15B11CI212 Theoretical Foundations of Computer Science

Course Objectives

- Apply the concepts of set theory, relations and functions in the context of various fields of computer science e.g. Database, Automata, Compiler etc.
- Evaluate Boolean functions and Analyze algebraic structure using the properties of Boolean algebra
- Convert formal statements to logical arguments and correlate these arguments to Boolean logic, truth tables, rules of propositional And predicate calculus
- Apply the fundamental principle of counting, combinatorics and recurrence relations to find the complex pattern and sequences in Given datasets
- Apply graph theory concepts for designing solutions of various computing problems e.g. shortest path, graph coloring, job Sequencing etc.
- Explain basic concepts of automata theory and formal languages e.g. Finite automata, regular expressions, context-free grammars etc.

Recommended Reading material

- Rosen, K. H., Discrete Mathematics and Its Applications with Combinatorics and Graph Theory, Tata McGraw-Hill, 2008.
- Liu, C. L., Elements of Discrete Mathematics, Tata McGraw-Hill, 2008.
- Ullman J. D. Foundations of Computer Science: C Edition, W. H. Freeman; 1994
- Tremblay and Manohar , Discrete Mathematical Structures, Tata McGraw Hill
- Linz, P, An Introduction To Formal Languages And Automata, Narosa Publishing House, 2007.
- Sipser, M., Introduction to the Theory of Computation, Second Edition, Thomson Course Technology, 2007
- Journal of Discrete Mathematics, Elsevier.

Course Outline

- Module 1 : Introduction to Discrete Mathematics and Set Theory
- Module 2: Relations
- Module 3: Functions and Recursion
- Module 4: Algebraic Structures
- Module 5: Logic
- Module 6: Counting and Combinatorics
- Module 7: Graph Theory
- Module 8: Automata Theory
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Module 1 – Introduction to Discrete Mathematics and Set Theory

Overview

Discrete Mathematics: A Brief Introduction, Set Notations, Cardinality of Sets; Some Standard Sets; Venn Diagrams; Operations on Sets; Principle of inclusion and exclusion; Disjoint Sets; Partition; Ordered Set; Cartesian Product of Sets; Algebra of Sets, Bit vector representation of sets.

Lecture 1

Outline of the Lecture 1

- Discrete Mathematics: A Brief Introduction
- Set Notations
- Cardinality of Sets
- Some Standard Sets

Discrete Mathematics: A Brief Introduction

The study of discrete, mathematical objects and structures

- **Why Study Discrete Math?**
- The basis of all of digital information processing: *Discrete manipulations of discrete structures represented in memory.*
- It's the basic language and conceptual foundation of all of computer science.
- Discrete concepts are also widely used throughout math, science, engineering, economics, biology, *etc.*, …
- A generally useful tool for rational thought!

Uses for Discrete Math in Computer Science

- Advanced algorithms & data structures
- Programming language compilers & interpreters.
- Computer networks
- Operating systems
- Computer architecture
- Database management systems
- Cryptography
- Error correction codes
- Graphics & animation algorithms, game engines
- Just about everything!

Set Theory

Introduction to Set Theory

- **Definition**: A **set** is a (unordered) collection of objects. These objects are sometimes called **elements** or **members** of the set.

Examples:

– **Vowels in the English alphabet**

V = { a, e, i, o, u }

– **First seven prime numbers**.

X = { 2, 3, 5, 7, 11, 13, 17 }

Set Notations

- **A set can be represented by :**

1) Listing (enumerating) the members of the set.

Example: Even integers between 50 and 63**.**

E = ?

2) Definition by property, using the set builder notation

$\{x \mid x \text{ has property } P\}$.

Example: E = ?

Set Notations

- **A set can be represented by :**

1) Listing (enumerating) the members of the set.

Example: Even integers between 50 and 63.

$E = \{50, 52, 54, 56, 58, 60, 62\}$

2) Definition by property, using the set builder notation

$\{x \mid x \text{ has property } P\}$.

Example: **$E = \{x \mid 50 <= x < 63, x \text{ is an even integer}\}$**

Basic properties of sets

- Sets are inherently *unordered*:
  - No matter what objects a, b, and c denote, {a, b, c} = {a, c, b} = {b, a, c} = {b, c, a} = {c, a, b} = {c, b, a}.
- All elements are *distinct* (unequal); multiple listings make no difference!
  - {a, b, c} = {a, a, b, a, b, c, c, c, c}.
  - This set contains at most 3 elements!

Definition of Set Equality

- Two sets are declared to be equal *if and only if* they contain <u>exactly the same</u> elements.
- In particular, it does not matter *how the set is defined or denoted*.
- For example: The set $\{1, 2, 3, 4\}$ =
  $\{x \mid x$ is an integer where $x>0$ and $x<5\}$ =
  $\{x \mid x$ is a positive integer whose square is >0 and $<25\}$

Infinite Sets

- Conceptually, sets may be *infinite* (*i.e.*, not *finite*, without end, unending).
- Symbols for some special infinite sets:
  N = {0, 1, 2, …} The **n**atural numbers.
  Z = {…, -2, -1, 0, 1, 2, …} The **i**ntegers.
  R = The "**r**eal" numbers, such as
  374.1828471929498181917281943125…
- Infinite sets come in different sizes!

Special sets

- Special sets:
- – The universal set is denoted by U: the set of all objects under the consideration.
- – The empty set is denoted as ∅ or { }.

