Statistical Approach for QPE- notes — hybrid of $\boxed{\text{VQA \& IPEA}}$

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OBJECTIVE: Given a unitary matrix U, we want to find the eigenphase & eigenvector pairs for that matrix.

We are essentially doing SPECTRAL DECOMP" of the matrix U.

i.e. $U = \sum_\lambda \lambda |\lambda\rangle\langle\lambda|$

⊛ Why this approach?

In the vanilla QPE & the iterative phase est. algos, there is a need for the preparation of the EIGENVECTOR of U with high accuracy.

They majorly rely on the fact that the phase is kicked back from the target register containing $|\psi\rangle$. This approach does not need an eigenvector prepared before hand.

U $\cancel{|\psi\rangle}$

⊛ **How?** In this approach, a classical computer is used along with a QC to get closer & closer to the desired (state - phase) pair.

$|\psi\rangle, e^{2\pi i\theta}$

Then, once it is found, we may run this algorithm again to find the full spectral-decomposition of the unitary.

NOTE: It does NOT SEARCH for all (state - phase) pairs at once. This algorithm optimizes for a $\boxed{\text{SINGLE}}$ eigenstate rather than diagonalizing the matrix directly.

$\boxed{\text{SECTION 2}}$ ✓

TRADITIONAL QPE

⊛ In the traditional QPE, the control

control register

CLASS for SPEA

This class would specifically eigenphase pair of our ...

resolution for θ, error ...

ATTRIBUTES

1. unitary : the unitary ... pair need ...
2. resolution for θ : How ... search ...
3. error threshold : how ...
4. max iters : maximum ... is not ...

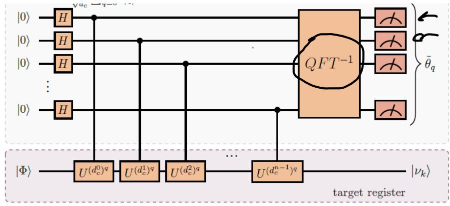
METHODS

1. get_basis_vectors() : ...
2. get_cost(angles, stat...
3. get_eigen_pair() : c...

register is transformed into the desired eigenstate after numerous phase kickbacks.

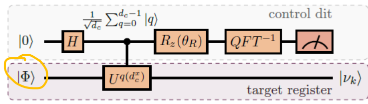
- For an arbitrary state in the target register i.e. $|\phi\rangle$, the prob of the circuit representing the eigenstate $|\psi_k\rangle$ is $|\langle\psi_k|\phi\rangle|^2$ (overlap of the 2 states squared)

- The eigenphase of the state is directly measured over the control qubits in the traditional approaches.



-> Circuit is run only once but no. of qubits required are of the O(precision).

IQPE



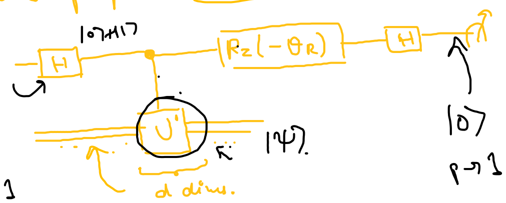
-> Circuit is run for n iters. & in each iter you find $\boxed{\text{A SINGLE BIT}}$ of your phase.

⇒ In both the cases, AN EIGENSTATE is reqd. in the target register for providing the phase kickback.

- Now, assume that we have done IPEA & have the eigenphase stored in the register as θ_R.

- If we have completed the IPEA for n steps & then go on to apply the unitary once again & measure. What happens?

① We would prepare the control as –



$|0\rangle+|1\rangle \rightarrow |0\rangle + e^{2\pi i\theta}|1\rangle$

$\downarrow$

$|0\rangle + e^{2\pi i(\theta-\theta_R)}|1\rangle$

$\frac{(1+e^{2\pi i(\theta-\theta_R)})}{2}|0\rangle \approx 1$

② the state before the second H gate would be –

$$P_0(\theta) \quad \left[\frac{|0\rangle + e^{2\pi i(\theta-\theta_R)}|1\rangle}{\sqrt{2}}\right]$$

Now, if we DID have the eigenvector in the target register & the θ_R was measured as θ only, the state would collapse to $|0\rangle$ with Prob 1, & if it weren't the current ... the prob would be < 1.

- What if the state of the target was not amongst one of the eigenvectors of U?

↳ We would then need to consider the prob. with which $|\phi\rangle$ would have been an eigenvector of U.

↳ If it were, then we can consider the prob. with which the control qubit would collapse to the $|0\rangle$ state

This how the cost f^n is 1 − Prob. of control collapsing to $|0\rangle$, is generated

↳ ① For all possible eigenvectors of U, ② if $|\phi\rangle$ was prepared as one of them, what is the ③ prob. of the state, the prob would ...

