Phenomenology of Chan–Paton Defects on Open Strings

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Abstract

This paper investigates the rôle of defects on extended quantum objects by generalizing the Chan–Paton factors of string theory as bimodules, wherein left- and right-actions encode brane dynamics. We introduce the concept of *Chan–Paton defects* on open strings, which partition the string worldsheet γ into stratified sectors γ_{in} and γ_{out}, separated by a defect modeled as a pseudoparticle. Both synthetic and analytic formulations are developed. Drawing on braid-theoretic considerations, we derive a generalized spin–statistics master equation that captures the behavior of fractional (anyonic) excitations.

We demonstrate that Chan–Paton defects suppress entanglement between external degrees of freedom, attributing this to the dominance of internal interactions between non-defective string sectors. Building on prior investigations of the authors [7, 8], we show that such nonlocal or fractional statistics break *epistemic harmony*. In particular, we prove that inverse-limit (pyknotic) constructions for anyonic observables fail generically, precluding the existence of a fully nonperturbative anyonic regime.

Two sources of this failure are identified: (a) measurement obstructions, and (b) breakdowns in domain-of-discourse-to-truth (DoD2T) mappings. These are addressed in the final section via categorical and modal completions. Taken together, these results suggest a unified framework for interpreting defects as both physical obstructions and semantic failures within extended quantum systems.

1 Introduction

Pseudo-particles are oftentimes realized as obstructions to lifting, in the most general sense. To name just two examples, *anyons*, in $2+1$-dimensional topological quantum field theory (TQFT), manifest as obstructions to the holonomy of braids [1, 2], while instantons are topological bundle obstructions [4]. These obstructions may arise from anomalies or fractional statistics.

If we treat the fundamental particles of our theory as extended (e.g., "stringy"), then such obstructions can appear on the *particle itself*, or at least on its worldsheet. The Freed–Witten anomaly is a canonical example of a defect occurring on an extended object, specifically on a brane; this is an example of a *Chan—Paton defect*. Such defects naturally arise in twisted K-theory [5], as studied by Atiyah and Segal.

In this section, we introduce the notion of Chan–Paton defects. We first introduce a *generalized Chan–Paton factor*, and define them in two ways. In subsequent sections, we exploit these two

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equivalent definitions: as a bimodule, and as a linear map between vector spaces, in order to interpret these defects physically.

1.1 Generalized Chan–Paton Factors

To understand these defects, we first recall the notion of Chan—Paton factors. Formally, let $\mathbb{I} \equiv [\alpha, \beta]$ be an interval[1] and assume it is smooth, so that $\mathbb{I}$ is modeled by an étale groupoid $\mathscr{I}$. The objects of $\mathscr{I}$ are points x_i, and the morphisms $f : x_i \to x_j$ correspond to paths $\gamma : \mathbb{I}_{x_i} \to \mathbb{I}_{x_j}$. Since $\mathscr{I}$ is a **Vec**-enriched category, each morphism space $\mathrm{Hom}_{\mathscr{I}}(x_i, x_j)$ is a vector space, allowing morphisms to be viewed as linear combinations of paths.

Interpreting γ physically, it corresponds to a string with endpoints at x_i and x_j. In the special case where the string is closed, then $\mathscr{I}$ is a *monad*. As string endpoints are confined to branes, we embed $\mathbb{I}$ into a manifold $\mathcal{M}^2$, modeling the target space. The branes themselves can be reconstructed by considering codimension-one submanifolds $\mathcal{N} \subset T^*_{x_\bullet}\mathcal{M}$. The Chan–Paton factor can then be thought of as such a submanifold, with the full brane recovered as a product $\mathcal{N} \times \mathcal{M}'$. If $\mathcal{M}' \neq \mathcal{N}$, the string is oriented; if $\mathcal{M}' = \mathcal{N}$, the string is unoriented.

Definition 1.1 (Generalized Chan–Paton Factor)**.** *Given an étale groupoid $\mathscr{I}$ modeling the interval $\mathbb{I}$, a* Chan–Paton factor *is a functor*

$$\mathcal{E} : \mathscr{I} \to \mathbf{Vect}_{\mathbb{K}}$$

assigning to each object $x_i \in \mathscr{I}$ a vector space $\mathcal{E}(x_i)$, and to each morphism $\gamma : x_i \to x_j$, a linear map $\mathcal{E}(\gamma) : \mathcal{E}(x_i) \to \mathcal{E}(x_j)$. Physically, this corresponds to a bundle of state spaces over the endpoints of open strings, with morphisms representing the transport of quantum states along the string.

