Solutions to Sheet 1

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Exercise 1

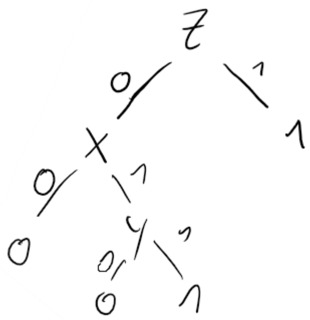
To calculate the class for every point we first need to find the k closest points (depending on the norm). To archive this, a script like the following can be used:

```
function [ nbs ] = knn( spc,pnt,k, norm_p )
%KNN Summary of this function goes here
%   Detailed explanation goes here
  nbs = [];
  nonb = 0;
  while nonb < k
    if nonb == 0
      candidates = spc;
    else
      candidates = transpose(setdiff(transpose(spc),transpose(nbs),'rows
    ↪ '));
    end
    closestp = [0,0,0];
    closestn = inf;
    for i = 1:length(candidates)
      d =transpose(pnt-candidates(:,i));
      n=norm(d,norm_p);
      if n < closestn
        closestn = n;
        closestp = candidates(:,i);
      end
    end
    nonb = nonb+1;
    nbs(:,nonb) = closestp;
  end
end
```

Using the results the following table can be created. Note that the ties can be arbirarily chosen.

Norm	Euclidian		Manhattan	
k	2	3	2	3
4,3,3	1	1	1	1
4,-1,1	1(tie)	-1	1(tie)	1
-2,4,5	-1	-1	1(tie)	1
-2,-6,1	1(tie)	1	1(tie)	1
6,0,2	1(tie)	1	-1	-1

Exercise2



The first split is done using z, because this is the only variable for wich the result stays the same for a value of the variable. The other two splits could be swapped because the influence on the result is the same for both variables.

Exercise 3

a)

We have

$$S = ((x_1, y_1), ..., (x_k, y_k))$$

with

$$x_i \in \{-1, 1\}^n, y_i \in \{-1, 1\} \, \forall i \in [k]$$

Let p be the number of updates needed for the Perceptron Algorithm.
So we have (1) :

$$\langle w_p, x_i y_i \rangle \geq 0 \ \forall i \in [k]$$

And we have also :

$$w_p - w_{p-1} = x_{p-1} y_{p-1}$$

So :

$$\langle w_p - w_{p-1}, x_{p-1} y_{p-1} \rangle = \langle x_{p-1} y_{p-1}, x_{p-1} y_{p-1} \rangle$$

$$\langle w_p - w_{p-1}, x_{p-1} y_{p-1} \rangle = n$$

$$\langle w_p, x_{p-1}y_{p-1}\rangle - \langle w_{p-1}, x_{p-1}y_{p-1}\rangle = n$$

Using (1) we have :

$$\langle w_{p-1}, x_{p-1}y_{p-1}\rangle + n \geq 0$$

$$\|w_{p-1}\| \, \|x_{p-1}y_{p-1}\| + n \geq \langle w_{p-1}, x_{p-1}y_{p-1}\rangle + n \geq 0$$

And we saw in the lecture that for each $i \leq p-1$:

$$\|w_i\| \leq \sqrt{i}$$

Thus :

$$n + n^2(p-1) \geq n + n^2\sqrt{p-1} \geq 0$$

Finally:

$$1 - \frac{1}{n} \leq p$$

b)

We have that the function:

$$maj(x_i) = \begin{cases} 1 & \text{if } \sum_{j=1}^{n} x_{ij} > 0 \\ -1 & \text{else} \end{cases}$$

If we have $\sum_{j=1}^{n} x_{ij} > 0$, we need to find w such as

$$\langle w, x_i\rangle \geq 0 \; \forall i$$

$$\langle w, x_i\rangle = \sum_{j=1}^{n} w_j x_{ij}$$

for w=(1,..,1) we have

$$\langle w, x_i\rangle = \sum_{j=1}^{n} x_{ij} > 0$$

We need to normalize this w, so finally our normalized vector w is :

$$w = (\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}})$$

To find an upper bound on the number of updates of w the Perceptron Algorithm performs for any training set for the function maj, we need to find the margin:

$$min_{(x,y)\in S} |\langle w, x\rangle|$$

And S is a normalised set We have for each p:

$$|\langle w, x_p\rangle| = \left|\sum_{j=1}^{n} \frac{x_{pj}}{n}\right|$$

And because n is an odd number so will must have :

$$\frac{1}{n} \leq \left| \sum_{j=1}^{n} \frac{x_{pj}}{n} \right|$$

So we found a lower bound, we just need to find a vector that verify it, and a vector that have the majority positive or negative by 1 elements verify it.
So the margin is:

$$\gamma = \frac{1}{n}$$

Thus (using theorem 1.10) we find that the upper bound is :

$$n^2 = \frac{1}{\gamma^2}$$

c)

Yes, we will still find a linear separator that realizes maj. Because the Perceptron Algorithm has no restriction on the set.

Exercise 4

a)

The Certer is underlined.

