MATHS PROJECT REPORT

Topic:Orthogonality of function and its Applications

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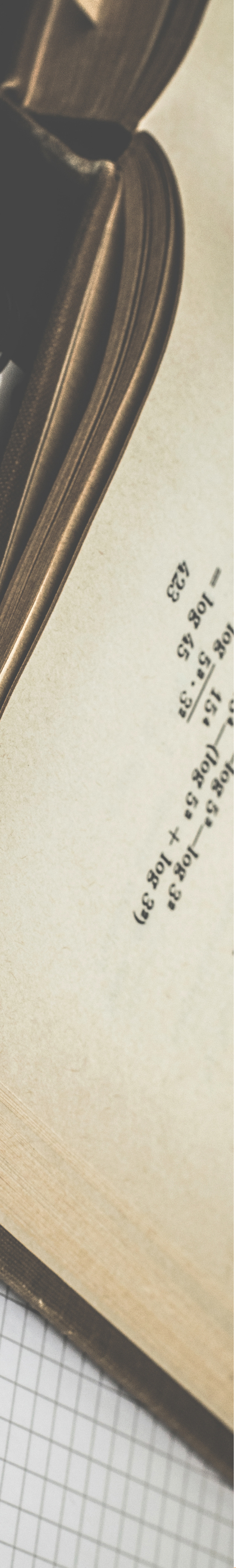

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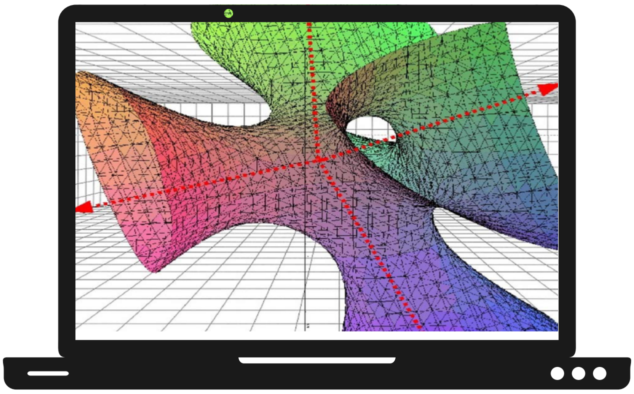
INTRODUCTION

Orthogonal is commonly used in mathematics, geometry, statistics, and software engineering. Most generally, it's used to describe things that have rectangular or right-angled elements. More technically, in the context of vectors and functions, orthogonal means “having a product equal to zero.



In mathematics, orthogonal functions belong to a function space that is a vector space equipped with a bilinear form. When the function space has an interval as the domain, the bilinear form may be the integral of the product of functions over the interval.

Two functions are orthogonal with respect to a weighted inner product if the integral of the product of the two functions and the weight function is identically zero on the chosen interval.

1.ORTHOGONAL FUNCTION

1. Two non-zero functions, $f(x)$ and $g(x)$, are said to be **orthogonal** on $a \le x \le b$ if,

$$\int_a^b f(x)\, g(x)\, dx = 0$$

2. A set of non-zero functions, $\{f_i(x)\}$, is said to be **mutually orthogonal** on $a \le x \le b$ (or just an **orthogonal set** if we're being lazy) if $f_i(x)$ and $f_j(x)$ are orthogonal for every $i \ne j$. In other words,

$$\int_a^b f_i(x)\, f_j(x)\, dx = \begin{cases} 0 & i \ne j \\ c > 0 & i = j \end{cases}$$

NOTE : THAT IN THE CASE OF I=J FOR THE SECOND DEFINITION WE KNOW THAT WE'LL GET A POSITIVE VALUE FROM THE INTEGRAL BECAUSE,

$$\int_a^b f_i(x)\, f_i(x)\, dx = \int_a^b [f_i(x)]^2\, dx > 0$$

Here are some important results;

1. $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ and $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ are mutually orthogonal on $-L \le x \le L$ as individual sets and as a combined set.
2. $\left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=0}^{\infty}$ is mutually orthogonal on $0 \le x \le L$.
3. $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$ is mutually orthogonal on $0 \le x \le L$.

HERE ARE SOME IMPORTANT RESULTS;

WE WILL ALSO BE NEEDING THE RESULTS OF THE INTEGRALS THEMSELVES, BOTH ON -L⩽X⩽L-L⩽X⩽L AND ON 0⩽X⩽L0⩽X⩽L SO LET'S ALSO SUMMARIZE THOSE UP HERE AS WELL SO WE CAN REFER TO THEM WHEN WE NEED TO.

