1. [11%] Suppose A and B are measurable sets in $\mathbb{R}^n$, show that

$$\int_{\mathbb{R}^n} m((A-x)\cap B))dx = m(A)m(B)$$

2. [12%] Does convergence in L^2 imply convergence almost everywhere? If yes, prove the conclusion. If no, provide a counter example and show that there exists a subsequence $\{f_{n_j}\}$ such that $f_{n_j} \to f$ almost everywhere if $f_n \in L^2(\mathbb{R})$.

3. [7%+9%] In $C^1[-1,1]$ define

$$||f|| = \left(\int_{-1}^{1} (|f(x)|^2 + |f'(x)|^2)dx\right)^{1/2}$$

   (1) Show that $||\cdot||$ is a norm on $C^1[-1,1]$.

   (2) Can the norm $||\cdot||$ be induced by an inner product? If yes, prove the conclusion and find the inner product. If no, give your reason.

4. [14%] Suppose f and g are nonnegative measurable functions on $(0,1)$, for any $\alpha > 0$, $m(\{f(x) > \alpha\}) = m(\{g(x) > \alpha\})$. Show that

$$\int_0^1 f(x)dx = \int_0^1 g(x)dx$$

5. [12%] Define $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$

$$T(x_1, \ldots, x_n, \ldots) \mapsto (\frac{3x_1 + x_2}{2}, \frac{4x_2 + x_3}{3}, \ldots, \frac{(n+2)x_n + x_{n+1}}{n+1}, \ldots)$$

   Is T a bounded linear operator? If yes, show your conclusion. If no, provide your reason.

6. [10%] $\mu = \mu_1 + \mu_2$ where $\mu_1 = Cauchy(0,1)$ $\mu_2 = Possion(1)$. $\nu = N(1,1)$ find the Lebesgue decomposition of μ w.r.t ν and the Radon-Nikodym derivative $d\mu_a/d\nu$, where $\mu_a \ll \nu$

7. [12%] Suppose f and g are real-valued measurable functions on a measurable set $E \subset \mathbb{R}^d$ with $m(E) < \infty$ Define a metric(Hint: You can use the properties of a metric directly without proof)

$$\rho(f,g) = \int_E \frac{|f-g|}{1+|f-g|}$$

   Then $\rho(f_n, f) \to 0$ if and only if f_n converges to f in measure on E

8. [13%] Is $L^p(X)$ a Banach space? If yes, prove the result. If no, find a counter example for some p. You should consider the two cases: (1) $p > 0$ (2) $p \geq 1$.

9. [Bonus 10%] Let $A = (a_{ij})_{n\times n}$ be a real positive definite matrix. Compute $\int_{\mathbb{R}^n} f(x_1, x_2, \ldots, x_n)dx$ where $f(x_1, x_2, \ldots, x_n) = \exp(-\frac{1}{2}\sum_{i,j=1}^n a_{ij}x_ix_j)$. [Hint: Cartan-Dieudonne Theorem: Every orthonormal transformation in Euclidean space of dimension n can be decomposed as a product of at most n reflections, where identity mapping is seen as the product of 0 reflection. You can use this conclusion directly without any proof.]