CONTINOUS FUNCTION

Defn: (ϵ-δ definition) Let $D \subseteq \mathbb{R}$ and $c \in D$. A function $f : D \to \mathbb{R}$ is said to be continuous at c if for $\epsilon > 0$ $\exists\ \delta > 0$ such that

$$|f(x) - f(c)| < \epsilon \quad \forall\ x \in D \cap (c-\delta, c+\delta)$$

Example: 1. Let $D = \{1, 2, \cdots, N\}$. Let $f : D \to \mathbb{R}$. Show that f is continuous at every pt. of D.

2. Any function $f : \mathbb{N} \to \mathbb{R}$ is continuous at every pt. of D.

3. **Dirichlet's fn:** Consider the fn. $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{Q}^c. \end{cases}$$

Then f is not continuous at any point.

4. Let $f: \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} x^2 & x \le 0, \\ x^3 & x > 0. \end{cases}$$

Show that f is continuous at each pt.

Defn (Sequential defn): Let $D \subseteq \mathbb{R}$ and $c \in D$. A function $f: D \to \mathbb{R}$ is said to be continuous at c if for every sequence $x_n \in D$ s.t. $x_n \to c$, we have $f(x_n) \to f(c)$.

Ex: Show that both the defns are equivalent.

Defn: Let $D \subseteq \mathbb{R}$. A fn $f: D \to \mathbb{R}$ is said to be continuous if f is continuous at every pt. of D.

Algebra of Continuous fns.

Let $D \subseteq \mathbb{R}$. Let $c \in D$. Let $f, g : D \to \mathbb{R}$ be continuous at c. Then we have the following.

(i) The function $(f+g) : D \to \mathbb{R}$ defined by
$$(f+g)(x) = f(x) + g(x) \quad x \in D,$$
is continuous at c.

(ii) Let $\alpha \in \mathbb{R}$. The function $(\alpha f) : D \to \mathbb{R}$ defined by
$$(\alpha f)(x) = \alpha f(x) \quad x \in D,$$
is continuous at $c \in D$.

(iii) The fn $(fg) : D \to \mathbb{R}$ defined by
$$(fg)(x) = f(x)\, g(x),$$
is conti at c.

(iv) The fn $|f|: D \to \mathbb{R}$ defined by.

$$|f|(x) = |f(x)| \quad x \in D,$$

is Conti. at c.

(v) The fn $\max\{f, g\}: D \to \mathbb{R}$ defined by

$$\max\{f,g\}(x) = \max\{f(x), g(x)\},$$

$$x \in D$$

is Conti. at c.

(vi) The fn $\min\{f, g\}: D \to \mathbb{R}$ defined by

$$\min\{f,g\}(x) = \min\{f(x), g(x)\}.$$

$$x \in D,$$

is Conti at c.

(vii) Let $f_1 : J_1 \to \mathbb{R}$ and $f_2 : J_2 \to \mathbb{R}$.

Assume that $f_1(J_1) \subseteq J_2$. Then

$f_2 \circ f_1 : J_1 \to \mathbb{R}$ is continuous at c

f_1 is conti. at c and f_2 is conti. at $f_1(c)$.

(viii) Let $f : J \to \mathbb{R}$ be a continuous fn. Let $J_1 \subseteq J$. Then the restriction fn $f\big|_{J_1} : J_1 \to \mathbb{R}$ defined by

$$f\big|_{J_1}(x) = f(x) \qquad x \in J_1$$

is continuous.

Thm: (Neighbourhood property): Let $f : (a,b) \to \mathbb{R}$ be continuous at $c \in (a,b)$. Assume that $f(c) > 0$, then $\exists\, \delta > 0$ such that

$$f(x) > 0 \qquad \forall\, x \in (c-\delta, c+\delta) \cap (a,b).$$

Proof: Let $\epsilon = \frac{1}{2} f(c) > 0$. Since f is continuous at c, $\exists\ \delta > 0$ such that

$$f(c) - \epsilon < f(x) < f(c) + \epsilon \quad \forall x \in (c-\delta, c+\delta) \cap (a,b).$$

Since $\epsilon = \frac{1}{2} f(c)$, $\forall x\, (c-\delta, c+\delta) \cap (a,b)$

$$f(x) > f(c) - \frac{1}{2} f(c) = \frac{1}{2} f(c) > 0.$$

This completes the proof.

