PH3102 Quantum Mechanics   Assignment 7

Instructor: Dr. Siddhartha Lal   Autumn Semester, 2024

Start Date: October 17, 2024   Submission Deadline: October 22, 2024 .

Submit your answers to the Tutor at the start of the tutorial.

Q1. Electron in a magnetic field: Landau levels [15 marks]

Consider a spinless particle of charge e and mass m confined to moving in a two-dimensional plane (say, the XY plane) and in the presence of an uniform external magnetic field in the z-direction ($\vec{B} = B\hat{z}$). Assume $c = 1$ throughout.

a. Construct a vector potential $\vec{A}(\vec{r})$ according to the symmetric gauge $\vec{A}(\vec{r}) = \frac{1}{2}(\vec{B} \times \vec{r})$. Write down the complete Hamiltonian for the system. [2 marks]

b. Construct a ladder operator a in which the Hamiltonian acquires the standard form of an SHO: $H = \hbar\omega_c \left(a^\dagger a + \frac{1}{2}\right)$, where $\omega_c = eB/mc$ is the cyclotron frequency. Make sure the ladder operator satisfies the proper commutation relation. [2 marks]

c. Once this is in place, write down the energy levels in terms of the cyclotron frequency. These states are referred to as the Landau levels. [1 mark]

d. In order to obtain the ground state, define a variable $r = x - iy$ and its complex conjugate $\bar{r}$. Also define the derivative operations $\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ and its complex conjugate $\bar{\partial}$. Rewrite the ladder operator a you obtained at the beginning in terms of r, ∂ and their complex conjugates. Solve the differential equation arising from the action of a on the ground state in order to obtain a functional form for the ground state wavefunction. [3 marks]

e. Let us now concentrate on the ground states. What do the probabilities (i.e., $|\psi|^2$) look like physically? [2 marks]

f. These eigenstates are highly degenerate. In order to reveal this, define a second ladder operator b by using the operators $\vec{\tilde{\pi}} = \vec{p} + e\vec{A}$. Make sure that these new ladder operators have the appropriate commutator, and also show that they commute with the Hamiltonian. a and b therefore form a set of commuting operators, and can be used to label the eigenstates of the Hamiltonian. [We can create new eigenstates by applying $b^\dagger$ on any given eigenstate of a, but since the energy depends only on the eigenvalue of $a^\dagger a$, these will all be degenerate.] [4 marks]

g. Can you identify which conserved observable is associated with this degeneracy? The symmetry of the Hamiltonian constructed in the first part should guide you; try to visualise the Hamiltonian in polar coordinates. [1 mark]