INTEGRATION.

Defn (partition): A finite set $P = \{x_0, x_1, \dots, x_n\}$ is said to be a partition of $[a,b]$ if

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Defn (Refinement): Let P, Q be two partition of $[a,b]$. We say that Q is a refinement of P if

$$P \subseteq Q.$$

Example: Consider the interval $[0,1]$. Let $n \in \mathbb{N}$. Define

$$h := \frac{1}{n}.$$

Define

$$x_i = ih \quad \text{for } i = 0, 1, 2, \dots, n.$$

Then $P_n = \{x_0, \dots, x_n\}$ is a partition of $[0,1]$. Also note that P_{2n+1} is a refinement of P_{2n}.

Qn: Does P_{n+1} is a refinement of P_n?

Assumption: To define the integration of a fn $f:[a,b]\to\mathbb{R}$, we will assume that f is bounded.

Notation: Let $f:[a,b]\to\mathbb{R}$ be a bdd fn.

Let $m = \inf\{f(x) : x\in[a,b]\}$

$M = \sup\{f(x) : x\in[a,b]\}$.

Let $P=\{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$. We define for $i=0,1,\dots,n-1$

$$m_i(f) = \inf\{f(x) : x\in[x_i, x_{i+1}]\}.$$

$$M_i(f) = \sup\{f(x) : x\in[x_i, x_{i+1}]\}.$$

Therefore $\boxed{m_i(f)\le M_i(f)}$ $i=0,1,2,\dots,n-1.$

Defn: (UPPER SUM & LOWER SUM).

Let $f:[a,b]\to\mathbb{R}$ be a bounded fn. Let $P=\{x_0,\dots,x_n\}$ be a partition of $[a,b]$. We define the upper sum $U(f,P)$ and lower sum $L(f,P)$ as follows.

$$U(f,P)=\sum_{i=0}^{n-1} M_i(f)\,(x_{i+1}-x_i)$$

$$L(f,P)=\sum_{i=0}^{n-1} m_i(f)\,(x_{i+1}-x_i)$$

Proposition: Let $f:[a,b]\to\mathbb{R}$ be a bdd fn. Then

$$\boxed{m(b-a)\le L(f,P)\le U(f,P)\le M(b-a)} \quad (1)$$

for all partition P of $[a,b]$.

Proof: Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a,b]$. Then proof follows from the fact

$$m \le m_i(f) \le M_i(f) \le M$$

$$\forall\ i = 0, 1, 2, \dots n-1.$$

Defn: (UPPER INTEGRAL & LOWER INTEGRAL)

We define the upper integral $U(f)$ and lower integral $L(f)$ by

$$U(f) = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$$

$$L(f) = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}.$$

The existence of sup and inf is guaranteed by (1).

Defn: (Darboux integrable).

We say that a bdd fn $f:[a,b]\to\mathbb{R}$ is Darboux integrable if

$$U(f) = L(f).$$

If it is integrable, we denote the integration by $\int_a^b f$ or $\int_a^b f(x)\,dx$,

i.e,

$$\boxed{U(f) = L(f) = \int_a^b f}$$

Lemma: (i) $L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$ for every partition P and Q such that $P \subseteq Q$.

(ii) $L(f,P) \leq U(f,Q)$ for every partitions P, Q of $[a,b]$.

(iii) $L(f) \le U(f)$.

Proof: (i) Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$. Let

$$c \in [a,b] \setminus P.$$

Let $Q = P \cup \{c\}$. Then Q is a partition and Q is a refinement of P. We will show that P and Q satisfies the desired inequality

$$\boxed{L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)}$$

Assume that

$$x_j < c < x_{j-1} \quad \text{for some } 0 \le j \le n-1.$$

Now

$$U(f,P) = \sum_{\substack{i=0 \\ i \ne j}}^{n-1} M_i(f)(x_{i+1} - x_i) + M_j(f)(x_{j+1} - x_j)$$

