Research Project

2D Ising Model

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Abstract

The Ising Model was introduced by Wilhelm Lenz along with his PhD student Ernest Ising. At that time it sought to explain the concept of ferromagnetism using a simple model that involved interaction with nearest neighbors. This report details out some of the models that were exactly solved in a quest for a solution to the Ising Model. Starting out with Ising's own Solution to the 1D Model,we move on the Mean field Model, and then finally arrive at a solution to the 2D Ising Model. The solution here is by Baxter [1], and is based on an article by Onsager in [2]. It solves the problem by using commuting transfer matrices obtained by constructing the lattice diagonally. This report arrives at the end to the free energy of the 2D Ising Model, and then uses that to calculate the critical temperature of the model. Finally, some Monte Carlo simulations are done to verify the results obtained theoretically. We also observe some numerical properties of the model using these simulations.

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On the very rare chance some reader has the dire misfortune of going through this report, I would like to thank them for their time and patience. Also on that note, I would like to apologise for any mistakes or errors that may have crept in noticed or unnoticed. The unfortunate reader is welcome to contact me at `iamsabarno[at]gmail[dot]com` for any clarifications, corrections or suggestions. Enjoy the ride!

1. Introduction

The Ising Model was introduced by Wilhelm Lenz in 1920 in an effort to describe ferromagnetism. It was then given as a problem to his PhD student, Ernest Ising, from where the name comes. In his 1924 PhD thesis [3], Ising solved the model in 1D using transfer matrices, and arrived a disheartening conclusion, the model does not show a phase transition at non-zero temperature in 1D. He then concluded incorrectly that the model does not show a phase transition at non-zero temperature in any dimension, much to the chagrin of a lot of people interested in explaining ferromagnetism at the time. Ironically, he stopped his research in physics thinking that his proof outlined that the Ising model cannot explain ferromagnets and was physically invalid. He took up teaching which he did for the rest of his life. 12 years later, this argument was disproved by Rudolf Peierls [4], who showed that the model does indeed show a phase transition at non-zero temperature in 2D and higher dimensions, using a shoreline proof. In 1944, a breakthrough arrived at the hands of Lars Onsager [5], who solved the 2D Ising Model exactly for a zero field Hamiltonian. An excellent review of the history of the Ising Model can be found at [6], [7].

In this report we aim to outline the exact results that have been obtained in the pursuit for the solution of the Ising Model. Most of the report will follow from the book "Exactly Solved Models in Statistical Physics" by R.J. Baxter [1]. Another Reference that I used while studying this is "A Students Guide to the Ising Model" by James S. Walker [8]. First we outline the solution to the 1D ising Model using Ising solution of transfer matrices and conclude why the 1D model does not show a phase transition at non-zero temperature, using spin correlations. Then we outline the solution of the mean field model and how that shows phase transitions at non-zero temperature.

Finally, we solve the 2D Ising model as done in [1], by constructing corner transfer matrices. For the ease of reading, after each section there is a summary of the results obtained in that section in a blue box.

2. The 1D Ising Model

Before we present the Hamiltonian of the 1D Ising system, it is quite important to note the discretizations of this model. We first have a spatial discretization, where we discretize the magnets into a grid of sites. Then comes our second discretization, the spin discretization, where we discretize the spins into two values, $+1$ and -1. The second discretization had little motivation unless you take into account the quantum nature of spins, we shall classically regard this property to be just some perk of the model.

The 1D Ising Model is defined by the Hamiltonian

$$\mathbb{H} = -J\sum_{j=1}^{N}\sigma_j\sigma_{j+1} - H\sum_{j=1}^{N}\sigma_j \tag{2.1}$$

where $\sigma_j = \pm 1$ is the spin at site j. J is the coupling constant here. $J > 0$ describes a ferromagnetic model here, because we want to lower energy so to lower it, neighbors spin should be aligned. $J < 0$ described an antiferromagnetic model, where we want to lower energy by having neighboring spins anti-aligned. $J = 0$ is fairly uninteresting model where we do not care about the alignment of neighboring spins. H is the external magnetic field, which decides how each spin affects the energy of the system. If $H > 0$, then we want to align spins in the positive direction, if $H < 0$ then we want to align spins in the negative direction. If $H = 0$, then there is no external field applied. In an attempt to make the system translationally invariant, we will assume periodic boundary conditions, i.e., for a system with N sites $\sigma_{N+1} = \sigma_1$. This translational invariance is important since the physics should not depend on where the analysis of the model is done.

2.1. Free Energy and Magnetization

The partition function of the 1D Ising Model is given by

$$Z_N = \sum_{\{\sigma\}} \exp(-\beta H) = \sum_{\{\sigma\}} \exp\left(k \sum_{j=1}^{N} \sigma_j \sigma_{j+1} + h \sum_{j=1}^{N} \sigma_j \right) \tag{2.2}$$

where $\beta = \frac{1}{k_B T}$, k_B is the Boltzmann constant and T is the temperature and $k = \beta J$ and $h = \beta H$ and N is the number of lattice. The sum is over all possible spin configurations. Let us define

$$V(\sigma, \sigma') = \exp\left(K\sigma\sigma' + \frac{1}{2} h(\sigma + \sigma') \right) \tag{2.3}$$

which simplifies our partition function to

$$Z_N = \sum_{\{\sigma\}} V(\sigma_1, \sigma_2) V(\sigma_2, \sigma_3) ... V(\sigma_N, \sigma_1)$$

To achieve these simplifications other forms of V exist, but this form is chosen to make V symmetric with respect to the arguments $V(\sigma, \sigma') = V(\sigma', \sigma)$. This choice will have some consequences later. Now one might question as to why this transformation is done. This transformation helps us in analytically computing the partition function. This helps us define and use *transfer matrices*. The transfer matrix is defined as

$$\mathbb{V} = \begin{pmatrix} V(+,+) & V(+,-) \\ V(-,+) & V(-,-) \end{pmatrix} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{-K-h} \end{pmatrix} \tag{2.4}$$

These transfer matrices are so-called because they determine the probability with which we get the next spin given our current spin. So we can see that $V(\sigma, \sigma') = \langle \sigma | V | \sigma' \rangle$. The partition function is then given as

$$Z_N = \sum_{\sigma_1} ... \sum_{\sigma_N} \langle \sigma_1 | \mathbb{V} | \sigma_2 \rangle ... \langle \sigma_N | \mathbb{V} | \sigma_1 \rangle \tag{2.5}$$

Note that the σ_j are independent of each other. Let us denote $|+\rangle = \binom{1}{0}$ and $|-\rangle = \binom{0}{1}$. From this we get that $\sum_{\sigma_i=0}^{1} |\sigma_i\rangle\langle\sigma_i| = \mathbb{I}$. This is our completeness relation with which we are quite familiar in QM. Using this, the partition function can be simplified,

$$\begin{aligned} Z_N &= \sum_{\sigma_1} ... \sum_{\sigma_N} \langle \sigma_1 | V | \sigma_2 \rangle ... \langle \sigma_N | V | \sigma_1 \rangle \\ &= \sum_{\sigma_1} \langle \sigma_1 | \mathbb{V} \left(\sum_{\sigma_2} |\sigma_2\rangle\langle\sigma_2| \right) \mathbb{V} \left(\sum_{\sigma_3} |\sigma_3\rangle\langle\sigma_3| \right) ... \left(\sum_{\sigma_N} |\sigma_N\rangle\langle\sigma_N| \right) \mathbb{V} |\sigma_1\rangle \\ &= \sum_{\sigma_1} \langle \sigma_1 | \mathbb{V}^N | \sigma_1 \rangle = \mathrm{Tr}[\mathbb{V}^N] \end{aligned} \tag{2.6}$$

Right here we realize why we chose V to have a symmetric form. As a consequence of the symmetric form V, the matrix $\mathbb{V}$ turns out to be a symmetric matrix. We will now use some basic linear algebra to get some properties of the matrix $\mathbb{V}$.

Using the spectral theorem for finite dimensional vector spaces [9], we can say that the hermitian matrix $\mathbb{V}$ is diagonalizable, and can represent in the form $\mathbb{V} = U^\dagger \Lambda U$, where U is a unitary matrix and Λ is a diagonal matrix with the eigenvalue as the diagonal entries. Then we obtain

$$\mathrm{Tr}[\mathbb{V}^N] = \mathrm{Tr}\big[U^\dagger \Lambda^N U\big] = \mathrm{Tr}[\Lambda^N] = \lambda_1^N + \lambda_2^N \tag{2.7}$$

, where λ_1, λ_2 are the eigenvalues of $\mathbb{V}$. We can also say something about the degeneracy of the eigenspectrum. Note that the matrix $\mathbb{V}$ has all positive entries for a fixed non zero T and non zero h, so the matrix is Irreducible and Primitive. For Irreducible and Primitive matrices, the Perron-Frobeius Theorem states that the Eigenspace of the principal or dominant eigenvalue will be of 1 Dimensional [10]. Thus we must necessarily have λ_1 and λ_2 to be distinct. There is however a special case. If $H = 0 = T$, then the matrix has only one eigenvalue, which is $\lambda_1 = \lambda_2 = 1$. From the point of view of correlations of the model this is important, as we shall see later.

Returning to the domain of physics, the partition function has a simple form of

$$Z_N = \lambda_1^N + \lambda_2^N = \lambda_1^N \left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^N\right) \tag{2.8}$$

where $\lambda_1 > \lambda_2$.

Note that our thermodynamic free energy makes sense only in the thermodynamic limit. So we get the free energy per site to be

$$\begin{aligned} f &= -\frac{1}{\beta}\lim_{N\longrightarrow\infty}\frac{1}{N}\ln Z_N = -\frac{1}{\beta}\ln(\lambda_1) \\ &= -k_B T \ln\Big[e^K \cosh(h) + (e^{2K}\sinh^2(h) + e^{-2K})^{\frac{1}{2}}\Big] \end{aligned} \tag{2.9}$$

Then the magnetization is defined as the average magnetic moment per site is defined as,

$$M = -\frac{\partial f}{\partial H} = \frac{e^K \sinh(h)}{[e^{2K}\sinh^2(h) + e^{-2K}]^{\frac{1}{2}}} \tag{2.10}$$

The Magnetization is an analytic function of h so we do not obtain a phase transition. A much nicer and much more intuitive argument is also provided using spin-spin correlations, but first a short word on correlations.

Result 1 (Free Energy and magnetization of the Ising model): The free energy per site of the 1D Ising Model is given by

$$f = -k_B T \ln\Big[e^K \cosh(h) + (e^{2K}\sinh^2(h) + e^{-2K})^{\frac{1}{2}}\Big] \tag{2.11}$$

The magnetization is given by

$$M = \frac{e^K \sinh(h)}{[e^{2K}\sinh^2(h) + e^{-2K}]^{\frac{1}{2}}} \tag{2.12}$$

The magnetization is an analytic function of h and does not exhibit a phase transition.

2.2. Correlations and Phase Transition

Let us understand correlations from the point of view of the Ising Model. Let us consider two Ising Spins $\sigma(x)$ and $\sigma(x')$ at positions x and x' respectively. Then the correlation function is defined as

$$g(d) = \langle\sigma(x)\sigma(x')\rangle - \langle\sigma(x)\rangle\langle\sigma(x')\rangle \tag{2.13}$$

where $d = |x - x'|$. The second term represents how they behave when they are isolated and independent. The first term shows how on an average the behavior of the one the spins influences the other spin. Here we thus quantify how the behavior changes between cases when are allowed to interact and when they are kept isolated. For more intuition and better explanations consider reading chapter 10 of [11]. Correlation calculations are quite important in understanding why, a more intuitive reason, the 1D Ising Model does not have a Phase Transition. The function we are interested in calculating is (2.13). Now let us calculate this function for the 1D Ising Model.

We start by calculating $\langle\sigma_j\rangle$.

$$\langle\sigma_j\rangle = Z_N^{-1}\sum_{\{\sigma\}}\sigma_j V(\sigma_1,\sigma_2)...V(\sigma_N,\sigma_1) \tag{2.14}$$

In similar fashion to (2.6), we use bra-ket notation here. However, all summations here will not give identity. Here for the j-th spin, we get

$$\sum_{\{\sigma_j=+,-\}} |\sigma_j\rangle\sigma_j\langle\sigma_j| = S = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{2.15}$$

in the representation of $|+\rangle, |-\rangle$ used here. We thus obtain the form using cyclicity of the trace or translational invariance of the 1D problem

$$\langle\sigma_j\rangle = Z_N^{-1}\,\mathrm{Tr}\big[SV^N\big] \tag{2.16}$$

Similarly, the form for $\langle\sigma_i\sigma_j\rangle$ follows and using the same cyclicity of the trace or translational invariance of the lattice, this quantity only depends on the distance between the sites i, j. WLOG suppose that $j \geq i$,thus $0 \leq j-i \leq N$ $\langle\sigma_i\sigma_j\rangle = Z_N^{-1}\,\mathrm{Tr}\big[SV^{j-i}SV^{N-(j-i)}\big]$

When we diagonalized our matrix $\mathbb{V}$, we obtained some unitary(in this case orthogonal, since we defined our problem over the real field) matrix U. The explicit form of U is given as

$$U = \begin{pmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{pmatrix} \tag{2.17}$$

, where $\cot(2\varphi) = e^{2K}\sinh(h)$, where $0 < \varphi < \frac{\pi}{2}$. . Therefore, these relations follow,

$$\begin{aligned} U^T\mathbb{V}U &= \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\\ U^TSU &= \begin{pmatrix}\cos(2\varphi) & -\sin(2\varphi)\\ -\sin(2\varphi) & -\cos(2\varphi)\end{pmatrix}\end{aligned} \tag{2.18}$$

Thus, the spin-spin average terms in the correlation function is given as

$$\begin{aligned}\langle\sigma_i\rangle &= \cos(2\varphi)\\ \langle\sigma_i\sigma_j\rangle &= \cos^2(2\varphi) + \sin^2(2\varphi)\left(\frac{\lambda_2}{\lambda_1}\right)^{j-i}\end{aligned} \tag{2.19}$$

Note two small and expected, but nonetheless interesting observations. $\langle\sigma_j\rangle$ is independent of the site, which we expect since each individual Ising spin at a site on its own is identical to any other Ising spin at any other site. Second, the quantity $\langle\sigma_i\sigma_j\rangle$ only depends on the distance between the spin sites. This also is to be expected. Since we have set up a translationally invariant problem, so we expect some function $f(i,j) = f(|j-i|)$. The correlation function $g(d) = g_{ij}$ follows,

$$g_{ij} = \sin^2\left(\frac{\lambda_2}{\lambda_1}\right)^{j-i} \tag{2.20}$$

We see that the correlation decays away quite fast. If some asymptotic behavior of $g(d)$ is considered,

$$g(d) \sim e^{-\frac{d}{\xi}} \tag{2.21}$$

where ξ is defined to be the correlation length. The correlation length comes out to be

$$\xi = \left[\ln\left(\frac{\lambda_2}{\lambda_1}\right)\right]^{-1} \tag{2.22}$$

Before we move on, a word on phase transitions is crucial. A phase transition is defined to an abrupt, discontinuous change in the properties of the system. For phase transition we define an order parameter, which takes a zero value in one phase (called the disordered phase) and a non-zero value in the other phase. This transition in the order parameter can be continuous or discontinuous, but there will be some discontinuous change in some property of the system to qualify it to be a phase transition. In our case, we get continuous phase transitions. But the derivative of Magnetization or susceptibility is discontinuous, since magnetization is not differentiable. We shall revisit this when we consider the mean field model. Here, we are trying to model ferromagnetism. Experimentally, we know below a certain critical Temperature the magnet still exhibits ferromagnetism even if we turn of the external magnetic field. Thus, we want to see when we take H to zero, what happens to the magnetization. If ferromagnetism is exhibited then the magnetization to go to a finite value m^* rather than 0.

The correlation length gives us a some kind of characteristic length over which the influence of a spin takes place. When we take the limit $H = 0$, we want to investigate the critical temperature T_c. Note when $H = 0$,

$$\mathbb{V} = \begin{pmatrix} e^K & 1 \\ 1 & e^{-K} \end{pmatrix}$$

If we take the limit, $T \longrightarrow 0+$, we obtain degeneracy giving us, $\lim_{T \longrightarrow 0+} \left(\frac{\lambda_2}{\lambda_1}\right) = 1$ Thus at $H = T = 0$, the correlation length becomes infinite and we can observe a different phase. However, we do not observe a phase transition, since the ordered state of the system doesn't exist.

Again, taking the formula obtained for magnetization using the derivative of the free energy per site, and applying the limit, we observe,

$$m^* = \lim_{H \longrightarrow 0} \frac{e^K \sinh(h)}{[e^{2K} \sinh^2(h) + e^{-2K}]^{\frac{1}{2}}} = 0 \tag{2.23}$$

Thus we see that our order parameter, Magnetization is identically zero over all $T > 0$.

Thus, observe by two different ways that the 1D Ising Model does not exhibit a phase transition. The first way is by observing that the free energy is analytic in H and T. The second way is by observing that the correlation length is finite for non-zero temperatures and the magnetization is identically zero over all $T > 0$.

