Tutorial - 9 :

The poisson bracket of two function $u(p, q, t)$ and $v(p, q, t)$ is defined as

$$[u, v] = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right)$$

$$= \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right)$$

coidn summation conversion

(i) Show that Anti-symmetric

$$\boxed{[u, v] = -[v, u]}$$

This is trivial

$$[u, v] = \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right)$$

$$= -\left(\frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} - \frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} \right)$$

$$= -[v, u]$$

(ii) Show that

$$[u+v, w] = [u, w] + [v, w]$$

$$\cancel{\frac{\partial}{\partial q_i}(u+w)\,\partial p_i}$$

$$\frac{\partial}{\partial q_i}(u+v)\frac{\partial w}{\partial p_i} - \frac{\partial}{\partial p_i}(u+v)\frac{\partial w}{\partial q_i}$$

$$= \left(\frac{\partial u}{\partial q_i} + \frac{\partial v}{\partial q_i}\right)\frac{\partial w}{\partial p_i} - \left(\frac{\partial u}{\partial p_i} + \frac{\partial v}{\partial p_i}\right)\frac{\partial w}{\partial q_i}$$

$$= \frac{\partial u}{\partial q_i}\frac{\partial w}{\partial p_i} + \frac{\partial v}{\partial q_i}\frac{\partial w}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial w}{\partial q_i} - \frac{\partial v}{\partial p_i}\frac{\partial w}{\partial q_i}$$

$$= [u, w] + [v, w]$$

(iii) Show that $[u, vw] = [u, v]w + v[u, w]$

$$[u, vw] = \frac{\partial u}{\partial q_i}\frac{\partial (vw)}{\partial p_i} + \frac{\partial u}{\partial p_i}\frac{\partial (vw)}{\partial q_i}$$

$$= \frac{\partial u}{\partial q_i}\left(v\frac{\partial w}{\partial p_i} + w\frac{\partial v}{\partial p_i}\right)$$

$$- \frac{\partial u}{\partial p_i}\left(v\frac{\partial w}{\partial q_i} + w\frac{\partial v}{\partial q_i}\right)$$

$$= v\frac{\partial u}{\partial q_i}\frac{\partial w}{\partial p_i} + w\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i}$$

$$- v\frac{\partial u_i}{\partial p_i}\frac{\partial w}{\partial q_i} - w\frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}$$

$$[u, vw] = v[u, w] + w[u, v]$$

$$\boxed{\text{(iv) show that } \frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial q_i} \cdot \dot{q}_i + \frac{\partial u}{\partial p_i} \cdot \dot{p}_i + \frac{\partial u}{\partial t}$$

using Hamiltonian EoM.

$$= \frac{\partial u}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i} \cdot \frac{\partial H}{\partial q_i} + \frac{\partial u}{\partial t}$$

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}$$

(v) $\dot{q}_i = [q_i, H]$

$$\dot{p}_i = [q_i, H]$$

~~[illegible]~~. $\frac{dH}{dt} = \frac{\partial H}{\partial t}$

These are trivial.

(VI) Home work:

Show that

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0$$