Vectors in $\mathbb{R}^n$

1 Vectors

A **vector** is an array of numbers:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

A vector with n numbers in it is called *n-dimensional*. Sometimes we call numbers *scalars* to distinguish them from vectors. Vectors are usually written with bold font, $\mathbf{x}$, and scalars unbolded, x.

The set of all n-dimensional vectors is called $\mathbb{R}^n$, or *n-dimensional space*. Familiar examples are the x-y plane, which is $\mathbb{R}^2$, and 3-dimensional space which is $\mathbb{R}^3$.

2 Addition and Multiplication with Vectors

Addition of two vectors is defined, but multiplication of vectors is not defined yet. Multiplication of a number (scalar) and a vector is called *scalar multiplication*. We also need to define the zero vector so the usual algebraic properties of addition will work as expected.

$$\textit{Zero:}\ \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Addition: $\mathbf{u}+\mathbf{v} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \\ \vdots \\ u_n+v_n \end{bmatrix}$

Scalar Multiplication: $c\mathbf{u} = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \\ \vdots \\ cu_n \end{bmatrix}$

The algebraic properties of vectors can be proven from these 3 definitions.

3 Geometry of Vectors

Vectors in $\mathbb{R}^2$ can be thought of as points in the plane, and in $\mathbb{R}^3$ as points in 3-dimensional space. For example, the vector $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ can be thought of as the point $(1, -3)$ in the x-y coordinate system. We will refer to vectors as points and points as vectors as it suits our purposes.

Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ can be added geometrically (graphically) using the *parallelogram rule.*

Parallelogram Rule: to add $\mathbf{u}$ and $\mathbf{v}$, draw a parallelogram with vertices at the origin, (v_1, v_2), and (u_1, u_2). The fourth vertex of the parallelogram corresponds to the point $\mathbf{u}+\mathbf{v}$.

4 Linear Combinations

A *linear combination* of a set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n\}$ is a vector that is formed by multiplying the vectors in the set by scalars and adding them up. We can write a linear combination using sigma notation:

$$\mathbf{u} = \sum_{i=1}^{n} c_i\mathbf{x}_i, \;\; c_i \in \mathbb{R}$$

We would say that $\mathbf{u}$ is a linear combination of the vectors in the set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n\}$.

The set of all linear combinations of a set of vectors is called the *span* of the set. We could write:

$$\text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n\} = \{\mathbf{u} \mid \mathbf{u} = \sum_{i=1}^{n} c_i \mathbf{x}_i,\ c_i \in \mathbb{R}\}$$

Important facts about the span of a set:

- The span of a set of vectors in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$.
- The span of any set of vectors always contains the zero vector. (Why?)
- The span of one vector in $\mathbb{R}^n$ is a line in $\mathbb{R}^n$.
- The span of two vectors in $\mathbb{R}^n$ is a plane in $\mathbb{R}^n$, as long as the vectors are not parallel.

5 Three Views of a System

A system of linear equations can be written as a matrix equation, and also as a vector equation.

System of Equations:

$$\begin{aligned} x_1 \qquad\quad - 3x_3 &= 8 \\ 2x_1 + 2x_2 + 9x_3 &= 7 \\ x_2 + 5x_3 &= -2 \end{aligned}$$

Matrix Equation:

$$\begin{bmatrix} 1 & 0 & -3 \\ 2 & 2 & 9 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ -2 \end{bmatrix}$$

Vector Equation:

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ -2 \end{bmatrix}$$