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Exercise Sheet 3

Due date: Friday, May 29 until 15:00

- **Note that the due date of this sheet is earlier than usual due to the excursion week.**
- Please upload your solutions to Moodle.
- Hand in your solutions in groups of **two to three students**.
- Please hand in the solutions of your group as a single PDF file.
- You will not be able to change your upload.
- The solutions of this exercise will be discussed live via Zoom on **Friday, June 12 at 12:30**.

Exercise 1 (MWU with Payoffs) 8 points

We consider the multiplicative weight update algorithm. In some situations it is easier to model the problem using *payoffs* (also called *gains* or *rewards*) instead of costs. For this we use the weight update rule

$$w_i^{(t+1)} := w_i^{(t)}\left(1 + \alpha \cdot r_i^{(t)}\right)$$

where $r_i^{(t)}$ is the *reward* of following expert i in round t. The choice which expert to follow is done randomly according to

$$p_i^{(t)} := \frac{w_i^{(t)}}{\sum_{j=1}^{n} w_j^{(t)}}.$$

Show that the MWU algorithm can be extended to work with a *payoff matrix* ($n \times t$, with entries $r_i^{(s)} \in [0,1]$) and is then able to achieve a bound on the *expected payoff in round t* of

$$\sum_{s=1}^{t}\sum_{i=1}^{n} r_i^{(s)} p_i^{(s)} \geq -\frac{\ln n}{\alpha} + (1-\alpha)\sum_{s=1}^{t} r_j^{(s)}$$

for all $t \geq 1$ and all $j \in [n]$.

Hint: You can follow the general idea of the proof of Theorem 4.2 in the lecture. The following inequalities may be useful for achieving the desired bound (you may use them without showing them to hold):

$$1 + \alpha x \geq (1+\alpha)^x \quad \text{for all } x \in [0,1] \text{ and } \alpha > -1. \tag{1}$$

$$\ln(1+\alpha) \geq \alpha - \alpha^2 \quad \text{for all } \alpha \geq 0. \tag{2}$$

$$1 + x \leq e^x \quad \text{for all } x \in \mathbb{R}. \tag{3}$$

Exercise 2 (Unit Balls of the L_1-Norm) **1+2+2=5 points**

Recall that the L_1-norm of a vector $\mathbf{x} = (x_1, \dots, x_d)^\mathsf{T} \in \mathbb{R}^d$ is defined as $\|x\|_1 := \sum_{i=1}^{d} |x_i|$. The *$d$-dimensional L_1 unit ball* is defined as $B_1^d := \{\mathbf{x} \in \mathbb{R}^d : \|x\|_1 \leq 1\}$.

a) Draw $B_1^2 \subseteq \mathbb{R}^2$ in the plane. Describe the shape of $B_1^3 \subseteq \mathbb{R}^3$.

b) Compute $\mathrm{vol}(B_1^2)$ and $\mathrm{vol}(B_1^3)$.

c) Show that $\lim_{d\to\infty} \mathrm{vol}(B_1^d) = 0$.

Exercise 3 (Balls in a Hypercube) **5 points**

Consider a d-dimensional *hypercube* Q of side length $\ell \in \mathbb{R}$, i. e. $|x_i - y_i| \leq \ell$ for all $\mathbf{x}, \mathbf{y} \in Q$ and all $i \in [d]$. Note that Q has 2^d corners. We fill Q with *(L_2-)hyperballs* the following way:

- We place 2^d hyperballs of radius $\frac{\ell}{4}$ close to the 2^d corners of Q, such that their distance to the center of Q is maximal while still being completely contained in Q.
- We place an additional single hyperball in the center of Q such that its radius is maximal with the property that it intersects with none of the other hyperballs' intereriors.

What is the minimal dimension d such that the the surface of the central hyperball peeks trough the surface of Q? Justify your answer.

Exercise 4 (Eigenvalues and Geometric Multiplicity) **5 points**

Let $A \in \mathbb{R}^{m\times n}$. Show that $A^\mathsf{T}A$ and AA^T have exactly the same *non-zero* eigenvalues, counting geometric multiplicities. That is, show that for all $\lambda \in \mathbb{R} \setminus \{0\}$ it holds that

(i) λ is a (non-zero) eigenvalue of $A^\mathsf{T}A$ with geometric multiplicity k

if and only if

(ii) λ is a (non-zero) eigenvalue of AA^T with geometric multiplicity k.

Exercise 5 (Positive Semi-Definite Matrices) **5+2=7 points**

Recall that a symmetric matrix $A \in \mathbb{R}^{n\times n}$ is *positive semi-definite* (or, *psd*) if (and only if) all of its eigenvalues are nonnegative.

a) Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Prove that A is psd if and only if there exists $B \in \mathbb{R}^{n\times n}$ such that $A = BB^\mathsf{T}$.

b) Find a matrix $A \in \mathbb{R}^{2\times 2}$ with only positive entries (i. e. $A \in \mathbb{R}^{2\times 2}_{>0}$) such that A is *not* psd. Justify your answer.