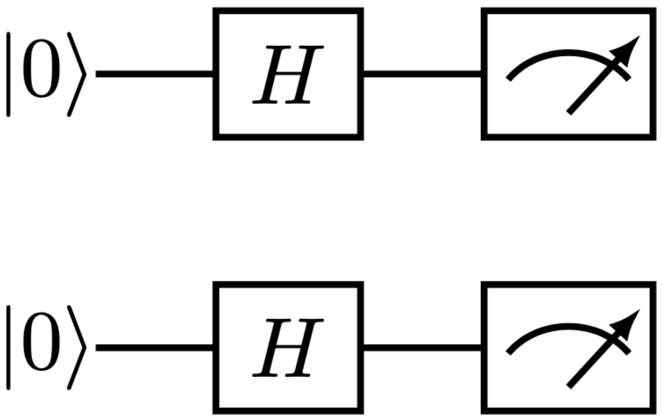
$P(|00\rangle) = |1/2|^2 = 1/4$ 25%

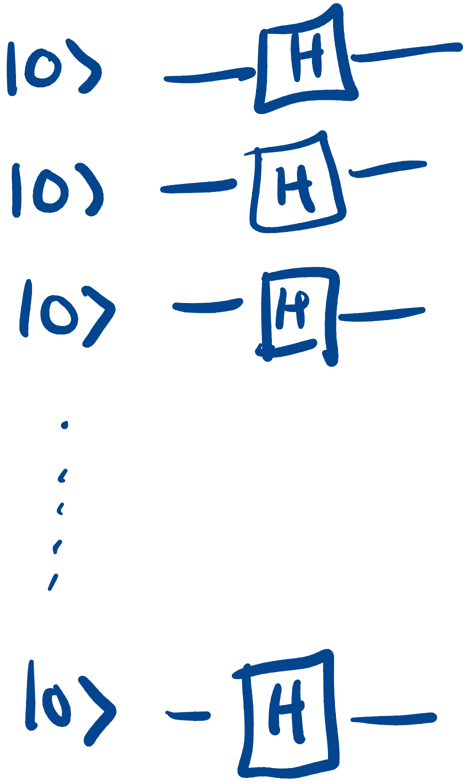
$P(|01\rangle) = |1/2|^2 = 1/4$ 25%

$P(|10\rangle) = |1/2|^2 = 1/4$ 25%

$P(|11\rangle) = |1/2|^2 = 1/4$ 25%

WORKED EXAMPLE



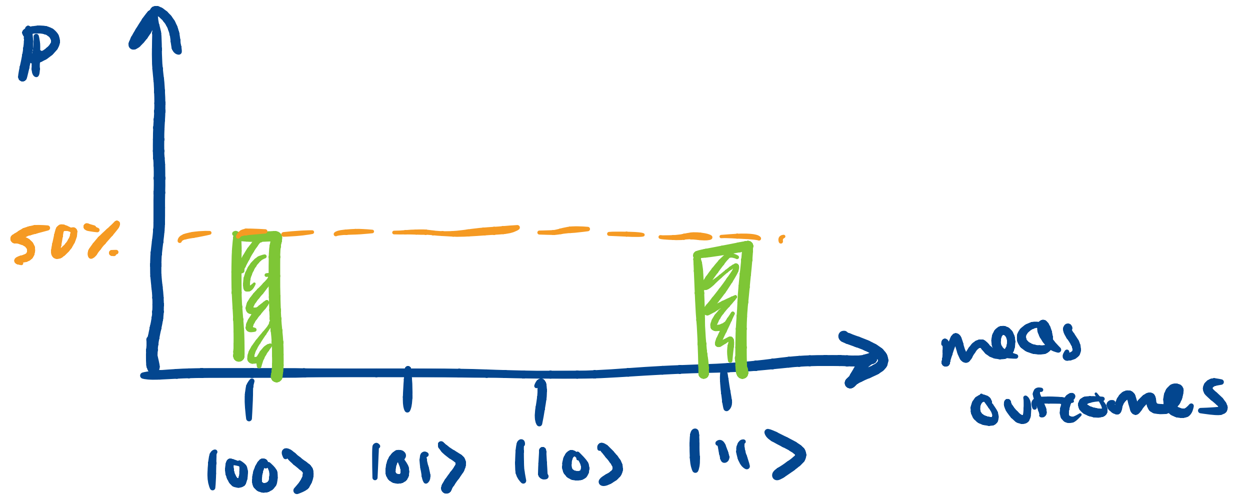


ENTANGLED STATES?

