PH 3101 : 2024, tutorial-1 Date 13/08/2024

Problem 1 : In a two-dimensional physical system, a particle is experiencing a force field given by

$$\vec{F} = -\frac{G(x\hat{i} + y\hat{j})}{(x^2+y^2)^{3/2}}$$

1) Write the equation of motion in Cartesian coordinate system

2) Is the system seperable in the given (cartesian) coordinate system

3) What are non-trivial constants of motion?

4) Is the system integrable and why?

The Newton's law of motion is given by

$$\frac{d\vec{p}}{dt} = \vec{F}$$

For the Euclidean basis $\hat{i}$ and $\hat{j}$, which are constants, we get

$$\frac{dp_i}{dt} = F_i \quad \text{for } i = \{1, 2\}$$

$$F_1 = \frac{-G\, x}{(x^2+y^2)^{3/2}} \;;\; F_2 = \frac{-Gy}{(x^2+y^2)^{3/2}}$$

Hence the EOM is given by

$$\frac{dP_1}{dt} = \frac{-Gx}{(x^2+y^2)^{3/2}} \qquad \frac{dP_2}{dt} = \frac{-Gy}{(x^2+y^2)^{3/2}}$$

(*) These equations are coupled in the cartesian coordinate and not separable

(*) conserved quantities

The force is given by

$$\vec{F} = \frac{-G}{(x^2+y^2)^{3/2}}\vec{r}$$

can be written as gradiant of $\phi = \frac{-G}{(x^2+y^2)^{1/2}}$

$$\vec{F} = \vec{\nabla}\left[\frac{-G}{(x^2+y^2)^{1/2}}\right]$$

So the force is conservative and time independent hence, the total energy

$$E = \phi + T = \text{const}$$

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$$

(a) let $\vec{r} = x\hat{i} + y\hat{j}$

we have $\vec{r} \times \vec{F} = \frac{-G}{(x^2+y^2)^{3/2}} \vec{r} \times \vec{r} = 0$

& torque $N = 0$ $\quad L = \vec{r} \times \vec{p}$ is const

$$L_z = m(x\dot{y} - y\dot{x}) \text{ is constant!}$$

(b) A system of N-degree of freedom is integrable if there are N-independent constant of motion. In the current problem there are 2-constants and Hence System is integrable.

Problem 2: Equation of motion in plane polar coordinate.

The Newton's law of motion for a point particle in a cartesian coordinate system is given by

$$\frac{dP_i}{dt} = F_i \qquad i = \{1, 2\}$$

or interms of acceleration as,

$$m\frac{d^2 x_i}{dt^2} = F_i \qquad i = \{1, 2\}$$

with $x_1 = x$, $x_2 = y$

Now transformation from cartesian to plane polar coordinate $\{r, \theta\}$ is given by

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r = (x^2+y^2)^{1/2} \qquad \tan\theta = y/x$$

(*) Write the expression for velocity and acceleration in polar coordinate

* Write the EOM, in terms of momentum as well as acceleration.

* How the basis vector in 2 coordinate are related

* Is the Newton's law is form invariant under this transformation

* Write the expression for Kinetic energy in polar coordinate.

We start with position vector

$$\vec{r} = x\hat{i} + y\hat{j}$$

using the transformation

$$\vec{r} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

The unit vector $\hat{r}$ is one of the basis vector $\Rightarrow$ $\boxed{\hat{r} = (\cos\theta\,\hat{i} + \sin\theta\,\hat{j})}$

Let $\hat{\theta}$ be other basis vector which is orthogonal to $\hat{r}$

Let $\hat{\theta} = \alpha_1\hat{i} + \alpha_2\hat{j}$

$\hat{\theta}$ is also unit vector and hence

$$\alpha_1^2 + \alpha_2^2 = 1$$

$$\hat{\theta} \cdot \hat{r} = 0 = \alpha_1 \cos\theta + \alpha_2 \sin\theta = 0$$

$$\Rightarrow \alpha_1 = -\sin\theta \quad \alpha_2 = \cos\theta$$

$$\boxed{\hat{\theta} = -\sin\theta\, \hat{i} + \cos\theta\, \hat{j}}$$

$\hat{i}$ and $\hat{j}$ are constant but $\hat{r}$ & $\hat{\theta}$ are

Not

$$\dot{\hat{r}} = \frac{d\hat{r}}{dt} = -\sin\theta\, \dot{\theta}\, \hat{i} + \cos\theta\, \dot{\theta}\, \hat{j}$$

$$\boxed{\dot{\hat{r}} = \dot{\theta}\, \hat{\theta}}$$

$$\frac{d\hat{\theta}}{dt} = -\cos\theta\, \dot{\theta}\, \hat{i} - \sin\theta\, \dot{\theta}\, \hat{j}$$

$$\boxed{\frac{d\hat{\theta}}{dt} = -\dot{\theta}\, \hat{r}}$$

Vector $\vec{A}$ can be written as

$$\vec{A} = A_x \hat{i} + A_y \hat{j} = A_r \hat{r} + A_\theta \hat{\theta}$$

$$= A_r \cos\theta\, \hat{i} + A_r \sin\theta\, \hat{j}$$

$$+ (-A_\theta) \sin\theta\, \hat{i} + A_\theta \cos\theta\, \hat{j}$$

$$(A_r\cos\theta - A_\theta\sin\theta)\hat{i} + (A_r\sin\theta + A_\theta\cos\theta)\hat{j}$$
$$= A_x\hat{i} + A_y\hat{j}$$

$$A_x = A_r\cos\theta - A_\theta\sin\theta$$
$$A_y = A_r\sin\theta + A_\theta\cos\theta$$

Similarly the inverse relation is given by

$$A_r = A_x\cos\theta + A_y\sin\theta$$
$$A_\theta = -A_x\sin\theta + A_y\cos\theta$$

The equation of motion

$$\vec{r} = r\hat{r}$$

$$\vec{v} = \dot{\vec{r}} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\hat{r}}$$
$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}}$$
$$= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

That is expression for Central force!

The Newton law of motion don't look same in polar coordinate. It is not form invariant under this transformation

⊗ The velocity of particle

$$\vec{v} = v_x \hat{e}_x + v_y \hat{e}_y = V_r \hat{e}_r + V_\theta \hat{e}_\theta$$

$$\frac{M}{2}(v_x^2 + v_y^2) = \frac{m}{2} \vec{v} \cdot \vec{v}$$

$$T = \frac{m}{2}(\dot{r}^2 + r^2 \dot{\theta}^2) \; !$$