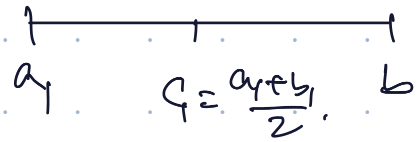
Thm (Existence of zero): Let $f : [a,b] \to \mathbb{R}$ be a continuous fn such that $f(a) < 0 < f(b)$. Then $\exists\ c \in (a,b)$ such that

$$f(c) = 0.$$

Proof: We will use the Cantor's intersection thm to prove this theorem.

Let $I_1 = [a, b] = [a_1, b_1]$, that is, $a_1 = a$, $b_1 = b$. We bisect I_1.

Let $c_1 = \frac{a_1 + b_1}{2}$.



If $f(c_1) = 0$, then we have the proof of the theorem.

If $f(c_1) \neq 0$, then either $f(c_1) > 0$ or $f(c_1) < 0$. We define $I_2 = [a_2, b_2]$ where $a_2 = a_1$, $b_2 = c_1$ if $f(c_1) > 0$. otherwise $a_2 = c_1$, $b_2 = b_1$ for $f(c_1) < 0$.

Next we bisect $I_2 = [a_2, b_2]$. Let $c_2 = \frac{a_2 + b_2}{2}$. If $f(c_2) = 0$, then proof is complete.

If $f(c_2) \neq 0$, then we construct
$I_3 = [a_3, b_3]$ s.t.

$$f(a_3) < 0 \quad f(b_3) > 0.$$

Again we bisect I_3 and continue.

If $f(c_n) = 0$ for some n then we have the proof of the theorem.

Suppose $f(c_n) \neq 0$ $\forall n$, then we have a seq. of $\{I_n\}$ s.t.

(i) $I_n \subseteq I_{n-1}$ $\forall\ n \geq 2$

(ii) $f(a_n) < 0$ and $f(b_n) > 0$ $\forall n$
where $I_n = [a_n, b_n]$.

(iii) $|I_n| = \frac{1}{2^{n-1}}(b-a)$.

Thus $\{I_n\}$ satisfies the hypothesis of Cantor's intersection theorem.
Hence $\exists\ c \in [a,b]$ such that

$$\bigcap_{n=1}^{\infty} I_n = \{c\}$$

and $\lim a_n = \lim b_n = c$.

Now since f is continuous at c,

$$f(c) = f(\lim a_n) = \lim f(a_n) \le 0$$

and $$f(c) = f(\lim b_n) = \lim f(b_n) \geq 0.$$

Thus $f(c) = 0$.

This completes the proof.

Corollary: Let $f:[a,b]\to\mathbb{R}$ be continuous such that $f(a)f(b)<0$. Then $\exists$ $c\in(a,b)$ s.t.

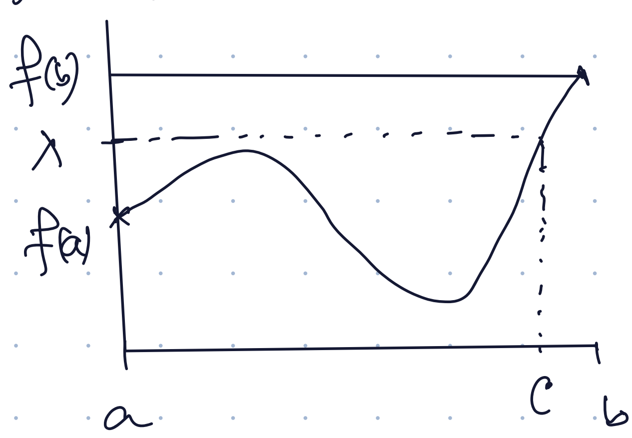
$$f(c)=0.$$

Proof: Exercise.

Thm: (Intermediate value theorem)
(in short IVT)

Let $f:[a,b]\to\mathbb{R}$ be a continuous fn. Let $\lambda\in\mathbb{R}$ be a number lies between $f(a)$ and $f(b)$. Then $\exists$ $c\in[a,b]$ s.t.

$$f(c)=\lambda$$



Proof: It is given that λ lies between $f(a)$ and $f(b)$. We will prove the theorem for the case

$$f(a) < \lambda < f(b), \longrightarrow (1)$$

Other cases can be proved similarly.

Define $g : [a,b] \to \mathbb{R}$ by

$$g(x) = f(x) - \lambda, \quad x \in [a,b]$$

Then g is a continuous fn. and by (1)

$$g(a) < 0, \quad g(b) > 0.$$

Therefore, by previous theorem, $\exists\, c \in (a,b)$ s.t.

$$g(c) = 0$$

$$\Rightarrow f(c) = \lambda.$$

This completes the proof.