In the context of open string theory, Chan–Paton factors represent additional algebraic data assigned to the endpoints of strings. When these endpoints lie on branes modeled by associative algebras, it is natural to interpret the corresponding state spaces as bimodules. Specifically, given with associated algebras A and B, a Chan–Paton factor corresponds to an (A, B)-bimodule. This formulation allows for left and right actions that capture how observables from each brane act on the shared string state space. The bimodule structure encodes the interaction between boundary conditions and internal degrees of freedom without requiring additional geometric assumptions. We have so far avoided reference to bimodules; however, they make the definition more elegant:

Definition 1.2 (Chan–Paton Factor as Bimodule)**.** *Let $R \subset \mathbb{K}$ be a commutative ring, and let A, B be unital R-algebras modeling two branes. A* Chan–Paton factor *for a string stretching from brane A to brane B is an (A, B)-bimodule*

$$\mathcal{E} \in {}_A\mathrm{Mod}_B$$

that is, an R-module $\mathcal{E}$ equipped with a compatible left A-action and right B-action satisfying

$$(a \cdot e) \cdot b = a \cdot (e \cdot b)$$

for all $a \in A, b \in B, e \in \mathcal{E}$.

[1] One could generalize intervals to posets or Heyting algebras, but for now we restrict to intervals, as they align better with topological intuition.

Adopting the bimodule perspective also permits a uniform algebraic treatment of oriented and unoriented strings, depending on whether the left and right module structures differ or coincide. Moreover, bimodules naturally compose through tensor products, aligning with the physical composition of strings. This framework also generalizes straightforwardly: in categorical and derived settings, bimodules become morphisms between module categories, and can accommodate more complex structures such as defects and anomalies [6]. Consequently, the bimodule model serves as a minimal algebraic setting in which Chan–Paton data may be consistently described and manipulated.

1.2 Chan-Paton Defects

Let $\mathbf{Oblv}_{\mu(\gamma)}$ be the category whose objects are paths γ and whose morphisms are discrete transformations $\gamma_i \xrightarrow{\mu} \gamma_f$ such that

Consilience $\equiv$ The endpoints of γ_i lie on the same branes as the endpoints of γ_f

Identity $\equiv$ The Chan–Paton factors $\mathcal{E}(\gamma_f)$ are related to the Chan–Paton factors $\mathcal{E}(\gamma_i)$ via a homomorphism of Lie groups $\tilde{\mu} \coloneqq \mathfrak{g}_{\mathcal{E}(\gamma_i)} \longrightarrow \mathfrak{g}_{\mathcal{E}(\gamma_j)}$ so that, for every point $\mathfrak{x} \in \gamma_i$, there exists an identity element $e \in R$ so that $e \cdot \tilde{\mu}(\mathfrak{x}) = \tilde{\mu}(e \cdot \mathfrak{x})$

both hold.

In this setting, $\mathbf{Oblv}_{\mu(\gamma)}$ assigns to each point $\mathfrak{x}$ of the string a *generalized truth-value* via a "measurement" μ, which is defined mathematically via $\tilde{\mu}$. According to Emmerson [8], in a completely torsion-free setting, the *entropy modal index* (EMI) will be zero, meaning that the string is "epistemically harmonic," as in [7]. However, in the case of a Chan–Paton defect, the string itself carries multiple incommensurable contexts due to its topology. This results phenomenologically as a *pseudo–particle*.

2 Anyonic Manifestations of Chan–Paton Defects

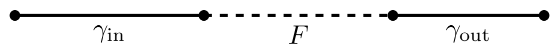
Let $\Phi \coloneqq \sum_{i=1}^{N} \int_{\mathcal{M}} d\phi_i$ be a field acting on a subspace of a spacetime manifold $\mathcal{M}$. We will not make assertions about the number of dimensions of the manifold; we only require that, for orientable strings/branes, the field Φ acts as a vector field, and otherwise it is a scalar field. Let us then label the Chan–Paton factors for each string by setting $\mathcal{E}_i(\gamma) = \mathcal{E}(\gamma_i)$, and $\phi_i \cdot \gamma = \phi \cdot \gamma_i$ so that our index aligns with the field.

