Vector Spaces

Two points

(I) Add^n or subtraction closed.

(II) Multiplication by scalar

Group $\longrightarrow$ Binary operation
↳ map from
$S \times S \rightarrow S$
$(s', s') \longrightarrow f(s', s') \in S$

f: A law of composition for a binary operation on $\mathbb{R}$.

$\boxed{\text{Group}}$

A set S with a binary operator $\circledast$ s.t. if its satisfies the following axioms.

(I) Associativity

$$a \circledast (b \circledast c) = (a \circledast b) \circledast c = a \circledast b \circledast c$$

(II) Identity

$$\exists\, i_G \in G \quad \text{s.t.} \quad \forall\, a \in G$$

$$a \circledast i_G = i_G \circledast a = a$$

(III) Inverse

$\forall\, a \in G \quad \exists\, b \in G$

$$a \circledast b = b \circledast a = i_G$$

Ex: $(\mathbb{Z}, +), (\mathbb{Q}, +)$ are subgroups of $(\mathbb{R}, +)$

A Subgroup of $(G, *)$ is a set $H \subseteq G$ s.t.

$\circledast : H \times H \longrightarrow H$

$i_G \in H$

Inverse $\in H$.

$\Rightarrow (H, *)$ is a group and $H \leq G$

Ex Every grp can have only one $i_G \in G$

Ans. Suppose i_{G_1} and $i_{G_2} \in G$ and $\circledast$

~~s.t. $\forall a$~~

~~$a \circledast i_{G_1} = a \circledast i_{G_2}$~~

Thus $\forall a \in G$

$a \circledast i_{G_1} = a = i_{G_2} \circledast a,$

Thus

$i_{G_2} \circledast i_{G_1} = i_{G_2}$

and

$i_{G_2} \circledast i_{G_2} = i_{G_1}$

Thus $i_{G_2} = i_{G_1}$

$\boxed{\text{Ex } \exists \text{ a unique Inverse}}$

Semigroup

↳ A set S s.t. "Associativity" is satisfied e.g. $(\mathbb{N}, +)$.

Monoid

A semigroup with an identity element

e.g. $(\mathbb{N}, \times)$.

Def^n

Given an $n \times n$ matrix $\in M_{\mathbb{C}}$ (complex) if $\exists$ a non zero vector v and $\lambda \in \mathbb{C}$ s.t. $Av = \lambda v$, then λ is called an eigen value of A and v is an ~~eigenvalue of~~ eigenvector of A.

$Av = \lambda v$

$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ $x = \begin{pmatrix} -3 \\ +3 \\ -1 \end{pmatrix}$ $-3+6-1$

$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -3 \\ 3 \\ -1 \end{pmatrix} = ($

Hw $\begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ 0 & & & -1 & 2 \end{pmatrix}_{n \times n}$

$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

$2^n +$

Unique inverse

Let there be 2 inverses.

b_1 and b_2.

$i_G = a \circledast b_1$, $a \circledast b_2 = i_G$

$= b_1 \circledast a$

$(b_1 \circledast a) \circledast b_2 = b_1$

$\Rightarrow b_2 = b_1$

$3 \quad 2 \quad 0$

$1, 0, 0, 0$ $k-2$

$3^A(k-2)$

$-2|A_{k-2}$

Cont'd class $(\mathbb{Z}_5, \oplus (\text{mod } 5))$

$(\mathbb{Z}/5\mathbb{Z}, \oplus)$

$\mathbb{Z}/5\mathbb{Z} \rightarrow$ pronounced $\mathbb{Z}$ mod $5\mathbb{Z}$

means take addition modulo 5.

Defn $(G, *)$ s.t. $\forall a, b \in G$

$a * b = b * a$.

we say $*$ is a commutative operator and G is a commutative or an abelian group.

Ring $(R, +, *)$

- $(R, +)$ is an abelian group $\rightarrow$ Identity 1_R
- $(R, *)$ is a monoid.
- $*$ is distributive over add^n.
  - $a * (b + c) = a * b + a * c$
  - and, $(b + c) * a = b * a + c * a$.

$\#(\mathbb{Z}/n\mathbb{Z} \setminus \{0\}, *)$ is a group iff n is a prime.

Commutative ring

A ring $(R, +, *)$ when $*$ is commutative.

Field

A commutative ring F s.t. $F^* := (F \setminus 0, *)$ is also an abelian grp.

Vector Space It is an abelian group $(V, +)$ s.t.

$\exists$ an operation called scalar multiplication from $F \times V \longrightarrow V$

$c \in F$

s.t. $cv = w \in V$

Vector Space

(i) $(V, +)$ is an abelian group.

(ii) Scalar multiplication.

$F \times V \longrightarrow V$ $c \in F, v \in V$

$cv = w \in V$.

Associativity

$a(bV) = (ab)V$

distributivity

$(a + b)V = aV + bV$.

Identity

$1_F V = V \quad \forall v \in V$.

$(\mathbb{Z}, +)$ is a group

What are the subgroups of $\mathbb{Z}$?

$n\mathbb{Z}, n \in \mathbb{Z}$

Proposition

Any subgroup of $(\mathbb{Z}, +)$ is of the form $(n\mathbb{Z}, +)$

Pf. Let $G \subseteq \mathbb{Z}$. Then if $G = \{0\}$ (Trivial)

Let $n \in G$ be the smallest (+) integer in G.

Case 1 If every $n \in G$ is divisible by n, then $G \subseteq n\mathbb{Z}$. — (i)

But $\forall k \in \mathbb{Z}$ we have

$$\text{sgn}(k)\underbrace{(n+n+n\ldots n)}_{|k| \text{ times}} \in G \text{ is } nk \in G$$

$nk \in G$ — (ii)

$n\mathbb{Z} \subseteq G$

$n\mathbb{Z} = G$ ← (i), (ii)

Case 2

$\exists\, l \in G$ s.t. l is not divisible by n.

then

$l \equiv t \pmod n$ $\quad t < n$

$l = qn + t$

$qn \in G$ Also $l \in G$.

$\Rightarrow l - qn = t \in G$. (group).

But $t < n$ where n is the smallest (+)ve integer in G contradiction

Thus G ...

$\mathbb{Z}(\in)$ $a \in \mathbb{Z}$

$b \in \mathbb{Z}$ $\quad a\mathbb{Z} + b\mathbb{Z}$ is a group.

So $\exists$ some n s.t.

$a\mathbb{Z} + b\mathbb{Z} = n\mathbb{Z} \rightarrow \gcd(a, b)$

n divides both a and b

$ar + bs = n$

$n' | a, b$

$a = n'a'$

$b = n'b'$

$ar + bs = n'a'r + n'b's$

$n = n'(a'r + b's)$

$n = n'$

$\Rightarrow n$ is the gcd of a & b.

$\{a_1, \ldots, a_n\} \in \mathbb{Z}$ and let $d = \gcd(a_1, \ldots, a_n)$

$a_1\mathbb{Z} + a_2\mathbb{Z} + \ldots + a_n\mathbb{Z} = d\mathbb{Z}$

$\Rightarrow a_1 r_1 + a_2 r_2 \ldots a_n r_n \equiv$

$\exists\, r_1 \ldots r_m \in \mathbb{Z}$ s.t. [Lagrange interpolation / Bezout's lemma.]

$a_1 r_1 + a_2 r_2 + \ldots + a_m r_m = d$

$\mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/p\mathbb{Z} \setminus \{\bar{0}\}$

$\{\bar{1}, \bar{2}, \ldots, \overline{p-1}\}$ $\quad m, p$ are coprime.

Let $m \in \mathbb{N}$ is s.t. $m < p$

$\Rightarrow \exists\, r, s \in \mathbb{Z}$ s.t. $mr + ps = 1$

$\Rightarrow \bar{m}\bar{r} \equiv 1 \pmod p$

RISHAB'S NICE PROOF

$\{\bar{1}, \bar{2}, \ldots, \overline{p-1}\}$

Take $x \in \mathbb{Z}$ and look at $\bar{x}$ is the residue class $x \pmod p$ s.t. $\bar{x} \neq \bar{0}$

$\{x, x^2, x^3, \ldots\}$

map given by reduction mod p.

Map cannot be injective.

Thus atleast two elements should go to one element in $\mathbb{Z}/p\mathbb{Z}$

$x^m \equiv x^n \pmod p$

$x^n(x^{m-n} - 1) \equiv 0 \pmod p$ $\quad m-n = p-1$

$x^m - x^n \equiv 0 \pmod p$

$x^n(x^{m-n} - 1) \equiv 0 \pmod p$

$x^{m-n} - 1 \equiv 0 \pmod p$

$x^{m-n} \equiv 1 \pmod p$

x^{m-n-1} is the multiplicative inverse of x.

$(2+3)+4$

Vector Space $V - \mathbb{F}$

$(V, +)$ is an abellian group

- Scalar multiplication.

- $\mathbb{R}^n$ is a vector space.
- $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$

$\mathbb{F}_p^n$

Set of all function $\mathbb{R} \to \mathbb{R}$

Set of all continuous functions $\mathbb{R} \to \mathbb{R}$

" " " diff functions $\mathbb{R} \to \mathbb{R}$

Set of all polynomials ≤ 4 $\mathbb{R} \to \mathbb{R}$

Subspace

Def-1 $(V, +)$ given vector space.

$(W, +) \subseteq (V, +)$

subgroup of this abelian group.

S.t. W is closed under scalar multiplication. by $n \in \mathbb{F}$

$\Rightarrow W$ is a subspace of V

Def-2 Given a vector space V over a field $\mathbb{F}$ if $W \subseteq$ is s.t. $\forall w, w' \in W$

$aw + bw' \in W \quad \forall a, b \in \mathbb{F}$

then we call W a subspace of V.

$\phi : V \longrightarrow V'$ (Linear map, homomorphism, morphism)

$\phi(av_1 + bv_2) = a\phi(v_1) + b\phi(v_2)$

Vector Spaces

$(V,+)$ ab grp

Scalar multiplication → elements of F

$F \times V \xrightarrow{\text{scalar}} V$

Subspacing vector spaces.

Linear maps / morphisms / homomorphism of vector space over a field F

$\phi : V \longrightarrow V'$

$$\phi(av_1 + bv_1) = a\phi(v_1) + b\phi(v_2) \quad v_1, v_2 \in V,\ a, b \in F.$$

isomorphism

A bijective linear map between vector spaces

$\mathbb{R}^2 \simeq \mathbb{C} = \{a + ib \mid a, b \in \mathbb{R}\}$

Span

$S = \{v_1, \ldots, v_n\}$

Span $S := \{c_1v_1 + c_2v_2 + \ldots c_nv_n\} \mid c \in F\}$

Proposition

Let V be vector spaces and $W \subseteq V$ a subspace. Let $S \subseteq V$ be a subset s.t. $S \subseteq W$. then span $S \subseteq W$.

Pf Since W is a subspace of V and since $v_1, v_2 \in W$

$\Rightarrow c_1v_1 + c_2v_2 \in W$

$\Rightarrow 1(c_1v_1 + c_2v_2) + c_3v_3 \in W$

Proof by induction

$c_1v_1 + c_2v_2 + \cdots c_nv_n \in W$

$\boxtimes$

Prop 2 Given a subset $S \subseteq V$. Span S is a subspace of V

Pf $w, w' \in$ Span S.

$\Rightarrow \exists\ a_1, \ldots, a_n \in F$

& $b_1, \ldots, b_n \in F$

s.t.

$a_1v_1 + \ldots + a_nv_n \in W$

$b_1v_1 + \ldots + b_nv_n \in W'$

a, b $a, b \in F$. Then $aw + bw'$

$= a(a_1v_1 + \ldots + a_nv_n) + \ldots (a_1a_1 + bb_1)v_1 + \ldots$ (and.) $\boxtimes$

$V = \mathbb{R}^2$

Terminologies

If $S \subseteq V$ is s.t.

Span $S = W$ → subspace gen by S.

A spanning set / generating set for W.

Linear ~~Terminate~~ Relation

Let $v_1, \dots, v_n \in W$

If $\exists\ c_1, \dots, c_n \in F$ s.t. not all the c_i are 0 and if

$$\underbrace{c_1 v_1 + \dots + c_n v_n = 0}$$

we call this a linear relation among the given set of vectors.

Linear Independence

Let $v_1, v_2 \dots v_n \in V$. If $\nexists$ any linear relation among $v_1, \dots, v_n$ we say the set of vectors are linear independent.

Basis A linearly independent ordered set which spans a vector space

For example, let V be a vector spanned by $\mathbb{B} = (v_1, \dots v_n)$, where B is a lin. independent set of vectors.

Standard basis in ~~$\mathbb{F}$~~ $(\mathbb{F})^n$

$$\mathbb{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\} \text{ is the standard basis}$$

$\Downarrow$

$\mathbb{B}$ are the columns of the identity matrix of $\mathbb{F}^{n \times n}$.

V: A vector space over $\mathbb{F}$

convention: span $\phi = \{0\}$ since 0 is linearly dependent

Proposition

Let V be a vector space with a basis $\mathbb{B} := (v_1, \dots, v_n)$. Then every vector in V has a "unique" expression of the form

$$c_1 v_1 + \dots + c_n v_n, \quad c_i \in \mathbb{F}.$$

~~If~~ suppose $\exists$ a vector $v \in V$ which has two expression. where $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \neq \begin{pmatrix} c_1' \\ \vdots \\ c_n' \end{pmatrix}$:

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$
$$= c_1' v_1 + c_2' v_2 + \dots + c_n' v_n$$

$$v - v = 0$$
$$= (c_1 - c_1') v_1 + (c_2 - c_2') v_2 \dots + (c_n - c_n') v_n$$
$$= 0$$

where not all $(c_i - c_i')$ are 0.

which is a contradiction to linear independence of $v_1, \dots, v_n = 0$

Prop -2

Let V be a vector space and let $S \subseteq V$ be an ordered set of vectors. Let $v \in V$ be any vector. Define $S' = (S, v) := (v_1, \ldots, v_n, v)$ where $S := (v_1, \ldots, v_n)$

Then $v \in \text{span}\, S \iff \text{span}\, S' = \text{span}\, S$.

