Solutions to Second-Order Linear Homogeneous Difference Functions with Constant Coefficients

Explanations of the Terms

We call a function *a difference function* if it is one of the form(we use the notations we have learned in the middle school)

$$a_n = f(a_1, a_2, \dots, a_{n-1})$$

A kth order difference function is one that can be written as

$$a_{n+k} = f(a_n, a_{n+1}, \dots, a_{n+k-1})$$

where the range between the maximum index and the minimum index is k.

Thus, a *second-order* difference function is one of the form

$$a_{n+2} = f(a_{n+1}, a_n)$$

The equation is called *linear* if it can be written as

$$a_{n+2} = b_{n+1}a_{n+1} + b_n a_n + t_n$$

where b_{n+1}, b_n, t_n may be a function of n.

If $t_n = 0$ we call the equation a *homogeneous* equation. Otherwise,it is called *inhomogeneous*.

If the coefficients b_{n+1}, b_n are constant which means that b_{n+1}, b_n is not related to the index n, we say the function is with *constant coefficients*.

Therefore, what we want to discuss in this article is a kind of difference equations which is one of the form

$$a_{n+2} = pa_{n+1} + qa_n$$

or we transform into the form (you will know why I want to do this in the following paragraphs)

$$a_{n+2} + pa_{n+1} + qa_n = 0$$

where $q \neq 0$(otherwise it won't be a second-order difference equation)

And if the function has an additional condition like $a_1 = 1, a_2 = 1$ so that we can solve the unique function ,we call the conditions *initial conditions*.

And now we can set about solving it.

Method 1: By Using the Geometric Sequences

Obviously, $a_n = 0$ is one of the solutions which satisfy the equation. But it may be a little boring and even useless if we only stare at the zero solution. What we mainly concern about is the general solution

We first consider a geometric sequence which has the form $\{a_{n+1} - ta_n\}$ and satisfies the equation. Suppose the common ratio is r and , so we have

$$a_{n+2} - ta_{n+1} = r(a_{n+1} - ta_n)$$

i.e.

$$a_{n+2}+(t+r)a_{n+1}+tra_n=0$$

So according to the Vieta's formulas we know that t and r are the solutions of the equation

$$\lambda^2+p\lambda+q=0$$

We can easily solve the equation so that we get

$$\lambda_{1,2}=\frac{-p\pm\sqrt{p^2-4q}}{2}$$

Let $b_n=a_{n+1}-ta_n$(t can be either λ_1 or λ_2).Then $\{b_n\}$ is a geometric sequence. Without loss of generality we let $b_n=a_{n+1}-\lambda_1 a_n$ then $b_{n+1}=\lambda_2 b_n$ so

$$a_{n+1}-\lambda_1 a_n=b_n=\lambda_2^{n-1}b_1$$

Since $q\neq 0$,$\lambda_1,\lambda_2\neq 0$ then we have

$$\frac{a_{n+1}}{\lambda_1^{n+1}}-\frac{a_n}{\lambda_1^n}=\frac{1}{\lambda_2}(\frac{\lambda_2}{\lambda_1})^n b_1$$

Thus

$$\frac{a_n}{\lambda_1^n}=\frac{a_{n-1}}{\lambda_1^{n-1}}+\frac{1}{\lambda_2}(\frac{\lambda_2}{\lambda_1})^{n-1}b_1=\cdots=\frac{a_1}{\lambda_1}+\frac{b_1}{\lambda_2}((\frac{\lambda_2}{\lambda_1})^{n-1}+(\frac{\lambda_2}{\lambda_1})^{n-2}+\cdots+(\frac{\lambda_2}{\lambda_1}))$$

If $\lambda_1\neq\lambda_2$ which means $\Delta=p^2-4q\neq 0$,we can get

$$\frac{a_n}{\lambda_1^n}=\frac{a_1}{\lambda_1}+\frac{b_1}{\lambda_2}\frac{\lambda_2/\lambda_1(1-(\lambda_2/\lambda_1)^{n-1})}{1-\lambda_2/\lambda_1}=\frac{a_1}{\lambda_1}+\frac{b_1(1-(\lambda_2/\lambda_1)^{n-1})}{\lambda_1(\lambda_1-\lambda_2)}$$

