Ordinary Differential Equations

Part-15

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Numerical solution to ODE

Here we solve of set of ordinary differential equations of the for

$$\frac{dy_i}{dx} = f(y_i, x) \quad \text{for} \quad i = 1, 2, \cdots N$$

Why only first order ODE?

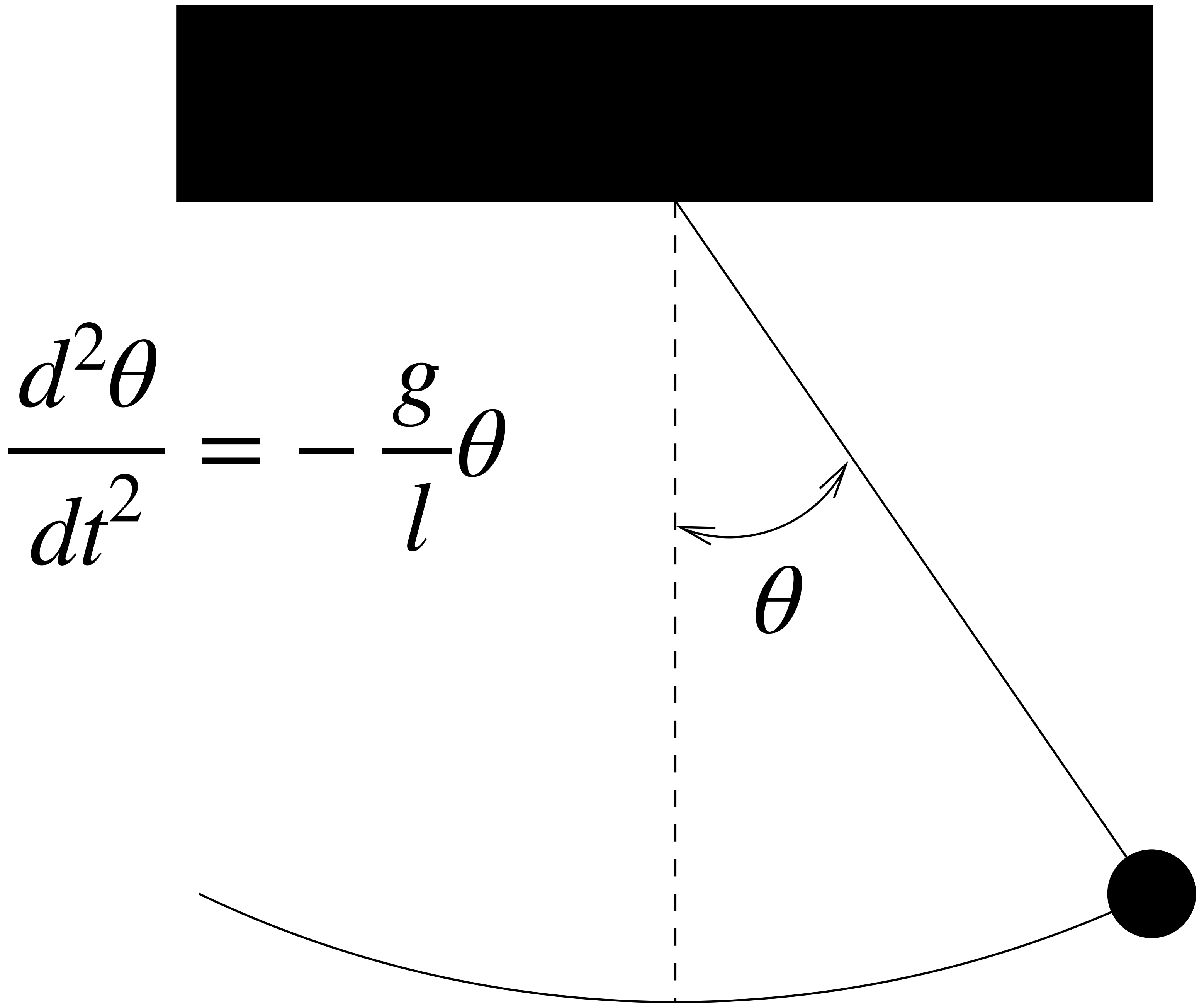
If we have second order ODE say

$$\frac{d^2y}{dx^2} = f(y, x)$$

We can write it as two first order differential equations

$$\frac{dy}{dx} = z \quad \text{and} \quad \frac{dz}{dx^2} = f(y, x)$$

Example



When we apply Newton's law of motion to a simple pendulum we get

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

Here given a $\theta(t=0)$ we have to find $\theta(t)$
Position at any given time

We rewrite the EOM, as

$$\frac{d\theta}{dt} = \omega$$

$$\frac{d\omega}{dt} = -\frac{g}{l}\theta$$

We have to solve two first order equations

Here we solve an ordinary differential equation

$$\frac{dy}{dx} = f(y, x)$$

Here, we are given $y(x_0)$ and $f(y, x)$, we have to solve for $y(x)$

And same method can be applied a larger set of equations

We apply our simplest 2-point formula $\dfrac{dy}{dx} \approx \dfrac{y(x+h) - y(x)}{h}$

