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Mathematics - 2 Tutorial sheet - 3

Q.1. (a) $\sum \frac{1}{(2n-1)^p} = 1 + \frac{1}{3^p} + \frac{1}{5^p} + \cdots + \frac{1}{(2n-1)^p} \cdots$

$U_n = \frac{1}{(2n-1)^p}$

let $V_n \geq U_n$ $\therefore V_n = \frac{1}{n^p}$

$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{n^p}{(2n-1)^p} = \lim_{n\to\infty} \frac{1}{\left(2-\frac{1}{n}\right)^p} = \frac{1}{2}$ unique & positive

$\therefore \sum V_n = \sum \frac{1}{n^p}$

$\therefore$ $p > 1 \longrightarrow V_n$ is convergent $\longrightarrow U_n$ is also convergent

$p \leq 1 \longrightarrow V_n$ is divergent $\longrightarrow U_n$ is also divergent.

(b) $\sum \left[1+\frac{1}{\sqrt{n}}\right]^{-n^{3/2}}$ $\quad U_n^{1/n} = \left(1+\frac{1}{\sqrt{n}}\right)^{-\cancel{n}\sqrt{n}/\cancel{n}} = \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}}$

By Cauchy's root test in $\sum U_n$:

$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{e^{\cancel{\sqrt{n}}\left(1+\frac{1}{\cancel{\sqrt{n}}}-1\right)}} = \frac{1}{e} < 1$

$\therefore$ U_n is convergent.

(c) $\sum \frac{n^n x^n}{n!}$ By using ~~higher ratio~~ ~~logarithmic~~ Test

$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!} \times \frac{n!}{n^n x^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n x$

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \cdot x = ex \begin{cases} < 1, & x < 1/e \text{ ; convergence} \\ > 1, & x > 1/e \text{ ; divergence} \\ = 1, & x = 1/e \text{ ; Test fail} \end{cases}$

if $x = \frac{1}{e}$ $\therefore \frac{U_n}{U_{n+1}} = \left(\frac{n}{n+1}\right)^n \frac{1}{x}$

$= \frac{e}{\left(1+\frac{1}{n}\right)^n}$

$$\log\left(\frac{u_n}{u_{n+1}}\right) = 1 - n\log\left(1+\frac{1}{n}\right)$$

$$\lim_{n\to\infty} n\left(\log\left(\frac{u_n}{u_{n+1}}\right)\right) = \lim_{n\to\infty} n - n^2\log\left(1+\frac{1}{n}\right)$$

$$\lim_{n\to\infty}\left(n - n^2\left[\frac{1}{n} - \frac{1}{n^2 2} + \frac{1}{n^3 3} + \dots\right]\right)$$

$$\lim_{n\to\infty}\left(\cancel{n} - \cancel{n} + \frac{1}{2} - \frac{1}{3n} + \dots\right) = \frac{1}{2} < 1 \quad \therefore \text{ divergent}$$

$x \geq 1/e$, divergent

$x < 1/e$, convergent

(d) $\sum \frac{1}{n^{(1+1/n)}} = \sum \frac{1}{n \cdot n^{1/n}}$ $\therefore U_n = \frac{1}{n \cdot n^{1/n}}$

let $V_n = \frac{1}{n}$

$\therefore \lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$ (unique & positive)

$\sum V_n = \sum \frac{1}{n^1}$ is divergent as $p = 1$

by comparison test $\sum U_n$ is divergent.

(e) $\sum \frac{x^n}{n(n+1)}$ By using ratio test

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^n}{n\cancel{(n+1)}} \times \frac{\cancel{(n+1)}(n+2)}{x^{n+1}} = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)\frac{1}{x}$$

$$= \frac{1}{x}\begin{cases} <1 , & x>1 ; \text{ divergent} \\ >1 , & x<1 ; \text{ convergent} \\ =1 , & x=1 \end{cases}$$

if $x=1$, $\frac{u_n}{u_{n+1}} = \frac{n+2}{n} \Rightarrow \frac{u_n}{u_{n+1}} - 1 = \frac{2}{n}$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = 2 > 1$$ So convergence at $x=1$

