NAME : Himanshu Dixit

ENROLL NO. : 21103262

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Mathematics - II (15B11MA211)

Tutorial 6

1\. (i) $\int_{-1}^{1} x^{2m} P_n dx = 0$ when n is odd

L.H.S $\int_{-1}^{1} x^{2m} P_n dx = \int_{-1}^{1} x^{2m} \frac{1}{2^n n!} \frac{d}{dx^n}(x^2-1)^n dx$

$$= \frac{1}{2^n n!} \int_{-1}^{1} \underbrace{x^{2m}}_{I} \underbrace{\frac{d^n}{dx^n}(x^2-1)^n}_{II} dx$$

$$\int_{-1}^{1} x^{2m} P_n dx = \frac{1}{2^n n!} \left[\left\{ x^{2m} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n \right\}_{-1}^{1} - \int_{-1}^{1} 2m x^{2m-1} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n dx \right.$$

$$= 0 - \frac{2m}{2^n n!} \int_{-1}^{1} x^{2m-1} \frac{d^{n-1}}{dx^{n-1}} (x^2-1)^n dx$$

$$= -[(-1) \frac{2m(2m-1)}{2^n n!} \int_{-1}^{1} x^{2m-2} \frac{d^{n-2}}{dx^{n-2}} (x^2-1)^n dx$$

integrating (2m-2) times, we get,

$$= (-1)^{2m} \frac{2m(2m-1)\dots 1}{2^n n!} \int_{-1}^{1} \frac{d^{n-2m}}{dx^{n-2m}} (x^2-1)^n dx$$

$$= \frac{(-1)^{2m} (2m)!}{2^n n!} \int_{-1}^{1} \frac{d^{n-2m}}{dx^{n-2m}} (x^2-1)^n dx$$

$$= \frac{(-1)^{2m} 2m!}{2^n n!} \left[\frac{d^{n-2m-1}}{dx^{n-2m-1}} (x^2-1)^n \right]_{-1}^{1} = 0$$

L.H.S = R.H.S

(ii) $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$

$$(x^2-1)^n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} x^{2n-2k}$$

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$$P_n(x) = \frac{1}{2^n n!} \sum_{k=0}^{N} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2n-2k)!}{(n-2k)!} x^{n-2k}$$

Put $n = 2m$, As n is even, $N = \frac{1}{2}n \rightarrow N = m$

$$P_{2m}(x) = \frac{1}{2^{2m}(2m)!} \sum_{k=0}^{m} \frac{(-1)^k (2m)!}{k!(2m-k)!} \frac{(4m-2k)!}{(2m-2k)!} x^{2(m-k)}$$

Here, we can see that we have $x^{2(m-k)}$ terms, so this means that only even terms will come, Hence $P_{2m}(x)$ has only even degree terms. Hence Proved

(iii) $P_n(x) = \frac{1}{n!2^n} \frac{d^n}{dx^n} (x^2-1)^n$

at $n=1$

$P_1(x) = x$

at $n=3$

$P_3(x) = \frac{1}{2}(5x^3-3x)$

So we can see that when n is odd, $P_n(x)$ always has 'x' common, so $x=0$ is always a root of $P_n(x)$ when n is odd, Hence Proved.

(iv) $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n$

$$P_4(x) = \frac{1}{384} \frac{d^4}{dx^4} (x^2-1)^4$$

$$= \frac{1}{384} \frac{d^4}{dx^4} (x^8 - 4x^6 + 6x^4 - 4x^2 + 1)$$

$$= \frac{35x^4 - 30x^2 + 3}{8}$$; hence Proved

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(V) To Prove :- $\sum_{r=0}^{m} \frac{(-1)^r (2n-2r)!}{r!(n-r)!(n-2r)!} = 2^n$

we know that, by generating function,

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) z^n$$

Put $x=1$, we get

$$(1-2z+z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(1) z^n$$

$$\frac{1}{1-z} = 1+z^2+z^3+\cdots+z^n+\cdots = \sum_{n=0}^{\infty} P_n(1) z^n$$

$$\boxed{P_n(1) = 1}$$

Now, we know that

$$P_n(x) = \frac{1}{2^n n!} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(2n-2k)!}{(n-2k)!} x^{n-2k}$$

Put $x=1$ on both side, we get

$$\frac{1}{2^n} \sum_{k=0}^{n} \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} = P_n(1)$$

$$\sum_{r=0}^{m} (-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} = 2^n$$

So L.H.S = R.H.S ; Hence Proved

Q.2. we know that,

$$J_n(x) = \sum_{p=0}^{\infty} \frac{(-1)^p (x/2)^{n+2p}}{p! \, \Gamma(n+p+1)}$$

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{v+2n}}{n! \, \Gamma(n+v+1)}$$

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{v+2n}}{n! \, (v+n)!}$$

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To Prove :- $J_v(x)$ converges for all value of x.

n^{th} term of $J_v(x)\,(U_n(x)) = \dfrac{(-1)^n (x/2)^{v+2n}}{n!\,(v+n)!}$

By D'Alembert's Ratio Test,

$$\frac{U_{n+1}}{U_n} = \frac{(-1)^{n+1}\, x^{v+2n+2}}{(n+1)!\,(v+n+1)!\,2^{v+2n+2}} \times \frac{n!\,(v+n)!\,2^{v+2n}}{(-1)^n\,(x)^{v+2n}}$$

$$\frac{U_{n+1}}{U_n} = \frac{x^2}{(n+1)(v+n+1)\,4}$$

So we can see that $\lim\limits_{n\to\infty} \dfrac{U_{n+1}}{U_n} = 0 < 1$ hence.