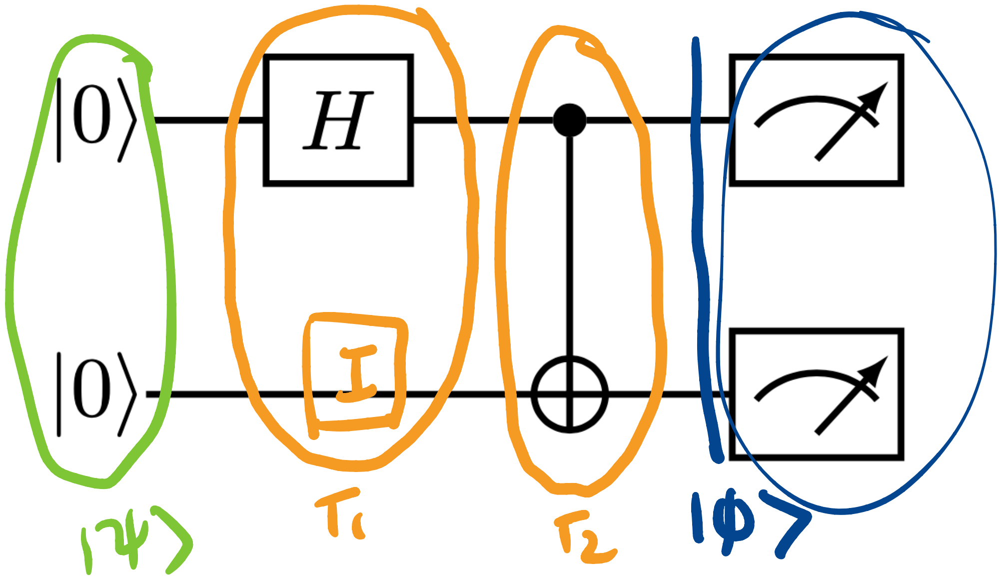
Bell's states (Bell's experiment)

↳ $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$, $\frac{|01\rangle + |10\rangle}{\sqrt{2}}$, $\frac{|01\rangle - |10\rangle}{\sqrt{2}}$, $\frac{|00\rangle - |11\rangle}{\sqrt{2}}$

ANTI



WORKED EXAMPLE



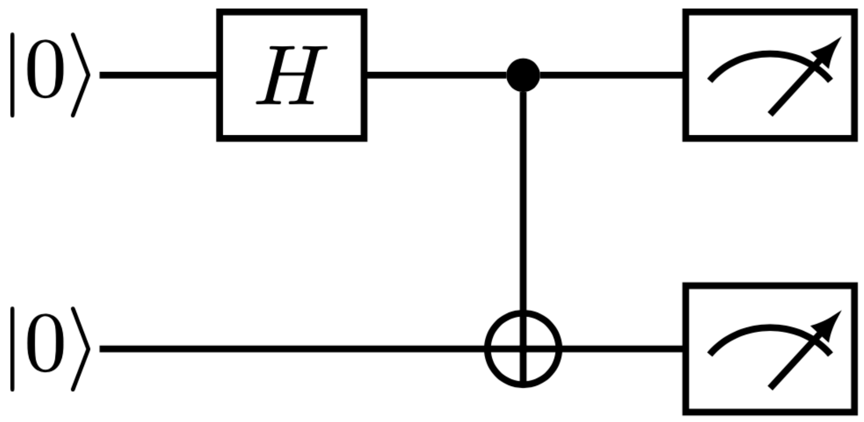
$$|0\rangle \otimes |0\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$T_1:\ H \otimes I = \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}$$

$$T_2:\ CNOT = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

$$|\phi\rangle = T_2\, T_1\, |\psi\rangle = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\underbrace{\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right)} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}$$

