Tutorial -5 11/09/2024

1. In Kepler problems $V(r) = -\frac{k}{r}$

(a) Show that there is stable circular orbit possible for every r

(b) find the orbital period of circular orbit at $r = r_e$

From class we have

$$\dot{\theta} = \frac{L_z}{mr^2} \quad \text{---} (1)$$

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{k}{r} \quad \text{---} (2)$$

From (2) we get

$$m\ddot{r} = \frac{L_z^2}{mr^3} - \frac{\partial V}{\partial r}$$

$$m\ddot{r} = -\frac{\partial V_{eff}}{\partial r}$$

$$V_{eff} = \frac{L_z^2}{2mr^2} - \frac{k}{r}$$

②

$$\frac{\partial V_{eff}}{\partial r} = \frac{k}{r^2} - \frac{L_z^2}{m r^3} = 0$$

$$\Rightarrow r_c = \frac{L_z^2}{km}$$ circular orbit

$$\frac{\partial^2 V_{eff}}{\partial r^2} = \left(\frac{3L_z^2}{m} - 2kr\right)\frac{1}{r^4}$$

put $r_c = \frac{L_z^2}{km}$

$$\Downarrow$$

$$\frac{L_z^2}{m}\frac{1}{r^4} > 0$$

it is stable

Now $\omega = \dot{\theta} = \frac{L_z}{m r_c^2}$

let's square

$$\omega^2 = \frac{L_z^2}{m^2 r_c^4} = \frac{k}{m r_c^3}$$

$$k = \frac{GMm}{\cancel{B}} \qquad = \frac{GM}{r_c^3}$$

is the kepler's law.

(2) In kepler problem $V(r) = -\frac{k}{r}$ when particle is moving in an orbit with ~~some~~ energy E and angular momentum L_z motion for 'r' is bount ~~for~~ between $r_{min} < r < r_{max}$

find the value of r_{min} & r_{max}

From class: $\dot{\theta} = \frac{L_z}{mr^2}$ —— (1)

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{k}{r} \quad \text{—— (2)}$$

from eq (2)

$$\frac{2}{m}\left(E + \frac{k}{r}\right) = \dot{r}^2 + \frac{L_z^2}{m^2r^2}$$

$$\Rightarrow \dot{r} = \left[\frac{2}{m}\left(E + \frac{k}{r} - \frac{L_z^2}{2mr^2}\right)\right]^{1/2}$$

for a physical system $\dot{r}$ have be

real and $\dot{r} = 0$ corresponds radial turning point

$$E + \frac{k}{r_0} - \frac{L_z^2}{2mr_0^2} = 0$$

$$Er^2 + kr - \frac{L_z^2}{2m} = 0$$

~~roots are~~ $\cancel{\frac{-k}{2E} \pm \sqrt{k^2 + 2L_z^2 E}}$

$$r^2 + \frac{kr}{\cancel{E}} - L$$

roots are

$$r_\pm = \frac{1}{2E}\left[-k \pm \sqrt{k^2 + \frac{2L_z^2 E}{m}}\right]$$

~~roots are imaginary~~ if $k < 0$

~~if E is +ve~~

one of the $k < 0$ if E is +ve

for both $k > 0$ we need E -ve