Thm: Let $f, g : [a,b] \to \mathbb{R}$ be integrable ~~fcs~~ such that

$$f \leq g.$$

Then $$\int_a^b f \leq \int_a^b g$$

Proof: Note that

$$U(f,P) \leq U(g,P) \quad \forall P.$$

Therefore, $U(f) \leq U(g)$,

hence,

$$\int_a^b f \leq \int_a^b g.$$

Thm: Let f be integrable on $[a,b]$. Assume that $m \leq f(t) \leq M$ $\forall t \in [a,b]$. Let $g : [m,M] \to \mathbb{R}$ be continuous fn. Then

$g \circ f : [a,b] \to \mathbb{R}$ is integrable.

Proof: Let $\epsilon > 0$. Since $g : [m,M] \to \mathbb{R}$ is uniformly continuous $\exists\, \delta_1 > 0$ such that

$$x, y \in [m,M],\ |x-y| < \delta_1 \Rightarrow |g(x) - g(y)| < \epsilon.$$

Choosing $\delta = \min\{\delta_1, \epsilon\}$, we have

$$\boxed{\forall x, y \in [m,M] \text{ with } |x-y| < \delta \Rightarrow |g(x) - g(y)| < \epsilon.} \longrightarrow (1)$$

Since $f : [a,b] \to \mathbb{R}$ is integrable, for $\eta > 0$, $\exists$ P s.t.

$$\boxed{\sum_{i=0}^{n-1} \left(M_i(f) - m_i(f)\right)(t_{i+1} - t_i) < \eta.} \longrightarrow (2)$$

Define

$$A = \{ i : M_i(f) - m_i(f) < \delta \}.$$

Let $i \in A$. Let $t, s \in [t_i, t_{i+1}]$. Then

$$f(t) - f(s) \leq M_i(f) - m_i(f)$$

and $$f(s) - f(t) \leq M_i(f) - m_i(f).$$

Therefore,

$$|f(s) - f(t)| < \delta \quad \forall s, t \in [t_i, t_{i+1}]$$

Then by the uniform continuity of g and f, for $i \in A$,

$\forall\, t, s \in [t_i, t_{i+1}]$,

$$|g \circ f(t) - g \circ f(s)| < \epsilon.$$

$$\Rightarrow \quad |M_i(g \circ f) - m_i(g \circ f)| \leq \epsilon.$$

Thus we have,

$$\boxed{i \in A \implies |M_i(g\circ f) - m_i(g\circ f)| \leq \epsilon.} \longrightarrow (3)$$

Now from (2),

$$\eta > \sum_{i=0}^{n-1} \left(M_i(f) - m_i(f)\right)(t_{i+1} - t_i)$$

$$\geq \sum_{i \in A^c} \left(M_i(f) - m_i(f)\right)(t_{i+1} - t_i)$$

$$\geq \delta \sum_{i \in A^c} (t_{i+1} - t_i)$$

$$\implies \boxed{\sum_{i \in A^c} (t_{i+1} - t_i) \leq \frac{\eta}{\delta}.} \longrightarrow (4).$$

Now

$$U(g\circ f, P) - L(g\circ f, P)$$
$$= \sum_{i=0}^{n-1} \left(M_i(g\circ f) - m_i(g\circ f)\right)(t_{i+1} - t_i)$$

$$= \sum_{i \in A} \left(M_i(g \circ f) - m_i(g \circ f)\right)(t_{i+1} - t_i)$$

$$+ \sum_{i \in A^c} \left(M_i(g \circ f) - m_i(g \circ f)\right)(t_{i+1} - t_i)$$

$$\leq \epsilon (b-a) + C \sum_{i \in A^c} (t_{i+1} - t_i)$$

When $M(g \circ f) - m(g \circ f) \leq C$,

Therefore,

$$U(g \circ f, P) - L(g \circ f, P)$$
$$\leq \epsilon (b-a) + C \cdot \frac{\eta}{\delta}.$$

Choose $\eta = \epsilon \delta$, Then we have

$$U(g \circ f, P) - L(g \circ f, P) \leq \epsilon (b-a+C).$$

This proves that $g \circ f$ is integrable.

<u>Remark</u>: In the above thm, if we drop the continuity condition on g. Then $g \circ f$ may not be integrable. For example, take

$$f = \text{Thomae's fn.}$$

$$g = \begin{cases} 0 & x = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Then $g \circ f : [0,1] \to \mathbb{R}$ is the Dirichlet's fn, which is not integrable.

<u>Thm</u>: Assume that $f_1, f_2 : [a,b] \to \mathbb{R}$ is integrable. Then the followings are

integrable.

(i) $|f_1|$ (ii) f_1^2 (iii) $f_1 f_2$

(iv) $\max\{f_1, f_2\}$ (v) $\min\{f_1, f_2\}$.

Proof: (i) choose $g(x) = |x|$

(ii) choose $g(x) = x^2$

(iii) Note
$$f_1 f_2 = \frac{1}{4}\left((f_1+f_2)^2 - (f_1-f_2)^2\right).$$
By (ii), $(f_1+f_2)^2, (f_1-f_2)^2 \in R([a,b])$.
Therefore $f_1 f_2 \in R([a,b])$.