1. $$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 2L & \text{if } n = m = 0 \\ L & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

2. $$\int_{0}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L & \text{if } n = m = 0 \\ \frac{L}{2} & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases}$$

3. $$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} L & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

4. $$\int_{0}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} \frac{L}{2} & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

5. $$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0$$

ORTHOGONAL SETS

A SET OF REAL-VALUED FUNCTIONS IS SAID TO BE ORTHOGONAL ON AN INTERVAL [A, B] IF

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x)\,\phi_n(x)\,dx = 0, \quad m \neq n.$$

EXAMPLE : ORTHOGONAL SETS:

SHOW THAT THE SET {1, COS X, COS 2X, . . .} IS ORTHOGONAL ON THE INTERVAL [Π, Π];

HERE ARE SOME IMPORTANT RESULTS;

SOLUTION If we make the identification $\phi_0(x) = 1$ and $\phi_n(x) = \cos nx$, we must then show that $\int_{-\pi}^{\pi} \phi_0(x)\,\phi_n(x)\,dx = 0$, $n \neq 0$, and $\int_{-\pi}^{\pi} \phi_m(x)\,\phi_n(x)\,dx = 0$, $m \neq n$. We have, in the first case,

$$(\phi_0, \phi_n) = \int_{-\pi}^{\pi} \phi_0(x)\,\phi_n(x)\,dx = \int_{-\pi}^{\pi} \cos nx\,dx$$

$$= \frac{1}{n}\sin nx\Big|_{-\pi}^{\pi} = \frac{1}{n}[\sin n\pi - \sin(-n\pi)] = 0, \qquad n \neq 0,$$

$$(\phi_m, \phi_n) = \int_{-\pi}^{\pi} \phi_m(x)\phi_n(x)\,dx$$

$$= \int_{-\pi}^{\pi} \cos mx \cos nx\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x]\,dx \qquad \leftarrow \text{trig identity}$$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0, \qquad m \neq n.$$

Orthogonality of series expansion

- Suppose $\{\phi_n(x)\}$ is an orthogonal set on $[a, b]$. If $f(x)$ is defined on $[a, b]$, we first write as

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots ? \qquad (6)$$

Then $\int_a^b f(x)\phi_m(x)\,dx$

$$= c_0\int_a^b \phi_0(x)\phi_m(x)\,dx + c_1\int_a^b \phi_1(x)\phi_m(x)\,dx + \cdots$$

$$+ c_n\int_a^b \phi_n(x)\phi_m(x)\,dx + \cdots$$

$$= c_0(\phi_0, \phi_m) + c_1(\phi_1, \phi_m) + \cdots + c_n(\phi_n, \phi_m) + \cdots$$

HERE ARE SOME IMPORTANT RESULTS;

ORTHOGONAL SERIES EXPANSION Suppose $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on an interval $[a, b]$. We ask: If $y = f(x)$ is a function defined on the interval $[a, b]$, is it possible to determine a set of coefficients c_n, $n = 0, 1, 2, \ldots$, for which

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots ?$$

As in the foregoing discussion on finding components of a vector we can find the coefficients c_n by utilizing the inner product. Multiplying (6) by $\phi_m(x)$ and integrating over the interval $[a, b]$ gives

$$\begin{aligned}\int_a^b f(x)\phi_m(x)\,dx &= c_0\int_a^b \phi_0(x)\phi_m(x)\,dx + c_1\int_a^b \phi_1(x)\phi_m(x)\,dx + \cdots + c_n\int_a^b \phi_n(x)\phi_m(x)\,dx + \cdots \\ &= c_0(\phi_0, \phi_m) + c_1(\phi_1, \phi_m) + \cdots + c_n(\phi_n, \phi_m) + \cdots.\end{aligned}$$

By orthogonality each term on the right-hand side of the last equation is zero *except* when $m = n$. In this case we have

$$\int_a^b f(x)\phi_n(x)\,dx = c_n\int_a^b \phi_n^2(x)\,dx.$$

It follows that the required coefficients are

$$c_n = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\int_a^b \phi_n^2(x)\,dx}, \qquad n = 0, 1, 2, \ldots.$$

In other words, $$f(x) = \sum_{n=0}^{\infty} c_n\phi_n(x),$$

where $$c_n = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\|\phi_n(x)\|^2}.$$

With inner product notation, (7) becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{\|\phi_n(x)\|^2}\phi_n(x).$$

