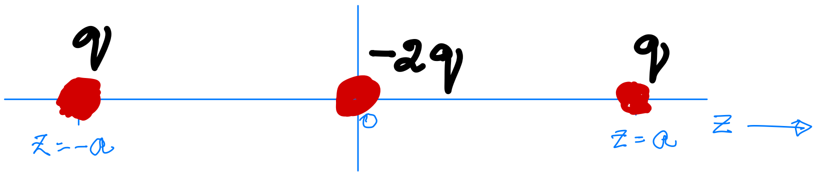
PH3202.2025 Tutorial 08 - 20/02/2025

Problem 1

Calculate first 3 multipole moment of the charge distribution



$$q_{00} = \frac{1}{\sqrt{4\pi}} \int \rho(x')\, d^3x' = \frac{q}{\sqrt{4\pi}}$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \int (x' - iy')\, \rho(x')\, d^3x' = -\sqrt{\frac{3}{8\pi}}\,(p_x - i p_y)$$

$$q_{10} = \sqrt{\frac{3}{4\pi}} \int z'\, \rho(x')\, d^3x' = \sqrt{\frac{3}{4\pi}}\, p_z$$

$$q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}} \int (3z'^2 - r'^2)\, \rho(x')\, d^3x' = \frac{1}{2}\sqrt{\frac{5}{4\pi}}\, Q_{33}$$

$$q_{21} = -\sqrt{\frac{15}{8\pi}} \int z'(x' - iy')\, \rho(x')\, d^3x' = -\frac{1}{3}\sqrt{\frac{15}{8\pi}}\,(Q_{13} - iQ_{23})$$

$$q_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \int (x' - iy')^2\, \rho(x')\, d^3x' = \frac{1}{12}\sqrt{\frac{15}{2\pi}}\,(Q_{11} - 2iQ_{12} - Q_{22})$$

$$p = \int x'\, \rho(x')\, d^3x' \qquad Q_{ij} = \int (3x'_i x'_j - r'^2 \delta_{ij})\, \rho(x')\, d^3x'$$

* The total charge in this case is zero and hence there is no monopole

$$q_{10} = \sqrt{\frac{3}{4\pi}} \sum_i^{1} z_i q_i$$

$$= d.q - d.q = 0$$

$$q_{11} = -\sqrt{\frac{3}{8\pi}} \int (x' - iy') \rho(x') = 0$$

because charges are arranged linearly on z-axis

Let us compute Quadrupole

$$q_{20} = \frac{1}{2}\sqrt{\frac{5}{4\pi}} \int (3z'^2 - r'^2) \rho(x)$$

$$= \frac{1}{2}\sqrt{\frac{5}{4\pi}} Q_{33}$$

$$Q_{ij} = \int (3 x_i' x_j' - r'^2 \delta_{ij}) \rho(x') d x'^3$$

$$Q_{33} = \int (3 z' z' - z'^2) \rho(x')$$

$$= \int 2 z' z' \rho(x') d x'^3$$

$$= 2d^2 q + 2d^2 q$$

$$= 4d^2 q$$

Problem 2

The Laplace eq^n in the cylindrical coordinate

$\{\rho, \phi, z\}$ $$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

1) Separate the above PDE into three ODE's

2) identify one of the ODE with following equations

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - m^2)\, y = 0$$

Find the locations and nature of singularity that may be present.

$$\underbrace{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2}}_{\text{function of } \rho,\phi \text{ only}} + \underbrace{\frac{\partial^2 \Phi}{\partial z^2}}_{\text{fun of } z \text{ only}} = 0$$

each of these two terms should be constant.

Since z goes from $-\infty$ to ∞, we can choose an exponential sol^n along z and constant of separation is such that

$$\frac{1}{\Phi_{xy}}\left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial \Phi_{xy}}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi_{xy}}{\partial \phi^2}\right] = -\alpha^2$$

$$\frac{1}{\Phi_z}\frac{\partial^2 \Phi_z}{\partial z^2} = +\alpha^2 \Rightarrow \frac{d^2\Phi_z}{dz^2} - \alpha^2\Phi_z = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi_{xy}}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\Phi_{xy}}{\partial\phi^2}+\alpha^2\Phi_{xy}=0$$

multiply by ρ^2

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi_{xy}}{\partial\rho}\right)+\frac{\partial^2\Phi_{xy}}{\partial\phi^2}+\rho^2\alpha^2\Phi_{xy}=0$$

$$\frac{1}{\Phi_x}\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi_x}{\partial\rho}\right)+\rho^2\alpha^2-m^2=0$$

$$\frac{1}{\Phi_y}\frac{\partial^2\Phi_y}{\partial\phi^2}+m^2=0$$

we get
$$\boxed{\frac{d^2\Phi_y}{d\phi^2}+m^2\Phi_y=0}$$

$$\boxed{\rho\frac{d}{d\rho}\left(\rho\frac{d\Phi_x}{d\rho}\right)+\left(\rho^2\alpha^2-m^2\right)\Phi_x=0}$$