Submitting for $\dfrac{dy}{dx}$ in our ODE, we get,

$$\frac{dy}{dx} \approx \frac{y(x+h) - y(x)}{h} = f(y, x)$$

Or

$$y(x+h) = y(x) + f(y, x) \times h$$

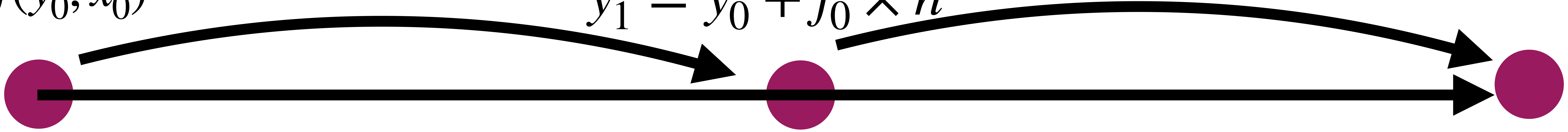
Initial value problem in ODE

We can compute

$f_0 = f(y_0, x_0)$

We can compute

$y_1 = y_0 + f_0 \times h$



x_0 Given y_0

$x_1 = x_0 + h$

$x_i = x_0 + i \times h$

We can go ahead a construction the solution $\{y_i\}$

When we apply Newton's law of motion to a simple pendulum we get

$$\frac{d\theta}{dt} = \omega$$

$$\frac{d\omega}{dt} = -\frac{g}{l}\theta$$

Our independent variable is t and dependent variables are $\{\theta, \omega\}$, and $h = \Delta t$

We are given $\theta_0 = \theta(t=0)$ and $\omega_0 = \omega(t)$

$$\frac{d\theta}{dt} = f_\theta = \omega \quad \text{and} \quad \frac{d\omega}{dt} = f_\omega = -\frac{g}{l}\theta$$

From this we get,

$$\theta_{n+1} = \theta_n + f_\theta \times \Delta t \quad \text{and} \quad \omega_{n+1} = \omega_n + f_\omega \times \Delta t$$

One we compute $\theta_{n+1}, \omega_{n+1}$ we can compute f_θ, f_ω using $\theta_{n+1}, \omega_{n+1}$

