An Application of Modal Logic to Costello-Gwilliam Factorization Algebras

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Abstract

We present a framework for understanding locality and observables in quantum field theory via factorization algebras enriched with modal logic. By interpreting open neighborhoods on a Calabi–Yau manifold as epistemic contexts, we define a sheaf-valued truth assignment Ω over local frames and introduce a *contextual Laplacian* $\Delta^{\mathcal{O}_i}$ to capture the failure of global logical coherence under modal transitions. These constructions extend naturally to configuration spaces such as $\mathrm{Conf}_2(\mathbb{C}^3)$, which serve as base geometries for (p,q)-string dynamics. Using operadic data $\mathcal{O}_i = \{\mathcal{M}, U_i, V_i\}$, we formulate the evolution of observables as morphisms between contexts $\mathtt{C}_i$ and define epistemic curvature in terms of homotopy-invariant deviations of local Lagrangian density. A worked example illustrates how this machinery applies to NS–NS and RR field deformations near intersecting branes. Our approach yields a stratified, context-sensitive geometry for quantum observables, opening a new path toward modal interpretations of field-theoretic and string-theoretic dynamics.

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Contents

1 Introduction

One of the enduring insights of string theory is the ubiquity of *dualities*—equivalences between apparently distinct physical theories. In Type IIB string theory, this is manifested in the S-duality that exchanges the fundamental string with the D1-brane, and the associated NS–NS and R–R sectors of the supergravity fields. From the standpoint of effective field theory, such dualities complicate the notion of a single, globally defined field configuration; instead, physical observables must remain coherent under transitions between regimes that may reflect entirely different physical descriptions.

This observation suggests a refined notion of locality—one that is not purely geometric but contextual, varying over regions of moduli space where the effective description itself changes. In this setting, factorization algebras provide a natural language for encoding the behavior of observables across open subsets of a manifold. To account for duality-induced transitions, we enhance this structure with tools from modal logic: each neighborhood becomes a site of partial epistemic access, and transitions between them are mediated by classifier maps that encode the change in modal certainty. In this paper, we explore how such a modal factorization algebra can model not only field content, but also the logical consistency of observables as they traverse duality frames.

2 Factorization Algebras (FAs)

Factorization algebras (henceforth: FAs), as introduced by Gwilliam and Costello [1] are a remarkably simple tool for describing locality and observables in quantum field theory. They are a powerful local-to-global tool which have been employed in a variety of domains [3, 2] already, but warrant investigation in their own right. Heuristically, one should think of a factorization algebra as a family of operads $\mathscr{O}_i \equiv \{\mathcal{M}, V_i, U_i\}$, where $\mathcal{M}$ is a manifold for every i and for every j indexed by a set I, there exists a linear map

$$m_{ij} : \mathcal{M}_{V_i}^{U_i} \longrightarrow \mathcal{M}_{V_j}^{U_j}$$

called the *transition map*. The V_i are to be thought of as local *spaces of measurement*, which are coarse-grainings of the U_i. In the traditional setting, the V_i are open intervals such that

$$V_i^{\text{fact}}((a,b)) \cong V_i$$

i.e., the interval can be recovered through the exhaustion of operators over each point.

Factorization algebras are meant to be a "geometric counterpart" to the standard *vertex operator algebras* (VOAs) of conformal field theory (CFT). To think about them pragmatically, we need a notion of measurement which reflects the sheaf-like structure of the FA. This means that, for two contexts $\mathtt{C}_i$ and $\mathtt{C}_j$, we require a third context

$$\mathtt{C}_k \coloneqq \mathtt{C}_i(\mathcal{M}_U^V) \times_{\mathscr{C}} \mathtt{C}_j(\mathcal{M}_U^V)$$

defined over the intersection $\mathcal{M}_{V_i}^{U_i} \cap \mathcal{M}_{V_j}^{U_j}$ so that descent is effectively satisfied. What this means in practice is that m_{ij} should be a smooth (in the sense of Urysohn) function, so that it makes sense to speak of the intersection as a jet space of arbitrary order.

