Problems on Continuous function

1. Let P be a polynomial of odd degree. Show that P has a root.

2. i. Show that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous iff for every open set O, $f^{-1}(O)$ is open. *(an alternative definition of continuity!)*

ii. Does the above result true if $f : [0,1] \to \mathbb{R}$ is continuous?

iii. Show that $\{x \in \mathbb{R} : x^2 + 5x + 6 > 0\}$ is open.

iv. Show that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous iff for every closed set C, $f^{-1}(C)$ is closed.

3. If $f : [a,b] \to \mathbb{R}$ is monotone on $[a,b]$, then the set of points of discontinuities in $[a,b]$ is a countable set.

4. Find all continuous functions $f : [a,b] \to \mathbb{R}$ assumes only rational values on $[a,b]$.

5. Let $f : [0,2\pi] \to [0,2\pi]$ be a continuous function such that $f(0) = f(2\pi)$. Show that there exists $c \in [0,\pi]$ such that

$$f(c) = f(c + \pi).$$

6. Show that the function $\sqrt{x}$ is uniformly continuous but not Lipschitz continuous on $(0,1)$.

6. Show that the function x^2 is uniform continuous on any bounded interval J but not on $\mathbb{R}$.