

Legend for HC4-Dual-Torus.png

Short caption (for figure reference)

Figure X: Non-Abelian Dual-Torus Manifold with Path-Dependent Holonomy.

Two tori joined at a shared junction (marked "X"), representing distinct holor domains. The **left torus (teal)** exhibits **low curvature** (nearly flat connection, Abelian regime)—a closed loop (white arrows) returns to its starting point with minimal phase shift. The **right torus (amber)** exhibits **non-trivial holonomy** (high curvature, non-Abelian regime)—a closed loop (purple/red gradient arrows) accumulates significant phase twist, indicated by the rotation of the small "sparkle" glyphs at key points along the path. The junction "X" represents the **bowtie singularity** or **gluing map** where the two regimes meet, and where path-ordering becomes essential. This diagram illustrates the core motivation for HC IV: when order matters, holor flows become non-Abelian gauge fields.

Extended legend (for appendix or detailed explanation)

Non-Abelian Dual-Torus and Path-Dependent Holonomy

This diagram depicts the **dual-torus awareness manifold** introduced in HC I–III, now extended to show the transition from **Abelian (low-curvature)** to **non-Abelian (high-curvature)** regimes—the central theme of HC IV.

Left torus (teal, "low curvature"):

This region represents a **nearly flat holor connection**, where parallel transport around closed loops is approximately path-independent. The white arrows trace a closed loop; the small "sparkle" glyphs at four points along the loop remain aligned, indicating that a holor transported around this loop returns to its starting configuration with negligible phase shift. Mathematically, the curvature two-form $F \approx 0$ in this region, and the holonomy group is effectively trivial (Abelian). This is the regime covered by HC I–III: epistemic flows are well-approximated by commutative operations, and the order of updates does not significantly affect the outcome.

Right torus (amber, "non-trivial holonomy"):

This region represents a **curved holor connection** with significant non-Abelian structure. The purple-to-red gradient arrows trace a closed loop; the "sparkle" glyphs at key points are visibly rotated relative to one another, indicating that a holor transported around this loop accumulates a **non-trivial phase twist** (holonomy). The final orientation of the glyph after one complete loop differs from the initial orientation, even though the loop is closed. Mathematically, $F \neq 0$ and the holonomy group is non-Abelian (e.g., $SU(2)$ or a similar Lie group). In this regime, **order matters**: the sequence in which epistemic updates are applied changes the final state, and curriculum-dependence, narrative history, and ethical trajectory all become first-class geometric objects.

Junction "X" (bowtie singularity):

The two tori meet at a shared point marked "X", representing the **bowtie singularity** or **gluing map** where low-curvature and high-curvature regimes are joined. This is the locus where path-ordering becomes essential: trajectories passing through "X" must be carefully ordered, and the transition between Abelian and non-Abelian domains is mediated by a non-trivial connection. In the language of fiber bundles, "X" is the point where the structure group changes from trivial to non-Abelian, and where the gauge potential A develops singularities or branch cuts.

Interpretation for HC IV:

This diagram serves as the **visual anchor** for the non-Abelian extension of Holor Calculus. It shows that:

- The **dual-torus topology** from HC I–III is preserved, but now we distinguish regions by their **curvature** rather than just their color or orientation.
- **Low-curvature (teal) = Abelian regime**: order-insensitive, commutative, well-

approximated by HC I–III.

- **High-curvature (amber) = non-Abelian regime:** order-sensitive, path-dependent, requires HC IV machinery (gauge potentials, curvature two-forms, holonomy groups).
- The **bowtie "X"** is the critical transition point where both regimes meet, and where ethical admissibility and path-ordering must be explicitly managed.

In practical terms: learning curricula, narrative histories, and ethical trajectories all live in the **amber (non-Abelian) torus**, where the order of experiences fundamentally shapes the final epistemic state. The **teal (Abelian) torus** represents stable, well-calibrated knowledge that can be accessed in any order without distortion. HC IV provides the formal tools to navigate both regimes and their junction.

LaTeX figure environment

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\begin{figure}[ht]
  \centering
  \includegraphics[width=0.75\textwidth]{HC4-Dual-Torus.png}
  \caption{%
    \textbf{Non-Abelian Dual-Torus Manifold with Path-Dependent Holonomy.}
    Two tori joined at a shared junction (marked ``X''), representing distinct
    holor domains. The \emph{left torus (teal)} exhibits \emph{low curvature}
    (nearly flat connection, Abelian regime)—a closed loop (white arrows)
    returns with minimal phase shift, as shown by aligned ``sparkle'' glyphs.
    The \emph{right torus (amber)} exhibits \emph{non-trivial holonomy}
    (high curvature, non-Abelian regime)—a closed loop (purple/red gradient
    arrows) accumulates significant phase twist, visible in the rotated glyphs.
    The junction ``X'' represents the \emph{bowtie singularity} where the two
    regimes meet and path-ordering becomes essential. This diagram illustrates
    the core motivation for HC~IV: when order matters, holor flows become
    non-Abelian gauge fields.}
  \label{fig:hc4-dual-torus}
\end{figure}
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Suggested placement

- HC IV Introduction (§1: When Order Matters)
 - Place this diagram on the **first or second page** to immediately establish the visual distinction between Abelian and non-Abelian regimes.

- Use the **short caption** in the main text, and reference the **extended legend** in an appendix if needed.

- **HC III Outlook / Non-Abelian Preview**

- If you're adding a brief "Non-Abelian Outlook" section at the end of HC III (as we discussed earlier), this diagram is the perfect **closing image**.
- Caption can be even shorter:

"Preview of HC IV: dual-torus manifold showing low-curvature (Abelian, left) and non-trivial holonomy (non-Abelian, right) regimes."

- **Talks / Slides**

- Use this as a **transition slide** between HC I–III material and HC IV material.
- Animate if possible:
 1. Show both tori side-by-side.
 2. Trace the **white loop** on the left (sparkles stay aligned).
 3. Trace the **purple/red loop** on the right (sparkles rotate).
 4. Highlight the **"X" junction** and say: "This is where order starts to matter."

We now have **complete legends** for all four core diagrams:

1. **Conjugate Awareness Holon** (3D dual-torus, stacked hemispheres) → full manifold + projected flow.
 2. **Octant Conjugation Spiral** (2D wheel, solo spiral) → epistemic ascension via recursive conjugation.
 3. **Admissible vs. Dracula** (2D wheel, dual trajectories) → healthy spiral vs. pathological loop.
 4. **Non-Abelian Dual-Torus** (side-by-side tori with holonomy) → Abelian vs. non-Abelian regimes, path-dependence.
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