# Fourier Series

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Jean-Baptiste Joseph Fourier was a pioneering French mathematician and physicist [1]. He is best known for developing the Fourier series and the Fourier transform, which have become fundamental tools in mathematical analysis and scientific applications. Fourier initially developed this formalism to investigate heat propagation, leading to significant advancements in the understanding of heat conduction. The Fourier series allows for the decomposition of periodic functions into sums of simpler sinusoidal components, making it invaluable for analyzing and understanding waveforms. The Fourier transform extends these concepts to non-periodic functions and is crucial in various fields, such as signal processing, heat transfer, and quantum mechanics. In this study, we delve into the properties of Fourier series and transforms, exploring their theoretical foundations as well as their diverse applications across different scientific and engineering domains.

## 1 Fourier Series

Any square-integrable function function, $f(x)$, in the interval $[-\pi, \pi]$ can be expressed as,

$$f(x) = \sum_{n=0}^{\infty} a_n \cos nx \; + \sum_{n=1}^{\infty} b_n \sin nx \,, \tag{1}$$

where the constants $a_n$ and $b_n$ are given by,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\, \cos nx\, dx \qquad n \geq 0 \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\, \sin nx\, dx \qquad n > 0 \end{aligned} \tag{2}$$

This representation of function $f(x)$ is called the Fourier series.

Example: Express the function

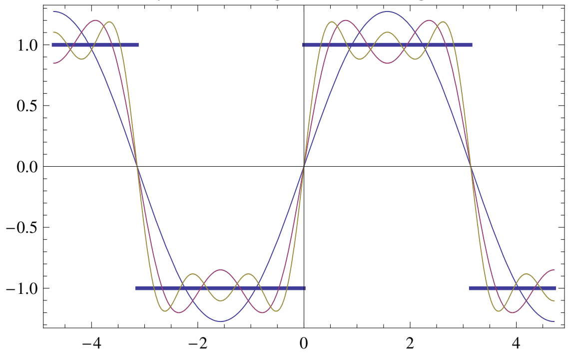
$$f(x) = \begin{cases} +1 & (x < 0) \\ -1 & (x \geq 0) \end{cases} \tag{3}$$

In the Fourier series.

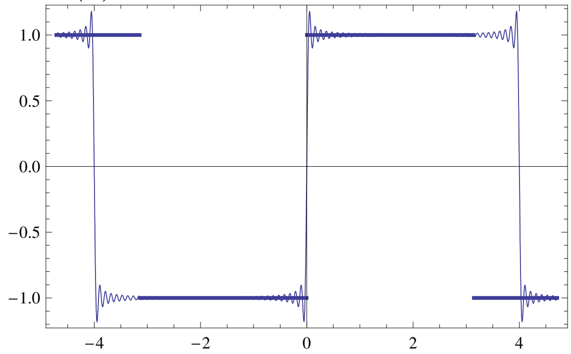
The coefficients $a_n$and $b_n$ can be determined by the eq. (2)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} (-1)\, dx + \frac{1}{\pi}\int_{0}^{+\pi} (+1)\, dx = 0,$$

Figure 1: Fourier series expansion of Square wave in the interval $[-\pi, \pi]$ given by the eq. (4). The plot (a) shows the first few terms of the series, as a large number of terms are taken, the series slowly converges to the given function as shown in plot(b)




(a)




(b)

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{0}(-\cos nx)\,dx + \frac{1}{\pi}\int_{0}^{+\pi}(+\cos nx)\,dx = 0, \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{0}(-\sin nx)\,dx + \frac{1}{\pi}\int_{0}^{+\pi}(+\cos nx)\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi}\sin nx\,dx \quad = \begin{cases} \frac{4}{n\pi} & (n = \text{odd}) \\ 0 & (n = \text{even}) \end{cases}
\end{aligned}
$$

The function $f(x)$ can be written as,

$$
f(x) = \frac{4}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{1}{n}\sin nx \tag{4}
$$

The plot of the series, by taking the different numbers to terms, is shown in figure 1.