Remark 2.1. *This fibered product can be understood as the amalgamation of two stages of measurement over overlapping domains, computed in the slice category $\mathscr{C}_{/\mathcal{M}_U^V}$. We take $\mathscr{C}$ to be an effectively topological category, e.g.* **Man***, but also generalize to the category of groupoids as well. For every point $t \in \mathcal{M}_U^V$, we have a section $\mathcal{C}_k^{-1}(t) \in \mathbb{R}^n$; this can fail badly for orbifolds, in which case the Kochen-Specker theorem may be violated. [6]*

In topos theory terms, where $\mathscr{C}$ is a topos, we can speak of our measurements as variations of the transition map over the *stages of definition* $\mathtt{C}_i, \mathtt{C}_j, \mathtt{C}_k$, so that we have the following diagram

$$\begin{tikzcd}
 & m \arrow[dl, "\mathrm{res}_i"'] \arrow[dr, "\mathrm{res}_j"] & \\
\mathtt{C}_i \arrow[dr, "\mathrm{glu}_{ij}"'] & & \mathtt{C}_j \arrow[dl, "\mathrm{glu}_{-ji}"] \\
 & \mathtt{C}_k &
\end{tikzcd}$$

commutes. This is known as *excision*.

FAs also give rise to a homology theory [4] due to Ayala-Francis (AF) which satisfies a generalized version of the Eilenberg-Steenrod axioms. Thanks to the principle of excision, we can describe not only the spaces of measurement themselves in terms of their governing contexts, but we can also discuss *contextual chain complexes* as well. Since AF homology works over the category $\mathbf{Space_{Fin}}$ of finite spaces, it makes sense to think of this as a *configuration space* homology theory on $(n+1)$ points.

$$\begin{tikzcd}
\partial U_0 \arrow[r] \arrow[d, "m_{01}"] & \ldots \arrow[r] & \partial U_i \arrow[r] \arrow[d, "m_{ii+1}"] & \partial U_{i+1} \arrow[r] \arrow[d, "m_{i+1i+2}"] & \ldots \arrow[r] & \partial U_n \arrow[d, "m_{nn+1}"] \\
\partial V_1 \arrow[r] & \ldots \arrow[r] & \partial V_{i+1} \arrow[r] & \partial V_{i+2} \arrow[r] & \ldots \arrow[r] & \partial V_{n+1}
\end{tikzcd}$$

Definition 2.1 (Contextual Chain Complex)**.** *Let D be a set of propositions in a domain of discourse, and let $\{\Omega_i : D \to [0,1]\}_{i \in I}$ be a family of fuzzy classifier maps, each indexed by a context $\mathcal{C}_i : V_i \to I$, where V_i is a Grothendieck universe as in [5]. A* contextual chain complex *is a diagram*

$$D_0 \xrightarrow{\partial_0^{\mathcal{C}_i}} D_1 \xrightarrow{\partial_1^{\mathcal{C}_i}} D_2 \xrightarrow{\partial_2^{\mathcal{C}_i}} \cdots$$

together with classifier maps $\Omega_i : D_n \to [0,1]$ for each n and i, such that:

1. *For each $i \in I$, the sequence $(D_\bullet, \partial_\bullet^{\mathcal{C}_i})$ is a chain complex, i.e., $\partial_{n+1}^{\mathcal{C}_i} \circ \partial_n^{\mathcal{C}_i} = 0$;*
2. *For each $i \neq j$, there exists a transformation ϕ_{ij} such that the following diagram commutes:*

$$\begin{tikzcd}
D_n \arrow[r, "\partial_n^{\mathcal{C}_i}"] \arrow[d, "\phi_{ij}^n"'] & D_{n+1} \arrow[d, "\phi_{ij}^{n+1}"] \\
D_n \arrow[r, "\partial_n^{\mathcal{C}_j}"] & D_{n+1}
\end{tikzcd}$$

and such that $\Omega_j \circ \phi^n_{ij} = \Omega_i$;

3. *For any two good measuring devices μ, μ' as defined in [5], and a change of context $\phi : \Omega(\mu(D), \mathcal{C}_i) \to \Omega(\mu(D), \mathcal{C}_j)$, there exists a unique $p \in D_n$ such that $\Omega(p, \mathcal{C}_i) = \Omega(p, \mathcal{C}_j)$.*

