Introduction to Computation (CS2201)

Lecture 2

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NUMERICAL SOLUTION OF EQUATIONS

Numerical Solution

Objective

Numerical computation of real roots of an equation of the form $f(x) = 0$, where f can be algebraic (e.g. polynomial) or transcendental (e.g. exponential, sin, log etc.)

Assumption

f is continuously differentiable

Theorem

If $f(x)$ is is continuous in the closed interval $[a, b]$ and $f(a)$, $f(b)$ are of opposite signs, then there is a real root α of the equation $f(x) = 0$ such that $a < \alpha < b$

If further $f(x)$ is differentiable in the open interval (a, b) and either $f'(x) < 0$ or $f' > 0$ in (a, b), then $f(x)$ is strictly monotonic in $[a, b]$ and then the root in unique.

Method of Tabulation

Description

- Method for finding a rough location (interval) of the roots of an equation using the sign change method
- We start with a rough interval, say, $[x_l, x_r]$ and moderate interval length (say h)
- We check for change in sign for the function $f(x)$ in each of the sub-intervals: $[x_l, x_l + h]$, $[x_l + h, x_l + 2h]$, ..., $[x_l + nh, x_r]$
- The above process is repeated with shorter intervals ($h/10$)
- Stop if the desired approximation is achieved

Drawback

Crude and computationally expensive process

Utility

Simple method to provide the initial approximation for starting an iterative method

Method of Tabulation (example)

Equation

$f(x) = 10^x + x - 4 = 0$

x	f(x)
0	-3
0.1	-2.64
0.2	-2.21
0.3	-1.70
0.4	-1.09
0.5	-0.34
0.6	0.58
0.7	1.71
0.8	3.11
0.9	4.84
1	7

Method of Tabulation (example)

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Method of Tabulation (example)

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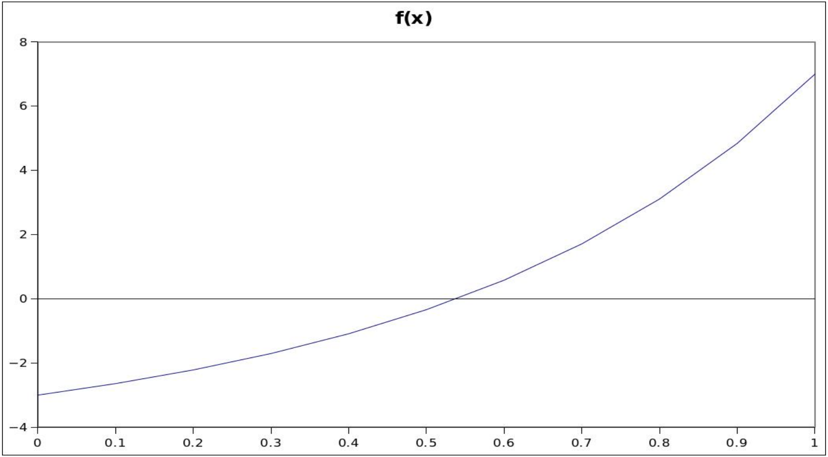
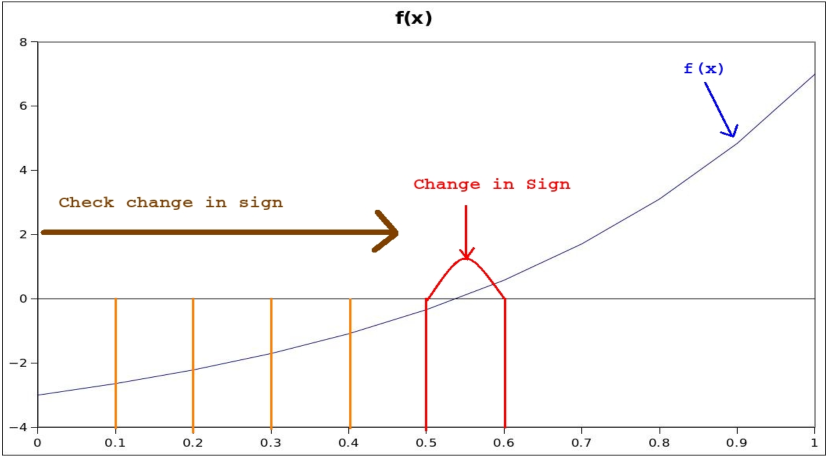