EXAMPLE : Express the function

$$
f(x) = x \qquad -\pi < x < \pi \tag{5}
$$

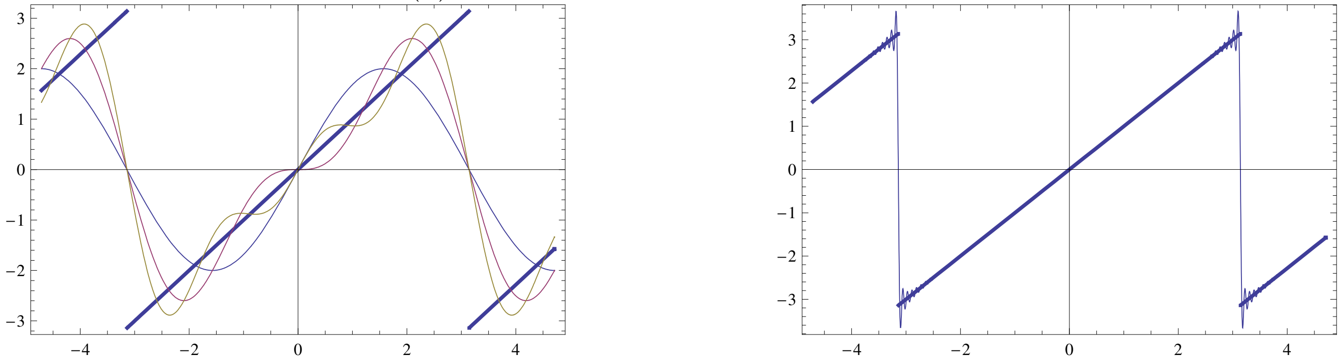
$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0 \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,\cos nx\,dx = 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\,\sin nx\,dx = -\frac{2\pi}{n}\cos n\pi \\
&= \frac{2}{n}(-1)^{n+1}
\end{aligned}
$$

and the function can be written as:

$$
f(x) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}\,. \tag{6}
$$

The plots of various terms of this series are shown in figure 2

Figure 2: Fourier series expansion of the function $f(x) = x$ in the interval $[-\pi, \pi]$. The Plot (a) corresponds to the only few terms from the series given by the eq. (6), while a large number of terms are taken in the plot(b)




(a) (b)

## 1.1 Extended domain

The Fourier series can be extended to arbitrary interval $[-\pi, \pi]$ to $[-L, L]$ by scaling. Any $L^2$ function in ter interval $[-L, L]$ can be expressed in terms of Fourier series as given in eq. (1):

$$f(x) = \sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \, .$$

where the coefficients $a_n$ and $b_n$ are given by,

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{+L} f(x) \, \cos\left(\frac{n\pi x}{L}\right) \, dx \, , \\ b_n &= \frac{1}{L}\int_{-L}^{+L} f(x) \, \sin\left(\frac{n\pi x}{L}\right) \, dx \end{aligned} \tag{7}$$

An even function $f(x)$ is such that,

$$f(x) = f(-x) \, .$$

For example, $\cos x$ is an even function. An odd function is such that

$$f(x) = -f(x) \, .$$

The function $\sin x$ is an example of an odd function. An even function can be expanded in the Fourier cosine series, and coefficients $b_n$ will be zero. An odd function can be expanded in terms of Fourier sine series and the coefficients $a_n$ will be identically zero.

Example: Express the function

$$f(x) = \begin{cases} x & 0 < x < L \\ -x & -L < x < 0 \end{cases} \tag{8}$$

Since the function is an even function, the coefficients $b_n$ are identically zero, after simple computation, we get:

$$f(x) = \frac{L}{2} - \frac{4}{L} \sum_{n=\text{odd}}^{\infty} \frac{\cos\left(\frac{n\pi x}{L}\right)}{n^2} \, , \tag{9}$$

and the plots are given in the figure 3.

Figure 3: Plot of function given in eq. (8) given by the Fourier series expansion is given by eq. (9). Because the function is continuous, it matches well with series expansion.

