Worksheet 3: Numerical integration and differentiation

If you are using Julia or Python, we recommend using a jupyter notebook. In WeLearn, you need to submit this file. Please clearly indicate in the markup cells, the number of the question for which you are writing the program. Also, please remember to add documentation through comments in your program.

You may also use scripts and use REPL to evaluate them. In that case, please keep all your files for a particular worksheet in a folder and you may upload the compressed archive of that folder.

Please feel free to ask for help!

1. (8 points) Numerically estimate $f'(x)$ for $f(x) = sin(x)$ at $x = 2\pi/5$, using
   (a) Forward difference: $f'_n \approx (f_{n+1} - f_n)/h$
   (b) Backward difference: $f'_n \approx (f_n - f_{n-1})/h$
   (c) Central difference: $f'_n \approx (f_{n+1} - f_{n-1})/(2h)$
   (d) Five-point approximation: $f'_n \approx (f_{n-2} - 8f_{n-1} + 8f_{n+1} - f_{n+2})/(12h)$

   for $h = [0.5, 0.2, 0.1, 0.05, 0.02, 0.01, 0.005, 0.002, 0.001, 0.0005, 0.0002, 0.0001]$.

   (a) (3 points) Create a table to record the estimate derivatives using different methods for each h.
   (b) (3 points) Plot the differences between exact values and the estimated values (the error).
   (c) (2 points) Fit the log of the error versus the log of h and compare the efficiencies of the methods.

2. (8 points) Numerically estimate $\int_0^1 f(x)dx$ for $f(x) = e^x$, using the following methods:
   (a) Linear (Trapezoidal): $\int_{x_1}^{x_3} f(x)dx = \frac{h}{2}(f_1 + 2f_2 + f_3) + \mathcal{O}(h^3)$
   (b) Quadratic (Simpson's $\frac{1}{3}$ rule): $\int_{x_1}^{x_3} f(x)dx = \frac{h}{3}(f_1 + 4f_2 + f_3) + \mathcal{O}(h^5)$
   (c) Cubic (Simpson's $\frac{3}{8}$ rule): $\int_{x_1}^{x_4} f(x)dx = \frac{3h}{8}(f_1 + 3f_2 + 3f_3 + f_4) + \mathcal{O}(h^5)$
   (d) Quartic (Bode's rule): $\int_{x_1}^{x_5} f(x)dx = \frac{2h}{45}(7f_1 + 32f_2 + 12f_3 + 32f_4 + 7f_5) + \mathcal{O}(h^7)$

   (a) (4 points) Divide the interval $[0, 1)$ into *approximately* $[20, 200, 2000, 20000]$ points to compare the efficiencies of the methods. Adjust N for each method to match the number of strips required for that particular method. Create a table to record the estimate integral using different methods for each $h = (1.0 - 0.0)/(N - 1)$ value.
   (b) (2 points) Plot the differences between exact values and the estimated values (the error).
   (c) (2 points) Fit the log of the error versus the log of h and compare the efficiencies of the methods.

3. (4 points) Use Simpson's 3/8 method to integrate $f(x) = x^{-2/3}(1 - x)^{-1/3}$ from 0 to 1. The exact integral is $2\pi/\sqrt{3}$. Use variable substitutions to handle the singularities.