Noticing that $b_1=a_2-\lambda_1 a_1$.Thus,

$$a_n=a_1\lambda_1^{n-1}+\frac{(a_2-\lambda_1 a_1)(\lambda_1^n-\lambda_1\lambda_2^{n-1})}{\lambda_1(\lambda_1-\lambda_2)}=\frac{a_2-\lambda_2 a_1}{\lambda_1(\lambda_1-\lambda_2)}\lambda_1^n+\frac{\lambda_1 a_1-a_2}{\lambda_2(\lambda_1-\lambda_2)}\lambda_2^n$$

where $\lambda_1,\lambda_2\in\mathbb{C}$

If $\Delta=p^2-4q>0$,then $\lambda_1,\lambda_2\in\mathbb{R}$.Thus,the solution is $c_1\lambda_1^n+c_2\lambda_2^n$,where c_1,c_2 can be solved by the initial conditions(This proof is not strict enough,we will discuss a more strict one in solution 2).

If $\Delta=p^2-4q<0$ then $\lambda_1,\lambda_2\in\mathbb{C}$ and can be written as $a\pm bi(a,b\in\mathbb{R})$.So by using the De Moivre's formula we can get

$$\begin{aligned}a_n&=c_1\lambda_1^n+c_2\lambda_2^n\\&=c_1(a+bi)^n+c_2(a-bi)^n\\&=c_1(\sqrt{a^2+b^2}(\cos\theta+i\sin\theta))^n+c_2((\sqrt{a^2+b^2}(\cos\theta-i\sin\theta))^n)\\&=M^n(c_1(\cos n\theta+i\sin n\theta)+c_2(\cos n\theta-i\sin n\theta))\\&=M^n((c_1+c_2)\cos n\theta+i(c_1-c_2)\sin n\theta)\end{aligned}$$

where $M=\sqrt{a^2+b^2}$. And the equality means that the solution is a form of $a_n=M^n(d_1\cos n\theta+d_2\sin n\theta)$ And we can solve d_1,d_2 by the initial conditions.

If $\lambda_1=\lambda_2=\lambda$ i.e. $\Delta=p^2-4q=0$ then

$$\frac{a_n}{\lambda^n}=\frac{a_1}{\lambda}+\frac{b_1}{\lambda}(n-1)=(\frac{a_1}{\lambda}-\frac{b_1}{\lambda})+\frac{b_1}{\lambda}n=\frac{(1+\lambda)a_1-a_2}{\lambda}+\frac{a_2-\lambda a_1}{\lambda}n$$

Thus,

$$a_n = (\frac{(1+\lambda)a_1 - a_2}{\lambda} + \frac{a_2 - \lambda a_1}{\lambda} n)\lambda^n$$

which means that the solution is the form $a_n = (c_1 + c_2 n)\lambda^n . c_1, c_2$ can be solved by using the initial conditions.

A Better Method to Calculate

When we get the formula $a_{n+1} - \lambda_1 a_n = b_n = \lambda_2^{n-1} b_1 = \lambda_2^{n-1}(a_2 - \lambda_1 a_1)$, we can get another similar equality

$$a_{n+1} - \lambda_2 a_n = \lambda_1^{n-1}(a_2 - \lambda_2 a_1)$$

Thus when $\lambda_1 \neq \lambda_2$

$$a_n = \frac{\begin{vmatrix} 1 & \lambda_2^{n-1}(a_2 - \lambda_1 a_1) \\ 1 & \lambda_1^{n-1}(a_2 - \lambda_2 a_1) \end{vmatrix}}{\begin{vmatrix} 1 & -\lambda_1 \\ 1 & -\lambda_2 \end{vmatrix}}$$

$$= \frac{\lambda_1^{n-1}(a_2 - \lambda_2 a_1) - \lambda_2^{n-1}(a_2 - \lambda_1 a_1)}{\lambda_1 - \lambda_2}$$
$$= \frac{a_2 - \lambda_2 a_1}{\lambda_1(\lambda_1 - \lambda_2)} \lambda_1^n + \frac{\lambda_1 a_1 - a_2}{\lambda_2(\lambda_1 - \lambda_2)} \lambda_2^n$$

And we get the same answer but the amount of calculation is highly declined!

Method 2: By Using the Characteristic Equation and Roots

After seeing solution 1, you probably have the view that the calculation is so complex and easy to make mistakes(maybe the author himself has some) and it's better if there is a simple solution which can decrease the amount of calculation and decline the error rate. Therefore, this method comes into being.