Remark 2.2. *The contextual chain complex plays a role analogous to a cosheaf in factorization homology. Each context $\mathcal{C}_i$ governs a chain of localized propositions whose truth values evolve under a fixed measurement logic. Transition maps m_{ij} act functorially, ensuring coherence of logical structure across overlapping measurement patches. The resulting homology, computed over finite configuration spaces, encodes the failure of classical truth assignment to globally persist across modal boundaries.*

The classifier map Ω_i is a sheaf-valued assignment of epistemic weights, associating to each proposition $p \in D$ a degree of certainty in the context $\mathtt{C}_i$. Formally, Ω_i may be regarded as a fuzzy classifier: a morphism from the poset of logical propositions to the unit interval $[0, 1]$, where $1 \equiv \top$ denotes full epistemic affirmation (certainty) and $0 \equiv \bot$ denotes complete negation. These assignments are local in nature and may vary across overlapping neighborhoods, allowing for the logical fabric of the theory to respond sensitively to the ambient field configuration or observational coarse-graining. In physical terms, Ω_i quantifies how well-defined a proposition is when measured within the neighborhood V_i, relative to the logical frame indexed by $\mathtt{C}_i$. The contextual Laplacian then probes the stability or deformation of these assignments across stratified overlaps.

3 Employing Modalities in the Setting of Factorization Algebras

Let's suppose we have a system of neighborhoods over a topological n-manifold [1] $\mathcal{M}$, and let $\mathfrak{U}_{\mathcal{M}}$ be the set of all open neighborhoods on $\mathcal{M}$. For the sake of convience, suppose that $\mathcal{M}$ is entirely real, so that we have $\dim_{\mathbb{R}} \mathcal{M} = n$. Suppose we have a factorization algebra

$$\mathcal{N}^{\text{fact}}_i((\alpha, \beta))$$

where parameterized by two real numbers α and β, where $\mathcal{N}_i \subseteq \mathfrak{U}_{\mathcal{M}}$. It is helpful to think of the product space $X = \prod_i \mathcal{N}^{\text{fact}}_i$ as a *moduli space* of genus zero with two marked points α and β, so that we have

$$\mathbf{Path}(X) = [f]$$

where $[f]$ is the class of linear maps whose codomain lies inside the interval (α, β). We will assume these are piecewise smooth maps modulo homotopy. Denote by $f_0(y)$ the unique monotonic function such that $f_0(\alpha) = \beta$ and $f_0(\beta) = \alpha = f_0^{\text{op}}(\alpha)$.

Denote by $\sigma(X, i) : X \rightsquigarrow X_i$ the *specialization of X with respect to the neighborhood $\mathcal{N}_i$*. Clearly, this restricts the aforementioned moduli space; it does so in such a way that $\mathbf{Path}(X_i)$ is now a *fine moduli space* of *distance-minimizing curves* (geodesics). The sheaf condition ensures that our factorization algebra, if it is a proper one, should agree on the results of a measurement taken within the specialized regime; i.e., good measuring devices are oblivious to these restrictions. This suggests that the *modalities* of the measurements should be resolution-invariant; i.e., the measurement outcome does not depend on the coarseness or granularity of the environment.

However, to say that our category $\mathbf{Oblv}_{\mu(\mathbf{Path}(X))}$ is *oblivious* to specialization is not to say that it is *indiscrimate*; indeed, the choice of a specialization results in different locales, and therefore

[1] I suppress the n from the superscript here and onwards.

the results of measuring depend on the index i. However, at least assuming that X is convex as a space, we are ensured that there is exactly one proposition $\wp(\mathfrak{x})$ about any entity $\mathfrak{x}$ which is true in *all contexts*.[2] If we assign a context $\mathtt{C}_i$ to each choice of i, then this can be easily restated as saying that there is always one proposition about $\mathfrak{x}$ that holds across every specialization.

