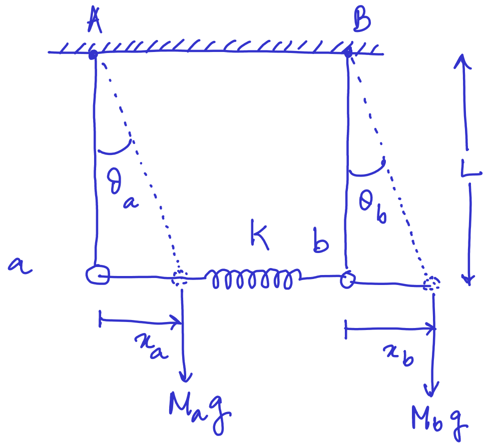
1.



(a) To find the equation of motion we equate moment of inertia times the angular acceleration to the total external torque.

For mass M_a, the moment of inertia is $M_a L^2$ (assuming point mass and a rotation about the point A).

So, we have,

$$M_a L^2 \ddot{\theta}_a = \underbrace{- M_a g L \sin\theta_a}_{\text{Torque due to the weight}} - \underbrace{K L (\sin\theta_a - \sin\theta_b) L \cos\theta_b}_{\text{Torque due to the compression of the spring}}$$

Using small angle approximation we get,

$$\ddot{\theta}_a = -\frac{g}{L}\theta_a - \frac{K}{M_a}\theta_a + \frac{K}{M_a}\theta_b \quad \ldots (1)$$

Similarly for the mass b, we have,

$$\ddot{\theta}_b = -\frac{g}{L}\theta_b - \frac{K}{M_b}\theta_b + \frac{K}{M_b}\theta_a \quad \ldots (2)$$

(b) Assuming a trial solution for a particular normal mode

$$\left.\begin{aligned} \theta_a &= \psi_a e^{i\omega t} \\ \theta_b &= \psi_b e^{i\omega t} \end{aligned}\right\}$$ where ψ_a and ψ_b are amplitudes of oscillators a and b,

and substituting the trial solution in (1) and (2) we obtain,

$$-\omega^2 \psi_a e^{i\omega t} = -\left(\frac{g}{L} + \frac{K}{M_a}\right)\psi_a e^{i\omega t} + \frac{K}{M_a}\psi_b e^{i\omega t}$$

and

$$-\omega^2 \psi_b e^{i\omega t} = -\left(\frac{g}{L} + \frac{K}{M_b}\right)\psi_b e^{i\omega t} + \frac{K}{M_b}\psi_a e^{i\omega t}$$

As, the above holds for all values of t, we must have

$$\begin{pmatrix} \omega^2 - \left(\frac{g}{L} + \frac{K}{M_a}\right) & \frac{K}{M_a} \\ \frac{K}{M_b} & \omega^2 - \left(\frac{g}{L} + \frac{K}{M_b}\right) \end{pmatrix} \begin{pmatrix} \psi_a \\ \psi_b \end{pmatrix} = 0 \quad \ldots (3)$$

For nontrivial solutions of ψ_a and ψ_b, we must have,

$$\begin{vmatrix} \omega^2 - \left(\frac{g}{L} + \frac{K}{M_a}\right) & \frac{K}{M_a} \\ \frac{K}{M_b} & \omega^2 - \left(\frac{g}{L} + \frac{K}{M_b}\right) \end{vmatrix} = 0 \quad \ldots\ldots \quad (4)$$

$$\Rightarrow \omega^4 - \omega^2\left[\frac{2g}{L} + K\left(\frac{1}{M_a} + \frac{1}{M_b}\right)\right] + \left(\frac{g}{L}\right)^2 + \frac{g}{L}K\left(\frac{1}{M_a} + \frac{1}{M_b}\right) = 0$$

$$\Rightarrow \omega^2 = \frac{1}{2}\left\{\frac{2g}{L} + K\left(\frac{1}{M_a} + \frac{1}{M_b}\right)\right\} \pm \frac{1}{2}\sqrt{4\left(\frac{g}{L}\right)^2 + 4\frac{g}{L}\cdot K\left(\frac{1}{M_a} + \frac{1}{M_b}\right) + K^2\left(\frac{1}{M_a} + \frac{1}{M_b}\right)^2 - 4\left(\frac{g}{L}\right)^2 - 4\frac{g}{L}K\left(\frac{1}{M_a} + \frac{1}{M_b}\right)}$$

$$= \frac{g}{L} + \frac{K}{2}\left(\frac{1}{M_a} + \frac{1}{M_b}\right) \pm \frac{K}{2}\left(\frac{1}{M_a} + \frac{1}{M_b}\right)$$

$$= \boxed{\frac{g}{L}} \text{ or } \boxed{\frac{g}{L} + K\left(\frac{1}{M_a} + \frac{1}{M_b}\right)}$$