Cardinality

- **Definition:** Let S be a set. If there are exactly n distinct elements in S, where n is a nonnegative integer, we say S is a finite set and that n is the **cardinality of S**. The cardinality of S is denoted by **$|S|$.**

Examples:

- **$V=\{1\ 2\ 3\ 4\ 5\}$**

$|V| = 5$

- **$A=\{1,2,3,4, \ldots, 20\}$**

$|A| = 20$

- **$|\emptyset| = 0$**

Power set

- **Definition:** Given a set S, the **power set** of S is the set of all subsets of S. The power set is denoted by **P(S).**

Examples:

- Assume an empty set Ø
- What is the power set of Ø?

$P(\emptyset) = \{\emptyset\}$

What is the cardinality of P(Ø) ?

$|P(\emptyset)| = 1.$

Assume set {1}

- $P(\{1\}) =$

$\{\emptyset, \{1\}\}$

- $|P(\{1\})| = 2$

Power set

- **Definition:** Given a set S, the **power set** of S is the set of all subsets of S. The power set is denoted by **P(S).**

Examples:

- Assume an empty set Ø
- What is the power set of Ø?

$P(\emptyset) = \{\emptyset\}$

What is the cardinality of P(Ø) ?

$|P(\emptyset)| = 1.$

Assume set {1}

- P({1}) =

$\{\emptyset, \{1\}\}$

- $|P(\{1\})| = 2$

Power set

- $P(\{1\}) = \{ \varnothing, \{1\} \}$
- $|P(\{1\})| = 2$

- Assume $\{1,2\}$
- $P(\{1,2\}) = \{ \varnothing, \{1\}, \{2\}, \{1,2\} \}$
- $|P(\{1,2\})| = 4$

- Assume $\{1,2,3\}$
- $P(\{1,2,3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$
- $|P(\{1,2,3\}| = 8$

- **If S is a set with $|S| = n$ then $| P(S) | = ?$**

- **If S is a set with $|S| = n$ then $| P(S) | = 2^n$**

Important sets in discrete math

- **Natural numbers:**
  - **N** = {0,1,2,3, …}
- **Integers**

– **Z** = {…, -2,-1,0,1,2, …}

- **Positive integers**

– **Z+** = {1,2, 3.…}

- **Rational numbers**

– **Q** = $\{p/q \mid p \in Z, q \in Z, q \neq 0\}$

- **Real numbers**

– **R**

Exercises

Q1. Describe the following sets by enumerating their members.

a. The set of all the non-negative even numbers less than 20.

b. The set of the first eight prime numbers.

c. The set of the names of the continents in the world.

d. The set of the smallest five squares of integers.

Q2. Write the following sets by enumerating their elements.

a. S = {x | Z+ : x^2 < 25}.

b. S = {x | N : x^2 < 3x + 5}.

c. S = {x | R : (x - 1)(x - 5)(x -9.5) = 0}.

d. S = {x | N : (x - 1)(x - 5)(x – 9.5) = 0}.

Exercises

Q3. Describe the following sets by identifying the correct property , P(x), and a reasonable universal set .

a. The set of all pairs of non-negative of integers that add up to more than 12.

b. The set of real numbers whose square is at least 12.

c. The set of all points on a circle of radius 1 centered at (0,0).

d. The set of all real numbers whose square-root is a positive integer.

Q4. Write the following sets in roster form (enumerating elements)

a. $A=\{n|n\in Z,|n|<5\}$

b. $p|p$ is prime,$p<25\}$

c. $B=\{x\in N|x2+4x-5=0\}$

d. $C=\{x|x\in Z,x2-2x-8\leq 0\}$

e. $D=\{(x,y)|x\in N,y\in N, (x-1)2+y2\leq 1\}$

Exercises

- Q5. How many elements in each of the sets: $\emptyset$

1. $\{\emptyset\}$
2. $\{\emptyset,\{\emptyset\}\}$
3. $\{\{\emptyset,\{\emptyset\}\}\}$
4. $\{\emptyset,\{\emptyset\},\emptyset\}$

Q6. Find the power set of the following sets:

1. $A=\{3,\{3\}\}$
2. $B=\{3,3,4,5\}$
3. $C=\{3,\{3,4\},5\}$
4. $D=\{x \in Z \mid x^2<2\}$
5. What is the cardinality of the sets in Q1,Q2,Q5 and Q6