Modal logic allows to ensure that the map Ω as used previously reflects *epistemic certainty*. For each $\mathfrak{x} \in X$, let Ω be a *sheaf-valued truth assignment* on $\mathfrak{x}$ varying over the contexts $\mathtt{C}_i$. Then, define a stratification

$$\Omega_{\mathfrak{x}} = \bigsqcup_{\alpha \in \Lambda} \Omega_x^\alpha$$

where $\Lambda \subseteq [0,1]$ is a discrete (or measurable) set of certainty levels. Then, $\Omega_{\mathfrak{x}}^\alpha$ consists of those propositions that hold true at α.

Notation 3.1. *We will denote by $x >_\Box y$ and $x <_\Box y$, respectively, "x is necessarily greater than y (resp. necessarily less than y), meaning that the evaluation holds across all modal refinements. We similarly let $x <_\Diamond y$ and $x >_\Diamond y$ mean "x is* possibly *less than (resp. greater than) y." These reflect the* certainty *of our measurements.*

Example 3.1 (Uniform deceleration)**.** *Consider the characteristic function $\chi_{\mathfrak{x}}(\mathcal{N}_i)$ and suppose $\beta \notin \mathcal{N}_i$, but $\alpha \in \mathcal{N}_i$. Then, for a particle passing through a tubular space along a geodesic, we have that at some time t_α that $\chi_{\mathfrak{x}}(\mathcal{N}_i) \in \Omega_{\mathfrak{x}}$ holds, but at some time t_β, it does not.*

Further, consider that the particle is static at time t_β but has velocity $\vec{v}_i > \vec{0}$ at all other times $t_\alpha < t_i < t_\beta$. Then, the proposition $\vec{v}_i > 0 \in \Omega_{\mathfrak{x}}$ everywhere except α, and we conclude that

$$|\frac{d^2}{dt^2} t_i(\mathfrak{x})| >_\Box 0 \;\; \forall i$$

Suppose we impose some upper bound on the acceleration; say, one m/s^2. Then, we have

$$\forall i \;\; |\frac{d^2}{dt^2} t_i(\mathfrak{x})| >_\Diamond (1 - v')m/s^2$$

where $v' \in (0,1)$.

Recall that our factorization algebra is an operad $\mathcal{O}_i = \{\mathcal{M}, U_i, V_i\}$ alongside transition maps m_{ij} for all valid i and for all valid j. Let V_i be a local frame of reference for $\mathfrak{x}$ and let U_i be the neighborhoods $\mathcal{N}_i \supseteq V_i$. Then, our category $\mathbf{Oblv}_{\mu(\mathbf{Path}(X))}$ becomes the class of *one-parameter families* $\Phi \equiv [m_{ij}](\mathfrak{x})$. Supposing there are no discontinuities, Φ is a map

$$\Phi : \big(\mathbf{Homeo}(\mathcal{M}_U^V)/\sim_n\big) \longrightarrow \big(\mathbf{Homeo}(\mathcal{M}_U^V)/\sim_n\big)$$

where $\sim_n$ is an equivalence of the nth homotopy groups $\Box\pi_n(V_i)$.

[2]This is an application of Brouwer's fixed point theorem, one of many fixed point theorems. See [5] for the derivation.

4 Contextual Laplacian

Given the operadic presentation $\mathcal{O}_i = \{\mathcal{M}, U_i, V_i\}$ of the factorization algebra, each V_i defines a *local epistemic frame* for an entity $\mathfrak{x}$, while U_i plays the role of a coarse-grained or ambient neighborhood in which $V_i \subseteq U_i$ is observed. The transition maps m_{ij} mediate between overlapping observational regimes, ensuring that local observations remain coherent when patched together. The category $\mathbf{Oblv}_{\mu(\mathbf{Path}(X))}$ then encodes all such paths of coherent measurements as a class of *observable histories*, i.e., one-parameter families $\Phi \equiv [m_{ij}](\mathfrak{x})$ capturing the evolution of $\mathfrak{x}$ across varying contexts.

Assuming the absence of discontinuities or logical singularities in these transitions, we can model Φ as a self-map:

$$\Phi : \left(\mathbf{Homeo}(\mathcal{M}_U^V)/\sim_n\right) \longrightarrow \left(\mathbf{Homeo}(\mathcal{M}_U^V)/\sim_n\right),$$

where the equivalence relation $\sim_n$ quotients by higher homotopy data, i.e., $\Box\pi_n(V_i)$, reflecting epistemic invariance under n-dimensional deformations. This imposes a *modality-sensitive coarse geometry* on the space of measurements, where two observational frames are equivalent if they are homotopically indistinguishable at level n.