(iv) Exercise

(v) Exercise.

Thm: Let $f \in R([a,b])$. Then

$|f| \in R([a,b])$ and

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Proof: By last result, $|f| \in R([a,b])$.

Now $f(x) \leq |f(x)| \quad \forall x \in [a,b]$

and $-f(x) \leq |f(x)| \quad \forall x \in [a,b]$.

Therefore,
$$\int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx \longrightarrow (1)$$

and
$$\int_a^b -f(x)\,dx \leq \int_a^b |f(x)|\,dx$$

$$\Rightarrow \quad -\int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx \longrightarrow (2)$$

From (1) and (2) we have,

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

This completes the proof.

Theorem: If f is integrable on $[a,b]$, then f is integrable on $[c,d]$, where $[c,d] \subseteq [a,b]$. Moreover,

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof: Let $\epsilon > 0$. Then $\exists$ a partition P on $[a,b]$ such that

$$U(f,P) - L(f,P) < \epsilon. \longrightarrow (1)$$

Let $Q = P \cup \{c,d\}$. Then Q is a refinement of P. Therefore,

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$$

Thus from (1), we have

$$U(f,Q) - L(f,Q) < \epsilon$$

Let $\tilde{Q} = Q \cap [c,d]$. Then $\tilde{Q}$ is a partition on $[c,d]$ and

$$U(f,\tilde{Q}) - L(f,\tilde{Q}) \leq U(f,Q) - L(f,Q) < \epsilon.$$

This proves that f is integrable on $[c,d]$.

Let $c \in [a,b]$. Let P be a partition on $[a,b]$. Let $Q = P \cup \{c\}$. Define $Q_1 = [a,c] \cap Q$ and $Q_2 = [c,b] \cap Q$.

Note that Q_1 is a partition on $[a,c]$ and Q_2 is a partition on $[c,b]$.

Now $U(f,P) \geq U(f,Q) = U(f,Q_1) + U(f,Q_2)$

$$\geq \int_a^c f + \int_c^b f.$$

$$\Rightarrow \boxed{\int_a^b f \geq \int_a^c f + \int_c^b f} \longrightarrow (1)$$

Again

$L(f,P) \leq L(f,Q) = L(f,Q_1) + L(f,Q_2)$

$$\leq \int_a^c f + \int_c^b f$$

$$\Rightarrow \boxed{\int_a^b f \leq \int_a^c f + \int_c^b f} \longrightarrow (2)$$

From ① & ② we have.

$$\int_a^b f = \int_a^c f + \int_c^b f$$

This completes the proof.

Defn: Let $f:[a,b]\to\mathbb{R}$ be integrable.

We define

$$\int_b^a f = -\int_a^b f \quad , \quad \int_\delta^\delta f = 0 \quad \forall\, \delta \in [a,b].$$

Using the above definition, it is easy to see the following:

Thm: Let $f:[a,b]\to\mathbb{R}$ integrable. Then

$$\int_x^y f = \int_x^z f + \int_z^y f \qquad \forall\, x,y,z \in [a,b].$$

Proof: Exercise.

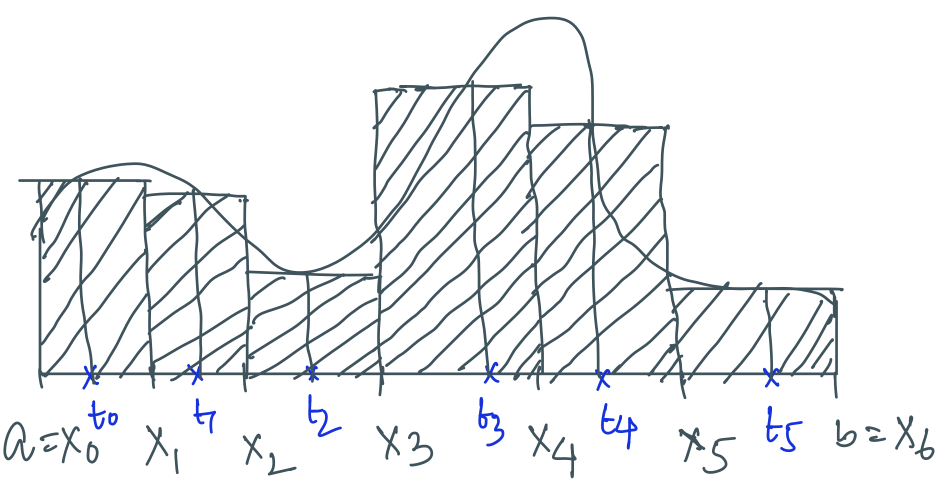
Riemann's definition of integration.