We first guess(I don't know the thought of the man who came up with the idea, but we can just follow his steps) the solution has the form $a_n = \lambda^n$ And we substitute it in the equation we will obtain

$$\lambda^{n+2} + p\lambda^{n+1} + q\lambda^n = \lambda^n(\lambda^2 + p\lambda + q) = 0$$

We suppose that $\lambda \neq 0$ then we can get $\lambda^2 + p\lambda + q = 0$ This equation is what we have seen in method 1, but now the equation has its new meaning that the solution of which is directly the solution of the initial difference equation. We call the equation $\lambda^2 + p\lambda + q = 0$ the *characteristic equation* of the difference equation $a_{n+2} + pa_{n+1} + qa_n = 0$. And the solution of characteristic equation is called *characteristic roots*.

Now we have the question: in most cases, the equation $\lambda^2 + p\lambda + q = 0$ has two distinct roots, do both of roots consist of the general solution? You may find the answer in the lemma as follows:

Lemma: the Superposition for Homogeneous Equations

Lemma: If $a_n^{(1)}, a_n^{(2)}$ are the solutions of the difference equation $a_{n+2} + pa_{n+1} + qa_n = 0$, then $c_1 a_n^{(1)} + c_2 a_n^{(2)}$ is also the solution of the equation.

Proof: Based on the conditions, we have

$$a_{n+2}^{(1)}+pa_{n+1}^{(1)}+qa_n^{(1)}=0 \qquad a_{n+2}^{(2)}+pa_{n+1}^{(2)}+qa_n^{(2)}=0$$

Then

$$\begin{aligned}&c_1a_{n+2}^{(1)}+c_2a_{n+2}^{(2)}+p(c_1a_{n+1}^{(1)}+c_2a_{n+1}^{(2)})+q(c_1a_n^{(1)}+c_2a_n^{(2)})\\=&c_1(a_{n+2}^{(1)}+pa_{n+1}^{(1)}+qa_n^{(1)})+c_2(a_{n+2}^{(2)}+pa_{n+1}^{(2)}+qa_n^{(2)})\\=&0\end{aligned}$$

Q.E.D.

Now a new question occurs what is the form of the general solution. Take the equation $a_{n+2}-3a_{n+1}+2a_n=0$ for example, which is the general solution of it, $a_n=c_1 1^n$, $a_n=c_1 2^n+c_2 2^n$ or else? Theorem 1 tells the answer.

Theorem 1: Solutions for Distinct Characteristic Roots

Theorem: If λ_1,λ_2 are the distinct roots(which means $\Delta=p^2-4q\neq 0$) of the characteristic equation $\lambda^2+p\lambda+q=0$ then the general solution of the difference equation is

$$a_n=c_1\lambda_1^n+c_2\lambda_2^n$$

where c_1,c_2 are constants, $\lambda_1,\lambda_2\in\mathbb{C}$

Proof: Based on the lemma, $a_n=c_1\lambda_1^n+c_2\lambda_2^n$ is the solution of difference equation $a_{n+2}+pa_{n+1}+qa_n=0$

Thus, what we should prove is that given any initial value a_1 and a_2 we can get the unique specific solution of the difference equation.

Given the initial value a_1 and a_2 we can get the linear systems:

$$\begin{cases}c_1\lambda_1+c_2\lambda_2=a_1\\c_1\lambda_1^2+c_2\lambda_2^2=a_2\end{cases}$$

Since $\lambda_1\neq\lambda_2$ the determinant of coefficient

$$\begin{vmatrix}\lambda_1 & \lambda_2\\ \lambda_1^2 & \lambda_2^2\end{vmatrix}\neq 0$$

Thus, the linear systems has the unique solution c_1,c_2. Therefore,the general solution of the equation is $a_n=c_1\lambda_1^n+c_2\lambda_2^n$. Q.E.D.

We still have one thing to do: what if $\Delta=p^2-4q=0$. Still, there is a theorem providing us the answer:).

Theorem 2: Solutions for Duplicate Characteristic Roots

Theorem: If the two roots of the characteristic roots are identical ($\lambda_1=\lambda_2$) ,they can be both denoted as λ and the general solution of the difference function is

$$a_n=(c_1+c_2n)\lambda^n$$

The proof is not difficult, so you can work on it by yourself.

Method 3: By Using the Generating Functions

Until now we have not used the calculus method. To confirm that calculus is really really useful, we decide to use the series theory to solve the equation.

We first consider a series(here we assume a_0 exists)

$$S(x) = \sum_{n=0}^{\infty} a_n x^n$$

where $S(x)$ is called *generating function*.

