

# Solutions to Second-Order Linear Homogeneous Difference Functions with Constant Coefficients

## Explanations of the Terms

We call a function a *difference function* if it is one of the form (we use the symbols we have learned in the middle school)

$$a_n = f(a_1, a_2, \dots, a_{n-1})$$

A  $k$ th order difference function is one that can be written as

$$a_{n+k} = f(a_n, a_{n+1}, \dots, a_{n+k-1})$$

where the range between the maximum index and the minimum index is  $k$ .

Thus, a *second-order* difference function is one of the form

$$a_{n+2} = f(a_{n+1}, a_n)$$

The equation is called *linear* if it can be written as

$$a_{n+2} = b_{n+1}a_{n+1} + b_n a_n + t_n$$

where  $b_{n+1}, b_n, t_n$  may be a function of  $n$ .

If  $t_n = 0$  we call the equation a *homogeneous* equation. Otherwise, it is called *inhomogeneous*.

If the coefficients  $b_{n+1}, b_n$  are constant which means that  $b_{n+1}, b_n$  is not related to the index  $n$ , we say the function is with *constant coefficients*.

Therefore, what we want to discuss in this article is a kind of difference equations which is one of the form

$$a_{n+2} = pa_{n+1} + qa_n$$

or we transform into the form (you will know why I want to do this in the following paragraphs)

$$a_{n+2} + pa_{n+1} + qa_n = 0$$

where  $q \neq 0$  (otherwise it won't be a second-order difference equation)

And if the function has an additional condition like  $a_1 = 1, a_2 = 1$  so that we can solve the unique function, we call the conditions *initial conditions*.

And now we can set about solving it.

## Method 1: By Using the Geometric Sequences

Obviously,  $a_n = 0$  is one of the solutions which satisfy the equation. But it may be a little boring and even useless if we only stare at the zero solution. What we mainly concern about is the general solution

We first consider a geometric sequence which has the form  $\{a_{n+1} - ta_n\}$  and satisfies the equation. Suppose the common ratio is  $r$  and, so we have

$$a_{n+2} - ta_{n+1} = r(a_{n+1} - ta_n)$$

i.e.

$$a_{n+2} + (t + r)a_{n+1} + tra_n = 0$$

So according to the Vieta's formulas we know that  $t$  and  $r$  are the solutions of the equation

$$\lambda^2 + p\lambda + q = 0$$

We can easily solve the equation so that we get

$$\lambda_{1,2} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

Let  $b_n = a_{n+1} - ta_n$  ( $t$  can be either  $\lambda_1$  or  $\lambda_2$ ). Then  $\{b_n\}$  is a geometric sequence. Without loss of generality we let  $b_n = a_{n+1} - \lambda_1 a_n$  then  $b_{n+1} = \lambda_2 b_n$  so

$$a_{n+1} - \lambda_1 a_n = b_n = \lambda_2^{n-1} b_1$$

Since  $q \neq 0, \lambda_1, \lambda_2 \neq 0$  then we have

$$\frac{a_{n+1}}{\lambda_1^{n+1}} - \frac{a_n}{\lambda_1^n} = \frac{1}{\lambda_2} \left( \frac{\lambda_2}{\lambda_1} \right)^n b_1$$

Thus

$$\frac{a_n}{\lambda_1^n} = \frac{a_{n-1}}{\lambda_1^{n-1}} + \frac{1}{\lambda_2} \left( \frac{\lambda_2}{\lambda_1} \right)^{n-1} b_1 = \dots = \frac{a_1}{\lambda_1} + \frac{b_1}{\lambda_2} \left( \left( \frac{\lambda_2}{\lambda_1} \right)^{n-1} + \left( \frac{\lambda_2}{\lambda_1} \right)^{n-2} + \dots + \left( \frac{\lambda_2}{\lambda_1} \right) \right)$$

