SETS IN $\mathbb{R}$.

Defn: (a) NEIGHBOURHOOD (nbd): Let $p \in \mathbb{R}$ and $\delta > 0$.
Define
$$N_\delta(p) = (p-\delta, p+\delta)$$
$$= \{x \in \mathbb{R} : |x-p| < \delta\}.$$

The set $N_\delta(p)$ is called a δ-neighbourhood of p.

A nbd of p is a set $N_\delta(p)$ for some $\delta > 0$.

EXAMPLE: The set $(-1, 1)$ is a nbd of 0, with $\delta = 1$.

(b) INTERIOR PT: Let $E \subseteq \mathbb{R}$. A pt. $p \in \mathbb{R}$ is an interior pt. of E, if $\exists\ \delta > 0$ s.t.
$$N_\delta(p) \subseteq E.$$

NOTE: If p is an interior pt. then $p \in E$.

(c) $\text{Int}(E) =: \{x \in \mathbb{R} : x$ is an interior pt of $E\}$.

EXAMPLE: (i) $E = [-1, 1]$, then any $x \in (-1, 1)$ is an interior pt of E. But 1 and -1 are NOT interior pts of E. $\text{Int}(E) = (-1, 1)$

(ii) $E = (-1, 1) \cap \mathbb{Q}$. Then no pt. of E is an interior pt.

$$\text{Int}(E) = \phi.$$

(d) OPEN SET: A subset E of $\mathbb{R}$ is said to be open if every pt of E is an interior pt.

Exercise: Show that a set is open iff
$$\mathrm{Int}(E) = E.$$

EXAMPLE: (i) any open interval is an open set.

(ii) ϕ, $\mathbb{R}$ are open sets.

(iii) $[a,b]$ is not open set.

(iv) $\mathbb{Q}$ is NOT an open set.

Defn (LIMIT PT.) A pt. $p \in \mathbb{R}$ is said to be a limit pt. of E, if
$$N'_\delta(p) \cap E \neq \phi \quad \forall\, \delta > 0.$$
Where $N'_\delta(p) = N_\delta(p) \setminus \{p\}$.

Notation: $E' =: \{x \in \mathbb{R} : x \text{ is a limit pt. of } E\}$.

Exercise: Show that $\text{Int}(E) \subseteq E'$

Example: 1. $E = \{ \frac{1}{n} : n \in \mathbb{N} \}$.

Then 0 is a limit pt of E.

In fact, $E' = \{0\}$.

2. $E = (0,1)$, $E' = [0,1]$.

3. $E = \mathbb{N}$, then $E' = \emptyset$.

4. $E = \mathbb{Q}$, $E' = \mathbb{R}$.

Defn (CLOSED SET) A set E is closed if every limit pt. of E is a pt. of E, i.e.

$$E' \subseteq E.$$

Thm: Let p be a limit pt of E. Then every nbd of p contains infinitely many elements of E.

Pf: Suppose N be a nbd of p s.t. $N \cap E$ is a finite set. Let

$$N \cap E = \{q_1, \dots, q_k\}. \text{ for some } k \in \mathbb{N}.$$

Let $\delta = \frac{1}{2} \min \{ |p - q_i| : i = 1, 2, \dots, k \}$.

Then $N_\delta(p) \cap E = \phi$

Therefore, p is not a limit pt. of E. This proves the thm.

Ex: A finite set has no limit pt.

PROPERTIES OF OPEN SETS:

Thm: (i) union of open sets is open.

(ii) let $\{A_\alpha\}_{\alpha \in I}$ be a collection of open sets. Then $\bigcup_{\alpha \in I} A_\alpha$ is open.

Pf: let $x \in \bigcup_{\alpha \in I} A_\alpha$. We will show that x is an interior pt. of $\bigcup_{\alpha \in I} A_\alpha$.

Now $x \in \bigcup_{\alpha \in I} A_\alpha$

$\Rightarrow x \in A_\beta$ for some $\beta \in I$.

$\Rightarrow x$ is an interior pt. of A_β.
(As A_β is an open set).

$\Rightarrow \exists\, \delta > 0$, s.t.
$(x-\delta, x+\delta) \subseteq A_\beta$

$\Rightarrow (x-\delta, x+\delta) \subset A_\beta \subseteq \bigcup_{\alpha \in I} A_\alpha$

$\Rightarrow$ x is an interior pt. of $\bigcup_{\alpha \in I} A_\alpha$.

Since x is arbitrary, every pt. of $\bigcup_\alpha A_\alpha$ is an interior pt. of $\bigcup_\alpha A_\alpha$.