Since $a_{n+2} + pa_{n+1} + qa_n = 0(n \geq 0)$

$$\begin{aligned}
(qx^2 + px + 1)S(x) &= qx^2 S(x) + pxS(x) + S(x) \\
&= qx^2 \sum_{n=0}^{\infty} a_n x^n + px \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} qa_n x^{n+2} + \sum_{n=0}^{\infty} pa_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} qa_n x^{n+2} + \sum_{n=0}^{\infty} pa_{n+1} x^{n+2} + \sum_{n=0}^{\infty} a_{n+2} x^{n+2} + pa_1 x + a_1 x + a_0 \\
&= \sum_{n=1}^{\infty} (qa_n + pa_{n+1} + a_{n+2}) x^{n+2} + pa_1 x + a_1 x + a_0 \\
&= pa_1 x + a_1 x + a_0
\end{aligned}$$

Thus,

$$S(x) = \frac{pa_1 x + a_1 x + a_0}{qx^2 + px + 1}$$

If $\Delta = p^2 - 4q \neq 0$ $S(x)$ can be decomposed into two partial fractions:

$$S(x) = \frac{A_1}{1 - \lambda_1 x} + \frac{A_2}{1 - \lambda_2 x}$$

Then we use the Taylor's expansion

$$\begin{aligned}
S(x) &= \frac{A_1}{1 - \lambda_1 x} + \frac{A_2}{1 - \lambda_2 x} \\
&= A_1 \sum_{n=0}^{\infty} (\lambda_1 x)^n + A_2 \sum_{n=0}^{\infty} (\lambda_2 x)^n \\
&= \sum_{n=0}^{\infty} (A_1 \lambda_1^n + A_2 \lambda_2^n) x^n
\end{aligned}$$

Once again we get the general form of the solution $a_n = A_1 \lambda_1^n + A_2 \lambda_2^n$ If $\Delta = p^2 - 4q = 0$,$S(x)$ can be written as

$$S(x) = \frac{A_1 x + A_2}{(1 - \lambda x)^2}$$

Then by using Taylor's expansion

$$\begin{aligned}
S(x) &= \frac{A_1 x + A_2}{(1 - \lambda x)^2} \\
&= A_1 x \sum_{n=1}^{\infty} n(\lambda x)^{n-1} + A_2 \sum_{n=1}^{\infty} n(\lambda x)^{n-1} \\
&= -A_2 + \sum_{n=1}^{\infty} (A_2 + (\frac{A_1}{\lambda_1} + A_2)n) \lambda^n x^n
\end{aligned}$$

Still, a_n is like the form $(c_1 + c_2 n)\lambda^n$

An Example: Fibonacci Sequence

Fibonacci Sequence is a sequence $\{f_n\}$ given the recurrence relation and initial values as follows:

$$f_n = f_{n-1} + f_{n-2} \quad f_1 = 1 \quad f_2 = 1$$

The characteristic function of the equation is

$$\lambda^2 - \lambda - 1 = 0$$

By solving this equation we can get characteristic roots

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

Thus the general solution is

$$f_n = c_1 (\frac{1+\sqrt{5}}{2})^n + c_2 (\frac{1-\sqrt{5}}{2})^n$$

Then we substitute $a_1 = 1, a_2 = 1$ in the equation and get

$$\begin{cases} c_1(\frac{1+\sqrt{5}}{2}) + c_2(\frac{1-\sqrt{5}}{2}) = 1 \\ c_1(\frac{1+\sqrt{5}}{2})^2 + c_2(\frac{1-\sqrt{5}}{2})^2 = 1 \end{cases}$$

Thus, $c_1 = \frac{1}{\sqrt{5}}, c_2 = -\frac{1}{\sqrt{5}}$

So we get the special solution

$$f_n = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$$

Summary

We have introduce the three methods to solve a second-order linear homogeneous difference functions with constant coefficients. Interestingly, all of the method can be used in solving nth order linear difference equations(Still, the attributes mentioned above are necessary). If you are interested enough you could have a try.

Reference

https://en.wikipedia.org/wiki/Linear_difference_equation

Richard A. Brualdi, *Introductory Combinatorics*. Madison: Pearson Education, 2012

C. Henry Edwards & David E. Penney, *Elementary Differential Equations*. Jersey: Pearson Education, 2008

If you find any mistakes or have any suggestion, please contact with the author as soon as possible.