Examples of Fourier Series

An $\mathbb{L}^2$ function $f(x)$ in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ can be written as

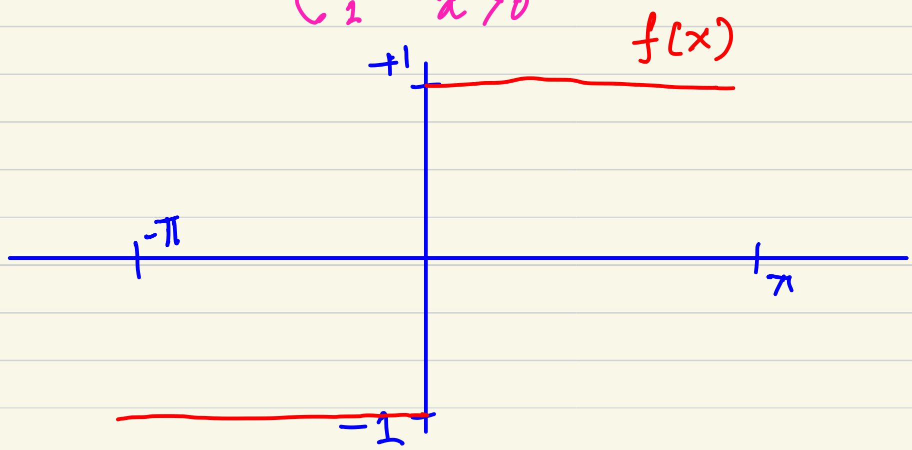
$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{m=1}^{\infty} b_m \sin mx$$

where

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_m &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}}$$

Example:

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases} \quad \text{in the range } [-\pi, \pi]$$



$f(x)$ is an odd function,

which $\Rightarrow$ $a_0 = 0$

and $a_n = 0 \;\forall\; n > 0$

We need to find b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\sin nx \, dx + \frac{1}{\pi}\int_{0}^{\pi} \sin nx \, dx$$

 ?

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin nx \, dx = -\frac{2}{\pi} \left. \frac{\cos nx}{n} \right|_{0}^{\pi}$$

