Fourier Series :

Let $f(x)$ be an $\mathbb{L}^2$ function in the close interval $[-\pi, \pi]$

Then $f(x)$ can be written as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{m=1}^{\infty} b_m \sin mx \quad \text{——} \quad (1)$$

This representation is known as "Fourier Series" or Fourier representation.

Based on orthonormality of $\{\sin mx, \cos nx\}$, one can easily determine Fourier coefficients $\{a_n, b_n\}$ as follows.

Take eq (1) and integrate from $-\pi$ to $+\pi$ on both sides:

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \sum_{n=1}^{\infty} a_n \cancel{\int_{-\pi}^{\pi} \cos nx\,dx} = 0$$

Because of property of $\cos nx$

$$+ \sum_{m=1}^{\infty} \cancel{\int_{-\pi}^{\pi} \sin mx\,dx} = 0$$

because of property of $\sin mx$

we get $$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0$$

or

$$\boxed{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx}$$

Let's multiply eq (1) by $\cos mx$ and integrate from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)\cos lx\,dx = \underbrace{\cancel{\int_{-\pi}^{\pi} a_0 \cos lx}}_{=0} + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos lx\,dx$$

because of periodicity of $\cos lx$

$$+ \sum_{m=1}^{\infty} b_m \underbrace{\cancel{\int_{-\pi}^{\pi} \sin mx \cos lx\,dx}}_{=0}$$

because of orthogonality of sin and cos

finally we get

$$\int_{-\pi}^{\pi} f(x)\cos lx\,dx = \pi a_l$$

or

$$\boxed{a_l = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos lx\,dx}$$

Similarly we get

$$\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx}$$

changing boundary :

We can extend the Domain of Fourier Series from $[-\pi, \pi]$ to $[-L, L]$

This can be done by mapping $x \longleftrightarrow \frac{\pi x}{L}$

finally we have

Any $\mathcal{L}^2$ function in the interval $[-L, L]$ can be expressed as Fourier Series, as:

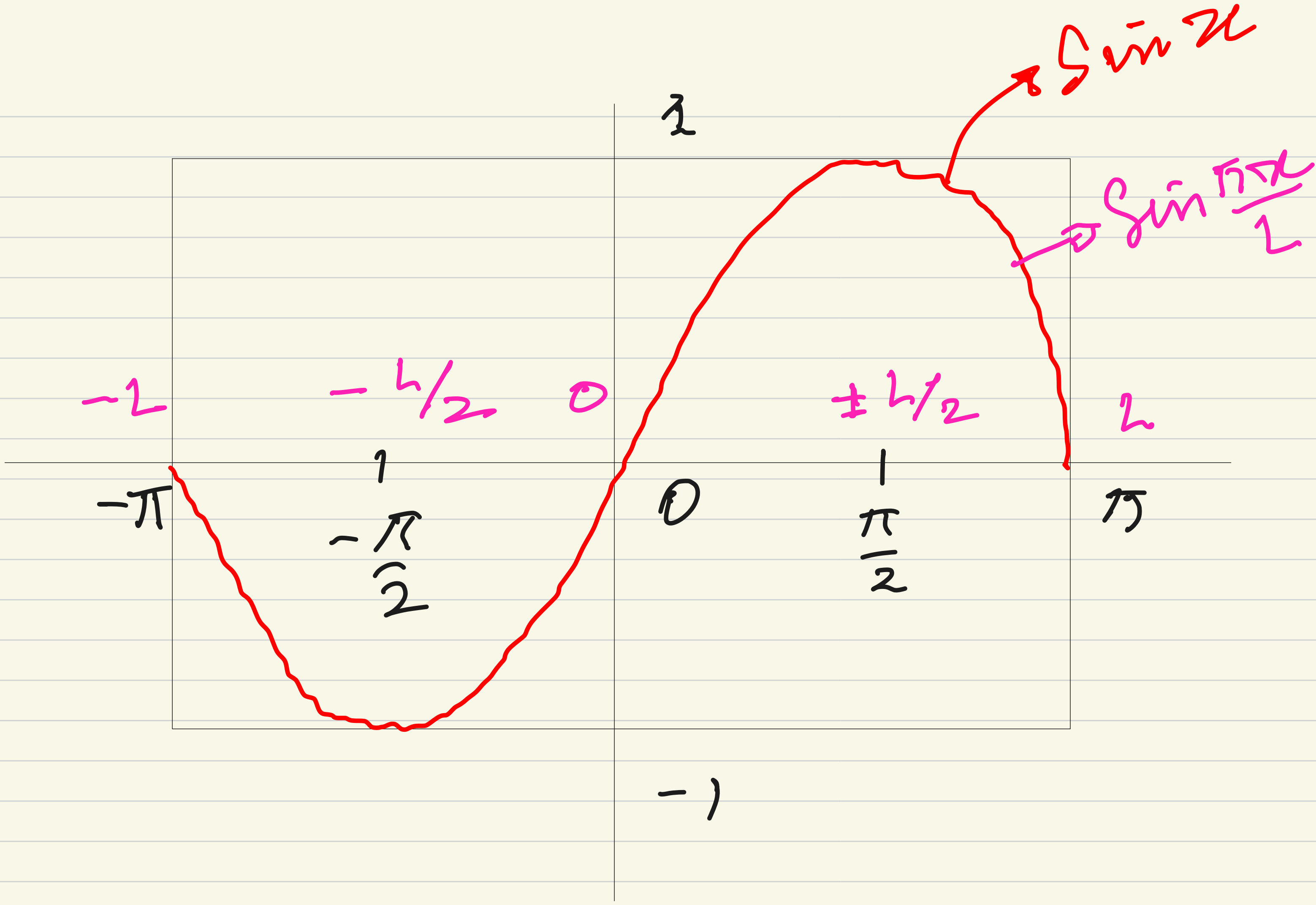
$$f(x) = a_0 + \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

As before, we can determine coefficients using orthonormality of $\sin mx$ and $\cos mx$ as

$$a_0 = \frac{1}{2\pi} \int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{\pi} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$



The function $\sin x$ in region $[-\pi, \pi]$ can be shown as function $\sin \frac{\pi x}{2}$ in the interval $[-2, 2]$