LIMIT

Defn. Let $D \subseteq \mathbb{R}$. Let c be a limit pt. of $\mathbb{R}$. Let $f: D \to \mathbb{R}$. A no. l is said to be a limit of f at c, if for each $\epsilon > 0$, $\exists\, \delta > 0$ s.t.

$$|f(x) - l| < \epsilon \quad \forall\, x \in \{x \in D : 0 < |x-c| < \delta\}.$$

Remark: Note that to define the limit of f at c, f is not required to be defined at c.

Thm: (uniqueness of limit). Let l_1 and l_2 be the limit of f at c. Then

$$l_1 = l_2.$$

Proof: Let $\epsilon > 0$. Since l_1 is a limit of f at c, $\exists\ \delta_1 > 0$ such that

$$|f(x) - \ell_1| < \epsilon/2 \quad \forall x \in \{x \in D : 0 < |x-c| < \delta_1\}.$$

Similarly ℓ_2 is the limit of f at c,
$\exists\ \delta_2 > 0$ s.t.

$$|f(x) - \ell_2| < \epsilon/2 \quad \forall x \in \{x \in D : 0 < |x-c| < \delta_2\}.$$

Let $x_0 \in D$ such that $0 < |x_0 - c| < \delta_1$
and $0 < |x_0 - c| < \delta_2$. Then

$$|\ell_1 - \ell_2| \leq |f(x_0) - \ell_1| + |f(x_0) - \ell_2|.$$

$$< \epsilon/2 + \frac{\epsilon}{2} = \epsilon.$$

Thus for all $\epsilon > 0$, we have

$$|\ell_1 - \ell_2| < \epsilon$$

Therefore, $\ell_1 = \ell_2$.

Example. Take $f(x) = \dfrac{x^2-4}{x-2}$. Note that domain of f is

$$D = \mathbb{R} \setminus \{2\}.$$

Also note 2 is a limit pt. of D.

Claim: $\displaystyle\lim_{x \to 2} f(x) = 4$.

Let $\epsilon > 0$. Now for $x \in D$.

$$|f(x) - 4| = \left| \frac{x^2-4}{x-2} - 4 \right|$$

$$= |(x+2) - 4| \quad \text{as } x \neq 2$$

$$= |x-2|.$$

Choose $\delta = \epsilon$, then for $0 < |x-2| < \delta$,

$$|f(x) - 4| < \epsilon.$$

Therefore $\displaystyle\lim_{x \to 2} f(x) = 4$.

Sequential criteria

Thm: Let $D \subseteq \mathbb{R}$ and c be a limit pt. of D. Let $f: D \to \mathbb{R}$ be a function.

Then $\lim_{x \to c} f(x) = l$ iff

$f(x_n) \to l$ for every seq. $x_n \in D \setminus \{c\}$ s.t. $x_n \to c$.

Proof: ($\Rightarrow$) Let $\lim_{x \to c} f(x) = l$.

Let $\epsilon > 0$. Then $\exists\, \delta > 0$ s.t.

$$|f(x) - l| < \epsilon \quad \forall x \in \{x \in D : 0 < |x - c| < \delta\}. \longrightarrow (i)$$

Now let $x_n \in D \setminus \{c\}$ s.t. $x_n \to c$.

Then for $\delta > 0$, $\exists\, N \in \mathbb{N}$ s.t.

$$|x_n - c| < \delta \quad \forall\, n \geq N.$$

Since $x_n \neq c$ $\forall n$,

Thus

$$0 < |x_n - c| < \delta \quad \forall\, n \in \mathbb{N}$$

Thus by (i),

$$|f(x_n) - l| < \epsilon \qquad \forall\, n \geq N.$$

$$\Rightarrow \quad \lim_{n\to\infty} f(x_n) = l$$

($\Leftarrow$). Assume that for ~~every seq.~~ $x_n \in D\setminus\{c\}$, $x_n \to c \Rightarrow f(x_n) \to l$.

We will show that

$$\lim_{x\to c} f(x) = l.$$

Suppose not true, then there exists $\epsilon > 0$ s.t. for every $\delta > 0$, $\exists\, x \in D$

$0 < |x-c| < \delta$ but $|f(x) - l| \geq \epsilon$.

Let $\delta = 1$, then $\exists\ x_1 \in D$ s.t.

$0 < |x_1 - c| < 1$ but $|f(x_1) - l| \geq \epsilon$.

Let $\delta = \frac{1}{2}$, then $\exists\ x_2 \in D$ s.t.

$0 < |x_2 - c| < \frac{1}{2}$ but $|f(x_2) - l| \geq \epsilon$

In general, for $\delta = \frac{1}{n}$ $\exists\ x_n \in D$ s.t.

$0 < |x_n - c| < \frac{1}{n}$ but $|f(x_n) - l| \geq \epsilon$.

Thus this seq. $\{x_n\}$ satisfies

(i) $x_n \in D \setminus \{c\}$

(ii) $x_n \to c$

(iii) $f(x_n) \not\to l$.

This gives a contradiction. This completes the proof of the theorem.

