# PH3102 Quantum Mechanics    Assignment 9

**Instructor: Dr. Siddhartha Lal    Autumn Semester, 2024**

**Start Date: October 29, 2024    Submission Deadline: November 05, 2024 .**

**Submit your answers to the Tutor at the start of the tutorial.**

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## Q1. Perturbing the Infinite Square Well [15 marks]

Consider an infinite square well potential in one dimension defined by

$$\begin{aligned} V(x) &= 0 \quad \text{for } 0 < x < a \ , \\ &= \infty \quad \text{elsewhere} \ . \end{aligned}$$

(a) Will the addition of a small constant positive potential $V_0 > 0$ lying in the spatial range $a/3 \leq x \leq 2a/3$ to the infinite square well change the ground state energy and ground state wavefunction? If so, explain the reason for the changes without doing any explicit calculations? ( *Hint*: Note and indicate whether the ground state energy and the peak of the ground state wavefunction is expected to increase or decrease upon introducing the perturbation.)
(b) Treating the potential $V_0$ as a perturbation, calculate the first order correction to the ground state energy.
(c) By noting carefully the nature of perturbation potential $V_0$ (and without carrying out any explicit calculations!), can you anticipate which states of the infinite square well problem can possibly mix with one another due to the perturbation $V_0$? Using this insight, compute the second order correction to the ground state energy due to $V_0$ arising from mixing with *only* the lowest allowed excited state. Also, write down the complete ground state energy including the first and second order corrections you have computed.
(d) Now compute the first order correction to the ground state wavefunction due to $V_0$ arising from mixing with *only* the lowest allowed excited state. Also, write down the complete ground state wavefunction including the first order correction you have computed.
(e) Now consider a negative $V_0 < 0$ in the same problem, and using the results obtained above, write the complete energy and wavefunction for the ground state including all the corrections till the orders you computed earlier.
(f) Sketch in a single plot the perturbed ground state wavefunction for (i) $V_0 > 0$, (ii) $V_0 < 0$ and (iii) the unperturbed problem. What do you observe from the three wavefunctions?

## Q2. Perturbation theory for a 3-level problem. [25 marks]

The Hamiltonian for a 3-level system is given by the matrix

$$H = \begin{pmatrix} -u & v & 0 \\ v & u & 0 \\ 0 & 0 & u' \end{pmatrix} \quad ; \quad u, u', v > 0 \ . \tag{1}$$

The goal is to apply perturbation theory to approximate the eigenvalues of $H$.
(a) First, write this matrix as the sum of two matrices $H = H_0 + V$, such that $H_0$ is the diagonal part of the Hamiltonian matrix given above while $V$ is the off-diagonal part. We will consider $H_0$ as the zeroth Hamiltonian, and $V$ as the perturbation.
(b) Now, write down the eigenstates $|\psi_1^{(0)}\rangle, |\psi_2^{(0)}\rangle$ and $|\psi_3^{(0)}\rangle$ and eigenvalues $E_1^{(0)}, E_2^{(0)}$ and $E_3^{(0)}$ of $H_0$. [1 mark]
(c) Next, find the eigenvalues $E_1^{(2)}, E_2^{(2)}$ and $E_3^{(2)}$ of $H$ to second order in perturbation theory, using the expressions derived in the class.
(d) Also, find the eigenstates $\psi_1^{(1)}, \psi_2^{(1)}$ and $\psi_3^{(1)}$ of $H$ to first order in perturbation theory.
(e) Now, solve the problem exactly by diagonalising the matrix $H$. Obtain the perturbative result from the exact result by a suitable approximation. In which regime of parameters is this approximation valid? What does this say about the validity of perturbation theory in this problem?
(f) Calculate the wavefunction renormalisation $Z$ in the ground state. How does it behave as the parameters are tuned beyond the regime of validity of perturbation theory?