Method of Tabulation (example)

Equation

$f(x) = 10^x + x - 4 = 0$



Iteration Process

Premise

- Let x_n be the sequence of iterates of a required root α of the equation $f(x) = 0$, produced by a given method
- The error at the *nth* iteration (ϵ_n) is defined by $\epsilon_n = \alpha - x_n$

Convergence

The iteration converges if and only if $\epsilon_n \to 0$ as $n \to \infty$

Features

- Initial approximation (x_0) is chosen judiciously (e.g. using tabular method) – critical for convergence (bad initial guess may cause the iterative process to fail)
- Self-correcting: if an accidental error occurs in calculating the iterate, the erroneous iterate serves as the new initial guess and ultimately leads to the correct result

Bisection Method

Initial Interval

Use tabulation method to find a sufficiently small interval $[a_0, b_0]$ containing the root α of the equation $f(x) = 0$ such that $f(a_0)$ and $f(b_0)$ are of opposite signs.

Generation of iterates

1. Set $x_0 = a_0$ or b_0, $x_1 = (a_0 + b_0)/2$ and $f(x_1)$
2. If $f(a_0)$, $f(x_1)$ are of opposite signs, set $a_1 = a_0$, $b_1 = x_1$ ($[a_1, b_1] = [a_0, x_1]$)
3. Else, if $f(x_1)$, $f(b_0)$ are of opposite signs, set $a_1 = x_1$, $b_1 = b_0$ ($[a_1, b_1] = [x_1, b_0]$)
4. Set $x_2 = (a_1 + b_1)/2$
5. If the desired accuracy is reached end; else continue the above steps

Bisection Method (algorithm)

General

1. With the interval $[a_n, b_n]$ containing α such that $f(a_n)f(b_n) < 0$, set $x_{n+1} = (a_n + b_n)/2$
2. $a_{n+1} = a_n$, $b_{n+1} = x_{n+1}$, if $f(a_n)f(x_{n+1}) < 0$
   $a_{n+1} = x_{n+1}$, $b_{n+1} = b_n$, if $f(x_{n+1})f(b_n) < 0$
3. $[a_{n+1}, b_{n+1}]$ contains α $(f(a_{n+1})f(b_{n+1}) < 0)$

Convergence

$|\epsilon_n| = |\alpha - x_n| \leq b_n - a_n = (b_0 - a_0)/2^n \to 0$ as $n \to \infty$

Bisection Method (algorithm)

General

1. With the interval $[a_n, b_n]$ containing α such that $f(a_n)f(b_n) < 0$, set $x_{n+1} = (a_n + b_n)/2$
2. $a_{n+1} = a_n$, $b_{n+1} = x_{n+1}$, if $f(a_n)f(x_{n+1}) < 0$
   $a_{n+1} = x_{n+1}$, $b_{n+1} = b_n$, if $f(x_{n+1})f(b_n) < 0$
3. $[a_{n+1}, b_{n+1}]$ contains α $(f(a_{n+1})f(b_{n+1}) < 0)$

Convergence

$$|\epsilon_n| = |\alpha - x_n| \leq b_n - a_n = (b_0 - a_0)/2^n \to 0 \text{ as } n \to \infty$$

CONVERGENCE GUARANTEED!!

Bisection Method (example)