Pf

$v \in \text{Span}\, S$

$S' = (S, v)$

$S' = \text{span}\, v$

$S' \subseteq \text{Span}\, S$.

"S is a vector."

[recall. If ~~S~~ subspace s.t. $S' \subseteq \text{span}\, S$. Then $\text{span}\, S \subseteq \text{span}$

Also by defn. $\text{span}\, S \subseteq \text{span}\, S'$

$\text{span}\, S' = \text{span}\, S$.

$(\Leftarrow)$

$\text{span}\, S' = \text{span}\, S$.

then $v \in \text{span}\, S'$ [defn]

$\Rightarrow v \in \text{span}\, S$ $\boxed{}$

Prop

Let $L = (v_1, \ldots v_n)$ be a lin independent set and let $v \in A$ by any vector. Then the set $L' := (L, v)$ is a lin independent set

$\iff v \notin \text{span}\, L$

contrapositive

$v \in \text{span}\, L \iff$ set L' is linearly dependent.

Pf:=

If $v \in \text{Span}\, L$.

the $\exists c_1, \ldots, c_n \in \mathbb{F}$

s.t.

$c_1 v_1 + c_2 v_2 \ldots + c_n v_n = v$

$\Rightarrow c_1 v_1 + c_2 v_2 + \ldots + c_n v_n - v = 0$

Since coeff of v is (-1)

Linear relation.

Thus L' is linearly dependent.

$(\Leftarrow)$ L' is linearly ~~in~~dependent

$\Rightarrow \exists c_1, \ldots c_{n+1} \in \mathbb{F}$ s.t. $\exists c_i \neq 0$,

$c_1 v_1 + \ldots + c_n v_n + c_{n+1} v = 0$

Now $c_{n+1} = 0$ or $c_{n+1} \neq 0$

Let $c_{n+1} = 0$

$\Rightarrow c_1 v_1 + \ldots c_n v_n = 0$

Then $c_1 = \ldots = c_n = 0$ as L is lin ...ndent. ~~Thus~~ Since $\exists c_i \neq 0$

$$v = \frac{-c_1}{c_{n+1}} v_1 + \left(\frac{-c_2}{c_{n+1}}\right) v_2 + \dots + \left(-\frac{c_n}{c_{n+1}}\right) v_n$$

~~Since~~ Thus $v \in \operatorname{span} L$

Finite dimensional vector spaces

Every vector ~~will have~~ space with a finite spanning set is a finite dimensional

$$S = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \right)$$

$\operatorname{span} S = \mathbb{R}^2$

Prop^n 4 Let V be a finite dim vector space and let S be a finite and [ordered] set that spans V. Then S contains a basis of V.

optional.

Pf

Let $S = \{v_1, \dots, v_n\}$

If S is lin independent we are done,

If not, then $\exists$ a linear relation.

$c_1 v_1 + \dots + c_n v_n = 0$ s.t. not all c_i are 0,

say $c_k \neq 0$.

$$\Rightarrow v_k = -\frac{c_1}{c_k} v_1 \dots - \frac{c_{k-1}}{c_k} v_{k-1} - \frac{c_{k+1}}{c_k} v_{k+1} \dots - \frac{c_n}{c_k} v_n \in \operatorname{span}(v_1, \dots v_{n-1})$$

This process can recursively till we get a linearly independent set, which is the required basis V.

Thus by proposition 3 we have that the recursion will terminate since $v \notin \operatorname{span} L$.

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recall proposition

Let L be a lin independent set in vector space V. Then $\forall$ $v \in V$,

$$v \notin \operatorname{span} L \iff L' = (L, v) \text{ is a lin independent set.}$$

Props

Let V be a finite dimensional vector space. Then any lin ind subset $L \subseteq V$ is extendable to a basis of V.

Pf $\because$ V is finite dimensional, $\exists$ a finite set S s.t. $\operatorname{span} S = V$. If $S \subseteq L$, then

$$V = \operatorname{span} S \subseteq \operatorname{span} L \subseteq V$$

$\Rightarrow \operatorname{span} L = V$, $\Rightarrow$ ~~span~~

Hence L is a basis of V.

Otw $\operatorname{span} S \subseteq \operatorname{span} L$

$\Rightarrow \exists\, v \in S$ s.t. $v \notin \operatorname{span} L$

Why can we write?

$v \in S$ in general.

$S \subseteq W \Rightarrow \text{span } S \subseteq W$

if span $S \not\subseteq W \Rightarrow S \not\subseteq W = \text{span } L$

we know span $S \not\subseteq$ span S.

$\exists\, v \in S$ s.t. $v \notin \text{span } L$.

$\Rightarrow L' := (L, v)$ is a lin independent set

Continue this process till $v \in \text{span } L$.

Since $v \in S$, and S is finite cardinality, then the recursion will terminate.

Why will it terminate in a basis?

$\Rightarrow$ After the recursion if $\exists\, v \notin \text{span } L$ then we can continue the recursion.

Since for the previous if $v \in \text{span } L$ and we cannot find a new vector then we have found a lin ind spanning set.

Let S & L be two finite subsets of a vector space V s.t. span $S = V$ & L is linearly independent.

Then $|L| \leq |S|$.

Pf Let $|S| = m$ & $|L| = n$ and let

$S := (v_1, \ldots, v_m)$ & $L := (w_1, \ldots, w_n)$

A linear combination of the elements of L is of the form $c_1 w_1 + \ldots + c_n w_n$ $\forall c_i \in \mathbb{F}$.

Since span $S = V \supseteq L$

$\Rightarrow \forall j \in \{1, 2, \ldots, n\}$ we have

$w_j = a_{1j} v_1 + \ldots + a_{mj} v_m$.

$$(v_1 \cdots v_n)\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{m\times n} \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$= (w_1, \ldots, w_n)\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$Ax = 0$ has a non trivial solution if $n > m$

Thus $\exists \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ st. not all $c_1, \ldots, c_n$ are 0.

Thus L is not linearly independent.

$\Rightarrow$ Contradiction

$\Rightarrow m \geq n$

Corollary

Any two basis of the same vector space has the same no. of vectors.

Lm 2

If V is a finite dim vector space and if $S \subseteq V$ is a spanning set for V whereas $L \subseteq V$ is a lin ind set, ...

~~Corollary~~
Definition (dimension)

The no. of vectors in a basis of a finite dimensional vector space is called the dimension of a vector space ~~is~~

Corollary ($n \in \mathbb{N}$)

Every vector space dimension n over a field $\mathbb{F}$ is isomorphic (not canonical) to $\mathbb{F}^n$.
↳ Linear bijective map.

[Not canonical means depends on the basis provided]

Pf

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$$

notation

$$[\mathbb{B}] = \underbrace{(v_1, \dots, v_n)}_{\text{multivector}}.$$

$(\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n)$

$$\phi : \mathbb{F}^n \longrightarrow V$$
$$x \longmapsto [\mathbb{B}]x$$

$$[\mathbb{B}]\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 v_1 + x_2 v_2 \dots + x_n v_n$$

Since $\mathbb{B}$ is a basis every vector in V is uniquely represented by a unique linear combination of the elements of $\mathbb{B}$

⇓

Thus the mapping is injective.

~~Since~~
Linearity of transformation

$$(ax+bx') \longmapsto [\mathbb{B}](ax+bx')$$

$$a(x+bx') = a\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + b\begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}$$

$$[\mathbb{B}](ax+bx') = [\mathbb{B}]\begin{pmatrix} ax_1 \\ a \vdots \\ ax_n \end{pmatrix} + [\mathbb{B}]\begin{pmatrix} bx_1' \\ \vdots \\ bx_n' \end{pmatrix}$$

$$\vdots$$

$$= a[\mathbb{B}]x + b[\mathbb{B}]x' \quad \blacksquare$$

Propositions

Let $W \subseteq V$ (finite dim vector space) be a subspace. Then $\dim W \leq \dim V$ where equality holds $\iff W = V$

Pf

CLAIM

W is finite dimensional.

Let L be a linearly independent subset of W. Then L is also a independent subset of V

$$|L| \leq \dim(V)$$

We can extend L only if $\exists$ a $w \in W$ s.t. $w \notin \operatorname{span}(L)$. Since $\dim V < \infty$, this we can do only finitely many times. (Bound)

$\Rightarrow W$ has to be finite dimensional.

$W \subseteq V$
Let be a basis of W.
Let $\mathbb{B}$ be a basis of W.
~~Then we can extend $\mathbb{B}$ be a basis of V.~~
~~[illegible] extend~~

$\mathcal{B}$ to a basis $\mathcal{B}'$ of V.

$\Rightarrow \dim V = |\mathcal{B}'| > |\mathcal{B}| = \dim W$

otherwise $V = W \to$ don't have to extend.

$\boxtimes$

If $S = (v_1, \dots v_m)$ | If $S' \subseteq$ span S

$S' = (w_1, \dots, w_m)$ | Then $\exists A \in \mathbb{F}^{m \times n}$

$GL_n(\mathbb{R})$ $n \times n$ invertible matrices in $\mathbb{R}$.

General Linear Group under multiplication.

What is the order of $GL_2(\mathbb{F}_p)$

2x2 matrices

$\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$

They have to be invertible for it to form a group. Thus it has to $GL_n(\mathbb{R})$

$SL_n(\mathbb{F})$: Special linear group of order n of $\mathbb{F}$

$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$

$GL_n(\mathbb{R}) \longleftarrow$ preimage of $\mathbb{R} \setminus \{0\}$.

All determinants not equal 0.

$SL_n(\mathbb{F}) \subseteq GL_n(\mathbb{F})$

$n \times n$ matrices with $\det = 1$.

Coordinates of vectors

Let $\mathcal{B}$ be a basis $= (v_1, \dots, v_n)$ of vector space (finite dimensional) V. Then any $v \in V$ has unique expression of the form.

$$v = x_1 v_1 + \dots + x_n v_n$$

and $(x_1, \dots, x_n)$ is the coordinate vector of v.

$S = (w_1, \dots, w_n)$, $S' = (w_1', \dots, w_m') \subseteq V$

s.t. $S' \subseteq \text{Span } S$ Then

$$w_i' = a_{i1} w_1 + \dots + a_{im} w_m$$

$$(w_1 \; w_2 \; \dots \; w_n) A_{m \times n} = (w_1' \dots w_n')$$

Proposition

Let $\mathcal{B}'$ & $\mathcal{B}$ be two bases of a vector space V

The $P \in M_n(\mathbb{F})$ $\to$ All matrixs of $n \times n$ order s.t.

$$[\mathcal{B}'] P = [\mathcal{B}],$$

where $n = \dim(V)$. The matrix is called the matrix of change of basis from $\mathcal{B}$ to $\mathcal{B}'$ as we are expressing $\mathcal{B}$ vector in terms of $\mathcal{B}$

Pf

$\exists P' \in M_n(\mathbb{F})$ s.t.

$[\mathcal{B}] P' = [\mathcal{B}] \xrightarrow{\text{opposite}} \Rightarrow [\mathcal{B}'] P P' = [\mathcal{B}']$

Can only be done if ~~...~~ right P^{-1} exists.

Now take

$$[\mathbb{B}']P = [\mathbb{B}]$$

Now $\exists\ P'$ s.t.

$$[\mathbb{B}]P' = [\mathbb{B}']$$

$$[\mathbb{B}]P'P = [\mathbb{B}]\,\mathbb{I}$$

Thus we have $P'P = \mathbb{I}$.

Thus $PP' = P'P = \mathbb{I}$.

~~...~~ Since $[\mathbb{B}]PP'$ ~~..~~ has a unique linear combination $P'P$ has to be $\mathbb{I}$.

Observe that any invertible matrix $A \in GL_n(\mathbb{F})$ & a vector space V of dim n & a basis $\mathbb{B}$,

$[\mathbb{B}]\,A = \underbrace{[\mathbb{D}]}_{\text{hypervector}}$ also corresponds to a basis of V

Because

$$[\mathbb{D}]A^{-1} = [\mathbb{B}]$$

$\Rightarrow \{\mathbb{B}\} \subseteq$ span $\mathbb{D}$

$\Rightarrow$ span$(\mathbb{B}) \subseteq$ span$(\mathbb{D})$

$\Rightarrow V \subseteq$ span$(\mathbb{D})$

Since ~~$\mathbb{B}$~~ elements of $\mathbb{D}$ are from V,

span$(\mathbb{D}) \subseteq V$

$\Rightarrow$ span$(\mathbb{D}) = V$

Change of coordinates

V: F in dim vector space

$\mathbb{B}, \mathbb{B}'$ bases of V.

$$[\mathbb{B}']P = [\mathbb{B}]$$

↳ change of ~~...~~ basis matrix from $\mathbb{B} \to \mathbb{B}'$

change of coordinates $[X' = PX]$

$$[\mathbb{B}']P = [\mathbb{B}]$$
$$[\mathbb{B}]P' = [\mathbb{B}']$$

$\Rightarrow [\mathbb{B}]PP' = [\mathbb{B}]$ change of basis is invertible

$\Rightarrow PP' = \mathbb{I}$

why invertible?