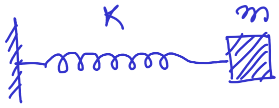
(c) Substituting $\omega^2 = g/L$ in equation (3) we get

$\psi_a = \psi_b$. Hence, for $\omega^2 = g/L$ we have in phase motion with both the pendulums oscillating with equal amplitude.

For $\omega^2 = g/L + K\left(1/M_a + \frac{1}{M_b}\right)$, we get,

$$\frac{\psi_a}{M_b} + \frac{\psi_b}{M_a} = 0 \Rightarrow \boxed{\frac{\psi_a}{\psi_b} = -\frac{M_b}{M_a} = -\frac{1/M_a}{1/M_b}}$$

So, we have an out of phase motion with amplitudes being inversely proportional to the respective mass.

2. 

The schematics are shown on the left.

1. From the given condition, we have

$$Kh = mg \Rightarrow K = mg/h.$$

2. The viscous force $= \gamma u = mg \Rightarrow \gamma = mg/u.$

(a) Hence the equation of motion is given by,

$$m\ddot{x} = -\gamma\dot{x} + Kx$$

$$\Rightarrow \quad m\ddot{x} = -\frac{mg}{u}\dot{x} + \frac{mg}{h} \Rightarrow \boxed{\ddot{x} + \underbrace{\frac{g}{u}}_{\boxed{\alpha}}\dot{x} + \underbrace{\frac{g}{h}}_{\boxed{\omega_0^2}}x = 0} \quad \text{--} (1)$$

(b) The angular frequency of the oscillation for this system is given by

$$\omega_1 = \sqrt{\omega_0^2 - \alpha^2/4} = \sqrt{\frac{g}{h} - \frac{1}{4}\cdot\frac{g^2}{u^2}} = \sqrt{\frac{g}{h} - \frac{1}{\cancel{4}}\cdot\frac{g^2}{\frac{1}{\cancel{4}}3gh}}$$

$$= \sqrt{\frac{g}{h}}\left(1 - \frac{1}{3}\right)^{1/2} = \boxed{\sqrt{\frac{2g}{3h}}}$$

(c) The solution in its generic form is

$$x = Ce^{-\alpha t/2}\cos(\omega_1 t + \phi)$$

Given, $x(0) = \frac{3}{2}h$ and $\dot{x}(0) = -u$.

Now, $\dot{x}(t) = -\frac{\alpha}{2}Ce^{-\alpha t/2}\cos(\omega_1 t + \phi) - \omega_1 C e^{-\alpha t/2}\sin(\omega_1 t + \phi)$

So, $C\cos\phi = \frac{3}{2}h$

and $\frac{\alpha}{2}C\cos\phi + \omega_1 C\sin\phi = u$

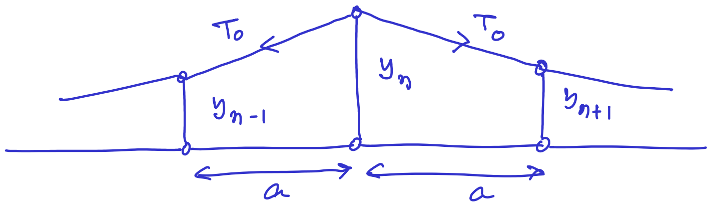
$$\Rightarrow \quad C\sin\phi = \frac{1}{\omega_1}\left[u - \frac{\alpha}{2}\,\frac{3}{2}h\right] = \frac{1}{\omega_1}\left[\frac{1}{2}\sqrt{3gh} - \frac{g}{\frac{1}{2}\sqrt{3gh}}\cdot\frac{1}{2}\,\frac{3}{2}h\right]$$

$$= 0 \quad \Rightarrow \quad \phi = 2n\pi \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\Rightarrow C = \frac{3}{2}h$$

$$\Rightarrow \boxed{x(t) = \frac{3}{2}h\,e^{-\alpha t/2}\cos\left(\sqrt{\frac{2g}{3h}}\,t\right)} \quad \text{with} \quad \alpha = \frac{2g}{\sqrt{3gh}} = 2\sqrt{\frac{g}{3h}}$$

