Note to show orthogonality of sin and cos :

Let us look at the differential equation

$$\frac{d^2y}{dx^2} + \omega^2 y = 0$$

for a given ω has two sol^n

$$y_1 = \cos \omega x \qquad y_2 = \sin \omega x$$

y_1 and y_2 are linearly independent, that can be tested by Wronskian of $y_1 y_2$

$$W[y_1, y_2] = y_1 y_2' - y_2 y_1'$$

$\Rightarrow$ $y_1 = \cos \omega x$ $\quad y_1' = -\omega \sin \omega x$

$y_2 = \sin \omega x$, $y_2' = \omega \cos \omega x$

$y_1 y_2' - y_2 y_1' =$

$\omega \cos^2 \omega x + \omega \sin^2 \omega x = \omega \neq 0$

if $\omega \neq 0$ they are linearly independent

we have 2 sol^n $y_1 = \cos \omega x$ and $y_2 = \sin \omega x$. A general sol^n can be written as

$$y = A \cos \omega x + B \sin \omega x$$

Let us relook ode as Eigen value problem

$$\frac{d^2y}{dx^2} = \lambda y$$

where $\lambda = -\omega^2$

In this case λ changes continuously for each λ

Let y_1 be Eigen function for Eigenvalue λ_1 and let y_2 be Eigenvector for Eigen value λ_2

$$\frac{d^2y_1}{dx^2} + \lambda_1 y_1 = 0 \quad (1)$$

$$\frac{d^2y_2}{dx^2} + \lambda_2 y_2 = 0 \quad (2)$$

Let us multiply eq (1) by y_2 and eq (2) by y_1

$$y_2 \frac{d^2 y_1}{dx^2} = \lambda_1 y_1 y_2 \quad \text{(3)}$$

$$y_1 \frac{d^2 y_2}{dx^2} = \lambda_2 y_1 y_2 \quad \text{(4)}$$

rewrite (3) and (4) as

$$\frac{d}{dx}\left[y_2 \frac{dy_1}{dx}\right] - \frac{dy_1}{dx}\frac{dy_2}{dx} = \lambda_1 y_1 y_2 \quad \text{(5)}$$

$$\frac{d}{dx}\left[y_1 \frac{dy_2}{dx}\right] - \frac{dy_1}{dx}\frac{dy_2}{dx} = \lambda_2 y_1 y_2 \quad \text{(6)}$$

eq (5) − eq (6) we get

$$\frac{d}{dx}\left[y_2 y_1'\right] - \frac{d}{dx}\left[y_1 y_2'\right] = (\lambda_1 - \lambda_2) y_1 y_2$$

$$\frac{d}{dx}\left[y_2 y_1' - y_1 y_2'\right] = \lambda_1 - \lambda_2 \, y_1 y_2$$

integrating in the interval

$[a, b]$

$$\int_a^b \frac{d}{dx}\left[y_2 y_1' - y_1 y_2'\right] dx = (\lambda_1 - \lambda_2) \int_a^b y_1 y_2 \, dx$$

$$\left[y_2 y_1' - y_1 y_2'\right]\Big|_a^b = (\lambda_1 - \lambda_2) \int_a^b y_1 y_2 \, dx$$

$$\int_a^b y_1 y_2 \, dx$$ is dot product

between vectors y_1 and y_2

if the functions

$$y_1(a) = y_2(a) = 0$$

and $y_1(b) = y_2(b) = 0$

Eigen functions y_1, y_2 corres-ponding to different Eigenvalues are orthogonal !

Same will happen if

$$y_1'(a) = y_2'(a) = y_1'(b) = y_2'(b) = 0$$