$$= -\frac{2}{\pi n}\left[\cos n\pi - 1\right]$$

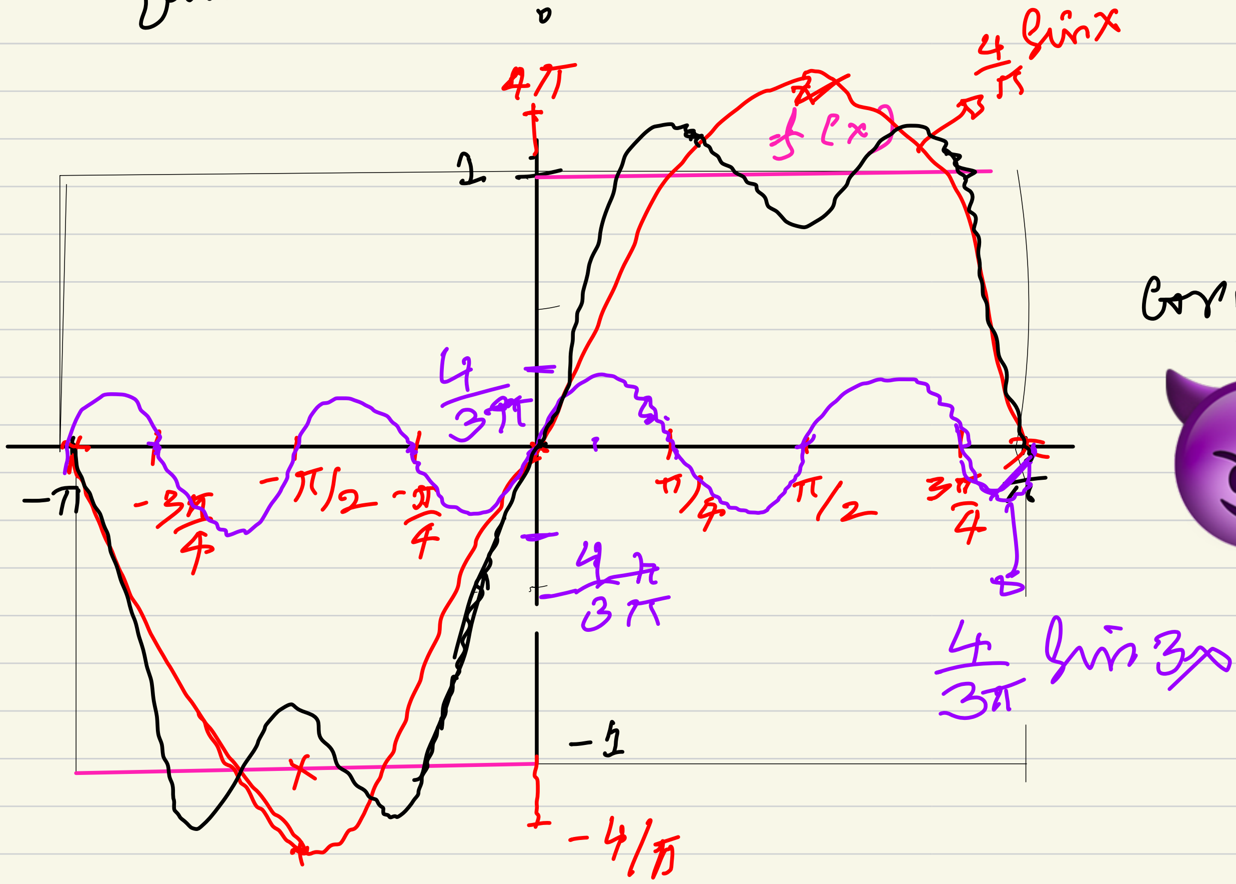
$$= \frac{4}{n\pi} \text{ for odd}$$

$$0 \text{ for even}$$

$$b_n = \frac{4}{(2n+1)\pi}$$

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n+1)x}{(2n+1)}$$

is the Fourier representation for square wave ?



Correct if it is wrong

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0

A plot of first two term

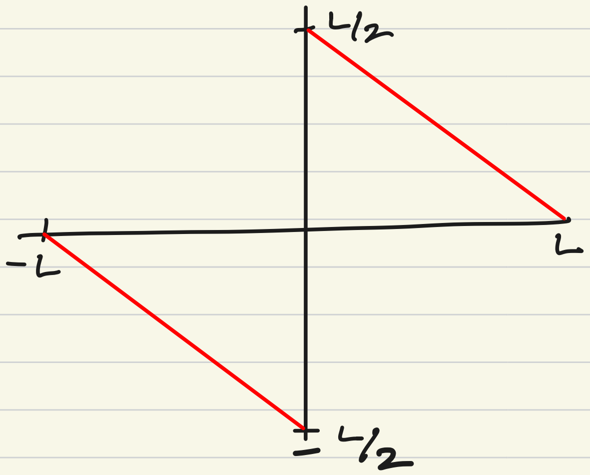
$$f_1(x) = \frac{4}{\pi}\sin x$$

$$f_2(x) = \frac{4}{3\pi}\sin 3x$$

Example:

express the function given below in term of Fourier series

$$f(x) = \begin{cases} \frac{1}{2}(L-x) & 0<x<L \\ -\frac{1}{2}(L+x) & -L<x<0 \end{cases}$$



f(x) is odd function since

$a_0 = 0$ and $a_n = 0 \; \forall n$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