One way to interpret a Chan–Paton defect is as follows. We will let $\mathscr{O}(\gamma)$ be a *conserved quantity* which is then measured via the morphism μ, so that the following diagram

$$\begin{tikzcd} \gamma_i \arrow[r, "\mu"] \arrow[d] & \gamma_f \arrow[d] \\ \mathscr{O}_i(\gamma) \arrow[r, "\nu"'] & \mathscr{O}_f(\gamma) \end{tikzcd}$$

commutes.

We now must define what we mean by ν. To distinguish between global transformations and local constraints, we interpret ν as a pointwise endomorphism of the Chan–Paton fiber, so that $\nu(\mathfrak{x}) \in \mathrm{End}_R(\mathcal{E}_{\mathfrak{x}})$ for each $\mathfrak{x} \in \gamma$. This allows us to describe internal modifications to the fiber, such as defect insertions or symmetry constraints, independently of the morphism $\tilde{\mu} : \mathfrak{g}_{\mathcal{E}(\gamma_i)} \to \mathfrak{g}_{\mathcal{E}(\gamma_j)}$, which captures induced transformations at the level of the structure algebra. In particular, we interpret ν as encoding localized modifications that may obstruct or stabilize the global transport defined by $\tilde{\mu}$. These may manifest physically as quantum flux terms localized at a point $\mathfrak{x}_0 \in \gamma$, where γ denotes an open string. The presence of such a defect partitions the string into two segments, γ_{in} and γ_{out}, with the defect mediating a coupling between them. Concretely, we model the defect by an endomorphism $\nu(\mathfrak{x}_0) \in \text{End}_R(\mathcal{E}_{\mathfrak{x}_0})$ satisfying $\nu^2 = \nu$, where the image of ν enforces a coupling structure modeled on the tensor product $\mathcal{E}(\gamma_{\text{in}}) \otimes_R \mathcal{E}(\gamma_{\text{out}})$, identifying the allowed joint states of the two segments.



Since the string has been partitioned into two sectors, it may be possible that they may become entangled with one another; thus, external degrees of freedom which are related to external entanglement scenarios may be suppressed. Let $\Phi_0 \subset \mu(\Phi)$, meaning that Φ_0 acts on the spacetime in the same way as Φ, but acts only on subobjects; i.e., substrings or limited subspaces of branes. The presence of such a pseudoparticle demands a correction term, which reads

$$\boxed{\Phi_{\text{anyon}} = \sum_{i=1}^{N} \int_{\mathcal{M}} (d\phi_i - F)}$$

where the term $F = \frac{d\Phi_0}{N}$ encodes the fractional anyon statistics.

2.1 Physical Braiding and Anyonic Coupling

We consider a collection of Chan–Paton defects located at ordered points $\mathfrak{x}_0, \ldots, \mathfrak{x}_n$ along an open string. Each defect represents a disruption in the locality of observables and imposes a constraint on how state spaces may factorize across the string. While the observable map ν characterizes how measurements are transported across these regions, it does not by itself serve as a label for the defects. Rather, the defects may be labeled either by their associated fractional flux term, $F_i = \frac{d\Phi_0}{N}$, or by the local behavior of a factorization algebra $\mathcal{F}$ supported on a punctured neighborhood of $\mathfrak{x}_i$.

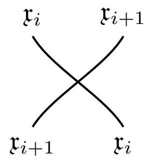
These defect labels encode intrinsic features of the string that influence global observables without directly participating in measurement. As the positions $\mathfrak{x}_i$ are varied along the string, the topological class of the configuration may change. We model this process by an action of the braid group B_n on the coupled state space

$$\mathcal{E}_{\text{total}} = \bigotimes_{i=0}^{n} \mathcal{F}_{\mathfrak{x}_i}$$

where each $\mathcal{F}_{\mathfrak{x}_i}$ denotes the fiber of observables localized near the defect. The braid group acts by permuting these defect regions, inducing automorphisms of the total observable algebra:

$$\rho : B_n \to \text{Aut}_R(\mathcal{E}_{\text{total}})$$

where B_n is the braid group on n strands. This action captures the nonlocal statistical behavior of the system and encodes how the presence and ordering of defects affects physical quantities across the string. To illustrate this, we depict below the action of a braid generator $\sigma_i \in B_n$, which exchanges the defect points $\mathfrak{x}_i$ and $\mathfrak{x}_{i+1}$. This induces a corresponding automorphism of the total state space via the representation ρ.



