Example For PDE with Boundary Condition

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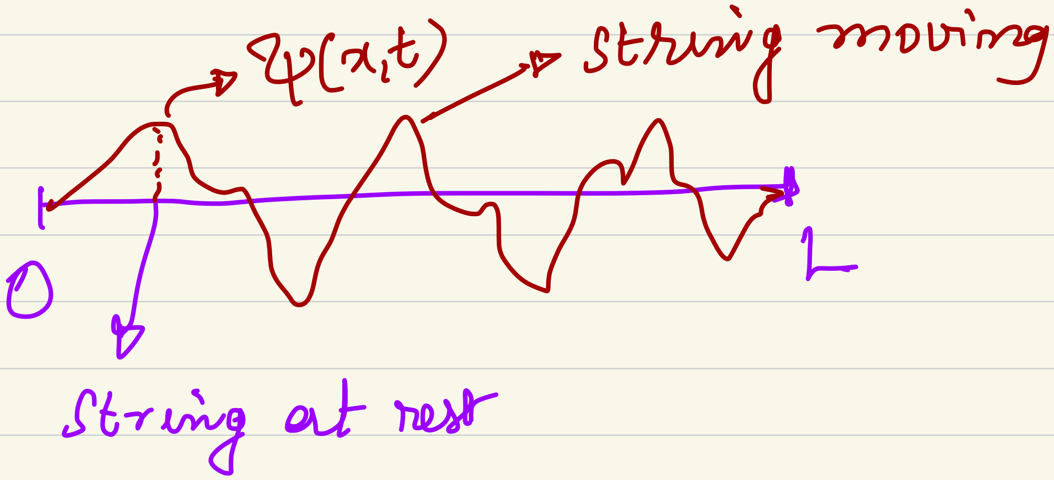
A Vibrating String

A string of Length 'L' is fixed at both end is left with an initial shape of $\psi_0(x)$ find the motion of string as function of x, t.



This motion of string can be modelled by wave equation

$$\frac{\partial^2 \psi}{\partial x^2} - \underbrace{\frac{1}{c^2}}_{\text{speed of the wave}} \frac{\partial^2 \psi}{\partial t^2} = 0$$

the boundary condition given are

$$\psi(x=0, t) = 0 \quad \text{and} \quad \psi(x=L, t) = 0$$

both end of string in at rest all the time

Also $\psi(x, t=0) = \psi_0(x)$

we also take $\left.\frac{\partial \psi}{\partial t}\right|_{t=0} = 0$

we start with PDE

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

$\psi(x,t)$ is assummed to be Separable

i.e $\psi(x,t) = \psi_x \psi_t$

Substituting in PDE and after some simplification, we get

$$\underbrace{\frac{1}{\psi_x}\frac{\partial^2 \psi_x}{\partial x^2}}_{1} - \underbrace{\frac{1}{c^2 \psi_t}\frac{\partial^2 \psi}{\partial t^2}}_{2} = 0$$

since term 1 is purely function of x, and term 2 is purely a function of 't' both must be a Const.

Let $-\alpha^2$ be constant of Separability ?

From 1 we get

$$\frac{1}{\psi_x}\frac{\partial^2 \psi_x}{\partial x^2} + \alpha^2 = 0$$

$$\frac{\partial^2 \psi_x}{\partial x^2} + \alpha^2 \psi_x = 0$$

from 2 we get

$$\frac{1}{c^2 \psi_t}\frac{\partial^2 \psi}{\partial t^2} + \alpha^2 = 0$$

Since ψ_x is a smooth sol^n in the interval, it can be expressed in terms of Fourier Series in interval $[0, L]$ as

$$\psi_x = a_0 + \sum_{n=1}^{\infty}\left\{a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right)\right\}$$

Since $\psi_x(0) = \psi(L) = 0$ $\quad a_0 = 0$, also $a_n = 0 \ \forall n$

We left with

$$\psi_x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{L}x\right)$$

To fix the constant, we substitute ψ_x in the ODE

$$\frac{d^2\psi_x}{dx^2} + \alpha^2 \psi_x = 0$$

$$\sum \left\{ -a_n \left(\frac{2\pi n}{L}\right)^2 \sin\left(\frac{2\pi n}{L}x\right) + \alpha^2 a_n \sin\left(\frac{2\pi n}{L}x\right) \right\} = 0$$

$$\sum_{n=1} \left[\alpha^2 - \left(\frac{2\pi n}{L}\right)^2\right] a_n \sin\left(\frac{2\pi n}{L}x\right) = 0$$

This can happen only iff

$$\alpha^2 = \left(\frac{2\pi n}{L}\right)^2$$ i.e α dependent on n

$$\alpha_n = \frac{2\pi n}{L}$$

Now we look at time part

$$\frac{d^2\psi_t}{dt^2} + \alpha^2 c^2 \psi_t = 0$$

and the sol^n is

$$A_n \cos(\alpha_n c t) + B_n \sin(\alpha_n c t)$$

At $t=0$ we need $\left.\frac{\partial \psi}{\partial t}\right|_{t=0}$ should be '0'

This condition gives us $B_n = 0$

$$\psi_t = A_n \cos(\alpha_n c t)$$

The complete sol^n is given by

$$\psi(x,t) = \sum_{n=1}^{\infty} a_n' \sin(\alpha_n x) \cos(\alpha_n c t)$$

$$\alpha_n = \frac{2\pi n}{L}$$

finally at $t=0$

$$\psi(x,t=0) = \psi_0(x) = \sum_{n=1} a_n' \sin\alpha_n x$$

from this we get

$$a_n' = \frac{2}{L}\int_0^L \sin(\alpha_n x)\, \psi_0(x)\, dx$$

This is the end !