Thm: (Fixed pt. Thm)
Let $f:[a,b]\to[a,b]$ be continuous fn. Then $\exists\, c\in[a,b]$ such that

$$f(c)=c.$$

[This c is called a fixed pt. of f].

Proof: It is given that

$$a\le f(x)\le b \quad \forall\, x\in[a,b].$$

If $f(a)=a$ or $f(b)=b$, then there is nothing to prove.

We assume that $f(a)\neq a$ and $f(b)\neq b$. Thus

$$a<f(a) \text{ and } f(b)<b.$$

We Consider the fn $g : [a,b] \to \mathbb{R}$

by

$$g(x) = x - f(x)$$

Then g is Continuous and

$$g(a) = a - f(a) < 0 \quad \text{and}$$

$$g(b) = b - f(b) > 0.$$

Therefore $\exists$ $c \in (a,b)$ s.t.

$$g(c) = 0$$

$$\Rightarrow \boxed{f(c) = c}$$

Thm (Extreme Value Thm) (Weierstrass Thm)

Let $f : [a,b] \to \mathbb{R}$ be Continuous. Then f is bounded. There exists $x_1, x_2 \in [a,b]$ such that

$$f(x_1) = \max\{f(x) : x \in [a,b]\}$$

$$f(x_2) = \min\{f(x) ; x \in [a,b]\}.$$

Proof: Let f is not bounded. Then $\exists\ x_n \in [a,b]$ s.t.

$$|f(x_n)| > n. \quad \forall n \in \mathbb{N}. \longrightarrow (1)$$

Since $x_n \in [a,b]$, by Bolzano-Weirestrass Thm, $\{x_n\}$ has a convergent subseq. Say $\{x_{n_k}\}_k$. Let $x_{n_k} \to x$. Since $x_{n_k} \in [a,b]$ and $[a,b]$ is closed $x \in [a,b]$. Since f is conti. at x, we have

$$f(x_{n_k}) \longrightarrow f(x).$$

Therefore, $\{f(x_{n_k})\}$ is cgt. Hence $\{f(x_{n_k})\}$ is bdd but (1) shows that $\{f(x_{n_k})\}$ is unbdd. This gives a contradiction. Therefore.

f is bounded.

Let $\alpha = \sup\{f(x) : x \in [a,b]\}$.

Claim: $\exists\ p \in [a,b]$ s.t.

$$f(p) = \alpha.$$

Since $\alpha = \sup\{f(x) : x \in [a,b]\}$

$\exists\ x_n \in [a,b]$ s.t.

$$f(x_n) \longrightarrow \alpha$$

Now $x_n \in [a,b]$ and $[a,b]$ is cpt,

$\exists\ x_{n_k}, p \in [a,b]$ s.t.

$$x_{n_k} \longrightarrow p.$$

Since f is - at p, hence

$$f(p) = \alpha.$$

This proves that $\sup f$ is attained. Similarly we can show that $\inf f$ is attained. This completes the proof.

Monotone fns and continuity

Thm: Let $J \subseteq \mathbb{R}$ be an interval. Let $f: J \to \mathbb{R}$ be a continuous ~~Continous~~ and injective. Let $a, b, c \in J$ and $a < b < c$. Then

either $f(a) < f(b) < f(c)$

or $f(a) > f(b) > f(c)$

Proof: Since f is injective and $a < c$, then $f(a) \neq f(c)$. Then either $f(a) < f(c)$ or

$f(a) > f(c)$

Assume that $f(a) < f(c)$. Will show that $f(a) < f(b) < f(c)$.

Suppose not, then either

$f(b) < f(a) < f(c)$

or $f(a) < f(c) < f(b)$.

Let $f(b) < f(a) < f(c)$.

Let $\lambda \in (f(b), f(a)) \subseteq (f(b), f(c))$.

By IVT, $\exists$ $x_1 \in (a, b)$, $x_2 \in (b, c)$

s.t. $f(x_1) = f(x_2) = \lambda$

Since $x_1 \neq x_2$, this contradicts the fact f is injective.

Similarly $f(a) < f(c) < f(b)$ is not possible.

Therefore, we have

$$f(a) < f(b) < f(c).$$

This completes the proof.

Thm: Let $J \subseteq \mathbb{R}$ be an interval. Let $f: J \to \mathbb{R}$ be a continous and one-one. Then f is strictly monotone.