$$\sigma_i : (\mathfrak{x}_i, \mathfrak{x}_{i+1}) \mapsto (\mathfrak{x}_{i+1}, \mathfrak{x}_i) \quad \rightsquigarrow \quad \rho(\sigma_i) \in \mathrm{Aut}_R(\mathcal{F}_{\mathfrak{x}_i} \otimes \mathcal{F}_{\mathfrak{x}_{i+1}})$$

Definition 2.1 (Nonlocal Statistics)**.** *We refer to* nonlocal statistics *as those arising from constraints on the entanglement structure of the state space, where the observable algebra $\mathcal{E}(\mathfrak{x}_i)$ induces a module-like distribution of degrees of freedom that resists global factorization. This includes but is not limited to anyonic statistics in 1+1D systems.*

Definition 2.2 (Adjoint Modules)**.** *Let $\Psi : \Phi^{\mathrm{op}} \to$ **Sets** be a functor from the space of input fields (statistical modes). Depending on the underlying particle statistics, the image of Ψ obeys either Bose-Einstein, Fermi-Dirac, or nonlocal statistical constraints. In each case, the following holds:*

$$\forall \phi \in \Psi, \quad \exists \mathbb{I} \text{ such that } \phi^{i \in \mathbb{I}} \subseteq \rho(\sigma_i \cdot \phi)$$

where $\rho \in \mathrm{Aut}_R$ acts as the intertwiner*. For fermionic or nonlocal statistics, this action is nontrivial, encoding a homotopical obstruction to decomposing the observable algebra into tensor factors.*

Definition 2.3 (Nontrivial Homotopy)**.** *A homotopy is said to be* nontrivial *if*

$$\left(\phi^0 \times \phi^1 \times \mathbb{I}\right) / \mathbb{I} \ncong *$$

i.e., if the resulting space carries a non-contractible factorization algebra $\mathcal{F}(\Phi)$. This nontriviality obstructs reduction to trivial topological charge and is associated with conserved structure in the state space.

Proposition 2.1. *The rank of a nontrivial adjoint bundle over the interval $\mathbb{I}$ is given by the fiber dimension n of $\mathbb{I}$, where nontriviality corresponds to a nonvanishing topological charge induced by a defect.*

Proof. Let $\mathbb{I} \subset [0,1]$ be an interval in a 1-manifold, stratified by a defect at a point $\mathfrak{x}_i \in \mathbb{I}$. Consider a principal G-bundle $P \to \mathbb{I} \setminus \{\mathfrak{x}_i\}$ whose transition function across $\mathfrak{x}_i$ induces a nontrivial gluing datum. The associated adjoint bundle $\mathrm{Ad}(P) \to \mathbb{I}$ has typical fiber isomorphic to $\mathfrak{g} = \mathrm{Lie}(G)$, hence its rank is $n = \dim \mathfrak{g}$.

In the absence of the defect, the interval $\mathbb{I}$ is contractible, so any bundle over it would be trivial. However, the defect stratifies the manifold into three pieces:

$$\mathbb{I} = (0, \mathfrak{x}_i - \varepsilon] \cup \{\mathfrak{x}_i\} \cup [\mathfrak{x}_i + \varepsilon, 1)$$

This stratification upgrades $\mathbb{I}$ into a constructible stratified space. To this space we may associate the exit-path ∞-category $\mathrm{Exit}(\mathbb{I})$, which encodes how observables (or bundles) behave relative to the singularity at $\mathfrak{x}_i$. d By the theory of constructible sheaves [9, 10], representations of $\mathrm{Exit}(\mathbb{I})$ correspond to factorization algebras or constructible bundles compatible with the stratification. The defect then induces a nontrivial monodromy around $\mathfrak{x}_i$, realized by a nontrivial automorphism in the image of the exit functor:

$$\mathrm{Exit}(\mathbb{I}) \longrightarrow \mathrm{Vect}_R$$

Thus, even though the total space is contractible, the bundle is nontrivial due to its singular behavior near $\mathfrak{x}_i$.

