NAME: Himanshu Dixit

BATCH: B10

ENROLL NO.: 21103262

Date ____

Page ____

Mathematics - 2 Tutorial Sheet - 2

Q.1. (i) $y'' + y = \sec x$

for CF: characterstic eq: $m^2+1 = 0$

$m = \pm i$

$\therefore \; y = A\cos x + B\sin x$

part of CF: is $\cos x$

So, Solution is $y = uv$, $u = \cos x$, $p=0$, $q=1$

$R = \sec x$

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + 0 + \frac{2}{\cos x}(-\sin x)\frac{dv}{dx} = \frac{\sec x}{\cos x}$$

$$\frac{d^2v}{dx^2} + (-2\tan x)\frac{dv}{dx} = \sec^2 x$$

$$\therefore \; I.F = e^{\int -2\tan x\, dx} = \cos^2 x$$

$$\frac{dv}{dx} \times \cos^2 x = \int 1 \, \cancel{\sec x} \, dx + C$$

$$\frac{dv}{dx} \times \cos^2 x = \cancel{\sec x \tan x} \; x + C$$

$$\frac{dv}{dx} = x\sec^2 x + C\sec^2 x$$

$$v = \int x\sec^2 x \, dx + C\int \sec^2 x \, dx$$

$$v = x\tan x - \log|\sec x| + C\tan x + C'$$

$$\therefore \; y = uv$$

$$y = x\sin x - \cos x \log|\sin x| + C\sin x + C'\cos x$$ Ans

(ii) $x^2y'' + xy' - y = 2x^2$

for CF : characterstic eq^n is : ?

On solving by cauchy euler method : $x = e^z$

$(D_1(D_1-1) + D_1 - 1)y = 2e^{2z}$

$(D_1^2 - 1)y = 2e^{2z}$

characterstic eq : $m^2 - 1 = 0$

$m = \pm 1$ $\therefore$ $y = Ae^z + Be^{-z}$

$y = Ax + \frac{B}{x}$

$\therefore$ $u = x$:

$y = uv = xv$

$y' = xv' + v$

$y'' = xv'' + v' + v'$

$x^2y'' + xy' - y = x^3v'' + x^2v' + x^2v' + x^2v' + \cancel{xv} - \cancel{xv} = 2x^2$

$x^3v'' + (2x^2 + x^2)v' = 2x^2$

$\therefore$ $I \cdot f = e^{\int \frac{3x^2}{x^3} dx} = e^{\int \frac{3}{x} dx} = x^3$

$v' \times x^3 = \int 2x^2 \, dx + c$

$v' x^3 = \frac{2}{3}x^3 + c$

$v' = \frac{2}{3} + \frac{c}{x^3}$

$v = \frac{2x}{3} + \frac{-c}{2x^2} + c'$

$\therefore$ $y = \frac{2}{3}x^2 - \frac{c}{2x} + c'x$ Ans

Date
Page

(iii) $xy'' + (1-x)y' - y = e^x$

$$y'' + \left(\frac{1}{x}-1\right)y' + \left(\frac{-1}{x}\right)y = e^x$$

$\therefore$ $1+P+Q = \cancel{1}+\cancel{\frac{1}{x}}-\cancel{1}-\cancel{\frac{1}{x}} = 0$ $\therefore u = e^x$

$\therefore$ $y = uv = e^x v$

$y' = e^x v' + v e^x$

$y'' = e^x v'' + v' e^x + v e^x + e^x v'$

$xy'' + (1-x)y' - y = xe^x v'' + 2xe^x v' + xe^x v +$
$(1-x)e^x v' + (1-x)e^x v - y = e^x$