Hence $\bigcup_\alpha A_\alpha$ is an open set.

<u>Thm</u>: Finite intersection of open sets is open.

<u>Pf</u>: Let $A_1, \ldots, A_k$ be open sets.

<u>claim</u>: $\bigcap_{i=1}^{k} A_i$ is open.

Let $x \in \bigcap_{i=1}^{k} A_i$

$\Rightarrow$ $x \in A_i$ $\forall\ i = 1, 2, \ldots, k$

$\Rightarrow$ $\exists\ \delta_i > 0$ s.t.

$$(x-\delta_i, x+\delta_i) \subseteq A_i$$

Let $\delta = \min\{\delta_1, \dots, \delta_k\}$. Then $\delta > 0$ and

$$(x-\delta, x+\delta) \subseteq (x-\delta_i, x+\delta_i) \subseteq A_i \quad \forall i.$$

$$\Rightarrow (x-\delta, x+\delta) \subseteq A_i \quad \forall i$$

$$\Rightarrow (x-\delta, x+\delta) \subseteq \bigcap_{i=1}^{k} A_i$$

$\Rightarrow$ x is an interior pt. of $\bigcap_{i=1}^{k} A_i$

Since x is arbitrary, every pt. of $\bigcap_{i=1}^{k} A_i$ is an interior pt. Hence proved.

EXAMPLE: Let $A_i = \left(-\frac{1}{i}, \frac{1}{i}\right)$ $i \in \mathbb{N}$.

Then each A_i is an open set.

Now $\bigcap_{i=1}^{\infty} A_i = \{0\}$, which is NOT open set.

PROPERTIES OF CLOSED SETS.

Thm: (i) Finite union of closed sets closed

(ii) intersection of closed sets is closed.

Pf: Ex.

Ex: Give an example of closed sets A_i, $i \in \mathbb{N}$, such that $\bigcup_{i \in \mathbb{N}} A_i$ is NOT a closed sets.

Thm: A set E is open iff E^c is closed set.

Pf: Let E be an open set. We will show that E^c is closed set. Let x be a limit pt. of E^c.

$\Rightarrow$ for each $\epsilon > 0$,

$$(x-\epsilon, x+\epsilon) \cap E^c \neq \phi$$

$\Rightarrow \quad \forall \epsilon > 0, \; (x-\epsilon, x+\epsilon) \not\subseteq E$

$\Rightarrow$ x is not an interior pt. of E.

$\Rightarrow$ $x \notin E$ (As E is open)

$\Rightarrow$ $x \in E^c$.

Hence E^c is a closed set.

Let E^c be a closed set. We will show that E is an open set.

Let $x \in E$. Then

$x \notin E^c$

$\Rightarrow$ x is not a limit pt of E^c (As E^c is closed set)

$\Rightarrow \exists \delta > 0$, s.t.

$(x-\delta, x+\delta) \cap E^c = \phi$

$\Rightarrow \quad (x-\delta, x+\delta) \subseteq E.$

$\Rightarrow$ x is an interior pt. of E

$\Rightarrow$ E is open set.

This completes the proof.

Thm: A set E is closed if and only if for every sequence $\{x_n\} \subseteq E$, $x \in \mathbb{R}$, and $x_n \to x$ then $x \in E$.

Pf: ($\Rightarrow$) Let E be a closed set. Assume that $\{x_n\}$ be a sequence s.t.

(i) $x_n \in E \quad \forall n$

(ii) $x_n \longrightarrow x$, for some $x \in \mathbb{R}$.

Claim. $x \in E$.

Let $\epsilon > 0$. Since $x_n \to x$ as $n \to \infty$, there exists a $N_\epsilon \in \mathbb{N}$, s.t.

$$x_n \in (x-\epsilon, x+\epsilon) \quad \forall \ n \geq N_\epsilon.$$

$\Rightarrow \quad N'_\epsilon(x) \cap E \neq \phi.$

$\Rightarrow$ x is a limit pt. of E.

$\Rightarrow$ $x \in E$ as x is closed.

($\Leftarrow$) We assume the sequential property of E. We will show that E is closed set.

Let $x \in E'$.

<u>Claim:</u> $x \in E$.

Since $x \in E'$, for each $n \in \mathbb{N}$

$\exists$ $x_n \in E$ s.t.

$$x - \frac{1}{n} < x_n < x + \frac{1}{n}.$$

Thus $\{x_n\}$ satisfy the following properties:

(i) $x_n \in E \quad \forall n \in \mathbb{N}$.

(ii) $x_n \to x$ by Sandwich.

Thus by the hypothesis,

$$x \in E.$$

This proves that E is closed.