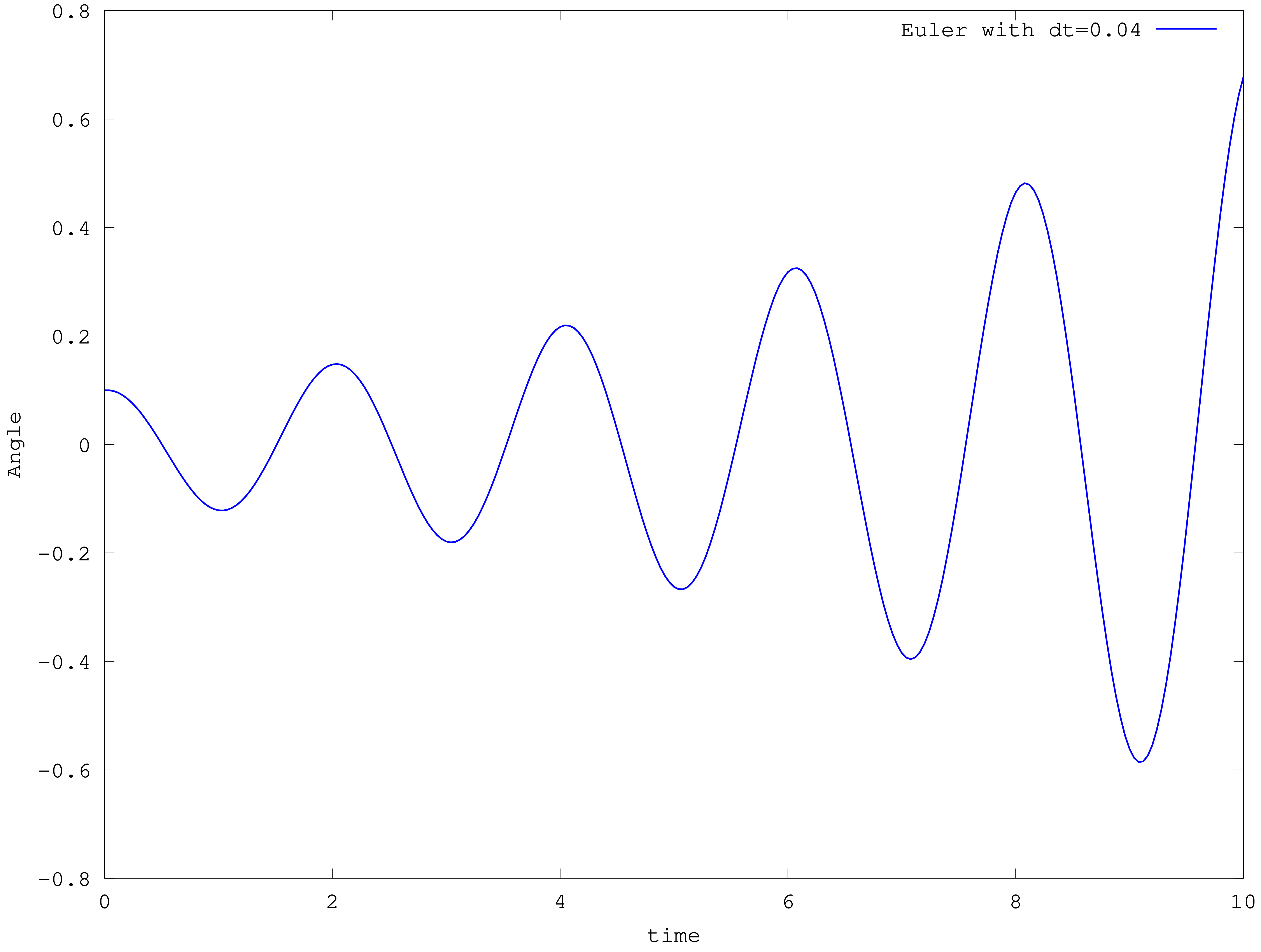
We can proceed and can compute the solution at all points!

Algorithm

1. Given t_0, θ_0 and ω_0. $n = 0$ and compute $f_{\theta_0} = \omega_0$ and $f_{\omega_0} = -\frac{g}{l}\theta_0$.
2. Compute $\theta_{n+1} = \theta_n + f_{\theta_n} \times \Delta t$
3. Compute $\omega_{n+1} = \omega_n + f_{\omega_n} \times \Delta t$
4. Compute $f_{\theta_{n+1}}, f_{\omega_{n+1}}$ using $\theta_{n+1}, \omega_{n+1}$
5. For next step; $n = n + 1$ and $t_{n+1} = t_0 + (n + 1) \times \Delta t$
6. repeat steps 2 to 5, till $n = N$

We should have $\{t_n\}$, $\{\theta_n\}$ and $\{\omega_n\}$

The method is called Euler method!