$v_i = 0v_1 + \dots + 1v_i + \dots + 0v_n$

~~...~~ Since v_i can be expressed through the linear combinations of the basis uniquely. ~~...~~ If v_i had some other expression in $\{0, \dots v_n\}$ then v_i would not be linearly independent. Thus PP' has to be $\mathbb{I}$

Qn If we start with the standard basis of $\mathbb{F}^n$, we take the set of columns of an invertible $n \times n$ matrix, we get a basis of $\mathbb{F}_n$.

$GL_2(\mathbb{F}_p)$ The basis of $\mathbb{F}_p^2$

$\dim \mathbb{F}_p^2 = 2$

$\mathbb{B} = (v_1, v_2)$

$v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$ ← choices
$a \to p$
$b \to p$

$v_2 = \begin{pmatrix} a' \\ b' \end{pmatrix}$ $\Rightarrow p^2 - 1$
↳ -1 because $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ won't be allowed

↳ p^2 again for a', b'

order
$= (p^2-p)(p^2-1)$

$p^2 - p$ ↪ p multiples of v_1

Sum (or Span) of a set of subspaces of a fin dim vector space V.

$W_1, \dots, W_n$ are subspaces of V.

$W_1 + \dots + W_n = \{ w_1 + \dots + w_n \mid w_i \in W_i \}$

Independence of Subspace

$W_1, \dots, W_n$ subspaces of fin dim vector subspace V.

W_i's are independent if $\nexists$ any set of $w_i \in W_i$, w_i s.t.

$w_1 + w_2 + \dots + w_n = 0$

unique representation the ovectors.

If $W = W_1 + \dots + W_n$ (Direct Sum Defn)
and if $W_1, \dots, W_n$ are independent, then we write $W = W_1 \oplus W_2 \oplus \dots \oplus W_n$ and say this W is a 'direct sum' of W_i's.

CLAIM

If $W = W_1 \oplus \dots \oplus W_n$ then for $\exists$ a unique sum of the form $\omega_1 + \omega_2 + \dots + \omega_n$ that equals ω.

Proof by contradiction

$\omega = \omega_1 + \dots + \omega_n = \omega_1' + \dots + \omega_n'$ where not all ω_i and ω_i's are the same.

$$\Rightarrow \underbrace{(\omega_1 - \omega_1')}_{\in W_1} + \underbrace{(\omega_2 - \omega_2')}_{\in W_2} + \dots + \underbrace{(\omega_n - \omega_n')}_{\in W_n} = 0$$

That contradicts the independence of $W_1, W_2 \dots W_n$. [Proved] $\blacksquare$

Prop^n

i) Every W_i is independent. [Trivial]

ii) W_1 & W_2 are independent subspaces of a finite dimensional vector space $V \iff W_1 \cap W_2 = \{0\}$.

Pf ($\Rightarrow$) Intersection $\neq \{0\}$ (contradiction).
Assume
Let $\omega \neq 0$ and $\omega \in W_1 \cap W_2$.
as $\omega \in W_1 \cap W_2$ $\Rightarrow -\omega \in W_1 \cap W_2$

$$\Rightarrow \underset{W_1}{\underset{\Uparrow}{(\omega)}} + \underset{W_2}{\underset{\Uparrow}{(-\omega)}} = 0$$

Thus they are not linearly independent.

($\Leftarrow$)

Suppose W_1 and W_2 are not independent.

$\Rightarrow \exists\ w_1, w_2$ s.t. $w_1 \neq 0, w_2 \neq 0$

$$\begin{matrix} w_1 & w_2 \\ \cap & \cap \\ W_1 & W_2 \end{matrix}$$

$w_1 + w_2 = 0$

$w_1 = -w_2 \in W_1$

$w_2 = -w_1 \in W_2$

since $-w_1 \in W_2$, $w_1 \in W_2$.

$w_1 \overset{0}{\neq} \in W_1 \cap W_2$

Thus contradiction that Intersection is $\{0\}$

Prop^n i) Let $B_1, \ldots, B_n$ be bases of $W_1, \ldots, W_n$ and let $W = W_1 \oplus \ldots \oplus W_n$

$\Rightarrow B = (B_1, \ldots, B_n)$ a basis of W.

ii) If $W = W_1 + \ldots + W_n$, then.

$\dim(W) \leq \dim W_1 + \ldots + \dim W_n$

[If $W = W_1 \oplus \ldots \oplus W_n \Leftrightarrow$

$\dim(W) = \dim(W_1) + \ldots + \dim(W_n)$]

Proposition-1

V is a finite dimensional vector space.

Given W, a subspace of V $\exists W'$. Another subspace V

$$V = W \oplus W'$$

If start with a basis of B of W. $(w_1, \ldots, w_r)$ and extend it to a basis of V by adjoining $(v_1, \ldots, v_{n-r})$.

Then $B' = (w_1, \ldots, w_r, v_1, \ldots, v_{n-r})$ is a basis of V.

$W' := \text{Span}(v_1, \ldots, v_{n-r})$

$W + W' = V$

$W \cap W' = \{0\}$

If not $\exists\ v \neq 0 \in W \cap W'$

$\Rightarrow v = c_1 w_1 + \ldots + c_r w_r = c_1' v_1 + \ldots + c_{n-r}' v_{n-r}$

$\Rightarrow c_1 w_1 + \ldots + c_r w_r - c_1' v_1 - \ldots - c_{n-r}' v_{n-r} = 0$

$\Rightarrow c_1 = c_2 = \ldots = -c_1' = -c_2' = \ldots = -c_{n-r}' = 0$ Since they forms a basis of V.

$\Rightarrow W \cap W' = \{0\}$

$\Rightarrow v = 0$ [contradiction]

Prop (A) W: Subspace of V, W_1: Another subspace of V

$W_1 \cap W_2$ is also a subspace of V.

Then $\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$

If $(u_1, \ldots, u_k)$: Basis of $W_1 \cap W_2$

$(u_1, \ldots, u_k, u_1', \ldots, u_m') \rightarrow$ Basis of W_1 (B)

$(u_1, \ldots, u_k, u_1'', \ldots, u_m'') \rightarrow B$ of W_2

By defn W_1+W_2 is spaned by $B \cup B'$.

Claim

$B \cup B'$ is a basis for W_1+W_2

If not, $\exists$ a non trivial linear combination

$$(c_1u_1 + \dots c_\ell u_\ell) + c_1'u_1' \dots + c_m'u_m' + c_1''u_1'' + \dots + c_n''u_n'' = 0$$

$$\Leftrightarrow \underbrace{c_1'u_1' + \dots + c_m'u_m'}_{\in W_1} = \underbrace{-c_1u_1 - \dots - c_\ell u_\ell - c_1''u_1'' - \dots - c_n''u_n}_{\in W_2}$$

Thus $v \in W_1 \cap W_2$

$\Rightarrow \exists\, d_1, \dots, d_\ell \in \mathbb{F}$ s.t.

$v = d_1u_1 + \dots + d_\ell u_\ell = c_1'u_1' + \dots + c_m'u_m'$

$\Rightarrow d_1u_1 + \dots + d_\ell u_\ell - c_1'u_1' - \dots - c_m'u_m' = 0$

$\Rightarrow d_1 = d_\ell = \dots = -c_1' = \dots = -c_m' = 0$

$\Rightarrow v = 0$

$\Rightarrow c_1u_1 + \dots + c_\ell u_\ell + c_1''u_1'' + \dots + c_n''u_n'' = 0$

Since $(u_1, \dots, u_\ell, u_1'', \dots, u_n'')$ is a B.

$\Rightarrow c_i, c_i'' \;\forall i = 0$

Thus $\nexists$ a non trivial linear combination of $B \cup B'$.

Thus $B \cup B'$ is a basis for W_1+W_2

$\dim(W_1+W_2) = n(B \cup B')$

$\dim(W_1) = n(B)$ $\quad \dim(W_1 \cap W_2) = n(B \cap B')$

$\dim(W_2) = n(B')$

1

We know

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$\Rightarrow n(A \cup B) + n(A \cap B) = n(A) + n(B)$

$\Rightarrow \dim(W_1+W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2)$

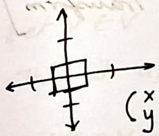
Linear Maps (Transformations)

$\operatorname{Ker} T = \{v \in V \mid Tv = 0\}$ → check that both are subspace.

$\operatorname{Im} T = \{w \in W \mid Tv = w\}$

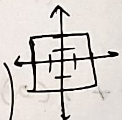
Examples of Linear Transforms.

Enlarging / Resizing Pictures



$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto 2\begin{pmatrix} x \\ y \end{pmatrix}$

↳ Enlarging



$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}$

↳ compression

$V_4 :=$ Real Polynomials of degree ≤ 4.

$V_3 =$ Real polynomials of degree ≤ 3

$V_4 \to V_3$ [Differentiation]

Let D be the differential map.

$$D : V_4 \to V_3$$

$$D(af_1 + bf_2) = aD(f_1) + bD(f_2)$$

Thus D is linear Transform.

! Again Integration is a Linear Transform.

Linear operator (Linear Transform)

$\Downarrow$

Line transform from $V \longrightarrow V$

$$\mathbb{F}^{m\times n} \longrightarrow \mathbb{F}^{m\times n}$$

$$P_{m\times m} A_{m\times n} Q_{n\times n} \longrightarrow (PAQ)_{m\times n}$$

$T: V \longrightarrow W$

$$\boxed{Im(T) = \{ \omega \in W / T(v) = \omega \}}$$

$T(av + bv') = aT(v) + bT(v')$ [As T is a linear Transform]

CLAIM

$T(0) = 0$

$\hookrightarrow T(0+0) = T(0) + T(0)$

$\Rightarrow T(0) = 2T(0)$

$\Rightarrow T(0)(1-2) = 0$

$1 \neq 2 \Rightarrow T(0) = 0$

$\ker(T) = \{ v \in V / T(v) = 0 \}$

> Again a vector space.

~~$av + bv' \in Ker(T)$~~

Let $v \in \ker(T)$

$Tv = 0$ Also $v = 0$ $T(0) = 0$

$0 \in Ker T$

Also if $v \in \ker(T)$, $v' \in \ker(T)$

$T(av) = aT(v) = 0$

$T(av + bv') = aT(v) + bT(v') = 0$

rank$(T) = \dim(Img(T))$

nullity(T), null$(T) = \dim(Ker(T))$

RANK NULLITY THEOREM

Thm: rank(T) + null$(T) = \dim(V)$

Pf: We already proved $\dim(W_1) + \dim(W_2)$

$= \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$

$(u_1, \dots, u_r)$ be a basis of T

Extend $(u_1, \dots, u_r)$ to a basis of V

Let $B := \{(u_1, \dots, u_r, v_1, \dots, v_s)\}$ be a basis of V

Let $v \in V$

Then $\exists\ a_1, \dots, a_r, b_1, \dots, b_s$ s.t.

$v = a_1u_1 + \dots a_r u_r + b_1v_1 + \dots + b_s v_s$

$T(v) = b_1 T(v_1) + \dots + b_s T(v_s)$

Let $T(v_i) := \omega_i$ $\quad T(v) = \omega$

$\omega = b_1\omega_1 + \dots + b_s\omega_s$

$Im(T) = \text{Span}\{\omega_1, \dots, \omega_s\}$

now we show $\omega_1, \dots, \omega_2$

Let $c_1, \dots, c_s \in \mathbb{F}$ s.t.

$c_1\omega_1 + \dots + c_s\omega_s = 0$

Let $v' := c_1v_1 + \dots + c_sv_s$ $\quad T(v') = 0$, $v' \in Ker(T)$

$\exists d_1, \dots, d_r$ s.t. $v' = d_1u_1 + \dots + d_ru_r$

$c_1v_1 + \dots + c_sv_s - d_1u_1 - \dots - d_ru_r = 0$

As $\mathcal{B}$ is a basis for V, we have

$c_i = 0 \quad \forall i$

$-d_j = 0 \quad \forall j$

Thus
$(w_1, \ldots, w_s)$ is a basis of $Im(T)$

$dim(Im(T)) = s$

$dim(Ker(T)) = r$

$dim(V) = r+s$ [As $\mathcal{B}$ is a basis

A map T is injective $\Leftrightarrow$ $Ker(T) = \{0\}$

$T(v_1) = T(v_2) \quad v_1 \neq v_2$

$\Leftrightarrow T(\underbrace{v_1 - v_2}) = 0$

$\Leftrightarrow$ Non triviality implies $v_1 - v_2 \neq 0$

Thus $v_1 \neq v_2 \neq T(v_1) \neq T(v_2)$ As $T(v_1) = T(v_2)$

Corollary

$T: V \rightarrow W$

$dim\, V > dim\, W$

CLAIM

T can't be injective

$\Rightarrow dim\, V = rank(T) + null(T)$

$null(T) = dim\, V - ran(T) > 0$

$rank(T) \leq dim\, W < dim\, V$

$\Rightarrow dim\, V - rank(T) > 0$

$\Leftrightarrow dim\, ker(T) > 0 \Rightarrow$ ...

Corollary

If $T: V \rightarrow W$ and $dim\, W > dim\, V$,

then T cannot be surjective.

Pf Suppose T is surjective.

Implies $Im(T) = W$

$\Leftrightarrow rank(T) = dim(W)$

$rank(T) > dim(V)$

$rank(T) > rank(T) + ker(T) \geq rank(T)$

$\Rightarrow rank(T) > rank(T)$

$\Rightarrow$ Contradiction.

