Problems on Sequence and Series

1. Show that a sequence is bounded iff every subsequence of $\{x_n\}$ has a convergent subsequence.

2. Show that a monotone sequence has a convergent subsequence, then it is convergent.

3. Prove the convergence of the following sequences the sequence $\{x_n\}$ be defined by

$$\begin{cases} x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) & n = 1, 2, ... \\ x_1 = 2. \end{cases} \tag{1}$$

$$\begin{cases} x_{n+1} = \sqrt{2x_n} & n = 1, 2, ... \\ x_1 = 1. \end{cases} \tag{2}$$

$$\begin{cases} x_{n+1} = \sqrt{2 + \sqrt{x_n}} & n = 1, 2, ... \\ x_1 = \sqrt{2}. \end{cases} \tag{3}$$

4. Determine whether the following series is convergent or divergent :

1. $\sum \frac{2^n n!}{n^n}$
2. $\sum \frac{n!}{n^n}$
3. $\sum \frac{1}{2^n - n}$
4. $\sum \frac{1}{(n+2)\log(n+2)}$
5. $\sum \frac{\log n}{n^p}$ where $p > 0$
6. $\sum \frac{1}{(\log n)^p}$ where $p > 0$
7. $\sum (2^{n+(-1)^n})^{-1}$
8. $\sum (-1)^{n+1} \frac{1}{n}$
9. $\sum (-1)^{n+1} \frac{1}{\sqrt{n}}$

5. Let $\{a_n\}$ be a sequence such that $a_n \to 0$, then show that there exists a subsequence $\{a_{n_k}\}$ such that $\sum_k a_{n_k}$ is convergent.

6. Let $\sum a_n^2$ be convergent. Show that $\sum a_n^3$ is convergent.

7. If $\sum a_n^2$ and $\sum b_n^2$ are convergent, then show that $\sum a_n b_n$ is convergent.