PH3103 Mathematical Methods of Physics
Autumn Semester - 2024
Indian Institute of Science Education and Research, Kolkata
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End-Semester Examination Examination Date: 12/12/2024

1. (i) By explicitly calculating the integration (not using Residue theorem) over a convenient contour show that

$$\oint dz \frac{z^m}{z^n} = (2\pi i)\delta_{m+1,n} \ , \tag{1}$$

where m and n are integers. **Marks: 3**

(ii) Interpret the result in terms of Cauchy's Residue Theorem. **Marks: 2**

2. Evaluate the following integral:

$$\oint \frac{1}{z-1}\frac{1}{(z-2)^2}dz \tag{2}$$

(i) The closed contour is a circle $\mathcal{C}_1 = 4e^{i\theta}$ for $0 \leq \theta \leq 2\pi$. **Marks: 4**

(ii) Deform the contour into a contour $\mathcal{C}_2 = Re^{i\theta}$ with $R \to \infty$, and estimate the value of the integration. Note that you are interested only in the limit of $R \to \infty$. **Marks: 2**

(iii) Now, comment on two values in part (i) and (ii), in particular, why they are same/different! **Marks: 1**

3. Evaluate the following real integral using the complex integration techniques:

$$I = \int_0^\infty \frac{cos\ x}{q^2 + x^2}dx \tag{3}$$

Marks: 6

4. Find out the Fourier transformation $g(x)$ of $f(k) = 1/(k^2 + m^2)^2$. **Marks: 9**

5. Rodrigues formula for the Legendre polynomials of n-th order is given by the following formula (discussed before the mid-semester)

$$P_n(x) = \frac{1}{2^n n!}\left(\frac{d}{dx}\right)^n (x^2-1)^n \ . \tag{4}$$

(i) Show that it can be written as

$$P_n(z) = \frac{1}{2^n 2\pi i}\oint \frac{(z'^2-1)^n}{(z'-z)^{n+1}}dz' \tag{5}$$

where the contour encircles the complex point z counterclockwise. **Marks: 3**

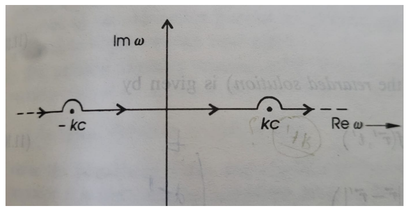
(ii) From the above relation, show that $P_n(1) = 1$. **Marks: 3**

6. Show that eigenvectors of a unitary operator corresponding to different eigenvalues are orthogonal. **Marks: 5**

7. Obtain the retarded Green's function $G(X,T)$ (see the appropriate contour shown bellow) for the (1+1) dimensional wave operator

$$D = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \ , \tag{6}$$

where $X = x - x'$ and $T = t - t'$ (assuming translationally invariant boundary conditions). You need to find expressions in terms of integral over k, but do not need to evaluate it. But, surely, you need to perform the w integration! **Marks: 9**



8. The time-independent Schrodinger equation for a non-relativistic particle in a potential $V(r)$ can be written in the following form

$$(\nabla^2 + k^2)\psi(\vec{r}) = \lambda U(r)\psi(\vec{r}), \quad with \ \ U(r) = (2m/\hbar^2)V(r), \tag{7}$$

and λ is a 'small' parameter.

(i) Assuming free particle scattering state to be $\psi_0 = e^{i\vec{k}\cdot\vec{r}}$, write down the general solution of the above equation treating the R.H.S as the source term. For appropriate boundary conditions, it is given that

$$(\nabla^2 + k^2)\left(-\frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}\right) = \delta^3(\vec{r}-\vec{r}'). \tag{8}$$

Marks: 2

(ii) Write down the approximate solution for $\psi(\vec{r})$ up to order λ. **Marks: 1**