Equation

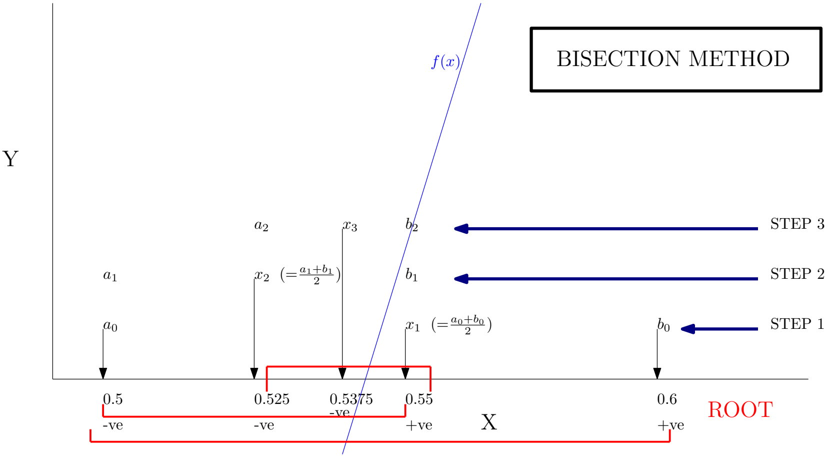
$f(x) = 10^x + x - 4 = 0$

n	a_n	$f(a_n)$	b_n	$f(b_n)$	x_{n+1}	$f(x_{n+1})$
0	0.500000	-0.337722	0.600000	0.581072	0.550000	0.098134
1	0.500000	-0.337722	0.550000	0.098134	0.525000	-0.125346
2	0.525000	-0.125346	0.550000	0.098134	0.537500	-0.015034
3	0.537500	-0.015034	0.550000	0.098134	0.543750	0.041188
4	0.537500	-0.015034	0.543750	0.041188	0.540625	0.012987
5	0.537500	-0.015034	0.540625	0.012987	0.539063	-0.001046
6	0.539063	-0.001046	0.540625	0.012987	0.539844	0.005965
7	0.539063	-0.001046	0.539844	0.005965	0.539453	0.002458
8	0.539063	-0.001046	0.539453	0.002458	0.539258	0.000706
9	0.539063	-0.001046	0.539258	0.000706	0.539160	-0.000170

Bisection (example)

Equation

$f(x) = 10^x + x - 4 = 0$



Newton-Raphson Method

Initial guess

1. Use tabulation method to find a sufficiently small interval $[a_0, b_0]$ containing the root α of the equation $f(x) = 0$ such that $f(a_0)$ and $f(b_0)$ are of opposite signs.
2. Choose a_0 or b_0 as the initial guess x_0 for the root

Generation of iterates

1. Set $h_n = -\frac{f(x_n)}{f'(x_n)}$
2. $x_{n+1} = x_n + h_n = x_n - \frac{f(x_n)}{f'(x_n)}$
3. If the desired accuracy is reached end; else continue the above steps

Convergence Condition

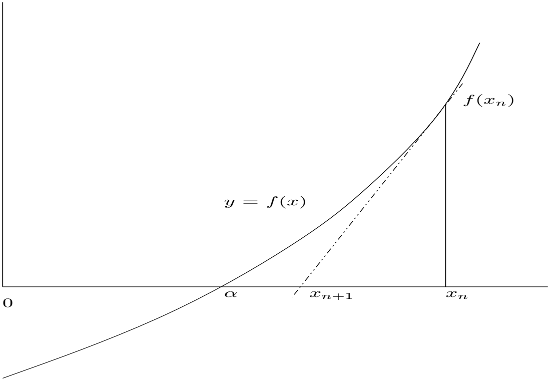
$|f(x)f''(x_n)| < |f'(x_n)|^2$

Newton-Raphson Method (geometric interpretation)

The point at which the tangent to the curve $y = f(x)$ at the point $(x_n, f(x_n))$ cuts the x-axis is x_{n+1}, for the said tangent has the equation

$$y - f(x_n) = f'(x_n)(x - x_n)$$

and intersects $y = 0$ at $x = x_n$ - $\frac{f(x_n)}{f'(x_n)} = x_{n+1}$



Newton-Raphson Method (example)

Equation

$f(x) = 10^x + x - 4 = 0$

n	x_n	$f(x_n)$	$f'(x_n)$	h_n
0	0.540781	0.014388	8.998277	-0.040781
1	0.539182	0.000024	8.968883	0.001599
2	0.539179	0.000000	8.968835	0.000003

Newton-Raphson Method: Features

Advantages

- If $|f(x)|$ is small and $|f'(x)|$ is of moderate magnitude, the iteration converges
- Rapidly convergent

Disadvantages

- If the initial guess (x_0) is bad, the iteration may not converge
- Computationally expensive: both $f(x)$ and $f'(x)$ need to be computed

Regula Falsi Method

Initial Interval

Use tabulation method to find a sufficiently small interval $[a_0, b_0]$ containing the root α of the equation $f(x) = 0$ such that $f(a_0)$ and $f(b_0)$ are of opposite signs.