and

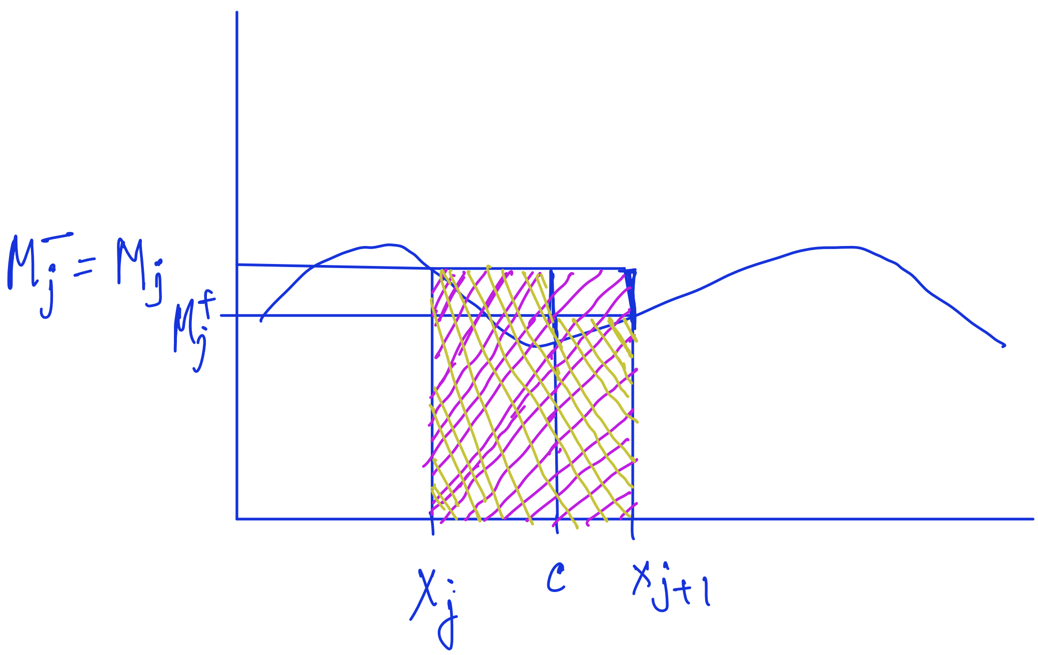
$$U(f,Q) = \sum_{\substack{i=0 \\ i\neq j}}^{n-1} M_i(f)\,(x_{i+1}-x_i)$$

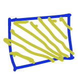
$$+\, M_j^-\,(c-x_j) + M_j^+\,(x_{j+1}-c)$$

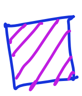
Where

$$M_j^- = \sup\{f(x) : x \in [x_j, c]\}$$

$$M_j^+ = \sup\{f(x) : x \in [c, x_{j+1}]\}.$$



 $= M_j^-\,(c-x_j) + M_j^+\,(x_{j+1}-c)$

 $= M_j(f)\,(x_{j+1}-x_j)$

Now

$$U(f,P) - U(f,Q)$$
$$= \left(M_j(f) - M_j^-\right)(c - x_j) + \left(M_j(f) - M_j^+\right)(x_{j+1} - c)$$

As $M_j(f) \geq M_j^-$ and $M_j(f) \geq M_j^+$, therefore

$$\boxed{U(f,P) \geq U(f,Q)}.$$

Now, if Q is a refinement of P then by induction we conclude that

$$U(f,P) \geq U(f,Q).$$

as $Q \setminus P$ is finite.

Then similarly, we prove

$$L(f,P) \leq L(f,Q)$$

$\forall$ P, Q satisfying $P \subseteq Q$.

This Completes the proof of (i).

(ii). Let P, Q be two partition of $[a,b]$.

Let $\tilde{P} = P \cup Q$. Then $\tilde{P}$ is a refinement of both P and Q. Then by (i)

$$L(f,P) \leq L(f, P\cup Q) \leq U(f, P \cup Q) \leq U(f,Q).$$

This proves (ii).

(iii) Since

$$L(f,P) \leq U(f,Q),$$

$\forall P, Q$, $U(f,Q)$ is an upper bound of the set

$$\{L(f,P) : P \text{ is a partition}\}.$$

Therefore

$$L(f) \leq U(f,Q) \quad \forall Q.$$

Therefore,

$$L(f) \leq U(f).$$

Theorem: A bdd fn $f:[a,b] \to \mathbb{R}$ is integrable iff for each $\epsilon > 0$ $\exists$ a partition P_ϵ such that

$$U(f,P_\epsilon) - L(f,P_\epsilon) < \epsilon.$$

Proof: Assume that f is integrable, then

$$U(f) = L(f).$$

Since $\quad U(f) = \inf\{U(f,P) : P \text{ is a partition}\}$

for $\epsilon > 0$, $\exists$ P_1 s.t.

$$U(f) \leq U(f,P_1) < U(f) + \epsilon. \longrightarrow ①$$

Similarly, $\exists$ P_2 such that

$$L(f)-\epsilon < L(f,P_2) < L(f) \longrightarrow ②$$

Let $P = P_1 \cup P_2$, then P is a refinement of P_1 and P_2. Thus we obtain

$$L(f)-\epsilon \leq L(f,P_2) \leq L(f,P) \leq U(f,P)$$
$$\leq U(f,P_1) \leq U(f)+\epsilon.$$

