Transfer Matrices for the 2D Ising Model Problem

Only algebra is done, writing I'll do after

Draw the square lattice diagonally. Let us consider a set of three rows, one with black circles and one with white ones. The spin vectors of each of the rows from bottom to top is $\varphi, \varphi'', \varphi'$ in that order. The partition function then, has the term $V_{\varphi,\varphi''} W_{\varphi'',\varphi'}$ Then we define the transfer matrices as,

$$\langle \varphi | \mathbb{V} | \varphi'' \rangle = V_{\varphi,\varphi''} = \sum_{j=1}^{n} \left[K\sigma_{j+1}\sigma''_j + L\sigma_j\sigma'_j \right]$$

$$\langle \varphi'' | \mathbb{W} | \varphi' \rangle = W_{\varphi'',\varphi} = \sum_{j=1}^{n} \left[K\sigma'''_j\sigma'_j + L\sigma''_j\sigma'_{j+1} \right]$$

For translational invariance, we will implement toroidal boundary conditions, which entails $\sigma_{n+1} = \sigma_1$, along with proper row placement.

Let us consider $m \times n$ lattice for this. Then using these, we can easily construct the partition function in terms of the transfer matrices,

$$\begin{aligned} Z_N &= \sum_{\varphi_1} \dots \sum_{\varphi_m} \langle \varphi_1 | \mathbb{V} | \varphi_2 \rangle \langle \varphi_2 | \mathbb{W} | \varphi_3 \rangle \dots \langle \varphi_{m-1} | \mathbb{V} | \varphi_m \rangle \langle \varphi_m | \mathbb{W} | \varphi_1 \rangle \\ &= \sum_{\varphi} \langle \varphi | \mathbb{V}\mathbb{W} \dots \mathbb{V}\mathbb{W} | \varphi \rangle = \mathrm{Tr}(\mathbb{V}\mathbb{W} \dots \mathbb{V}\mathbb{W}) \\ &= \mathrm{Tr}\left((VW)^{\frac{m}{2}} \right) = \sum_{j=1}^{m} \Lambda_j^{m/2} \end{aligned}$$

where Λ_j is the jth eigenvalue of the matrix $\mathbb{V}\mathbb{W}$. Note that for the last equality we use Schur's Decomposition theorem, because in general $\mathbb{V}\mathbb{W}$ is not symmetric.

Commutation Relations

Note that the transfer matrices $\mathbb{V}, \mathbb{W}$ can be regarded as functions of the variables K, L. Thus we have $\mathbb{V} = V(K, L)$ and $\mathbb{W} = W(K, L)$. Suppose we consider two sets of interaction coefficients K, L and K', L'. We are interested in what happens when we switch the $K \longleftrightarrow K'$ and $L \longleftrightarrow L'$. Thus we want to know that when the following relation holds,

$$V(K, L)W(K', L') = V(K', L')W(K, L)$$

Again using our notation from the previous section of the three rows from the bottom to the top row being denoted by $\varphi = \{\sigma_1, ..., \sigma_n\}, \varphi'' = \{\sigma''_1, ..., \sigma''_n\}, \varphi' = \{\sigma'_1, ..., \sigma'_n\}$, we are interested in the matrix element $V_{K,L}W_{K',L'}(\varphi, \varphi') = \langle \varphi | \mathbb{V}(K, L) | \varphi'' \rangle \langle \varphi'' | \mathbb{W}_{K',L'} | \varphi' \rangle$. This is given by

$$VW(\varphi, \varphi') = \sum_{\sigma''_1} \dots \sum_{\sigma''_n} \prod_{j=1}^{n} \exp\left(\sigma''_j \left(K\sigma_{j+1} + L\sigma_j + K'\sigma'_j + L'\sigma'_{j+1} \right) \right)$$

Note that this expression inside the product containing the σ''_j is unique to one summand in the whole sum. Thus we can write the whole sum in the form of,

$$VW(\varphi, \varphi') = \prod_{j=1}^{n} X\left(\sigma_j, \sigma_{j+1}; \sigma'_j, \sigma'_{j+1} \right) \text{ where } X \text{ is given by}$$

$$X(a, b, c, d) = \sum_{f=\pm 1} \exp[f(La + Kb + K'c + L'd)]$$

where the variables a, b, c, d take the values ± 1. Note that for any number M, the expression

Inversion Relations

Symmetries

Relation between $\mathbb{V}$ and $\mathbb{W}$

Eigenvalues