Thus T cannot be surjective. $\square$

Corollary-3

If $T: V \rightarrow W$ and $dim\, W = dim\, V$.

then for any T the following are equivalent

i) T is injective ii) T is surjective.

Pf (i) T is injective

$\Leftrightarrow ker(T) = 0$

CLAIM

$rank(T) = dim\, W$

Pf [Rank - Nullity Theorem]

$rank(T) + ker(T) = dim\, V$

$\Rightarrow rank(T) = dim\, V$

$\Rightarrow rank(T) = dim\, W$ [Proved]

$\square$

CLAIM T is injective

(ii) T is surjective

$\Rightarrow$ rank(T) = dim W

$\Rightarrow$ rank(T) = dim V

[Rank-Nullity Theorem]

$\Rightarrow$ rank(T) = rank(T) + ker(T)

$\Rightarrow$ ker(T) = 0

$\square$

Condition for injectivity and surjectivity

(i) T is injective $\iff$ $\ker(T) = \{0\}$ [Easy PF]

(ii) T is surjective $\iff$ ~~rank dim~~ $\mathrm{Img}(T) = W$ [Defn]

Cor 3 if dim V = dim W

HW/- Proving T is injective or T is subjective.

$T^{-1}: W \to V$ $\quad$ T^{-1} is also bijective

T^{-1} must be linear map

The matrix of a Linear Transform

$T: V \to W$

$\mathbb{D} = (w_1, \dots, w_n)$ a basis

$\mathbb{B} = (v_1, \dots, v_n)$ a basis.

$T(\mathbb{B}) = w_1' + \dots + w_n'$

can be represented as a linear combinations of w [as $\mathbb{D}$ is a basis of W]

$= a_1 w_1 + \dots + a_n w_n$

$[T(\mathbb{B})] = A[\mathbb{D}]$

$\Rightarrow \begin{pmatrix} w_1' \\ w_2' \\ \vdots \\ w_n' \end{pmatrix} = A \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$ or $(w_1' \dots w_n') = (w_1, \dots w_m) A$

$m = \dim(W)$

$n = \dim(V)$

$A \Rightarrow m \times n$ or $n \times m$

A is called the matrix of Linear Transformation w.r.t the basis $\mathbb{B}$ of V & $\mathbb{D}$ of W.

$\hookrightarrow$ [Like the change of basis matrix]

The matrix of Linear Transform

$\hookrightarrow$ change of coordinates

$v \in V$. Then $\exists\, x \in F^n$ $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

s.t.

$v = x_1 v_1 + \dots x_n v_n$ when $[\mathbb{B}] = (v_1, \dots, v_n)$

$\Rightarrow v = [\mathbb{B}] X$

Applying T on bothsides

$T(v) = x_1 T(v_1) + \dots + x_n T(v_n)$

$= \cancel{[\mathbb{D}] A x}$

$= [T(\mathbb{B})] x$

$= [\mathbb{D}] A x$

$x \longmapsto Ax$

Ax is the coordinate vector of $T(v)$ w.r.t. the [...] of W.

Propn $T: V \to W$ is injective

Then for any lin. independent subset $\{v_1, \dots v_n\}$ of V we have $\{T(v_1), \dots, T(v_n)\}$ is linearly independent

Pf Suppose we have $c_1, \dots c_n \in \mathbb{F}$ s.t.

$c_1 T(v_1) + \dots + c_n T(v_n) = 0$

$\Rightarrow T(c_1 v_1 + \dots + c_n v_n) = 0$

$\Rightarrow c_1 v_1 + \dots + c_n v_n = 0$ (as T is injective)

$\Rightarrow c_1 = c_2 = \dots = c_n = 0$

$\Rightarrow \{T(v_1), \dots, T(v_n)\}$ is a lin independent set $\square$

$T: V \to W$

$\mathbb{B} \subseteq V$ is a basis

$\mathbb{D} \subseteq V$ is a basis

$[\mathbb{D}] = [T(\mathbb{B})]$

$\quad \parallel$

$(T(v_1), \dots, T(v_n))$

$\rightarrow$ matrix of linear transform.

$[\mathbb{D}] A = [T(\mathbb{B})]$

HW $T: V \to W$ bijective linear Transform,

$w_1, w_2 \in W \Rightarrow \exists v_1, v_2 \in V$

s.t. $T^{-1}(w_1) = v_1$

$\alpha, \beta \in \mathbb{F}$ & $T^{-1}(w_2) = v_2$

$T^{-1}(\alpha w_1 + \beta w_2)$

$= T^{-1}(T(\alpha v_1 + \beta v_2))$

$= \alpha v_1 + \beta v_2 = \alpha T^{-1}(w_1) + \beta T^{-1}(w_2)$

Propn Given a linear transform $T: V \to W$, $\exists$ basis $\mathbb{B}$ & $\mathbb{D}$ of V & W resp, s.t. the matrix of linear transf of T w.r.t. $\mathbb{B}$ and $\mathbb{D}$ is of the form $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$, $r = \text{rank}(\text{matrix}(T))$

Pf Let $(u_1, \dots, u_k)$ basis of $\ker(T)$

Let's extend it to a basis of V as follows

$(u_1, \dots, u_k, v_1, \dots, v_r) \rightarrow \text{rank}(T) + \text{null}(T) = \dim(V)$

$\Rightarrow \mathbb{B} = (v_1, \dots, v_r, u_1, \dots, u_k)$ is a basis of V.

$T(\mathbb{B}) = (w_1, \dots, w_r, 0, \dots 0)$

Now $(w_1, \dots, w_r)$ are linearly independent

as

$c_1 w_1 + \dots + c_r w_r = 0$

$\Rightarrow T(c_1 v_1 + \dots + c_r v_r) = 0$ Thus $(c_1 v_1 + \dots + c_r v_r) \in \ker T$

$\Rightarrow d_1 u_1 + \dots d_k u_k = 0$

$\Rightarrow d_1 u_1 + \dots + d_k u_k - c_1 w_1 \dots - c_r w_r = 0$

$\Rightarrow d_1 = d_2 = \dots = d_k = -c_1 = \dots = -c_r = 0$

Thus $(w_1, \dots, w_r)$ is lin independent

Let us extend $(w_1, \dots, w_r)$ to a basis of W

$D = (w_1, \dots, w_r, z_1, \dots, z_d)$

$[D] \overset{\text{matrix}}{A} = [T(B)]$

$\Rightarrow (w_1, \dots, w_r, z_1, \dots, z_d) A = (w_1, \dots, w_r, 0, \dots, 0)$

$$A = \left(\begin{array}{ccc|c} 1 & & 0 & \\ 0 & \ddots & & 0 \\ 0 & \dots & 1_r & \\ \hline & 0 & & 0 \end{array}\right)$$ (Proved) ∎

A mx of the linear ~~...~~ transform T

$[D]A = [B]$

$v \in V \;\; \exists\, x \in \mathbb{F}^n$ s.t.

$v = [B]x$

$T(v) = T([B])x$

$= [D]Ax$

$T : V \longrightarrow W$

$A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$

$x \longmapsto Ax$

$[B]x \longmapsto [D]Ax$

$$\begin{tikzcd} V \arrow[r, "T"] & W \\ \mathbb{F}^n \arrow[u] \arrow[r, "A"] & \mathbb{F}^m \arrow[u] \\ x \arrow[r, mapsto] & Ax \end{tikzcd}$$

Cor.

Let $A \in \mathbb{F}^{m \times n}$. Then

$\exists\, P \in GL_n(\mathbb{F})$ & $Q \in GL_m(\mathbb{F})$

s.t. QAP^{-1} is of the shape

$$\left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & 0 \end{array}\right)$$

Proposition

If $A \in \mathbb{F}^{m \times n}$, then $\exists\, P \in GL_n(\mathbb{F})$ and $Q \in GL_m(\mathbb{F})$ s.t.

$QAP^{-1} = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$ for some $r \in \mathbb{N}$

Pf. Let $T : V \longrightarrow W$ be a linear transform

Let $[B]$ be a basis for V and $[D']$ be a basis for W s.t. A' is of the shape $= \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$

We using prev theorem we name $\exists$ exists bases s.t. the matrix of linear transform w.r.t those bases will be of the shape $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$

Then

$[T(B')] = [D']A'$

Let $[B']P = [B] \rightarrow$ where $[B]$ is the standard basis

$[D']Q = [D] \rightarrow$ where $[D]$ is the standard basis.

Let A be the matrix of linear transform be A

$[T(\mathbb{B})] = [\mathbb{D}]A$

$= [\mathbb{D}']QA$

Now

$[\mathbb{B}']P = [\mathbb{B}]$

$T([\mathbb{B}']P) = T([\mathbb{B}])$

$\Downarrow$

Here

$T([\mathbb{B}']P) = T([\mathbb{B}'])P$

since P is a matrix of linear combinations of T.

$T([\mathbb{B}'])P = T([\mathbb{B}])$

$= [\mathbb{D}']QA.$

$T(\mathbb{B}') = [\mathbb{D}']QAP^{-1}$

since A' is of the shape $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$ and A' is unique.

$$A' = QAP^{-1} = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

Elementary matrices

(1) Row addition

$$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & \cdots c & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$$

jth row added i

(2) Row scaling

$$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & c & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix}$$

ith row scaled.

For row reduction we premultiply Elementary matrix

For col reduction we post multiply Elementary matrix

(3) Row permutation

$$\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & \ddots \; 1 \end{pmatrix}$$

$\to$ ith and jth row exchange.

Column space

The column space of a matrix A is the span of the columns of A.

column rank = dim(column space)

Row space

The row space of a matrix A is the span of the rows of A / columns of A^T.

Corollary

An $m \times n$ matrix over a field $\mathbb{F}$ has the same column and row rank.

Pf $QAP = \left(\begin{array}{c|c} I_r & B \\ \hline 0 & B' \end{array}\right)$

$\Rightarrow AP = Q^{-1}\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$

Since Q is invertible, left multiplication by Q defines an isomorphism from $\mathbb{F}^n \longrightarrow \mathbb{F}^m$.

$\Rightarrow$ The $img(A)$ is mapped isomorphically to $img(A')$

under basis $\mathcal{B}$ and $\mathcal{B}'$

$x \longmapsto Ax \longmapsto QAx$ (since Q is invertible)

$colspace(A) \cong colspace(A')$

$\Rightarrow \dim(col(A)) = \dim(col(A'))$

$\Rightarrow$ col rank(A) = col rank(A')

col sp$(A^t) \cong$ col sp(A'^t)

$\Rightarrow$ col rank(A^t) = col rank(A'^t)

$\Rightarrow$ row rank(A') = row rank(A)

col rank(A) = row rank(A)

A' is of the form $\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$

trivially col rank(A') = row rank(A')

Invariant Subspaces

Let $T: V_{(\mathcal{B})} \longrightarrow V_{(\mathcal{B})}$ be a linear operator.

$$[T(\mathcal{B})] = [\mathcal{B}]A$$

The matrix of linear operator T w.r.t to the basis $\mathcal{B}$.

$$\begin{tikzcd} T: V_{(\mathcal{B})} \arrow[r, "A"] \arrow[d, "P"] & V_{(\mathcal{B})} \arrow[d, "P"] \\ T: V_{(\mathcal{B}')} \arrow[r, "A'"] & V_{(\mathcal{B}')} \end{tikzcd}$$

where $A' = PAP^{-1}$

Similar Matrices / Conjugate matrix
A & B are called similar if $\exists$ invertible matrix P s.t. $B = PAP^{-1}$.
<They have the same det>

Thm Let $T: V \to V$ be a linear operator and let the matrix of the lin op T w.r.t a basis of V be A. (Let V be n-dimensional)

i) Then the matrix of T w.r.t any other basis is a matrix similar to A. (by defn)

ii) If $P \in GL_n(\mathbb{F})$ then PAP^{-1} is the matrix of the lin transform w.r.t some basis of V.

Pf

(ii) Given $P \in GL_n(\mathbb{F})$, $\mathcal{B}$ a basis of V.

Define $\mathcal{B}' \subseteq V$ by

$[\mathcal{B}'] = [\mathcal{B}]P^{-1}$ → change of basis matrix is P^{-1}

$T[\mathcal{B}'] = T([\mathcal{B}])P^{-1}$

$T[\mathcal{B}'] = [\mathcal{B}]AP^{-1}$

Also we have $[\mathcal{B}] = [\mathcal{B}']P$

$= [\mathcal{B}']PAP^{-1}$ Thus PAP^{-1}

Let $W \subseteq V$ be a subspace s.t.

$$T(W) = \{T(w) \mid w \in W\}$$

W is called an invariant subspace of T

if $T(W) \subseteq W$

Notes

Let $W \subseteq V$ be an invariant subspace of V.

Let T be a lin op. and let $\mathbb{B}_W (w_1, \ldots, w_n)$ be a basis of W. Let's extend $(w_1, \ldots, w_k, v_1, \ldots)$ to be a basis of V

$$\mathbb{B} := (w_1, \ldots, w_k, v_1, \ldots, v_r)$$

Consider the matrix of T w.r.t $\mathbb{B}$

$$[\mathbb{B}] M = T[\mathbb{B}]$$

$$(w_1, \ldots, w_k, v_1, \ldots, v_r) \left(\begin{array}{c|c} A & B \\ \hline 0 & D \end{array}\right) = \begin{matrix}(T(w_1), \ldots, T(w_k), \\ T(v_1), \ldots, T(v_r))\end{matrix}$$

Since ~~is~~ $\mathbb{B}_W$ ~~is~~ ~~be~~ the basis of an invariant subspaces

$T(w_1), \ldots, T(w_k)$ do not ~~depend~~ depend on $v_1, \ldots, v_r$ as $T(W) \subseteq W$.