3. The Mean Field Model

Monumental efforts have been invested in the solution of the Ising Model in 3 Dimensions without much progress. The model can be extended to a mean field model that can be solved analytically. There are two ways to go about it, using the approach that Baxter uses and another using an inequality called the Bogoliubov inequality and a variational method [12]. We will outline the solution by Baxter. This model is pretty nice because we can graphically see why the phase transition takes place.

3.1. The Mean Field Hamiltonian

The mean field Hamiltonian, interestingly, does not feature the classic nearest neighbor interactions in the Ising Model. Suppose each spin site on a lattice of N sites, has q neighbors,(2d for a d dimensional Ising Model). The total field acting on that is given by,

$$H + J \sum_{\{j\}} \sigma_j \tag{3.1}$$

where the summation is over the q neighbors of the spin site. For the mean field the interactions over all other $N - 1$ sites, averaged and then amplified over the number of neighbors q. The field for a spin at site i is given by

$$H + (N-1)^{-1} qJ \sum_{j \neq i} \sigma_j \tag{3.2}$$

where the sum is taken over all $N - 1$ sites. The Hamiltonian is thus given by,

$$\mathbb{H}(\{\sigma\}) = -\frac{qJ}{N-1} \sum_{i,j} \sigma_i \sigma_j - H \sum_i \sigma_i \tag{3.3}$$

where the first sum is over all the $\binom{N}{2} = \frac{1}{2} N(N-1)$ distinct pairs. There is an unphysical property here. The strength of coupling between spins depends on the number of spin sites. However, we can get some proper thermodynamic properties from the model. This analytically solvable model also exhibits phase transitions. The total magnetization is defined as

$$M = \sum_{j=1}^{N} \sigma_j \tag{3.4}$$

Let there be r down spins, then the number of up spins is given by $N-r$. The total magnetization for r down spins is given by $M_r = N-2r$. Note,

$$
\begin{aligned}
M_r^2 &= \sum_{i,j} \sigma_i \sigma_j = \sum_{i=1}^{N} \sigma^2 + \sum_{i \neq j} \sigma_i \sigma_j \\
\implies \sum_{i \neq j} \sigma_i \sigma_j &= M_r^2 - N = (N-2r)^2 - N
\end{aligned}
$$

A small note here, here the sum distinguishes (i,j) and (j,i), but the sum in the Hamiltonian only takes into account distinct pairs. The mean field Hamiltonian can be rewritten in terms of the magnetization and indexed by the number of down spins r.

$$
\begin{aligned}
\mathbb{H}_r &= -\frac{qJ}{N-1} \sum_{i,j} \sigma_i \sigma_j - H \sum_i \sigma_i = -\frac{qJ}{N-1}[M_r^2 - N] - HM_r \\
&= -\frac{qJ}{2(N-1)}[(N-2r)^2 - N] - H(N-2r)
\end{aligned}
\tag{3.5}
$$

3.2. Magnetization and Free Energy

The partition function is now expressed as,

$$
\begin{aligned}
Z_N &= \sum_{r=0}^{N} c_r \text{ where} \\
c_r &= \binom{N}{r} \exp\left(\frac{q\beta J}{2(N-1)}[(N-2r)^2 - N] + \beta H (N-2r) \right)
\end{aligned}
\tag{3.6}
$$

The properties of c_r can be analyzed using $d_r = \frac{c_{r+1}}{c_r}$. To motivate this expression, we consider this to characterize the sets where c_r is increasing and c_r is decreasing. Using that we can determine r where c_r takes the maximum value and some Laplace Method approximation can be applied. The graphs of these c_r and d_r will also be provided, which also helps in motivating whatever algebraic jugglery we might indulge in here. The expression for d_r is given by,

$$
\begin{aligned}
d_r &= \frac{c_{r+1}}{c_r} = \frac{\binom{N}{r+1}}{\binom{N}{r}} \frac{\exp\left(\frac{q\beta J}{2(N-1)}[(N-2r-2)^2 - N] + \beta H(N-2r-2)\right)}{\exp\left(\frac{q\beta J}{2(N-1)}[(N-2r)^2 - N] + \beta H(N-2r)\right)} \\
&= \frac{N-r}{r+1} \exp\left(-2\beta qJ\left(1 - \frac{2r}{N-1}\right) - 2\beta H\right) \\
&= A_H \frac{N-r}{r+1} \exp\left(-2\beta qJ\left(1 - \frac{2r}{N-1}\right)\right)
\end{aligned}
\tag{3.7}
$$

Let us consider a continuous extension to this for non-natural r. Let $\varphi(x)$ be a function such that for $r = \{0, 1, ..., N\}$, we have $\varphi(r) = d_r$. Let's ignore the initial A_H factor, since it's just a scaling factor which is the same for all x. A natural choice for this is given by,

$$
\varphi(x) = \frac{N-x}{x+1} \exp\left[-2\beta qJ\left(1 - \frac{2x}{N-1}\right)\right] \tag{3.8}
$$

The derivative of the function should be able to tell us something about the domains where the function rises and where it dips. Let us denote $\eta = 2\beta qJ$ for ease of algebra

$$
\varphi'(x) = -\frac{2\eta x^2 + 2\eta(1-N)x + (N^2 - 2\eta N - 1)}{(N-1)(x+1)^2} \exp\left(-\eta\left(1 - \frac{2x}{N-1}\right)\right) \tag{3.9}
$$

We can just focus on the quadratic term in the derivative. The other terms, namely the exponential and the denominator are always zero. The zeros of the quadratic define maxima and minima of the function. The function thus can attain either one or two stationary points. When the function has complex roots, we just get a monotonically decreasing function over the whole domain. Thus, d_r is a monotonically decreasing function. The quadratic term has complex roots when,

$$\eta \leq \frac{2(N-1)}{N+1} \implies q\beta J \leq \frac{N-1}{N+1} \tag{3.10}$$

As we shall see later, this condition will determine the critical temperature. The other case can be that the quadratic term has two positive or two negative roots, using Vieta's Theorem [13]. The roots are given by

$$x_{\pm} = \frac{\eta(N-1) \pm \sqrt{\eta^2(N-1)^2 - \eta(N^2 - 2\eta N - 1)}}{2\eta} \tag{3.11}$$

Since x_+ is positive, both roots must be positive. Thus, we have two cases, one where the function decreases monotonically, and one where it has two positive roots. Let us examine the case, where there is no root, $qJ \leq k_B T$. Note that for this condition the function d_r starts somewhere greater than 1 and goes to 0 when $r = N$. Thus, there is one point where the function $\varphi(x)$ crosses the line $y = 1$. Thus, in the discrete version of d_r,there is some $r_o \in \{0, ..., N\}$ such that we obtain,

$$\begin{aligned} d_r &> 1 \qquad r = \{0,, 1, ..., r_o - 1\} \\ d_{r_o} &\geq 1 \qquad r_o \\ d_r &< 1 \qquad r = \{r_o + 1, ..., N\} \end{aligned} \tag{3.12}$$

Thus, there is a maximum value of c_r which it attains at some point r_o. We shall exploit this later when we use a Saddle point approximation on our problem. When the case for double roots are considered, assuming you take high enough N for a given η, we will obtain three points where $\varphi(x) = 1$. This fact can be easily seen if you observe the fact that the two points where the derivative of $\varphi(x)$ is symmetric with respect to $x = (N-1)/2$. At that point x, the derivative of $\varphi(x)$ is greater than or equal to zero. If it is equal to zero, it corresponds to the repeated root condition and that also means there is only 1 point where $\varphi(x)$ crosses 1. If the slope is positive, then $\varphi(x_+)$ and $\varphi(x_-)$, are above and below 1 respectively. Thus, the function d_r crosses the line $y = 1$ three times. The graph of $\varphi(x)$ and d_r has been plotted below. Three graphs have been plotted, one where there are no stationary points, one where there is one stationary point and one where there is two. Our extremely tedious and completely unmotivated analysis has matched with computational results.

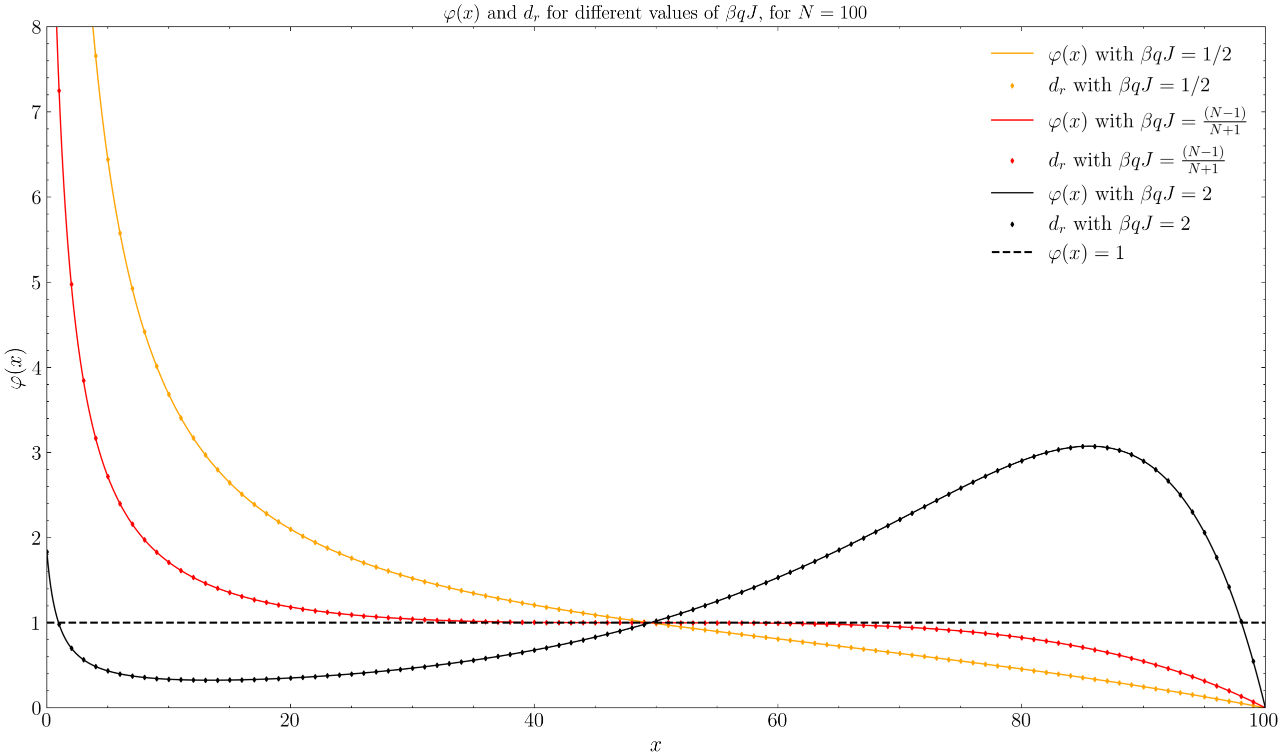


Figure 1: The dependence of $\varphi(x)$ and d_r on $q\beta J$

Before we move on to the critical temperature and the graphs of c_r, the magnetization needs to calculated. The average magnetization per site is then defined to be,

$$m = N^{-1}\langle M\rangle = \left\langle 1 - \frac{2r}{N}\right\rangle = Z^{-1}\sum_{r=0}^{N}\left(1 - \frac{2r}{N}\right)c_r \tag{3.13}$$

The height of the peak at c_{r_o} scales with N, and the width of the peak scales with $N^{\frac{1}{2}}$. So we can say that the maximum contribution of the sum is given by the maximum summand. Let us define another function, which we will motivate after we introduce the function.

$$d_r = \frac{c_{r+1}}{c_r} = \psi\left(1 - \frac{2r}{N}\right) \tag{3.14}$$

where we can manipulate the general continuous function $\psi(x)$, where $-1 < x < 1$, to remove all dependence on r and N. For this next step assume that we have $N \gg 1$, i.e. a large number of sites. The functional form is given by,

$$\begin{aligned} \psi\left(x = 1 - \frac{2r}{N}\right) &= \frac{N-r}{r+1}\exp(-2\beta qJx - 2\beta H) \\ \implies \psi\left(x = 1 - \frac{2r}{N}\right) &= \frac{1 + 1 - \frac{2r}{N}}{1 + \frac{2}{N} - (1 - \frac{2r}{N})}\exp(-2\beta qJx - 2\beta H) \\ \implies \psi(x) &= \frac{1+x}{\frac{2}{N} + 1 - x}\exp(-2\beta qJx - 2\beta H) \\ \implies \psi(x) &\approx \frac{1+x}{1-x}\exp(-2\beta qJx - 2\beta H) \end{aligned} \tag{3.15}$$

Note that $-1 < x < 1$. We are looking for the point r_o where c_r attains the maximum value at c_{r_o}. If N is large enough, c_{r_o+1} and c_{r_o} are close enough that we look for the point where $d_{r_o} = 1 = \psi\big(x_o =$

$1 - \frac{2r_o}{N}\Big)$. Note that the value of x_o only depends on βqJ and βH, and does not depend on N at large N. The average magnetization can then be given by,

$$\begin{aligned} m &= \lim_{N \gg 1} Z^{-1} \sum_{r=0}^{N} \left(1 - \frac{2r}{N}\right) c_r \\ &= \lim_{N \gg 1} Z^{-1} \sum_{r=0}^{N} x_r c_r \\ &= x_{r_o} \frac{c_{r_o}}{c_{r_o}} = x_{r_o} \end{aligned} \tag{3.16}$$

Thus, $\psi(m) = 1$. Again our average magnetization is Thus we can find the solution of m, in the form of a transcendental equation. The average magnetization m is given by,

$$\begin{aligned} &\frac{1+m}{1-m} = \exp(2\beta qJm + 2\beta H) \\ \implies &m(\beta) = \frac{\exp(\beta qJm + \beta H) - \exp(-\beta qJm - \beta H)}{\exp(\beta qJm + \beta H) + \exp(-\beta qJm - \beta H)} \\ \implies &m(\beta) = \tanh(\beta qJm + \beta H) \end{aligned} \tag{3.17}$$

The free energy per site is calculated for the system before we move onto the critical behavior and temperature for the model. The free energy per site, for infinitely large systems is given by,

$$-\beta f = \frac{1}{2} \ln\left(\frac{4}{1-m^2}\right) - \frac{1}{2} qJ\beta m^2 \tag{3.18}$$

Result 2 (Free Energy and Magnetization of the Mean Field model):

The free energy per site is given by

$$-\beta f = \frac{1}{2} \ln\left(\frac{4}{1-m^2}\right) - \frac{1}{2} qJ\beta m^2 \tag{3.19}$$

and the average magnetization is given by

$$m(\beta) = \tanh(\beta qJm + \beta H) \tag{3.20}$$

The free energy cannot be expressed in just in terms of H without dependence on m. So we look at the behavior of m as a function of H.

3.3. Critical Point

We can obtain the H as a function of m and then rotate the axes to find how m varies with H.

$$H(\beta) = qJ\left(m + \frac{1}{q\beta J} \tanh^{-1}(M)\right) \tag{3.21}$$

We can plot this and then reverse the graph to easily visualize the plot of m versus H and see how we get a phase transition. Furthermore, we know that when $T > \frac{qJ}{k_B}$, there is only one solution to $d_r = 1$, the graph is plotted below. There is no spontaneous magnetization, which we expect.

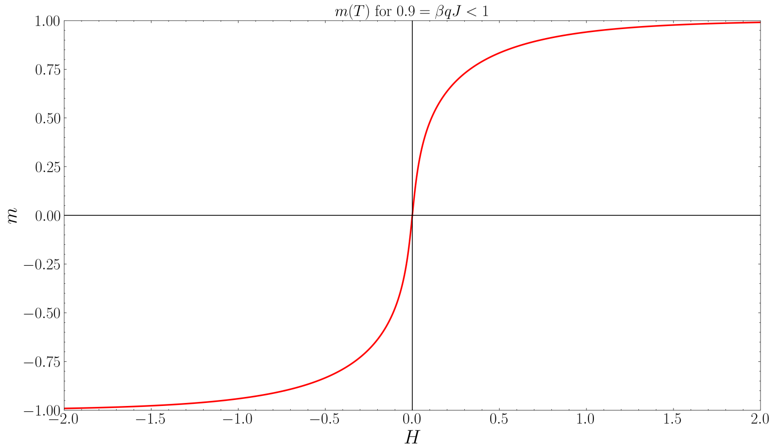
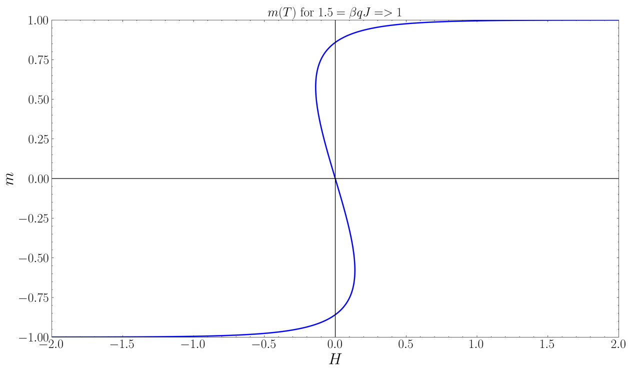
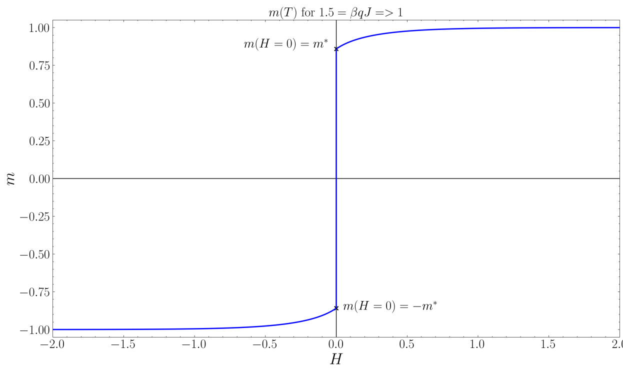


Figure 2: At higher temperatures, the graph of magnetization versus the applied magnetic field

At lower temperatures, $T < \frac{qJ}{k_B}$, the graph is quite frankly, ridiculous, because for a specific applied magnetic field H, there is a region, where 3 values of magnetization are possible. This apparent oddity comes from our approximation. In this lower temperature regime, there are three solutions to $d_r = 1$, as we have seen in the graph before. This directly follows the 2 stationary points of d_r These correspond to 3 stationary points in c_r. Instead of choosing local minima the maximum contribution comes from the absolute maximum. So the wrong solutions need to be rejected.



