# PH2101 : Waves and Optics

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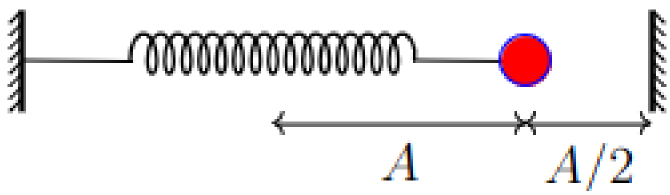
1. **Question 1** Consider a spring-mass oscillator of time period T as shown in the figure below. There is a wall $\frac{A}{2}$ distance to the right from the equilibrium position of the oscillator.




The oscillator is given an initial displacement A towards the left and released from the rest. Considering all collisions to be elastic, what is the time period of the oscillator?

**Solution:**
The general solution is given by

$$x(t) = A\sin(\omega t + \phi)$$

We have the time period to be T thus $\omega = \frac{2\pi}{T}$. Since the collision is elastic the energy and momentum is conserved and as a result the only effect it has is in the reduction of the time period. The collision being elastic just effectively reverses the velocity. As a result this just changes the phase of the system that what would have been at a normal non-truncated SHM at that time.
Then we know that the motion between the equilibrium position and the leftmost extreme and back takes $\frac{T}{2}$ to accomplish. The rest of the time is found out as t'.
The half of the oscillation is unrestricted and thus takes time $\frac{T}{2}$. Also since the motion starts from the equilibrium position $\phi = 0$. Let t be the time it takes to go from the equilibrium position to the wall at $\frac{A}{2}$. Thus the time period is given by $T' = \frac{T}{2} + 2t$ where t is given by

$$\begin{aligned}
&\frac{A}{2} = A\sin(\omega t) \\
\Rightarrow &\frac{1}{2} = \sin(\omega t) \\
\Rightarrow &\omega t = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \\
\Rightarrow &\frac{2\pi t}{T} = \frac{\pi}{6} \\
\Rightarrow &t = \frac{T}{12}
\end{aligned}$$

$$T' = \frac{T}{2} + 2t$$
$$\Rightarrow T' = \frac{2T}{3}$$

Thus the total time period comes out to be

$$\boxed{T' = \frac{2T}{3}}$$

2. **Question 2** For an oscillator $\omega = \pm\omega_0$. If it started with $x(0) = A$ and $\dot{x}(0) = \frac{\omega_0 A}{2}$, then find $x(t)$ Can you solve the problem using only $Ae^{i\omega_0 t}$? Why not?

**Solution:** The general solution is given by

$$\begin{aligned} x(t) &= C_1 \sin(\omega t) + C_2 cos(\omega t) \\ &= W \sin(\omega t + \phi) \end{aligned}$$

The boundary conditions give us

$$\begin{aligned} x(0) &= C_2 \cos(0) = A \\ \Rightarrow C_2 &= A \\ \dot{x}(t) &= C\omega_0 \cos(\omega t) - \omega_0 A \sin(\omega t) \\ \Rightarrow \dot{x}(0) &= C\omega_0 \cos(0) = \frac{\omega_0 A}{2} \\ \Rightarrow C_1 &= \frac{A}{2} \end{aligned}$$

We can now find the phase difference by making the the coefficients of $\cos(\omega t)$ and $\sin(\omega t)$, $\sin(\phi)$ and $\cos(\phi)$ respectively.

$$\begin{aligned} x(t) &= \frac{A}{2}\sin(\omega_0 t) + A\cos(\omega_0 t) \\ \Rightarrow x(t) &= \sqrt{A^2 + \frac{A^2}{4}}\left(\frac{A/2}{\sqrt{A^2 + \frac{A^2}{4}}}\sin(\omega_0 t) + \frac{A}{\sqrt{A^2 + \frac{A^2}{4}}}\cos(\omega_0 t)\right) \\ \Rightarrow x(t) &= \frac{\sqrt{5}A}{2}\left(\frac{A/2}{\frac{\sqrt{5}A}{2}}\sin(\omega_0 t) + \frac{A}{\frac{\sqrt{5}A}{2}}\cos(\omega_0 t)\right) \end{aligned}$$

Let us define $\cos(\phi)$ and $\sin(\phi)$ as follows:

$$\cos(\phi) = \frac{A/2}{\frac{\sqrt{5}A}{2}}$$
$$\sin(\phi) = \frac{A}{\frac{\sqrt{5}A}{2}}$$