If  $\lambda_1 \neq \lambda_2$  which means  $\Delta = p^2 - 4q \neq 0$ , we can get

$$\frac{a_n}{\lambda_1^n} = \frac{a_1}{\lambda_1} + \frac{b_1}{\lambda_2} \frac{\lambda_2/\lambda_1 (1 - (\lambda_2/\lambda_1)^{n-1})}{1 - \lambda_2/\lambda_1} = \frac{a_1}{\lambda_1} + \frac{b_1 (1 - (\lambda_2/\lambda_1)^{n-1})}{\lambda_1 - \lambda_2}$$

Noticing that  $b_1 = a_2 - \lambda_1 a_1$ . Thus,

$$a_n = a_1 \lambda_1^{n-1} + \frac{(a_2 - \lambda_1 a_1)(\lambda_1^n - \lambda_1 \lambda_2^{n-1})}{\lambda_1 - \lambda_2} = \left( \frac{a_1}{\lambda_1} + \frac{a_2 - \lambda_1 a_1}{\lambda_1 - \lambda_2} \right) \lambda_1^n + \frac{\lambda_1 (\lambda_1 a_1 - a_2)}{\lambda_2 (\lambda_1 - \lambda_2)} \lambda_2^n$$

where  $\lambda_1, \lambda_2 \in \mathbb{C}$

If  $\Delta = p^2 - 4q > 0$ , then  $\lambda_1, \lambda_2 \in \mathbb{R}$ . Thus, the solution is  $c_1 \lambda_1^n + c_2 \lambda_2^n$ , where  $c_1, c_2$  can be solved by the initial conditions (This proof is not strict enough, we will discuss a more strict one in solution 2).

If  $\Delta = p^2 - 4q < 0$  then  $\lambda_1, \lambda_2 \in \mathbb{C}$  and can be written as  $a \pm bi$  ( $a, b \in \mathbb{R}$ ). So by using the De Moivre's formula we can get

$$\begin{aligned} a_n &= c_1 \lambda_1^n + c_2 \lambda_2^n \\ &= c_1 (a + bi)^n + c_2 (a - bi)^n \\ &= c_1 (\sqrt{a^2 + b^2} (\cos \theta + i \sin \theta))^n + c_2 (\sqrt{a^2 + b^2} (\cos \theta - i \sin \theta))^n \\ &= M^n (c_1 (\cos n\theta + i \sin n\theta) + c_2 (\cos n\theta - i \sin n\theta)) \\ &= M^n ((c_1 + c_2) \cos n\theta + i(c_1 - c_2) \sin n\theta) \end{aligned}$$

where  $M = \sqrt{a^2 + b^2}$ . And the equality means that the solution is a form of  $a_n = M^n (d_1 \cos n\theta + d_2 \sin n\theta)$  And we can solve  $d_1, d_2$  by the initial conditions.

If  $\lambda_1 = \lambda_2 = \lambda$  i.e.  $\Delta = p^2 - 4q = 0$  then

$$\frac{a_n}{\lambda^n} = \frac{a_1}{\lambda} + \frac{b_1}{\lambda} (n - 1) = \left( \frac{a_1}{\lambda} - \frac{b_1}{\lambda} \right) + \frac{b_1}{\lambda} n = \frac{(1 + \lambda)a_1 - a_2}{\lambda} + \frac{a_2 - \lambda a_1}{\lambda} n$$

Thus,

$$a_n = \left( \frac{(1 + \lambda)a_1 - a_2}{\lambda} + \frac{a_2 - \lambda a_1}{\lambda} n \right) \lambda^n$$

which means that the solution is the form  $a_n = (c_1 + c_2 n) \lambda^n$ .  $c_1, c_2$  can be solved by using the initial conditions.

## Method 2: By Using the Characteristic Equation and Roots

After seeing solution 1, you probably have the view that the calculation is so complex and easy to make mistakes(maybe the author himself has some) and it's better if there is a simple solution which can decrease the amount of calculation and decline the error rate. Therefore, this method comes into being.

