Application of Fourier Series To Solve ODE

Let us say we are interested in Solving an ODE of the form

$$\alpha \frac{d^2y}{dt^2} + \beta \frac{dy}{dt} + \gamma y = 0$$

Here, α, β and γ are constants

Now we consider the solution which is Π^2 in the interval $[-\pi, \pi]$

The we should be able to express the Sol^n y in terms of Fourier Series

$$y = \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

we have to find constants a_n and b_n by substituting y in ODE

Differentiating y we get

$$y' = \sum_{n=1}^{\infty} -a_n n \sin nt + b_n n \cos nt$$

$$y'' = \sum_{n=1}^{\infty} -a_n n^2 \cos nt - b_n n^2 \sin nt$$

Putting back everything in ODE, we get

$$\sum_{n=1}^{\infty} \left(-a_n n^2 \alpha \cos nt - b_n n^2 \alpha \sin nt\right)$$
$$+ \left(-a_n n \beta \sin nt + b_n n \beta \cos nt\right)$$
$$+ \left(a_n \gamma \cos nt + b_n \gamma \sin nt\right)$$

Collecting sin and cos terms

we get

$$\sum_{n=1}^{\infty} \left(-a_n n^2 \alpha + b_n n \beta + a_n \gamma\right) \cos nt$$
$$+ \left(-b_n n^2 \alpha - a_n n \beta + b_n \gamma\right) \sin nt = 0$$

Further simplifying

$$\sum_{n=1}^{\infty}\left[\left(\gamma-n^2\alpha\right)a_n+b_n n\beta\right]\cos nt$$

$$\left[\left(\gamma-n^2\alpha\right)b_n-a_n n\beta\right]\sin nt=0$$

This summation can be zero only if individual coefficient of sin and cos identically zero

and, we get

$$① — \left(\gamma-n^2\alpha\right)a_n+b_n n\beta=0$$

$$② — \left(\gamma-n^2\alpha\right)b_n-a_n n\beta=0$$

from ② or $\quad b_n=\dfrac{a_n n\beta}{\left(\gamma-n^2\alpha\right)}$

Putting back in ① we get

$$\left(\gamma - n^2\alpha\right)a_n + \frac{a_n n^2\beta^2}{\left(\gamma^2 - n^2\alpha\right)} = 0$$

$$\Rightarrow \quad \left(\gamma - n^2\alpha\right)^2 a_n + n^2\beta^2 a_n = 0$$

two possibilities are

$a_n = 0$ or

$$\left(\gamma - n^2\alpha\right)^2 + n^2\beta^2 = 0$$

n for which $a_n \neq 0$

can be determine from above eqn for specific values of α, β & γ

Inhomogenous ODE

Let us look at a more complex problem

$$\alpha \frac{d^2y}{dt^2} + \beta \frac{dy}{dt} + \gamma y = f(t)$$

we are looking for sol^n in the interval $[-\pi, \pi]$

Let $f(t)$ is L^2 function;

we are looking for particular sol^n

First we expand $f(t)$ in Fourier series as

$$f(t) = \sum_{n=1}^{\infty} c_n \cos nx + d_n \sin nx$$

As before we write solution as

$$y = \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right)$$

again as before we substitute in ODE and simplify; Since the homogenous part is same

we can reuse the eq^n

$$\sum_{n=1}^{\infty}\left[(\gamma - n^2\alpha)a_n + b_n n\beta\right]\cos nt$$

$$\left[(\gamma - n^2\alpha)b_n - a_n n\beta\right]\sin nt =$$

$$\sum_{n=1}^{\infty} c_n \cos nx + d_n \sin nx$$

collecting the coefficient of sin and cos, we get

$$\sum_{n=1}^{\infty}\left[(\gamma - n^2\alpha)a_n + b_n n\beta - c_n\right]\cos nx$$

$$\left[(\gamma - n^2\alpha)b_n - a_n n\beta - d_n\right]\sin nx$$

$$= 0$$

in order to satisfy this eqn the coefficients of sin and cos should identically zero.

Or we get

$$(\nu - n^2\alpha)\, a_n + b_n n\beta = c_n$$

$$(\nu - n^2\alpha)\, b_n - a_n n\beta = d_n$$

This set of linear eqn can be solved to obtain a_n and b_n which provide the soln

Let us look at specific example : :

Solve the eqn

$$\frac{d^2y}{dt^2} + \gamma \frac{dy}{dt} + \omega^2 y = f(t)$$

where $f(t) = \begin{cases} -1 & \text{when } t < 0 \\ +1 & \text{when } t > 0 \end{cases}$

In the interval $[-\pi, \pi]$

From the worked out example in the class,

we know $f(t) = \sum_{n=1}^{\infty} \frac{4}{\pi n} \sin nx$

for odd n

We directly use the above result

We get :

$$\sum_{n=1}^{\infty} \left[(\nu - n^2\alpha) a_n + b_n n\beta - c_n \right] \cos nx$$

$$\left[(\nu - n^2\alpha) b_n - a_n n\beta - d_n \right] \sin nx$$

$= 0$

In this case c_n are identically zero and a_n are zero for even $n = 0$

$$(\nu - n^2\alpha)a_n + b_n n\beta = 0$$

$$(\nu - n^2\alpha)b_n - a_n n\beta = 0$$

for even n

$$(\nu - n^2\alpha)b_n - a_n n\beta - \frac{4}{n\pi} = 0$$

for odd n

or

$$(\nu - n^2\alpha)n b_n - a_n n^2\beta - \frac{4}{\pi} = 0$$

which may be solved in general