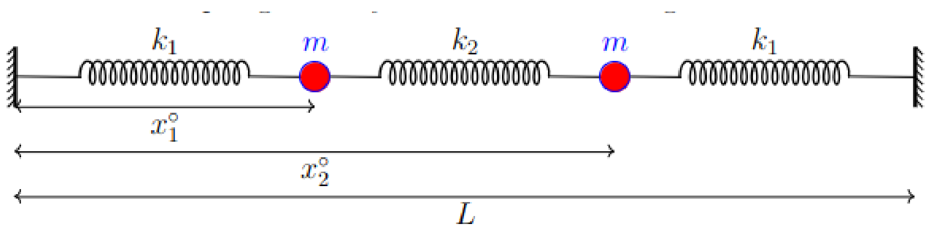
We thus have $\tan(\phi) = \frac{A}{A/2} = 2$. Thus we have $\phi = \arctan(2)$

$$x(t) = \frac{\sqrt{5A}}{2}\left(\frac{A/2}{\frac{\sqrt{5A}}{2}}\sin(\omega_0 t) + \frac{A}{\frac{\sqrt{5A}}{2}}\cos(\omega_0 t)\right)$$
$$\Rightarrow x(t) = \frac{\sqrt{5A}}{2}\sin\left(\omega_0 t + \arctan(2)\right)$$

$$\boxed{x(t) = \frac{\sqrt{5A}}{2}\sin\left(\omega_0 t + \arctan(2)\right)}$$

No we cannot only solve this using $Ae^{i\omega_0 t}$ because on differentiating and putting t=0 in the equation we get $i = \frac{1}{2}$ which is a mathematical absurdity. Also physically this does not make sense as the IVP does make this a real Equation of motion as plugging in the initial value still keeps the amplitude a complex number.

3. **Question 3** Consider the spring-mass system given below:




a) Find the equilibrium positions($x_1^o$ and $x_2^o$). Assume the equilibrium length of the spring $a_o$ to be $\frac{L}{10}$.

**Solution:** In the equilibrium position the force exerted by the middle and left springs on block $m_1$ are equal. The force due to the left spring on $m_1$ is $F_l = -k_1(x_1^o - a_o)$ and the force due to the middle spring on $m_1$ is $F_m = -k_2(x_2^o - x_1^o - a_o)$. Then we equate them

$$\begin{aligned}
&F_l = F_m \\
&\Rightarrow -k_1(x_1^o - a_o) = -k_2(x_2^o - x_1^o - a_o) \\
&\Rightarrow k_1(x_1^o - a_o) = k_2(x_2^o - x_1^o - a_o) \\
&\Rightarrow k_1 x_1^o - k_1 a_o = k_2 x_2^o - k_2 x_1^o - k_2 a_o \\
&\Rightarrow (k_1 + k_2)x_1^o - k_2 x_2^o = a_o(k_1 - k2)
\end{aligned}$$

We have from the diagram itself that $x_1^o + x_2^o = L \cdots (*)$.This equation is true because the system is symmetric about the $k_2$ spring.We also have $a_0 = \frac{L}{10}$. We then multiply $(*)$ with $k_2$ to get $k_2 x_1^o + k_2 x_2^o = k_2 L$ and then solve the system of equations.

$$(k_1 + k_2)x_1^o - k_2 x_2^o = \frac{L}{10}(k_1 - k_2) \tag{1}$$
$$k_2 x_1^o + k_2 x_2^o = k_2 L \tag{2}$$

Adding (2) and (1) we get

$$
\begin{aligned}
x_1^o(k_1+2k_2) &= \frac{L}{10}(k_1-k_2)+k_2L \\
\Rightarrow x_1^o &= \frac{L}{10}\left(\frac{k_1+9k_2}{k_1+2k_2}\right)
\end{aligned}
$$

Then putting this value of $x_1^o$ in $(*)$ we have

$$
\begin{aligned}
x_2^o &= L-\frac{L}{10}\left(\frac{k_1+9k_2}{k_1+2k_2}\right) \\
\Rightarrow x_2^o &= \frac{L}{10}\left(\frac{9k_1+11k_2}{k_1+2k_2}\right)
\end{aligned}
$$

Thus we have the equilibrium lengths to be

$$\boxed{x_1^o = \tfrac{L}{10}\left(\tfrac{k_1+9k_2}{k_1+2k_2}\right)}$$
$$\boxed{x_2^o = \tfrac{L}{10}\left(\tfrac{9k_1+11k_2}{k_1+2k_2}\right)}$$

**Question 3(b)** Assuming unequal masses $m_1$ and $m_2$ and $k_1=k_2=k$, find the longitudinal normal mode frequencies.