$P(|0\rangle) = P(-\theta_R) = \frac{1}{2}\left|1+e^{2\pi i(\theta-\theta_R)}\right|^2$

≈ 1. $\boxed{P(|0\rangle) = \cos^2(\theta-\theta_R)}$

... then, what ... control qubit collapsing to $|0\rangle$?

$\Rightarrow \sum_{i=1}^{d} |\langle v_i|\phi\rangle|^2 \times P(0_i)$ when we have ϕ considered as v_i

COST FUNCTION of our Algorithm

$$\boxed{C(|\phi\rangle, \theta_R) = \sum_{i=1}^{d} |\langle v_i|\phi\rangle|^2 \times \cos^2(\theta_i - \theta_R)}$$

this is just $P(0)$.

NOTE: $C(|\phi\rangle, \theta_R) = 1$ if and only if $|\phi\rangle$ got prepared as an eigenstate & θ_R its CORRESPONDING eigenvector.

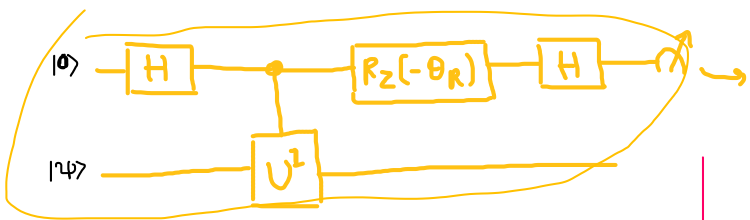
($\langle\ldots|\ldots\rangle$ evaluates to zero for other eigenvectors as they are orthogonal to $|\phi\rangle$, if it were an eigenstate).

SECTION 3 : What are we actually doing?

↳ We just need the LAST STAGE of our IQPE circuit i.e. single unitary & a single rotation gate.

→ Why though? ....

⊛ In the last stage, we have an estimate θ_R of our eigenphase. But if we once again apply a phase-kickback & an inv. rotation, we would surely be getting a 0 outcome.



$\rightarrow P(0) = \cos^2(\theta - \theta_R)$

NOTE: IPE fails when $|\psi\rangle$ is not an eigenvector of U but that is precisely the information that SPEA uses to tune the initial state $|\psi\rangle$ & the rotation angle θ_R.

Also, many iterations are needed but only ONE EXTRA QUBIT is needed for the algorithm.

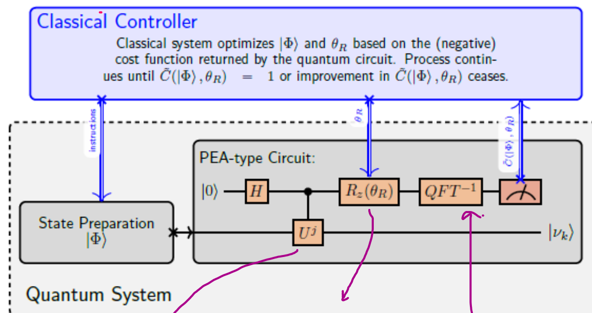
STOPPING CRITERIA: when $\tilde{C}(|\phi\rangle, \theta_R) \approx 1$ or the max computing time has been reached, the algorithm stops. If $\tilde{C}(|\phi\rangle, \theta_R) \approx 1$,

$\Rightarrow |\phi\rangle$ & θ_R found are an eigenstate – eigenphase pair.

HARDWARE USED

The circuit that SPEA uses resembles that of the last iteration of the IPEA algorithm.

Besides that, a classical controller is needed to tune the state $|\psi\rangle$ after each of the iterations & refine the angle θ_R.



Unitary

inverse rotation gate based on the partial θ_R estimate

This is an $\boxed{H}$ gate as QFT^{-1} for single qubit is $\boxed{H}$

P(0) → given that ϕ is an superposition of eigenvectors.

↳ Find post-measurement state of $|\phi\rangle$.

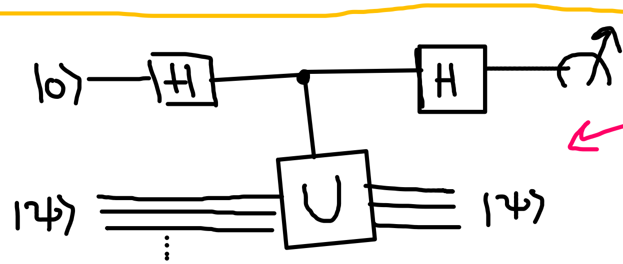
find θ, $\theta_R = 0$,

↓

converges.