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(f) $\Sigma\left((n^3+1)^{1/3}-n\right) = \Sigma n\left(\left(1+\frac{1}{n^3}\right)^{1/3}-1\right)$

$u_n = n\left(\cancel{1}+\frac{1}{3n^3}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{n^6}+\dots-\cancel{1}\right)$

$= \frac{1}{3n^2}+\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\frac{1}{n^5}+\dots$

let $V_n = \frac{1}{n^2}$

$\lim_{n\to\infty}\frac{u_n}{V_n} = \lim_{n\to\infty}\frac{1}{3}-\frac{2}{9n^3} = \frac{1}{3}$ (uniqe & finite)

$V_n = \frac{1}{n^2}$ $\therefore$ $P>1$ hence, ΣV_n is convergence

$\therefore$ ΣU_n is convergence

(g) $\frac{x}{1}+\frac{1}{2}\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\frac{x^7}{7}+\dots+$

$U_n = \frac{1.3.5\dots(2n-1)}{2.4.6\dots(2n)}\frac{x^{2n+1}}{(2n+1)}$

$U_{n+1} = \frac{1.3.5\dots(2n+1)\,x^{2n+3}}{2.4.6\dots(2n+2)(2n+3)}$

$\frac{U_n}{U_{n+1}} = \frac{(2n+2)}{(2n+1)}\cdot\frac{(2n+3)}{(2n+1)}\frac{1}{x^2} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{2n}\right)^2}\cdot\frac{1}{x^2}$

$$\lim_{n\to\infty}\frac{U_n}{U_{n+1}} = \frac{1}{x^2} = \begin{cases} x^2<1, & \text{convergence} \\ x^2>1, & \text{divergence} \\ x^2=1, & \text{failed} \end{cases}$$

$\lim_{n\to\infty} n\left[\frac{U_n}{U_{n+1}}-1\right] = \lim_{n\to\infty} n\left[\frac{6n+3}{4n^2+4n+1}\right] = \frac{6}{4} = \frac{3}{2} > 1$

hence, convergence

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(h) $\frac{1^2}{4^2} + \frac{5^2}{8^2} + \frac{9^2}{12^2} + \frac{13^2}{16^2} + \dots\dots$

$U_n = \left(\frac{4n-3}{4n}\right)^2 \quad \therefore \quad U_{n+1} = \left(\frac{4n+1}{4(n+1)}\right)^2$

$\frac{U_n}{U_{n+1}} = \left(\frac{4n-3}{4n}\right)^2 \times \left(\frac{4(n+1)}{4n+1}\right)^2$

$\lim_{n\to\infty} \left(\frac{4-3/n}{4+1/n}\right)^2 \times \left(1+\frac{1}{n}\right)^2 = 1$ Test fail.

By ~~Logarithmic~~ Raabe's Test Series :

$\cancel{\lim_{n\to\infty} n \log\left(\frac{U_n}{U_{n+1}}\right) = \lim_{n\to\infty} n\left[2\log(4-3/n) - 2\log(4+1/n) + 2\log(1+1/n)\right]}$

$\cancel{\lim_{n\to\infty} n\left[\log 4\right.}$

$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\left(\frac{4-3/n}{4+1/n}\right)^2\left(1+\frac{1}{n}\right)^2 - 1\right) < 1$

$\therefore$ it is divergent.

(i) $1 + \frac{x}{2} + \frac{2!}{3^2}.x^2 + \frac{3!}{4^3}.x^3 + \frac{4!}{5^4}x^4 + \dots$

$U_n = \frac{(n-1)!x^{(n-1)}}{n^{(n-1)}} = \frac{(n-1)!}{n^{(n-1)}} x^{n-1}$

$U_{n+1} = \frac{n!\, x^n}{(n+1)^n}$

$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{\cancel{n!}^{\,n}\, \cancel{x^n}^{\,x}}{(n+1)^n} \times \frac{n^{(n-1)}}{\cancel{(n-1)!}\, \cancel{x^{n-1}}}$

