Tutorial 02, 22/08/2024

Problem 1: Starting from D Alembert principle, derive the equation of motion in a curvilinear coordinate $\{q_i\}$

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta r_i$$

$\vec{F}_i$ is force field $\frac{d\vec{p}_i}{dt}$ is rate of change of momentum

$$p_i = m v_i = \frac{d r_i}{dt} \qquad r_i \text{ is position vector}$$

$$\frac{d r_i}{dt} = \sum_k \frac{\partial r_i}{\partial q_k} \dot{q}_k + \frac{\partial r_i}{\partial t}$$

q_k is generalized coordinate

The virtual displacement δr_i is

$$\delta r_i = \sum_j \frac{\partial r_i}{\partial q_j} \delta q_j \quad \text{because for}$$

virtual work $\frac{\partial r_i}{\partial t} = 0$

Work done $\sum_i F_i \cdot \delta r_i = \sum_{i,j} F_i \cdot \frac{\partial r_i}{\partial q_j} \delta q_j$

$$= \sum_j Q_j \delta q_j$$

Q_j is generalized force

$$Q_j = \sum_i F_i \frac{\partial r_i}{\partial q_j}$$

Now to the rate of change of momentum

$$\sum_i (\dot{p}_i \cdot \delta r_i) = \sum_i m_i \ddot{r}_i \cdot \delta r_i$$

$$= \sum_{i,j} m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} \delta q_j$$

Putting all together

$$\sum_j \left(Q_j - \sum_i m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} \right) \delta q_j = 0$$

we need to simplify the term

$$\sum_i \left(m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} \right) \text{ in terms of } q_i$$

$$\sum_i m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} = \sum_i \left[m_i \frac{d}{dt}\left(\dot{r}_i \cdot \frac{\partial r_i}{\partial q_j} \right) - m_i \dot{r}_i \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j} \right) \right]$$

Let us look at $\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right)$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial r_i}{\partial q_j}\right) = \frac{\partial \dot{r}_i}{\partial q_j} = \frac{\partial v_i}{\partial q_j}$$

This is $\sum_k \frac{\partial^2 r_i}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 r_i}{\partial q_j \partial t}$

Now $v_i = \sum_k \frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t}$

$$\frac{\partial v_i}{\partial \dot{q}_j} = \sum_k \frac{\partial r_i}{\partial q_k}\frac{\partial \dot{q}_k}{\partial \dot{q}_j} + \frac{\partial}{\partial t}\left(\frac{\partial r_i}{\partial q_j}\right)$$

$$\boxed{\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}}$$

$$\sum_i m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_j} = \sum m_i \frac{d}{dt}\left(v_i \cdot \frac{\partial v_i}{\partial \dot{q}_j}\right) - m_i v_i \cdot \frac{\partial v_i}{\partial q_j}$$

Substituting back

$$\sum_j \left[Q_j - \sum_i \frac{d}{dt}\left(m_i v_i \cdot \frac{\partial v_i}{\partial \dot{q}_j}\right) + \sum_i m_i v_i \frac{\partial v_i}{\partial q_j}\right]\delta q_j$$

$$\sum_j \left[Q_j - \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right) + \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right)\right]\delta q_j = 0$$

$$\sum_j \left[Q_j - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j}\right]\delta q_j = 0$$

This is the equation of motion;

for arbitrary δq_i

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j}$$