The 1D Ising Model

Before we present the Hamiltonian of the 1D Ising system, it is quite important to note the discretizations of this model. We first have a spatial discretization, where we discretize the magnets into a grid of sites. Then comes our second discretization, the spin discretization, where we discretize the spins into two values, $+1$ and -1. The second discretization had little motivation unless you take into account the quantum nature of spins, we shall classically regard this property to be just some perk of the model.

The 1D Ising Model is defined by the Hamiltonian

$$H = -J\sum_{j=1}^{N}\sigma_j\sigma_{j+1} - H\sum_{j=1}^{N}\sigma_j$$

where $\sigma_j = \pm 1$ is the spin at site j. J is the coupling constant here. $J > 0$ describes a ferromagnetic model here, because we want to lower energy so to lower it, neighbours spin should be aligned. $J < 0$ described an antiferromagnetic model, where we want to lower energy by having neighbouring spins anti aligned. $J = 0$ is fairly uninteresting model where we do not care about the alignment of neighbouring spins. H is the external magnetic field, which decides how each spin affects the energy of the system. If $H > 0$, then we want to align spins in the positive direction, if $H < 0$ then we want to align spins in the negative direction. If $H = 0$, then there is no external field applied. In an attempt to make the system translationally invariant, we will assume periodic boundary conditions, i.e., for a system with N sites $\sigma_{N+1} = \sigma_1$. This translational invariance is important since the physics should not depend on where the analysis of the model is done.

The partition function of the 1D Ising Model is given by

$$Z_N = \sum_{\{\sigma\}}\exp(-\beta H) = \sum_{\{\sigma\}}\exp\left(K\sum_{j=1}^{N}\sigma_j\sigma_{j+1} + h\sum_{j=1}^{N}\sigma_j\right)$$

where $\beta = \frac{1}{k_B T}$, k_B is the Boltzmann constant and T is the temperature and $K = \beta J$ and $h = \beta H$ and N is the number of lattice. The sum is over all possible spin configurations. Let us define

$$V(\sigma, \sigma') = \exp\left(K\sigma\sigma' + \frac{1}{2}h(\sigma + \sigma')\right)$$

which simplifies our partition function to

$$Z_N = \sum_{\{\sigma\}} V(\sigma_1, \sigma_2)V(\sigma_2, \sigma_3)...V(\sigma_N, \sigma_1)$$

To achieve this simplifications other forms of V exist, but this form is chosen to make V symmetric with respect to the arguments $V(\sigma, \sigma') = V(\sigma', \sigma)$. This choice will have some consequences later. Now one might question as to why this transformation is done. This transformation helps us in analytically computing the partition function. This helps us define and use *transfer matrices*. The transfer matrix is defined as

$$\mathbb{V} = \begin{pmatrix} V(+,+) & V(+,-) \\ V(-,+) & V(-,-) \end{pmatrix} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{-K-h} \end{pmatrix}$$

These transfer matrices are so called cause they determine the probability with which we get the next spin given our current spin. So we can see that $V(\sigma, \sigma') = \langle\sigma|V|\sigma'\rangle$. The partition function is then given as

$$Z_N = \sum_{\sigma_1} ... \sum_{\sigma_N} \langle \sigma_1 | \mathbb{V} | \sigma_2 \rangle ... \langle \sigma_N | \mathbb{V} | \sigma_1 \rangle$$

Note that the σ_j are independent of each other. Let us denote $|+\rangle = \binom{1}{0}$ and $|-\rangle = \binom{0}{1}$. From this we get that $\sum_{\sigma_i=0}^{1} |\sigma_i\rangle\langle\sigma_i| = \mathbb{I}$. This is our completeness relation with which we are quite familiar in QM. Using this, the partition function can be simplified,

$$\begin{aligned} Z_N &= \sum_{\sigma_1} ... \sum_{\sigma_N} \langle \sigma_1 | V | \sigma_2 \rangle ... \langle \sigma_N | V | \sigma_1 \rangle \\ &= \sum_{\sigma_1} \langle \sigma_1 | \mathbb{V} \left(\sum_{\sigma_2} |\sigma_2\rangle\langle\sigma_2| \right) \mathbb{V} \left(\sum_{\sigma_3} |\sigma_3\rangle\langle\sigma_3| \right) ... \left(\sum_{\sigma_N} |\sigma_N\rangle\langle\sigma_N| \right) \mathbb{V} | \sigma_1 \rangle \\ &= \sum_{\sigma_1} \langle \sigma_1 | \mathbb{V}^N | \sigma_1 \rangle = \mathrm{Tr}[\mathbb{V}^N] \end{aligned}$$

Right here we realise why we chose V to have a symmetric form. As a consequence of the symmetric form V, the matrix $\mathbb{V}$ turns out to be a symmetric matrix. Using the spectral theorem for finite dimensional vector spaces hoffman1971linear , we can say that the matrix $\mathbb{V}$ is diagonalizable, and can represented in the form $\mathbb{V} = U^\dagger \Lambda U$, where U is a unitary matrix and Λ is a diagonal matrix with the eigenvalue as the diagonal entries. Then we obtain $\mathrm{Tr}[\mathbb{V}^N] = \mathrm{Tr}\left[U^\dagger \Lambda^N U\right] = \mathrm{Tr}[\Lambda^N] = \lambda_1^N + \lambda_2^N$. We can also say something about the degeneracy of the eigenspectrum.