We first guess(I don't know the thought of the man who came up with the idea, but we can just follow his steps) the solution has the form  $a_n = \lambda^n$ . And we substitute it in the equation we will obtain

$$\lambda^{n+2} + p\lambda^{n+1} + q\lambda^n = \lambda^n(\lambda^2 + p\lambda + q) = 0$$

We suppose that  $\lambda \neq 0$  then we can get  $\lambda^2 + p\lambda + q = 0$ . This equation is what we have seen in method 1, but now the equation has its new meaning that the solution of which is directly the solution of the initial difference equation. We call the equation  $\lambda^2 + p\lambda + q = 0$  the *characteristic equation* of the difference equation  $a_{n+2} + pa_{n+1} + qa_n = 0$ . And the solution of characteristic equation is called *characteristic roots*.

Now we have the question: in most cases, the equation  $\lambda^2 + p\lambda + q = 0$  has two distinct roots, do both of roots consist of the general solution? You may find the answer in the lemma as follows:

### Lemma: the Superposition for Homogeneous Equations

*Lemma:* If  $a_n^{(1)}, a_n^{(2)}$  are the solutions of the difference equation  $a_{n+2} + pa_{n+1} + qa_n = 0$ , then  $c_1 a_n^{(1)} + c_2 a_n^{(2)}$  is also the solution of the equation.

*Proof:* Based on the conditions, we have

$$a_{n+2}^{(1)} + pa_{n+1}^{(1)} + qa_n^{(1)} = 0 \quad a_{n+2}^{(2)} + pa_{n+1}^{(2)} + qa_n^{(2)} = 0$$

Then

$$\begin{aligned} & c_1 a_{n+2}^{(1)} + c_2 a_{n+2}^{(2)} + p(c_1 a_{n+1}^{(1)} + c_2 a_{n+1}^{(2)}) + q(c_1 a_n^{(1)} + c_2 a_n^{(2)}) \\ &= c_1 (a_{n+2}^{(1)} + pa_{n+1}^{(1)} + qa_n^{(1)}) + c_2 (a_{n+2}^{(2)} + pa_{n+1}^{(2)} + qa_n^{(2)}) \\ &= 0 \end{aligned}$$

Q.E.D.

Now a new question occurs what is the form of the general solution. Take the equation  $a_{n+2} - 3a_{n+1} + 2a_n = 0$  for example, which is the general solution of it,  $a_n = c_1 1^n$ ,  $a_n = c_1 2^n + c_2 2^n$  or else? Theorem 1 tells the answer.

### Theorem 1: Solutions for Distinct Characteristic Roots

*Theorem:* If  $\lambda_1, \lambda_2$  are the distinct roots(which means  $\Delta = p^2 - 4q \neq 0$ ) of the characteristic equation  $\lambda^2 + p\lambda + q = 0$  then the general solution of the difference equation is

$$a_n = c_1 \lambda_1^n + c_2 \lambda_2^n$$

where  $c_1, c_2$  are constants,  $\lambda_1, \lambda_2 \in \mathbb{C}$

*Proof:* Based on the lemma,  $a_n = c_1 \lambda_1^n + c_2 \lambda_2^n$  is the solution of difference equation  $a_{n+2} + pa_{n+1} + qa_n = 0$

Thus, what we should prove is that given any initial value  $a_1$  and  $a_2$  we can get the unique specific solution of the difference equation.

Given the initial value  $a_1$  and  $a_2$  we can get the linear systems:

$$\begin{cases} c_1 \lambda_1 + c_2 \lambda_2 = a_1 \\ c_1 \lambda_1^2 + c_2 \lambda_2^2 = a_2 \end{cases}$$

Since  $\lambda_1 \neq \lambda_2$  the determinant of coefficient

$$\begin{vmatrix} \lambda_1 & \lambda_2 \\ \lambda_1^2 & \lambda_2^2 \end{vmatrix} \neq 0$$

Thus, the linear systems has the unique solution  $c_1, c_2$ . Therefore, the general solution of the equation is  $a_n = c_1 \lambda_1^n + c_2 \lambda_2^n$ . Q.E.D.

We still have one thing to do: what if  $\Delta = p^2 - 4q = 0$ . Still, there is a theorem providing us the answer: ).