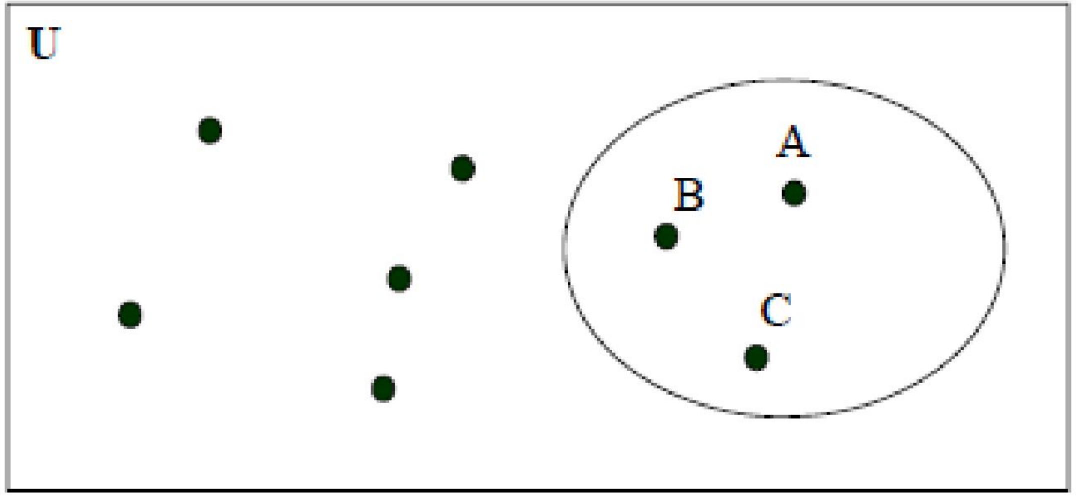
Lecture 2

Outline of the Lecture 2

- Venn Diagrams
- Operations on Sets
- Principle of inclusion and exclusion

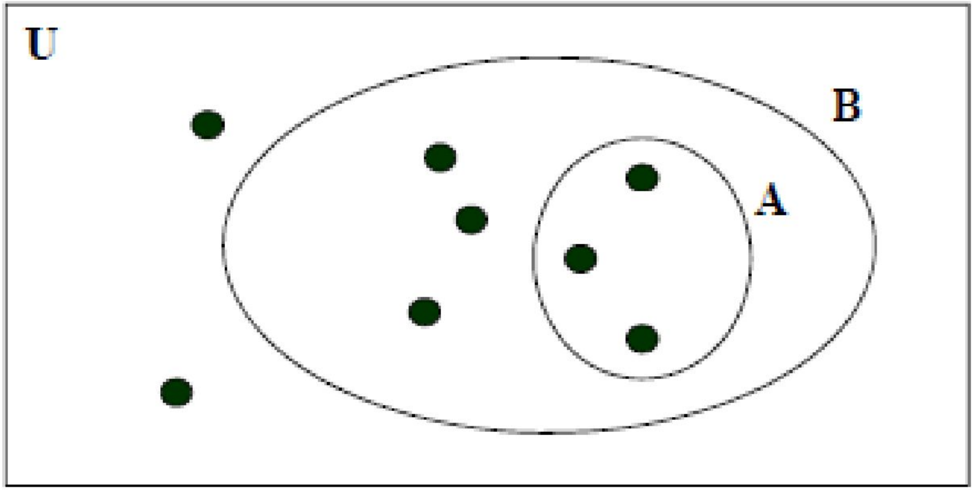
Venn diagrams

- A set can be visualized using **Venn Diagrams**:In a Venn diagram, a rectangle shows the universal set, and all other sets are usually represented by circles within the rectangle. The shaded region represents the result of the operation.
- V={ A, B, C }



A Subset

• **Definition:** A **set** A is said to be a **subset** of B **if and only if** every element of A is also an element of B. We use **A⊆B** to indicate **A is a subset of B**.



Alternate way to define A is a subset of B:

$$\forall x\ (x \in A) \rightarrow (x \in B)$$

Subset properties

Theorem $\varnothing \subseteq S$

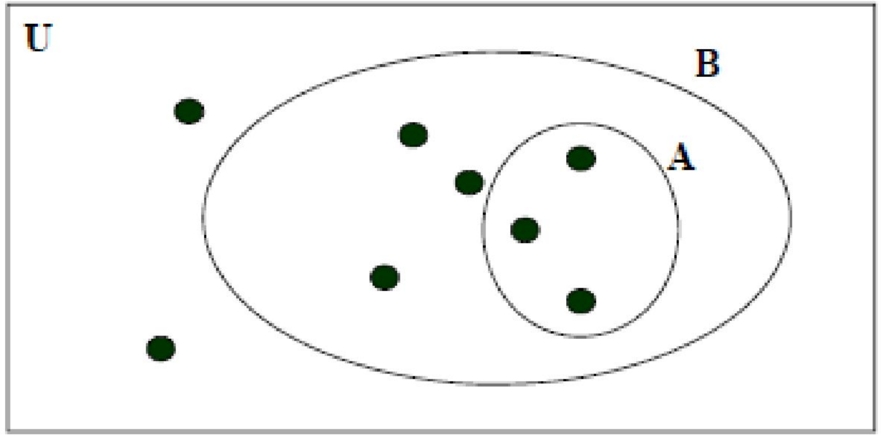
- **Empty set is a subset of any set.**

Theorem: $S \subseteq S$

- **Any set S is a subset of itself**

A proper subset

- **Definition:** A **set A** is said to be a **proper subset** of B if and only if **A ⊆B** and **A ≠ B**. We denote that A is a proper subset of B with the notation A ⊂ B.



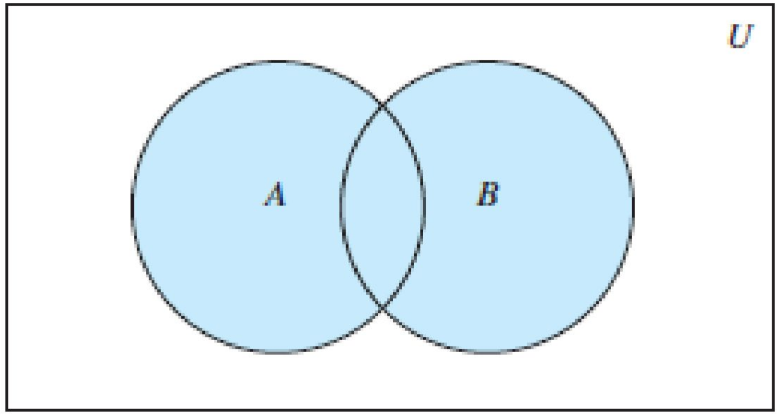
Example: A={1,2,3} B ={1,2,3,4,5}
Is: A ⊂ B ?