If we would write V as the direct sum of invariant subspaces

$$V = W_1 \oplus W_2$$

Then the matrix of linear transfor. w.r.t the Basis $\mathbb{B}$ $\{\underbrace{w_1, \ldots, w_k}_{\text{Basis of } W_1}; \underbrace{w_1', \ldots, w_m'}_{\text{Basis of } W_2}\}$

will be of the form $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$

Suppose W is a one dimensional subspace of V, $W = \{cw \mid c \in \mathbb{F}\}$

$$T(W) \subseteq W$$

$T(cw) = c'w$ i.e $\forall c \in \mathbb{F}, c' \in \mathbb{F}$, $w \in W$

$\Rightarrow T(w) = \lambda w$ for some $\lambda \in \mathbb{F}$

The non zero vectors $w \in V$ s.t. $T(w) = \lambda w$ for some $\lambda \in \mathbb{F}$ are called "eigenvectors" and λ is the corresponding eigenvalue.

Prop^n

Let $w_1, \ldots, w_k$ be eigenvectors of a lin op $T: V \to V$ s.t. the corresponding eigenvalues $\lambda_1, \ldots \lambda_k$ are all distinct. Then $w_1, \ldots w_k$ are lin independent.

Pf: Suppose $\exists\, c_1, \ldots, c_k \in \mathbb{F}$ s.t. not all $c_i = 0$

$$c_1 w_1 + \ldots + c_k w_k = 0$$

By induction

$k = 1$

$c_1 w_1 = 0$ $\quad w_1 \neq 0$ [By def^n] [Trivial]

Let it hold for $k-1$

$c_1 w_1 + \ldots + c_{k-1} w_{k-1} = 0$ [given]

~~$c_1 w_1 + \ldots + c_{k-1} w_{k-1} + c_k w_k$~~ $c_k w_k \neq 0$

Then

$T(c_1 w_1 + \dots c_k w_k) = 0$

$\Rightarrow c_1 \lambda_1 w_1 + \dots c_k \lambda_k w_k = 0$

now multiply

$c_1 w_1 \dots c_n w_n = 0$ by λ_n

$c_1 \lambda_n w_1 + \dots + c_n \lambda_n w_n = 0$

$c_1 \lambda_1 w_1 + \dots + c_n \lambda_n w_n = 0$

$c_1(\lambda_n - \lambda_1) w_1 + \dots + (\lambda_n - \lambda_{n-1}) c_{n-1} w_{n-1} = 0$

Since it holds for $k-1$

then

$c_1 (\lambda_n - \lambda_1) = \dots = (\lambda_n - \lambda_{n-1}) c_{n-1} = 0$

but eigen distinct values of λ

$\Rightarrow c_1 = c_2 = \dots = c_{n-1} = 0$

Let $T: V \to V$ be a linear operator.
$\uparrow$ finite dimensional

Thm The following are equivalent for a basis $\mathcal{B}$ of V.
(i) The max of linear transform is upper triangular w.r.t. $\mathcal{B}$.

(ii) $T(v_j) \in span(v_1, \ldots, v_j)$.

(iii) $\forall j$, $span(v_1, \ldots, v_j)$ is ~~[illegible]~~ invariant under T.

(Pf) $T[B] = [B]A$. $[(i) \Leftrightarrow (ii)]$ $(\Leftarrow)$ (Trivial)

$\Rightarrow T[(v_1, \ldots, v_n)] = (v_1, \ldots, v_n) A$

$\Rightarrow (T(v_1), T(v_2), \ldots, T(v_n)) = (v_1, \ldots, v_n) A$.

$$\Rightarrow (T(v_1), \ldots, T(v_n)) = (v_1, \ldots, v_n) \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{bmatrix}$$

[Given]

$T(v_1) = a_{11} v_1$

$\Rightarrow T(v_2) = a_{12} v_1 + a_{22} v_2$.

$\vdots$

$T(v_j) = a_{1j} v_1 + \cdots + a_{jj} v_j$

$\Rightarrow T(v_j) \in span(v_1, \ldots, v_j)$.

$[(ii) \Rightarrow (iii)]$

~~$T(span(v_j))$~~

$T(span(v_1, \ldots, v_j)) \subseteq span(v_1, \ldots, v_j)$.

We have

$T(v_j) \in span(v_1, \ldots, v_j)$.

$\Rightarrow T(c_1 v_1 + \cdots + c_j v_j) = c_1 T(v_1) + \cdots + c_j T(v_j)$

$\in span(v_1, \ldots, v_j)$.

$\Rightarrow T(span(v_1, \ldots, v_j)) \subseteq span(v_1, \ldots, v_j)$.

Thm Every linear operator over a finite dimensional vector space has a representation by an upper triangular matrix.

Pf The claim is true for $\dim(v) = 1$. Let's assume the claim holds $\forall$ dimension $\leq n \in \mathbb{N}$.

Since V is a complex vector space.

$\Rightarrow T$ has an eigenvalue $\lambda \in \mathbb{C}$

$(T - \lambda I)(V) = U$

U is the range of $(T - \lambda I)$

$\dim(U) < \dim(V)$ as $\ker(T - \lambda I) \neq \{0\}$

$\Rightarrow null(T - \lambda I) > 0$.

CLAIM U is an invariant subspace. $(T - \lambda I)$.

Pf Let $w \in U$

$w = (T - \lambda I) w + \lambda w$

here $w \in U$

$\Rightarrow \lambda w \in W$

and

$(T - \lambda I) w \in U$

$$\Rightarrow (T - \lambda I) w = \underbrace{w - \lambda w}_{\in U}$$

Eigenspace

Defⁿ

Eigenspace associated to an eigenvalue λ of a lin op $T: V \to V$ is the subspace of V spanned by the eigenvectors with the eigenvalues λ.

Note : $E(\lambda) = \ker(T - \lambda I)$

[Geometric multiplicity of λ]

$\dim(E(\lambda))$

[Algebraic multiplicity of λ_i]

is a_i where.

$$C_A(t) := \prod_{i=1}^{n} (t - \lambda_i)^{a_i} \quad , \sum_{i=1}^{n} a_i = \dim(V)$$

Theorem

For any eigenvalue of T, its algebraic multiplicity $\geq$ Geometric multiplicity

Pf: Let $T: V \to V$ be a linear operator with an eigenvalue λ with geo multiplicity k.

Since, $\dim(E(\lambda)) = k$. $\exists$ a basis of E_λ .

$(\bar{w}_1, \ldots, \bar{w}_k)$

Let's extend it to a basis of ~~$\mathbb{R}$~~. V

$\mathcal{B} := (\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k, \bar{v}_1, \ldots, \bar{v}_r)$

$[\mathcal{B}]A = [T(\mathcal{B})]$

$$A = \left[\begin{array}{c|c} \lambda I_k & C \\ \hline 0 & D \end{array}\right]$$

$$C_A(t) = (t - \lambda)^k \, C_D(t)$$

Thus the algebraic multiplicity is atleast k

Dual Spaces

Defⁿ [Linear functionals]

Linear maps from V to its underlying field $\mathbb{F}$.
↓
a finite dim vector space.

$$f: V \longrightarrow \mathbb{F}$$

Defⁿ [Dual Space]

The space of linear functionals over a vector space V and it is denoted by $V^* = \mathcal{L}(V, F)$

(*) $f: V \to \mathbb{F}$

$g: V \to \mathbb{F}$

$(af + bg): V \to \mathbb{F}$

Eg: Like the trace of a matrix.

Defⁿ [Dual Basis]

Given $\mathcal{B} := (\vec{v}_1, \ldots, \vec{v}_n)$ a basis of V. define $\mathcal{B}^*$ as a basis of V^*. $f_i \in \mathcal{B}^*$ by

$$f_i(\vec{v}_j) = \delta_{ij} \quad \text{[Study from H\&K]}$$

(*) Let

$c_1 f_1 + c_2 f_2 + \ldots + c_n f_n = 0$

we just have to show $c_i = 0 \ \forall i$

Let's apply this functional on the basis elements

$(c_1 f_1 + c_2 f_2 + \ldots + c_n f_n)(\vec{v}_j) = 0$

$\Rightarrow c_j f_j(v_j) = 0$

$\Rightarrow c_j = 0 \quad \forall j$

Thus $\{f_1, \ldots, f_n\}$ are linearly independent.

we need to show $\dim(V^*) = \dim(V)$.

we have if V is of dim n then $V \cong \mathbb{F}^n$

consider $f \in V^*$

$$f : V \longrightarrow \mathbb{F}$$
$$(\mathbb{B}) \quad (\mathbb{D}]$$

$$[\mathbb{D}]\, A = [f(\mathbb{B})]$$

say $\mathbb{D} = 1$

$\Rightarrow A = [f(\mathbb{B})]$

$= [f(v_1) \quad f(v_2) \quad \cdots \quad f(v_n)]$

$= [a_1 \quad \ldots \quad a_n]$

$\Rightarrow$ This Row vector uniquely represents f.

Thus V is of dim n, then

$V^* \cong \mathbb{F}^n$, $\dim(V^*) = n$

[Defn] [Dual Basis of B]

$B := (\vec{v}_1, \ldots, \vec{v}_n)$

$B^* = (f_1, \ldots, f_n)$

where $f_i(\vec{v}_j) = \delta_{ij}$ where B^* is called the dual basis of $\mathbb{B}$.

[Defn] [Anhilators]

$S \subseteq V$
(subset)

$$S^\circ := \{ f \in V^* \mid f(s) = 0 \quad \forall s \in S \}$$

Anhilator of S.

[Set of all lin functionals which are sending S to $\{0\}$]

(i) Analogous to $\ker(T)$

$\forall S \subseteq V$, $S^\circ \subseteq V^*$ is a subspace.

Let $f, g \in S^\circ$

$\Rightarrow af + bg \in S^\circ$

also '0' map $\in S^\circ \Rightarrow S^\circ$ is a subspace

Thm [Rank- Nullity]

Let $W \subseteq V$ subspace. Then
$$\dim(W) + \dim(W^\circ) = \dim V$$

Pf Let $W \subseteq V$ be a subspaces. Let $(v_1, \ldots, v_k)$ be a basis of W and extend it to V.

Let $B := \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$.

Let $\mathbb{B}^*$ be a dual of $\mathbb{B}$ s.t.

$\mathbb{B}^* := \{f_1, \ldots, f_n\} \qquad f_i(v_j) = \delta_{ij}$

If $l \in \{k+1, \ldots, n\}$ and $r \in \{1, \ldots, k\}$ then

$f_l(v_r) = 0$

$\Rightarrow f_{k+1}, f_{k+2}, \ldots, f_n \in W^\circ$

and being part of the basis $\mathcal{B}^*$ of V^*, they are lin-independent.

$\Rightarrow \dim W^0 \geq n-k$

Assume $f \in V^*$

$f = \sum_{i=1}^{n} c_i f_i$ where $c_i \in \mathbb{F}$

$\Rightarrow f(\vec{v}_j) = \sum_{i=1}^{n} c_i f_i(\vec{v}_j) = \sum_{i=1}^{n} c_i \delta_{ij} = c_j \ \forall j$

$\Rightarrow f = \sum_{j=1}^{n} f(\vec{v}_j) f_j$

If $f \in W^0$, then

$f(\vec{v}_i) = 0$ where $i \in \{1, \ldots, k\}$

$\Rightarrow f = \sum_{j=k+1}^{n} f(\vec{w}_j) f_j$

$\Rightarrow \dim W^0 = n-k$

$\Rightarrow \dim W + \dim W^0 = \dim V$

From Rank-Nullity, $f: V \to \mathbb{F}$, $f \in V^*$, $f \neq 0$,

$\operatorname{im}(f) = 1$

$\dim \ker(f) = n-1$

⊛ Hyperplane / Hyperspace : $(n-1)$ dim subspace of V

Cor. Every hyperplane of V is the kernel of a linear functional $(f \neq 0)$

Cor. Every d-dim proper subspace of V is the intersection of the kernel of $n-d$ non-zero functionals in V.

Double Dual

Let's consider the linear maps from

$V^* \longrightarrow \mathbb{F}$

This is the dual of V^*, which we denote by V^{**}.

We will show that

. $V = V^{**}$ [natural isomorphism]

For every $\vec{v} \in V$, we define, $L_{\vec{v}} : V^* \longrightarrow \mathbb{F}$

$$L_{\vec{v}}(f) := f(v)$$

Linearity of $L_{\vec{v}}$.

$$L_{\vec{v}}(af + bg) = (af + bg)(\vec{v}) = aL_{\vec{v}}(f) + bL_{\vec{v}}(g)$$

Thm

V and V^{**} are naturally isomorphic.

Pf. Since $\dim V = \dim V^{**}$, we have to show that the linear map $v \longrightarrow L_v$ is injective i.e. $\vec{v} \neq 0$ then $L_{\vec{v}} \neq 0$.

(to show

$\vec{v}_1 \neq \vec{v}_2 \Rightarrow L_{\vec{v}_1} \neq L_{\vec{v}_2}$

$\Rightarrow \vec{v}_s \neq 0 \Rightarrow L_{\vec{v}} \neq 0$

$\vec{v} \neq 0$.