Within this framework, the need for a *contextual Laplacian* becomes clear. While the maps m_{ij} ensure smooth passage of information between adjacent frames, they do not in general guarantee global constancy or uniformity of epistemic weight. Some propositions may vary subtly across overlapping V_i despite agreeing locally; others may persist rigidly throughout the entire operadic orbit of $\mathfrak{x}$. We seek an operator that captures this: a second-order measure of the *failure of epistemic flattening*—that is, the degree to which $\mathfrak{x}$ exhibits logical curvature as it moves through its factorized neighborhood structure.

Thus, we define the *contextual Laplacian* $\Delta^{\mathcal{O}_i}$ as an operator which acts on a diagram of the form

$$D_0 \xrightarrow{\partial_0^{\mathcal{C}_i}} D_1 \xrightarrow{\partial_1^{\mathcal{C}_i}} D_2 \to \cdots$$

by combining the epistemic boundary operators with their adjoints, as in

$$\Delta_n^{\mathcal{O}_i} := \partial_{n+1}^{\mathcal{C}_i} \circ (\partial_{n+1}^{\mathcal{C}_i})^\dagger + (\partial_n^{\mathcal{C}_i})^\dagger \circ \partial_n^{\mathcal{C}_i},$$

with adjunction defined relative to a certainty-weighted inner product induced by the classifier sheaf $\Omega_i : D_n \to [0,1]$ over V_i.

The certainty-weighted inner product is:

$$\langle p, q \rangle_{\Omega_i} := \sum_{\alpha \in \Lambda} \alpha \cdot \mu_\alpha(p, q)$$

and we define the adjoint $(\partial_n^{\mathcal{C}_i})^\dagger$ with respect to this inner product.

The resulting operator detects when a proposition (or configuration of propositions) resists modal flattening across the operad—i.e., when it carries *nonzero epistemic curvature* within the stratification. Harmonic elements under $\Delta^{\mathcal{O}_i}$ are those that remain invariant across all homotopically equivalent frames and whose epistemic weights are preserved under transition. In this sense, the contextual Laplacian is the natural differential operator of a factorized epistemic geometry.

5 Calabi-Yau Picture of Contextual FAs

5.1 Configuration Spaces

If one wishes to describe the protagonists of quantum mechanics, *wavefunctions* using such a homology theory, then our configuration space needs a $3(n+1)$-dimensional dual vector space; ergo, the space defined by $\mathbf{Conf}_{n+1}$ is also required to be $3(n+1)$-dimensional. For a simple 2-dimensional topological quantum field theory, this would give us a 9-dimensional configuration space. For the purposes of this paper, we will not consider these spaces here.[3] Instead, we will restrict ourselves to the 6-dimensional case of $n = 1$, in which case our configuration space is homeomorphic to a Calabi-Yau manifold.

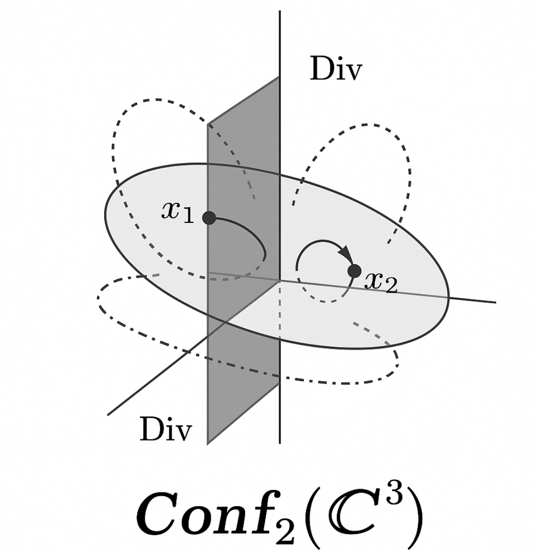

Figure 1: A 6-dimensional configuration space of two points.