Example: Let $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Define $x_n = \frac{1}{2n\pi}$ $n \in \mathbb{N}$. and

$$y_n = \frac{2}{(4n+1)\pi} \quad n \in \mathbb{N}.$$

Then x_n and y_n both converges to zero. Now

$$f(x_n) = \sin(2n\pi) = 0$$

and $f(y_n) = \sin\left((4n+1)\frac{\pi}{2}\right) = 1$

Therefore, $f(x_n) \longrightarrow 0$ and $f(y_n) \longrightarrow 1$.

Thus by the above theorem

$$\lim_{x \to 0} f(x)$$

does not exist.

Example: Consider the function

$$f(x) = \begin{cases} 5 & x \geq 0, \\ 4 & x < 0, \end{cases}$$

Then $\lim_{x \to 0} f(x)$ does not exists.

Take $x_n = \frac{1}{n}$ and $y_n = -\frac{1}{n}$.

Then both x_n and y_n converges to zero. but $f(x_n) \to 5$ and $f(y_n) \to 4$. Thus by sequential criteria, $\lim_{x \to 0} f(x)$ does not exist.

Algebra of limits:

Thm: Let $D \subseteq \mathbb{R}$ and c be a limit point of D. Let $f, g : D \to \mathbb{R}$ be two fns. Such that

$$\lim_{x \to c} f(x) \text{ and } \lim_{x \to c} g(x) \text{ exists.}$$

Then

(i) $\lim_{x\to c}(f(x)+g(x))$ exists and

$$\boxed{\lim_{x\to c}(f(x)+g(x)) = \lim_{x\to c} f(x) + \lim_{x\to} g(x)}$$

(ii) For every $\alpha \in \mathbb{R}$,

$\lim_{x\to c}(\alpha f(x))$ exists and

$$\boxed{\lim_{x\to c}(\alpha f(x)) = \alpha \lim_{x\to c} f(x).}$$

(iii) $\lim_{x\to c}(f(x)g(x))$ exists and

$$\boxed{\lim_{x\to c}(f(x)g(x)) = \left(\lim_{x\to c} f(x)\right)\left(\lim g(x)\right)}$$

(iv) Assume that $g(x) \neq 0 \ \forall\, x \in D$ and $\lim_{x\to c} g(x) \neq 0$. Then

$\lim_{x\to c} \frac{f(x)}{g(x)}$ exists and

$$\boxed{\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}}$$

Right limit: Let $D \subseteq \mathbb{R}$ and c be a point such that

$(c, c+\alpha) \subseteq D$ for some $\alpha > 0$.

Let $f: D \to \mathbb{R}$. We say that l is the right limit of f at c if for $\epsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$\boxed{|f(x) - l| < \epsilon \quad \forall\, x \in D \cap (c, c+\delta).}$$

We will denote this by

$$\lim_{x \to c+} f(x) = l.$$

Left limit: Let $D \subseteq \mathbb{R}$ and c be a point such that

$$(c-\alpha, c) \subseteq D \quad \text{for some } \alpha > 0.$$

Let $f : D \to \mathbb{R}$. We say that l is the left limit of f at c if for $\epsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$\boxed{|f(x) - l| < \epsilon \quad \forall\, x \in D \cap (c-\delta, c).}$$

We will denote this by

$$\lim_{x \to c-} f(x) = l.$$

EXAMPLE: Consider the fn $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x > 0, \\ 5 & x = 0, \\ 6 & x < 0. \end{cases}$$

Then $\lim_{x \to 0+} f(x) = 1$, $\lim_{x \to 0-} f(x) = 6$.

Example: Consider the fn $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin \frac{1}{x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Then Show that $\lim_{x \to 0+} f(x)$ does not exists but $\lim_{x \to 0-} f(x)$ exists.

MONOTONE FUNCTION & LIMIT:

Thm: Let $f:(a,b)\to\mathbb{R}$ be monotonically increasing function.

(i) If f is bdd above ie,

$$\{f(x) : x\in(a,b)\}$$

is bdd above, then

$$\lim_{x\to b^-} f(x) = \sup_{x\in(a,b)} f(x)$$

(ii) If f is bdd below i.e

$$\{f(x) : x\in(a,b)\}.$$

is bdd below, then

$$\lim_{x\to a^-} f(x) = \inf_{x\in(a,b)} f(x)$$

Proof: (i) Since $\{f(x) : x\in(a,b)\}$ is bdd above, $\sup\{f(x) : x\in(a,b)\}$ exists.

Let $\alpha = \sup\{f(x) : x \in (a,b)\}$

Let $\epsilon > 0$. $\exists\ x_0 \in (a,b)$ s.t.

$$\alpha - \epsilon < f(x_0) \leq \alpha$$

Choose $\delta > 0$. s.t.

$$x_0 < \alpha - \delta < \alpha.$$

Then $\forall\ y \in (\alpha - \delta, \alpha)$

$$\alpha - \epsilon < f(x_0) \leq f(y) \leq \alpha \leq \alpha + \epsilon$$

as f is increasing. This shows that

$$\lim_{x \to b-} f(x) = \alpha.$$

(ii) Proof is similar to (i).

This completes the proof.