Yes.

Set operations

- **Definition**: Let A and B be sets. The **union of A and B**, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

Alternate: $A \cup B = \{ x \mid x \in A \vee x \in B \}$.



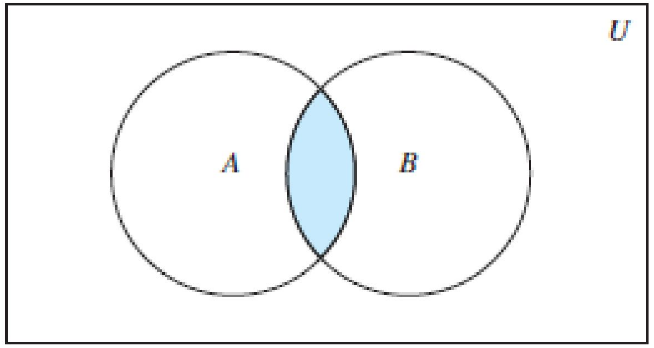
$A \cup B$ is shaded.

Example:

- $A = \{1,2,3,6\}$ $B = \{ 2,4,6,9\}$
- $A \cup B = \{ 1,2,3,4,6,9 \}$

Definition: Let A and B be sets. The **intersection of A and B**, denoted by $A \cap B$, is the set that contains those elements that are in both A and B.

- Alternate: $A \cap B = \{ x \mid \mid x \in A \wedge x \in B \}$.



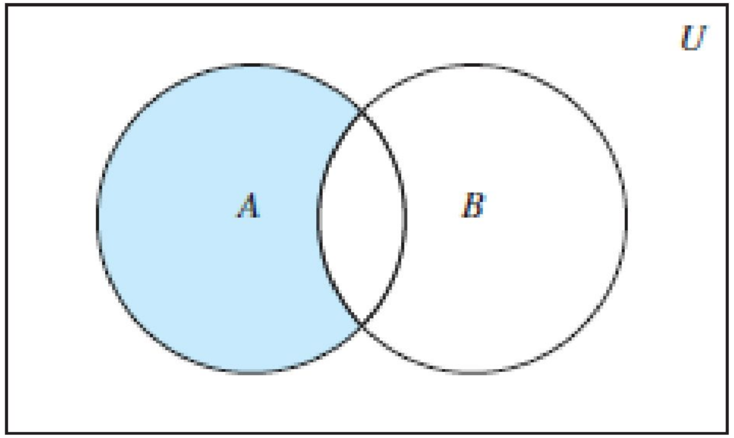
$A \cap B$ is shaded.

Example:

- $A = \{1,2,3,6\}$ $B = \{ 2, 4, 6, 9\}$
- $A \cap B = \{ 2, 6 \}$

Set difference

- **Definition**: Let A and B be sets. The **difference of A and B**, denoted by **$A\backslash B$ or $A-B$** , is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.
- • Alternate: $A - B = \{ x \mid x \in A \wedge x \notin B \}$.



$A - B$ is shaded.

Example: A= {1,2,3,5,7} B = {1,5,6,8}

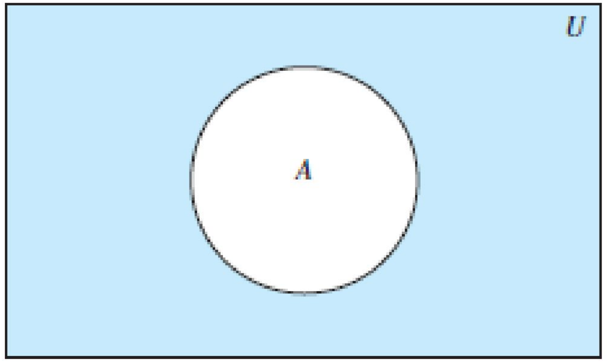
• A - B ={2,3,7}

Let A={a,b,c}, B={e,g,f}. A-B ?

A-B={a,b,c}.

Complement of a set

- **Definition:** Let U be the **universal set**: the set of all objects under the consideration.
- **Definition:** The **complement of the set A**, denoted by A, is the complement of A with respect to U.
- Alternate: $\bar{A}$ (or A^c)= { x | x ∉ A } it can also represented as U-A



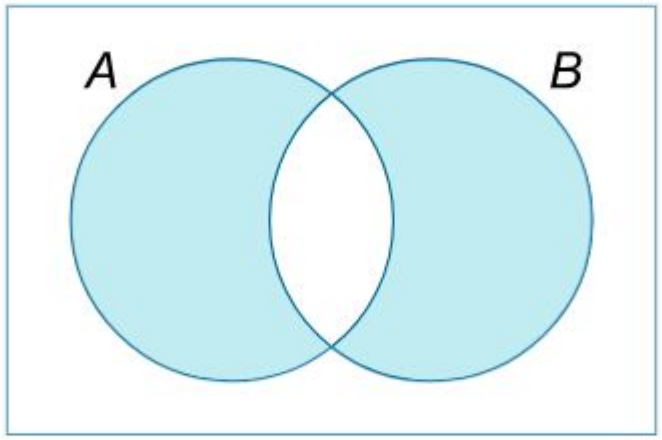
$\bar{A}$ is shaded.