### Theorem 2: Solutions for Duplicate Characteristic Roots

*Theorem:* If the two roots of the characteristic roots are identical ( $\lambda_1 = \lambda_2$ ), they can be both denoted as  $\lambda$  and the general solution of the difference function is

$$a_n = (c_1 + c_2 n) \lambda^n$$

The proof is not difficult, so you can work on it by yourself.

### Method 3: By Using the Generating Functions

Until now we have not used the calculus method. To confirm that calculus is really really useful, we decide to use the series theory to solve the equation.

We first consider a series (here we assume  $a_0$  exists)

$$S(x) = \sum_{n=0}^{\infty} a_n x^n$$

where  $S(x)$  is called *generating function*.

Since  $a_{n+2} + pa_{n+1} + qa_n = 0 (n \geq 0)$

$$\begin{aligned} (qx^2 + px + 1)S(x) &= qx^2 S(x) + px S(x) + S(x) \\ &= qx^2 \sum_{n=0}^{\infty} a_n x^n + px \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} qa_n x^{n+2} + \sum_{n=0}^{\infty} pa_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} qa_n x^{n+2} + \sum_{n=0}^{\infty} pa_{n+1} x^{n+2} + \sum_{n=0}^{\infty} a_{n+2} x^{n+2} + pa_1 x + a_1 x + a_0 \\ &= \sum_{n=1}^{\infty} (qa_n + pa_{n+1} + a_{n+2}) x^{n+2} + pa_1 x + a_1 x + a_0 \\ &= pa_1 x + a_1 x + a_0 \end{aligned}$$

Thus,

$$S(x) = \frac{pa_1x + a_1x + a_0}{qx^2 + px + 1}$$

If  $\Delta = p^2 - 4q \neq 0$   $S(x)$  can be decomposed into two partial fractions:

$$S(x) = \frac{A_1}{1 - \lambda_1 x} + \frac{A_2}{1 - \lambda_2 x}$$

Then we use the Taylor's expansion

$$\begin{aligned} S(x) &= \frac{A_1}{1 - \lambda_1 x} + \frac{A_2}{1 - \lambda_2 x} \\ &= A_1 \sum_{n=0}^{\infty} (\lambda_1 x)^n + A_2 \sum_{n=0}^{\infty} (\lambda_2 x)^n \\ &= \sum_{n=0}^{\infty} (A_1 \lambda_1^n + A_2 \lambda_2^n) x^n \end{aligned}$$

Once again we get the general form of the solution  $a_n = A_1 \lambda_1^n + A_2 \lambda_2^n$  If  $\Delta = p^2 - 4q = 0, S(x)$  can be written as

$$S(x) = \frac{A_1 x + A_2}{(1 - \lambda x)^2}$$

Then by using Taylor's expansion

$$\begin{aligned} S(x) &= \frac{A_1 x + A_2}{(1 - \lambda x)^2} \\ &= A_1 x \sum_{n=1}^{\infty} n(\lambda x)^{n-1} + A_2 \sum_{n=1}^{\infty} n(\lambda x)^{n-1} \\ &= -A_2 + \sum_{n=1}^{\infty} (A_2 + (\frac{A_1}{\lambda_1} + A_2)n) \lambda^n x^n \end{aligned}$$

Still,  $a_n$  is like the form  $(c_1 + c_2 n) \lambda^n$

## Summary

We have introduced the three methods to solve a second-order linear homogeneous difference functions with constant coefficients. Interestingly, all of the methods can be used in solving  $n$ th order linear difference equations (Still, the attributes mentioned above are necessary). If you are interested enough you could have a try.

## Reference

[https://en.wikipedia.org/wiki/Linear\\_difference\\_equation](https://en.wikipedia.org/wiki/Linear_difference_equation)

Richard A. Brualdi, *Introductory Combinatorics*. Madison: Pearson Education, 2012

C. Henry Edwards & David E. Penney, *Elementary Differential Equations*. Jersey: Pearson Education, 2008

**If you find any mistakes, please contact with the author as soon as possible.**