$= xe^x v'' + xe^x v' + e^x v' + \cancel{e^x} v - \cancel{e^x} v = e^x$

$xv'' + xv' + v' = 1$

$$v'' + \left(\frac{1}{x}+1\right)v' = \frac{1}{x}$$

$$I.f = e^{\int \frac{1}{x}+1\,dx} = e^{\log x + x} = x \cdot e^x$$

$$v' \times xe^x = \int e^x\,dx$$

$v' \cdot x\, xe^x = e^x + c$

$$\int dv = \int \frac{1}{x}\,dx + c\int \frac{1}{x} e^{-x}\,dx$$

it can not be further Integrate

$$v = \log x + c\int \frac{1}{xe^x}\,dx$$

$$\therefore y = e^x \log x + ce^x \int \frac{1}{xe^x}\,dx$$ Ans

Date
Page

Q.2. $y'' - 2\tan x\, y' + 8y = e^x \sec x$

$y'' + Py' + Qy = R \quad \therefore P = -2\tan x,\ Q = 8,\ R = e^x \sec x$

$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4} = 8 + \frac{1}{\cancel{2}} \cancel{2}\sec^2 x - \tan^2 x = 9$

$I = \text{const}$ ✓ $\left(\text{or } \frac{\text{const}}{x^2}\right)$

$\therefore$ part of C.f. $u = e^{-1/2\int P dx} = e^{-1/2 \cancel{\times 2 \int \tan x\, dx}} = \sec x$

$y = uv = \sec x\, v$

$\frac{d^2v}{dx^2} + Iv = \frac{R}{u}$

$\frac{d^2v}{dx^2} + 9v = \frac{e^x \cancel{\sec x}}{\cancel{\sec x}}$

$(D^2 + 9)v = e^x$

A.E: $m^2 + 9 = 0$

$m = \pm 3i \qquad y = A\cos 3x + B\sin 3x$

PI: $v = \frac{e^x}{D^2+9} = \frac{e^x}{10}$

$v = CF + PI$

$= A\cos 3x + B\sin 3x + \frac{e^x}{10}$

$y = uv = \sec x \left[A\cos 3x + B\sin 3x + \frac{e^x}{10} \right]$

Q.3. $xy'' - y' - 4x^3 y = 8x^3 \sin x^2$

$y'' + Py' + Qy = R$

changing independent variable x to z by Relation $z = f(x)$.

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}, \quad Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}, \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$$P = \frac{-1}{x}, \quad Q = -4x^2, \quad R = 8x^2 \sin x^2$$

let $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = -4$ (constant of Q)

$$\cancel{-4}x^2 = \cancel{-4}\left(\frac{dz}{dx}\right)^2$$

$$\boxed{z = \frac{x^2}{2}}$$

$$P_1 = \frac{1 + \left(\frac{-1}{\cancel{x}}\right)\left(\frac{\cancel{2x}}{\cancel{2}}\right)}{\frac{\cancel{4}x^2}{\cancel{4}}} = 0, \quad Q_1 = -4, \quad R_1 = \frac{\cancel{8x^2}\sin x^2}{\frac{\cancel{4x^2}}{4}}$$