Example: U={1,2,3,4,5,6,7,8} A ={1,3,5,7}

- $\bar{A}$ = {2,4,6,8}

Symmetric Difference

The symmetric difference of two sets A and B is the set of all elements which belong to exactly one of the two original sets. This operation is written as $A \triangle B$ or $A \oplus B$.



In terms of unions and intersections, the symmetric difference of two sets A and B can be expressed as

$A \oplus B = (A \cup B) - (A \cap B)$.

Example

A={a,b,c}, B={e,f,g}. The symmetric difference of two disjoint sets is equal to their union:

1. $A \oplus B = (A \cup B) - (A \cap B) = (A \cup B) - \varnothing = A \cup B = \{a,b,c,e,f,g\}$.

2. C={1,2,3,4}, D={2,4,6,7}. The symmetric difference of the sets C and D is given by

$C \oplus D = (C \cup D) - (C \cap D) = \{1,2,3,4,6,7\} - \{2,4\} = \{1,3,6,7\}$.

The Membership Relation

- Let A be a set and let x be some object.
- Notation: $x \in A$
- Meaning: x is a member of A, or x is an element of A, or x belongs to A.
- Negated by writing $x \notin A$
- Example:. $V = \{ a, e, i, o, u \}$ $e \in V$ $b \notin V$

Equality of Sets

- Two sets A and B are equal, denoted $A=B$, if they have the same elements.
- Otherwise, $A \neq B$.
- Example:
  - The set A of odd positive integers is not equal to the set B of prime numbers.
  - The set of odd integers between 4 and 8 is equal to the set of prime numbers between 4 and 8.

Principle of inclusion/Exclusion

The idea of inclusion-exclusion is:

- Ignore duplications at first and include all objects which are in it.
- Then exclude the duplications.

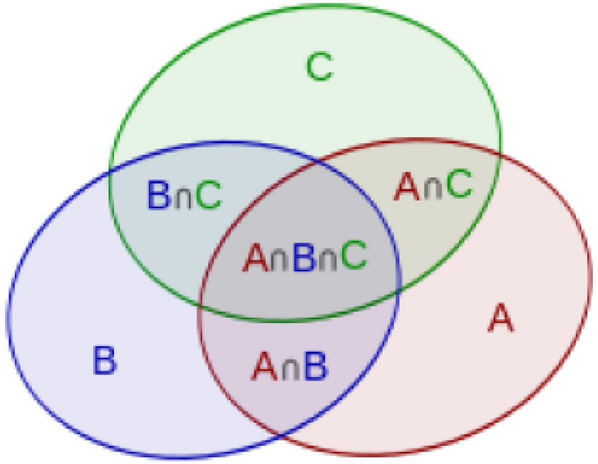
- Finally the result has no repetitions and no losses.

Example - The number of multiples of 2 or 3 in [1,20]
[Solution] Multiples of 2:
2，4，6，8，10，12，14，16，18，20。
10 numbers
Multiples of 3:
3，6，9，12，15，18。
6 numbers
The answer is not 10+6=16, as 6,12,18appears in both classes. So we need to subtract them. The answer is: 16-3=13

Principle of inclusion/Exclusion for 3 sets

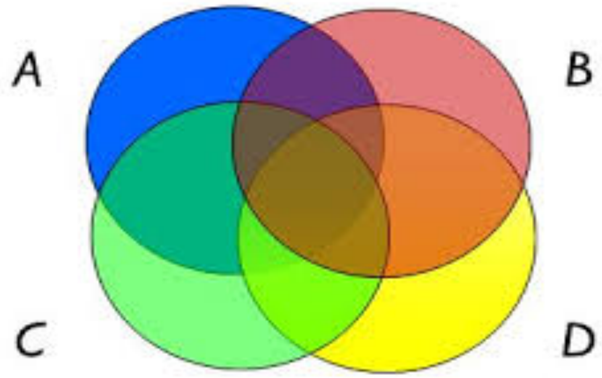
$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B|$$
$$- |\mathrm{A} \cap \mathrm{C}| - |B \cap C| + |A \cap B \cap C|$$



Principle of inclusion/Exclusion for 4 sets

$$
\begin{aligned}
|A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| \\
& - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\
& + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\
& - |A \cap B \cap C \cap D|
\end{aligned}
$$



$$\left|A_1 \cup A_2 \cup \ldots \cup A_n\right| = \sum_{i=1}^{n} \left|A_i\right| - \sum_{i=1}^{n} \sum_{j>i} \left|A_i \cap A_j\right|$$

$$+\sum_{i=1}^{n} \sum_{j>i} \sum_{k>j} \left|A_i \cap A_j \cap A_k\right| - \ldots$$

$$+(-1)^{n-1} \left|A_1 \cap A_2 \cap \ldots \cap A_n\right|$$

Examples

- **Calculate the number of multiples of 3 or 5 from 1 to 500.**

Solution: Let A be the set of multiples of 3 from 1 to 500.

B is the set of multiples of 5 from 1 to 500.

$$|A| = \left\lfloor \frac{500}{3} \right\rfloor = 166, |B| = \left\lfloor \frac{500}{5} \right\rfloor = 100;$$

$$|A \cap B| = \left\lfloor \frac{500}{15} \right\rfloor = 33$$

The number of multiples of 3 or 5 is:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 166 + 100 - 33 = 233$$

A large software development company employs 100 computer programmers. Of them, 45 are proficient in Java, 30 in C#, 20 in Python, six in C# and Java, one in Java and Python, five in C# and Python, and just one programmer is proficient in all three languages above.

