PH3102 Quantum Mechanics   Assignment 5

Instructor: Dr. Siddhartha Lal   Autumn Semester, 2024

Start Date: September 03, 2024   Submission Deadline: September 10, 2024 .

Submit your answers to the Tutor at the start of the tutorial.

Q1. Particle on a circle. (15 marks)

A quantum particle of mass M confined on a circle of radius R has a moment of inertia $I = MR^2$. The Hamiltonian of such a particle can be written purely in terms of its angular momentum operator L

$$H = \frac{L^2}{2I} \qquad , \text{ where } \; L^2 = -\hbar^2 \frac{d^2}{d\phi^2}$$

and $\phi \; \epsilon [0, 2\pi]$ is the angular coordinate defined on the circle. The eigenfunctions for this problem satisfy periodic boundary conditions: $\psi(\phi) = \psi(\phi + 2\pi)$.

(a) What are the eigenfunctions of this Hamiltonian and what are the quantum numbers that characterise them? Remember to normalise the eigenfunctions.

(b) What are the energy eigenvalues associated with these eigenfunctions?

(c) Draw the real part of the wavefunctions for the ground state and the first two excited states.

(d) Upon introducing a solenoid with flux Φ through the circle, the Hamiltonian becomes

$$H = \frac{\hbar^2}{2MR^2}(-i\frac{d}{d\phi} - \frac{\Phi}{\Phi_0})^2 \qquad , \text{ where } \; \Phi_0 = \frac{\hbar c}{e} \text{ , the flux quantum.}$$

What is the ground state wavefunction and energy for $\Phi = \Phi_0$? Note that the introduction of the flux Φ leads to *twisted* boundary conditions.

(e) What is the ground state wavefunction and energy for $\Phi = \Phi_0/2$?

(f) Plot the energy spectrum as a function of the flux Φ. This is a demonstration of *spectral flow*, i.e., the sensitivity of the eigenspectrum to changes in the boundary conditions.

Q2. Holstein-Primakoff transformation. (10 marks)

There is a striking similarity between the equal spacing of the eigenvalues of L_z (in any particular representation of angular momentum labelled by an integer or half-integer l) and the raising and lowering operators L_+ and L_- on the one hand, and the equal spacing of the energy levels of a simple harmonic oscillator and the raising and lowering operators $a^\dagger$ and a on the other hand.

(a) Show that if we define

$$L_z = \hbar(1 - a^\dagger a) \text{ , } L_+ = \hbar(\sqrt{2l - a^\dagger a})a \text{ , } L_- = \hbar a^\dagger(\sqrt{2l - a^\dagger a}) \text{ ,}$$

the above expressions satisfy the commutation relations $[L_i, L_j] = i\epsilon_{ijk}L_k \; (i,j,k) \in (x,y,z)$ and $\vec{L}^2 = l(l+1)\hbar^2$ as long as a and $a^\dagger$ obey the commutation relation $\left[a, a^\dagger\right] = 1$.

(Note: the square root of an operator is defined as a Taylor series expansion, e.g.,
$\sqrt{2l - a^\dagger a} = \sqrt{2l}\left[1 - \frac{1}{2}(\frac{a^\dagger a}{2l}) - \frac{1}{8}(\frac{a^\dagger a}{2l})^2) - \ldots\right]$.)

(b) What are the maximum and minimum possible eigenvalues of $a^\dagger a$ in this representation?