Extend $\vec{v}$ to a basis of V ~~...~~

$\mathcal{B} = (\vec{v}, \vec{v}_2, \ldots, \vec{v}_{n})$

$\vec{w} \in V$ is given by

$$\vec{w} = a\vec{v} + a_2\vec{v} + \ldots + a_n\vec{v}_n \xrightarrow{f} a \in \mathbb{F}$$

$$L_{\vec{v}}(f) = 1 = f(\vec{v}) = 1 \neq 0$$

Cor: Given $L : V^* \longrightarrow \mathbb{F}$ $\exists$ a $\vec{v} \in V$ s.t.

$$L = L_{\vec{v}}$$

Motivation

For any basis $\mathbb{D}$ of V^*, does there exist a basis $\mathcal{B}$ of V s.t

$$\mathbb{D} = \mathcal{B}^*.$$

Pf Let $\mathbb{D} = (f_1, f_2, \ldots, f_n)$ be a basis of V^*. ~~Let~~ Let

$$\mathbb{D}^* = (L_1, \ldots, L_n) \text{ in } V^{**} \text{ s.t}$$

$$L_i(f_j) = \delta_{ij}$$

Under the natural isomorphism, $\exists$ $\vec{v}_i \in V$ s.t.

$$L_i = L_{\vec{v}_i}$$

Claim $\mathcal{B} := (\vec{v}_1, \ldots, \vec{v}_n)$ is a basis of V s.t.

$$\mathbb{D} = \mathcal{B}^*$$

$\Rightarrow$ B is a basis. ~~by rank-nullity theorem~~

As v_i's are lin independent, if they were not lin independent, then L_i's won't be a basis

And n-vectors thus, $(\vec{v_1}, \dots, \vec{v_n})$ is a basis of V.

we just need to show

$$D = B^*$$

$$L_i(f) = f(\vec{v_i}) \quad \forall i \quad (\text{as } L_i = L_{\vec{v_i}})$$

for $f_j \in D$

$$\Rightarrow L_i(f_j) = f_j(\vec{v_i}) = \delta_{ji}$$

$$\Rightarrow D = B^*$$

Propn

Under the identification of V^{**} & V via the evaluation isomorphism, we have,

$$W = (W^\circ)^\circ = W^{\circ\circ}$$

for every subspace $W \subseteq V$

$$W^\circ = \{f \in V^* \mid f(\vec{w}) = \vec{0} \ \forall w \in W\} \subseteq V^*$$

$$W^{\circ\circ} = \{L \in V^{**} \mid L(f) = 0 \ \forall f \in W^\circ\} \subseteq V^{**}$$

$$\dim W + \dim W^\circ = \dim V$$

$$\Rightarrow \dim W^\circ + \dim W^{\circ\circ} = \dim V^*$$

$$\Rightarrow \dim W = \dim W^{\circ\circ}$$

$$\therefore L_{\vec{w}}(W^\circ) = 0 \quad \forall \vec{w} \in W$$

$$W \subseteq W^{\circ\circ} \Rightarrow W = W^{\circ\circ}$$

Lemma: If $f, g \in V^*$ s.t.

$\vec{v} \in V$ if $f(\vec{v}) = 0$, then $g(\vec{v}) = 0$ $\Rightarrow g = cf$.

$\boxed{\text{OR}}$

if $\ker(g) \supseteq \ker(f)$

Pf If f is the 0 map, $f \equiv 0$.

i.e. $f(\vec{v}) = 0 \quad \forall \vec{v} \in V$

$\Rightarrow g(\vec{v}) = 0 \quad \forall \vec{v} \in V$

$\Rightarrow g \equiv 0$

otherwise $\exists \vec{v} \in V$ s.t. $f(\vec{v}) \neq 0$

Let $c := \dfrac{g(\vec{v})}{f(\vec{v})}$

Define

$h : V \longrightarrow F$ by $h := g - cf$

$h|_{\ker(f)} = 0$

as $\ker(f) \subseteq \ker(g)$

$\dim(\ker(f)) = n-1$

$\Rightarrow B$: basis of ker of f.

$\Rightarrow (B, \vec{v})$: A lin independent set of cardinality n.

as $\vec{v} \notin \ker(f)$.

$\Rightarrow (B, \vec{v})$ is a basis of V.

$$h(\vec{v}) = g(\vec{v}) - c\, f(\vec{v})$$

$$= g(\vec{v}) - \frac{g(\vec{v})}{f(\vec{v})} f(\vec{v}) \quad (\text{as } f(\vec{v}) \neq 0)$$

$$= 0$$

$\Rightarrow h \equiv 0$ on V

$\Rightarrow g = cf$

Generalized from previous theorem.

If $f_1, f_2, \dots, f_r, g : V \longrightarrow \mathbb{F}$ s.t.

$$\ker f_1 \cap \ker f_2 \cap \dots \cap \ker f_r \subseteq \ker g$$

$\Rightarrow g = c f_1 + c f_2 + \dots + c_r f_r$ for some $c_1, c_2, \dots, c_r \in \mathbb{F}$

Pf We know that the claim holds for $r=1$.
{From previous theorem}.

Induction hypothesis

The claim holds for $r = k \geq 1$. (We will show that it also holds for $r = k+1$

Let $V' := \ker(f_{k+1})$

$\hookrightarrow \ker(f_1|_{V'}) \cap \ker(f_2|_{V'}) \cap \dots \cap \ker(f_r|_{V'}) \subseteq \ker(g|_{V'})$

$g|_{V'} = c_1 f_1|_{V'} + c_2 f_2|_{V'} + \dots + c_k f_k|_{V'}$

By induction hypothesis we have the prev eqn.

Let h be defined st.

$$h := g - c_1 f_1 - c_2 f_2 - \dots - c_k f_k$$

Let $v \in V'$

$\therefore h(v) = g(v) - c_1 f(v) - \dots - c_k f_k$ as $v \in V'$

$= 0$

$\Rightarrow \ker f_{k+1} \subseteq \ker h$

$\Rightarrow h = c_{k+1} f_{k+1}$ (By base step hypothesis)

$\Rightarrow g = c_1 f_1 + \dots + c_{k+1} f_{k+1}$

Thus we have

$$g = c_1 f_1 + \dots + c_r f_r \quad \forall r \quad \text{by induction.}$$

Bilinear Form

$$\langle , \rangle : V \times V \longrightarrow \mathbb{F}$$

- The bilinear form is linear in two arguments.

$\to \langle av + bv', w \rangle = a \langle \vec{v}, \vec{w} \rangle + b \langle \vec{v}', \vec{w} \rangle$

$\to \langle \vec{v}, a\vec{w} + b\vec{w}' \rangle = a \langle \vec{v}, \vec{w} \rangle + b \langle \vec{v}, \vec{w}' \rangle$

Symmetric Bilinear form

$$\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$$

Positive definite bilinear form

$\langle v, v \rangle > 0 \quad \forall v \in V$ s.t. $v \neq 0$

iff $v = 0$

Positive semi-definite bilinear form

$\langle v, v\rangle \geq 0 \quad \forall \vec{v} \in V.$

Matrix of a bilinear form

Let V be a vector space with basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$. Let $\langle , \rangle : V \times V \to F$ be a bilinear form. Then the matrix of the bilinear form $\langle , \rangle$ wrt to the basis $\mathcal{B}$ is

$$A = \begin{pmatrix} \langle \vec{v}_1, \vec{v}_1\rangle & \langle v_1, v_2\rangle & \cdots & \langle v_1, v_n\rangle \\ \langle \vec{v}_2, \vec{v}_1\rangle & \langle v_2, v_2\rangle & & \vdots \\ \vdots & \vdots & \ddots & \\ \langle \vec{v}_n, \vec{v}_1\rangle & \langle \vec{v}_n, \vec{v}_2\rangle & \cdots & \langle v_n, v_n\rangle \end{pmatrix}$$

$$A_{ij} = \langle \vec{v}_i, \vec{v}_j\rangle$$

Prop If $\vec{v}, \vec{w} \in V$ are s.t. $\vec{v} = [\mathcal{B}]x$ and $\vec{w} = [\mathcal{B}]y$

and if A is the gram matrix of $\langle , \rangle$ w.r.t $\mathcal{B}$ then

$$\langle \vec{v}, \vec{w}\rangle = x^T A y$$

Pf Let $\mathcal{B} := (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$

$v = x_1 v_1 + x_2 v_2 + \dots + x_n v_n$

$w = y_1 v_1 + y_2 v_2 + \dots + y_n v_n$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$\langle \vec{v}, \vec{w}\rangle = \left\langle \sum_{i=1}^n x_i \vec{v}_i , \sum_{j=1}^n y_j \vec{v}_j \right\rangle$$

$$= \sum_{i=1}^n \sum_{j=1}^n x_i \langle \vec{v}_i, \vec{v}_j\rangle y_j$$

$$\Rightarrow \sum_{i=1}^n \sum_{j=1}^n x_i A_{ij} y_j$$

$$= x^T A y$$

Prop^n

If a bilinear form $\langle , \rangle$ is symmetric, then so is its Gram matrix w.r.t any basis and vice-versa.

Pf ($\Leftarrow$) Let A be symmetric.

$$\langle v, w\rangle = \underbrace{x^T A y}_{\text{It's a scalar so } c^T = c} = y^T A^T x = y^T A x = \langle w, v\rangle$$

Thus $\langle , \rangle$ is symmetric

($\Rightarrow$)

$$A_{ij} = \langle v_i, v_j\rangle = \langle v_j, v_i\rangle = A_{ji}$$

$$\Rightarrow A_{ij} = A_{ji} \Rightarrow A = A^T$$

The effect of change of basis on Gram matrix

$[\mathcal{B}'] P = [\mathcal{B}]$

$\vec{v} = [\mathcal{B}]x \qquad \vec{w} = [\mathcal{B}']y'$

$= [\mathcal{B}']Px \qquad \Rightarrow y' = Py$

$= [\mathcal{B}']x' \qquad x' = Px \qquad y' = Py$

If A & A' are Gram matrices which represent the basis form $\langle , \rangle$ w.r.t $\mathcal{B}$ and $\mathcal{B}'$.

$$x^T A y = \langle \vec{v}, \vec{w} \rangle$$
$$= x'^T A' y'$$
$$= (Px)^T A' (Py)$$
$$= x^T \underbrace{P^T A' P}\, y$$

- Let $x = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ i^{th} posⁿ and $y = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ j^{th} position.

Observation

Given a Gram matrix A of a bilinear form w.r.t. some basis $\mathcal{B}$ of V all ~~the~~ the other matrix representation of $\langle , \rangle$ is given by

$$Q^T A Q \text{ for } Q \in GL_n(\mathbb{F})$$
$\hookrightarrow \dim V$

Corollary

All the matrix representations of dot product over $\mathbb{R}^n$ is given by $Q^T Q$ where $Q \in GL_n(\mathbb{R})$

Note: A bilinear form is not linear

$V \longrightarrow \mathbb{F}$

$W: V \times V \longrightarrow \mathbb{F}$

$W \ni (x, y) \longrightarrow x^T y$

$c(x,y) = (cx, cy) \rightarrow c^2 x^T y$

$\boxed{*}$ $\langle , \rangle : V \times V \longrightarrow \mathbb{F}$

Let $W := V \times V$

$\Rightarrow \langle a\vec{v} + b\vec{w} \rangle \neq \langle \vec{v} \rangle + b \langle \vec{w} \rangle$ where $\vec{v}, \vec{w} \in W$

$\Rightarrow \vec{v} = (v_1, v_2)$

and $\vec{w} = (w_1, w_2)$ where $v_1, v_2, w_1, w_2 \in V$.

$$= \langle a\vec{v} + b\vec{w} \rangle$$
$$= \langle a(\vec{v_1}, \vec{v_2}) + b(\vec{w_1}, \vec{w_2}) \rangle$$
$$= \langle a\vec{v_1} + b\vec{w_1}, a\vec{v_2} + b\vec{w_2} \rangle$$
$$\Rightarrow \langle a\vec{v_1}, a\vec{v_2} + b\vec{w_2} \rangle + b \langle \vec{w_1}, a\vec{v_2} + b\vec{w_2} \rangle$$
$$\Rightarrow a^2 \langle \vec{v_1}, \vec{v_2} \rangle + ab \langle \vec{v_1}, \vec{w_2} \rangle + ab \langle \vec{v_2}, \vec{w_1} \rangle + b^2 \langle \vec{w_1}, \vec{w_2} \rangle$$

Skew Symmetric form

$$\langle \vec{v}, \vec{w} \rangle = -\langle \vec{w}, \vec{v} \rangle$$

and

$$\langle \vec{v}, \vec{v} \rangle = -\langle \vec{v}, \vec{v} \rangle$$
$$\Rightarrow \langle \vec{v}, \vec{v} \rangle = 0 \quad \forall \vec{v} \in V$$

Negative definite form

$$\langle \vec{v}, \vec{v} \rangle < 0 \quad \forall \vec{v} \neq 0 \in V$$

Negative semi-definite form

$$\langle \vec{v}, \vec{v} \rangle \leq 0 \quad \forall v \in V$$

Indefinite form

If $\langle , \rangle$ is neither definite or semi-definite

Eg Lorentz form [Inner product in Minkowski Space]

$x, y \in \mathbb{R}^4$

$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4$

$$A = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{bmatrix}$$

Thm [The follo.. are equivalent]

(i) A represents dot product w.r.t some basis of $\mathbb{R}^n$

(ii) $A = Q^T Q$ for some matrix $Q \in GL_n(\mathbb{R})$

(iii) A is symmetric and positive definite.

Pf (i) $\Leftrightarrow$ (ii) already done.

(ii) $\Rightarrow$ (iii)

$A = Q^T Q$

$\Rightarrow A^T = (Q^T Q)^T = Q^T Q = A$

Thus A is symmetric.

