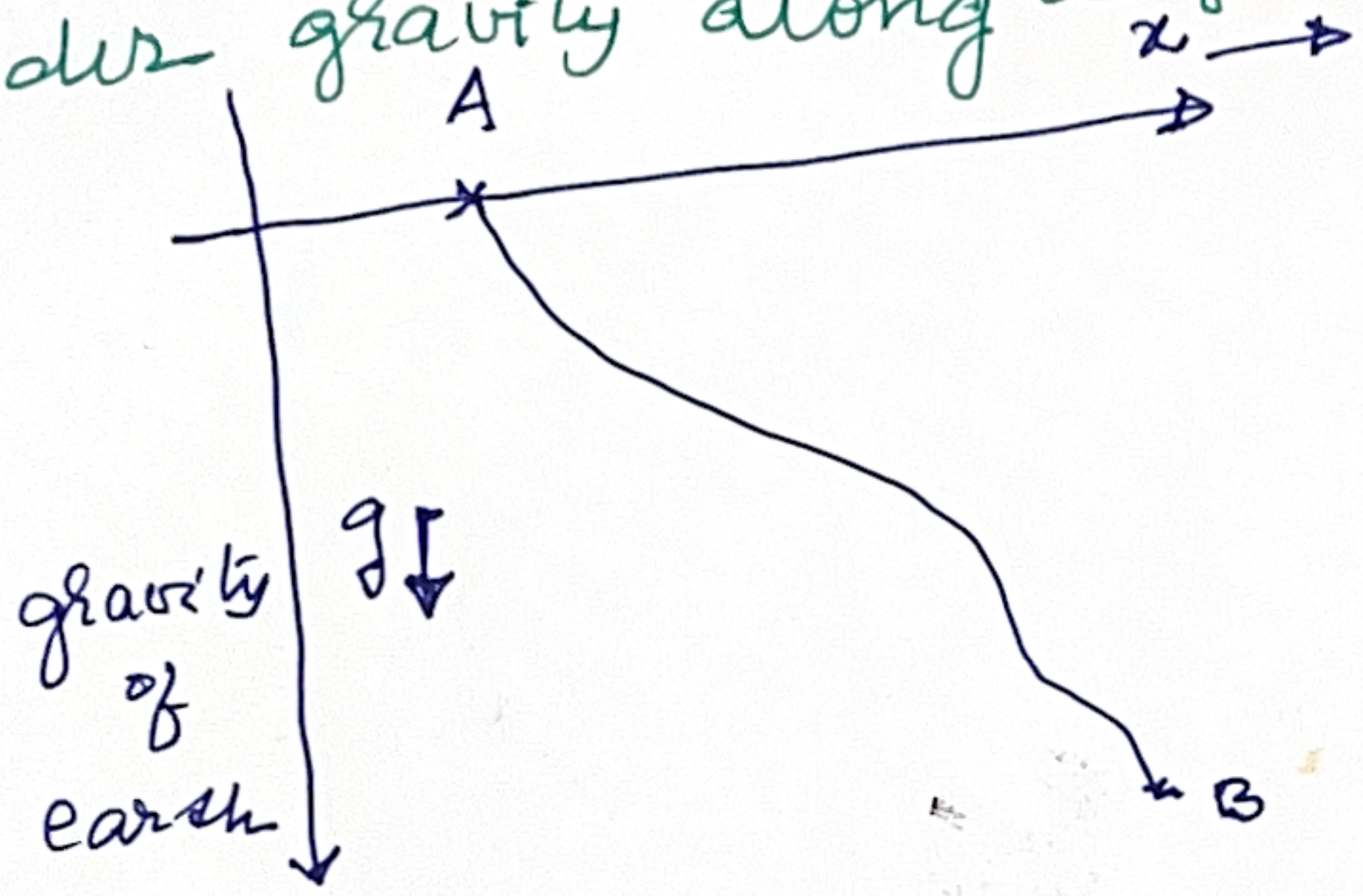
Brachistochrone problem :

A particle is made to freely fall under gravity along a given curve



find the shape of this curve which provide shortest travel time from A to B.

This problem was first formulated by John Bernoulli (1696)!.

is a good application for Calculus of Variations.

The time taken by a particle is

$$T_{AB} = \int_A^B \frac{ds}{v}$$

ds is length element,

v is the velocity of the particle.

for this problem,

If we start from rest at A

Then we have $\frac{1}{2}\cancel{m}v^2 = \cancel{m}gy$! energy conservation

$$v^2 = 2gy \Rightarrow v = \sqrt{2gy}$$

$$ds^2 = dx^2 + dy^2$$

$$ds = \left(1 + \frac{dy^2}{dx}\right)^{1/2} dx$$

$$\cancel{= \left(1 + y'^2\right)^{1/2}}$$

$$ds = \left[1 + \left(y'\right)^2\right]^{1/2} dx$$

$$T_{AB} = \int_A^B \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}\, dx$$

$$\mathcal{L}(y, y', x) \longrightarrow \mathcal{L}(q, \dot{q}, t)$$

LEOM

$$\mathcal{L} = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}$$

$$\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) - \frac{\partial \mathcal{L}}{\partial y} = 0$$

$$\mathcal{L} - y'\frac{\partial \mathcal{L}}{\partial y'} = C$$ ← energy fn!

$$\frac{\partial \mathcal{L}}{\partial y'} = \frac{y'}{(1+y'^2)^{-1/2}} \cdot \frac{1}{\sqrt{2gy}}$$

$$\frac{1}{\sqrt{(1+y'^2)}} \cdot \frac{1}{\sqrt{2gy}} = \text{const} = C$$

~~Squaring~~. Rearrange

$$\frac{1}{\sqrt{2gC^2}} = (1+y'^2)^{1/2}$$

$$\left(1+\frac{dy}{dx}^2\right)y = \frac{1}{2gc}$$

one need to solve this. That is HW problem

(2) $$I = \int_A^B L \, dt$$

find the least action sol^n such that

$\delta I = 0$

for $L = p_i \dot{q}_i - H(p, q, t)$

here L is function of $q, \dot{q}, p, \dot{p}$

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we get two sol^n

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad \text{--- (1)}$$

$$\Rightarrow \quad \frac{d}{dt}(p_i) + \frac{\partial H}{\partial q_i} = 0$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Similarly

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_i}\right) - \frac{\partial L}{\partial p_i} = 0$$

$$0 \quad \dot{q}_i - \frac{\partial H}{\partial p_i} = 0$$