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Exercise Sheet 4

Due date: Monday, June 15 until 15:00

- Please upload your solutions to Moodle.
- Hand in your solutions in groups of **two to three students**.
- Please hand in the solutions of your group as a single PDF file.
- Mark your answers clearly.
- The topics needed for exercise 5 will be published in week 7.
- The solutions of this exercise will be discussed live via Zoom on **Friday, June 19 at 12:30**.

Exercise 1 (Principal Component Analysis) 5 points

Consider the data points given below. Compute the first and second principal component of the data points via Spectral Decomposition of the Covariance Matrix. Compute the best-fit 1- and 2-dimensional subspace. Give exact results.

$A = (3, 1, 1)$; $B = (6.5, 1, 1.5)$; $C = (5.5, 3, 2.5)$; $D = (0.5, -1, -0.5)$; $E = (-0.5, 1, 0.5)$

Exercise 2 (Positive Semi-Definite Matrices II) 4+1=5 points

The following is an equivalent characterisation of positive semi-definiteness:
A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive semi-definite if (and only if) for all $\mathbf{x} \in \mathbb{R}^n$ it holds that $\mathbf{x}^\mathsf{T} A \mathbf{x} \geq 0$.

a) Show that the two definitions are indeed equivalent. That is, show that for every symmetric matrix $A \in \mathbb{R}^{n \times n}$ the following holds: all eigenvalues of A are non-negative if and only if $\mathbf{x}^\mathsf{T} A \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

b) Find a matrix $A \in \mathbb{R}^{2 \times 2}$ with only positive entries and some $\mathbf{x} \in \mathbb{R}^2$ such that $\mathbf{x}^\mathsf{T} A \mathbf{x} < 0$.

Exercise 3 (Power Iteration) **2+2+2+1=7 points**

Power iteration works since the successive multiplication with the same matrix shifts a randomly generated vector towards the eigenvector which belongs to the eigenvalue of the largest magnitude (the dominant eigenvalue). The algorithm starts with a random vector and terminates when this vector does not change anymore.
For the computations of the power iteration, we always start with the appropriate vector $\mathbf{x}$ consisting only of ones. For three dimensions this is $\mathbf{x} = (1, 1, 1)^T$.

a) Compute the power iteration on M_1 and M_2 for 5 iterations.

b) Compute three iterations of the power iteration procedure for M_3. Will the Power Iteration converge? If not, why does Power Iteration fail on this matrix? Justify your answer.

c) Observe that, if A is non-singular, then from $A\mathbf{x} = \lambda\mathbf{x}$ we get $A^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}$. Use this to compute the eigenvalue with the smallest magnitude and corresponding eigenvector of M_1. What is the exact result?

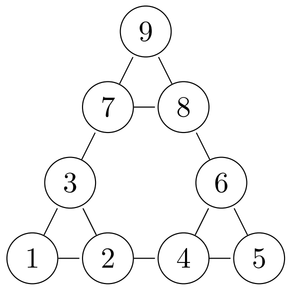
d) How many iterations does the algorithm need until the eigenvector becomes stable for up to 3 significant digits for the matrices M_4 and M_5?

$$M_1 = \begin{pmatrix} 2 & -12 \\ 1 & -5 \end{pmatrix};\ M_2 = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix};\ M_3 = \begin{pmatrix} 1 & 1 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix};\ M_4 = \begin{pmatrix} 4 & 5 \\ 6 & 5 \end{pmatrix};\ M_5 = \begin{pmatrix} -4 & 10 \\ 7 & 5 \end{pmatrix}$$

Hint: You do not need to hand in any code, if used. It suffices to give the results up to 3 significant digits, for task a)-c), and 4 significant digits for task d).

Exercise 4 (Spectral Clustering) **2+3+2=7 points**

Consider the following graph G



and the similarity measure $s\colon [9] \times [9] \to \mathbb{R}_{\geq 0}$ defined by $s(v, w) = 1$ if vw is an edge and $s(v, w) = 0$ otherwise. We want to cluster the vertices of the graph using spectral clustering methods.

a) Compute the Laplacian of the similarity matrix S.

b) Compute the three smallest eigenvalues and their eigenvectors (of the Laplacian computed in part a)).

c) Cluster the vertices into three sets using the results from part b) (applying the spectral clustering from the lecture).

Exercise 5 (Study Times) **1+2+3=6 points**

We model the behavior of a student in the exam preparation period by the three states studying, watching series and cleaning his/her room. We assume that the student switches between those activities hourly. If the student was studying, he/she will continue studying in 40% of the cases, in 10% of the cases he/she tries distraction by cleaning and in 50% he/she gives into watching series. If the student was watching series, he/she continues watching in half the cases, start studying in 30% the cases and start cleaning in the other cases. The student hates cleaning. He/she will never continue cleaning, but start studying in 80% of the cases and watch series in 20% of the cases.

a) Model the behaviour of the student as a Markov Chain. Draw the corresponding graph.

b) Assume the student is currently done cleaning. How probable is it that he/she starts cleaning again after 2 hours pass?

c) Which fraction of the time is this student actually studying (in the long term)?