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Linear Algebra I (MA2102)

Assignment 3

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Exercise 1. (10 points)
Let V be a finite dimensional vector space and let $T : V \to V$ be a linear operator such that $\operatorname{rank}(T) = \operatorname{rank}(T^2)$. Show that $V = \ker(T) \oplus \operatorname{im}(T)$.

Exercise 2. (10 points)
Let p be a prime, let V be a finite dimensional vector space over $\mathbb{F}_p$ and let $T : V \to V$ be a surjective linear operator. For $n \in \mathbb{N}$, define $T^n : V \to V$ inductively by

$$T^n(v) := T(T^{n-1}(v))$$

for all $v \in V$, where T^0 denotes the identity map of V. Show that there exists an integer $m > 0$ such that $T^m = T^0$.

Exercise 3. (12 points)
Let $n \in \mathbb{N}$, let V be a vector space of dimension n and let $T : V \to V$ be a linear operator such that there exists a $v \in V$ with $T^n(v) = 0$ and $T^{n-1}(v) \neq 0$. Show that

$$\mathbb{B} := (v, T(v), T^2(v), \ldots, T^{n-1}(v))$$

is a basis of V and write down the matrix representation of T w.r.t. the basis $\mathbb{B}$.

Exercise 4. (10 points)
Prove that every $m \times n$ matrix of rank 1 with entries from a field $\mathbb{F}$ has the form $A = XY^{\mathrm{T}}$, where $X \in \mathbb{F}^m$ and $Y \in \mathbb{F}^n$.

Exercise 5. (12 points)
Let M be an $n \times n$ real matrix. Show that the $\operatorname{rank}(M) = \operatorname{rank}(MM^{\mathrm{T}})$.

Maximum score: 50 points. You may answer as many questions as you wish. If the sum of your total score exceeds 49, you shall get the maximum score. Please mention your name, roll no. and **group** in your answersheet. Please submit your answersheet by 11:59 p.m. on **14.09.2023** in the DMS mailbox for MA2102, which is designated with your group name.