MA2103 IISER Kolkata

Mid Semester Exam

Instructor: Dr. Rajesh Kumble Nayak

10:00 AM, 7th October 2023.

Duration: 60 + 30 Minutes.

- Answer all the question.
- No calculators are allowed!
- Good luck

Q - 1: [20 Marks] Plot the function

$$f(x) = e^{-\frac{1}{x}},$$

In an appropriate domain.

The domain is $-\infty, \infty$

The special point $-\infty, 0^-, 0^+, +\infty$

at $x \to -\infty \quad f(x) \to 1$

at $x \to +\infty \quad f(x) \to 1$

0^-: as $x \to 0$ from -ve x axis

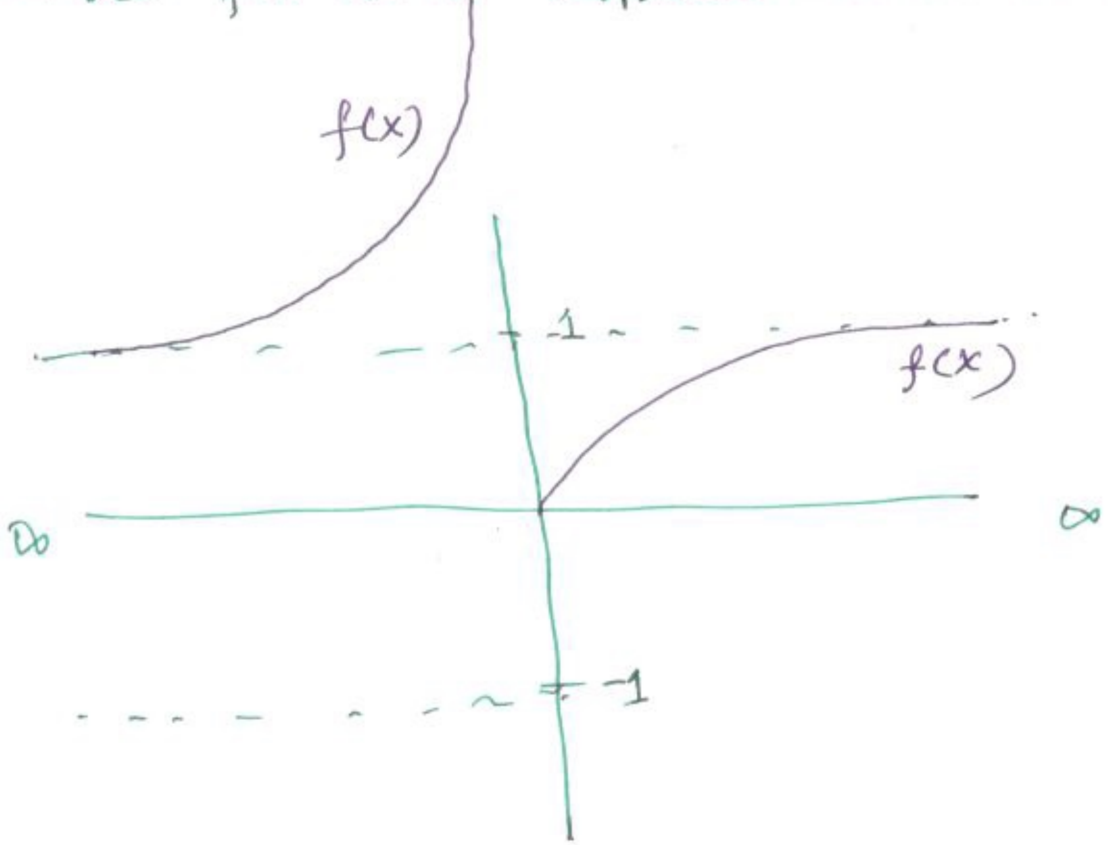
$e^{+\infty} \Rightarrow f(x) \to \infty$

0^+: as $x \to 0$ from +ve x axis

$e^{-\infty} \Rightarrow f(x) \to 0$

$$\frac{dy}{dx} = e^{-\frac{1}{x}} \cdot \frac{1}{x^2}$$

+ ve for all x other than $x = 0$

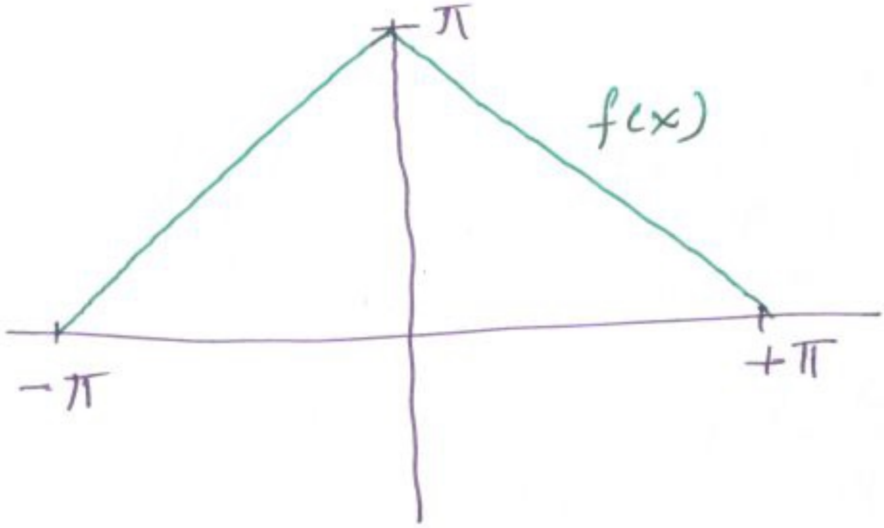


Q - 2: [20 Marks] Find the Fourier series expansion of the function

$$f(x) = \begin{cases} \pi + x & -\pi \leq x < 0 \\ \pi - x & 0 < x \leq \pi \end{cases},$$

Make plot of function $f(x)$, first two individual terms and the partial sum of two terms.