(a) Complete solution of m vs H.



(b) After rejection of idiotic solutions.

Figure 3: The complete and corrected graph of magnetization vs applied magnetic field.

A nice thing can be noted if we plot, for a given qJ and $T < \frac{qJ}{k_B}$, the graph of c_r and for two values $H = -h$ and $H = h$. We observe in the graph of c_r there are 2 maxima and 1 minima. One maximum is the absolute maximum when $H = h$ and the other maxima is attains the maximum value when $H = -h$. This is the core property which causes the phase transition to occur for this model. The maximum value of c_{r_o} comes for different values of r_o if the limit of magnetic field is taken $H \longrightarrow 0-$ and $H \longrightarrow 0+$. The graph of c_r vs r for a low enough temperature is plotted below for small values $H = -h, h$. From this we determine the critical temperature to be,

$$T_c = \frac{qJ}{k_B} \tag{3.22}$$

Here are some supplemental graphs to cement the fact that the peak scales with N and the width of the peak scales as $N^{\frac{1}{2}}$, for both $T < T_c$ and $T > T_c$. These graphs are not really important, but since I spent some time coding and beautifying the graph, it would be a waste not to put it in the article.

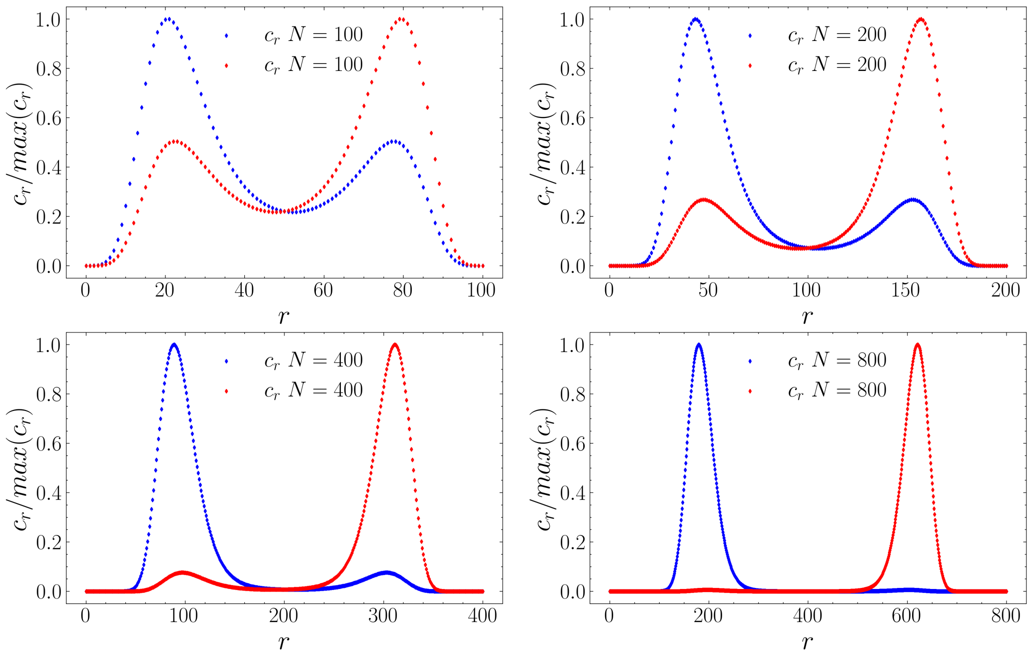


Figure 4: The graph of c_r vs r for $T < T_c$ for different values of $N = 100, 200, 400, 800$

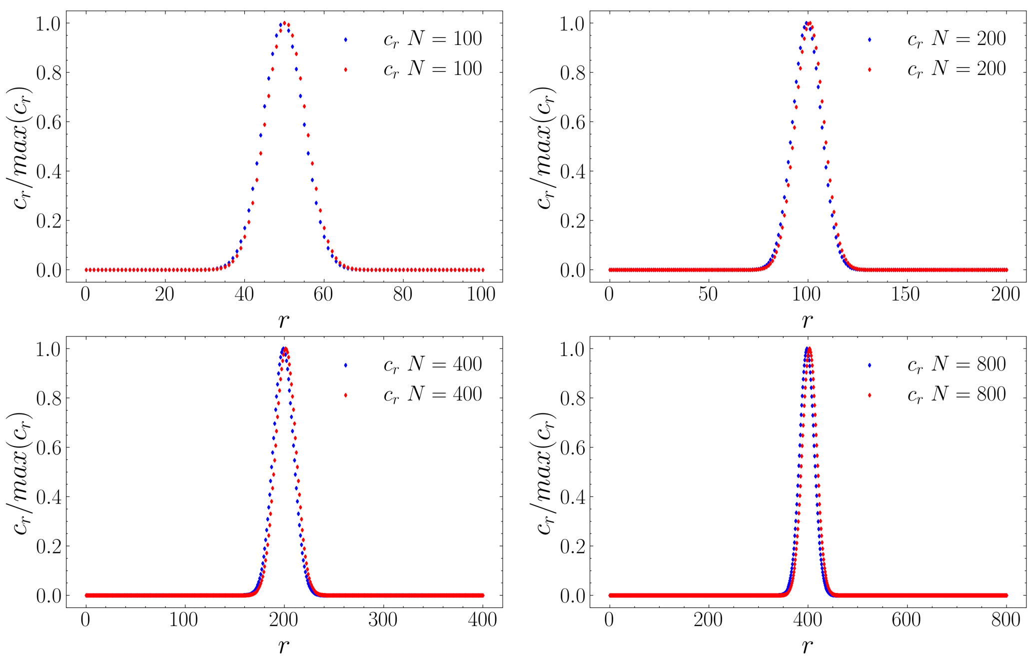


Figure 5: The graph of c_r vs r for $T > T_c$ for different values of $N = 100, 200, 400, 800$

Intermission

The report has talked about models that have been mathematically quite intuitive to solve. We move onto the biggest task of the article. The big behemoth known as the 2D Ising Model. The solution outlined here is the one given at [1]. The original solution was by Onsager in [5]. It was simplified later by his PhD student, Bruria Kaufman, [14] by analyzing spinors. Fisher [15] presented a solution by analyzing something called a decorated lattice, using the number of Dimer configurations in a lattice given by Kastelyn [16]. The solutions outlined in Baxter require some mathematical heavy lifting, using complex functions and complex variables. Coming up with a solution of this form is quite hard but is easy to follow once you have seen it. We will try to "motivate" the solution of the model. For ease of understanding the major results of each important subsection will be summarized in Result boxes at the end.

The format of the second part of the article will begin with Rudolf Peierls's Proof [4] that a phase transition takes place at a non-zero temperature for lattice with dimension $d > 1$. *Then using duality relations of Lattices, Kramers and Wannier [17] found out the critical temperature of the 2D Ising Model. We then outline the solution of the Ising Model at the critical temperature. We start with this, since at critical temperature we deal with fairly easy trigonometric functions, however, lower than critical temperatures, we resort to elliptic functions. Let's brew a cup of coffee and begin on the perilous journey to Mordor in our quest to understand the 2D Ising Model.*

4. Peierls's proof for Phase transition

Ising showed using spin-spin correlations that the 1D Ising model does not exhibit a phase transition, and incorrectly assumed that the same holds for higher dimensions. However, in a very short paper Rudolf Peierls [4] showed that the 2D Ising model does exhibit a phase transition, using shorelines. We shall outline the proof here. This proof and subsequent explanations are taken from [18].

The way Peierls's proof works is by showing that if we take a lattice and fix all spins on the boundary to be up, if the phase transition occurs then the probability of a site having spin antiparallel to the boundary inside the lattice should diverge to go to zero. Define ν as a configuration of spins in our system. We consider three sets of spins configurations,

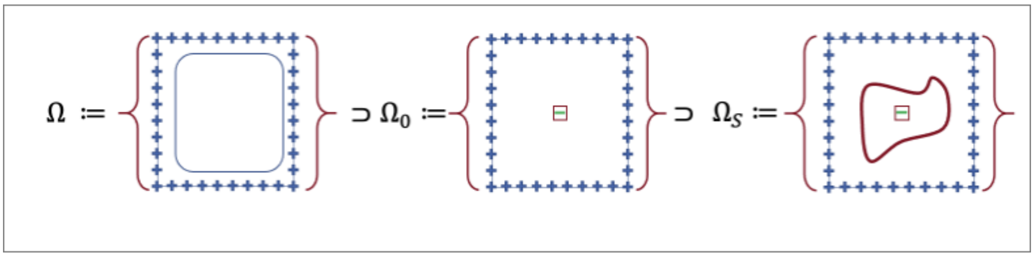


Figure 6: The three sets of configurations considered in Peierls's proof. Ω is the set of all configurations with up boundary conditions, Ω_0 is the set of configurations with one spin inside to be antiparallel to the boundary and Ω_S is the set of configurations with a shoreline, which seperates islands of down spins in seas on up spins. (Source: [18])

We want to show that the probability of a configuration in Ω_0 goes to zero. To do this directly is hard, so we will instead find the probability of configurations in Ω_S. In Ω_S, we have shorelines which separate unlike spins from each other. Thus, if two spins σ_i and σ_j are separated by a shoreline, then their product is $\sigma_i\sigma_j = -1$. Then Ω_S separates a configuration into islands of down spins and seas of up spins with length s. Then, the probability of a configuration in Ω_0 is given by the sum over all possible shorelines S. The probability of $\nu \in \Omega_S$,

$$\begin{aligned} P(\nu \in \Omega_S) &= \frac{1}{Z_\Omega} \sum_{\nu \in \Omega_S} \exp(\beta H_\nu) \\ &= \frac{1}{Z_\Omega} \sum_{\nu \in \Omega_S} \exp\left(\beta J \sum_{i,j} \sigma_i \sigma_j\right) \end{aligned} \tag{4.1}$$

We can separate the sum over the shorelines and the spin-spin contribution not over the shorelines,

$$P(\nu \in \Omega_S) = \frac{1}{Z_\Omega} \sum_{\nu \in \Omega_S} \exp(\beta J n(S)) \exp\left(\beta J \sum_{i,j \notin S} \sigma_i \sigma_j\right) \tag{4.2}$$

Consider a spin configuration ν in Ω_S and flip all the spins in the shoreline S, surrounding the center spin. Let this system be denoted as Ω'_S.

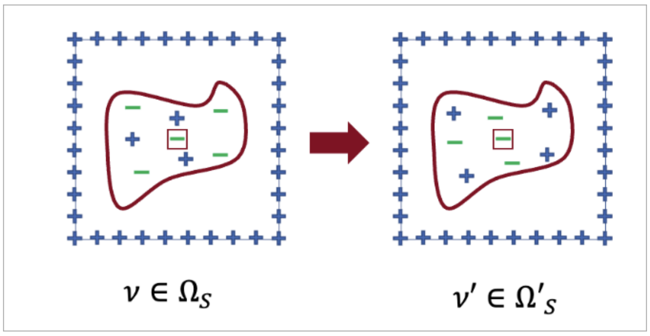


Figure 7: The configuration Ω'_S obtained by flipping all spins in the shoreline S of the configuration Ω_S. (Source: [18])

For all $\nu' \in \Omega'_S$,

$$\sum_{i,j} \sigma'_i\sigma'_j = \sum_{i,j\notin S} \sigma'_i\sigma'_j + \sum_{i,j\in S} \sigma'_i\sigma'_j = \sum_{i,j\notin S} \sigma'_i\sigma'_j + n(S) \tag{4.3}$$

Note that flipping inside spins means $\sigma'_i = -\sigma_i$. Using this we obtain,

$$\sum_{i,j\notin S} \sigma'_i\sigma'_j = \sum_{i,j\notin S} \sigma_i\sigma_j \tag{4.4}$$

Thus, we can also say that,

$$\sum_{i,j\notin S} \sigma_i\sigma_j = \sum_{i,j\notin S} \sigma'_i\sigma'_j - n(S) < \sum_{i,j\notin S} \sigma'_i\sigma'_j \tag{4.5}$$

Following the calculations in [18], we can then write,

$$P(\nu \in \Omega_S) < e^{-\beta J n(S)} \tag{4.6}$$

Let η be the set of all shorelines. Then the probability of $\nu \in \Omega_0$ is obtained by summing over all shorelines S,

$$P(\nu \in \Omega_0) = \sum_{S\in\eta} P(\nu \in \Omega_S) < \sum_{n=1}^{\infty} \aleph(n)e^{-\beta J n(S)} \tag{4.7}$$

where $\aleph(n)$ is the number of shorelines of length n. A very crude estimate can be found as,

$$\aleph(n) < 4^n \tag{4.8}$$

Using this we can write,

$$P(\nu \in \Omega_0) < \sum_{n=1}^{\infty} \aleph(n)^{-\beta J n(S)} < \sum_{n=1}^{\infty} \left(4e^{-\beta J}\right)^n \underset{\lim \beta\longrightarrow\infty}{\longrightarrow} 0 \tag{4.9}$$

Thus, for sufficiently low temperatures the probability of a configuration in Ω_0 goes to zero, which means that the probability of a spin being antiparallel to the boundary condition goes to zero. This shows that the 2D Ising model does exhibit a phase transition. For more details refer to [4] and [18].

5. Kramers and Wannier Duality relations

Before the complete solution to the 2D Ising model was published by Onsager , Kramers and Wannier provided the critical temperature of the model, using the duality relations of the model. They showed a relation between the high and low temperature expansions of the model.

5.1. Duality relation on a square lattice

Let us consider the duality relations of the 2D ising model on a square lattice. We can locate this temperature by considering the duality relations of the model.

Consider the 2D Ising model on a square lattice $\mathfrak{L}$ with the Hamiltonian with periodic boundary conditions,

$$H = -J \sum_{\{i,j\}} \sigma_i \sigma_j - J' \sum_{\{i,k\}} \sigma_i \sigma_k \tag{5.1}$$

where the first sum of spins is over nearest horizontal neighbors (i, j) and the second sum is over nearest vertical neighbors (i, k). The periodic boundary conditions imply that the first row is just below the last row and the first column is just to the right of the last column. This is equivalent to a torus. The partition function can be given as,

$$\begin{aligned} Z &= \sum_{\{\sigma\}} \exp(-\beta H) \\ &= \sum_{\{\sigma\}} \exp\left(K \sum_{\{i,j\}} \sigma_i \sigma_j + L \sum_{\{i,k\}} \sigma_i \sigma_k \right) \end{aligned} \tag{5.2}$$

We now represent the partition into two similar representations and then find the relation between them, from which we get the critical temperature of the model.

5.1.1. Low temperature expansion

Consider the lattice $\mathfrak{L}$. To make the lattice a square one, we force a constraint which makes the number of rows as the same as the number of rows M. Then draw an edge if the spins σ_i and σ_j are the same where i, j are nearest neighbors. Let the number of unlike horizontal pairs be s and the number of unlike vertical pairs be r.

Then draw an edge if the spins σ_i and σ_j are the same where i, j are nearest neighbors. Then the graph of $\mathfrak{L}$ has $M - s$ horizontal edges and $M - r$ vertical edges. Each summand in the partition function can then be written as,

$$\exp[K(M - 2s) + L(M - 2r)] \tag{5.3}$$

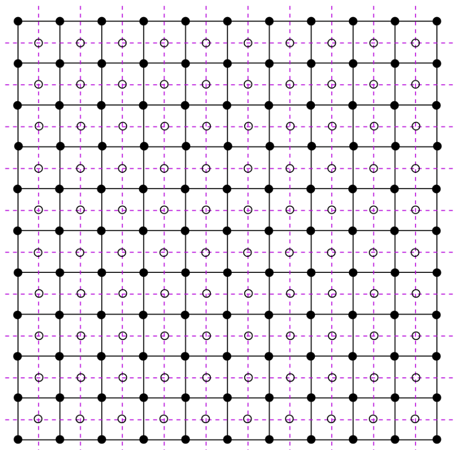


Figure 8: The lattice $\mathfrak{L}$ in dark with filled-in sites and the dual lattice $\mathfrak{L}_D$ in light with empty sites.