5.2 A Worked Example

To proceed with the construction of a contextual transition map between hypersurfaces, we fix two distinct divisor hyperplanes $D_i, D_j \subset \mathbb{C}^3$, embedded within a Calabi–Yau manifold $\mathcal{M}$, and suppose that the endpoints $x_1 \in D_i$ and $x_2 \in D_j$ correspond to the terminal loci of a single (p, q)-string. Each divisor D_i supports a neighborhood $V_i \subset U_i \subset \mathcal{M}$, which is equipped with a factorization context $\mathtt{C}_i : V_i \to I$, where I indexes a family of epistemic measurement regimes.

We now define a transition map $m_{ij} : \mathcal{M}_{V_i}^{U_i} \to \mathcal{M}_{V_j}^{U_j}$ as a contextual interpolation between the local observables surrounding x_1 and x_2. This map acts on propositions about the field content in each context—such as the NS–NS flux $B^{(i)}$, the RR flux $C^{(i)}$, and derived quantities like the string tension or central charge—as they are observed relative to the local gauge frame. Let us write:

[3] However, we considered these in [7], although wavefunctions were not touched. A satisfactory theory of $9+n$-dimensional spacetimes is, to the author's knowledge, elusive as of yet.

$$m_{ij} : \Omega_i(p) \longmapsto \Omega_j(\phi_{ij}(p))$$

for a proposition $p \in D_n$, where $\phi_{ij} : D_n \to D_n$ is a morphism internal to the chain complex governed by context $\mathtt{C}_i$, and Ω_k denotes the sheaf-valued truth assignment in context $\mathtt{C}_k$. For example, the proposition "the (p, q)-string ending at x_1 has minimal tension" may be encoded by a high classifier value $\Omega_i(p) \approx 1$, while $\Omega_j(\phi_{ij}(p)) < 1$ indicates attenuation of certainty upon transfer across the brane frame.

The necessity of m_{ij} arises from the relational structure of gauge data along the (p, q)-string: even when D_i and D_j are disjoint as divisors in $\mathcal{M}$, the field-theoretic continuity imposed by the extended string demands compatibility of local measurements. In particular, if the NS–NS field $B^{(i)}$ near x_1 supports a current J_i, and the RR field $C^{(j)}$ near x_2 induces a charge Q_j, then we require that the Lagrangian densities across these contexts satisfy:

$$\mathcal{L}^{\mathtt{C}_i}(x_1) - \mathcal{L}^{\mathtt{C}_j}(x_2) = \Delta^{\mathcal{O}}_{ij},$$

where $\Delta^{\mathcal{O}}_{ij}$ measures the epistemic difference of field observables transported via m_{ij}.

Remark 5.1. *This quantity $\Delta^{\mathcal{O}}_{ij}$ can be interpreted as a form of* contextual curvature*, in the sense that it measures the failure of the contextual Lagrangian to remain invariant under transition between measurement regimes. Just as geometric curvature quantifies the noncommutativity of parallel transport, this term encodes the nontriviality of gluing field-theoretic information across stratified contexts.*

Suppose x_1 and x_2 are endpoints of a (p, q)-string lying on divisors $D_i \subset U_i$ and $D_j \subset U_j$, respectively, on a Calabi–Yau threefold $\mathcal{M}$. We fix two open neighborhoods $V_i \subset U_i$, $V_j \subset U_j$, each equipped with local fields and contexts $\mathtt{C}_i$, $\mathtt{C}_j$. We define *real propositions* in terms of local gauge field strengths and string charges, specifically:

- Let $F^{\mathrm{NS}} \in \Omega^2(V_i)$ denote the Neveu–Schwarz field strength.
- Let $F^{\mathrm{RR}} \in \Omega^2(V_j)$ denote the Ramond–Ramond field strength.
- Let $T^{(p,q)} \in \mathbb{R}_{>0}$ denote the (constant) string tension of the (p, q)-string.

