Classtest 1

If you are using Julia or Python, we recommend using a jupyter notebook. In WeLearn, you need to submit this file. Please clearly indicate in the markup cells, the number of the question for which you are writing the program. Also, please remember to add documentation through comments in your program.

You may also use scripts and use REPL to evaluate them. In that case, please keep all your files for a particular worksheet in a folder and you may upload the compressed archive of that folder.

Please feel free to ask for help!

Answer any three questions

1. **Planck's Law for Radiance** Planck's law for spectral radiance $B(\nu)$ can be integrated over frequency ν to find the total radiated power of a blackbody. A simplified integral you might use is: [10]

$$\int_0^\infty \frac{x^3}{e^x - 1}\, dx = \frac{\pi^4}{15}.$$

(This integral appears when converting Planck's law to a dimensionless form.)

Task

1. Define the function
$$f(x) = \frac{x^3}{e^x - 1}.$$
2. Use a numerical integration method, either Simpson's 3/8 or Bode, to approximate $\int_0^\infty f(x)\, dx$.
3. Since the upper limit is infinite, integrate up to some large $x_{\max}$ (e.g., 10 or 12) where the integrand becomes negligible.
4. Compare your result to the known value $\frac{\pi^4}{15} \approx 6.493939$.
5. Report the numerical result and the error.

Hints

- Use a moderate step size at first, then refine it to see how quickly the result converges.
- This integral converges nicely, so standard methods work well.

2. **Root Finding: Lennard-Jones Potential Minimum** The Lennard-Jones potential (often used to model intermolecular interactions) is given by [10]

$$V(r) = 4\epsilon\left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right].$$

The equilibrium bond distance occurs where the force

$$F(r) = -\frac{dV}{dr}$$

is zero.

Task

1. Write an expression for the force $F(r)$.
2. Use a *root-finding* method (e.g., bisection, Newton, or secant) to find r^* such that $F(r^*) = 0$.
3. Assume *epsilon* $= 1$ and *sigma* $= 1$ for simplicity.
4. Use an initial guess around $r \approx 1$.
5. Print out the converged value of r^*.
6. *(Optional)* Compare it to the known analytical result $r^* = 2^{1/6} \approx 1.122462$.

3. **Simple Planetary Orbit (Velocity Verlet)** [10]

Consider a planet orbiting a star (assume the star is fixed at the origin, and we work in two dimensions). The gravitational force on the planet is

$$\mathbf{F} = -\frac{GMm}{r^2}\,\hat{\mathbf{r}}, \quad r = \sqrt{x^2 + y^2}.$$

Here, G is the gravitational constant, M the mass of the star, and m the mass of the planet.

Task

1. Convert this second-order ODE into four first-order ODEs for the variables (x, y, v_x, v_y).
2. Implement the *velocity Verlet* algorithm:

$$\mathbf{r}_{n+1} = \mathbf{r}_n + \mathbf{v}_n\,\Delta t + \tfrac{1}{2}\,\mathbf{a}_n\,(\Delta t)^2,$$

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \tfrac{1}{2}\,[\,\mathbf{a}_n + \mathbf{a}_{n+1}\,]\,\Delta t,$$

   where $\mathbf{a}_n = \mathbf{F}(\mathbf{r}_n)/m$.
3. Start with an initial circular orbit, e.g., $r_0 = 1\,(\mathrm{AU})$, and $v_0 = \sqrt{\dfrac{GM}{r_0}}$.
4. Evolve the system for one or more orbital periods.
5. Track the *total energy*, $E = \tfrac{1}{2}\,m\,|\mathbf{v}|^2 - \dfrac{GMm}{r}$, at each timestep. Observe how well the symplectic integrator conserves energy over many orbits.

Hints

- For simplicity, you might set $GM = 1$ in some unit system.
- Use a sufficiently small timestep (e.g., $\Delta t = 0.01$) to maintain stability and conserve orbital energy over many cycles.

4. **Projectile Motion with Quadratic Drag (ODE)**: A projectile of mass m is launched at speed v_0 and angle θ_0, subject to gravity and a drag force proportional to v^2. In two dimensions, we have the system [10]

$$\begin{cases} \dfrac{dx}{dt} = v_x, \\ \dfrac{dy}{dt} = v_y, \\ \dfrac{dv_x}{dt} = -\dfrac{C_d}{m} \left|\mathbf{v}\right| (v_x), \\ \dfrac{dv_y}{dt} = -g - \dfrac{C_d}{m} \left|\mathbf{v}\right| (v_y), \end{cases}$$

where $|\mathbf{v}| = \sqrt{v_x^2 + v_y^2}$ and C_d is a drag coefficient.

Task

1. Convert this system into a set of first-order ODEs in the variables (x, y, v_x, v_y).
2. Implement an ODE solver *Runge-Kutta 4*.
3. Integrate from $t = 0$ until the projectile returns to $y = 0$ (assuming flat ground).
4. Record the *range* (the final x position when $y = 0$ again).
5. Print out the range for a chosen v_0 and θ_0.

Hints

- A small time step (e.g., $\Delta t = 0.01$) may be needed for stability and good accuracy.
- You can experiment with different drag coefficients C_d and observe the effect on the range.

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