$J_v(x)$ convergence for all x ; hence Proved

Q.3. (i) To Prove :- $\dfrac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)$

L.H.S $$\frac{d}{dx}(x^n J_n(x)) = \frac{d}{dx}\left(\sum_{p=0}^{\infty} \frac{(-1)^p x^{2n+2p}}{p!\,(n+p)!\,2^{n+2p}}\right)$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^p (2n+2p)}{p!\,(n+p)!\,2^{n+2p}}\, x^{2n+2p-1}$$

$$= \sum_{p=0}^{\infty} \frac{(-1)^p\, 2(n+p)}{p!\,(n+p)!\,2^{n+2p}}\, x^{2n+2p-1}$$

$$= x^n \underbrace{\sum_{p=0}^{\infty} \frac{(-1)^p}{p!\,(n+p-1)!} \left(\frac{x}{2}\right)^{n-1+2p}}_{J_{n-1}(x)} = x^n J_{n-1}(x)$$

R.H.S

; hence Proved

(ii) To Prove :- $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\sin x$

$$J_{1/2}(x) = \sum_{p=0}^{\infty} \frac{(-1)^p (x/2)^{1/2+2p}}{p!\,(p+\frac{1}{2})!}$$

$$= \frac{(x/2)^{1/2}}{\lceil 3/2} - \frac{(x/2)^{1/2+2}}{1!\,\lceil 5/2} + \frac{(x)^{1/2+4}}{2!\,\lceil 7/2} - \cdots\cdots$$

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$$= \left(\frac{x}{2}\right)^{1/2} \frac{1}{x} \left[\frac{x}{\frac{1}{2}\sqrt{\pi}} - \frac{(x/2)^2 x}{1!\, 3/2\, 1/2\sqrt{\pi}} + \frac{(x/2)^4 x}{2!\, 5/2\, 3/2\, 1/2\sqrt{\pi}} - \cdots \right]$$

$$= \left(\frac{2}{\pi x}\right)^{1/2} \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right] = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{R.H.S}$$

; hence Proved

(iii) we know that,

$$e^{x(t-1/t)/2} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$$

$t = \cos\theta + i\sin\theta$, $1/t = \cos\theta - i\sin\theta$

$$e^{x(2i\sin\theta)/2} = e^{xi\sin\theta} = \cos(x\sin\theta) + i\sin(x\sin\theta)$$

$$= J_0(x) + J_1(x)t + J_{-1}(x)\frac{1}{t} + J_2(x)t^2 + J_{-2}(x)\frac{1}{t^2} + \cdots$$

$$= J_0(x) + J_1(x)(2i\sin\theta) + J_2(x)(2\cos 2\theta) + J_3(2\sin 3\theta) + \cdots$$

So, by this by comparing real part on both sides, we get

$$\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \cdots$$

put $\theta = \pi/2$, we get

$$\cos\theta = J_0(x) + 2J_2(x) - 2J_4(x) + \cdots$$

hence, $\cos\theta = J_0(x) + 2J_2(x) - 2J_4(x) + \cdots$

; hence Proved

(iv) To Prove :- $\int_0^x t^n J_{n-1}(t)\,dt = x^n J_n(x)$ — (1)

we know that ;

$$x^n J_{n-1}(x) = \frac{d}{dx}\left(x^n J_n(x)\right) \quad \text{— (2)}$$

So taking L.H.S from eq^n (1),

$$\int_0^x t^n J_{n-1}(t)\,dt = \int_0^x \frac{d}{dt}\left(t^n J_n(t)\right)dt = \Big[t^n (J_n(t))\Big]_0^x$$

$$= x^n J_n(x)$$; hence Proved

(V) To Prove :- $J_0'' = \frac{J_2 - J_0}{2}$

we know that,

$$2J_n' = J_{n-1} - J_{n+1} \quad \text{— (1)}$$
$$xJ_n' = nJ_n - xJ_{n+1} \quad \text{— (2)}$$
$$xJ_n' = xJ_{n-1} - nJ_n \quad \text{— (3)}$$

So, $2J_n' = J_{n-1} - J_{n+1}$ —

differentiating both sides w.r.t x, we get

$$2J_n'' = J'_{n-1} - J'_{n+1} \quad \text{— (4)}$$

$$2J_n'' = \frac{(n-1)J_{n-1}}{x} - \frac{xJ_n}{x} - \left[\frac{-(n+1)J_{n+1} + xJ_n}{x}\right]$$

$$2J_n'' = \frac{(n-1)J_{n-1} + (n+1)J_{n+1}}{x} - 2J_n \quad \left(\text{diff. (2) \& (3) \& put in (4)}\right)$$

put $n = 0$, we get

$$2J_0'' = \frac{-J_{-1} + J_1}{x} - 2J_0$$

$$2J_0'' = \frac{2J_1}{x} - 2J_0 \quad \left(J_{-n} = (-1)^n J_n\right)$$

$$2J_0'' = \frac{2J_1 - xJ_0 - xJ_0}{x} \quad \text{— (5)}$$

we know that,

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

here, put $n = 1$, we get

$$J_2 = \frac{2J_1}{x} - J_0$$

put value of J_1 from here to eq^n (5), we get

$$2J_0'' = \underbrace{\left[\frac{2J_1}{x} - J_0\right]}_{J_2} - J_0$$

$$2J_0'' = J_2 - J_0$$

$$J_0'' = \frac{J_2 - J_0}{2} \quad \text{; hence Proved}$$