Now define the *contextual Lagrangians* as propositional functionals:

$$\mathcal{L}^{\mathtt{C}_i}(x_1) := p \cdot \int_{V_i} F^{\mathrm{NS}} + q \cdot \int_{V_i} F^{\mathrm{RR}} + T^{(p,q)}$$

$$\mathcal{L}^{\mathtt{C}_j}(x_2) := p \cdot \int_{V_j} F^{\mathrm{NS}} + q \cdot \int_{V_j} F^{\mathrm{RR}} + T^{(p,q)}$$

These encode propositions of the form: *"The (p, q)-string within V_i experiences an action given by $\mathcal{L}^{\mathtt{C}_i}$ due to its coupling to local fluxes."*

Then the *contextual Laplacian* $\Delta^{\mathcal{O}}_{ij}$ compares the coarse-curved epistemic difference across the factorized frame of reference:

$$\Delta^{\mathcal{O}}_{ij} := \left(\partial^{\mathtt{C}_i}_{n+1} \circ (\partial^{\mathtt{C}_i}_{n+1})^{\dagger} + (\partial^{\mathtt{C}_i}_{n})^{\dagger} \circ \partial^{\mathtt{C}_i}_{n}\right)(\wp)$$

for some proposition $\wp \in D_n$, such as: *"Flux lines form a consistent closed loop around both endpoints."*

Proposition 5.1 (Contextual Epistemic Invariance). *Let $\mathcal{L}^{\mathtt{C}_i}(x_1)$ and $\mathcal{L}^{\mathtt{C}_j}(x_2)$ be contextual Lagrangians as defined above. If $\Delta^{\mathcal{O}}_{ij} = 0$, then any proposition $p \in D_n$ with $\Omega_i(p) = 1$ satisfies $\Omega_j(\phi_{ij}(p)) = 1$.*

5.3 Exact Lagrangian Solution

To solve:

$$\mathcal{L}^{\mathtt{C}_i}(x_1) - \mathcal{L}^{\mathtt{C}_j}(x_2) = \Delta^{\mathcal{O}}_{ij},$$

plug in the expressions:

$$p \cdot \left(\int_{V_i} F^{\mathrm{NS}} - \int_{V_j} F^{\mathrm{NS}} \right) + q \cdot \left(\int_{V_i} F^{\mathrm{RR}} - \int_{V_j} F^{\mathrm{RR}} \right) = \Delta^{\mathcal{O}}_{ij} - (T^{(p,q)} - T^{(p,q)})$$

$$\Rightarrow \quad \Delta^{\mathcal{O}}_{ij} = p \cdot \int_{\delta_{ij}} F^{\mathrm{NS}} + q \cdot \int_{\delta_{ij}} F^{\mathrm{RR}},$$

where $\delta_{ij} := V_i - V_j$ is the signed region of difference in flux support (which may overlap or be disjoint).

This means: if the NS and RR fluxes are constant across V_i and V_j, then $\Delta^{\mathcal{O}}_{ij} = 0$, and the proposition is *epistemically harmonic*. If the fluxes differ across contexts, then $\Delta^{\mathcal{O}}_{ij}$ measures the *contextual curvature* of the proposition: *"This string experiences unequal field strengths at each endpoint."*[4]

6 Modal Configuration Spaces and Contextual Dynamics

To further interpret the action of the contextual Laplacian and the structure of observables in our factorization algebra, it is natural to examine the underlying configuration spaces on which our propositions and modal logic are defined. In particular, we are interested in the configuration space of distinguishable particles on a Calabi–Yau threefold $\mathcal{M}$, modeled locally as $\mathbb{C}^3$. This leads us to consider $\mathrm{Conf}_2(\mathbb{C}^3)$, the space of ordered pairs of distinct points in $\mathbb{C}^3$.

$$\mathrm{Conf}_2(\mathbb{C}^3) = \{(x_1, x_2) \in \mathbb{C}^3 \times \mathbb{C}^3 \mid x_1 \neq x_2\}. \tag{6.1}$$

This space is naturally a complex 6-manifold, which inherits a rich geometric and topological structure from $\mathbb{C}^3$. In the context of Type IIB string theory, we interpret x_1 and x_2 as the endpoints of a (p, q)-string, embedded within a Calabi–Yau compactification. The local geometry around each point is determined by the neighborhoods V_i and V_j, and the ambient brane structure is described by the coarse-grained regions $U_i, U_j \subset \mathcal{M}$.