Round 1

Cluster 1: $\underline{A}$
Cluster 2: $\underline{B}$
Cluster 3: $\underline{C}, D, E, F, G, H$

Round 2

Cluster 1: $\underline{A}$
Cluster 2: $\underline{B}, C$
Cluster 3: $\underline{(7.5, 6)}, D, E, F, G, H$

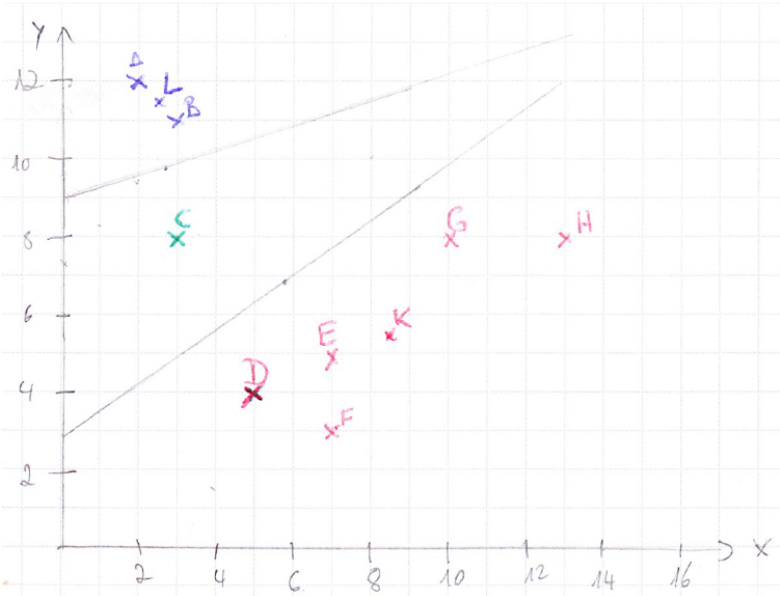
Round 3

Cluster 1: $\underline{A}, B$
Cluster 2: $\underline{(3, 9.5)}, C$
Cluster 3: $\underline{(8.5, 5.6)}, D, E, F, G, H$

Round 4

Cluster 1: $\underline{(3.5, 11.5)}, A, B$
Cluster 2: $\underline{C}$
Cluster 3: $\underline{(8.5, 5.6)}, D, E, F, G, H$
The clustering of the Points stays the same as the roud before so the algorithm halts.

b)



L is mean of cluster 1: $(2.5, 11.5)$, C is mean of cluster 2: $(3, 8)$, K is mean of cluster 3: $(8.5, 5.6)$

The dividing lines between the clusters are the lines that contain all points with the same distance to two of the points.
So the line seperating the clusters with mean A and B are defined by

$$|| (x, f(x)) - A|| = || (x, f(x)) - B||$$

$$\Leftrightarrow ||(x - A_1, f(x) - A_2)|| = ||(x - B_1, f(x) - B_2)||$$

$$\Leftrightarrow \sqrt{(x - A_1)^2 + (f(x) - A_2)^2} = \sqrt{(x - B_1)^2 + (f(x) - B_2)^2}$$

$$\Leftrightarrow (x - A_1)^2 + (f(x) - A_2)^2 = (x - B_1)^2 + (f(x) - B_2)^2$$

$$\Leftrightarrow f(x) = \frac{(x - A_1)^2 - (x - B_1)^2 - A_2^2 - B_2^2}{2(A_2 - B_2)}$$

$$\Leftrightarrow f(x) = \frac{B_1 - A_1}{A_2 - B_2} x + \frac{A_1^2 + B_1^2 - A_2^2 - B_2^2}{2(A_2 - B_2)}$$

The seperation of the clusters can be described by three functions:
Cluster 1&2: $f(x) = 1/7x + \frac{181}{7}$
Cluster 2&3: $f(x) = \frac{11}{4.8}x + \frac{14.11}{4.8}$
Cluster 1&3: $f(x) = \frac{12}{11.8}x + \frac{85.11}{11.8}$

c)

Choosa A, F and H as initial cluster means.

$||B-A|| = \sqrt{2}, ||C-A|| = \sqrt{17} \rightarrow A, ||D-A|| = \sqrt{73}, ||E-A|| = \sqrt{74}, ||G-A|| = 4\sqrt{5}$

$||B-F|| = 4\sqrt{5}, ||C-F|| = \sqrt{41}, ||D-F|| = \sqrt{5} \rightarrow F, ||E-F|| = 2 \rightarrow F, ||G-F|| = \sqrt{34}$

$||B-H|| = \sqrt{109}, ||C-H|| = 10 \rightarrow A, ||D-H|| = 4\sqrt{5}, ||E-H|| = 3\sqrt{5}, ||G-H|| = 3 \rightarrow H$

Cluster 1: $\underline{A}, B, C$ Center is underlined
Cluster 2: $D, E, \underline{F}$
Cluster 3: $G, \underline{H}$
Using $z^j \leftarrow \frac{\sum_{x \in C^j} x}{|C^j|}$ the new centroids are: $C_1 : (2.667, 10.333), C_2 : (6.333, 4), C_3 : (11.5, 8)$.
Using the new centroids leads to the following clusters:
$C_1 : \underline{(2.667, 10.333)}, A, B, C$
$C_2 : \underline{(6.333, 4)}, D, E, F$
$C_3 : \underline{(11.5, 8)}, G, H$
Th clusters are the same as in the prevois step so thete starting centroids produce a different result than a).

d)

$$||A-G|| = 4\sqrt{5}, ||B-G|| = \sqrt{58}, ||B-A|| = \sqrt{2}$$

The shortest distance of all the points to eachother is the distance between A and B. So A and B have to be in a cluster except A and B are centers of a cluster each. Then the distance between A and G is longer then B ang G. Therfore G would be in the cluster of B. It's impossible for k-means to return $\{A, G\}$ as a cluster.