Pf: Let $a, b \in J$ and $a < b$. Since f is one-one $f(a) \neq f(b)$. Thus either $f(a) < f(b)$ or $f(a) > f(b)$.

We will prove that

(i) $f(a) < f(b) \Rightarrow f$ is strictly increasing

(ii) $f(a) > f(b) \Rightarrow f$ is str. decreasing.

<u>Proof of (i)</u>: Let $f(a) < f(b)$

Let $x, y \in J$ and $x < y$. Then x satisfies one of the following

(i) $x < a < b$

(ii) $a < x < b$

(iii) $a < b < x$.

In each cases we have

(i) $x < a < b \Rightarrow f(x) < f(a) < f(b)$

(ii) $a < x < b \Rightarrow f(a) < f(x) < f(b)$

(iii) $a < b < x \Rightarrow f(a) < f(b) < f(x)$

by last thm, as $f(a) < f(b)$

Similarly for y, we have

(a) $y < a < b \Rightarrow f(y) < f(a) < f(b)$

(b) $a < y < b \Rightarrow f(a) < f(y) < f(b)$

(c) $a < b < y \Rightarrow f(a) < f(b) < f(y)$

Now Combining (i), (ii), (iii) and (a), (b), (c) and using last theorem, we have

$$f(x) < f(y) \quad \text{for all } x < y.$$

For example if $x < y < a$, then by (i), $f(x) < f(a)$. Thus by last theorem, $f(x) < f(y) < f(a)$. Similar analysis can be done for other cases. This proves that f is increasing.

Similarly if $f(a) > f(b)$, we can prove that f is strictly decreasing.

<u>Thm:</u> Let J be an interval. Let $f: J \to \mathbb{R}$ be monotone and $f(J) = I$ be an interval. Then f is continuous.

<u>Proof:</u> Without loss of generality, we assume that f is strictly increasing fn.

Let $x \in J$. Assume that x is not an end pt. of J. Then $\exists\ x_1, x_2 \in J$ s.t.

$$x_1 < x < x_2$$

Then $$f(x_1) < f(x) < f(x_2)$$

$\exists\ \eta > 0$ s.t.

$$f(x_1) < f(x) - \eta < f(x) < f(x) + \eta < f(x_2)$$

Since $f(J)$ is an interval,

$$f(x) - \eta,\ f(x) + \eta \in f(J)$$

$\exists\ s_1, s_2 \in J$, with $s_1 < x < s_2$ s.t.

$$f(s_1) = f(x) - \eta$$
$$f(s_2) = f(x) + \eta$$

Choose $\delta > 0$ s.t.

$$s_1 < x - \delta < x + \delta < s_2.$$

Then $\forall\ y \in (x - \delta, x + \delta)$,

$$f(s_1) < f(y) < f(s_2)$$

$$\Rightarrow \quad f(x) - \eta < f(y) < f(x) + \eta$$

Thus $\boxed{\begin{array}{l} \forall\ y \in (x-\delta, x+\delta) \\ |f(x)-f(y)| < \eta. \end{array}}$ $\longrightarrow$ (*)

Let $\epsilon > 0$. If $\epsilon > \eta$, then for the above δ, we have

$$y \in (x-\delta, x+\delta) \Rightarrow |f(x)-f(y)| < \eta < \epsilon. \longrightarrow (*1)$$

If $\epsilon < \eta$, then

$$f(x)-\eta < f(x)-\epsilon \leq f(x) \leq f(x)+\epsilon < f(x)+\eta$$

then as before, $\exists$ $\delta_1^* < x < \delta_2^*$ s.t.

$$f(\delta_1^*) = f(x)-\epsilon, \quad f(\delta_2^*) = f(x)+\epsilon$$

Choose $\delta_1 > 0$ s.t

$$\delta_1^* \leq x-\delta_1 \leq x < x+\delta_1 \leq \delta_2^*.$$

Then $\forall\ y \in (x-\delta_1, x+\delta_1)$, we have

$$|f(x)-f(y)| < \epsilon.$$

This proves that for $\epsilon>0$, $\exists\ \delta>0$ s.t.

$$y \in (x-\delta, x+\delta) \Rightarrow |f(x)-f(y)|<\epsilon.$$

Thus f is continuous at x. This completes the proof.

Corollary: Let $f: J \to \mathbb{R}$ be an strictly increasing continuous function. Then

(i) $f(J)$ is an interval

(ii) $f: J \to f(J)$ is invertible

(iii) $f^{-1}: f(J) \to J$ is continuous.

Proof: Exercise.