Let us start with a plot



$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx.$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{0}(\pi+x)\,dx + \frac{1}{2\pi}\int_{0}^{\pi}(\pi-x)\,dx$$

$$= \frac{1}{2\pi}\left[\pi x + \frac{x^2}{2}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_{0}^{\pi}$$

$$= \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi-x)\cos nx\,dx$$

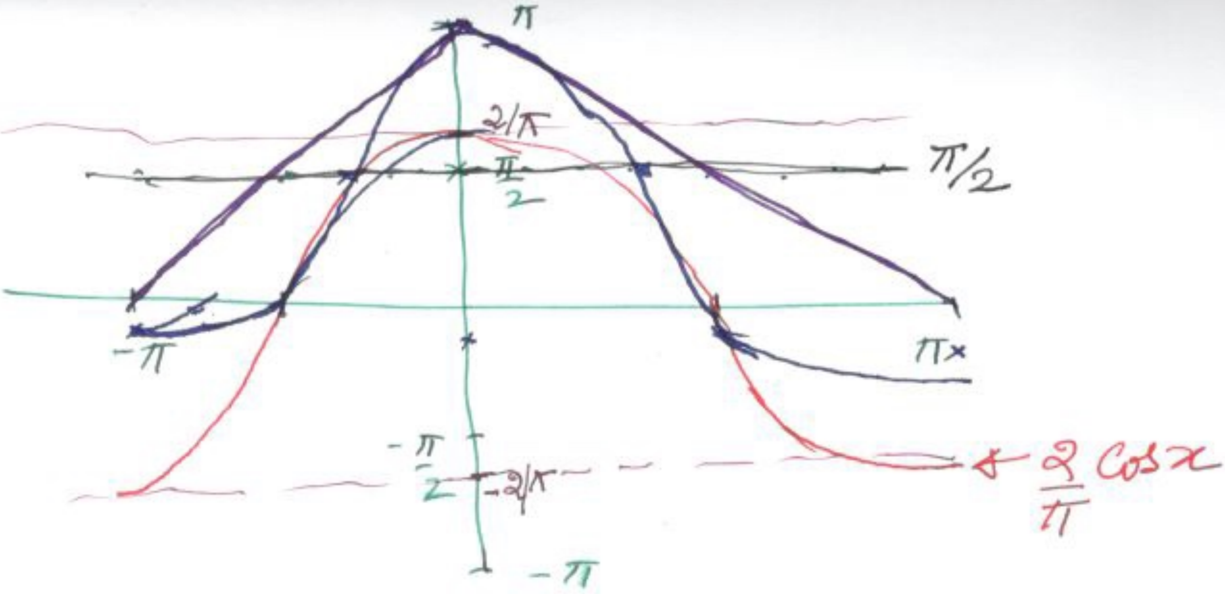
$$= \frac{1-\cos n\pi}{2n^2\pi} + \frac{1-\cos n\pi}{2n^2\pi}$$

$$a_n = \frac{1-\cos n\pi}{n^2\pi} = \frac{1-(-1)^n}{n^2\pi}$$

$$= \begin{cases} 0 & \text{for even } n \\ \dfrac{2}{n^2\pi} & \text{for odd } n \end{cases}$$

$b_n = 0$ because $f(x)$ is odd function

$$f(x) = \frac{\pi}{2} + \sum_{n=0}^{\infty}\frac{2}{(2n+1)^2\pi}\cos(2n+1)x$$



$f(x)_0 = \frac{\pi}{2}$ first term

$f(x)_1 = \frac{2}{\pi} \cos x$

$f(x) = \frac{\pi}{2} + \frac{2}{\pi} \cos x.$

Q - 3: [20 Marks] A string of length L is fixed at both end is left with an initial shape given by the function $\psi_0(x)$, the initial velocity of the string is zero *i.e.* $\left.\frac{\partial\psi(x,t)}{\partial t}\right|_{t=0} = 0$. Find the amplitude of string, $\psi(x,t)$ as function of position and time.

Solved in the class!

Q - 4: [20 Marks] If $p(x)$ is the probability density function, then the mean of the distribution is given by

$$\mu = \int_{-\infty}^{+\infty} x p(x)\, dx \,.$$

Explicitly compute the mean for the distribution given below

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-2)^2} \,.$$

It is given that $\int_{-\infty}^{+\infty} p(x) dx = 1$.

$$\mu = \int_{-\infty}^{\infty} x\, p(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-2)^2} dx$$

Let $y = x - 2 \Rightarrow x = 2 + y$

$dx = dy$

$$= \int_{-\infty}^{\infty} \frac{2+y}{\sqrt{2\pi}} e^{-\frac{1}{2}(y^2)} dy$$

$$= 2 \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} dx}_{=1} + \underbrace{\int y \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy}_{=0}$$

$$= 2 .$$

mean $\mu = 2$.!