**Solution:** Let us move $m_1$ by $x_1$ and $m_2$ by $x_2$ both towards the right. Then we have two equations of motion for both of the masses

$$
\begin{aligned}
m_1\ddot{x_1} &= -kx_1-k(x_1-x_2) \\
\Rightarrow m_1\ddot{x_1} &= -2kx_1+kx_2) \qquad (3) \\
m_2\ddot{x_2} &= -kx_2-k(x_2-x_1) \\
\Rightarrow m_2\ddot{x_2} &= -2kx_2+kx_1 \qquad (4)
\end{aligned}
$$

Putting test solutions $x_1=Ae^{i\omega t}$ and $x_2=Be^{i\omega t}$ and eliminating $e^{i\omega t}$ from both sides we get

$$
\begin{aligned}
(m_1\omega^2-2k)A+kB=0 \\
(m_1\omega^2-2k)B+kA=0
\end{aligned}
$$

Writing in matrix form we have the a matrix equation of the form $Ax=0$. Here we are looking for a non trivial solution thus the determinant of A has to be 0.

$$\begin{pmatrix} m_1\omega^2-2k & k \\ k & m_1\omega^2-2k \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} m_1\omega^2-2k & k \\ k & m_1\omega^2-2k \end{vmatrix}=0$$

$$
\begin{aligned}
&(m_1\omega^2 - 2k)(m_2\omega^2 - 2k) - k^2 = 0 \\
\Rightarrow & m_1m_2\omega^4 - 2k\omega^2(m_1+m_2) + 3k^2 = 0 \\
\Rightarrow & \omega^2 = \frac{2k(m_1+m_2) \pm \sqrt{4k^2(m_1+m_2)^2 - 12k^2m_1m_2}}{2m_1m_2} \\
\Rightarrow & \omega^2 = \frac{k(m_1+m_2) \pm \ k\sqrt{m_1^2+m_2^2-m_1m_2}}{m_1m_2} \\
\Rightarrow & \omega = \pm\sqrt{\frac{k(m_1+m_2) \pm k\sqrt{m_1^2+m_2^2-m_1m_2}}{m_1m_2}}
\end{aligned}
$$

Thus we have found the normal mode frequencies for a coupled spring mass system with different masses

**Question 5** For longitudinal modes, assuming equal masses $m_1 = m_2$ (you may start from the known solutions), find the $x_1(t)$ and $x_2(t)$ for motion starting from the rest with an initial displacement $x_1(0) = A$ and $x_2(0) = \frac{A}{2}$?

**Solution:** Again making the normal mode displacements but keeping the mass same this time while keeping the spring constants different we get the equations

$$
\begin{aligned}
m\ddot{x_1} &= -k_1x_1 - k_2(x_1 - x_2) \\
\Rightarrow m_1\ddot{x_1} &= -x_1(k_1+k_2) + k_2x_2 \quad (5) \\
m\ddot{x_2} &= -k_1x_2 - k_2(x_2 - x_1) \\
\Rightarrow m_2\ddot{x_2} &= -k_1(x_1+x_2) + k_2x_1 \quad (6)
\end{aligned}
$$

Thus we can write as SHMs by algeabraic manipulation

$$
\begin{aligned}
m(\ddot{x_1^2} + \ddot{x_2^2}) &= -k_1(x_1+x_2) \cdots (5) + (6) \\
m(\ddot{x_1^2} + \ddot{x_2^2}) &= -(k_1+2k_2)(x_1-x_2) \cdots (5) - (6)
\end{aligned}
$$

Thus they are second order differential equations with the cosine and sine functions. On differentiating and putting 0 in place of t only the coefficient of the original sine terms remain. And since the motion starts from rest, the coefficients of the original sin terms go to 0.

$$
x_1 + x_2 = A_1\cos(\omega_+t) \quad \text{where } \omega_+ = \pm\sqrt{\frac{k_1}{m}} \quad (7)
$$

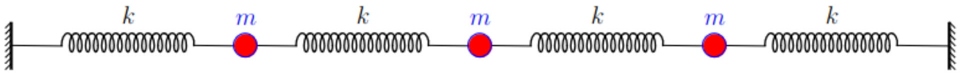
$$
x_1 - x_2 = A_2\cos(\omega_-t) \quad \text{where } \omega_- = \pm\sqrt{\frac{k_1+2k_2}{m}} \quad (8)
$$

Putting in boundary values given to us we obtain $A_1 = \frac{3A}{2}$ and $A_2 = \frac{A}{2}$. On doing (7) + (8) we get function $x_1(t)$ and on doing (7) - (8) we get the function $x_2(t)$ we get