We now map a configuration on the lattice $\mathfrak{L}$ to its dual $\mathfrak{L}_D$ (see above Figure 8). The lattice $\mathfrak{L}$ has edges connecting like spins. The sites of $\mathfrak{L}_D$ are drawn at the centers of the squares in $\mathfrak{L}$. If there is no edge between two sites in $\mathfrak{L}$, draw the corresponding edge in $\mathfrak{L}_D$ which crosses the hypothetical edge in $\mathfrak{L}$ perpendicularly. Note that this edge in $\mathfrak{L}_D$ separates two unlike spins. This generates a set of edges in $\mathfrak{L}_D$ have a one-to-one correspondence with the edges in $\mathfrak{L}$. Thus, there are $M - s$ horizontal edges and $M - r$ vertical edges in $\mathfrak{L}_D$. We can regard the spins in $\mathfrak{L}$ to be on the faces of $\mathfrak{L}_D$.

One can observe that in any specific row of $\mathfrak{L}_D$, there must be an even number of edges. This can be shown to be true, by contradiction. Suppose there is only one edge in a row. Thus, the spins on the left and right of this edge must be different. The lattice $\mathfrak{L}$ has periodic boundary conditions. This implies that the spins on the left and right of this edge must be the same. This is a contradiction. One can extend this argument to any number of odd spins, by noting that the spins left and right of sections which have an even number of edges between them are the same. Thus, there must be an even number of edges in each row of $\mathfrak{L}_D$. The square lattice $\mathfrak{L}$ and correspondingly the dual lattice $\mathfrak{L}_D$ is the same if you rotate by 90 degrees. Thus, the number of edges in each column of $\mathfrak{L}_D$ must also be even.

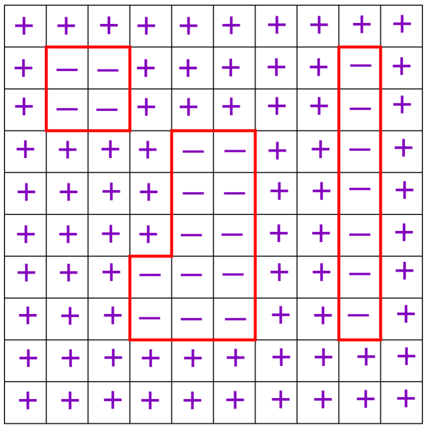


Figure 9: The polygons in $\mathfrak{L}_D$ formed by the edges. Observe that regions of up (+) spins and down (−) spins are separated by these polygons.

As a result of this, the edges can be connected to form polygons. A subtle edge case where you cannot form polygons would be when you have three consecutive rows of alternating spins. One way to form polygons from this without violating anything in the previous argument. Consider that the end column has two pseudo-edges on the left and the right, on after the last spin and one before the first spin. Since the three rows alternate in the signs of the spins, we can form a rectangle using the edges above and below the first row and the edges to the left of the first row and to the right of the last row. This rectangle is a polygon in $\mathfrak{L}_D$ and does not violate the argument that there must be an even number of edges in each row and column.

Thus these polygons in $\mathfrak{L}_D$ separate this lattice into regions of up spins and down spins. Given a configuration of polygons, there are two spin configurations corresponding to it. The two spin configurations are obtained by flipping the spins in the regions of up spins and down spins. The partition function can be written as,

$$Z = 2\exp(M(K+L)) \sum_{P \in \mathfrak{L}_D} \exp(-2Lr - 2Ks) \tag{5.4}$$

where P is the set of polygons in $\mathfrak{L}_D$ and s and r are the number of horizontal and vertical edges in P. This is called the low temperature expansion of the partition function because for low temperatures the maximum contribution comes from $r = s = 0$, because K, L are large and positive.

Result 3 (Low Temperature Expansion): The low temperature expansion of the partition function is given by,

$$Z = 2\exp(M(K+L)) \sum_{\{P \in \mathfrak{L}_D\}} \exp(-2Lr - 2Ks) \tag{5.5}$$

where P is the set of polygons in $\mathfrak{L}_D$ and s and r are the number of horizontal and vertical edges in the polygon P.

5.1.2. High temperature expansion

Now we consider the high temperature expansion of the partition function. Let us consider $\exp\left(K\sigma_i\sigma_j\right)$. σ_i, σ_j can only take the values ± 1. Observe that $\left(\sigma_i\sigma_j\right)^2 = 1$. Taking the exponential and expand the Taylor series for the exponential, we obtain,

$$\exp\left(K\sigma_i\sigma_j\right) = \cosh(K) + \sigma_i\sigma_j \sinh(K) \tag{5.6}$$

This means that we can write the partition function (5.2) as,

$$Z = (\cosh(K)\cosh(L))^M \sum_{\{\sigma\}} \prod_{i,j} \left(1 + v\sigma_i\sigma_j\right) \prod_{i,k} (1 + w\sigma_i\sigma_k) \tag{5.7}$$

where $v = \tanh(K)$ and $w = \tanh(L)$. The first product is over the horizontal nearest neighbors and the second product is over the vertical nearest neighbors. There is an excellent representation of the terms in the partition function in terms of a graph.

Consider the two products,

$$\prod_{i,j} \left(1 + v\sigma_i\sigma_j\right) \prod_{i,k} (1 + w\sigma_i\sigma_k)$$

Note that there are 2^M terms in the expanded product. We can create a graph on the lattice $\mathfrak{L}$ for each term in the product. Take any term in the product and then made a graph on the lattice $\mathfrak{L}$ by following the following rules,

1. If the term contains $v\sigma_i\sigma_j$, then draw a horizontal edge between the sites i and j.
2. If the term contains $w\sigma_i\sigma_k$, then draw a vertical edge between the sites i and k.

Each horizontal and vertical nearest neighbor spin pair contributes to each term in the expansion. A vertical edge (i,k) can provide a contribution of either $w\sigma_i\sigma_k$ or 1 to the term. Similarly, a horizontal edge (i,j) can provide a contribution of either $v\sigma_i\sigma_j$ or 1 to the term. Thus, each term in the expansion corresponds to a graph on the lattice $\mathfrak{L}$. Then each term in the expansion can be written in the form

$$v^r w^s \sigma_1^{n_1} \sigma_2^{n_2} \ldots \sigma_N^{n_N} \tag{5.8}$$

where r is the number of horizontal edges, s is the number of vertical edges, n_i is the number of edges that go into the site i. We can now sum over all the spin configurations $\{\sigma\}$ and each σ_i can only take values ± 1. This implies upon summing over all the spin configurations only the terms where n_i is even will contribute to the sum. If any n_i is odd, consider the two spin configurations $\sigma_i = +1$ and $\sigma_i = -1$, keeping the rest of the spins the same. The two terms will cancel each other out. Thus, only the terms are considered which have an even number of edges going into each site. We can reduce this further. In the remaining terms, flipping one spin will not change the value of the term. If the spin site already contributes nothing to the term, flipping it does not matter and if the spin site contributes an even edge number to the term flipping the spin also remains the same because $\sigma_i^{2n} = 1$. Thus, we have a degeneracy of 2^N for each term.

Thus the partition function can be written as,

$$Z = 2^N (\cosh(K)\cosh(L))^M \sum_{\{r,s\}} v^r w^s$$

where the sum is over all the graphs on the lattice $\mathfrak{L}$ with r horizontal edges and s vertical edges. Note the terms only contribute if the number of edges going into each site is even. These are constructed into polygons as described in the low temperature expansion. The sum over the horizontal and vertical edges then becomes a sum over the polygons in the lattice $\mathfrak{L}$. The high temperature expansion of the partition function can then be written as,

$$Z = 2^N (\cosh(K)\cosh(L))^M \sum_{\{P\in\mathfrak{L}\}} v^{r(P)} w^{s(P)} \tag{5.9}$$

Result 4 (High Temperature Expansion): The high temperature expansion of the partition function is given by,

$$Z = 2^N (\cosh(K)\cosh(L))^M \sum_{\{P\in\mathfrak{L}\}} v^{r(P)} w^{s(P)} \tag{5.10}$$

where P is the set of polygons in $\mathfrak{L}$ and $r(P)$ and $s(P)$ are the number of horizontal and vertical edges in the polygon P.

5.2. Duality Relation

The two expansions of the partition function can be equated to obtain a relation between the low and high temperature expansions. The low temperature expansion Result 3 and the high temperature expansion Result 4 are very similar except that the low temperature expansion has a summation over the polygons in the dual lattice $\mathfrak{L}_D$ and the high temperature expansion has a summation over the polygons in the lattice $\mathfrak{L}$. The two lattices $\mathfrak{L}$ and $\mathfrak{L}_D$ only differ at the boundary conditions, which should not matter in the thermodynamic limit. Thus, the free energy per site obtained using either expression must be equal.

Let the rescaled free energy per site be represented with $k_B T\varphi$,

$$-\varphi = \lim_{\{N\longrightarrow\infty\}} N^{-1} \ln Z_N \tag{5.11}$$

Note that if we consider the lattice $\mathfrak{L}$ then the number of horizontal or vertical edges M is equal to $N - \sqrt{N}$. For the lattice with periodic boundary conditions M is equal to N. In both cases if we take the thermodynamic limit $N \longrightarrow \infty$, the limit $M/N \longrightarrow 1$. Using Result 3 and Result 4, we can write the free energy per site as,

$$\begin{aligned} -\varphi &= K + L + \Phi(e^{-2L}, e^{-2K}) \\ &= \ln(2\cosh(K)\cosh(L)) + \Phi(\tanh(K), \tanh(L)) \end{aligned} \tag{5.12}$$

where $\Phi(x, y)$ is the function defined as,

$$\Phi(x, y) = \lim_{N\longrightarrow\infty} N^{-1} \ln \sum_{\{P\in\mathfrak{L}\}} x^{r(P)} y^{s(P)} \tag{5.13}$$

If we perform a suitable variable substitution, the two free energies are equal.

$$\begin{aligned} \tanh(K^*) &= \exp(-2L) \\ \tanh(L^*) &= \exp(-2K) \end{aligned} \tag{5.14}$$

We obtain from (5.12) and (5.14), the duality relation,

$$\begin{aligned} \varphi(K, L) + K + L &= \varphi(K^*, L^*) + \ln(2\cosh(K^*)\cosh(L^*)) \\ \implies \varphi(K^*, L^*) &= \varphi(K, L) + K + L - \ln(2\cosh(K^*)\cosh(L^*)) \end{aligned} \tag{5.15}$$

Observe the relation (5.14) maps high temperature coefficients to low temperature coefficients and vice versa. $\tanh(x)$ is a monotonically increasing function. If K^*, L^* are high (which means low temperature), then K, L are low (which means high temperature). This is the duality relation of the 2D Ising model. We can cast this into a much more symmetric and useful form.

Using the first relation in (5.14)

$$\begin{aligned}\tanh(K^*) &= \exp(-2L)\\ \implies \sinh(2K^*) &= \frac{2e^{-2L}}{1-e^{-4L}}\\ \implies \sinh(2K^*)\sinh(2L) &= 2\frac{e^{-2L}}{1-e^{-4L}}\frac{e^{2L}-e^{-2L}}{2}\\ \implies \sinh(2K^*)\sinh(2L) &= 1\end{aligned} \tag{5.16}$$

A similar calculation can also be done for the second relation in (5.14). We obtain the more useful form of the duality relations,

$$\begin{aligned}\sinh(2K^*)\sinh(2L) &= 1\\ \sinh(2K)\sinh(2L^*) &= 1\end{aligned} \tag{5.17}$$

Also observe $\sinh(2K^*)\sinh(2L^*) = (\sinh(2K)\sinh(2L))^{-1}$. Using this we can recast (5.15) into a much nicer form, the usefulness of which we will see just later.

$$\begin{aligned}\varphi(K^*, L^*) &= \varphi(K, L) + K + L - \ln(2\cosh(K^*)\cosh(L^*))\\ \implies \varphi(K^*, L^*) &= \varphi(K, L) - \frac{1}{2}\ln(4\cosh^2(K^*)\cosh^2(L^*)\tanh(K^*)\tanh(L^*))\\ \implies \varphi(K^*, L^*) &= \varphi(K, L) - \frac{1}{2}\ln(\sinh(K^*)\sinh(L^*))\\ \implies \varphi(K^*, L^*) &= \varphi(K, L) + \frac{1}{2}\ln(\sinh(2K)\sinh(2L))\end{aligned} \tag{5.18}$$

The duality relations between the dual lattice $\mathfrak{L}_D$ and the lattice $\mathfrak{L}$ in terms of the variables K ,L, K^* and L^* are given by,

$$\begin{aligned}&\sinh(2K^*)\sinh(2L) = 1\\ &\sinh(2K)\sinh(2L^*) = 1\\ &\varphi(K^*, L^*) = \varphi(K, L) + \frac{1}{2}\ln(\sinh(2K)\sinh(2L))\end{aligned} \tag{5.19}$$

Now we arrive at our main point, the argument to find the critical temperature of the 2D Ising model. Consider the Isotropic case when $J = J'$ which implies $K = L$ and $K^* = L^*$. At the critical point the free energy per site should be non-analytic function of T and thus of K. If the free energy is non-analytic for some $K = K_c$, then it should also mean that the free energy is non-analytic for $K^* = K_c^*$. Generally this should correspond to a different value of K^*, but if there is only one critical temperature, then that would be at $K^* = K_c^* = K_c$. Note that the free energy functionals at the critical point should be equal, even for the anisotropic case, because $\sinh(2K)\sinh(2L) = 1$. The critical temperature is then given by the condition,

$$\begin{aligned}&\sinh(2K_c) = 1\\ \implies &e^{2K_c} = 1 + \sqrt{2}\\ \implies &K_c = \frac{\ln\left(1+\sqrt{2}\right)}{2}\\ \implies &T_c = \frac{J}{k_B \ln\left(1+\sqrt{2}\right)/2}\\ \implies &T_c \approx 2.69185\frac{J}{k_B}\end{aligned} \tag{5.20}$$

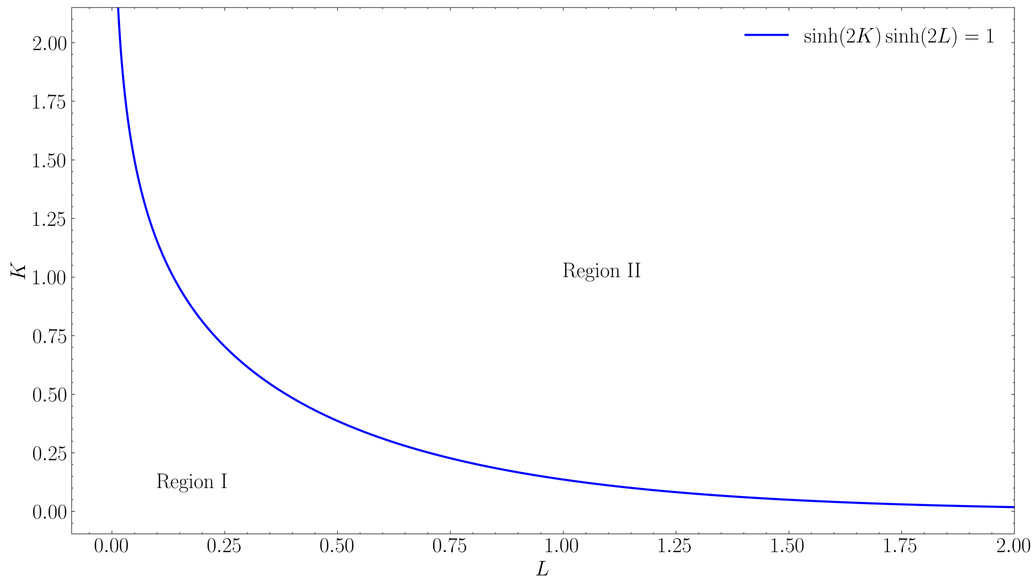


Figure 10: The duality relation $\sinh(2K)\sinh(2L) = 1$ in the $K - L$ plane. The critical line is shown in blue. The duality relation maps the region I to the region II and vice versa. The critical line is the line where the free energy remains the same under the duality transformation.

For the anisotropic case, see the figure below the duality relation maps region I to region II and vice versa. The free energy remains the same under the duality transformation on the line

$$\sinh(2K)\sinh(2L) = 1 \tag{5.21}$$

Thus, if there is only one line of critical points for the anisotropic 2D Ising model, it must be on this line.

Result 5 (Critical Temperature of the 2D Ising Model): The duality relations between the high and low temperature expansions are given by,

$$\sinh(2K^*)\sinh(2L) = 1$$
$$\sinh(2K)\sinh(2L^*) = 1$$
$$\varphi(K^*, L^*) = \varphi(K, L) + \frac{1}{2}\ln(\sinh(2K)\sinh(2L))$$

Using these, if there is only critical point the critical temperature of the 2D Ising model is given by,

$$\begin{aligned} T_c &= \frac{J}{k_B \ln\left(1+\sqrt{2}\right)/2} \\ &\approx 2.69185\frac{J}{k_B} \end{aligned} \tag{5.22}$$

.

We can extend these duality relations on the square lattice to other types of lattices as well. The solution to the 2D Ising model by Baxter relies on the Honeycomb-Triangular duality relations and the star-triangle relations.

6. Honeycomb-Triangular Duality

The Ising model can be constructed on any lattice, the ones of our concern here are the honeycomb and the triangle lattices. Unlike the square lattice which is self-dual, the honeycomb lattice is dual to the triangular lattice and vice versa.

