Expression for Kinetic energy in an arbitrary generalize coordinate

The expression for K.E in an inertial frame is given by

$$T = \frac{1}{2}\sum_i m_i v_i^2$$

where $v_i = \frac{dr_i}{dt}$ is velocity

Now we do a transformation from

$$\{r_i\} \longrightarrow \{q_j\}$$

we have already derived the expression for velocity under the transformation

$$v_i \equiv \frac{dr_i}{dt} = \sum_k \frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t}$$

Now the Kinetic energy is

$$T = \frac{1}{2}\sum_i m_i\left(\sum_j \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}\right)^2$$

$$= \sum_i \frac{1}{2} m_i \left(\sum_j \frac{\partial r_i}{\partial q_j} \dot{q}_j + \frac{\partial r_i}{\partial t} \right) \left(\sum_k \frac{\partial r_i}{\partial q_k} \dot{q}_k + \frac{\partial r_i}{\partial t} \right)$$

$$= \sum_i \frac{m_i}{2} \left[\left(\frac{\partial r_i}{\partial t} \right)^2 + \frac{\partial r_i}{\partial t} \left(\sum_j \frac{\partial r_i}{\partial q_j} \dot{q}_j + \sum_k \frac{\partial r_i}{\partial q_k} \dot{q}_k \right) + \sum_{jk} \frac{\partial r_i}{\partial q_j} \frac{\partial r_i}{\partial q_k} \dot{q}_j \dot{q}_k \right]$$

$$= \sum_i \frac{m_i}{2} \left[\left(\frac{\partial r_i}{\partial t} \right)^2 + 2 \frac{\partial r_i}{\partial t} \sum_j \frac{\partial r_i}{\partial q_j} \dot{q}_j + \sum_{jk} \frac{\partial r_i}{\partial q_j} \frac{\partial r_i}{\partial q_k} \dot{q}_j \dot{q}_k \right]$$

$$T = M_0 + \sum_j M_j \dot{q}_j + \frac{1}{2} \sum_{j,k} M_{jk} \dot{q}_j \dot{q}_k$$

Here $M_0 = \sum_i \frac{1}{2} m_i \left(\frac{\partial r_i}{\partial t} \right)^2$

$$M_j = \sum_i m_i \frac{\partial r_i}{\partial t} \cdot \frac{\partial r_i}{\partial q_j}$$

$$M_{jk} = \sum_i m_i \frac{\partial r_i}{\partial q_j} \frac{\partial r_i}{\partial q_k}$$

$$T = T_0 + T_1 + T_2$$

$T_0 = M_0$ is function of q, and t only

$T_1 = \sum_j M_j \dot{q}_j$ is function of q, t but linear function of $\dot{q}$

$$T_2 = \frac{1}{2} \sum_{j,k}' M_{jk} \dot{q}_j \dot{q}_k$$

function q, t and homogeneous function of $\dot{q}$, of order 2