$\lim_{n\to\infty} \frac{x}{\left(1+\frac{1}{n}\right)^n} = \frac{x}{e} = \begin{cases} <1, & x<e \text{ . Conver} \\ >1, & x>e \text{ diverg} \\ =1, & x=e \text{ fail} \end{cases}$

for $x=e$ using logarithmic Test Series :

$$\lim_{n\to\infty} n\left(\log\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty} n\log\left[\left(1+\frac{1}{n}\right)^n\cdot\frac{1}{x}\right]$$

$$\lim_{n\to\infty} n\left[n\log\left(1+\frac{1}{n}\right)-\log e\right]$$

$$\lim_{n\to\infty} n\left[n\log\left(1+\frac{1}{n}\right)-1\right]$$

$$\lim_{n\to\infty} n^2\left[\log\left(1+\frac{1}{n}\right)\right]-n$$

$$\lim_{n\to\infty} n^2\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\cdots\right]-n$$

$= \frac{-1}{2} < 1$, divergent

$x<e \rightarrow$ conv.
$x \geq e \rightarrow$ div.] Ans

Q.2 (a) $\frac{a^n}{x^n+a^n}$ Case 1: if $x=a$

$u_n = \frac{1}{2} < 1$ $\lim_{n\to\infty} u_n \neq 0$

hence, divergence.

Case 2: if $x/a<1$ $u_n = \frac{1}{\left(\frac{x}{a}\right)^n+1}$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{1}{1+\left(\frac{x}{a}\right)^n} = 1 \neq 0$$

hence, divergence.

~~(b)~~ case 3: if $x/a>1 \Rightarrow a/x<1$

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{(a/x)^n}{1+\left(\frac{a}{x}\right)^n} = 0$$

let $v_n = \left(\frac{a}{x}\right)^n$

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$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{(a/x)^n}{1+(a/x)^n} \times \frac{1}{(a/x)^n} = \lim_{n\to\infty} \frac{1}{1+(a/x)^n}$$

$$= 1$$

Now ΣU_n & ΣV_n cgs & dgs together

$\Sigma V_n = \Sigma \left(\frac{a}{x}\right)^n$ is cgs as $a/x < 1$

$\therefore$ by limit comparison test,

ΣU_n is cgs where $\frac{a}{x} < 1$ or $x > a$ ✓

(b) $\frac{1}{\sqrt{n}+\sqrt{n+1}} = U_n$

let $V_n = \frac{1}{\sqrt{n}}$

$$\frac{U_n}{V_n} = \frac{1}{\sqrt{n}+\sqrt{n+1}} \times \sqrt{n}$$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{1}{1+\sqrt{1+1/n}} = \frac{1}{2} \neq 0$$

ΣV_n divergence $\because p < 1$.

So ΣU_n is also divergence.

(c) $$\frac{\sqrt{n+1}-1}{(n+2)^3-1} = \frac{\sqrt{n}}{n^3}\left(\frac{\sqrt{1+1/n} - 1/\sqrt{n}}{\left(1+\frac{2}{n}\right)^3 - 1/n^3}\right)$$

$$= \frac{1}{n^{5/2}}\left(\frac{\sqrt{1+1/n} - 1/\sqrt{n}}{\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}}\right)$$

$V_n = \frac{1}{n^{5/2}}$ ΣV_n is convergence $p > 1$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{\sqrt{1+1/n} - 1/\sqrt{n}}{\left(1+\frac{2}{n}\right)^3 - \frac{1}{n^3}} = 1 \neq 0$$

ΣU_n is convergence by limit comparison test.

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(d) $\dfrac{(a+nx)^n}{n!} = U_n$ let

$$\frac{U_{n+1}}{U_n} = \frac{(a+(n+1)x)^n \cdot (a+(n+1)x)}{(n+1)!} \times \frac{n!}{(a+nx)^n}$$

$$= \frac{(a+nx+x)^n (a+nx+x)}{(n+1)(a+nx)^n}$$

$$= \frac{(a+nx)^n (n+1)\left(1+\frac{x}{a+nx}\right)^n \left(\frac{a}{n+1}+x\right)}{(n+1)(a+nx)^n}$$

$$= \left(1+\frac{x}{a+nx}\right)^{n+\frac{a}{x}-\frac{a}{x}} \left(x+\frac{a}{n+1}\right)$$

