1. Suppose $f(x, y)$ is continuous on $\mathbb{R}^2$ and bounded by M and there exists a constant $L \in (0, 1)$ such that for all x, y_1, y_2 we have

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2|$$

Consider the first order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = f(x, y)$$

(1)Show that $y = \phi(x)$ is a bounded solution of above equation iff $\phi(x)$ is a continuous and bounded solution of

$$\phi(x) = \int_{-\infty}^{x} e^{t-x} f(t, \phi(t))\mathrm{d}t$$

(2)Show that the above first-order differential equation is a unique bounded solution in $(-\infty, +\infty)$

2. Show that the homogeneous equation $P(x, y)\mathrm{d}x + Q(x, y)\mathrm{d}y = 0$ has integrating factor

$$\rho = \frac{1}{xP(x, y) + yQ(x, y)}$$

And solve the following differential equation using two different methods.

$$xy\mathrm{d}x - (x^2 + y^2)\mathrm{d}y = 0$$

3. Consider the following inequality

$$F'(t) \geq t^2 F^3(t) \quad F(t) \geq t, \quad \forall t \geq 1$$

Does there exist a function $F(t)$ satisfying the above inequality such that $F(t)$ is continuously differentiable on $[1. + \infty)$? If your answer is yes, please give the solution. If your answer is no, prove it.

4. Suppose $f(x, y)$ is continuously differentiable on $(-\infty, +\infty)$ solve the differential equation

$$y(1 + y^2 f(xy))\mathrm{d}x + x(y^2 f(xy) - 1)\mathrm{d}y = 0 \, (y > 0)$$

Show that any solution curve of the differential equation has at most one intersection point with $xy = 2$.

5. Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 + y^2 + \sin x \quad y(x_0) = y_0$$

where $(x_0, y_0) \in \mathbb{R}^2$. Denote $J^+ = [x_0, b)$ is the maximum interval of definition on the right side of x_0. Show that b is finite.

6. Solve at least four of the following differential equations:

(1) $(x^3y - 3x^2y + y^3)\mathrm{d}x + 2x^3\mathrm{d}y$=0

(2) $e^x \sin^3 y\mathrm{d}x + (1 + e^{2x}) \cos y\mathrm{d}y$=0

(3) $(x + y^3)\mathrm{d}x + 3(y^3 - x)y^2\mathrm{d}y = 0$

(4) (bonus) $y' = \frac{1}{2}\sqrt{x} + \sqrt[3]{y}$

(5) $(e^y - y')x = 2$

(6) $2xy\ln y\mathrm{d}x + (x^2 + y^2\sqrt{y^2+1})\mathrm{d}y = 0$

(7) $p^3 - x^3(1-p) = 0$