- Determine the number of computer programmers that are not proficient in any of these three languages.

- Let U denote the set of all employed computer programmers and let J, C and P denote the set of programmers proficient in Java, C# and Python, respectively. Thus:
- $|U| = 100$ $|J| = 45$ $|C| = 30$ $|P| = 20$
- $|J \cap C| = 6$ $|J \cap P| = 1$ $|C \cap P| = 5$
- $|J \cap C \cap P| = 1$

- Determine the number of computer programmers that are not proficient in any of these three languages.

- we need to determine the cardinality of the complement of the set $J \cup C \cup P$. (This is denoted as $|(J \cup C \cup P)'|$). Calculate $|J \cup C \cup P|$ first before determining the complement value:
- $|J \cup C \cup P| = 39 + 5 + 20 + 4 + 15 + 1 = 84$
- Now calculate the complement:
- $|(J \cup C \cup P)'| = |U| - |J \cup C \cup P|$
- $= 100 - 84$
- $= 16$
- 16 programmers are not proficient in any of the three languages.

Exercises

Q1. Given A={2,3,4,5,6,7} and B={0,1,2,3}. List the elements of the following sets:

1) $A \cap B$ 2) A-B 3) B-A 4) $A \triangle B$

Q2. Let the universal set be $U=\{x \in N \mid x \leq 10\}$. Its subsets A and B are given by $A=\{x \mid x$ is even$\}$, $B=\{x \in N \mid 4 \leq x < 9\}$. Find the following sets:

1) $A \cup B^c$ 2) $(A \cap B)^c$ 3) $(A-B)^c$

Q3. Find the elements of sets A and B if

A-B={a,b,d}, $A \cap B$={c,e} and $A \cup B$={a,b,c,d,e,g}

Q4.Let A,B, and C be sets. Draw the Venn diagram for the set combination $A \cap (B-C)$.

Q5. In a foreign language spoken survey taken out of 100 students, . 45 students learn Spanish, 28 spoke French, and 22 spoke Japanese. 12 students learn Spanish and French, 8 spoke Spanish and Japanese, and 10 spoke French and Japanese. 30 students learn no language. How many students learn three languages?

Q6. Let S be a finite set of natural numbers. It is known that there are 60 numbers among them multiple of 2, 75 numbers multiple of 3, 50 numbers multiple of 5, 20 numbers multiple of 6, 33 numbers multiple of 10, 15 numbers multiple of 15, and 11 numbers multiple of 30. Find the cardinality of the set S.

Lecture 3

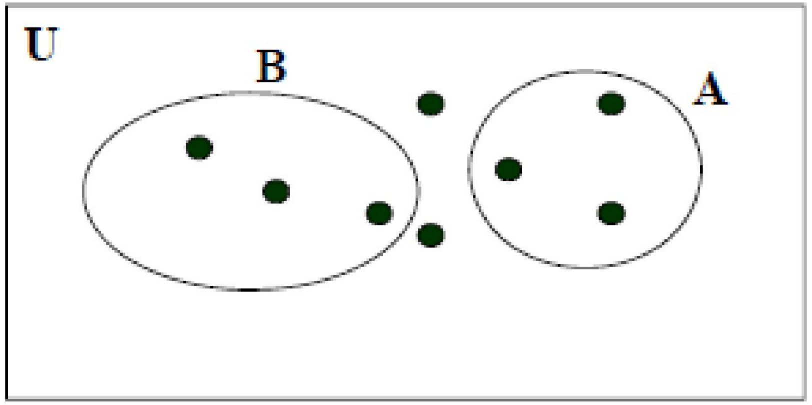
Outline of the Lecture 3

- Disjoint Sets
- Partition; Ordered Set
- Cartesian Product of Sets

Disjoint sets

- **Definition**: Two sets are called **disjoint** if their intersection is empty.

Alternate: A and B are disjoint **if and only if** $A \cap B = \emptyset$.



Example:

- A={1,2,3,6} B={4,7,8} Are these disjoint?

Yes.

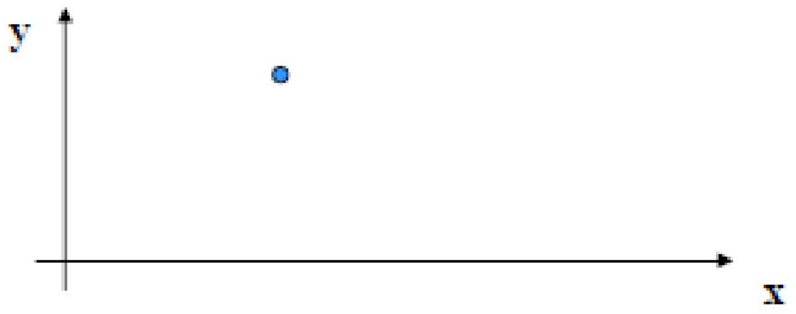
- $A \cap B = \emptyset$

N-tuple

Sets are used to represent unordered collections.

An ordered pair is defined as a set of two objects together with an order associated with them. Ordered pairs are usually written in parentheses (as opposed to curly braces, which are used for writing sets).

Example:



- Coordinates of a point in the 2-D plane (12, 16)

Cartesian product

- **Definition:** Let S and T be sets. The **Cartesian product of S and T**, denoted by **S x T,** is the set of all ordered pairs (s,t), where $s \in S$ and $t \in T$. Hence,
- $S \times T = \{ (s,t) \mid s \in S \land t \in T \}$.

