

Jaypee Institute of Information Technology
ODD Semester 2022
Theoretical Foundations of Computer Science(15B11CI212)
Tutorial 6

Topic: Induction and Recurrence

Ques 1: Prove that $1^2 + 3^2 + 5^2 + \dots + (2n + 1)^2 = \frac{(n+1)(2n+1)(3n+1)}{3}$ whenever n is a nonnegative integer.

Solution:

" $1^2 + 3^2 + \dots + (2n + 1)^2 = (n + 1)(2n + 1)(2n + 3)/3$."
Basis step: $P(0)$ is true because $1^2 = 1 = (0 + 1)(2 \cdot 0 + 1)(2 \cdot 0 + 3)/3$. *Inductive step:* Assume that $P(k)$ is true. Then $1^2 + 3^2 + \dots + (2k + 1)^2 + [2(k + 1) + 1]^2 = (k + 1)(2k + 1)(2k + 3)/3 + (2k + 3)^2 = (2k + 3)[(k + 1)(2k + 1)/3 + (2k + 3)] = (2k + 3)(2k^2 + 9k + 10)/3 = (2k + 3)(2k + 5)(k + 2)/3 = [(k + 1) + 1][2(k + 1) + 1][2(k + 1) + 3]/3$.

Ques 2: Prove that $2^n > n^2$ is n is an integer greater than 4.

Solution:

Let $P(n)$ be " $2^n > n^2$." *Basis step:* $P(5)$ is true because $2^5 = 32 > 25 = 5^2$. *Inductive step:* Assume that $P(k)$ is true, that is, $2^k > k^2$. Then $2^{k+1} = 2 \cdot 2^k > k^2 + k^2 > k^2 + 4k \geq k^2 + 2k + 1 = (k + 1)^2$ because $k > 4$.

Ques 3: Give the recursive definition of the following sequence a_n $n = 1, 2, 3$

- | | |
|--------------------------|----------------------------|
| a) $a_n = 6n$. | b) $a_n = 2n + 1$. |
| c) $a_n = 10^n$. | d) $a_n = 5$. |

Solution:

- a) $a_{n+1} = a_n + 6$ for $n > 1$ $a_1 = 6$
- b) $a_{n+1} = a_n + 2$ for $n > 1$ $a_1 = 3$
- c) $a_{n+1} = 10a_n$ for $n > 1$ $a_1 = 10$
- d) $a_{n+1} = a_n$ for $n > 1$ $a_1 = 5$

Ques 4: Find the solution for the recurrence relation

$$a_n = \begin{cases} 2 & \text{if } n = 0, \\ 7 & \text{if } n = 1, \\ a_{n-1} + 2a_{n-2} & \text{otherwise.} \end{cases}$$

Solution:

Characteristic equation:

$$r^2 - r - 2 = 0$$

Roots $r = -1, 2$

General Solution:

$$c_1 2^n + c_2 (-1)^n$$

Solve for c_1, c_2

$$c_1 + c_2 = 2$$

$$2c_1 - c_2 = 7$$

$$c_1 = 3, c_2 = -1$$

$$3 \cdot 2^n - (-1)^n$$

Ques 5: Find the solution for the recurrence relation

$$x_n = 6x_{n-1} - 9x_{n-2}$$

$$x_0 = 2$$

$$x_1 = 3$$

Solution. The characteristic equation

$$r^2 - 6r + 9 = 0 \iff (r - 3)^2 = 0$$

has only one root $r = 3$. Then the general solution is

$$x_n = c_1 3^n + c_2 n 3^n.$$

The initial conditions $x_0 = 2$ and $x_1 = 3$ imply that $c_1 = 2$ and $c_2 = -1$. Thus the solution is

$$x_n = 2 \cdot 3^n - n \cdot 3^n = (2 - n)3^n, \quad n \geq 0.$$

Ques 6: Find the solution for the recurrence relation

$$\begin{cases} x_n &= 10x_{n-1} - 25x_{n-2} + 8 \cdot 5^n \\ x_0 &= 6 \\ x_1 &= 10 \end{cases}$$

Solution. The characteristic equation is

$$t^2 - 10t + 25 = 0 \iff (t - 5)^2 = 0.$$

We have roots $r_1 = r_2 = 5$, then $r = r_1 = r_2 = 5$. A special solution can be of the type $x_n = An^25^n$. Put the solution into the non-homogeneous relation. We have

$$An^25^n = 10A(n-1)^25^{n-1} - 25A(n-2)^25^{n-2} + 8 \cdot 5^n$$

Dividing both sides by 5^{n-2} ,

$$An^25^2 = 10A(n-1)^25 - 25A(n-2)^2 + 8 \cdot 5^2.$$

Since $An^25^2 = 10An^25 - 25n^2$, we have

$$10A(-2n+1)5 - 25A(-4n+4) + 8 \cdot 5^2 = 0 \implies A = 4.$$

So a nonhomogeneous solution is

$$x_n = 4n^25^n.$$

The general solution is

$$x_n = 4n^25^n + c_15^n + c_2n5^n.$$

The initial condition implies $c_1 = 6$ and $c_2 = -8$. Therefore

$$x_n = (4n^2 - 8n + 6)5^n.$$