A SET OF REAL-VALUED FUNCTIONS

$$\{\phi_0(x), \ \phi_1(x), \ \phi_2(x), \ldots\}$$

IS SAID TO BE ORTHOGONAL WITH RESPECT TO A WEIGHT FUNCTION W(X) ON AN INTERVAL [A, B] IF

HERE ARE SOME IMPORTANT RESULTS;

THE USUAL ASSUMPTION IS THAT W(X) 0 ON THE INTERVAL OF ORTHOGONALITY [A, B]. THE SET {1, COS X, COS 2X, . . .} IN EXAMPLE 1 IS ORTHOGONAL WITH RESPECT TO THE WEIGHT FUNCTION W(X) 1 ON THE INTERVAL [P, P]. IF {FN(X)} IS ORTHOGONAL WITH RESPECT TO A WEIGHT FUNCTION W(X) ON THE INTERVAL [A, B], THEN MULTIPLYING (6) BY W(X)FN(X) AND INTEGRATING YIELDS

$$c_n = \frac{\int_a^b f(x)\, w(x)\, \phi_n(x)\, dx}{\|\phi_n(x)\|^2},$$

WHERE

$$\|\phi_n(x)\|^2 = \int_a^b w(x)\, \phi_n^2(x)\, dx.$$

THE SERIES WITH COEFFICIENTS GIVEN BY EITHER IS SAID TO BE AN ORTHOGONAL SERIES EXPANSION OF F OR A GENERALIZED FOURIER SERIES

ORTHOGONALITY OF EIGENFUNCTIONS THEOREM:

Precise statement: suppose $X_n'' + \lambda_n X_n = 0$ and $X_m'' + \lambda_m X_m = 0$ on $a < x < b$, and that X_n and X_m both satisfy the **same type of BC**. If $\lambda_n \neq \lambda_m$ then X_n and X_m are orthogonal:

$$\int_a^b X_n(x) X_m(x)\, dx = 0.$$

Proof. By the Integration Lemma, we have

$$\begin{aligned}
&\int_a^b X_n(x) X_m(x)\, dx \\
&\qquad = \frac{1}{\lambda_n - \lambda_m} [X_n(x) X_m'(x) - X_n'(x) X_m(x)]_a^b \\
&\qquad = 0 \quad \text{under Dirichlet BCs, because } X_n(a) = X_n(b) = 0 \text{ and } X_m(a) = X_m(b) = 0 \\
&\qquad = 0 \quad \text{under Neumann BCs, because } X_n'(a) = X_n'(b) = 0 \text{ and } X_m'(a) = X_m'(b) = 0 \\
&\qquad = 0 \quad \text{under Mixed BCs, for similar reasons.}
\end{aligned}$$

For periodic BCs, we use that $X_n(b) = X_n(a)$ and $X_m'(b) = X_m'(a)$ and so on, to see

$$\begin{aligned}
[X_n(x) X_m'(x) - X_n'(x) X_m(x)]_a^b &= [X_n(b) X_m'(b) - X_n'(b) X_m(b)] \\
&\quad - [X_n(a) X_m'(a) - X_n'(a) X_m(a)] = 0.
\end{aligned}$$

HERE ARE SOME IMPORTANT RESULTS;

$$\int_{-1}^{1} f_m(x) P_n(x)\ dx = \frac{1}{2^n n!} \int_{-1}^{1} f_m(x) \frac{d^n}{dx^n}(x^2-1)^n\, dx$$
$$= \frac{1}{2^n n!} \Big/_{-1}^{1} f_m(x) \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n - \frac{1}{2^n n!} \int_{-1}^{1} f_m'(x) \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\, dx$$
$$\cdots \qquad \cdots$$
$$= (-1)^m \frac{a_m m!}{2^n n!} \int_{-1}^{1} \frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n\, dx$$
$$= (-1)^m \frac{a_m m!}{2^n n!} \Big/_{-1}^{1} f_m(x) \frac{d^{n-m-1}}{dx^{n-m-1}}(x^2-1)^n = 0,$$

ORTHOGONALITY OF BESSEL FUNCTION

$$\int_0^1 xJ_p(ax)J_p(bx)dx = \begin{cases} 0 & \text{if} \quad a \neq b \\ \frac{1}{2}J_{p+1}^2(a) = \frac{1}{2}J_{p-1}^2(a) = \frac{1}{2}J_p'^2(a)) & \text{if} \quad a = b \end{cases},$$

WHERE A AND B ARE CALLED ZERO'S OF JP (X).