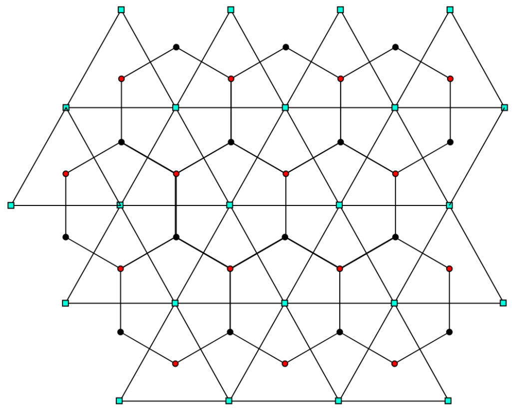


Figure 11: The honeycomb lattice with sites in red and black and the triangular lattice sites in turqoise formed by duality.

The different colors on the honeycomb lattice in Figure 11 show the fact that it is a bipartite graph.

Definition 6.1 (Bipartite Graph): A bipartite graph is a graph whose vertices can be divided into two disjoint sets such that every edge connects a vertex in one set to a vertex in the other set.

In the case of the honeycomb lattice, the vertices can be colored red and black such that no two adjacent vertices have the same color. This just means that the red vertices are only connected to the black vertices and vice versa.

The honeycomb lattice $\mathfrak{H}$ is defined on N sites and in general depends on three interaction coefficients. The square lattice $\mathfrak{L}$ has two interaction coefficients in two directions J and J'. The edges in $\mathfrak{H}$ has edges in three directions with interaction coefficients $-k_BTL_i$ where $i \in \{1, 2, 3\}$. The labels in Figure 12 show the directions of the edges in the honeycomb lattice. The 2D Ising model on the honeycomb lattice $\mathfrak{H}$ takes the following partition function for N sites,

$$Z_N^{\mathfrak{H}}(\{\sigma\}) = \sum_{\sigma} \exp\left(L_1 \sum_{i,j} \sigma_i\sigma_j + L_2 \sum_{i,k} \sigma_i\sigma_k + L_3 \sum_{i,l} \sigma_i\sigma_l \right) \tag{6.1}$$

The first summation is over the edges in the direction labelled by L_1, the second summation is over the edges in the direction labelled by L_2 and the third summation is over the edges in the direction labelled by L_3.

The partition function for the triangular lattice $\mathfrak{T}$ is defined similarly with interaction coefficients $-k_BTK_i$ where $i \in \{1, 2, 3\}$. The edges in the triangular lattice are labelled by K_1, K_2, and K_3 as shown in Figure 12. The partition function for the triangular lattice is given by,

$$Z_N^{\mathfrak{T}}(\{\sigma\}) = \sum_{\sigma} \exp\left(K_1 \sum_{i,j} \sigma_i \sigma_j + K_2 \sum_{i,k} \sigma_i \sigma_k + K_3 \sum_{i,l} \sigma_i \sigma_l \right) \tag{6.2}$$

The first summation is over the edges in the direction labelled by K_1, the second summation is over the edges in the direction labelled by K_2 and the third summation is over the edges in the direction labelled by K_3. The indexing is done so as the edges labelled by L_i are in a direction perpendicular to the edges labelled by K_i.

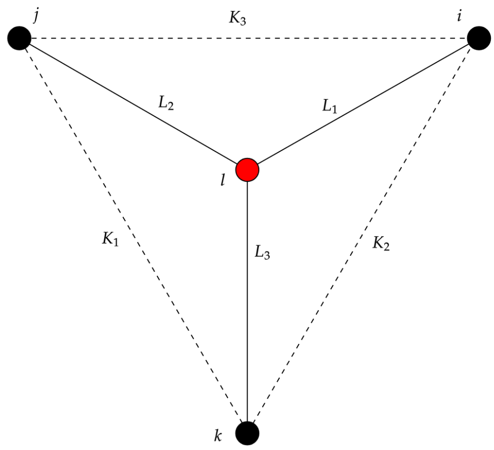


Figure 12: The directions are in the edges labelled by L_1, L_2, and L_3 for the honeycomb lattice. The red and black vertices are the two sets of the bipartite graph. The directions are in the edges labelled by K_1, K_2, and K_3 for the triangular lattice.

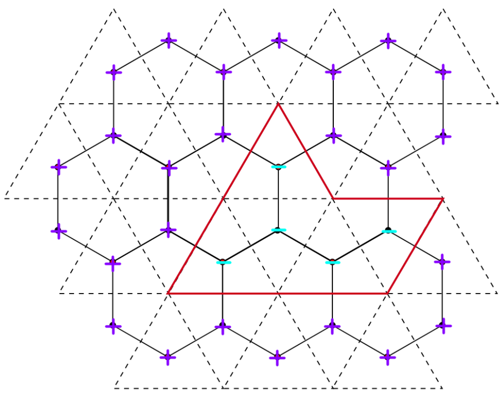


Figure 13: Polygons in the triangular lattice seperate up spin regions from the down spin regions in the honeycomb lattice.

Following a similar argument to that of the square lattice ising model, we can form polygons on the dual triangular lattice $\mathfrak{T}$ of the honeycomb lattice $\mathfrak{H}$. Observe from Figure 14 that each black vertex can map to one vertex in the triangular lattice. There is certain row-like structure in the honeycomb lattice $\mathfrak{H}$, with alternating red and black vertices. Thus, a triangular lattice of N is the dual lattice of a honeycomb lattice of $2N$ vertices. We now perform a low temperature expansion of the honeycomb lattice with $2N$ sites,

$$Z_{2N}^{\mathfrak{H}}(\{K\}) = 2\exp(N(L_1 + L_2 + L_3)) \sum_{P\in\mathfrak{T}} \exp(-2L_1r_1 - 2L_2r_2 - 2L_3r_3) \tag{6.3}$$

The summation is over the polygons in the triangular lattice $\mathfrak{H}$ with r_i being the number of edges in the polygon in the direction labelled by L_i. Note that for $2N$ sites the number of horizontal edges is N.

Performing a high temperature expansion for the triangular lattice $\mathfrak{T}$ for N sites, we obtain,

$$Z_N^{\mathfrak{T}}(\{K\}) = (2\cosh(K_1)\cosh(K_2)\cosh(K_3))^N \sum_{\{P\in\mathfrak{T}\}} v_1^{r_1} v_2^{r_2} v_3^{r_3} \tag{6.4}$$

where $v_i = \tanh(K_i)$ for $i \in \{1,2,3\}$. In the thermodynamic limit, this summation over polygons will be the same as the summation over polygons in (6.3). Comparing the two expansions we obtain the following relation holds true,

$$Z_{2N}^{\mathfrak{H}}(\{L\}) = (2s_1s_2s_3)^{N/2} Z_N^{\mathfrak{T}}(\{K\}) \tag{6.5}$$

where

$$\begin{aligned} s_i &= \frac{1}{2}\exp(2L)\operatorname{sech}^2\left(K_j\right) \\ &= \sinh\left(2L_j\right) = \left(\sinh\left(2K_j\right)\right)^{-1} \qquad j \in \{1,2,3\} \end{aligned} \tag{6.6}$$

if the following relations hold,

$$\tanh\left(K_j\right) = \exp\left(-2L_j\right) \qquad j \in \{1,2,3\} \tag{6.7}$$

Similar to the square lattice, this is the duality relations between the honeycomb and triangular lattices. This maps the high temperature expansion of the honeycomb lattice to the low temperature expansion of the triangular lattice and vice versa.

Result 6 (Honeycomb-Triangular Duality): The honeycomb lattice $\mathfrak{H}$ is dual to the triangular lattice $\mathfrak{T}$ and vice versa. The high temperature expansion of the honeycomb lattice with $2N$ sites is equal to the low temperature expansion of the triangular lattice with N sites and vice versa.

The relation between the two partition functions on the honeycomb $\mathfrak{H}$ and triangular $\mathfrak{T}$ lattices is given by,

$$Z_{2N}^{\mathfrak{H}}(\{L\}) = (2s_1s_2s_3)^{N/2} Z_N^{\mathfrak{T}}(\{K\}) \tag{6.8}$$

where

$$\begin{aligned} s_i &= \frac{1}{2}\exp(2L)\operatorname{sech}^2\left(K_j\right) \\ &= \sinh\left(2L_j\right) = \left(\sinh\left(2K_j\right)\right)^{-1} \qquad j \in \{1,2,3\} \end{aligned} \tag{6.9}$$

if the following relations hold,

$$\tanh\left(K_j\right) = \exp\left(-2L_j\right) \qquad j \in \{1,2,3\} \tag{6.10}$$

This is not enough for us to locate the critical point of the Ising model on $\mathfrak{H}$ or $\mathfrak{T}$. This maps a high temperature expansion of $\mathfrak{H}$ to a low temperature expansion of $\mathfrak{T}$ and vice versa. The star triangle relation derived just after this maps a low temperature expansion of $\mathfrak{H}$ to a low temperature expansion of $\mathfrak{T}$ and vice versa. Using the relations in this section and the star triangle relations in the next section, we can also locate the critical point of the 2D Ising model on $\mathfrak{H}$ and $\mathfrak{T}$.

7. Star-Triangle Relation

There is another relation relating the partition functions of the honeycomb $\mathfrak{H}$ and triangle $\mathfrak{T}$ lattices. This is the star-triangle relation, by which we relate the Boltzmann weights of the star and triangle units. The honeycomb lattice $\mathfrak{H}$ can be comprised of repeating star cells and the triangular lattice $\mathfrak{T}$ can be comprised of repeating triangle cells[cf. Figure 12]. The triangle is obtained by summing over the red vertex l in Figure 12, which is the vertex in the center of the star.

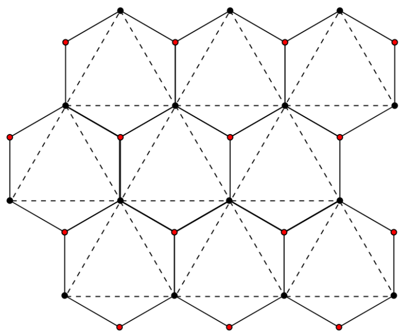


Figure 14: The triangular lattice generated by the star triangle relation from the honeycomb lattice.

Note that the honeycomb lattice $\mathfrak{H}$ is bipartite. The summand in (6.1) can be written as,

$$\prod_{l} W\big(\sigma_l|\sigma_i, \sigma_j, \sigma_k\big) \tag{7.1}$$

where the product is over the red vertices l in Figure 14 and the Boltzmann weight W is defined as,

$$W\big(\sigma_l|\sigma_i, \sigma_j, \sigma_k\big) = \exp\big[\sigma_l\big(L_1\sigma_i + L_2\sigma_j + L_3\sigma_k\big)\big] \tag{7.2}$$

Let us denote the set of black vertices by A and the set of red vertices by B. Note during the summation written in the form of (7.1), the spin σ_l appears in only one factor of the product. We can then sum over the spins in the set B to obtain the partition function as,

$$Z_N^{\mathfrak{H}}(\{L\}) = \sum_{\sigma\in A} w\big(\sigma_i, \sigma_j, \sigma_k\big) \tag{7.3}$$

where the Boltzmann weight w is defined as,

$$\begin{aligned} w\big(\sigma_i, \sigma_j, \sigma_k\big) &= \sum_{\sigma_l\in B} W\big(\sigma_l|\sigma_i, \sigma_j, \sigma_k\big) \\ &= 2\cosh\big(L_1\sigma_i + L_2\sigma_j + L_3\sigma_k\big) \end{aligned} \tag{7.4}$$

The sum in (7.3) is over the spins in the set A which are the black vertices in Figure 14. $\sigma_i, \sigma_j, \sigma_k$ are the spins in the three edges connected to the red vertex l in each star in $\mathfrak{H}$. Note from (7.4) that $w\big(-\sigma_i, -\sigma_j, -\sigma_k\big) = w\big(\sigma_i, \sigma_j, \sigma_k\big)$. This means that there we can write $w\big(\sigma_i, \sigma_j, \sigma_k\big)$ as,

$$w\big(\sigma_i, \sigma_j, \sigma_k\big) = R\exp\big(K_1\sigma_j\sigma_k + K_2\sigma_i\sigma_k + K_3\sigma_i\sigma_j\big) \tag{7.5}$$

where R, K_1, K_2, K_3 are found later, which causes this identity to hold true for all spins $\sigma_i, \sigma_j, \sigma_k$. The partition function for $\mathfrak{H}$ is then given by,

$$Z_N^{\mathfrak{H}}(\{L\}) = R^{N/2} \sum_{\sigma\in A} \prod_{\{i,j,k\}} \exp\big(K_1\sigma_i\sigma_j + K_2\sigma_i\sigma_k + K_3\sigma_j\sigma_k\big) \tag{7.6}$$

This is exactly the partition function for the triangular lattice $\mathfrak{T}$ with $N/2$ sites (6.2). Then we can relate the partition function of $\mathfrak{H}$ on $2N$ spins with the partition function of $\mathfrak{T}$ on N spins as,

$$Z_{2N}^{\mathfrak{H}}(\{L\}) = R^N Z_N^{\mathfrak{T}}(\{K\}) \tag{7.7}$$

This is known as the star triangle relation. The relation holds true for all spins $\sigma_i, \sigma_j, \sigma_k$ and thus the Boltzmann weights R, K_1, K_2, K_3 are independent of the spins.

Result 7 (Star Triangle Relation): The star triangle relation relates a honeycomb cell in $\mathfrak{H}$ to a triangular cell in $\mathfrak{T}$[cf. Figure 12]. There exists R, K_1, K_2, K_3 such the following relation holds true independent of the spins $\sigma_i, \sigma_j, \sigma_k$,

$$2\cosh\big(L_1\sigma_i + L_2\sigma_j + L_3\sigma_k\big) = R\exp\big(K_1\sigma_i\sigma_j + K_2\sigma_i\sigma_k + K_3\sigma_j\sigma_k\big) \tag{7.8}$$

See the next section (result) for the determination of the interaction coefficients R, K_1, K_2, K_3.

7.1. Determination of Interaction coefficients

We now determine R, K_1, K_2, K_3 in (7.5). Considering all cases in (7.4) and (7.5), we can write the following equations,

$$\begin{aligned}
2c\ \ &= 2\cosh(L_1 + L_2 + L_3) = R\exp(K_1 + K_2 + K_3) \\
2c_1 &= 2\cosh(-L_1 + L_2 + L_3) = R\exp(K_1 - K_2 - K_3) \\
2c_2 &= 2\cosh(L_1 - L_2 + L_3) = R\exp(-K_1 + K_2 - K_3) \\
2c_3 &= 2\cosh(L_1 + L_2 - L_3) = R\exp(-K_1 - K_2 + K_3)
\end{aligned} \tag{7.9}$$

We can obtain K_i just from these equations and some hyperbolic identities. Furthermore, we outline the procedure to obtain K_1 here, the others can be obtained similarly.

$$\frac{cc_1}{c_2c_3} = \exp(4K_1) \tag{7.10}$$

We obtain,

$$\begin{aligned}
c_2c_3(\exp(4K_1) - 1) &= \cosh(L_1 + L_2 + L_3)\cosh(-L_1 + L_2 + L_3) \\
&\qquad - \cosh(L_1 + L_2 - L_3)\cosh(L_1 - L_2 + L_3) \\
&= 4(\sinh^2(L_2 + L_3) - \sinh^2(L_2 - L_3)) \\
&= \sinh(2L_2)\sinh(2L_3)
\end{aligned} \tag{7.11}$$

As a result of this we obtain,

$$\sinh(2K_1)\sinh(2L_1) = \frac{\sinh(2L_1)\sinh(2L_2)\sinh(2L_3)}{2(cc_1c_2c_3)^{1/2}} \tag{7.12}$$

Note that the (7.9) do not change if we permute the suffices $1, 2, 3$. The other two interaction coefficients K_2, K_3 can be obtained similarly, just by permuting the indices. The RHS of (7.12) does not change if we permute the indices. Thus, we can write the following relation,

$$\sinh(2K_i)\sinh(2L_i) = k^{-1} \qquad i \in \{1, 2, 3\} \tag{7.13}$$

This is a property of star triangle relations. The products $\sinh(2K_i)\sinh(2L_i)$ are independent of the indices i. The coefficient R is given as, Using (7.9) (7.12) (7.13), we can write the following relation,

$$\begin{aligned}
R &= 2k\sinh(2L_1)\sinh(2L_2)\sinh(2L_3) \\
&= \frac{2}{k^2\sinh(2K_1)\sinh(2K_2)\sinh(2K_3)}
\end{aligned} \tag{7.14}$$

Result 8 (Star-Triangle Relation Partition Function):

The star triangle relation relates the partition function of the honeycomb lattice $\mathfrak{H}$ with $2N$ sites to the partition function of the triangular lattice $\mathfrak{T}$ with N sites. The relation is given by,

$$Z_{2N}^{\mathfrak{H}}(\{L\}) = R^N Z_N^{\mathfrak{T}}(\{K\}) \tag{7.15}$$

where the interaction coefficients R, K_1, K_2, K_3 are given by,

$$\begin{aligned} R &= \frac{2}{k^2 \sinh(2K_1)\sinh(2K_2)\sinh(2K_3)} \\ \sinh(2K_i)\sinh(2L_i) &= k^{-1} \quad \text{for } i \in \{1,2,3\} \end{aligned} \tag{7.16}$$

and where

$$\frac{\sinh(2L_1)\sinh(2L_2)\sinh(2L_3)}{2(cc_1c_2c_3)^{1/2}} \tag{7.17}$$

The value of k can also be written in terms of $\{K\}$ as,

$$k = \frac{(1-v_1^2)(1-v_2^2)(1-v_3^2)}{4[(1+v_1v_2v_3)(v_1+v_2v_3)(v_2+v_1v_3)(v_3+v_1v_2)]^{1/2}} \tag{7.18}$$

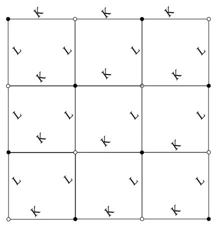
We will be using these relations show some commuting properties of the transfer matrices of the 2D Ising Model. Onsager mentioned this in his paper [5], and later on in an article in [2]. In his words, the whole solution of the 2D Ising Model derived below was something he tried to solve for some "variety" after working on 2D Ising model. He mentions that if the square lattice is constructed diagonally, the diagonal transfer matrices commute.