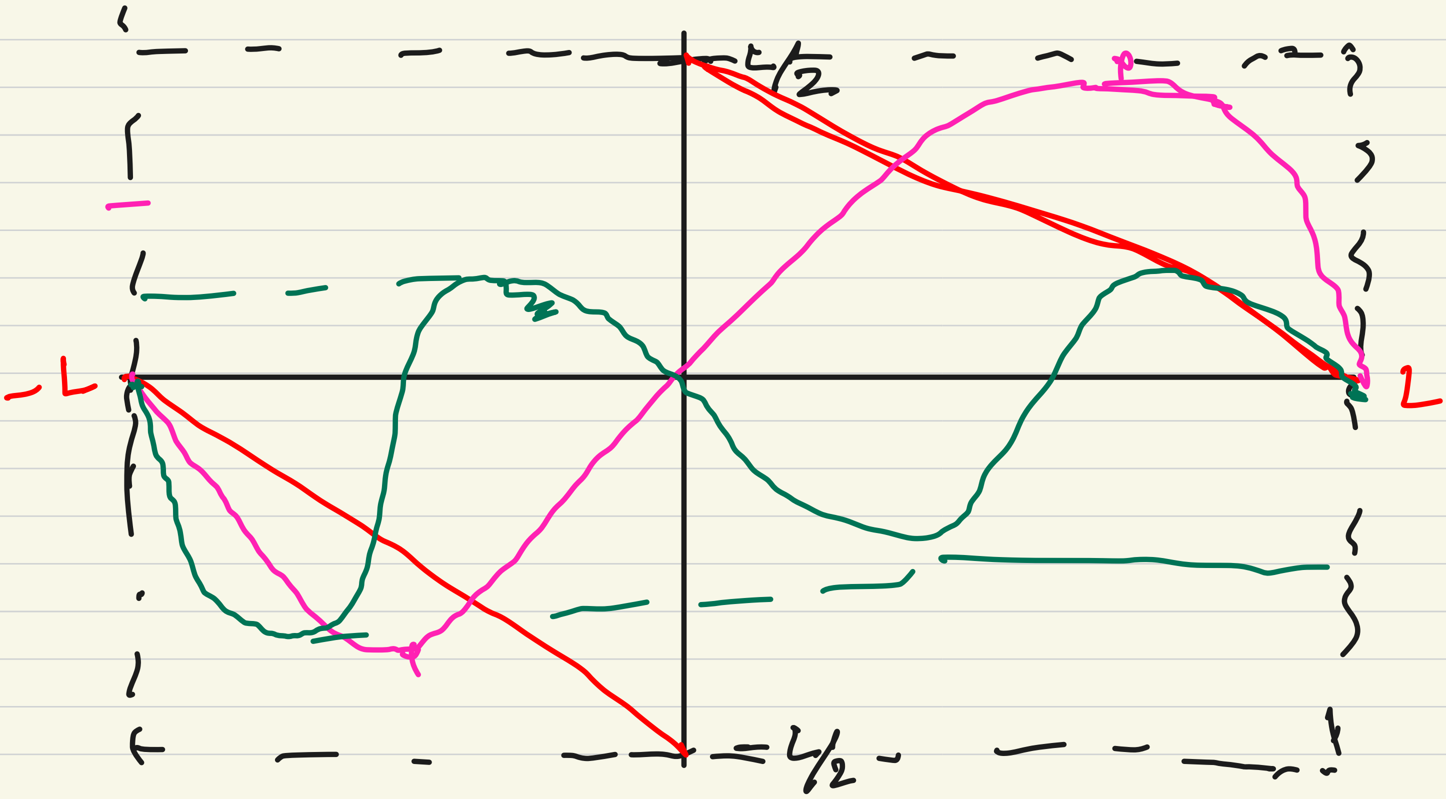
$$= \frac{1}{L}\int_{-L}^{0} -\frac{1}{2}(L+x)\sin\frac{n\pi x}{L}\,dx$$

$$+ \frac{1}{L}\int_{0}^{L}\frac{1}{2}(L-x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\int_0^L \frac{1}{2}(L-x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{L^2}{n\pi L} = \frac{L}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$$



$$f_1 = \frac{L}{\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{L}{2\pi}\sin\left(\frac{2\pi x}{L}\right)$$