Therefore,

$$L(f)-\epsilon \leq L(f,P) \leq U(f,P) \leq U(f)+\epsilon.$$

Since $U(f) = L(f)$

Therefore,

$$\boxed{U(f,P) - L(f,P) < 2\epsilon}$$

This proves that if f is integrable then for each $\epsilon > 0$, $\exists$ P such that

$$U(f,P) - L(f,P) < \epsilon.$$

Suppose for every $\epsilon > 0$, $\exists$ P_ϵ such that

$$U(f,P_\epsilon) - L(f,P_\epsilon) < \epsilon.$$

Then $U(f) \leq U(f,P_\epsilon) < \epsilon + L(f,P_\epsilon)$
$\leq \epsilon + L(f).$

Therefore,

$$0 \leq U(f) - L(f) < \epsilon \quad \forall\, \epsilon > 0.$$

Then $U(f) = L(f).$

This proves that f is integrable.

Example: 1. Let $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \in [0,1) \\ 100 & x = 1. \end{cases}$$

Then f is integrable and

$$\int_0^1 f = 1.$$

Sol$\underline{n}$: Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition, Then

$$U(f,P) = \sum_{i=0}^{n-1} M_i(f)\,(x_{i+1} - x_i)$$

and
$$L(f,P) = \sum_{i=0}^{n-1} m_i(f)\,(x_{i+1} - x_i)$$

Note that

$$M_i(f) = \begin{cases} 1. & i = 0, 1, \ldots, n-2 \\ 100 & i = n-1 \end{cases}$$

$$m_i(f) = 1 \qquad i = 0, 1, \ldots, n-1.$$

Thus

$$U(f,P) = (x_{n-1} - x_0) + 100\,(x_n - x_{n-1})$$

and

$$L(f,P) = 1.$$

Choosing P such that $x_{n-1} = 1-\frac{1}{n}$, $x_n = 1$
we see that

$$U(f) = 1$$

Thus $U(f) = L(f)$.

This proves that f is integrable
and

$$\boxed{\int_0^1 f = 1.}$$

Example 2: Let f is defined by

$$f(x) = \begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ 2 & \frac{1}{2} < x < 1. \end{cases}$$

Then f is integrable and

$$\int_0^1 f\,dx = \frac{1}{2} + 1 = \frac{3}{2}.$$

Sol^n: Let $\epsilon > 0$. Choose the partition

$$P_\epsilon = \{x_0, \dots, x_j, x_{j+1} \dots x_n\} \text{ s.t.}$$

if $x_j < \frac{1}{2} < x_{j+1}$, then

$$x_{j+1} - x_j < \epsilon.$$

Then one can show that

$$U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon.$$

Thm: Let $f : [a,b] \to \mathbb{R}$ be a monotone fn. Then f is integrable.

Proof: Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$.

Then

$$U(f,P) - L(f,P)$$
$$= \sum_{i=0}^{n-1} (M_i(f) - m_i(f))(x_{i+1} - x_i)$$

Since f is monotone

$$U(f,P)-L(f,P)=\sum_{i=0}^{n-1}\left(f(x_{i+1})-f(x_i)\right)(x_{i+1}-x_i)$$

Now, we choose P such that

$$\sup_i (x_{i+1}-x_i) < \epsilon/f(b)-f(a)$$

Then

$$U(f,P)-L(f,P) < \left(\sup(x_{i+1}-x_i)\right)\sum_{i=0}^{n-1}\left(f(x_{i+1})-f(x_i)\right)$$

$$\leq \epsilon/\left(f(b)-f(a)\right)\cdot\left(f(b)-f(a)\right)$$

$$=\epsilon.$$

Thm: Let $f:[a,b]\to\mathbb{R}$ be continuous. Then f is integrable.

Proof: Since f is continuous on the interval $[a,b]$, f is bdd on $[a,b]$.

