Solutions to Sheet 4

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Exercise 1

From the given points the centered datamatrix

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 3.5 & 0 & 0.5 \\ 2.5 & 2 & 1.5 \\ -2.5 & -2 & -1.5 \\ -3.5 & 0 & -0.5 \end{bmatrix}$$

can be created. The covariance matrix

$$C = A^T A = \begin{bmatrix} 37 & 10 & 11 \\ 10 & 8 & 6 \\ 11 & 6 & 5 \end{bmatrix}$$

is derived from the datamatrix A. From the characterisitc polynominal of C $-\lambda^3 + 50\lambda^2 - 264\lambda$ the eigenvalues

$$\lambda_1 = 44, \lambda_2 = 6, \lambda_3 = 0$$

can be calculated. Using the corresponding eigenvectors

$$v_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ -4 \\ 7 \end{pmatrix}$$

the PCA-transformation

$$U = \begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{bmatrix}$$

of A can be calculated. Therefore the first and second principal components of A are

$$\mathbb{P}_1 = \text{span}\left(\begin{pmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{pmatrix}\right), \mathbb{P}_2 = \text{span}\left(\begin{pmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}\right)$$

. That leads to the best fit subspaces

$$\mathbb{U}_1 = \mathbb{P}_1, \mathbb{U}_2 = \text{span}\left(1/\sqrt{11}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, 1/\sqrt{6}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}\right)$$

Exercise 2

a)

$A \in \mathbb{R}^{nxn}$ is a symetric matrix. Let $(a_i)_{i\in[n]}$ be the eigenvectors and $(\lambda_i)_{\lambda\in[n]}$ the eigenvalues.
if $\lambda_i \geq 0$:
We have $A = U\text{diag}(\lambda_i))U^T$ with $U \in \mathbb{R}^{nxn}$ ortogonal matrix. Let $x \in \mathbb{R}^n$

$$x^TAx = x^TU\text{diag}(\lambda_i))U^Tx = \sum_{i=1}^{n} \lambda_i \left([U^Tx]_i\right)^2 \geq 0$$

.
if $x^TAx \geq 0$:
for $x = a_i$

$$a_i^TAa_i = \lambda_i a_i^T a_i \geq 0 \text{and} a_i^T a_i \geq 0$$

. So $\lambda_i \geq 0$.

b)

For

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$x^TAx = \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -3 < 0$$

Exercise 3

a) If we compute the power iteration on M_1 and M_2 for 5 iterations, we find the following eigenvectors:
   - for M_1 : (-0.948, -0.318)
   - for M_2 : (0.410, 0.404, 0.817)

b) if we compute three iterations of the power iteration procedure for M_3, we find : (0.577, 0.577, -0.577).

The Power Iteration will not converge, because if we calculate the eigenvalues of M_3, we will find 2 values -2 and 2. So we don't have the assumption 5.24. That's why the Power Iteration Algo fails.

c) M_1 have two eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -1$

If we apply the power iteration algo on M_1^{-1} , we will find the eigenvector of M_1 associated with λ_2 We find :(0.970, 0.239)

d)
- for M_4 it needs 3 iterations and the eigenvector is :

  (0.6401, 0.7682)
- for M_5 it needs 60 iterations and the eigenvector is :

  (0.5814, 0.8135)

Exercise 4

a)

Compute S:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$L = D - S$

$$D = \begin{pmatrix} 2 & 0 & & & \cdots & & & & 0 \\ 0 & 3 & 0 & & & & & & \\ & 0 & 2 & 0 & & & & & \\ & & 0 & 3 & 0 & & & & \\ \vdots & & & 0 & 2 & 0 & & & \vdots \\ & & & & 0 & 3 & 0 & & \\ & & & & & 0 & 3 & 0 & \\ & & & & & & 0 & 3 & 0 \\ 0 & & & & \cdots & & & 0 & 2 \end{pmatrix}$$

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 3 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

b)