WORKED EXAMPLE

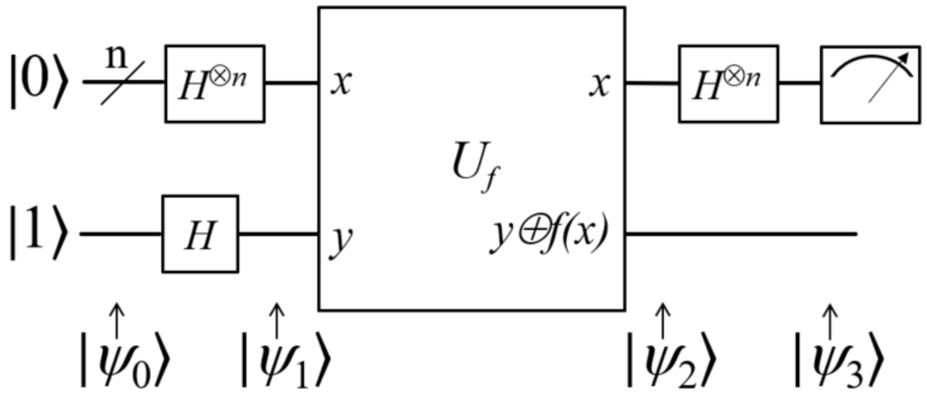


$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{array}{l} \rightarrow P(|00\rangle) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \quad 50\% \\ \rightarrow P(|01\rangle) = |0|^2 = 0 \\ \rightarrow P(|10\rangle) = |0|^2 = 0 \\ \rightarrow P(|11\rangle) = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \quad 50\% \end{array}$$

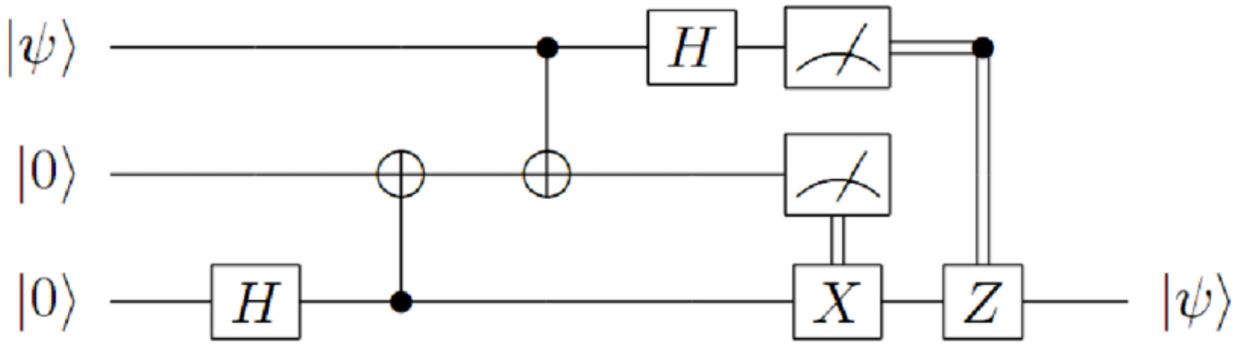
$|00\rangle$ $|11\rangle$

SOME MULTI-QUBIT CIRCUITS TO COME…

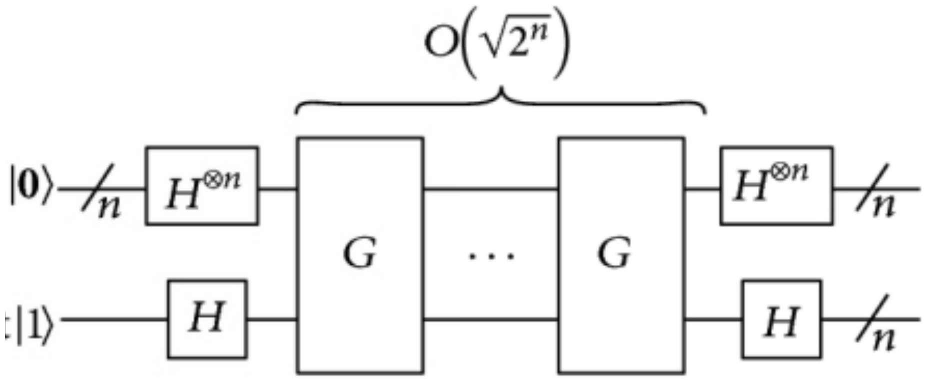
DEUTSCH-JOZSA



QUANTUM TELEPORTATION



GROVER'S



QAOA