

# Solutions to Sheet 2

Friedrich May, 355487; Markus Moll, 406263; Mariem Mounir, 415862

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## Exercise 1

**a)**

Let

$$X_2 = Y_1 Y_2$$

with  $Y_1, Y_2$  representing one throw each. Because  $E(XY) = E(X)E(Y)$  for independent  $X, Y$

$$E(X_2) = E(Y_1 Y_2) = E(Y_1)E(Y_2)$$

holds. With  $E(Y_1) = E(Y_2) = 3.5$  this lead to

$$E(X_2) = 3.5^2 = 12.25$$

. We further know that  $\text{Var}(X_2) = E(X_2^2) - E(X_2)^2$ . Therefore

$$\text{Var}(X_2) = E(Y_1^2 Y_2^2) - 12.25^2 = E(Y_1^2)E(Y_2^2) - 12.25^2 = \frac{91^2}{36} - 12.25^2 \approx 79.97$$

, beacuse  $E(Y_1^2) = E(Y_2^2) = \frac{91}{6}$ .

**b)**

Let

$$X_n = Y_1 Y_2 \dots Y_n$$

with  $Y_1, \dots, Y_n$  independent and describing one throw each. Then

$$E(X_n) = \prod_{i=1}^n E(Y_i)$$

. Using  $E(Y_i) = 3.5 \forall i$

$$E(X_n) = 3.5^n$$

. As above

$$\text{Var}(X_n) = E(X_n^2) - E(X_n)^2 = \prod_{i=1}^n E(Y_i^2) - 3.5^{2n} = \frac{91^n - 73.5^n}{6^n}$$

.

c)

Let  $X_i \in \{0; 1\}$  with  $X_i = \begin{cases} 1 & \text{throw results in one} \\ 0 & \text{otherwise} \end{cases}$  then

$$X = \sum_{i=1}^{200} X_i$$

is the number of ones in 200 throws. Observe that

$$\Pr(X_i = 1) = \frac{1}{6} = 1 - \Pr(X_i = 0) \forall i \in [n]$$

Then

$$\begin{aligned} E(X) &= \sum_{i=1}^{200} E(X_i) = \sum_{i=1}^{200} \frac{1}{6} = \frac{200}{6} \\ E(X^2) &= \left( \sum_{i=1}^{200} E(X_i) \right)^2 = E(X)^2 \end{aligned}$$

This leads to

$$\text{Var}(X) = E(X^2) - E(X)^2 = 0$$

The Probability to encounter at most 7 ones in 200 throws is

$$\Pr(X \leq 7) = \Pr(\text{"193 times not 1"}) = \prod_{i=1}^{193} \frac{5}{6} \leq 7 \times 10^{-16}$$

The inequalities yield the following borders:

**Chebyshev**

$$\Pr(|X - E(X)| \geq \frac{158}{6}) \leq \frac{6^2 \text{Var}(X)}{158} = 0$$

where  $\frac{158}{6}$  is the minimum distance 7 ones are from the Expectation.

**Chernoff**

$$\Pr(X \leq \left(1 - \frac{158}{200}\right) E(X)) \leq e^{-\frac{200 \cdot 158^2}{2}} = 0.0000303818$$

where  $\left(1 - \frac{158}{200}\right) E(X) = 7$ .

**Hoeffding**

$$\Pr(X \leq \frac{200}{6} - \left(\frac{1}{6} - \frac{7}{200}\right) 200) \leq e^{-2 \cdot 200 \cdot \left(\frac{1}{6} - \frac{7}{200}\right)} = 0.009736638$$

In this case Chernoff's inequality gives the best bound. Chebyshev's inequality does not work here because  $E(X)^2 = E(X^2) \Rightarrow \text{Var}(X) = 0$ .