$$= \frac{\left(1+\frac{1}{n+\frac{a}{x}}\right)^{n+\frac{a}{x}} \left(x+\frac{a}{n+1}\right)}{\left(1+\frac{1}{n+\frac{a}{x}}\right)^{a/x}}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n+\frac{a}{x}}\right)^{n+\frac{a}{x}} \left(x+\frac{a}{n+1}\right)}{\left(1+\frac{1}{n+\frac{a}{x}}\right)^{a/x}}$$

$$= ex \begin{cases} <1, & x<1/e \quad \text{cgs} \\ >1, & x>1/e \quad \text{dgs} \\ =1, & x=1/e \quad \text{test fail.} \end{cases}$$

At $x=e$

$$U_n = \frac{\left(a+\frac{n}{e}\right)^n}{n!}$$

$$U_{n+1} = \frac{\left(a+\frac{(n+1)}{e}\right)^{n+1}}{(n+1)!}$$

$$\frac{U_n}{U_{n+1}} = \frac{\left(a+\frac{n}{e}\right)^n}{n!} \times \frac{(n+1)!}{\left(a+\frac{(n+1)}{e}\right)^{n+1}}$$

$$= \left(\frac{n}{e}\right)^n (n+1) \left(1+\frac{ae}{n}\right)^n \Big/ \left(\frac{n+1}{e}\right)^{n+1} \left(1+\frac{ae}{n+1}\right)^{n+1}$$

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$$= \frac{e}{\left(1+\frac{1}{n}\right)^n} \left[\frac{1+\frac{ae}{n}}{1+\frac{ae}{n+1}}\right]^n \frac{1}{\left(1+\frac{ae}{n+1}\right)}$$

$$\log \frac{U_n}{U_{n+1}} = 1 - n\log\left(1+\frac{1}{n}\right) + n\log\left(1+\frac{ae}{n}\right) - (n+1)\log\left(1+\frac{ae}{n+1}\right)$$

$$= 1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \dots\right) + n\left(\frac{ae}{n} - \frac{a^2e^2}{2n^2} + \frac{a^3e^3}{3n^3} + \dots\right)$$

$$-(n+1)\left(\frac{ae}{n+1} - \frac{a^2e^2}{2(n+1)^2} + \frac{1}{3}\frac{a^3e^3}{(n+1)^3} + \dots\right)$$

$$n\log \frac{U_n}{U_{n+1}} = n\left\{\left[\frac{1}{2n} - \frac{1}{3n^2} + \dots\right] + \left[\cancel{ae} - \frac{a^2e^2}{2n} + \frac{a^3e^3}{3n^2} + \dots\right]\right.$$

$$\left. - \left(\cancel{ae} - \frac{a^2e^2}{2(n+1)} + \frac{a^3e^3}{3(n+1)^2} + \dots\right)\right\}$$

$$= \left(\frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^2} + \dots\right) - \frac{a^2e^2}{2} + \frac{a^3e^3}{3n} - \dots +$$

$$\left[\frac{a^2e^2}{2\left(1+\frac{1}{n}\right)} - \frac{a^3e^3}{3\left(1+\frac{1}{n}\right)(n+1)} + \dots\right]$$

$$\lim_{n\to\infty} n\log\frac{U_n}{U_{n+1}} = \frac{1}{2} - \cancel{\frac{a^2e^2}{2}} + \cancel{\frac{a^2e^2}{2}} = \frac{1}{2} < 1$$

So . divergence.

(e) $\frac{1}{\sqrt{n}}\tan\left(\frac{1}{n}\right) = U_n$ let

$V_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$ & ΣV_n is covergence $p>1$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \frac{\tan(1/n)}{(1/n)} = 1 \text{ (finite \& unique)}$$

ΣU_n is also convergence by Comparison test.

(f) $3^{-n-(-1)^n} \rightarrow u_n$ let

$\Sigma U_n = 1 + \frac{1}{3^3} + \frac{1}{3^2} + \frac{1}{3^5} + \frac{1}{3^4} + \ldots$

$= 1 + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \ldots > 1 \therefore$

it is convergence.

or

$r \cong \frac{1}{3} \therefore r < 1$ it is convergence