Let $x \in \mathbb{R}^n$

$\langle x, x \rangle = x^T A x = x^T Q^T Q x$

$= (Qx)^T Q x$ since $x \in \mathbb{R}^n \Rightarrow Qx \in \mathbb{R}^n \Rightarrow$ Let $y = Qx$

$= y^T y$

Let $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ $= y_1^2 + \dots + y_n^2 \geq 0$

also $y_1^2 + \dots + y_n^2 = 0 \Leftrightarrow y_1 = y_2 = \dots = y_n = 0$

(iii) $\Rightarrow$ (ii) To show,

$\exists$ a basis of $\mathbb{R}^n$ w.r.t which the Gram matrix of the form $\langle x, y \rangle = x^T A y$ would be I

$\Rightarrow$ Thus we need to show that given a symmetric and bilinear form on a real vector space V, $\exists$ an orthonormal basis of V w.r.t to that form.

[Defn] [Orthogonality]

Let $\langle , \rangle : V \times V \to \mathbb{R}$

v is said to be orthogonal to w iff $\langle v, w \rangle = 0$

[Defn] [Orthogonal basis of V]

Let $B = (\vec{v}_1, \dots, \vec{v}_n)$ is an orthogonal basis

$\Leftrightarrow \langle v_i, v_j \rangle = 0 \quad \forall i \neq j$

[Defn] [Orthonormal basis of V]

Let $B = (\vec{v}_1, \dots, \vec{v}_n)$ be an orthonormal basis

$\Leftrightarrow \langle \vec{v}_i, \vec{v}_j \rangle = \delta_{ij} \quad \forall i, j$

Defn [Orthogonal group] O_n

$O_n := \{ P \in GL_n(\mathbb{R}) : P^T P = I \}$

If $P \in O_n$, $\det P = \pm 1$.

$SO_n := \{ P \in O_n : \det P = 1 \}$

Def^n [Unitary Grp] U_n

$U_n := \{P \in GL_n(\mathbb{C}) : (\bar{P})^T P = \mathbb{I}\}$

$(\bar{P})^T \equiv P^* \equiv P^t$ [Conjugate Transpose]

$\det(P^* P) = 1$

$\Rightarrow \det(P^*) \det(P) = 1 \Rightarrow (\det(P))^* \det(P) = 1.$

$\Rightarrow |z|^2 = 1 \Rightarrow z = e^{i\theta}$ [Unit circle]

$SU_n := \{P \in U_n : \det P = 1\}$

Def^n [Hermitian Form]

Let V be a finite dimension vector space on $\mathbb{C}$.

$\langle , \rangle : V \times V \longrightarrow \mathbb{C}$ is hermitian if it satisfies the following.

(i) $\langle a\vec{v}, \vec{w}\rangle = \bar{a}\langle \vec{v}, \vec{w}\rangle$ } conjugate in 1st coord.

(ii) $\langle \vec{v}, b\vec{w}\rangle = b\langle \vec{v}, \vec{w}\rangle$

(iii) $\langle \vec{v}+\vec{v}', \vec{w}\rangle = \langle \vec{v}, \vec{w}\rangle + \langle \vec{v}', \vec{w}\rangle$

(iv) $\langle \vec{v}, \vec{w}+\vec{w}'\rangle = \langle \vec{v}, \vec{w}\rangle + \langle \vec{v}, \vec{w}'\rangle$

(v) $\langle \vec{v}, \vec{w}\rangle = \overline{\langle \vec{w}, \vec{v}\rangle}$

$\langle , \rangle : V \times V \longrightarrow \mathbb{C}$

$\vec{v} \in V$

$\langle \vec{v}, \vec{v}\rangle = \overline{\langle \vec{v}, \vec{v}\rangle}$

Thus $\langle \vec{v}, \vec{v}\rangle = \|\vec{v}\|^2 \in \mathbb{R}$

Def^n (+ve definite Hermitian form)

Let $\langle , \rangle : V \times V \longrightarrow \mathbb{C}$ be a hermitian form.

if $\langle v, v\rangle = \|v\|^2 > 0 \ \forall v \in V$

→ Standard hermitian form on $\mathbb{C}^n$

$x, y \in \mathbb{C}^n$

$\langle x, y\rangle = x^* y$

$= \bar{x}_1 y_1 + \bar{x}_2 y_2 + \dots + \bar{x}_n y_n$

Its Gram matrix w.r.t the standard basis is $\mathbb{I}$.

$\boxed{\text{Prop}}$ Given a hermitian form $\langle , \rangle : V \times V \longrightarrow \mathbb{C}$ and a basis $\mathcal{B}$ of V, $\exists$ a $n \times n$ complex matrix A s.t. for $\vec{v}, \vec{w} \in V$,

$\langle \vec{v}, \vec{w}\rangle = (x)^* A y$

where

$\vec{v} = [\mathcal{B}]x, \quad \vec{w} = [\mathcal{B}]y$

Then $A = A^*$

Pf Let $\vec{v}, \vec{w} \in V$

$\vec{v} = [\mathcal{B}]x, \quad \vec{w} = [\mathcal{B}]y$

$\langle \vec{v}, \vec{w}\rangle = x^* A y$

$\langle \vec{w}, \vec{v}\rangle = y^* A x = \overline{\langle \vec{v}, \vec{w}\rangle}$

$\Rightarrow x^* A y = \overline{(y^* A x)}$

$\Rightarrow x^* A y = y^T A^* x^*$

$x = \overline{e_i}$, $\vec{y} = \vec{e_j}$ i^{th} and j^{th} of standard basis.

$\Rightarrow A_{ij} = \overline{A_{ji}}$ *??

$\Rightarrow A = A^*$

Defn [Adjoint] $A^* := (\overline{A})^T$

Properties of Adjoint

- $(cA)^* = \overline{c} A^*$
- $(A+B)^* = A^* + B^*$
- $(AB)^* = B^* A^*$

Propn Given a hermitian form $\langle , \rangle : V \times V \to \mathbb{C}$.

It is +ve definite $\Leftrightarrow$ the gram matrix of the form w.r.t to any basis is ⊕ve definite.

Pf !!

Thm The eigenvalue of a Hermitian matrix are real

Pf Let λ be an eigenvalue of a Hermitian matrix A and $x \neq 0$ be a corresponding eigenvector.

$$Ax = \lambda x$$

$$\Rightarrow \langle x, x \rangle = x^* A x$$

$$= x^t \lambda x = \lambda x^t x$$

$$\langle x, x \rangle = x^* A x \qquad A = A^*$$

$$= x^* A^* x$$

$$= (Ax)^* x$$

$$= \overline{\lambda} x^* x$$

$$\overline{\lambda} \cancel{(x^* x)} = \lambda \cancel{(x^* x)}$$

$$\Rightarrow \overline{\lambda} = \lambda$$

$$\Rightarrow \lambda \in \mathbb{R} \qquad \text{[Proved]}$$

Defn [Orthogonal Subspaces]

Let $W, W' \subseteq V$ are two subspaces of V. We say they are orthogonal to each other $W \perp W'$ if

$$\langle \overline{w}, \overline{w}' \rangle = 0 \quad \forall \overline{w} \in W, \overline{w}' \in W'$$

Defn [Orthogonal complement]

$W \subseteq V$.

$$W^\perp := \{ \vec{v} \in V : \langle \vec{v}, \vec{w} \rangle = 0 \ \forall w \in W \}$$

Defn [Self orthogonal complement]

Space of null vectors w.r.t. to a given Hermitian form $\langle , \rangle$ on $V \times V$. ↳ $\langle \vec{v}, \vec{v} \rangle = 0$

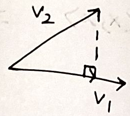
Non-degenerate forms; The forms s.t. $V^\perp = \{0\}$

Prop^n A form $\langle , \rangle : V \times V \to \mathbb{C}$ is non-degenerate
$\iff$ The Gram matrix of the form w.r.t any basis of B is non-singular.

Gram Schmidt Orthonormalization

Let $B := \{v_1, v_2, \ldots, v_n\}$
we'll construct a basis which is orthonormal out of B.

$\|v\| = \sqrt{\langle v, v\rangle}$



$\hat{\omega}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$

$\vec{\omega}_2 = \vec{v}_2 - \langle \vec{v}_2, \hat{\omega}_1\rangle \hat{\omega}_1$

$\hat{\omega}_2 = \dfrac{\vec{\omega}_2}{\|\omega_2\|}$

$\vec{\omega}_3 = \vec{v}_3 - \langle \vec{v}_3, \hat{\omega}_2\rangle \hat{\omega}_2 - \langle \vec{v}_3, \hat{\omega}_1\rangle \hat{\omega}_1$

$\hat{\omega}_3 = \dfrac{\vec{\omega}_3}{\|\vec{\omega}_3\|}$

we have for any $i \in \mathbb{N}$

$\vec{\omega}_i = \vec{v}_i - \sum_{j=1}^{i-1} \langle \vec{v}_i, \hat{\omega}_j\rangle \hat{\omega}_j$

and $\hat{\omega}_i = \vec{\omega}_i / \|\vec{\omega}_i\|$

We will later show that

$\hat{B} := (\hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_n)$ is orthonormal

---

Thm [TFAE] [An extension from $GL_n(\mathbb{R})$]

(i) An $n \times n$ complex matrix A represents the standard Hermitian form w.r.t to some basis

(ii) $\exists P \in GL_n(\mathbb{C})$ s.t. $A = P^* P$

(iii) A is hermitian and $\oplus$ve definite.

(Pf) (i) (i) $\Rightarrow$ (ii)

$\exists$ some basis B' of $\mathbb{C}^n$ st. if $\vec{v} = [B']x$ and $\vec{w} = [B']y$ then

$s(\vec{v}, \vec{w}) = x^* y$

Let $[B'] = [B]P$
Let $[B] = \mathbb{I}$

$\Rightarrow \vec{v}^* A w = x^* y$

$\Rightarrow x^* y = (Px)^* A (Py) \Rightarrow [B'] = P$

$= x^* (P^* A P) y$

this holds for all x, y, thus

let $x = \hat{e}_i$ and $y = \hat{e}_j \in B$

Thus $\mathbb{I}_{ij} = (P^* A P)_{ij}$

$\Rightarrow \boxed{P^* A P = \mathbb{I}}$

P is invertible.

$P^* A P = \mathbb{I} \Rightarrow A = (P^*)^{-1} (P)^{-1}$

Let $(P^{-1}) = Q \in GL_n(\mathbb{C})$

$\boxed{A = Q^* Q}$ for some $Q \in GL_n(\mathbb{C})$

(ii) $\Rightarrow$ (i) (trivial)

(ii) $\Rightarrow$ (iii)

$A = Q^* Q$

$A^* = Q^* Q = A$

and Let $x \in V$

$\langle x, x\rangle = x^* A x$

$= (Qx)^* (Qx) > 0$

(iii) $\Rightarrow$ (ii)

By Gram-Schmidt

Defn - Euclidean Space

A finite dimensional vector space (real) with a $\oplus$ve definite symmetric bilinear form.

Defn - ~~Eucl~~ Hermitian Space

A finite dimensional complex vector space with a $\oplus$ve definite Hermitian form.

Gram-Schmidt

Let V be a hermitian space with

$$\langle , \rangle : V \times V \longrightarrow \mathbb{C}$$

Let $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$ be a basis of V

~~We~~ we construct a basis $\hat{\mathcal{B}} = (\vec{\omega}_1, \vec{\omega}_2, \dots, \vec{\omega}_n)$

s.t.

i) $\langle \omega_i, \omega_j \rangle = \delta_{ij}$

ii) $\text{span}(\vec{\omega}_1, \dots, \vec{\omega}_j) = \text{span}(\vec{v}_1, \dots, \vec{v}_j)$

Base - Case

$$\vec{\omega}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}$$

$$\vec{v}_2{}' = \vec{v}_2 - \langle \vec{v}_2, \vec{\omega}_1 \rangle \vec{\omega}_1$$

$$\vec{\omega}_2 = \frac{\vec{v}_2{}'}{\|\vec{v}_2{}'\|}$$

$$\langle \vec{\omega}_1, \vec{\omega}_1 \rangle = \frac{1}{\|\vec{v}_1\|^2} \langle \vec{v}_1, \vec{v}_1 \rangle = 1$$

$$\langle \vec{\omega}_2, \vec{\omega}_2 \rangle = \frac{1}{\|\vec{v}_2{}'\|^2} \langle \vec{v}_2{}', \vec{v}_2{}' \rangle = 1$$

$\langle \vec{w}_2, \vec{w}_1 \rangle = 0$

$\Leftrightarrow \quad \langle \vec{v}_2', \vec{w}_1 \rangle = 0$

$\langle \vec{v}_2', \vec{w}_1 \rangle = 0$

$\Rightarrow \langle \vec{v}_2 - \langle \vec{v}_2, \vec{w}_1 \rangle \vec{w}_1, \vec{w}_1 \rangle$

Real scalar so $r^* = r$

$\Rightarrow \langle \vec{v}_2, \vec{w}_1 \rangle - (\langle \vec{v}_2, \vec{w}_1 \rangle)^* \cancel{\langle \vec{w}_1, \vec{w}_1 \rangle}^{1}$

$\Rightarrow$

$\langle \vec{v}_2, \vec{w}_1 \rangle - \langle \vec{v}_2, \vec{w}_1 \rangle = 0$

