Solutions to Sheet 5

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June 24, 2020

Exercise 1

We have $\mathcal{Q}$ a symmetric Markov chain. Let Q be it's transition matrix. From Lemma 6.4 we have π is the stationary distribution then $\pi_i a_{ij} = \pi_j a_{ji}$ for all $i, j \in [n]$ and $a_{ij} = a_{ji}$. So $\pi_i = \pi_j$ for all $i \in [n]$. So π is uniform over $[n, n]$.

So for every connected,m symmetric Markov chain we have the same stationary distribution π and we know that π is unique from Theorem 6.2.

Exercise 2

In this case we have a connected Markov chain because in the given directed graph G_Q there is a path from i to j for all $i, j \in [n]$. So the Lemma 6.4 holds ($\pi_i q_{ij} = \pi_j q_{ij}$). The transition matrix is

$$Q = \begin{pmatrix} \frac{13}{24} & \frac{1}{12} & \frac{3}{8} & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{4} & \frac{1}{12} & 0 & \frac{1}{6} \\ 0 & \frac{1}{8} & \frac{1}{4} & \frac{5}{8} \end{pmatrix}$$

Argument (1): The transition matrix Q is a stochastic matrix, that is, all entries are non-negative and all row sums are 1.
Entry q_{24}: argument (1): $q_{24} = 1 - 0.5 - 0.25 = 0.25 \Rightarrow q_{24} = 0.25$
Entry q_{32}: $\pi_3 q_{32} = \pi_2 q_{23} \Leftrightarrow 0.25 q_{32} = \frac{1}{12} \cdot \frac{1}{4} \Leftrightarrow q_{32} = \frac{1}{12}$
Entry q_{34}: argument (1): $q_{34} = 1 - 0.75 - \frac{1}{12} \Rightarrow q_{34} = \frac{1}{6}$
Entry q_{42}: $\pi_4 q_{42} = \pi_2 q_{24} \Leftrightarrow \frac{1}{6} q_{42} = \frac{1}{12} \cdot \frac{1}{4} \Leftrightarrow q_{42} = \frac{1}{8}$
Entry q_{43}: $\pi_4 q_{43} = \pi_3 q_{34} \Leftrightarrow \frac{1}{6} q_{43} = \frac{1}{6} \cdot \frac{1}{4} \Leftrightarrow q_{43} = \frac{1}{4}$
Entry q_{44}: argument (1): $q_{44} = 1 - \frac{1}{8} - \frac{1}{12} \Rightarrow q_{44} = \frac{5}{8}$
Entry q_{13}: $\pi_1 q_{13} = \pi_3 q_{31} \Leftrightarrow 0.5 q_{13} = \frac{1}{4} \cdot \frac{3}{4} \Leftrightarrow q_{13} = \frac{3}{8}$
Entry q_{12}: $\pi_1 q_{12} = \pi_2 q_{21} \Leftrightarrow 0.5 q_{12} = \frac{1}{12} \cdot \frac{1}{2} \Leftrightarrow q_{12} = \frac{1}{12}$
Entry q_{11}: argument (1): $q_{11} = 1 - 0 - \frac{3}{8} - \frac{1}{12} \Rightarrow q_{11} = \frac{13}{24}$

Exercise 4

We want to sample from the probability space (U,P), but now we assume that $U = D^l$ for some finite set D.

We define the transition matrix $Q = (q_{uv}) \in R^{U \times U}$ of a Markov chain as follows. Let $u = (u_1, ..., u_l), v = (v_1, ..., v_l) \in U$.

- If u and v differ in more than one elements then $q_{uv} = q_{vu} = 0$.

  So $P(u)q_{uv} = P(v)q_{vu} = 0$.

- If there is exactly one $i \in [l]$ such that $u_i \neq v_i$ then $q_{uv} = \frac{1}{l}P(v_i/u_1, .., u_{i-1}, u_{i+1}, ..., u_l)$ and $q_{vu} = \frac{1}{l}P(u_i/v_1, .., v_{i-1}, v_{i+1}, ..., v_l)$. So :

  $P(u)q_{uv} = \frac{1}{l}P(u)P(v_i/u_1, .., u_{i-1}, u_{i+1}, ..., u_l)$

  $P(u)q_{vu} = \frac{1}{l}P(u)\frac{P(v)}{\sum_{v \in D} P(u_1,..,u_{i-1},v,u_{i+1},...,u_l)}$

  $P(u)q_{vu} = \frac{1}{l}P(v)\frac{P(u)}{\sum_{v \in D} P(v_1,..,v_{i-1},v,v_{i+1},...,v_l)}$

  $P(u)q_{uv} = \frac{1}{l}P(v)P(u_i/v_1, .., v_{i-1}, v_{i+1}, ..., v_l)$

  $P(u)q_{uv} = P(v)q_{vu}$

- If u=v then $P(u)q_{uv} = P(v)q_{vu}$

So from Lemma 6.4 we conclude that the probability distribution P is the unique stationary distribution of the Markov chain.

Exercise 5

To calculate the expected total aggregated runtime for a job of n tasks we need to know the expected runtime of a task.

$$E(R) = nE(R_1)$$

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The expected runtime of a task consists of the time the task itself takes t and the expected additional recovery time A_1

$$E(R_1) = t + E(A_1)$$

. We know that a failure during the execution or the recovery of a task happens with probability $p_f \in [0, 1)$. The probability for a task to fail k times therefore is p_f^k. With every recovery takeing $10t$ this yields a expected additional recovery time

$$E(A_1) = 10t\sum_{k=1}^{\infty} kp^k = 10t\frac{p_f}{(p_f - 1)^2}$$

. Therefore the expected total accumulated runtime is

$$E(R) = n\left(t + 10t\frac{p_f}{(p_f - 1)^2}\right)$$

.

Exercise 6

a)

Map: Input:

$$(P, \text{Friends}_P)$$

Output:

$$(\{P_1, P_2\}, \{\text{Friends}_{P_1}, \text{Friends}_{P_2}\})$$

Reduce: Input:

$$(\{P_1, P_2\}, \{\text{Friends}_{P_1}, \text{Friends}_{P_2}\})$$

Output:

$$(\{P_1, P_2\}, \{P | P \in \text{Friends}_{P_1} \wedge P \in \text{Friends}_{P_2}\})$$

b)

Map: Input:

$$(P, \text{Friends}_P)$$

Output:

$$(P, \text{Friends}_F) \forall F \in \text{Friends}_P$$

Reduce: Input:

$$(P, \text{Friends}_F)$$

Select 10 values Friends_F. From every value select one person $F_i \neq P$.
Output:

$$(P, \{F_i\})$$