This star triangle relation is derived in this section can also be used to locate the critical points of the Ising model on $\mathfrak{H}$ and $\mathfrak{T}$. This star triangle relation is also important in determining the critical point of the Ising model on $\mathfrak{H}$ and $\mathfrak{T}$. For more details on the criticality of $\mathfrak{H}$ and $\mathfrak{T}$, see [1].

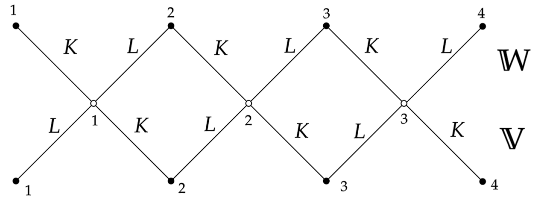
8. Transfer Matrices for the 2D Ising Model Problem

Remark (Motivation): This derivation is done in a bottom up approach, meaning we build up will the tools and details that are required to solve the model later on. There are properties of the Transfer matrix that we will derive later. I will try to provide justification as and when possible. One can also start by reading Section 9, and then refer to the relevant properties in Section 8 to better understand the motivation for the properties.

The way we will solve for the eigenvalue is by using the Transfer Matrix method. The square lattice will be taken and rotated by $\frac{\pi}{4}$,[cf Figure 15]. Then we will find some properties of the matrices which will help us determine the eigenvalues.



(a)



(b)

Figure 15: (a) The square lattice $\mathfrak{L}$ with interaction coefficients K and L in different directions. (b) The same square lattice drawn diagonally $\mathfrak{L}_R$. The filled and not filled in circles are drawn in both the pictures to show how $\mathfrak{L}$ to the rotated version

The model will have periodic boundary conditions enforced. Note that $\mathfrak{L}_R$ is a bipartite graph, with the filled circles being one set and the hollow ones being the other set. To enforce periodic boundary conditions we put a constraint on $\mathfrak{L}_R$ that the number of rows are even. If the first row starts with the filled circles and the top row is with hollow circles. Now if periodic boundary conditions are enforced, the graph will still be bipartite. The free energy is evaluated in the thermodynamic limit, which does not depend on how the number of rows go to infinity. Similarly enforce this conditions on the columns. The rightmost column will be to the left of the first column. Note that if the number of sites in each row is the same, the graph will still be bipartite.

Let us consider a set of three rows, one with black circles and one with white ones. The spin vectors of each of the rows from bottom to top is $\varphi, \varphi'', \varphi'$ in that order. The partition function then, has the term $V_{\varphi,\varphi''}W_{\varphi'',\varphi'}$ Then we define the transfer matrices as,

$$\langle\varphi|\mathbb{V}|\varphi'\rangle = V_{\varphi,\varphi'} = \exp\left(\sum_{j=1}^{n}\left[K\sigma_{j+1}\sigma'_j + L\sigma_j\sigma'_j\right]\right) \tag{8.1}$$

$$\langle\varphi|\mathbb{W}|\varphi'\rangle = W_{\varphi,\varphi} = \exp\left(\sum_{j=1}^{n}\left[K\sigma_j\sigma'_j + L\sigma_j\sigma'_{j+1}\right]\right) \tag{8.2}$$

For translational invariance, we will implement toroidal boundary conditions, which entails $\sigma_{n+1} = \sigma_1$, along with proper row placement, as discussed before.

Let us consider $m \times n$ lattice for this. Then using these, we can easily construct the partition function in terms of the transfer matrices,

$$\begin{aligned} Z_N &= \sum_{\varphi_1}\dots\sum_{\varphi_m}\langle\varphi_1|\mathbb{V}|\varphi_2\rangle\langle\varphi_2|\mathbb{W}|\varphi_3\rangle\dots\langle\varphi_{m-1}|\mathbb{V}|\varphi_m\rangle\langle\varphi_m|\mathbb{W}|\varphi_1\rangle \\ &= \sum_{\varphi}\langle\varphi|\mathbb{VW}\dots\mathbb{VW}|\varphi\rangle = \mathrm{Tr}(\mathbb{VW}\dots\mathbb{VW}) \\ &= \mathrm{Tr}\big((VW)^{\frac{m}{2}}\big) = \sum_{j=1}^{m}\Lambda_j^m \end{aligned} \tag{8.3}$$

where Λ_j^2 is the j-th eigenvalue of the matrix $\mathbb{VW}$. Note that for the last equality we use Schur's Decomposition theorem, because in general $\mathbb{VW}$ is not symmetric.

This is where the constraint of even number of rows comes into consequence. If the number of rows is odd, we will have $(\mathbb{VW})^k\mathbb{V}$ or $(\mathbb{VW})^k\mathbb{W}$ in the end, not the nice form $(\mathbb{VW})^k$ which we are using. Note that all the elements of the matrix $\mathbb{VW}$ are non-negative. Thus, the Perron Frobenius Theorem applies, which states that the largest eigenvalue is positive and its eigenspace in non-degenerate [10]. In the thermodynamic limit we then obtain,

$$Z_N \sim \Lambda_{max}^m \tag{8.4}$$

where Λ_{max}^2 is numerically the largest eigenvalue of the matrix $\mathbb{V}\mathbb{W}$.

8.1. Interchanging Relations

Note that the transfer matrices $\mathbb{V}, \mathbb{W}$ can be regarded as functions of the variables K, L. Thus, we have $\mathbb{V} = V(K, L)$ and $\mathbb{W} = W(K, L)$. Suppose we consider two sets of interaction coefficients K, L and K', L'. We are interested in what happens when we switch the $K \longleftrightarrow K'$ and $L \longleftrightarrow L'$. Thus, we want to know that when the following relation holds,

$$V(K, L)W(K', L') = V(K', L')W(K, L) \tag{8.5}$$

Again using our notation from the previous section of the three rows from the bottom to the top row being denoted by $\varphi = \{\sigma_1, \ldots, \sigma_n\}, \varphi'' = \{\sigma_1'', \ldots, \sigma_n''\}, \varphi' = \{\sigma_1', \ldots, \sigma_n'\}$, we are interested in the matrix element $V_{K,L}W_{K',L'}(\varphi, \varphi') = \langle\varphi|\mathbb{V}(K, L)|\varphi''\rangle\langle\varphi''|\mathbb{W}_{K',L'}|\varphi'\rangle$. This is given by

$$VW(\varphi, \varphi') = \sum_{\sigma_1''} \ldots \sum_{\sigma_n''} \prod_{j=1}^{n} \exp\left(\sigma_j''\left(K\sigma_{j+1} + L\sigma_j + K'\sigma_j' + L'\sigma_{j+1}'\right)\right) \tag{8.6}$$

Note that this expression inside the product containing the σ_j'' is unique to one summand in the whole sum. Thus, we can write the whole sum in the form of,

$$\begin{aligned} VW(\varphi, \varphi') &= \prod_{j=1}^{n} X\left(\sigma_j, \sigma_{j+1}; \sigma_j', \sigma_{j+1}'\right) \text{ where } X \text{ is given by} \\ X(a, b, c, d) &= \sum_{f=\pm 1} \exp[f(La + Kb + K'c + L'd)] \end{aligned} \tag{8.7}$$

where the variables a, b, c, d take the values ± 1. Note that for any number M, the expression (8.6), remains unchanged under the following transformation,

$$X(a, b, c, d) \mapsto e^{Mac} X(a, b, c, d) e^{-Mbd} \tag{8.8}$$

This is because when we take the product in (8.6), the term $e^{Mac} = \exp\left(M\sigma_j\sigma_j'\right)$ is cancelled by the factor in the previous term, $e^{-Mbd} = \exp\left(M\sigma_{j-1+1}\sigma_{j-1+1}'\right) = \exp\left(-M\sigma_j\sigma_j'\right)$. Thus, if we exchange the terms $K \longleftrightarrow K'$ and $L \longleftrightarrow L'$, if we find a number M such that

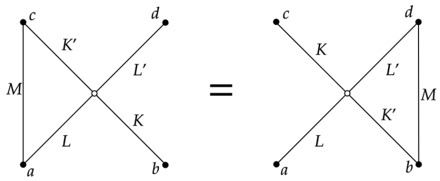
$$X'(a, b, c, d) = e^{Mac} X(a, b, c, d) e^{-Mbd} \tag{8.9}$$

where X' is of the form (8.7) with the interaction coefficients swapped. This can be viewed as a nice graph and from this visualization we can easily infer what the number M should be. From star-triangle relation (7.4) and (7.5), given three spins $\{\sigma_1, \sigma_2, \sigma_3\}$, we get the following relation,

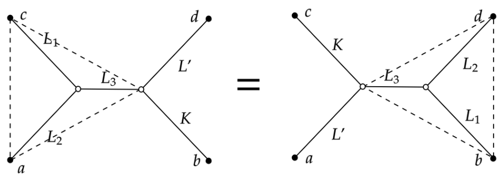
$$\begin{aligned} w\left(\sigma_i, \sigma_j, \sigma_k\right) &= \exp\left(K_1\sigma_j\sigma_k + K_2\sigma_k\sigma_i + K_3\sigma_i\sigma_j\right) \\ &= R\cosh\left(L_1\sigma_i + L_2\sigma_j + L_3\sigma_k\right) \end{aligned} \tag{8.10}$$

for some number R. Then if the both the graphs are equivalent, on transformation from the triangle to star, the corresponding transformed graphs also must be the equivalent. On both the graphs we can see a (L, K', M) triangle. Using the star-triangle relation above, set $K_1 = L, K_2 = K', K_3 = M$, we get the following relations,

$$\begin{aligned} w(c, a, f) &= \exp(Lfa + K'fc + Mac) = R\cosh(L_1c + L_2a + L_3f) \\ w(b, d, f) &= \exp(Lfb + K'fd + Mbd) = R\cosh(L_1b + L_2d + L_3f) \end{aligned} \tag{8.11}$$



(a)



(b)

Figure 16: (a)The L, K', M triangles and their corresponding (b) stars

If we consider the equivalence of the graphs and thus, if $L_1 = K$ and $L_2 = L'$, there exists a number M such that (8.9) holds true. This holds if,

$$\sinh(2K)\sinh(2L) = \sinh(2K')\sinh(2L') \tag{8.12}$$

Thus, (8.5) holds true if (8.12) is satisfied.

8.2. Inversion Relations

Now we move onto the Inversion Relations. Suppose, K, L, K', L' are given. We are interested in when $V_{K,L}(\varphi, \varphi'')W_{K',L'}(\varphi'', \varphi')$ is a diagonal or a near diagonal matrix. Using (8.7), for this to be a diagonal matrix if $a \neq c$ or $b \neq d$, $X(a, b; c, d) = 0$. In general this cannot be proved to be true. However, what can be shown is

$$X(a, b; c, d) = 0 \quad \text{if} \quad a \neq c \quad \text{and} \quad b = d \tag{8.13}$$

for certain constraints on K, L, K', L'

There are then 4 possibilities

$$\begin{aligned} \{a, b, c, d\} &= \{1, 1, -1, 1\} \\ &= \{1, -1, -1, -1\} \\ &= \{-1, 1, 1, 1\} \\ &= \{-1, -1, 1, -1\} \end{aligned} \tag{8.14}$$

These result in the following two equations once plugged into (8.7) and letting X vanish,

$$\cosh(L + K - K' + L') = 0 \tag{8.15}$$

$$\cosh(L - K - K' - L') = 0 \tag{8.16}$$

Adding and subtracting (8.15) and (8.15), we get,

$$\begin{aligned} \sinh(L - K')\sinh(L' + K) &= 0 \\ \cosh(L - K')\cosh(L' + K) &= 0 \end{aligned} \tag{8.17}$$

There are no real solutions. There are two complex solutions. The two solutions are

$$L = K' + i\frac{\pi}{2} \qquad L' = -K \tag{8.18}$$

or

$$L = K' \qquad L' = -K + i\frac{\pi}{2} \tag{8.19}$$

We choose, following Baxter, the solution (8.18). This condition now implies a stronger constraint on the spin chains. Let us only consider the non-zero elements of this matrix. This implies that if σ_j, σ'_j are unlike, then $\sigma_{j+1}, \sigma'_{j+1}$ also has to be unlike, for a non-zero element of $V_{K,L}W_{K',L'}$. Note in our model we assume periodic boundary conditions. Let us set a pair of spins as our reference, and label it as (σ_0, σ'_0).

There are two cases for :

1. $\boldsymbol{(\sigma_0, \sigma'_0)}$ **are unlike**. This, in turn implies all spin pairs (σ_j, σ'_j) must all be unlike. Otherwise, this particular matrix entry in VW is 0.
2. $\boldsymbol{(\sigma_0, \sigma'_0)}$ **are like**. There are, then 2 possibilities for $\boldsymbol{(\sigma_1, \sigma'_1)}$. They can be like or they can be unlike. This choice goes on till we encounter the last pair of spins. Suppose at some $0 < t \leq n$, (σ_t, σ'_t) is unlike and for all $0 \leq j < t$ the spin pairs are like. This implies from the previous condition that for all $j > t$, the spin pairs must be unlike. Then we encounter a problem at the last spin. (σ_n, σ'_n) must be unlike. However, (σ_0, σ'_0) is a like spin pair. This means that the matrix element will be zero. We appear to be at a contradiction. All of this is resolved if no t, where for $t-1$, we have like spins and t we have unlike spins. Thus, it means that if (σ_0, σ'_0) are like spins, then the whole spin chain must take the same values.

Using (8.18) and all like spins, we get,

$$\begin{aligned} X_{\text{like}} &= \cosh\left(2L + i\frac{\pi}{2}\right) \\ &= 2i\sinh(2L) \end{aligned} \tag{8.20}$$

Again using (8.18) and all unlike spins($a \neq c$ and $b \neq d$), for a choice of a, b, we obtain,

$$X_{\text{unlike}} = -2iab\sinh(2K) \tag{8.21}$$

As a result of this, we can write out the matrix $\mathbb{V}(K,L)\mathbb{W}(L + i\pi/2, -K)$. To this consideration, let us define a matrix whose non-zero elements are those with unlike spins. Thus, we get

$$R = \delta(\sigma_0, -\sigma'_0)...\delta(\sigma_n, -\sigma'_n) \tag{8.22}$$

It is also worth noting that $R^2 = \mathbb{I}$. Each non-zero element is of the form (8.7). For the unlike spin elements note that each spin is taken into consideration twice, as a result of which the dependence on a and b vanish. A non-zero element of unlike spins looks like,

$$\begin{aligned} &(-2i\sinh(2K))^n \sigma_0\sigma'_0\sigma_1\sigma'_1\sigma_1\sigma'_1...\sigma_n\sigma'_n\sigma_0\sigma'_0 \\ &= (-2i\sinh(2K))^n \prod_{j=0}^{n} \sigma_j^2\sigma_j'^2 \\ &= (-2i\sinh(2K))^n \end{aligned} \tag{8.23}$$

Thus, we obtain, from (8.23), (8.22), (8.18), (8.20) and (8.21),

$$\mathbb{V}(K,L)\mathbb{W}(L + i\pi/2, -K) = (2i\sinh(2L))^n\mathbb{I} + (-2i\sinh(2K))^n R \tag{8.24}$$

8.3. Symmetries

We will also be looking for the symmetries of the matrices $\mathbb{V}$ and $\mathbb{W}$. If we look at a form of V and W from (8.1) and (8.2), we can see that if we $(K,L) \longleftrightarrow (L,K)$ and swap φ and φ', it would be the same as exchanging V and W. Swapping φ and φ', constitutes taking a transpose of the matrix. Thus, we obtain,

$$\begin{aligned} V(K,L) &= W^T(L,K) \\ W(K,L) &= V^T(L,K) \end{aligned} \tag{8.25}$$

Negating K and L is the same as negating $-\varphi$ or $-\varphi'$. Note that $-\varphi = R\varphi$ and $-\varphi' = R\varphi'$. We obtain,

$$V(-K,-L) = RV(K,L) = V(K,L)R \tag{8.26}$$

This also implies that V and R commute. A similar relation also holds for $W(K,L)$.