Each such pair defines a context $\mathtt{C_i}$ for measurements made in a local frame V_i, giving rise to sheaf-valued truth assignments $\Omega_i \colon D_n \to [0, 1]$. The epistemic configuration space is then not simply $\mathrm{Conf}_2(\mathbb{C}^3)$, but rather a derived object that encodes modal variation across contexts:

$$\widetilde{\mathrm{Conf}}_2^{\,\mathrm{modal}}(\mathbb{C}^3) := \mathrm{Conf}_2(\mathbb{C}^3) \times_{\mathscr{C}} \prod_i \mathtt{C}_i. \tag{6.2}$$

[4]This can be interpreted as contextual curvature, in the sense that epistemic variation arises across factorized frames due to differences in observable field content.

This fibered construction allows the contextual Laplacian $\Delta_{ij}^{\mathcal{O}_i}$ to act on sections of chain complexes defined over epistemically enriched points in configuration space. For example, variations in RR field strength or NS-NS curvature near the string endpoints are interpreted as nonzero modal curvature:[5]

$$\mathcal{L}^{\mathsf{C}_i}(x_1) - \mathcal{L}^{\mathsf{C}_j}(x_2) = \Delta_{ij}^{\mathcal{O}} \in D_0. \tag{6.3}$$

In this way, the configuration space becomes not only a geometric stage but a logical one: a scaffold for contextual interaction, deformation, and epistemic propagation. Future work may endow $\widetilde{\mathrm{Conf}_2}^{\mathrm{modal}}(\mathbb{C}^3)$ with a higher stack structure or employ factorization homology to compute invariants reflective of entangled modal histories.

[5] That is, the deviation of a proposition's truth weight under transitions m_{ij} across overlapping neighborhoods.

A Speculation and Future Directions

The contextual Laplacian $\Delta^{\mathcal{O}}$ developed in this work serves as a first attempt to articulate the epistemic geometry of a quantum field theory localized over a factorized Calabi–Yau manifold. However, several open directions remain. Chief among them is the integration of the Laplacian into a broader contextual variational framework—one that enables the construction of discrete Euler–Lagrange equations over stratified configuration spaces.

One speculative possibility lies in the formulation of a contextual master equation that governs the dynamics of brane-coupled (p, q)-strings. This would involve a Lagrangian density $\mathcal{L}^{\mathtt{C}_i}(\mathfrak{x})$ expressed not merely as a local field term, but as an object internal to a category of sheaf-valued classifiers. Such an equation might take the schematic form

$$\delta\left(\mathcal{L}^{\mathtt{C}_i}(\mathfrak{x}) - \mathcal{L}^{\mathtt{C}_j}(\mathfrak{y})\right) = \Delta^{\mathcal{O}}_{ij},$$

where the operator $\Delta^{\mathcal{O}}_{ij}$ measures epistemic curvature across overlapping contextual regimes. The aim here would be to capture how microphysical field variations (e.g., deformations in the B-field or RR-potential) respond not simply to changes in spacetime configuration, but to shifts in logical or observational context. This aligns with recent efforts in quantum foundations to model decoherence and entanglement via higher sheaf theory and topos-theoretic constructions.

Another direction is the incorporation of **derived stacks** or ∞-topoi to refine the notion of context. If each context $\mathtt{C}_i$ is modeled not merely as a sheaf over a site but as a geometric object in a derived category, then transitions between them—especially across homotopy equivalence classes—may reveal new types of symmetry, including perhaps noninvertible or higher categorical dualities.

Finally, we anticipate applications to holography. Factorization algebras have already shown promise in encoding boundary data in topological field theories, and the contextual Laplacian could serve as a tool for describing "logical descent" from bulk dynamics to boundary observables. In this light, epistemic curvature may acquire a holographic dual: variations in logical consistency on the boundary corresponding to topological or categorical features in the bulk. This would open an avenue toward a logic-encoded AdS/CFT correspondence, where modal truths in one regime are encoded by physical field configurations in another.

In short, while the present work introduces only a localized, finite configuration of contexts, we believe it gestures toward a more ambitious program: one where logic, locality, and geometry are unified through a stratified, modal homotopy theory.

B References

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