Let $\epsilon > 0$. Since f is unif. continuous on $[a,b]$, there exists a $\delta > 0$ such that

$$x, y \in [a,b], \ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

Let P be a partition such that

$$P = \{x_0, x_1, \dots, x_n\}$$

and $x_{i+1} - x_i < \delta \quad i = 0, 1, 2, \dots, n-1.$

Now

$$U(f,P) - L(f,P)$$

$$= \sum_{i=0}^{n-1} \left(M_i(f) - m_i(f)\right)(x_{i+1} - x_i)$$

Now $|f(x) - f(y)| < \epsilon \quad \forall x, y \in [x_i, x_{i+1}]$.

by UC of f. Since f is continuous on $[x_i, x_{i+1}]$, for some $\tilde{x}, \tilde{y} \in [x_i, x_{i+1}]$

$$f(\tilde{x}) = m_i(f)$$
$$f(\tilde{y}) = M_i(f).$$

Therefore,
$$M_i(f) - m_i(f) < \epsilon.$$

Now,
$$U(f,P) - L(f,P)$$
$$< \epsilon \sum_{i=0}^{n-1} (x_{i+1} - x_i) = \epsilon (b-a).$$

This proves that f is integrable on $[a,b]$.

Thm: Let $f:[a,b] \to \mathbb{R}$ be bounded fn such that f is continuous on $[a,b]$ except $c \in (a,b)$. Then f is integrable on $[a,b]$.

Proof: Let $\epsilon > 0$. Since f is continuous on $[a, c-\epsilon/2]$, $\exists$ a partition P_1 on $[a, c-\epsilon/2]$ such that

$$U(f,P_1) - L(f,P_1) < \epsilon.$$

Similarly $\exists$ a partition on $[c+\epsilon/2, b]$ such that

$$U(f,P_2) - L(f,P_2) < \epsilon.$$

Let $P = P_1 \cup P_2$. Then P is a partition on P. Also,

$$U(f,P) - L(f,P) \leq 2\epsilon + \epsilon(M-m).$$

This shows that f is integrable.

Similarly we prove the following theorem.

Thm: Let $f:[a,b] \to \mathbb{R}$ be a bdd fn. Assume that f is continuous except finitely many pts in $[a,b]$.
Then f is integrable.

Next we discuss the Thomae's fn, which is discontinuous at infinite points but integrable.

Thomae's fn: consider the function $f:[0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x = 0 \\ 0 & x \in [0,1] \setminus \mathbb{Q}. \\ \frac{1}{q} & x = \frac{p}{q}, \ \gcd(p,q) = 1. \end{cases}$$

Proposition:
Then (i) f is continuous at 0.
(ii) f is discontinuous at non-zero

rationals.

(iii) f is continuous at irrational.

Solution: (i) Note that

$$0 \leq f(x) \leq x \quad \forall x \in [0,1].$$

This shows that

$$\lim_{x \to 0+} f(x) = f(0) = 0.$$

Therefore f is continuous at 0.

(ii) Let $r \in (0,1) \cap \mathbb{Q}$. Then there exists a sequence of irrationals r_n such that

$$r_n \longrightarrow r.$$

Now $f(r_n) = 0$, but $f(r) \neq 0$.

Therefore, $f(r_n) \not\to f(r)$.

This proves that f is discontinuous at non-zero rationals.

(iii) Let $c \in [0,1] \setminus \mathbb{Q}$. Let $\epsilon > 0$. We need to find a $\delta > 0$ such that

$$x \in (c-\delta, c+\delta) \subseteq (0,1) \Rightarrow |f(x) - f(c)| < \epsilon.$$

Now, note that $f(c) = 0$. Thus we need to find a $\delta > 0$ s.t.

$$\boxed{x \in (c-\delta, c+\delta) \Rightarrow f(x) < \epsilon.} \longrightarrow (1)$$

Observe the following property of f:

(i) $f : [0,1] \longrightarrow \{1, \frac{1}{2}, \frac{1}{3}, \dots, \} \cup \{0\}$.

(ii) Let $k \in \mathbb{N}$. Consider the set $\{x : f(x) \geq \frac{1}{k}\}$ is finite set. Also

Note

$$\boxed{\{x : f(x) \geq \tfrac{1}{k}\} = \bigcup_{m=1}^{k} \{x : f(x) = \tfrac{1}{m}\}}$$

Let $\epsilon > 0$. Then by Archimedean property, $\exists\, N \in \mathbb{N}$ such that

$$\frac{1}{N} < \epsilon$$

Since the set $\{x : f(x) \geq \frac{1}{N}\}$ is finite and $c \notin \{x : f(x) \geq \frac{1}{N}\}$, we can choose a $\delta > 0$ such that

$$(c-\delta, c+\delta) \cap \{x : f(x) \geq \tfrac{1}{N}\} = \phi.$$

Therefore,

$$(c-\delta, c+\delta) \subseteq \{x : f(x) < \tfrac{1}{N}\}$$

Hence

$$\forall x \in (c-\delta, c+\delta) \Rightarrow f(x) < \tfrac{1}{N} < \epsilon.$$

This proves that f is continuous at irrationals.