Part 2 put $x = \lambda r$ and $\phi_x = y$

we get

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - m^2)y = 0$$

rewriting

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{m^2}{x^2}\right)y = 0$$

clearly at $x \to 0$ is a singular point

and it is regular singular point

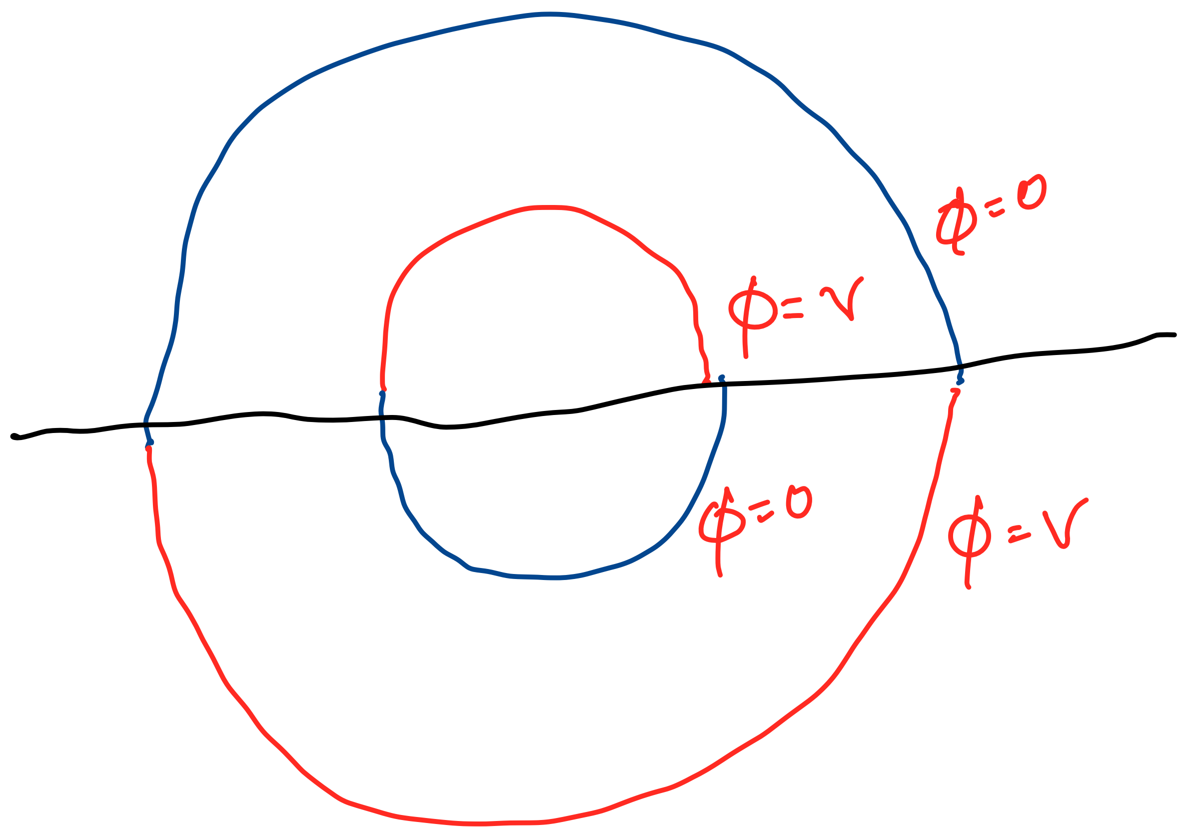
because $p(x)$ has simple pole at $x = 0$

and $q(x)$ has pole of order 2 at $x = 0$

Problem 3

Two concentric spheres have radii $a, b\ (b > a)$ and each is divided into two hemispheres by the same horizontal plane, The Upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V. The other hemisphere are at zero potential.

Determine the potential inside ~~the~~ region $a \le r \le b$



basic eqn is Laplacian in spherical polar coordinate with axial or azimuthal symmetry, the Soln can be written as

$$\phi = \sum_{l=0}^{\infty} \alpha_l r^l P_l(\cos\theta) + \beta_l \frac{1}{r^{l+1}} P_l(\cos\theta)$$

$$\Phi = \sum_{l=0}^{\infty} \left(\alpha_l r^l + \beta_l \cdot \frac{1}{r^{l+1}} \right) P_l \cos\theta .$$

$$\Phi(a,\theta) = \sum_l \left[A_l a^l + B_l a^{-l-1} \right] P_l(\cos\theta)$$

$$= \begin{cases} V & \cos\theta \geq 0 \\ 0 & \cos\theta \leq 0 \end{cases}$$

$$\phi(b,\theta) = \sum_l \left[A_l b^l + B_l b^{-l-1} \right] P_l(\cos\theta)$$

$$= \begin{cases} 0 & \cos\theta \geq 0 \\ V & \cos\theta \leq 0 \end{cases}$$

$$A_l a^l + B_l a^{-l-1} = \frac{2l+1}{2} V \int_0^1 P_l(x)\,dx$$

$$A_l a^l + B_l b^{-l-1} = \frac{2l+1}{2} V \int^0 P_l(x)\,dx$$

$$= (-1)^l \frac{2l+1}{2} V \int_0^{-1} P_l(x)\,dx$$

we need to solve for A_l & B_l

$$A_l a^l + B_l a^{-l-1} = K_1$$

$$A_l b^l + B_l b^{-l-1} = K_2$$

$$K_1 = \frac{2l+1}{2} V \int_0^1 P_l(x)\,dx$$

$$K_2 = \frac{2l+1}{2} V (-1)^l \int_0^1 P_l(x)\,dx$$

$$A_l = \left(\frac{2l+1}{2}\right) \cancel{V} \frac{\left((-1)^l b^{l+1} - a^{l+1}\right)}{b^{2l+1} - a^{2l+1}} \int_0^1 P_l(x)\,d$$

$$B_l = \left(\frac{2l+1}{2}\right) V \frac{(ab)^{l+1}\left(b^l + (-1)^{l+1} a^l\right)}{b^{2l+1} - a^{2l+1}} \times \int_0^1 P_l(x)\,dx$$