V and W also have some symmetries because of the exponential form. This can be seen if we consider a different formulation. Let r be the number of unlike pairs of spins $\left(\sigma_{j+1}, \sigma_j'\right)$ and let s be the number of unlike pairs $\left(\sigma_j, \sigma_j'\right)$. Noting periodic boundary conditions, we can see that for each r there will be a corresponding s. This can be easily seen, when one considers the Ising chain with periodic boundary conditions to be a ring of spins. A proof by contradiction can be easily, by assuming that $r+s$ be odd, and then the contradiction should follow due to the choice of boundary conditions. This implies that $r+s$ should be even.

We can reformulate (8.1) as

We can relax our condition on n. Since we later obtain the free energy functional, which is obtained in the thermodynamic limit, it does not matter how we approach infinity. Thus, we can set

$$n = 2p \tag{8.27}$$

and only allow even length of Ising Chains. This allows us to introduce an extra level of symmetry which will help us later.

When we approach infinity with only even length chains the elements of the $\mathbb{V}$ matrix can be written as,

$$V(K,L) = \exp[\pm 2s'K \pm 2r'L] \tag{8.28}$$

where $s' = \frac{n-2s}{2}$ and $r' = \frac{n-2r}{2}$. Note that s' and r' are integers in the range $(0,p)$. Note if we negate $\exp(2K)$ or $\exp(2L)$, it would be the same, under the condition (8.28). If we displaced K and L by $\pm i\frac{\pi}{2}$ in the complex plane (which is the same as $\exp(\pm 2K) \longrightarrow -\exp(\pm 2K)$ and similarly for $\exp(\pm 2L)$), the expression would remain unchanged iff $(n-2s)$ and $(n-2r)$ are even numbers. Thus, we obtain,

$$\begin{aligned} V\left(K \pm i\frac{\pi}{2}, L \pm i\frac{\pi}{2}\right) &= V(K,L) \\ W\left(K \pm i\frac{\pi}{2}, L \pm i\frac{\pi}{2}\right) &= W(K,L) \end{aligned} \tag{8.29}$$

This form of the matrix (8.28) allows us to infer more about the eigenvalues of the transfer matrices.

8.4. Commuting Relations

This is the penultimate stop before we embark onto solving the Ising Model. We want to find commuting relations for the transfer matrices $\mathbb{V}$ and $\mathbb{W}$.

Let us again note (8.1) and (8.2). Their forms are very similar. If we take line of column on this tilted lattice, and then shift by one column. Let us define C to be a matrix which does this. This can be defined as a $2^n \times 2^n$ matrix, s.t.,

$$C = \delta(\sigma_1, \sigma_2')...\delta(\sigma_n, \sigma_1') \tag{8.30}$$

This shifts the columns by one. Let's see what this does for the spin vector φ,

$$\begin{aligned} \varphi &= (\sigma_1, \sigma_2, ..., \sigma_{n-1}, \sigma_n)^T \\ C\varphi &= (\sigma_2, \sigma_3, ..., \sigma_n, \sigma_0)^T \end{aligned} \tag{8.31}$$

Note another property of the matrix C, which is $C^n = \mathbb{I}$. Our choice of boundary conditions, along with (8.1) and (8.2), we obtain that

$$\begin{aligned} \mathbb{V}(K,L) &= C^{-1}\mathbb{V}(K,L)C \\ \mathbb{W}(K,L) &= C^{-1}\mathbb{W}(K,L)C \end{aligned} \tag{8.32}$$

From (8.1) and (8.2), (8.30), we can infer a relation between the matrices $\mathbb{V}$ and $\mathbb{W}$. Shifting the columns and then rearranging we obtain,

$$\begin{aligned}\langle\varphi|\mathbb{V}|C\varphi'\rangle = V_{\varphi,C\varphi'} &= \exp\left(\sum_{j=1}^{n}\left[K\sigma_{j+1}\sigma'_{j+1} + L\sigma_j\sigma'_{j+1}\right]\right)\\ &= \exp\left(\sum_{j=1}^{n}\left[K\sigma_j\sigma'_j + L\sigma_j\sigma'_{j+1}\right]\right) = \langle\varphi|\mathbb{W}|\varphi\rangle\end{aligned} \tag{8.33}$$

This directly implies that $\mathbb{V}$ and $\mathbb{W}$ commute, since $\mathbb{V}$ and C commute.

$$\mathbb{W}(K,L) = \mathbb{V}(K,L)C \tag{8.34}$$

This shows that $\mathbb{V}$, $\mathbb{W}$ and C all mutually commute. Using this we can also see that if (8.12) is satisfied,

$$V(K,L)V(K',L') = V(K',L')V(K,L) \tag{8.35}$$

This, along with (8.34), also implies that

$$W(K,L)W(K',L') = W(K',L')W(K,L) \tag{8.36}$$

Also using, (8.26), we obtain that $\mathbb{V}$ and $\mathbb{W}$, also commute with the negating spin matrix R,

$$\begin{aligned}\mathbb{V}(K,L)R &= R\mathbb{V}(K,L)\\ \mathbb{W}(K,L)R &= R\mathbb{W}(K,L)\end{aligned} \tag{8.37}$$

So far we have found a set of matrices that mutually commute with each other. Assuming the condition (8.12) is satisfied, the set of operators that commute with each other is given below. Let $(K_i, L_i) \quad i \in I$,where I is some indexing set, be s.t.,

$$\sinh(K_i)\sinh(L_i) = k^{-1} \tag{8.38}$$

for some constant k. Then the set of operators that all mutually commute is given as,

$$\{V(K_i,L_i)\}_{\{i\in I\}}\bigcup\{W(K_i,L_i)\}_{\{i\in I\}}\bigcup\{R,C\} \tag{8.39}$$

Finally, we can eliminate the matrix $\mathbb{W}$ from (8.24), to obtain,

$$\mathbb{V}(K,L)\mathbb{V}(L+i\pi/2,-K)C = (2i\sinh(2L))^n\mathbb{I} + (-2i\sinh(2K))^nR \tag{8.40}$$

8.5. Functional relation of the Eigenvalues.

Note the set in (8.39). Since they all commute, they share a common set of eigenvectors. Note that the set of eigenvectors need not be an eigenbasis. Let us choose a common eigenvector $\boldsymbol{x}$. Note that this eigenvector cannot depend on K_i, L_i. This makes sense if we think of it in a different light. Let (8.38) be satisfied for a fixed value of k. Then we get a set of ordered pairs (K_i, L_i) all satisfying (8.38). Then let's define a family of matrices $M_i = V(K_i, L_i)$. Note that even if we change (K_i, L_i) constrained to (8.38), $\boldsymbol{x}$ is still an eigenvector of M_i. Thus, if we think of it as just a family of commuting set of operators, without any dependence on K_i, L_i, $\boldsymbol{x}$ would still be an eigenvector. However, the family $\{V\}_{\{i\in I\}}$ completely changes if k changes in (8.38). Thus, naturally the eigenvector $\boldsymbol{x}$ in general should change. As a result, it is natural that the eigenvector $\boldsymbol{x}$ does not depend on K_i, L_i, but it does depend on k. The eigenvalue of the matrix $V(K_i, L_i)$ corresponding to $\boldsymbol{x}(k)$ will in general depend on (K_i, L_i). We obtain the following relations,

$$\begin{aligned}V(K,L)\boldsymbol{x}(k) &= v(K,L)\boldsymbol{x}(k)\\ C\boldsymbol{x}(k) &= c\boldsymbol{x}(k)\\ R\boldsymbol{x}(k) &= r\boldsymbol{x}(k)\end{aligned} \tag{8.41}$$

Note that from (8.22), R is a symmetric matrix. As a result, R can only assume real eigenvalues [9]. Note from the crucial properties defined under (8.22) and (8.30), we obtain that,

$$c^n = r^2 = 1 \tag{8.42}$$

As a consequence, r can only assume the values $r \pm 1$. Using (8.40) , we can rewrite the eigenvalues $v(K,L)$ as a function of K, L and a choice of $r = \pm 1$.

$$v(K,L)v(L+i\pi/2,-K)c = (2i\sinh(2L))^n + (-2i\sinh(2K))^n r \tag{8.43}$$

Let us now define the eigenvalues of the matrix $\mathbb{VW}$, on which rests our final calculation. Let $\Lambda^2(K, L)$ be the eigenvalue of $\mathbb{V}(K, L)\mathbb{W}(K, L)$. Thus, we can write,

$$\begin{aligned} \Lambda^2(K, L) &= v^2(K, L)c \\ \Lambda(K, L) &= v(K, L)c^{\frac{1}{2}} \end{aligned} \tag{8.44}$$

Let us identify $\boldsymbol{x}(k)$ with some eigenvector of $\mathbb{VW}$, say $\boldsymbol{x}_j$. Let the eigenvalue of $\mathbb{VW}$ corresponding to $\boldsymbol{x}_j$ is Λ_j^2. We now identify $\Lambda^2(K, L)$ with the corresponding Λ_j. This identification complete, we will now work with the notation of $\Lambda(K, L)$. This allows us to write (8.43) as ,

$$\Lambda(K, L)\Lambda(L + i\pi/2, -K) = (2i\sinh(2L))^n + (-2i\sinh(2K))^n r \tag{8.45}$$

This will later on help us completely find an expression of $\Lambda_j = \Lambda(K, L)$. The whole ordeal of finding the solution of Λ_j, will first be shown for $k = 1$, which we know from (5.20) by Kramers and Wannier [17], to be the critical temperature. A better explanation of why the case of $k = 1$ is shown first, will be clear from the next section. The choice of $k = 1$, allows us to express $\Lambda(K, L)$ in the form of simple trigonometric functions. Other choices of k requires the use of some rather eccentric functions called Elliptic Functions.

9. Solution to the Eigenvalues at $T = T_c$

As shown by Kramers and Wannier [17], and will be shown later after the complete solution of the 2D Ising Model that the critical temperature of the model corresponds to $k = 1$ in (8.38). This allows us to write $\sinh(2K)$ and $\sinh(2L)$ in terms of simple trigonometric functions.

We choose the parameterization,

$$\begin{aligned} \sinh(2K) &= \tan(u) \\ \sinh(2L) &= \cot(u) \end{aligned} \tag{9.1}$$

Note that $\sinh(2K)$ and $\sinh(2L)$ are both positive for $K, L > 0$. This means that u can take values in the range $[0, \frac{\pi}{2})$.

Note that this is the only simple trigonometric parameterization that is possible. $\sinh(2K)$ and $\sinh(2L)$ can take any value in the set $[0, \infty)$. So $\sin(u)$, $\cos(u)$ and corresponding $\csc(u)$ and $\sec(u)$ are not possible. Now our functional equation for the eigenvalues can be rewritten in the variable u. Note that if we take $K \longrightarrow L + i\frac{\pi}{2}$ and $L \longrightarrow -K$. This allows us to write,

$$\begin{aligned} \sinh(2L + i\pi) &= \tan(u') & \sinh(-2K) &= \cot(u') \\ \Rightarrow -\sinh(2L) &= \tan(u') & \Longrightarrow -\sinh(2K) &= \cot(u') \end{aligned} \tag{9.2}$$

As a result, we can define

$$u' = u + \frac{\pi}{2} \tag{9.3}$$

Using (8.45), (9.1) and (9.3), we obtain,

$$\begin{aligned} \Lambda(u)\Lambda\left(u + \frac{\pi}{2}\right) &= (2i\sinh(2L))^n + (-2i\sinh(2K))^n r \\ &= (2i\cot(u))^n + (-2i\tan(u))^n r \end{aligned} \tag{9.4}$$

As a result of (9.1), we can write $\exp(\pm 2K)$ and $\exp(\pm 2L)$ as single valued functions of u. This along with (8.28) allows us to write the eigenvalues in terms of u and infer their forms. The parameterizations are given by,

$$\begin{aligned} \exp(2K) &= \frac{1+\sin(u)}{\cos(u)} \\ \exp(-2K) &= \frac{1-\sin(u)}{\cos(u)} \\ \exp(2L) &= \frac{1+\cos(u)}{\sin(u)} \\ \exp(-2L) &= \frac{1-\cos(u)}{\sin(u)} \end{aligned} \tag{9.5}$$

This parameterization is useful because $\exp(2K)$ and $\exp(2L)$,

1. Are single valued functions of u.
2. Are meromorphic functions whose singularities are poles and they form an isolated set.
3. Are periodic with period 2π.

9.1. The form of $\Lambda(u)$

Note that we have,

$$\begin{aligned} &\mathbb{V}(u)\boldsymbol{x}(k) = \Lambda(u)\boldsymbol{x}(k) \\ \implies &\sum_j \mathbb{V}_{ij}(u)\boldsymbol{x}_j(k) = \Lambda(u)\boldsymbol{x}_i(k) \\ \implies &\Lambda(u) = \sum_j \mathbb{V}_{ij}(u)\frac{\boldsymbol{x}_j(k)}{\boldsymbol{x}_i(k)} \end{aligned} \tag{9.6}$$

The eigenvalue can be written as function of the elements of the transfer matrix whose coefficients are ratios of the eigenvectors. Observe that these coefficients do not depend on the parameter u but only on the parameter k. Using (8.28) and (9.1), we obtain this following form of the elements of the transfer matrix,

$$V_{\varphi,\varphi'} = \frac{t(u)}{(\sin(u)\cos(u))^p} \tag{9.7}$$

where

$$t(u) = e^{-2ipu}\sum_{\alpha=0}^{2n} c_\alpha e^{2i\alpha u} \tag{9.8}$$

To drive this point home that the elements of the transfer matrix $\mathbb{V}$ can be written like this, we can continue (9.1) into (8.28) and continuing the algebraic monstrosity, we get,

$$\begin{aligned} V_{\varphi,\varphi'} &= \frac{(1+c\sin(u))^{s'}(1+c\cos(u))^{r'}}{\sin(u)^{r'}\cos(u)^{s'}} \\ &= \frac{(1+c\sin(u))^{s'}(1+c\cos(u))^{r'}\sin(u)^{p-r'}\cos(u)^{p-s'}}{(\sin(u)\cos(u))^p} \\ &= \frac{\left(1+c\frac{e^{iu}-e^{-iu}}{2i}\right)^{s'}\left(1+c\frac{e^{iu}+e^{-iu}}{2}\right)^{r'}\left(\frac{e^{iu}-e^{-iu}}{2i}\right)^{p-r'}\left(\frac{e^{iu}+e^{-iu}}{2}\right)^{p-s'}}{(\sin(u)\cos(u))^p} \\ &= \left[\frac{\left(e^{-iu}+c\left(\frac{e^{2iu}-1}{2i}\right)\right)^{s'}\left(e^{-iu}+c\left(\frac{e^{2iu}+1}{2}\right)\right)^{r'}\left(e^{-iu}+c\left(\frac{e^{2iu}-1}{2i}\right)\right)^{p-r'}\left(\left(e^{-iu}+\frac{c(e^{2iu}+1)}{2}\right)\right)^{p-s'}}{(e^{2iu}\sin(u)\cos(u))^p}\right] \\ &= \frac{t(u)}{(\sin(u)\cos(u))^p} \end{aligned} \tag{9.9}$$

We know that the eigenvalues of this matrix can be written as a linear combination of the matrix elements of the transfer matrix. We explore some symmetry properties of the transfer matrix to obtain a form of the eigenvalue form which we obtain the solution to the eigenvalues at $T = T_c$. Before we do that note that for a fixed the linear combinations of the matrix elements form the eigenvalue which do not depend on the parameter u. Mapping $u \longrightarrow u + \pi$, is the same as mapping $K \longrightarrow -K \pm \frac{i\pi}{2}$ and $L \longrightarrow -L \pm \frac{i\pi}{2}$. Using this mapping, (8.26) and (8.29), we can write,

$$V(K, L)R\boldsymbol{x}(k) = v(u + \pi)\boldsymbol{x}(k) \tag{9.10}$$

Using this we can write the eigenvalue as,

$$\Lambda(u + \pi) = r\Lambda(u) \tag{9.11}$$

Since the coefficients are constant for a fixed k, the eigenvalue $\Lambda(u)$ is linear combination of the matrix elements of the transfer matrix with constant coefficients. This means that the eigenvalue is a periodic function of u with the periodicity condition (9.11).

Using this periodicity condition it is obvious that when $r = 1$ only the even coefficients in (9.8) can have non-zero coefficients, to satisfy (9.11). When the parameter $r = -1$, only the odd coefficients in (9.8) can have non-zero coefficients, to satisfy (9.11). We can factor $t(u)$ in (9.8) to be in terms of the sine functions with appropriate roots. One can simply convince themselves by considering a product of two sine functions and using some simple algebraic manipulations we obtain $t(u)$ for the case of even coefficients,

$$\begin{aligned}
&\sin(u - u_1)\sin(u - u_2) \\
&= [\exp(i(u - u_1)) - \exp(-i(u - u_1))][\exp(i(u - u_2)) - \exp(-i(u - u_2))](4i^2)^{-1} \\
&= \exp(-2iu)[d_{01} + d_{11}\exp(2iu)][d_{02} + d_{12}\exp(2iu)](4i^2)^{-1} \\
&= \exp(-2iu)[c_0 + c_2\exp(2iu) + c_4\exp(4iu)]
\end{aligned} \tag{9.12}$$

The logic follows similarly for odd coefficients as well. As a result of the above symmetry relations, we can write the eigenvalue in terms of undetermined zeros of this function,

$$\Lambda(u) = \rho(\sin(u)\cos(u))^{-p}\prod_{j=1}^{l}\sin(u - u_j) \tag{9.13}$$

where ρ is a constant and u_j are the zeros of the function(which we will determine now). The indexing set's maximum index l is given as,

$$\begin{aligned}
l &= 2p \qquad &\text{if} \quad r = +1 \\
l &= 2p - 1 \qquad &\text{if} \quad r = -1
\end{aligned}$$

We now get to work on the zeros of this functional form. We can get the zeros of $\Lambda(u)$ using the functional equation (9.4).

