Solutions to Sheet 3

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Exercise 1

Let $R = (r_i^{(s)})$ a payoff matrix with $R^{(t)} = \sum_{i=1}^{n} p_i^{(t)} r_i^{(s)}$

We see that the weight update rule depends on the payoff matrix :

$$\omega_i^{(t+1)} = \omega_i^{(t)}(1 + \alpha R_{it})$$

We have the potential function:

$$\Phi^{(t)} = \sum_{i=1}^{n} \omega_i^{(t)}$$

So :

$$\Phi^{(t+1)} = \sum_{i=1}^{n} \omega_i^{(t)}(1 + \alpha R_{it})$$

$$\Phi^{(t+1)} = \sum_{i=1}^{n} \omega_i^{(t)} + \sum_{i=1}^{n} \omega_i^{(t)} \alpha R_{it}$$

$$\Phi^{(t+1)} \leqslant \Phi^{(t)} + \alpha \sum_{i=1}^{n} \omega_i^{(t)}$$

$$\Phi^{(t+1)} \leqslant (1 + \alpha)\Phi^{(t)}$$

By induction we can find:

$$\Phi^{(t+1)} \leqslant n(1 + \alpha)^t \, (*)$$

So:

$$\sum_{i=1}^{n} \omega_i^{(t+1)} \leqslant n(1 + \alpha)^t$$

$$\omega_i^{(t+1)} \leqslant n(1 + \alpha)^t$$

$$\prod_{s=1}^{t}(1+\alpha R_{it}) \leqslant n(1+\alpha)^t$$

$$ln(\prod_{s=1}^{t}(1+\alpha R_{it})) \leqslant ln(n(1+\alpha)^t)$$

$$\sum_{s=1}^{t} ln((1+\alpha R_{it})) \leqslant ln(n) + t * ln(1+\alpha)$$

$$\frac{\sum_{s=1}^{t} ln((1+\alpha R_{it})) - ln(n)}{ln(1+\alpha)} \leqslant t$$

From the inequality (3) given in the hint we find:

$$\frac{1}{ln(1+\alpha)} \geqslant \frac{1}{\alpha}$$

From the inequality (2) given in the hint we find:

$$ln((1+\alpha R_{it})) \geqslant \alpha R_{it}(1-\alpha R_{it})$$

$$ln((1+\alpha R_{it})) \geqslant \alpha R_{it}(1-\alpha)$$

So finally we find :

$$\frac{\sum_{s=1}^{t} \alpha R_{it}(1-\alpha) - ln(n)}{\alpha} \leqslant t$$

It's easy to see that :

$$t \leqslant \sum_{s=1}^{t} R^{(t)}$$

Thus:

$$\sum_{s=1}^{t} R_{it}(1-\alpha) + \frac{-ln(n)}{\alpha} \leqslant \sum_{s=1}^{t} R^{(t)}$$

Excercise 2

a)

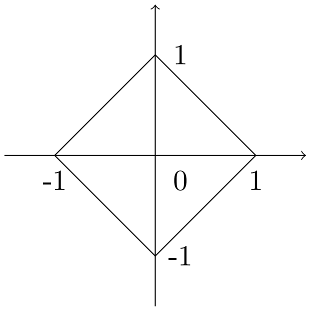

Figure 1: B_1^2

In $\mathbb{R}^3$ B_1^3 is a double pyramid. It's corners are located weher exactly one entry of the point is 1 or -1 and all others 0.

b)

$$\text{vol}(B_1^2) = 2\frac{1}{2} \cdot 2 \cdot 1 = 2$$

$$\text{vol}(B_1^3) = 2\frac{1}{3} \cdot \sqrt[2]{2}^2 \cdot 1 = \frac{4}{3}$$

c)

We can derive theat $vol(B_1^n) = 2\frac{1}{n} \sqrt[n-1]{2}^{n-1}$.

$$\lim_{n\to\infty} vol(B_1^n) = 2 \lim_{n\to\infty} \frac{1}{n} \cdot 2 = 0$$

Exercise 3

We know that for a hypercube of dimension d and edge length l the spacial diagonal has length

$$s = l\sqrt{d}$$

. We also have the radius of the outer hyperballs given as

$$r_o = \frac{l}{4}$$

. So the radius of the inner hyperball is half the difference of the spacial diagonal and two times the diameter of the outer hyperballs. That leads to

$$r_i = \frac{1}{2}s - 4 * r_o = \frac{1}{2}l\sqrt{d} - l$$

. Using this measure we can we can require the diameter of the inner hyperball to be larger than the sides of the hypercube.

$$l\sqrt{d} - l > l$$

which resolves to

$$d > 4$$

.

So a dimension of at least five results in the hyperball 'sticking out'.

Exercise 4

$$\begin{aligned} \mu \neq 0 \text{ is an eigenvalue of } A^T A \\ \Leftrightarrow \det\left(A^T A - \mu I\right) = 0 \\ \Leftrightarrow \det\left(I + \frac{-1}{\mu} A^T A\right) = 0 \\ \Leftrightarrow \det\left(I + A \frac{-1}{\mu} A^T\right) = 0 \\ \Leftrightarrow \det\left(AA^T - \mu I\right) = 0 \\ \mu \text{ is an eigenvalue of } AA^T \end{aligned}$$

Exercise 5

a)

if $A = BB^T$: observe that A i s symmetric.

$$x^T A x = x^T B B^T x = (B^T x)^T B^T x = ||B^T x|| \geq 0$$

so A is psd.

if A is psd: THere is an orthogonal matrix U such taht $A = UDU^T$ with $D = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$. We can write $D = R \times R$ with $R = \mathrm{diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n})$. So

$$A = UR \times RU^T = UR(UR)^T$$

. So there is a $B = UR$ such that $A = BB^T$.

b)

Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. A is not psd because $\begin{pmatrix} -1 & 1 \end{pmatrix} A \begin{pmatrix} -1 & 1 \end{pmatrix}^T = -2 < 0$.