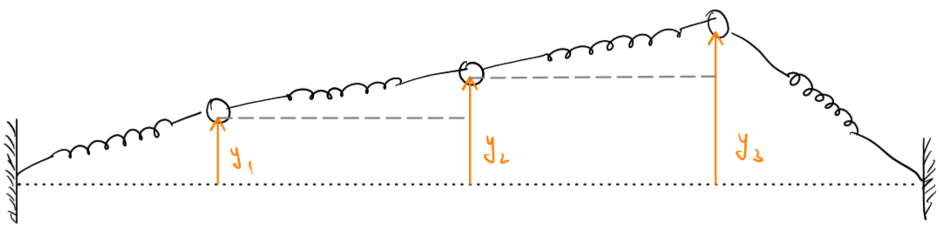
$$
\begin{aligned}
x_1(t) &= \frac{A}{4}\left(3\cos(\omega_+t) + \cos(\omega_-t)\right) \\
x_2(t) &= \frac{A}{4}\left(3\cos(\omega_+t) - \cos(\omega_-t)\right)
\end{aligned}
$$

4. **Question 4** Find the normal modes (frequencies and ratios of amplitude) for the transverse oscillation

of the following system:




**Solution:** Let us displace the first mass by $y_1$ the second by $y_2$ and the third by $y_3$.




We assume that the displacements are small and let $T = k(a - a_0)$ where a is the equilibrium length of the spring when kept in a straight line and $a_0$ is the natural length of a spring

$$
\begin{aligned}
m\ddot{y}_1 &= -\frac{Ty_1}{a} + \frac{T(y_2 - y_1)}{a} \\
\Rightarrow \ddot{y}_1 &= -\frac{2Ty_1}{ma} + \frac{Ty_2}{ma} \\
m\ddot{y}_2 &= -\frac{T(y_2 - y_1)}{a} + \frac{T(y_3 - y_2)}{a} \\
\Rightarrow \ddot{y}_1 &= \frac{Ty_1}{ma} - \frac{2Ty_2}{ma} + \frac{Ty_3}{ma} \\
m\ddot{y}_3 &= -\frac{(Ty_3}{a} - \frac{T(y_3 - y_2)}{a} \\
\Rightarrow \ddot{y}_1 &= -\frac{2Ty_3}{ma} + \frac{Ty_2}{ma}
\end{aligned}
$$

Then let $\frac{T}{ma} = \omega_0^2$ and as previously done use guessed solutions $y_1 = Ae^{i\omega t}, y_2 = Be^{i\omega t}, y_3 = Ce^{i\omega t}$.

$$\ddot{y}_1 = -2\omega_0^2 y_1 + \omega_0^2 y_2 \tag{9}$$

$$\ddot{y}_2 = \omega_0^2 y_1 - 2\omega_0^2 y_2 + \omega_0^2 y_3 \tag{10}$$

$$\ddot{y}_3 = -2\omega_0^2 y_1 + \omega_0^2 y_2 \tag{11}$$

Putting these in the above equations and cancelling the $e^{i\omega t}$ terms on both sides for all the equations we get the matrix equation ;

$$
\begin{pmatrix} \omega^2 - 2\omega_0^2 & \omega_0^2 & 0 \\ \omega_0^2 & \omega^2 - 2\omega_0^2 & \omega_0^2 \\ 1 & \omega_0^2 & \omega^2 - 2\omega_0^2 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
\begin{vmatrix} \omega^2 - 2\omega_0^2 & \omega_0^2 & 0 \\ \omega_0^2 & \omega^2 - 2\omega_0^2 & \omega_0^2 \\ 1 & \omega_0^2 & \omega^2 - 2\omega_0^2 \end{vmatrix} = 0
$$

Opening the determinant and simplifying we get

$$
\begin{aligned}
&(\omega^2 - 2\omega_0^2)[(\omega^2 - 2\omega_0^2)^2 - 2\omega_0^4] = 0 \\
\Rightarrow &(\omega^2 - 2\omega_0^2) = 0 \text{ or } [(\omega^2 - 2\omega_0^2)^2 - 2\omega_0^4] = 0 \\
\Rightarrow &\omega = \pm\sqrt{2}\omega_0 \text{ or } \omega^2 = \pm\sqrt{2 \pm \sqrt{2}}\omega_0
\end{aligned}
$$

Now putting the values in the matrix and then getting their corresponding equations we get three :
**Case 1:** $\omega^2 = 2\omega_0^2$

1. eqn (9) $B = 0$
2. eqn(10) $A + C = 0$

Solving gives us $A = -C$ and $B = 0$
**Case 2:** $\omega^2 = (2 + \sqrt{2})\omega_0^2$