Euler with dt=0.04
Angle
time
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
0
2
4
6
8
10

Error Analysis for Euler

We have used first order formula of the derivative. Hence error in $\{\theta, \omega\}$ is of the order $\mathcal{O}\left(h = \Delta t\right)$

That is not enough for this problem! ?

These simple problems are of type conservative system.

Which says total energy must be conserved. $E = \frac{1}{2}mgl\left(\omega^2 + \frac{g}{l}\theta^2\right)$

or one can write as, $E_{i+1} = E_i + \frac{1}{2}mgl\left(\omega_i^2 + \frac{g}{l}\theta_i^2\right)\Delta t^2$

First order method can not be used for conservative systems because energy is of the order $\mathcal{O}\left(\Delta t^2\right)$

How do we solve this problem ?

We know $\frac{d\theta}{dt} = \omega$ $E = \frac{1}{2}mgl\left[\omega^2 + \frac{g}{l}\theta^2\right]$

$$E = \frac{1}{2}mgl\left[\left(\frac{d\theta}{dt}\right)^2 + \frac{g}{l}\theta^2\right]$$

$$\left[\frac{2E}{mgl} - \frac{g}{l}\theta^2\right]^{\frac{1}{2}} = \frac{d\theta}{dt}$$

This is first order equation in θ with E is constant !