Defn: Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$. A set $t = \{t_0, t_1, \dots, t_{n-1}\}$ of $n-1$ points is a tag of P if

$$t_i \in [x_i, x_{i+1}] \quad i = 0, 1, \dots, n-1.$$

Defn: (Riemann Sum): Let $f : [a,b] \to \mathbb{R}$ be a given fn. Let P be a partition of $[a,b]$ and t be a partition of P. Then the Riemann Sum is defined by

$$S(f, P, t) = \sum_{i=0}^{n-1} f(t_i)\,(x_{i+1} - x_i)$$



$S(f, P, t)$

Defn: A fn $f : [a, b] \to \mathbb{R}$ is said to be Riemann integrable if $\exists\, A \in \mathbb{R}$ such that, for each $\epsilon > 0$, $\exists$ a partition P_ϵ of $[a, b]$ such that

$$|S(f,Q,t) - A| < \epsilon$$

for all partition $Q \supseteq P$ and for all tag t of Q.

Thm: Let $f:[a,b] \to \mathbb{R}$ be Riemann integrable. Let A_1 and A_2 satisfy the definition of Riemann integrability of f. Then

$$A_1 = A_2.$$

Proof: Exercise.

Not^n If $f:[a,b] \to \mathbb{R}$ is Riemann integrable, then we define

$$A := \mathcal{R}(f);$$

This is the integral of f as per the definition of Riemann integration.

Thm: Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable fn. Then f is bdd.

Proof: By the defn $\exists$ a partition P such that

$$|S(f,P,t) - A| < 1.$$

$\forall$ tag t of P.

Let $P = \{x_0, x_1, \dots, x_n\}$.

Note that $t = \{x_0, \dots, x_{n-1}\}$.

Let $s \in [a,b] \setminus t$. Then $s \in (x_j, x_{j+1}]$ for some j. Consider the tag

$$t^* = \{x_0, x_1, \dots, x_{j-1}, s, x_{j+1}, \dots x_{n-1}\}.$$

Then $|S(f,P,t) - A| < 1$

and $|S(f,P,t^*) - A| < 1$

Therefore,

$$|S(f,P,t) - S(f,P,t^*)| < 1$$

$$\Rightarrow \quad |f(s) - f(x_j)|\,(x_{j+1} - x_j) < 1.$$

$$\Rightarrow \quad |f(s)| \leq |f(x_j)| + \frac{1}{x_{j+1} - x_j}$$

$$\leq \max_j \left\{ |f(x_j)| + \frac{1}{x_{j+1} - x_j} \right\}$$

This proves that f is bdd.

Thm: A bdd fn $f : [a,b] \to \mathbb{R}$ is integrable iff it is Riemann integrable. Moreover $R(f) = \int_a^b f$.

Pf: Assume that $f : [a,b] \to \mathbb{R}$ is Darboux integrable.

Let $\epsilon > 0$. Then $\exists$ a partition P_ϵ s.t.

$$U(f, P_\epsilon) < \int_a^b f + \epsilon$$

and

$$L(f, P_\epsilon) > \int_a^b f - \epsilon.$$

Thus

$$\int_a^b f - \epsilon < L(f, P_\epsilon) \leq U(f, P_\epsilon) < \int_a^b f + \epsilon$$

Let $Q \supseteq P_\epsilon$. Then.

$$\int_a^b f - \epsilon < L(f, P_\epsilon) \leq L(f, Q) \leq U(f, Q) \leq U(f, P_\epsilon) \leq \int_a^b f + \epsilon \longrightarrow (1)$$

Let t be any tag of Q. Then.

$$L(f, Q) \leq S(f, Q, t) \leq U(f, Q) \longrightarrow (2)$$

Combining (1) and (2), we have

$$\left| S(f, Q, t) - \int_a^b f \right| < \epsilon. \longrightarrow (3)$$

This shows that f is Riemann integrable and $R(f) = \int_a^b f$.

Now we assume that f is Riemann integrable. Let $\epsilon > 0$. Then from the definition $\exists$ a partition P s.t. that

$$|S(f, P, t) - R(f)| < \epsilon \longrightarrow 4$$

$\forall$ tag t of P.

Let $P = \{x_0, x_1, \dots, x_n\}$. Since

$$m_i(f) = \inf\{f(t) : t \in [t_i, t_{i+1}]\}.$$

and $M_i(f) = \sup\{f(t) : t \in [t_i, t_{i+1}]\}$.

Then $\exists$ $s_i, \tilde{s}_i \in [t_i, t_{i+1}]$ s.t.

$$m_i(f) \leq f(s_i) < m_i(f) + \epsilon.$$

and

$$M_i(f) - \epsilon < f(\tilde{s}_i) \leq M_i(f)$$

This implies

$$\begin{cases} L(f,P) \leq S(f,P,\delta) < L(f,P) + \epsilon(b-a) \\ U(f,P) - \epsilon(b-a) < S(f,P,\tilde{\delta}) \leq U(f,P) \end{cases} \longrightarrow 5$$

Therefore,

$$|U(f,P) - L(f,P)|$$

$$\leq |U(f,P) - S(f,P,\tilde{\delta})| + |S(f,P,\tilde{\delta}) - S(f,P,\delta)|$$

$$+ |S(f,P,\delta) - L(f,P)|$$

$$\leq \epsilon(b-a) + \epsilon(b-a) + 2\epsilon.$$

using 4 and 5.

This proves that f is integrable in the sense by Darboux.

Then (3) shows that

$$\int_a^b f = R(f).$$

This completes the proof.