UNIFORM CONTINUITY

Defn: A fn. $f: J \to \mathbb{R}$ is uniform conti. if for each $\epsilon > 0$, $\exists\, \delta > 0$ such that

$$|f(x) - f(y)| < \epsilon$$

for all $x, y \in J$ satisfying $|x-y| < \delta$.

Remark: A uniformly continuous fn $f: J \to \mathbb{R}$ is continuous on J. (Exercise)

Thm (Sequential defn)
A fn $f: J \to \mathbb{R}$ is uniformly conti. iff for every pair of sequence $x_n, y_n \in J$ such that $|x_n - y_n| \to 0$, satisfies

$$|f(x_n) - f(y_n)| \to 0.$$

Proof: (Exercise)

Thm: Let $f: J \to \mathbb{R}$ is uniformly conti. Then $\{f(x_n)\}$ is Cauchy for every Cauchy sequence $\{x_n\}$ in J.

Proof: Exercise

Exercise: Show that the converse is NOT true.

Example: 1. $f(x) = x$ $x \in (0,1)$ is uniform continuous

2. $f(x) = \frac{1}{x}$, $x \in (0,1)$ is NOT uniformly continuous.

3. $f(x) = x^2$ is uniform continuous on $(0,1)$ but NOT on $\mathbb{R}$.

Thm: Let $f:[a,b]\to\mathbb{R}$ be continuous. Then f is uniform continuous.

Proof: Suppose f is not uniform cont. Then $\exists\ \epsilon>0$ such that

$$|x_n-y_n|<\frac{1}{n} \quad \forall\, n\in\mathbb{N} \longrightarrow (1)$$

and

$$|f(x_n)-f(y_n)|\geq\epsilon \quad \forall\, n\in\mathbb{N} \longrightarrow (2)$$

for some $x_n, y_n\in[a,b]$.

Since $[a,b]$ is cpt, $\exists$ a subseq $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$x_{n_k}\longrightarrow x \quad \text{as } k\to\infty,$$

for some $x\in[a,b]$.

Then by (1),

$$y_{n_k}\longrightarrow x \quad \text{as } k\to\infty$$

Since f is conti. at x, we have

$$f(x_{n_k}) \to f(x),$$

and $$f(y_{n_k}) \to f(x).$$

This implies

$$f(x_{n_k}) - f(y_{n_k}) \to 0 \quad \text{as } k \to \infty.$$

which contradicts (2). This proves the thm.

Following the above proof, we can prove the following thm.

Thm: Let $K \subseteq \mathbb{R}$ be a cpt. set. Let $f: K \to \mathbb{R}$ be continuous. Then f is uniform continuous.

Corollary: Let K be a cpt. set and $f: K \to \mathbb{R}$ be continuous. Let $J \subseteq K$. Then f is unifor continuous on J, that is, the restriction fn $f|_J : J \to \mathbb{R}$ is uniformly continuous.

Thm: A continuous fn $f: (a,b) \to \mathbb{R}$ is uniformly continuous iff $\lim_{x \to a+} f(x)$ and $\lim_{x \to b-} f(x)$ exists.

Pf: ($\Rightarrow$) Let $f: (a,b) \to \mathbb{R}$ be uniform continuous. We shall show that $\lim_{x \to a+} f(x)$ exists.

To show that it is enough to prove the following:

(i) $\{f(x_n)\}$ is cgt for every seq. $\{x_n\} \subseteq (a,b)$ s.t. $x_n \to a$.

(ii) If $x_n, y_n \in (a,b)$ and $\lim x_n = a = \lim y_n$ then $\lim f(x_n) = \lim f(y_n)$

<u>Proof of (i)</u>

$\{x_n\}$ is cgt

$\Rightarrow$ $\{x_n\}$ is Cauchy

$\Rightarrow$ $\{f(x_n)\}$ is Cauchy (as f is uni. cont.)

$\Rightarrow$ $\{f(x_n)\}$ is cgt.

Proof of (ii): $x_n \to a$ & $y_n \to a$

$\Rightarrow |x_n - y_n| \to 0$

$\Rightarrow |f(x_n) - f(y_n)| \to 0$ as f is uniform conti.

$\Rightarrow \lim f(x_n) = \lim f(y_n)$

(since $\{f(x_n)\}$ and $\{f(y_n)\}$ are cgt).

This proves that

$$\lim_{x \to a+} f(x) \text{ exists}$$

Similarly, one can prove that

$$\lim_{x \to b-} f(x) \text{ exists.}$$

($\Leftarrow$) Let us assume both the limit exists. We define

$$g : [a, b] \to \mathbb{R}$$

by

$$g(x) = \begin{cases} f(x) & x \in (a,b) \\ \lim_{x\to a+} f(x) & x = a \\ \lim_{x\to b-} f(x) & x = b. \end{cases}$$

Then $g : [a,b] \to \mathbb{R}$ is conti.
Therefore, g is uniformly conti.
Hence f is unif. conts.
This completes the proof.

<u>Thm:</u> (Continuous extension)
Let $f : (a,b) \to \mathbb{R}$ be unifm. conti. Then there exists a <u>unique</u> <u>continuous</u> fn $g : [a,b] \to \mathbb{R}$ st.

$$g(x) = f(x) \quad \forall \; x \in (a,b).$$

Proof: Exercise.

Lipschitz Continuous: A fn $f: J \to \mathbb{R}$ is Lipschitz continuous if $\exists\ c > 0$ s.t.

$$|f(x) - f(y)| \leq c\,|x - y| \quad \forall x, y \in J.$$

Ex. Show that a Lipschitz continuous fn $f: J \to \mathbb{R}$ is uniform continuous.