Problems on Differentiable functions

1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

Show that $f'(0)$ exists. Find $f'(0)$.

2. Let $n \in \mathbb{N}$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^n & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

For which values of n,

i. is f continuous at 0 ?

ii. is f differentiable at 0 ?

iii. is f' continuous at 0 ?

iv. is f' differentiable at 0 ?

3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at c. let $x_n < c < y_n$ be such that $y_n - x_n \to 0$. Show that

$$\lim_{n\to\infty} \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(c).$$

4. Find the values of r, for which the following function

$$f(x) = \begin{cases} x^r \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

is differentiable at 0.

5. Prove the following :

i. $e^x \geq ex$ for all $x \in \mathbb{R}$.

ii. $\frac{x}{1+x} < \log(1 + x) < x$ for all $x > 0$.

iii. $\frac{\sin x}{x}$ is strictly decreasing on $(0, \pi/2)$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all $x, y \in \mathbb{R}$. Show that f is a constant.

7. Show that the function $f(x) = \exp(\sin x)$ is uniformly continuous on $\mathbb{R}$.

8. Let $b \in \mathbb{R}$. Show that the equation $x^3 - 3x^2 + b = 0$ has at most one root in $[0, 1]$.

9. Let $f : [a, b] \to \mathbb{R}$ be differentiable function such that $f(a) = f(b) = 0$. Then show that there exists $c \in (a, b)$ such that

$$f'(c) + f(c)g'(c) = 0.$$

10. Let $f : (0, \infty) \to \mathbb{R}$ be differentiable. If $f'(x) \to l$ as $x \to \infty$, then show that

$$\lim_{x\to\infty} \frac{f(x)}{x} = l.$$

11. Let $f : [0, \infty) \to \mathbb{R}$ be differentiable. Assume that $f(0) = 0$ and f' is increasing. Prove that $f(x)/x$ is increasing.

12. Let $f : [a, b] \to \mathbb{R}$ be differentiable. Assume that there exists no $x \in [a, b]$ such that $f(x) = 0 = f'(x)$. Prove that the set

$$\{t \in [a, b] : f(t) = 0\}$$

is finite.

13. Let $f : [a, b] \to [a, b]$ be differentiable. Assume that $f'(x) \neq 1$ for all $x \in [a, b]$. Prove that f has a unique fixed point in $[a, b]$.

14. **Convex functions :** A function $f : (a, b) \to \mathbb{R}$ is said to be convex if, the following inequality holds :

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) \quad \forall t \in [0, 1], \forall x, y \in (a, b).$$

i. A function $f : (a, b) \to \mathbb{R}$ is convex iff for all $x_1, x_2, x_3 \in (a, b)$ with $x_1 < x_2 < x_3$, f satisfies

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_1)}{x_3 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}.$$

ii. A differentiable function $f : (a, b) \to \mathbb{R}$ is convex iff f' is increasing.

iii. A differentiable function $f : (a, b) \to \mathbb{R}$ is convex iff for all $x, y \in (a, b)$

$$f(y) \geq f(x) + f'(x)(y - x).$$