Examples:

- S = {1,2} and T = {a,b,c}
- S x T = { (1,a), (1,b), (1,c), (2,a), (2,b), (2,c) }
- T x S = { (a,1), (a, 2), (b,1), (b,2), (c,1), (c,2) }
- Note: $S \times T \neq T \times S$ **!!!!**

Cartesian Product of Several Sets

- Cartesian products may also be defined on more than two sets.
- Let $A1,\ldots,An$ be n non-empty sets. The Cartesian product $A1\times\ldots\times An$ is defined as the set of all possible ordered $n-$tuples $(a1,\ldots,an)$, where $ai\in Ai$ and $i=1,\ldots,n$.
- If $A1=\ldots=An=A$, then $A1\times\ldots\times An$ is called the nth Cartesian power of the set A and is denoted by A^n.

Some Properties of Cartesian Product

1. The Cartesian product is non-commutative: $A\times B \neq B\times A$
2. $A\times B=B\times A$, if only $A=B$.
3. $A\times B=\varnothing$, if either $A=\varnothing$ or $B=\varnothing$
4. The Cartesian product is non-associative: $(A\times B)\times C \neq A\times(B\times C)$
5. Distributive property over set intersection: $A\times(B\cap C)=(A\times B)\cap(A\times C)$
6. Distributive property over set union: $A\times(B\cup C)=(A\times B)\cup(A\times C)$
7. Distributive property over set difference: $A\times(B-C)=(A\times B)-(A\times C)$
8. If $A\subseteq B$, then $A\times C \subseteq B\times C$ for any set C.

Cardinality of the Cartesian product

The cardinality of a Cartesian product of two sets is equal to the product of the cardinalities of the sets:

$|S \times T| = |S| * |T|$.

Similarly,

$|A1 \times \ldots \times An| = |A1| \times \ldots \times |An|$.

Example:

- A= {J, P, M}
- B ={R, A, L}
- A x B= {(J, R),(J, A) , (J, L), (P, R),(P, A) , (P, L) , (M, R) , (M, A) ,(M, L)}
- $|A \times B| = 9$
- $|A|=3$, $|B|=3$

$|A| * |B| = 9$

Partition of a set

- Partition of a set A is a collection of disjoint, non-empty subsets that have A as their union.

The union of the subsets in P is equal to A.

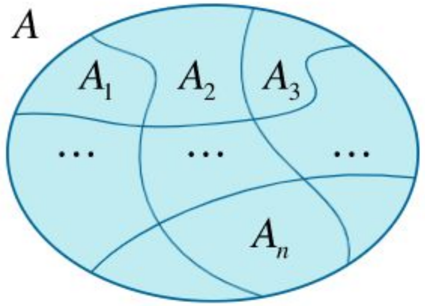
$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n = A$$

The partition P does not contain the empty set $\varnothing$.

$$A_i \neq \varnothing \ \forall i$$

The intersection of any distinct subsets in P is empty.

$$A_i \cap A_j = \varnothing \ \forall i \neq j$$



Example

A={1,2,3 } List all possible partitions:

P1: {1},{2},{3}
P2: {1,2},{3}
P3:{1,3},{2}
P4:{1},{2,3}
P5: {1,2,3}

Exercises

Q1.Given A={1,2,3} and B={1,2}. Find the following sets:

1)A×B 2) B×A 3) A^2

Q2. Given A={d,e,f} and B={e,f}. Find the following sets:

1) (A×B)∩(B×A) 2)(A×B)∪(B×A)

Q3. Find the Cartesian product A×P(A) if A={a,b}.

Q4. Let A={b_1,b_2,…b_n}. Compute the cardinality of the set P(A^m).

Exercise

Q5. Which of the following collections of subsets are partitions of {0,1,2,3,4,5}?

1. {0,1,2},{4,3},{5,4}
2. {},{0,2,1},{4,3,5}
3. {5,4,0,3},{2},{1}
4. {5},{4,3},{0,2}
5. {2},{1},{5},{3},{0},{4}

Lecture 4

Outline of the Lecture 4

- Algebra of Sets
- Bit vector representation of sets.

Algebra of sets

- The algebra of sets is the fundamental properties of set operations and set relations.
- It is the algebra of the set-theoretic operations of union, intersection and complementation, and the relations of equality and inclusion

Algebra of sets – Contd..

PROPOSITON 1: For any sets *A*, *B*, and *C*, the following identities hold	
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative Law
$(A \cup B) \cup C = A \cup (B \cup C)$ $(A \cap B) \cap C = A \cap (B \cap C)$	Associative Law
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive Law

PROPOSITON 2: For any subset *A* of universal set **U**, the following identities hold:	
$A \cup \varnothing = A$ $A \cap \mathbf{U} = A$	Identity Law
$A \cup A' = \mathbf{U}$ $A \cap A' = \varnothing$	compliment Law

PROPOSITON 3: For any subsets A and B of a universal set $\mathbf{U}$, the following identities hold:	
$A \cup A = A$ $A \cap \mathbf{A} = A$	Idempotent Law
$A \cup \mathbf{U} = \mathbf{U}$ $A \cap \varnothing = \varnothing$	Dominant Law
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption Law