ALTERNATE APPROACH for finding C^* & best θ i.e. θ^*.

Circuit



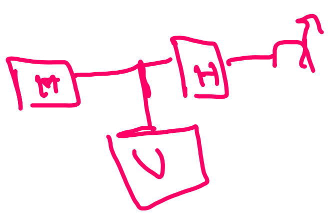
⌗ If $|\psi\rangle$ were an eigenstate of U, then at the end of our circuit, $P(0)$ & $P(1)$ for the control qubit are $P(0) = \cos^2(\pi\theta)$ & $P(1) = \sin^2(\pi\theta)$

assuming $\theta \in [0,1]$

⌗ Now, how to get θ? The estimate that was mentioned in the paper?

Note, θ is the angle which minimizes S. 10

where $S = \sum_{i=0}^{n-1} \left(P_i - C(i,\theta)\right)^2$ theoretical prob.



where $\boxed{P_i}$ is the probability of measuring i as the output on the control qubit

& $\boxed{C(i,\theta)}$ is the theoretical

⌗ How does this change our

algorithm ?

↳ Well, now to determine the Best phi, we do not have to generate new QCs.

↳ iterative procedure is executed classically.

↳ Only 2 quantum circuits are made in any iteration —

1. To determine P_i s
2. To run the circuit again with θ, to generate $\boxed{C^*(|\psi\rangle, \theta)}$.

probability of obtaining i, given θ was kicked back to the control.

Now, the get-circuit f^n is nullified as the circuit is only executed once to determine P_i s.

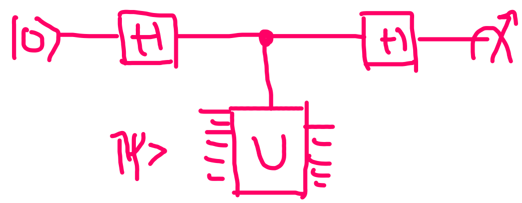
get-cost f^n changes only in the sense that it does not send many circuits to the backend not only 2 circuits.

After circuit 1, gets the results and calculates θ.

Then uses that θ to get C^*. ← circuit 2.

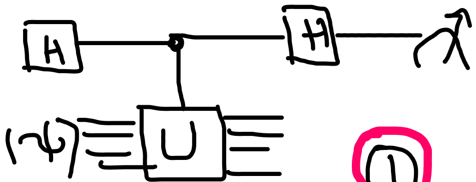
MATHEMATICS for determining P_i when $|\psi\rangle$ is not an eigenvector!

CIRCUIT



Let $|\psi\rangle$ be a random quantum state, so since eigenvectors of U form an orthonormal basis, $\Rightarrow$

Now, we have $\boxed{|\psi\rangle = \sum_i \alpha_i |v_i\rangle}$ where $|v_i\rangle$ are eigenvectors of U & α_i are prob. amplitudes.



(1) $|0\rangle \otimes |\psi\rangle$

(2) $\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |\psi\rangle \xrightarrow{C-U}$ (3) $\frac{|0\rangle|\psi\rangle + |1\rangle U|\psi\rangle}{\sqrt{2}} \rightarrow$ (4) $\frac{|0\rangle \sum_k \alpha_k |v_k\rangle + |1\rangle \sum_k \alpha_k e^{2\pi i \phi_k} |v_k\rangle}{\sqrt{2}}$

H ⊗ I

$$\sum_k \frac{(1+e^{2\pi i \phi_k})\alpha_k |0\rangle|v_k\rangle}{2} + \sum_k \frac{(1-e^{2\pi i \phi_k})\alpha_k |1\rangle|v_k\rangle}{2} \xleftarrow{\text{simplifies to}} \frac{(|0\rangle+|1\rangle)\sum_k \alpha_k |v_k\rangle + (|0\rangle - |1\rangle)\sum_k \alpha_k e^{2\pi i \phi_k}|v_k\rangle}{2}.$$

which implies

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$$P(0) = \sum_k |\alpha_k|^2 \left|\frac{1+e^{2\pi i\phi_k}}{2}\right|^2 = \sum_k |\alpha_k|^2 \cos^2(\pi\phi_k)$$

$$P(1) = \sum_k |\alpha_k|^2 \left|\frac{1-e^{2\pi i\phi_k}}{2}\right|^2 = \sum_k |\alpha_k|^2 \sin^2(\pi\phi_k).$$

The doubt is that -

If $S = (P_0 - C(0,\theta))^2 + (P_1 - C(1,\theta))^2$ is a metric, isn't the metric only valid when $|\psi\rangle$ is an eigenvector?

how to ensure that even when $|\psi\rangle$ is not an eigenvector, S is a good metric?