The rank of this adjoint bundle is given by the fiber dimension $n = \dim \mathfrak{g}$. The nontriviality of the defect ensures that the bundle is not globally trivial (i.e., not isomorphic to $\mathbb{I} \times \mathfrak{g}$), even though it remains locally trivial away from $\mathfrak{x}_i$. Therefore, the rank reflects the underlying Lie algebra dimension of the symmetry broken or twisted at the defect. □

Axiom 2.1. *Let $(\mathcal{F}(\Phi)^{\mathrm{op}})^n \to \mathbf{Sets}$ be a functor of cohomological codimension n. Then $\mathbb{I}$ must support a nontrivial n-dimensional structure, implying the presence of stratification or obstruction.*

To define a nontrivial homotopy more explicitly, recall that a homotopy encodes a smooth deformation between field configurations. A trivial homotopy corresponds to a retraction onto a constant map; the closer a homotopy is to trivial, the lower the rank of the associated bundle. We view

$$\Phi^{\mathrm{op}} \to \mathbf{Sets} \cong \mathrm{Hom}([0,1]^n, \mathbf{Sets})$$

as a representation of observables on a cubical parameter space.

A string defect introduces a puncture in the domain, partitioning $[0,1]$ as

$$(0, \tfrac{1}{m} - \varepsilon] \,\sqcup\, \mathfrak{x}_0 \,\sqcup\, [\tfrac{1}{m} + \varepsilon, 1)$$

The operator $\mathfrak{x}_0$ acts as a creation operator, with $\mathfrak{x}_0^\dagger$ its adjoint (annihilation). We then define a parity observable:

$$\mathscr{F} = \sum_{i=0}^{N-1} (-1)^i \mathfrak{x}_0 + \tfrac{1}{2}\hbar$$

This parity governs intrinsic string orientation or spin. In supersymmetric models $\mathcal{N} = *$, the index i may be interpreted as a fermion count, and the coefficient $\frac{1}{2}\hbar$ can be generalized to $\frac{1}{s}\hbar$, where $s \in \mathbb{N}$ denotes the spin of the excitation. When $s > 2$, the system enters a higher-spin regime, potentially describing phenomena such as neutrino flavor oscillation or spin–statistics anomalies [11].

The Kuramoto phase shift associated to the defect is then given by:

$$\theta = \theta_j - \theta_i = e^{\mathscr{F}} - e^{\star\mathscr{F}}$$

where $\star\mathscr{F}$ denotes the dual (right-action) component of the bimodular frame $\mathscr{F} \subseteq R$. This phase shift quantifies the obstruction to symmetric factorization across a defect. The full bimodule structure is then captured by a left $\mathscr{F}$-action and a right $\star\mathscr{F}$-action. This quantity θ, defined as the phase shift $\theta = e^{\mathscr{F}} - e^{\star\mathscr{F}}$, governs a form of parity-induced entanglement across the defect. While not necessarily equivalent to the Weinberg angle of electroweak theory, it shares a formal resemblance as a mixing parameter between dualizable sectors. In speculative models, such a parameter could be interpreted as a flavor-mixing angle or an emergent symmetry phase.

Taking a note from Emmerson [8]:

$$\underbrace{\underbrace{\theta}_{\text{Kuramoto Resonance}} = \underbrace{\underbrace{\alpha^n e^{\mathscr{F}}}_{\text{Obstacle}} - \underbrace{\beta^m e^{\star\mathscr{F}}}_{Defect}}_{\text{Bose-Einstein}} \underbrace{\underbrace{\pm}_{\text{Parity}} \underbrace{e^{i\frac{2}{p}\pi}}_{\text{Fractional Phase}} \cdot \underbrace{\omega_{\text{wind}}^{-1}}_{\text{inflow}}}_{\text{Dirac}}}_{\text{Anyonic}} \tag{2.1}$$

we get a quit rich vibrational structure where α^n and β^m are quadratic terms accounting for phenomenological velocity, and the inflow term is the inverse of the winding number. The Kuramoto resonance describes, in a non-probabilistic way, the essentially statistical phenomenon of *spontaneous synchronization*. Note that this equation is rather curious because it lacks an epistemic term. So, let

$$\mu(\mathscr{O}_\theta) = \omega_{\text{wind}}^{-1} \pi^{-1} \arg\theta$$

be a measurement inside the category $\mathbf{Oblv}_{\mu(\gamma)}$ of geodesic measurements. Suppose for a moment that there exists no defect on the string γ, and thus, its vibration carries epistemically harmonic information; i.e., $H^2_{\text{AF}} = 0$; i.e., the second Ayala-Francis cohomology is a nullhomotopy. Then, by reverse engineering, a *braid map*:

$$\rho : \mu(\mathscr{O}_\theta) \longrightarrow \mu(\mathscr{O}_{\theta'})$$

should be such that

$$H^2_{\text{AF}}(\rho) \subseteq \mathbb{Z}_p$$

for some prime $p \in \mathbb{N}_1$. This means the *contextual Laplacian* [7] is *p-adic*, and therefore there exists a non-zero modal entropy.