$$R_1 = 8\sin x^2$$

NOW; $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$

$$\frac{d^2y}{dz^2} + 0 - 4y = 8\sin 2z$$

$$(D^2-4)y = 8\sin 2z$$

CF: $m^2 - 4 = 0 \quad \therefore \quad m = \pm 2$

$$y = Ae^{2z} + Be^{-2z}$$
$$= Ae^{x^2} + Be^{-x^2}$$

PI: $y = \frac{8\sin 2z}{(D^2-4)} = -\sin 2z = -\sin x^2$

$$y(x) = CF + PI = Ae^{x^2} + Be^{-x^2} - \sin x^2 \quad \text{Ans}$$

Date ______
Page ______

Q.4. $y'' - 6y' + 9y = \frac{e^{3x}}{x}$

$y'' + Py' + Qy = R$

CF : $y = C_1 u + C_2 v$

Complete solⁿ : $y = Au + Bv$

then : $A'u + B'v = 0$ — ①
$A'u' + B'v' = R$ — ② $\rightarrow A', B'$ ✓

on integrating we find A & B ✓

$(D^2 - 6D + 9)y = e^{3x}$

CF : $m^2 - 6m + 9 = 0$

$(m-3)^2 = 0 \rightarrow m = 3, 3$

$y = C_1 e^{3x} + C_2 x e^{3x}$

Complete solⁿ : $y = Ae^{3x} + Bxe^{3x}$

$A'u + B'v = 0 \rightarrow A'e^{3x} + B'xe^{3x} = 0$ — ①

$A'u' + B'v' = e^{3x} \rightarrow A'\frac{e^{3x}}{3} + B'\frac{(xe^{3x} + e^{3x})}{3} = \frac{e^{3x}}{\cancel{3x}}$

$\frac{A'e^{3x}}{3} + \frac{B'xe^{3x}}{3} + B'e^{3x} = \frac{e^{3x}}{x}$ — ②

from ① & ② :

$B'e^{3x} = e^{3x}/x$

$B' = 1/x, \quad A' = -1$

$\int dA = \int -1 \, dx = -x + C_1$

$\int dB = \int \frac{1}{x} dx = \log x + C_2$

$y = Au + Bv = Ae^{3x} + Bxe^{3x}$

$= -xe^{3x} + C_1 e^{3x} + xe^{3x}\log x + xe^{3x} C_2$

$= \cancel{\frac{e^{3x}}{4}} \cancel{+ x C_1 e^{3x}} \cancel{+ x C_2 x e^{3x}}$ Ay

$e^{3x}(C_1 + xC_2) + xe^{3x}(\log x - 1)$ ✓

Date ____
Page ____

Q.5. $(x^2+1)y'' - 2xy' + 2y = 6(x^2+1)^2$

$$y'' - \frac{2x}{x^2+1}y' + \frac{2}{x^2+1}y = 6(x^2+1)$$

$$P = \frac{-2x}{x^2+1}, \quad Q = \frac{2}{x^2+1}, \quad R = 6(x^2+1)$$

for homogeneous diff. eq'n :

$$y'' - \frac{2x}{x^2+1}y' + \frac{2}{x^2+1}y = 0 \quad \text{—— ①}$$

$P + Qx = 0 \quad \therefore \quad u = x$

$\therefore$ Solution of $y = uv = xv$

$$y' = xv' + v$$

$$y'' = xv'' + 2v'$$

$$xv'' + 2v' - \frac{2x^2}{x^2+1}\cancel{y'} - \cancel{\frac{2xv}{x^2+1}} + \cancel{\frac{2xv}{x^2+1}} = 0$$

$$xv'' + \frac{2}{x^2+1}v' = 0 \quad \text{—— ②}$$

$$v'' + v'\left(\frac{2}{x(x^2+1)}\right) = 0$$

let $v' = w$

$$\omega' + \omega\left(\frac{2}{x(x^2+1)}\right) = 0$$

$$\frac{d\omega}{dx} = 2\left(\frac{x}{x^2+1} - \frac{1}{x}\right)\omega$$

$$\int\frac{d\omega}{\omega} = 2\int \frac{x}{x^2+1} - \frac{1}{x}\,dx$$

$$\log\omega = 2\left[\frac{1}{2}\log(x^2+1) - \log x\right] + \log a$$

$$\omega = \frac{x^2+1(a)}{x^2}$$

$$\frac{dv}{dx} = \left(1 + \frac{1}{x^2}\right)a$$

Date

Page

$$\int dv = \int a + \frac{a}{x^2}\, dx$$

$$v = a\left(x - \frac{1}{x}\right) + b$$

$$\therefore \quad y = uv$$

$$y = a(x^2-1) + bx$$

hence, two linearly independent solution of $\left(y'' - \frac{2x}{x^2+1} y' + \frac{2}{x^2+1} y = 0 \right)$ are ϕ & ψ :

$\phi = x^2-1$, $\psi = x$. Ans