Euler-Cromer method Algorithm

1. Given t_0, θ_0 and ω_0. $n = 0$ and compute $f_{\theta_0} = \omega_0$ and $f_{\omega_0} = -\frac{g}{l}\theta_0$.

2. Compute $\theta_{n+1} = \theta_n + f_{\theta_n} \times \Delta t$

3. Compute $\bar{f}_{\theta_n}, \bar{f}_{\omega_n}$ using θ_{n+1}, ω_n

4. Compute $\omega_{n+1} = \omega_n + \bar{f}_{\omega_n} \times \Delta t$

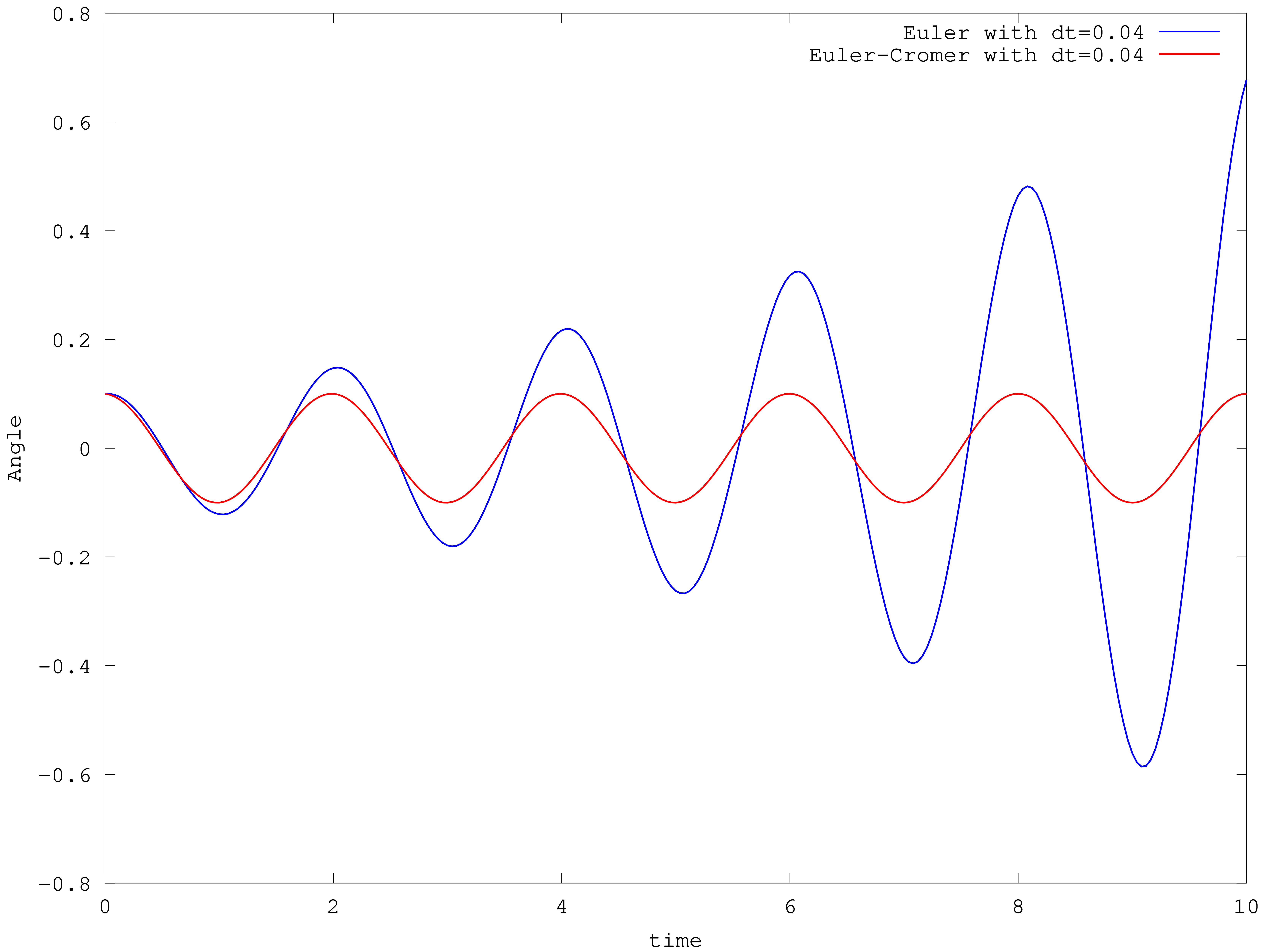
5. Compute $f_{\theta_{n+1}}, f_{\omega_{n+1}}$ using $\theta_{n+1}, \omega_{n+1}$

6. For next step; $n = n + 1$ and $t_{n+1} = t_0 + (n+1) \times \Delta t$

7. repeat steps 2 to 5, till $n = N$

We should have $\{t_n\}$, $\{\theta_n\}$ and $\{\omega_n\}$

The method is called Euler method!


Euler with dt=0.04
Euler-Cromer with dt=0.04
Angle
time
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
2
4
6
8
10

Let's check carefully !

Let us use x, y with

$$\dot{x} = f_1(x, , y, t) \qquad \dot{y} = f_2(x, , y, t)$$

Euler

1. Compute $x_{n+1} = x_n + f_1\left(x_n, y_n, t\right) \times \Delta t$
2. Compute $y_{n+1} = y_n + f_2\left(x_n, y_n, t\right) \times \Delta t$

Euler - Cromer

1. Compute $x_{n+1} = x_n + f_1\left(x_n, y_n, t\right) \times \Delta t$
2. Compute $\bar{f}_2$, using x_{n+1}, y_n

$$x_{n+1} = x_n + f_1\left(x_n, y_n, t\right)\Delta t + \frac{\partial}{\partial x} f_1\left(x_n, y_n, t\right)\Delta x$$

3. Compute $y_{n+1} = y_n + f_2\left(x_n + 1, y_n, t\right) \times \Delta t$

$$y_{n+1} = y_n + f_2\left(x_n, y_n, t\right) \times \Delta t + \frac{\partial}{\partial x} f_1\left(x_n, y_n, t\right)\Delta x \Delta t$$