3. The schematics of the beaded string is



$$a = \frac{L}{N+1}$$

(a) The equation of motion is

$$m\ddot{y}_n = -T_0\left(\frac{y_n - y_{n-1}}{a}\right) - T_0\left(\frac{y_n - y_{n+1}}{a}\right) - 2m\alpha\dot{y}_n$$

damping

$$\Rightarrow \boxed{\ddot{y}_n + 2\alpha\dot{y}_n = -\frac{T_0}{ma}\left[2y_n - y_{n-1} - y_{n+1}\right]} \quad \ldots \text{(1)}$$

(b) To find the continuum limit, we rearrange the terms as,

$$\ddot{y}_n + 2\alpha\dot{y}_n = \frac{T_0 a}{m}\left[\frac{\frac{y_{n+1} - y_n}{a} - \frac{y_n - y_{n-1}}{a}}{a}\right]$$

In the continuum limit, the right hand side is simply $\frac{T_0 a}{m}\frac{\partial^2 y}{\partial x^2}$ and we get

$$\boxed{\frac{\partial^2 y}{\partial t^2} + 2\alpha\frac{\partial y}{\partial t} = \frac{T_0 a}{m}\frac{\partial^2 y}{\partial x^2}}$$

(c) We can use the separation of variables if the string is (i) bound at both the ends or (ii) it is oscillating in the steady-state when one end of the string is forced.

(d) For $\alpha = 0$, we have,

$$\frac{\partial^2 y}{\partial t^2} = \frac{T_0 a}{m}\frac{\partial^2 y}{\partial t^2}$$

With a trial solution of either $c\sin kx(\cos\omega t + \phi)$ [bound string] or a traveling wave form $c\sin(kx - \omega t)$ we get

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 y \quad \text{and} \quad \frac{T_0 a}{m}\frac{\partial^2 y}{\partial x^2} = -\frac{T_0 a}{m}k^2$$

$$\Rightarrow \quad \omega^2 = \frac{T_0 a}{m} k^2$$

$$\Rightarrow \quad \boxed{\omega = \sqrt{\frac{T_0 a}{m}}\, k}$$ is the dispersion relation.

4. (a) From the question,

$$[2m\alpha v] = [F] = MLT^{-2}$$

$$\Rightarrow \quad M[\alpha]LT^{-1} = MLT^{-2}$$

$$\Rightarrow \quad [\alpha] = T^{-1}$$

So, α has the dimension of frequency.

(b)

$$m\ddot{x} = -2m\alpha\dot{x} - k(x-u)$$

$$\Rightarrow \quad m\ddot{x} = -2m\alpha\dot{x} - kx + kA\sin\omega t$$

$$\Rightarrow \quad \boxed{\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = \omega_0^2 A\sin\omega t} \quad \ldots (1)$$

(c) We find the steady state solution for

$$\ddot{x} + 2\alpha\dot{x} + \omega_0^2 x = \omega_0^2 A e^{i\omega t}$$

and take the imaginary component of x as the final solution.

Assuming, a steady state $x = Ce^{i\omega t}$ we get

$$(-\omega^2 + 2i\alpha\omega + \omega_0^2)C = \omega_0^2 A$$

$$\Rightarrow \quad C = A \cdot \frac{\omega_0^2}{2i\alpha\omega + (\omega_0^2 - \omega^2)}$$

$$= A \cdot \frac{\omega_0^2}{\sqrt{4\alpha^2\omega^2 + (\omega_0^2 - \omega^2)^2}} \cdot e^{-i\phi},$$

where $\tan\phi = \dfrac{2\omega\alpha}{\omega_0^2 - \omega^2}$

$\Rightarrow$ Final solution is

$$\boxed{x = A \cdot \frac{\omega_0^2}{\sqrt{4\alpha^2\omega^2 + (\omega_0^2 - \omega^2)^2}} \cdot \sin(\omega t - \phi)},$$ where $\boxed{\phi = \tan^{-1}\left(\dfrac{2\omega\alpha}{\omega_0^2 - \omega^2}\right)}$