Consider the Bessel Function $u = J_p(x)$, $p \geq 0$.
Then it satisfies the Bessel's equation

$$x^2 u'' + x u' + (x^2 - p^2) u = 0 \qquad \text{...(1)}$$

We put $x = at$, where $J_p(a) = 0$ in (1), to get

$$t^2 \frac{d^2u}{dt^2} + t \frac{du}{dt} + (a^2t^2 - p^2) u = 0 \qquad \text{--(2)}$$

Similarly choosing another member of the family
$v = J_p(bt)$ with $J_p(b) = 0$, we get

$$t^2 \frac{d^2v}{dt^2} + t \frac{dv}{dt} + (b^2t^2 - p^2) v = 0 \qquad \text{... (3)}$$

Multiplying (2) by v and (3) by u and then subtracting , we get

$$t^2(u''v - v''u) + t (u'v - v'u) + (a^2 - b^2)t^2 uv = 0$$

Dividing by t, we get

$$t(u''v - v''u) + (u'v - v'u) + (a^2 - b^2)t\, uv = 0$$

This can be written as

$$[t(u'v - v'u)]' + (a^2 - b^2)t\, uv = 0 \qquad \text{....(4)}$$

Integrating equation (4) on [0, 1], we get

$$t(u'v - v'u) \mid t= 0 \text{ to } t=1 + (a^2 - b^2) \int_0^1 tuv\, dt = 0 \qquad \text{.... (5)}$$

v(b) = 0, therefore first term in equation (5) is 0 and we get

$$(a^2 - b^2) \int_0^1 tuv\, dt = 0$$

For a ≠ b, we have

$$\int_0^1 tuv\, dt = 0 \qquad \text{...(6)}$$

This means u(t) and v(t) are orthogonal with weight w = t on [0, 1]. In (6), t is a dummy variable we can say that J_p (ax) and J_p (bx) are orthogonal on [0,1] with weight w = x.

Note1 – *It may be noted that derivation (6) is independent of p, therefore the result is true for any family $\mathcal{F}_p$, $p \geq 0$.*

Now we show that when the zero b -> a, we get

$$\int_0^1 tuv\, dt \;\; \to \;\; \| u \|^2 = \frac{(J_{p+1}(a))^2}{2} \quad \text{....(9)}$$

Proof- Consider relation (5) without assuming a and b are zeros, we have

$$\int_0^1 tuv\, dt = (J'_p(a)\, J_p(b) - J'_p(b)\, J_p(a))/(b^2 - a^2)$$

Proof- Consider relation (5) without assuming a and b are zeros, we have

$$\int_0^1 tuv\, dt = (J'_p(a)\, J_p(b) - J'_p(b)\, J_p(a))/(b^2 - a^2)$$

Taking limit as b -> a where a is a zero, using Hospital's rule, we get

$$\int_0^1 tu^2\, dt = J'_p(a) . J'_p(a) /(2a)$$

$$= \frac{(J_{p+1}(a))^2}{2}$$

(Using Bessel's identity for derivative and x=a is a zero of $J_p(x)$)

Note 2 – We have shown orthogonality in basic interval [0, 1]. This can be expanded to [0,R] by taking simple scaling transformation.

EXAMPLE :

Prove that $\int_{-1}^{1} xP_nP_{n-1}(x)dx = \frac{2n}{4n^2-1}$.

Solution: We recall the recurrence relation I:

$$(n+1)P_{n+1} = (2n+1)xP_n - (n-1)P_{n-1}$$

$$xP_n(x) = \frac{n+1}{2n+1}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x). \quad (19)$$

We now multiply both sides of (19) by $P_{n-1}(x)$, then integrating with respect to x from -1 to 1 and obtain

$$\int_{-1}^{1} x\ P_n(x)P_{n-1}(x)dx = \frac{n+1}{2n+1}\int_{-1}^{1} P_{n+1}(x)P_{n-1}(x)dx + \frac{n}{2n+1}\int_{-1}^{1}[P_{n-1}(x)]^2dx$$

$$= \ 0 + \frac{n}{2n+1} \times \frac{2}{2(n-1)+1} = \frac{2n}{(2n+1)(2n-1)} = \frac{2n}{4n^2-1}$$

REFERNCES

- https://www.researchgate.net/publication/250342226_Orthogonal_Function_Techniques_for_the_Identification
- https://en.wikipedia.org/wiki/Orthogonality_principle
- https://tutorial.math.lamar.edu/classes/de/PeriodicOrthogonal.aspx
George B. Arfken & H
- ans J. Weber (2005) Mathematical Methods for Physicists, 6th edition, chapter 10: Sturm-Liouville Theory — Orthogonal Functions


thank you!