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Linear Algebra I (MA2102)

Assignment 4

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Exercise 1. (10 points)
A linear operator $\pi : V \to V$ is called a *projection* on V if $\pi^2 = \pi$. Suppose $\pi_1, \pi_2, \dots, \pi_r$ are projections on V satisfying

(a) $\pi + \pi_2 + \cdots + \pi_r = I$, and
(b) $\pi_i \pi_j = 0$ for $i \neq j$.

Prove that $V = U_1 \oplus U_2 \oplus \cdots \oplus U_r$, where $U_i = \mathrm{Im}(\pi_i)$.

Exercise 2. (10 points)
For an eigenvalue λ of a linear operator, the *eigenspace* corresponding to λ is defined as the span of all the eigenvectors of λ. Let T be a linear operator on finite dimensional vector space V such that $T^2 = I$ but $T \neq I$, where I is the identity operator. Let $E(\lambda)$ denotes the eigenspace corresponding to the eigenvalue λ of T. Prove that $V = E(1) \oplus E(-1)$.

Exercise 3. (10 points)
A linear operator T is called *nilpotent* if there exists $m \in \mathbb{N}$ such that $T^m = 0$. Let $\mathbb{P}_n[x]$ be the vector space of all the polynomials of degree at most n. Let $D : \mathbb{P}_n[x] \to P_n[x]$ be the linear operator defined as $D(p(x)) = \frac{d}{dx}p(x)$.

(1) Prove that D is a nilpotent operator.
(2) Find the matrix of D with respect to the basis $\{1, 1+x, 1+x+x^2, \dots, 1+x+\cdots+x^n\}$.

Exercise 4. (10 points)

(1) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $U : \mathbb{R}^m \to \mathbb{R}^n$ are linear transformations where $n > m$. Then, UT is not invertible.
(2) Find two linear operators T and U on $\mathbb{R}^2$ such that $TU = 0$ but $UT \neq 0$.

Exercise 5. (15 points)
The *trace* of a matrix is the sum of its diagonal entries. Let W be the vector space of $n \times n$ real matrices and let W_0 be the subspace of W spanned by the matrices M of the form $M = AB - BA$, where A and B are arbitrary $n \times n$ real matrices. Prove that W_0 contains all the $n \times n$ real matrices whose traces are equal to zero.

Maximum score: 50 points. You may answer as many questions as you wish. If the sum of your total score exceeds 49, you shall get the maximum score. Please mention your name, roll no. and **group** in your answersheet. Please submit your answersheet by 11:59 p.m. on **28.09.2023** in the DMS mailbox for MA2102, which is designated with your group name.