PH3102 Quantum Mechanics Assignment 10

Instructor: Dr. Siddhartha Lal   Autumn Semester, 2024

Start Date: November 05, 2024   Submission Deadline: November 12, 2024 .

Submit your answers to the Tutor at the start of the tutorial.

Q1. Degenerate Perturbation Theory on 2-Dimensional Simple Harmonic Oscillator [18 marks]

The two-dimensional simple harmonic oscillator (SHO) has Hamiltonian

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}m\omega^2(x^2 + y^2) = H_x + H_y,$$

so that it is the sum of two one-dimensional simple harmonic oscillators (SHO).

(a) Show that $[H, H_x] = 0$. Hence, we can choose eigenstates of H to be eigenstates of H_x as well (and hence H_y — explain). The normalized wavefunctions for the ground state and first excited state of the one-dimensional SHO are $\phi_0(x) = N_0 e^{-\alpha x^2/2}$ and $\phi_1(x) = \sqrt{2\alpha}x\phi_0(x)$, respectively. Let $\Phi_{nm}(x, y) = \phi_n(x)\phi_m(y)$.

(b) The angular momentum in the x-y plane is $L = xp_y - yp_x$. Show that $[L, H_x] \neq 0$, but $[L, H] = 0$. What do you conclude about possible bases of eigenstates of H?

(c) Explain why the ground state wavefunction for the 2D SHO is $\Phi_{00}(x, y)$ and the first excited state is doubly degenerate with wavefunctions $\Phi_{10}(x, y)$ and $\Phi_{01}(x, y)$.

(d) A small perturbation $H_1 = \lambda xy$ is now added to the Hamiltonian. Show that to first order in λ, the perturbed ground state energy does not change.

(e) Now, using degenerate perturbation theory, show that the degeneracy of the first excited state is lifted, and that the wavefunctions of the two resulting states are $(\Phi_{01}(x, y) \pm \Phi_{10}(x, y))/\sqrt{2}$. What are the corresponding energies?

(f) Show that $[H_x, H_1] \neq 0$, and also $[L, H_1] \neq 0$. So neither the H_x basis used here, nor the angular momentum basis, will avoid the diagonalization. Consider the operation S of reflection about the line $x = y$, so that $Sx = yS$ and $Sy = xS$. Show that $[S, H_1] = 0$. Consider your eigenfunctions from part (e) in terms of behavior under S.

Q2. Variational Method for a Bouncing Ball Potential [7 marks]

Apply the variational method to find an upper limit on the ground state of a particle in the potential

$$V(x) = \begin{cases} mgx, & x > 0, \\ \infty, & x < 0. \end{cases}$$

This is the "bouncing ball" potential: the particle falls under gravity to $x = 0$ and then bounces back again.

(a) What are the boundary conditions for the wavefunction at $x = 0$ and $x \to \infty$? Show that a suitable wavefunction form is

$$\psi(x) \propto \begin{cases} xe^{-ax}, & x > 0, \\ 0, & x < 0. \end{cases}$$

Find the expectation value of the Hamiltonian. (Don't forget to normalize the wavefunction!)

(b) Complete the variational calculation to find an upper bound for the ground state energy. You should find

$$E_0 \leq 3\left(\frac{9\hbar^2 g^2 m}{32}\right)^{1/3}.$$