PROPOSITON 4: Let A and B be subsets of a universe $\mathbf{U}$, then:	
$(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$	De Morgan's Law
$A'' = A$	Double complementation or involution Law
$\varnothing' = \mathbf{U}$ $\mathbf{U}' = \varnothing$	Complement laws for the universal set and empty set
If $A \cup B = \mathbf{U}$, and $A \cap B = \varnothing$ then $B = A'$.	uniqueness of complements:

PROPOSITON 5: If A, B and C are sets then the following hold:

$A \subseteq A$	reflexivity
$A \subseteq B$ and $B \subseteq A$ if and only if $A = B$	Antisymmetric
If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$	Transitivity

PROPOSITON 6: If A, B and C are subsets of a set S then the following hold:

$\varnothing \subseteq A \subseteq S$	Existence of a least element and a greatest element
$A \subseteq A \cup B$ If $A \subseteq C$ and $B \subseteq C$ then $A \cup B \subseteq C$	Existence of joins
$A \cap B \subseteq A$ If $C \subseteq A$ and $C \subseteq B$ then $C \subseteq A \cap B$	Existence of meets

The algebra of relative complements

PROPOSITION 9: For any universe **U** and subsets A, B, and C of **U**, the following identities hold:

$C - (A \cap B) = (C - A) \cup (C - B)$

$C - (A \cup B) = (C - A) \cap (C - B)$

$C - (B - A) = (A \cap C) \cup (C - B)$

$(B - A) \cap C = (B \cap C) - A = B \cap (C - A)$

$(B - A) \cup C = (B \cup C) - (A - C)$

$A - A = \varnothing$

$\varnothing - A = \varnothing$

$A - \varnothing = A$

$B - A = A' \cap B$

$(B - A)' = A \cup B'$

$\mathbf{U} - A = A'$

$A - \mathbf{U} = \varnothing$

Prove that $A - (A - B) = A \cap B,$

Let us start from the left-hand side,

$$\begin{aligned} A - (A - B) &= A - (A \cap \overline{B}), \\ &= A \cap \overline{(A \cap \overline{B})}, \\ &= A \cap (\overline{A} \cup B), \\ &= (A \cap \overline{A}) \cup (A \cap B), \\ &= \emptyset \cup (A \cap B), \\ &= A \cap B. \end{aligned}$$

Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} \\
&= \{x \mid \neg(x \in (A \cap B))\} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} \\
&= \{x \mid x \notin A \vee x \notin B\} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} \\
&= \overline{A} \cup \overline{B}
\end{aligned}
$$

Prove the following statement

$$(U \cap A) \cup (B \cap A) = A$$

Proof:

$(U \cap A) \cup (B \cap A) \stackrel{?}{=} A$	Given
$(A \cap U) \cup (A \cap B) \stackrel{?}{=} A$	Commutative Law
$A \cap (U \cup B) \stackrel{?}{=} A$	Distributive Law
$A \cap U \stackrel{?}{=} A$	Identity Law
$A = A$	Identity Law

Prove the following statement

$$(A \cap U) \cap (\phi \cup A^{c}) = \phi$$

Proof:

$(A \cap U) \cap (\phi \cup A^{c}) \stackrel{?}{=} \phi$	Given
$A \cap (\phi \cup A^{c}) \stackrel{?}{=} \phi$	Identity Law
$(A \cap \phi) \cup (A \cap A^{c}) \stackrel{?}{=} \phi$	Distributive Law
$\phi \cup (A \cap A^{c}) \stackrel{?}{=} \phi$	Identity Law
$\phi \cup \phi \stackrel{?}{=} \phi$	Complement Law
$\phi = \phi$	Idempotent Law

Prove the following statement

$$(A-B)^C = A^C \cup B$$

Proof:

$(A-B)^C \stackrel{?}{=} A^C \cup B$	Given
$(A \cap B^C)^C \stackrel{?}{=} A^C \cup B$	Difference Law
$A^C \cup (B^C)^C \stackrel{?}{=} A^C \cup B$	De Morgan's Law
$A^C \cup B = A^C \cup B$	Involution Law

Computer representation of sets

- **How to represent sets in the computer?**
- Assign a bit in a bit string to each element in the universal set and set the bit to 1 if the element is present otherwise use 0

Example:

All possible elements: U={1 2 3 4 5}

Assume A={2,5}

– Computer representation: A = 01001

Assume B={1,5}

– Computer representation: B = 10001

Computer representation of sets

- **Example:**
- •A = 01001
- B = 10001

The **union** is modeled with a bitwise **or**

- $A \cup B = 11001$
- The **intersection** is modeled with a bitwise **and**
- $A \sqcap B = 00001$
- The **complement** is modeled with a bitwise **negation**
- $\bar{A}$ =10110

Exercises

Q1. Let A and B two sets. Show that

a. $A \cap B$ and $A \cap B^c$ are disjoint.

b. $A = (A \cap B) \cup (A \cap B^c)$

Q2. Show that $A \oplus B = (A \cup B) - (A \cap B)$.

Q3. What can you say about the sets A and B if $A \oplus B = A$?

Q4. Suppose that the universal set is $U = \{1, 2, 3, 4,5, 6, 7, 8, 9, 10\}$. Express each of these sets with bit strings where the ith bit in the string is 1 if i is in the set and 0 otherwise.

a) $\{3, 4, 5\}$

b) $\{1, 3, 6, 10\}$

c) $\{2, 3, 4, 7, 8, 9\}$

Q5. Using the same universal set as in the last problem, find the set specified by each of these bit strings.

a) 11 1100 1111

b) 01 0111 1000

c) 10 0000 0001