2.2 Pyknotic Failure and Epistemic Obstructions

One might hope to make computations using inverse (co)limits for some of the physical observables. These work quite neatly on flag manifolds in particular [12], where for every brane $\mathcal{B}^n$, there exists an ascending chain $\mathcal{B}^0 \subseteq \mathcal{B}^1 \subseteq ... \subseteq \mathcal{B}^n$, even without taking into account the topology of the branes. Then, if we have two *soldering forms* $\mathfrak{s}_A = \epsilon^\mu_{\rho\sigma}$ and $\mathfrak{s}_B = \epsilon^\nu_{\sigma\mu}$, then we can define a Čech cocycle $\mathfrak{s}_{BA}$ obeying all the usual transition conditions, allowing us to partition the space into partition the R-modules into *R-submodules* $S \lhd (R \curvearrowleft (B \circ A))$. Then, we can choose any $s \in S$, and construct it as

$$Q = \varprojlim_{\ t_i \in T} T \lhd (S(B' \circ A'))$$

so that $\mathfrak{s}_{CB} \simeq Q \times \mathbb{I}$; i.e., there is a *homotopy* between all submodules of a given stratum. On a brane, this means that any two string endpoints are indistinguishable.

However, one cannot take this approach when passing to the case of modal entropy; that is, we can study closed systems in this way, but in an evolving universe, the inverse limit methods are insufficient. This is especially true in the case of anyons with non-trivial braidings

$$\rho : \mu(\mathscr{O}_\theta) \longrightarrow \mu(\mathscr{O}_{\theta'})$$

which introduce epistemic curvature. If we consider the branes purely topologically, we lose information about the interaction of the Ramond–Ramond and Neveu–Schwartz fields [7]. That is to say, we ignore the ways in which the vibrations of the strings can perturb the local geometry. If we truly had a theory of quantum gravity, we could pinpoint precisely how these fields interact with one-another in a purely nonperturbative fashion, but unfortunately we cannot yet do so.

In the very best cases, we get

$$\mu(\mathscr{O}_{\theta'}) = \Delta_{ij}^{\mathscr{O}} = \mu(\mathscr{O}_{\theta})$$

and the entropy remains unaffected. This is effectively the *adiabatic case*, and this occurs across spacially distinct but temporally identical points of spacetime. Indeed, while there exist spontaneous transitions in quantum theory, such as the collapse of the wave-function, we can also expect that some quantum processes, such as fractional bifurcation and anyonic braiding, require an *evolution operator*, or *time-step map* $\tau : T^*_{x_\bullet}\mathcal{M} \longrightarrow T^*_{y_\bullet}$ with $y \succ x$, meaning y *succeeds* x. The time-step operator, under ordinary circumstances, will select two separate resonances $\theta_{\mathtt{A}}$ and $\theta_{\mathtt{B}}$, and *causually order them* via

$$(x \prec y) \implies \theta_{\mathtt{A}\ni x} \prec \theta_{\mathtt{B}\ni y}$$

so that the set of contexts is partially ordered. This culminates as a *Chan–Paton defect*, where $\mathtt{A} \prec mathfrak x_0 \prec \mathtt{B}$, the *present moment* of a string, becomes the defective entity. In this case, there is missing data $? \in \square_{/x_0}$ that remains undetected, either because the measuring device is faulty, or the classification (i.e., *domain-of-discourse-to-truth-mapping*) (DoD2T Map) is.

3 Modal Healing

We now turn from the topological and field-theoretic manifestations of epistemic breakdown to their logical and semantic counterparts. In particular, we examine how defects disrupt mappings between domains of discourse and truth values, and how this disruption may be remedied via categorical and lattice-theoretic constructions. Given that defects may introduce both physical obstructions and semantic gaps in our inference structures, we require a way to restore contextual coherence. We refer to this process as modal healing.