<u>Proposition</u>: Thomae's fn is integrable on $[0,1]$.

<u>Proof</u>: Let $\epsilon > 0$. We want to find a partition P such that

$$U(f,P) - L(f,P) < \epsilon.$$

Now, let $P = \{x_0, x_1, \ldots, x_n\}$. Then it is obvious that

$$L(f,P) = 0.$$

Therefore

$$U(f,P) = \sum_{i=0}^{n-1} M_i(f)(x_{i+1} - x_i)$$

Since $\epsilon > 0$, $\exists\, K \in \mathbb{N}$ s.t.

$$\frac{1}{K} < \epsilon.$$

Let $\delta > 0$. Now we choose P such that

$$x_{i+1} - x_i < \delta.$$

Consider

$$J =: \{ i : [x_i, x_{i+1}] \cap A_K \neq \emptyset \}.$$

Where

$$A_K = \{ x : f(x) \geq \tfrac{1}{K} \}.$$

Note that J contains finite no. pts, say N. Then

$$U(f,P)-L(f,P) = \sum M_i(f)(x_{i+1}-x_i)$$

$$= \sum_{i\in J} M_i(f)(x_{i+1}-x_i) + \sum_{i\notin J} M_i(f)(x_{i+1}-x_i)$$

$$\leq \sum_{i\in J}(x_{i+1}-x_i) + \sum_{i\notin J}\frac{1}{k}(x_{i+1}-x_i)$$

$$\leq N\delta + \frac{1}{k}.$$

Choose $\delta = \frac{\epsilon}{N}$, then we have

$$U(f,P)-L(f,P) \leq 2\epsilon.$$

This proves that f is integrable. This completes the proof.

Notation: $R([a,b]) =: \{f:[a,b]\to\mathbb{R} : f \text{ is bdd \& integrable}\}$.

Thm: (i) $f, g \in R([a,b]) \Rightarrow f+g \in R([a,b])$

(ii) $f \in R([a,b]) \Rightarrow cf \in R([a,b]) \quad \forall c \in \mathbb{R}$.

(iii) The map $T: R([a,b]) \to \mathbb{R}$ defined by

$$T(f) = \int_a^b f$$

is linear.

Proof: (i) Since $f, g \in R([a,b])$,

$$U(f) = L(f) \longrightarrow (1)$$

$$U(g) = L(g). \longrightarrow (2)$$

Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$. Now

$$M_i(f+g) \leq M_i(f) + M_i(g)$$

$$m_i(f+g) \geq m_i(f) + m_i(g)$$

Therefore

$$U(f+g,P) \leq U(f,P) + U(g,P)$$

$$L(f+g,P) \geq L(f,P) + L(g,P)$$

Hence,

$$U(f+g) \leq U(f,P) + U(g,P)$$

as $U(f+g)$ is the infimum of $U(f+g,P)$. Now using the property

$$\boxed{\inf_P \left(U(f,P) + U(g,P)\right) = \inf_P U(f,P) + \inf_P U(g,P)}$$

We have

$$U(f+g) \leq U(f) + U(g) \longrightarrow (3)$$

Similarly, we have

$$L(f) + L(g) \leq L(f+g) \longrightarrow (4)$$

Using ① and ②, combining with ③ and ④, we obtain,

$$U(f+g) = L(f+g) = U(f) + U(g) = L(f) + L(g).$$

This proves (i).

(ii) First note that

$$M_i(cf) = \sup_{[x_i, x_{i+1}]} cf(x)$$

$$= \begin{cases} c \sup_{[x_i, x_{i+1}]} f(x) & c > 0 \\ c \inf_{[x_i, x_{i+1}]} f(x) & c < 0 \end{cases}$$

$$= \begin{cases} c\, M_i(f) & c > 0 \\ c\, m_i(f) & c < 0 \end{cases}$$

Therefore

$$U(cf, P) = \begin{cases} c\,U(f,P) & \text{if } c>0 \\ c\,L(f,P) & \text{if } c<0. \end{cases}$$

Hence $U(cf) = \begin{cases} c\,U(f) & \text{if } c>0 \\ c\,L(f) & \text{if } c<0. \end{cases}$

Similarly

$$L(cf) = \begin{cases} c\,L(f) & \text{if } c>0 \\ c\,U(f) & \text{if } c<0. \end{cases}$$

Therefore, if $U(f) = L(f)$, then

$$U(cf) = L(cf) = c\,U(f) = c\,L(f).$$

This proves that cf is integrable

(iii) The last result follows from (i) & (ii).