PH3102 Quantum Mechanics   Assignment 4

Instructor: Dr. Siddhartha Lal   Autumn Semester, 2024

Start Date: August 27, 2024   Submission Deadline: September 03, 2024 .

Submit your answers to the Tutor at the start of the tutorial.

Q1. Position and Momentum Fluctuations in Free Particle [5 marks]

Consider a one-dimensional free particle of mass m whose position and momentum at time $t = 0$ are given by x_0 and p_0 respectively.

(a) Calculate $\langle \hat{p}(t) \rangle$ and show that

$$\langle \hat{x}(t) \rangle = \frac{p_0 t^2}{m} + x_0 \ . \tag{1}$$

(b) Show that the following two relations are satisfied for the same quantum system as above

$$\frac{d\langle \hat{x}^2 \rangle}{dt} = 2\frac{\langle \hat{p}\hat{x} \rangle}{m} + i\frac{\hbar}{m} \ , \tag{2}$$

$$\frac{d\langle \hat{p}^2 \rangle}{dt} = 0 \ . \tag{3}$$

(c) Show that the position and momentum fluctuations are related by

$$\frac{d^2(\Delta x)^2}{dt^2} = 2\frac{(\Delta p)^2}{m^2} \tag{4}$$

and that the solution to the above relation is given by

$$(\Delta x)^2 = \frac{(\Delta p_0)^2 \ t^2}{m^2} + (\Delta x_0)^2 \ , \tag{5}$$

where (Δx_0) and (Δp_0) are the initial fluctuations (i.e., those at time $t = 0$).

Q2. Heisenberg equation of motion for a Spin in a Magnetic Field [5 marks]

For a spin-1/2 electron with mass m and charge $-e$ in an external magnetic field, we consider the Zeeman Hamiltonian given by

$$H = \frac{e}{m}\vec{S} \cdot \vec{B} \ , \tag{6}$$

where $\vec{S}$ is the spin angular momentum (and we have ignored the orbital effects of the charged particle in the external magnetic field). Now show using Heisenberg's equation of motion, show that the equation of motion for the spin $\vec{S}$ is given by

$$\frac{d\vec{S}}{dt} = -\frac{e}{m}\vec{S} \times \vec{B} \ .$$

Q2. Some more on Coherent States [15 marks]

In an earlier assignment, you became familiar with the notion of coherent states as being the eigenstates of the lowering operator $\hat{a}$ with eigenvalue α (a given complex number), i.e.,

$$\hat{a}\psi(x, t = 0) = \alpha\psi(x, t = 0) \ ? \tag{7}$$

You obtained the explicit form of the wavefunction at $\psi(x, t = 0)$, and ensured that it is correctly normalised.

(a) Now, let $\psi(x, 0)$ evolve in time according to the Schrödinger equation

$$i\hbar\frac{\partial \psi(x, t)}{\partial t} = H\psi(x, t) \ . \tag{8}$$

Show that $\psi(x,t)$ remains a coherent state at all times, except that the eigenvalue of $\hat{a}$ changes with time (how does it change?).

(b) The mean position $\langle x \rangle$ and width Δx of the wavefunction $\psi(x,t)$ are defined as

$$\langle x \rangle = \int_{-\infty}^{\infty} dx \psi^* x \psi \quad , \quad (\Delta x)^2 = \int_{-\infty}^{\infty} dx \psi^* (x - \langle x \rangle)^2 \psi \ , \tag{9}$$

assuming that $\psi(x,t)$ is normalised. Show that $\langle x \rangle$ varies with time according to the classical (Ehrenfest) equation of motion, while Δx does not vary at all.

(c) Calculate the mean momentum $\langle p \rangle$ and width Δp, and show that they have similar properties as $\langle x \rangle$ and Δx (as obtained by you in part (b)). All of these are important properties of coherent states.