Generation of iterates

1. Set $x_0 = a_0$ or b_0, $x_1 = a_0$ - $f(a_0)\ \frac{(b_0 - a_0)}{f(b_0) - f(a_0)}$
2. If $f(a_0)$, $f(x_1)$ are of opposite signs, set $a_1 = a_0$, $b_1 = x_1$ ($[a_1, b_1] = [a_0, x_1]$)
3. Else, if $f(x_1)$, $f(b_0)$ are of opposite signs, set $a_1 = x_1$, $b_1 = b_0$ ($[a_1, b_1] = [x_1, b_0]$)
4. If the desired accuracy is reached end; else continue the above steps

Regula Falsi Method (algorithm)

General

1. With the interval $[a_n, b_n]$ containing α such that $f(a_n)f(b_n) < 0$, set $x_{n+1} = a_n$ - $f(a_n)\ \frac{(b_n - a_n)}{f(b_n) - f(a_n)}$
2. $a_{n+1} = a_n,\ b_{n+1} = x_{n+1}$, if $f(a_n)f(x_{n+1}) < 0$
   $a_{n+1} = x_{n+1},\ b_{n+1} = b_n$, if $f(x_{n+1})f(b_n) < 0$
3. $[a_{n+1}, b_{n+1}]$ contains α $(f(a_{n+1})f(b_{n+1}) < 0)$

Convergence

$|\epsilon_n| \leq (\frac{M_1 - m_1}{m_1})^n\ |\epsilon_0| \to 0$ as $n \to \infty$ (M_1, m_1 denote respectively the maximum and minimum of $|f'(x)|$ in $[a_0, b_0]$ and $M_1 - m_1 < m_1$)

Regula Falsi Method (algorithm)

General

1. With the interval $[a_n, b_n]$ containing α such that $f(a_n)f(b_n) < 0$,
   set $x_{n+1} = a_n$ - $f(a_n)\ \frac{(b_n - a_n)}{f(b_n) - f(a_n)}$
2. $a_{n+1} = a_n,\ b_{n+1} = x_{n+1}$, if $f(a_n)f(x_{n+1}) < 0$
   $a_{n+1} = x_{n+1},\ b_{n+1} = b_n$, if $f(x_{n+1})f(b_n) < 0$
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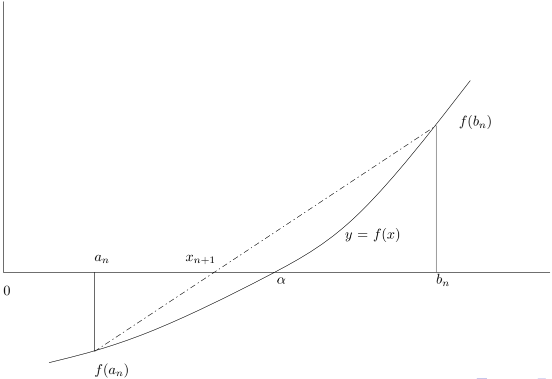
CONVERGENCE GUARANTEED!!

Regula Falsi Method (geometric interpretation)

The chord joining the points $(a_n, f(a_n))$ and $(b_n, f(b_n))$ has the equation

$$\frac{y - f(a_n)}{f(b_n) - f(a_n)} = \frac{x - a_n}{b_n - a_n}$$

which and intersects (x-axis) $y = 0$ at $x = a_n$ - $f(a_n)$ $\frac{(b_n - a_n)}{f(b_n) - f(a_n)} = x_{n+1}$



Regula Falsi Method (example)

Equation

$f(x) = 10^x + x - 4 = 0$

n	a_n	b_n	x_{n+1}	$f(x_{n+1})$
0	0.500000	0.600000	0.536757	-0.021669
1	0.536757	0.600000	0.539031	-0.001331
2	0.539031	0.600000	0.539170	-0.000081

THANK YOU !!!