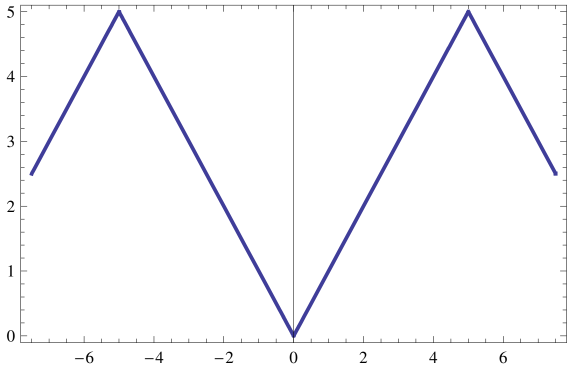
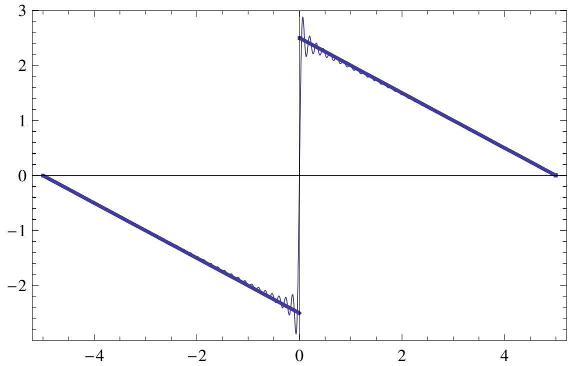


Figure 4: Plot of the odd function given in the eq. (10) and its Fourier series are given by the eq. (11)



EXAMPLE: Give Fourier series expansion for the function

$$f(x) = \begin{cases} \frac{1}{2}(L-x) & (0 < x < L) \\ -\frac{1}{2}(L+x) & -L < x < 0 \end{cases} \tag{10}$$

The Fourier series expansion is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right) \tag{11}$$

The plots are given in the figure 4,

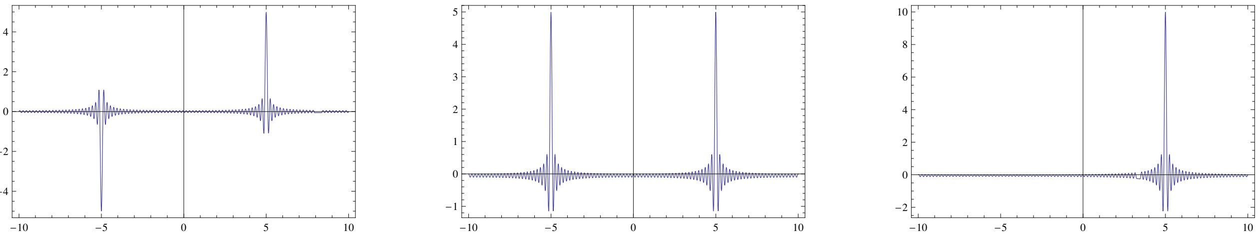
EXAMPLE: The Dirac delta function $\delta(x - x')$is such that,

$$f(x') = \int_a^b f(x)\,\delta(x-x')\,dx = \begin{cases} f(x') & x' \in [a,b] \\ 0 & x' \notin [a,b] \end{cases} \tag{12}$$

can be expressed in terms of Fourier series as follows

$$\begin{aligned} \delta(x-x') &= \left(\frac{1}{L}\right) \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x'}{L}\right), \\ \delta(x-x') &= \left(\frac{1}{L}\right) \sum_{n=0}^{\infty} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x'}{L}\right), \end{aligned} \tag{13}$$

where $L = b - a$. The coefficients $\{a_n, b_n\}$ can easily determined from the eq. (12) and eq. (7).



(a) (b) (c)

Figure 5: Plots of Dirac delta function as (a) Fourier sine series, (b) Fourier cosine series and (c) adding both sine and cosine series given in eq. (13). Here we have taken $x' = 5$ and $L = 10$ in the interval $[-L,\, L]$.

### 1.2 Fourier transform

As limit $L \to \infty$, the Fourier series becomes the Fourier transform.

## References

[1] https://en.wikipedia.org/wiki/Joseph_Fourier