The calculation of the eigenvalues of L yielded the eigenvalues 0,0.6972,0.6972,3,3,3,4.3028,4.3028,5. The three smallest eigenvalues therefore are $\lambda_1 = 0, \lambda_2 = 0.6972 \text{and} \lambda_3 = 0.6972$. The corresponding eigenvectors are
$e_1 = (-1/3 \quad -1/3 \quad -1/3 \quad -1/3 \quad -1/3 \quad -1/3 \quad -1/3 \quad -1/3 \quad -1/3)^T$, $e_2 = (-0.2331 \ -0.2605 \ -0.0432 \ -0.3234 \ -0.3507 \ -0.1335 \ 0.3940 \ 0.3666 \ 0.5838)^T$, $e_3 = (0.5396 \ 0.3046 \ 0.3984 \ -0.2367 \ -0.4717 \ -0.3778 \ 0.0733 \ -0.1617 \ -0.0679)^T$

c)

From the eigenvectors the matrix U can be derived

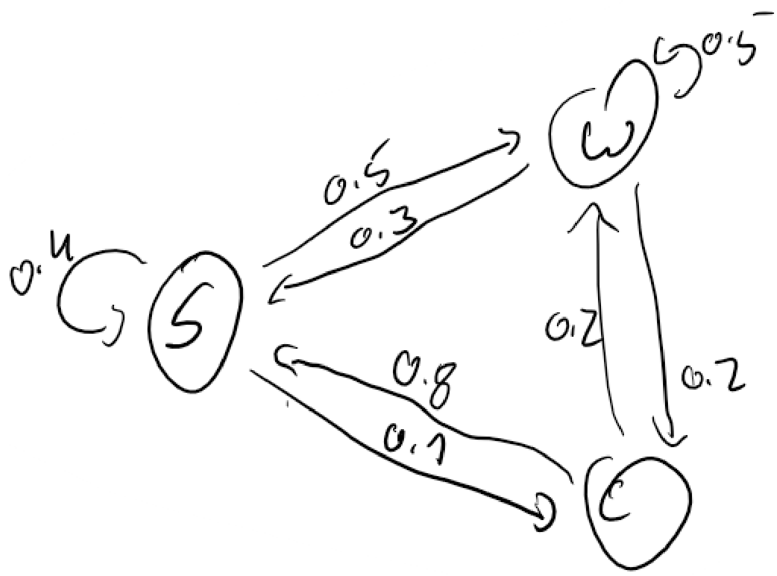
$$U = \begin{pmatrix} -0.3333 & -0.2331 & 0.5396 \\ -0.3333 & -0.2605 & 0.3046 \\ -0.3333 & -0.0432 & 0.3984 \\ -0.3333 & -0.3234 & -0.2367 \\ -0.3333 & -0.3507 & -0.4717 \\ -0.3333 & -0.1335 & -0.3778 \\ -0.3333 & 0.3940 & 0.0733 \\ -0.3333 & 0.3666 & -0.1617 \\ -0.3333 & 0.5838 & -0.0679 \end{pmatrix}$$

Splitting the rows of U using k-Means clustering yields the clusters $\{U_1, U_2, u_3\}, \{U_4, U_5, U_6\}, \{U_7, U_8, U_9\}$.

Exercise 5

a)

The Markov Chain for the sudents behaviour can be described by the following graph:



b)

We can derive the transition matrix

$$\mathcal{Q} = \begin{pmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0.8 & 0.2 & 0 \end{pmatrix}$$

The initial distribution is the distribution after leaving the state c. So $p_0 = \begin{pmatrix} 0.8 & 0.2 & 0 \end{pmatrix}$. From this distribution the propability distribution after 2 hours can be dervied:

$$p_2 = p_0 * \mathcal{Q}^2 = (0.398\ 0.464\ 0.138)$$

Therefore the proibaility of the student starting to clean two hours later is 13.8%.

c)

To find the long-term fraction of time the student is studying we need to find the stationary distribution of the Markov Chain. The Martkov Chain obviously is ergodic. Therefore we can find a stationary distribution π by calculating $p_t = p_0 * \mathcal{Q}^t$ for any starting distribution until $p_t = p_{t+1}$.

t	p_t
0	(0.398 0.464 0.138)
1	(0.4088 0.4586 0.1326)
2	(0.40718 0.46022 0.1326)
3	(0.407018 0.46022 0.1326762)
4	(0.407083 0.460171 0.1326746)
5	(0.407081 0.460176 0.1326743)
6	(0.407079 0.460177 0.1326743)
7	(0.407078 0.460177 0.1326743)
8	(0.407078 0.460177 0.1326743)

The fraction of time the student is studying in the long run is eqivalent to the stationary propability of the student choosing to study wich is 40.708%.