In order to address faulty DoD2T Maps, we can consider to obvious paths toward remediation. Recall that a contextual classifier is a map:

$$\Omega(p) : p \in D_n \longrightarrow \mathbf{LatComp}$$

sending a proposition to a complete lattice $\mathfrak{L}$. Recall further that a lattice $\mathfrak{L}$ complete if and only if $\mathfrak{L}$ is a partially-ordered set and every subset $\mathfrak{l} \subset \mathfrak{L}$ has both a greatest lower bound and least upper bound; i.e, an infimum and supremum. It is traditional in topos theory to denote these elements by

$$\top = \sup(\mathfrak{l})$$

and

$$\bot = \inf(\mathfrak{l})$$

The sources of DoD2T would then appear to have a simple typology:

- *Propositional failure*: p is not a valid proposition in D_n; i.e., there exists no inclusion ι and context $\mathtt{C}$ such that $\iota_{\mathtt{C}} : p \hookrightarrow D_n$ which is injective.
- *Discourse failure*: There exists no surjection $D_n \twoheadrightarrow p$
- *Context failure*: The lattice $\mathfrak{L}$ is not complete in the context $\mathtt{C}$.

Propositional and Discourse Failures To fix the first two of these problems, we essentially need to choose a *Grothendieck universe* $\mathfrak{V}$ and some smallest inaccessible cardinal $\{\delta_{\mathfrak{V}} \mid |\delta_{\mathfrak{V}}| > v \ \ \forall v \in \mathfrak{V}\}$ so that the universe is *downward closed*; i.e., the image of every function $F(\omega \subseteq \Omega)$ lies strictly within Ω. Let $\Omega \downarrow F$ denote the class of all propositions which are downward-closed in this way. The map

$$\Omega(p) \longrightarrow [\Omega \downarrow F](p')$$

yields the desired class of propositions; thus, if we find some suitable replacement $p' \sim p$, such that we can agree that the truth of p' should entail the truth of p, then we can avoid propositional and discourse failures, at the cost of inexact semantics.

Context Failure We now address the third case, *context failure*, in which the lattice $\mathfrak{L} \in$ **LatComp** is not actually complete in the relevant context $\mathtt{C}$. A general method for repairing such a defect is given by the *MacNeille completion* (also called the Dedekind–MacNeille completion), which universally embeds any partially ordered set into the smallest complete lattice containing it.

Let $\mathfrak{L}$ be a lattice or poset arising from a contextual classifier $\Omega(p) : p \in D_n \to \mathbf{LatComp}$. If $\mathfrak{L}$ lacks arbitrary joins or meets in context $\mathtt{C}$, we define the set of *lower bounds* for any subset $A \subseteq \mathfrak{L}$ as

$$A^\ell := \{x \in \mathfrak{L} \mid x \leq a \text{ for all } a \in A\},$$

and the *upper bounds* as

$$A^u := \{x \in \mathfrak{L} \mid x \geq a \text{ for all } a \in A\}.$$

The composition $A^{\ell u} := (A^\ell)^u$ defines a closure under cuts. The collection of all such sets

$$\overline{\mathfrak{L}} := \{A^{\ell u} \mid A \subseteq \mathfrak{L}\}$$

forms a complete lattice under inclusion, known as the MacNeille completion of $\mathfrak{L}$. The original lattice embeds faithfully into this completion via the map

$$x \mapsto \{x\}^{\ell u},$$

preserving its order structure and any existing joins and meets.

This construction serves as a canonical remedy for incomplete contexts. That is, if a discourse context $\mathtt{C}$ presents a lattice of propositions $\mathfrak{L}$ that lacks certain suprema or infima required by inference rules or modal reifications, we may pass to its MacNeille completion $\overline{\mathfrak{L}} \in \mathbf{LatComp}$. This allows the contextual classifier Ω to retain a valid interpretation, now defined over a space that supports complete entailment structure.

In epistemic or modal models where lattice incompleteness reflects informational gaps or inference breakdowns, the MacNeille completion can be interpreted as a form of *semantic closure*: an enrichment of the context to one in which latent entailments or interpolations become expressible. In this way, modal healing proceeds not only by repairing propositions, but by extending the semantic infrastructure over which those propositions range.

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