$\Rightarrow \langle \vec{w}_2, \vec{w}_1 \rangle = 0$

By defn

$\vec{w}_1, \vec{w}_2 \in \text{span}(\vec{v}_1, \vec{v}_2)$

$\vec{w}_1 = \vec{v}_1 \left(\frac{1}{\|\vec{v}_1\|} \right)$

$\vec{w}_2 = \frac{1}{\|\vec{w}_2\|} \left(\vec{v}_2 - \langle \vec{v}_2, \vec{w}_1 \rangle \frac{\vec{v}_1}{\|\vec{v}_1\|} \right).$

Thus $w_1, w_2 \in \text{span}(\vec{v}_1, \vec{v}_2)$

$\vec{v}_1 = c\vec{w}_1 \in \text{span}(\vec{w}_1, \vec{w}_2)$

$\vec{v}_2 = \vec{v}_2' + \langle \vec{v}_2, \vec{w}_1 \rangle \vec{w}_1$

$= c\vec{w}_2 + \langle \vec{v}_2, \vec{w}_1 \rangle \vec{w}_1 \in \text{span}(\vec{w}_1, \vec{w}_2)$

$\Rightarrow \vec{v}_1, \vec{v}_2 \in \text{span}(\vec{w}_1, \vec{w}_2)$

$\Rightarrow \text{span}(\vec{v}_1, \vec{v}_2) \subseteq \text{span}(\vec{w}_1, \vec{w}_2)$

and $\text{span}(\vec{w}_1, \vec{w}_2) \subseteq \text{span}(\vec{v}_1, \vec{v}_2)$

$\Rightarrow \text{span}(\vec{v}_1, \vec{v}_2) = \text{span}(\vec{w}_1, \vec{w}_2)$

Induction Hypothesis

Let's assume that the claim holds for all +ve integers upto $k \in \mathbb{N}$ and $\exists$ $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \in V$ satisfying conditions (i) and (ii)

$\vec{v}_{k+1}' = \vec{v}_{k+1} - \sum_{j=1}^{k} \langle v_{k+1}, w_j \rangle w_j$

$\vec{w}_{k+1} = \frac{\vec{v}_{k+1}'}{\|\vec{v}_{k+1}'\|}$

① To show $\langle \vec{w}_{k+1}, \vec{w}_{k+1} \rangle = 1$

$= \frac{1}{\|\vec{v}_{k+1}'\|^2} \langle \vec{v}_{k+1}', \vec{v}_{k+1}' \rangle = 1$

$$\langle \vec{w}_{k+1}, \vec{w}_j \rangle = 0 \qquad \forall j \leq k+1$$

$$\Leftrightarrow \langle \vec{v}'_{k+1}, \vec{w}_{m} \rangle = 0$$

$$\langle \vec{v}'_{k+1}, \vec{w}_m \rangle = \langle \vec{v}_{k+1} - \sum_{j=1}^{k} \langle \vec{v}_{k+1}, w_j \rangle \vec{w}_j$$

$$\Rightarrow \langle \vec{v}'_{k+1}, \vec{w}_m \rangle = \langle \vec{v}_{k+1}, \vec{w}_m \rangle - \sum_{j=1}^{k} \langle \vec{v}_{k+1}, \vec{w} \ldots \underbrace{\langle \vec{w}_j \ldots}_{\delta_{ij}}$$

choos... $j = \ldots$

$$= \langle \vec{v}_{k+1}, \vec{w}_m \rangle - \langle \vec{v}_{k+1}, \vec{w}_m \rangle \langle \ldots$$

$$= 0$$

(ii) $\text{Span}(\vec{v}_1, \ldots, \vec{v}_{k+1}) = \text{Span}(\vec{w}_1, \ldots, \vec{w}_{k+1})$

Pf We know

$\text{Span}(\vec{v}_1, \ldots, \vec{v}_k) = \text{span}(\vec{w}_1, \ldots, \vec{w}_k)$

also

$\vec{w}_{k+1} \in \text{span}(\vec{v}_1, \ldots, \vec{v}_{k+1})$

$\Rightarrow \vec{w}_1, \ldots, \vec{w}_{k+1} \in \text{span}(\vec{v}_1, \ldots, \vec{v}_{k+1})$

$\Rightarrow \text{span}(\vec{w}_1, \ldots, \vec{w}_{k+1}) \subseteq \text{span}(\vec{v}_1, \ldots, \vec{v}_{k+1})$

also

$$\vec{v}_{k+1} = \vec{v}'_{k+1} + \sum_{j=1}^{k} \langle \vec{v}_{k+1}, \vec{w}_j \rangle \vec{w}_j$$

$$= c\vec{w}_{k+1} + \sum_{j=1}^{k} \langle \vec{v}_{k+1}, \vec{w}_j \rangle \vec{w}_j$$

$$\in \text{span}(\vec{w}_1, \ldots, \vec{w}_{k+1})$$

Thus

$$\text{span}(\vec{v}_1, \dots, \vec{v}_{k+1}) = \text{span}(\vec{w}_1, \dots, \vec{w}_{k+1})$$

Corollary [Extension of Orthonormal basis of Subspace]

Given an orthonormal basis of a subspace W of V, we can extend it to an orthonormal basis of V via Gram Schmidt Process.

Prop If V is a Hermitian space $W \subseteq V$, then

$$V = W \oplus W^\perp$$

Pf

Let $\mathbb{B}_0 := (\vec{w}_1, \dots, \vec{w}_k)$ be an orthonormal basis of W. Extend it to an orthonormal basis

$$\mathbb{B} := (\vec{w}_1, \vec{w}_2, \dots, \vec{w}_k, \vec{w}_{k+1}, \dots, \vec{w}_n)$$

Claim $W^\perp = \text{span}(\vec{w}_{k+1}, \dots, \vec{w}_n)$

→ $\vec{w}_j \in W^\perp \quad \forall j \in \{k+1, \dots, n\}$

Because $\mathbb{B}$ is an orthonormal basis of V

$$\langle \vec{w}_j, \vec{w}_i \rangle = 0 \quad \forall j \in \{k+1, \dots, n\}$$
$$\forall i \in \{1, \dots, k\}$$

$$\Rightarrow \langle \vec{w}_j, \vec{w} \rangle = 0, \quad \forall \vec{w} \in W$$

$$\Rightarrow \text{span}(\vec{w}_{k+1}, \dots, \vec{w}_n) \subseteq W^\perp$$

Let $\vec{v} \in V$

$$\vec{v} = \sum_{i=1}^{n} c_i \vec{w}_i \qquad \langle \vec{v}, \vec{w}_j \rangle = \sum_{i=1}^{n} c_i \langle \vec{w}_i, \vec{w}_j \rangle$$
$$= c_j$$

$$\Rightarrow \vec{v} = \sum_{i=1}^{n} \langle \vec{v}, \vec{\omega}_i \rangle \vec{\omega}_i$$

if $\vec{v} \in W^\perp$

$$\Rightarrow \vec{v} = \sum_{j=k+1}^{n} \langle \vec{v}, \vec{\omega}_j \rangle \vec{\omega}_i$$

$$\Rightarrow \vec{v} \in \text{span}(\vec{\omega}_{k+1}, \dots, \vec{\omega}_n)$$

$$\Rightarrow W^\perp \subseteq \text{span}(\vec{\omega}_{k+1}, \dots, \vec{\omega}_n)$$

$$\Rightarrow W^\perp = \text{span}(\vec{\omega}_{k+1}, \dots, \vec{\omega}_n)$$

Spectral Theorem

V: Hermitian space with $\langle , \rangle : V \times V \longrightarrow \mathbb{C}$

$T: V \to V$ is a linear operator.

[Defn] [Adjoint op]

$$T^*: V \longrightarrow V$$

Since V is a hermitian space, $\exists$ an orthonormal basis $\mathcal{B}$ of V.

Let A be the matrix of T w.r.t $\mathcal{B}$

$$A^t := (\bar{A})^T = A^*$$

$T^*: V \longrightarrow V$ is defined to be the operator whose matrix w.r.t $\mathcal{B}$ is A^*

lations w.r.t to the Hermitian form	Operators	Matrices of the operator
$\langle T\vec{v}, \vec{w} \rangle = \langle \vec{v}, T\vec{w} \rangle$	$T^* = T$ [Hermitian]	$A^t = A = A^*$
$\langle T\vec{v}, T\vec{w} \rangle = \langle \vec{v}, \vec{w} \rangle$	$T^* T = \mathbb{I}$ [Unitary op]	$A^* A = \mathbb{I}$
$\langle T\vec{v}, T\vec{w} \rangle = \langle T^*\vec{v}, T^*\vec{w} \rangle$	$T^* T = T T^*$ [Normal]	$A^* . A = A A^*$

Lemma-1

If $W \subseteq V$ is T-invariant (i.e. $T(W) \subseteq W$) then $W^\perp$ is T^* invariant on a hermitian space.

Pf

$T(W) \subseteq W$, $W^\perp = \{ \vec{v} \in W , \langle \vec{w}, \vec{v} \rangle = 0 \} \forall \vec{w} \in W \}$

so it suffices to show for $\vec{v} \in W^\perp$, $T^* v \in W^\perp$

Let $\vec{w} \in W$

$$\langle \vec{w} , T^* \vec{v} \rangle = \langle T\vec{w} , \vec{v} \rangle \qquad \left[\langle T^*\vec{v}, \vec{w} \rangle = \langle \vec{v}, T\vec{w} \rangle = \langle T\vec{w}, \vec{v} \rangle \right]$$

Note
$T\vec{w} \in W$

$\Rightarrow \langle T\vec{w} , \vec{v} \rangle = 0$ as $\vec{v} \in W^\perp$

$\Rightarrow \langle \vec{w} , T^* \vec{v} \rangle = 0$

$\Rightarrow T^* \vec{v} \in W^\perp \Rightarrow T^*(W^\perp) \subseteq W^\perp$

Lemma-2 If $\vec{v} \in V$ is an eigenvector of a normal op $T: V \to V$ with eigenvalue λ. Then $\vec{v} \in V$ is also an eigenvector of T^* with eigenvalue $\bar{\lambda}$.

Pf

case ① $[\lambda = 0]$

so $\bar{\lambda} = 0$

$T\vec{v} = \vec{0}$

$\circledast$ $\langle T^* \vec{v}, T^* \vec{v} \rangle = \langle T\vec{v}, T\vec{v} \rangle = 0$

as $\langle , \rangle$ is +ve definite.

$\Rightarrow T^* V = 0$

$\Rightarrow$ $\vec{v}$ is an eigenvector of T^* with eigenvalue $\bar{\lambda}$.

case ⑪ $[\lambda \neq 0]$

Define $S := T - \lambda \mathbb{I}$

Claim $S^* S = S S^*$ [Normal op]

$S = T - \lambda \mathbb{I}$ $S^* = T^* - \bar{\lambda} \mathbb{I}^*$

$SS^* = (T - \lambda \mathbb{I})(T^* - \bar{\lambda}\mathbb{I}^*)$

$= TT^* - \bar{\lambda} T \cancel{\mathbb{I}} - \bar{\lambda} T + |\lambda|^2 \mathbb{I}$

$= TT^* - \bar{\lambda} T^* - \bar{\lambda} T + |\lambda|^2 \mathbb{I}$

$= T^*(T - \bar{\lambda} \mathbb{I}) - \bar{\lambda} \mathbb{I} (T - \lambda \mathbb{I}) = (T^* - \bar{\lambda} \mathbb{I})(T - \lambda \mathbb{I})$

$= S^* S$

then ~~the~~ ~~is~~ $\vec{v}$ ~~is~~ is an eigenvector of S with eigenvalue 0 as $Tv = \lambda v$.

$S\vec{v} = 0$

using case -1

$S^* \vec{v} = 0$

$= T^* \vec{v} - \vec{\bar{\lambda}} v = 0$

$\Rightarrow T^* \vec{v} = \bar{\lambda} \vec{v}$ [Proved] $\square$

Thm [Spectral Thm]

If V is a Hermitian space and $T: V \to V$ a normal operator, then $\exists$ an orthonormal basis of V consisting of eigenvectors of T.

Pf: Let $\vec{v}$ be an eigenvector of T.

$\Rightarrow \vec{v}$ is an eigenvector of T^*

Since $\vec{v}$ is an eigenvector, $\vec{v} \neq \vec{0} \Rightarrow \sqrt{\langle \vec{v}, \vec{v} \rangle} = \|\vec{v}\| \neq 0$

Define,

$$\vec{w} = \frac{\vec{v}}{\|\vec{v}\|}$$

Base step

Note, that, claim is trivial if $\dim(V) = 1$

Induction Hypothesis

Let's assume that the claim holds for all vector spaces of dim (k) for some $k \in \mathbb{N}$

Now assume that our vector space V has dim $k+1$

Let $\vec{\omega}$ be an eigenvector of T^*

Let $W := \text{span}(\vec{\omega})$

$\Rightarrow T^*(W) \subseteq W$

$\Rightarrow T(W^\perp) \subseteq W^\perp$ [Lemma-1]

Note

$T|_{W^\perp} : W^\perp \longrightarrow W^\perp$ is a normal operator.

By induction hypothesis $\dim(W^\perp) = \dim(V) - \dim(W) = k+1-1 = \underline{\underline{k}}$

$\exists$ an orthonormal basis

$\mathcal{B}_0 := (\vec{\omega}_2, \ldots, \vec{\omega}_{k+1})$ of $W^\perp$

We know

$V = W \oplus W^\perp$

$\Rightarrow \mathcal{B} = (\vec{\omega}, \mathcal{B}_0) = (\vec{\omega}, \ldots, \vec{\omega}_{n+1})$ is an orthonormal basis V of eigenvectors of T.

Cor: If $AA^* = A^*A$ for a complex matrix, then $\exists\ P \in U_n(\mathbb{C})$ s.t. P^*AP is diagonal.