Tutorial - 06/ → 06/02/2025

Problem 1

Solve the problem $\frac{d^2y(x)}{dx^2} + y(x) = x$ in the region, $0 \leq x \leq 1$

using the method of greens function with the boundary condition, $y(0) = y(1) = 0$

Compare the Sol^n with Sandard method.

Let us first get the sol^n in standar method

The differential Eq is $\frac{d^2y}{dx^2} + y(x) = x$ with bdry condition

$y(0) = y(1) = 0$

The Sol^n can be written as

$$y = \underbrace{A \sin x + B \cos x}_{y_g} + \underbrace{x}_{y_p}$$

general sol^n Homogenous part + particular sol^n to full Eq

Now to pick boundary condition at $x=0$ bdry condition demand

$$y = x_0 + A \sin x$$

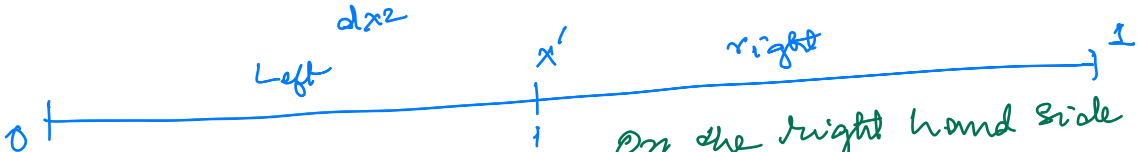
by choosing $A = \frac{-1}{\sin(1)}$ we get

$$y = x - \frac{\sin(x)}{\sin(1)}$$ which satisfy both bdry condition!

Now to the method of Green's fn:

we need to solve to get the Green's fn

$$\frac{d^2 G(x,x')}{dx^2} + G(x,x') = \delta(x-x')$$



on the left side $x < x'$

$$\frac{d^2 G_l}{dx^2} + G_l(x) = 0$$

with the boundary condition

$$G(0) = 0$$

$$G(x) = A_l \sin x$$

On the right hand side $x > x'$

$$\frac{d^2 G_r(x)}{dx^2} + G_l(x) = 0$$

with the boundary condition

$$G(1) = 0$$

$$G(x) = A_r \sin(x-1)$$

$G_l(x) = \sin x$

$G_r(x) = \sin(x-1)$

to match $G_l(x)$ and $G_r(x)$ at $x = x'$ we set

$$A_l = C\, G_r(x') \quad \text{and} \quad A_r = C\, G_l(x')$$

$$G(x) = A\, G_r(x')\, G_l(x) \quad \text{for } x < x'$$

$$G(x) = A\, G_l(x')\, G_r(x) \quad \text{for } x > x'$$

Summary

$$G(x,x') = \begin{cases} A\, G_l(x)\, G_r(x') & x < x' \\ A\, G_r(x')\, G_l(x) & x > x' \end{cases}$$

or

$$G(x,x') = \begin{cases} A \sin x \sin(x'-1) & x < x' \\ A \sin(x-1) \sin x' & x > x' \end{cases}$$

$$G(x,x') = \begin{cases} A \sin x \sin(x'-1) & x < x' \\ A \sin(x-1) \sin x' & x > x' \end{cases}$$

To find A we go back to differential Eqn

$$\frac{d^2 G(x,x')}{dx^2} + G(x,x') = \delta(x-x')$$

and integrate in a small interval around x' i.e. $[x'-\epsilon, x'+\epsilon]$

and take limit $\epsilon \longrightarrow 0$

$$\int_{x'-\epsilon}^{x'+\epsilon} \frac{d^2 G(x,x')}{dx^2} dx + \underbrace{\int_{x'-\epsilon}^{x'+\epsilon} G(x,x')\, dx}_{\text{goes to zero as } \epsilon \to 0} = \underbrace{\int_{x'-\epsilon}^{x+\epsilon} \delta(x-x')\, dx}_{1 \text{ as limit } \epsilon \to 0}$$

from comparison

$$\int_{x'-\epsilon}^{x'+\epsilon} \frac{d^2 G(x',x)}{dx^2} dx = \left. \frac{d G(x',x)}{dx} \right|_{x'-\epsilon}^{x'+\epsilon} = 1$$

$$\Rightarrow \left. \frac{d G(x',x)}{dx} \right|_{x'+\epsilon} - \left. \frac{d G(x',x)}{dx} \right|_{x'-\epsilon} = 1$$

After substituting for $G(x\ x')$ in terms of $G_l(x)$ $G_r(x)$

we have $A \left[G_r'(x) G_l(x') - G_l'(x) G_r(x') \right]_{x=x'} = 1$

$$A = \frac{1}{W[G_l(x) G_r(x)]_{x=x'}}$$

we can not determine consts of G_l and G_r are linearly dependent

Now $G_l = \sin x$ $\quad G_r = \sin(x-1)$

$$A = \frac{1}{\left[G_r'(x')G_l(x') - G_l'(x')G_r(x')\right]}$$

$$W[G_l, G_r] = \sin 1 \qquad A = \frac{1}{\sin 1}$$

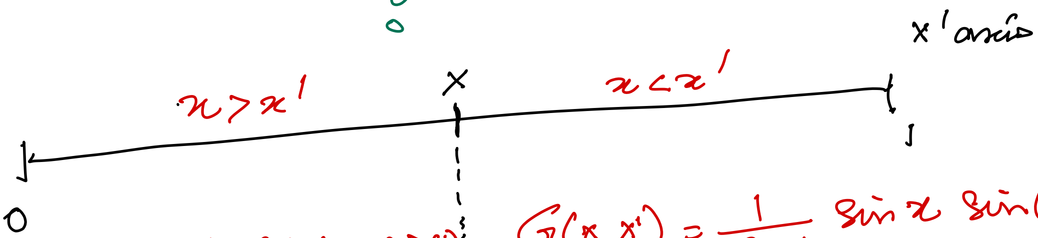
Finally

$$G(x,x') = \begin{cases} \dfrac{1}{\sin 1}\sin x \sin(x'-1) & x < x' \\[2ex] \dfrac{1}{\sin 1}\sin(x-1)\sin(x') & x > x' \end{cases}$$

is the Green's function for the problem.

The Soln $\quad y(x) = \int_a^b f(x')\,G(x,x')\,dx' \qquad f(x') = x'$

$$= \int_0^1 x'\,G(x,x')$$



$G(x,x') = \frac{1}{\sin 1}\sin(x-1)\sin(x') \qquad G(x,x') = \frac{1}{\sin 1}\sin x \sin(x'-1)$

$$y(x) = \int_0^x x'\,G(x,x')\,dx' + \int_x^1 x'\,G(x,x')\,dx'$$

$$y(x) = \int_0^x x' \frac{1}{\sin 1}\sin(x-1)\sin x'\,dx' + \int_0^1 x' \frac{1}{\sin 1}\sin x \sin(x'-1)\,dx'$$

$$y(x) = \frac{1}{\sin 1}\left[\sin(x-1)\int_0^x x'\sin x'\,dx' + \sin x\int_x^1 x'\sin(x'-1)\,dx'\right]$$

after some calculation

$$y(x) = x - \frac{\sin x}{\sin 1}$$