9.2. The zeros of $\Lambda(u)$

Plugging (9.13) into (9.4), we obtain,

$$\begin{aligned}
&\Lambda(u)\Lambda\left(u + \frac{\pi}{2}\right) = (2i\sinh(2L))^n + (-2i\sinh(2K))^n r \\
&\implies (-1)^p\rho^2(\sin(u)\cos(u))^{-2p}\prod_{j=1}^{l}\sin(u - u_j)\cos(u - u_j) = (2i\cot(u))^n + (-2i\tan(u))^n r \\
&\implies \rho^2\prod_{j=1}^{l}\sin(u - u_j)\cos(u - u_j) = (-1)^p(2i)^{2p}[\cos^{4p}(u) + r\sin^{4p}(u)] \\
&\implies \rho^2\prod_{j=1}^{l}\sin(u - u_j)\cos(u - u_j) = 2^{2p}[\cos^{4p}(u) + r\sin^{4p}(u)]
\end{aligned} \tag{9.14}$$

This must be an identity in u. The left-hand side must have the same zeros as the right-hand side, which are u_j.

Theorem 9.2.1 (Zeros of the eigenvalue at critical temperature):
The zeros of the eigenvalue $\Lambda(u)$ at critical temperature $k = 1$ are given by,

$$u_j = \mp\frac{\pi}{4} - i\varphi_j, \quad j = \{1, ..., l\}$$

where

$$\varphi_j = \frac{1}{2}\ln\left[\tan\left(\frac{\theta_j}{2}\right)\right]$$

and θ_j are given by,

$$\begin{aligned} \theta_j &= \left(j - \frac{1}{2}\right)\frac{\pi}{2p} \qquad && j = 1, ..., 2p \qquad \text{if} \quad r = +1 \\ \theta_j &= \frac{\pi j}{2p} && j = 1, ..., 2p - 1 \ \text{if} \quad r = -1 \end{aligned}$$

Proof: One way to find the zeros of the RHS can be done by variable substitution to make the identity a polynomial. Let the new variable be

$$\begin{aligned} z &= \exp(2iu) \\ z_j &= \exp(2iu_j) \end{aligned} \tag{9.15}$$

Note the following two relations,

$$\begin{aligned} \sin(u - u_j) &= \frac{\exp(i(u-u_j)) - \exp(-i(u-u_j))}{2i} \\ &= \exp(-i(u-u_j))\left[\frac{\exp(2i(u-u_j)) - 1}{2i}\right] \\ &= \sqrt{\frac{z_j}{z}}\left[\frac{z/z_j - 1}{2i}\right] \\ \cos(u - u_j) &= \frac{\exp(i(u-u_j)) + \exp(-i(u-u_j))}{2} \\ &= \exp(-i(u-u_j))\left[\frac{\exp(2i(u-u_j)) + 1}{2}\right] \\ &= \sqrt{\frac{z_j}{z}}\left[\frac{z/z_j + 1}{2}\right] \end{aligned}$$

Substituting this into (9.14), we obtain,

$$\begin{aligned} \rho^2 \prod_{j=1}^{l}\left(\frac{z_j}{z}\right)\left[\frac{z/z_j - 1}{2i}\right]\left[\frac{z/z_j + 1}{2}\right] &= 2^{-2p}\left(\frac{1}{z}\right)^{2p}[(z+1)^{4p} + r(z-1)^{4p}] \\ \implies \rho^2\left(-\frac{i}{4}\right)^{l}\prod_{j=1}^{l}\left[\frac{z^2 - z_j^2}{z_j}\right] &= 2^{-2p}z^{l-2p}[(z+1)^{4p} + r(z-1)^{4p}] \end{aligned} \tag{9.16}$$

The zeros of the LHS are z_j^2. Let us find the zeros of the RHS by substituting $z = z_j$ and using the two cases $r = \pm 1$. Consider the first case $r = 1$ and $l = 2p$,

$$\begin{aligned} z_j^{2p-2p}\left[\left(z_j+1\right)^{4p}+\left(z_j-1\right)^{4p}\right] &= 0 \\ \Rightarrow \left(z_j+1\right)^{4p}+\left(z_j-1\right)^{4p} &= 0 \end{aligned} \tag{9.17}$$

Note that $z_j \neq 1$. Thus, we can divide by the second term $\left(z_j-1\right)^{4p}$,

$$\begin{aligned} &\left(\frac{z+1}{z-1}\right)^{4p}+1=0 \\ &\Rightarrow \frac{z+1}{z-1}=\exp\left(\frac{\pm i\pi(2j-1)}{4p}\right) \quad \text{where } j=\{1,\ldots,2p\} \\ &\Rightarrow z=\pm i\tan\left(\frac{\theta_j}{2}\right) \quad \text{where } \theta_j=\left(j-\frac{1}{2}\right)\frac{\pi}{2p} \qquad j=1,\ldots,2p \end{aligned} \tag{9.18}$$

The roots are then given by,

$$\begin{aligned} z_j^2 &= -\tan^2\left(\frac{\theta_j}{2}\right) \quad \text{where} \\ \theta_j &= \left(j-\frac{1}{2}\right)\frac{\pi}{2p} \qquad j=1,\ldots,2p \end{aligned}$$

Similarly, for the case of $r=-1$ and $l=2p-1$,

$$\begin{aligned} z_j^{2p-1-2p}\left[\left(z_j+1\right)^{4p}+\left(z_j-1\right)^{4p}\right] &= 0 \\ \Rightarrow z_j^{-1}\left[\left(z_j+1\right)^{4p-1}+\left(z_j-1\right)^{4p-1}\right] &= 0 \end{aligned} \tag{9.19}$$

Note that z_j cannot be zero, because of (9.15). Following similar calculations to (9.18), we obtain the solution,

$$\begin{aligned} z_j^2 &= -\tan^2\left(\frac{\theta_j}{2}\right) \quad \text{where} \\ \theta_j &= \frac{\pi j}{2p} \qquad j=1,\ldots,2p-1 \end{aligned}$$

The roots are then given by,

$$\begin{aligned} z_j &= -\tan\left(\frac{\theta_j}{2}\right) \quad \text{where} \\ \theta_j &= \left(j-\frac{1}{2}\right)\frac{\pi}{2p} \qquad j=1,\ldots,2p \qquad \text{if} \quad r=+1 \\ \theta_j &= \frac{\pi j}{2p} \qquad\qquad\quad j=1,\ldots,2p-1 \text{ if} \quad r=-1 \end{aligned} \tag{9.20}$$

Note that the roots θ_j lie in the range $(0,\pi)$. Define $\varphi_1,\ldots,\varphi_l$,

$$\varphi_j=\frac{1}{2}\ln\left[\tan\left(\frac{\theta_j}{2}\right)\right] \qquad \text{where} \quad j=\{1,\ldots,l\} \tag{9.21}$$

Then inverting the variable substitution (9.15) and using (9.20), we obtain

$$u_j=\gamma_j\frac{\pi}{4}-i\varphi_j, \quad j=\{1,\ldots,l\} \tag{9.22}$$

where γ_j takes the values ± 1. ■

The number of roots is 2^l, which is $2^{\{2p\}}$ if $r=+1$ and $2^{\{2p-1\}}$ if $r=-1$. However, the total number of roots must be 2^{2p}, because the transfer matrix $\mathbb{VW}$ is $2^{2p}\times 2^{2p}$. So we are overcounting. Since we obtain $2^{2p} + 2^{2p-1} = 3 \times 2^{2p-1}$, which is more than the expected value. The problem lies with the fact that for $r = +1$, we have 2^{2p} solutions. We will now show that there is a constraint on the roots u_j if $r = +1$.

Consider the limit $u \longrightarrow i\infty$. The parameterizations of the matrix elements $\exp(\pm 2K)$ and $\exp(\pm 2L)$ in (9.5) show that in this limit,

$$\begin{aligned} \lim_{u \longrightarrow i\infty} \exp(\pm 2K) &= \pm i \\ \lim_{u \longrightarrow i\infty} \exp(\pm 2L) &= \pm i \end{aligned} \tag{9.23}$$

We also know that if we negate $\exp(\pm 2K)$ and $\exp(\pm 2L)$, we obtain the same transfer matrix using (8.29). Thus,

$$\lim_{u \longrightarrow i\infty} \Lambda(u) = \lim_{u \longrightarrow -i\infty} \Lambda(u) \tag{9.24}$$

Let us first consider the case $r = -1$. In this case, we have $l = 2p - 1$. Let us consider any one of the factors in the product of (9.13) and evaluate in the above limit and let $t_j = \exp(iu_j)$,

$$\lim_{u \longrightarrow \infty} \frac{\sinh(u - u_j)}{\sinh(u)} = \lim_{u \longrightarrow \infty} t_j \frac{1 - t_j^{-2} e^{-2u}}{1 - e^{-2u}} = t_j \tag{9.25}$$

This evaluates to some constant limit even if the denominator is $\cosh(u)$. Thus, each factor in the product goes to some constant limit for each $l = 2p - 1$ factors in the product. If the denominator is $\sinh(u)$, it goes to a limit with the same sign as t_j for both cases of $u \longrightarrow \infty$ and $u \longrightarrow -\infty$. However, if the denominator is $\cosh(u)$, it goes to a limit with the opposite sign as t_j if $u \longrightarrow -\infty$ and same sign of t_j otherwise. These small subtleties do not matter, however they are still noted here. They do not matter here since there is an extra factor of either $\sinh(u)^{-1}$ or $\cosh(u)^{-1}$ in (9.13). Thus, it anyway goes to zero, for either case of the limit being taken to both infinities.

Now we move onto the other case, $r = +1$. In this case, we have $l = 2p$. The same argument as above does not work anymore, and we need to examine further. In a similar vein let us expand the product in (9.13) and evaluate the limit,

$$\Lambda(iu) = D \prod_{j=1}^{2p} \frac{\exp(u - u_j) - \exp(-(u - u_j))}{\exp(u) \pm \exp(-u)} \tag{9.26}$$

where D is a constant. Note that the denominator is either $\sinh(u)$ or $\cosh(u)$. Let us now evaluate the limit of the product in (9.13) as $u \longrightarrow \infty$,

$$\begin{aligned} \lim_{u \longrightarrow \infty} \Lambda(iu) &= \lim_{u \longrightarrow \infty} D \prod_{j=1}^{2p} \frac{\exp(u - iu_j) - \exp(-(u - iu_j))}{\exp(u) \pm \exp(-u)} \\ &= D \prod_{j=1}^{2p} \exp(iu_j) \end{aligned} \tag{9.27}$$

Similarly for the limit $u \longrightarrow -\infty$,

$$\begin{aligned} \lim_{u \longrightarrow -\infty} \Lambda(iu) &= \lim_{u \longrightarrow -\infty} D \prod_{j=1}^{2p} \frac{\exp(u - iu_j) - \exp(-(u - iu_j))}{\exp(u) \pm \exp(-u)} \\ &= D(-1)^p \prod_{j=1}^{2p} \exp(-iu_j) \end{aligned} \tag{9.28}$$

These two limits must be equal. Thus, we obtain the constraint,

$$\prod_{j=1}^{2p} \exp(iu_j) = (-1)^p \prod_{j=1}^{2p} \exp(-iu_j) \implies \exp\left(2i \sum_{j=1}^{2p} u_j\right) = (-1)^p$$
$$\implies \exp\left(2i \sum_{j=1}^{2p} u_j\right) = \exp(i\pi(2n+1)p) \tag{9.29}$$
$$\implies \sum_{j=1}^{2p} u_j = \left(n + \frac{p}{2}\right)\pi$$

where n is an integer. This is a constraint on the roots u_j if $r = +1$. Thus, all γ_j cannot be independently chosen We obtain a constraint on the roots u_j if $r = +1$, which gives us

$$\frac{u_1 + \ldots + u_{2p}}{\pi} = \text{integer} + \frac{p}{2} \tag{9.30}$$

In (9.22), we can reframe the constraints in terms of γ_j. Before we do that, note the indexing set J of u_j is $j = 1, \ldots, 2p$ for this current case. We can divide the indexing set into smaller subsets,

$$J = \{j : j \in \mathbb{N}, 1 \leq j \leq p\} \cup \{2p - j + 1 : j \in \mathbb{N}, 1 \leq j \leq p\} \tag{9.31}$$

Using (9.20), (9.22), (9.21) and (9.30), we can write the roots u_j as,

$$\sum_j u_j = \frac{\pi}{4} \sum_{j=1}^{p} \gamma_j - \frac{i}{2} \ln\left[\prod_{j=1}^{p} \tan\left(\frac{\theta_j}{2}\right) \tan\left(\frac{\theta_{2p-j+1}}{2}\right)\right] \tag{9.32}$$

The term in the logarithm goes to zero and as a result of that, the reframed constraint equation in terms of γ_j, is given by,

$$\gamma_1 + \ldots + \gamma_{2p} = 2p - 4 \times \text{integer} \tag{9.33}$$

It is worth our time to recount and summarize the algebraic jugglery that has unfolded over the last section. Thus, we obtain the eigenvalue and determine the roots after all this calculation.

Result 9 (Eigenvalues at critical temperature):

$$\Lambda(u) = \rho(\sin(u)\cos(u))^{-p} \prod_{j=1}^{l} \sin\left(u + i\varphi_j + \frac{1}{4}\gamma_j \pi\right) \tag{9.34}$$

where $\gamma_1, \ldots, \gamma_j$ can take the values ± 1 and for $r = 1$ has an additional constraint,

$$\gamma_1 + \ldots + \gamma_{2p} = 2p - 4 \times \text{integer}$$

ρ can be obtained by then plugging in (9.34) into (9.4).

The eigenvalue at critical temperature is given by (9.34). Note that the eigenvalue is a function of u and the roots u_j. The roots u_j are determined by the functional equation (9.4). This is a limiting case of the more general solution given in the next section. This was outlined because the parameterization can be done in terms of elementary functions we are familiar with.

10. Elliptic Functions

A large list of identities for the elliptic functions can be found in Chapter 15 [1]. The relevant ones for the calculation of the 2D ising model shall be outlined here.

Definition 10.1 (Doubly Periodic): A function $f(u)$ is said to be doubly periodic if it satisfies the following relations,

$$\begin{aligned} f(u+2I) &= \pm f(u) \\ f(u+2iI') &= \pm f(u) \end{aligned} \tag{10.1}$$

where I and I' are the half periods functions. Any upright rectangle of height $2iI'$ and width $2I$ in the complex plane is called the period rectangle.

The elliptic functions are functions of two variables the *Nome* q and the *argument* u. In our case the Nome q is real and restricted to $0 < q < 1$. The Half periods of the Jacobian elliptic functions are given by,

$$\begin{aligned} I &= \frac{\pi}{2}\prod_{n=1}^{\infty}\left[\frac{1+q^{2n-1}}{1-q^{2n-1}}\cdot\frac{1-q^{2n}}{1+q^{2n}}\right]^2 \\ I' &= \pi^{-1} I \ln(q^{-1}) \end{aligned} \tag{10.2}$$

and thus q is given by,

$$q = \exp(-\pi I'/I) \tag{10.3}$$

The modulus k and the conjugate modulus k' are then defined as,

$$\begin{aligned} k &= 4q^{\frac{1}{2}}\prod_{n=1}^{\infty}\left[\frac{1+q^{2n}}{1+q^{2n-1}}\right]^4 \\ k' &= \prod_{n=1}^{\infty}\left[\frac{1-q^{2n-1}}{1+q^{2n-1}}\right]^4 \end{aligned} \tag{10.4}$$

In the 2D Ising model we will identify the modulus with (8.38). The theta functions are defined in EQ.15.1.5 of Chapter 15 in [1]. Using those the Jacobian elliptic functions are then defined as,

$$\begin{aligned} \mathrm{sn}(u) &= k^{-\frac{1}{2}}\frac{H(u)}{\Theta(u)} \\ \mathrm{cn}(u) &= \left(\frac{k'}{k}\right)^{\frac{1}{2}} H_1\frac{u}{\Theta(u)} \\ \mathrm{dn}(u) &= k'^{\frac{1}{2}}\frac{\Theta_1(u)}{\Theta(u)} \end{aligned} \tag{10.5}$$

where k' is the conjugate modulus and k is the modulus. These are doubly periodic or anti-periodic, as we shall see a few identities below.

The zeros of the theta functions are given by,

$$\begin{aligned} H(u) = 0 &\quad \text{when} \quad u = 2mI + 2niI' \\ \Theta(u) = 0 &\quad \text{when} \quad u = 2mI + (2n+1)iI' \end{aligned} \tag{10.6}$$

where m, n are any integers. Thus, the Jacobian elliptic functions are meropmorphic whose only singularities are poles which are zeros of the theta functions.

The theta function $H(u)$ satisfies the quasi-periodic relations,

$$\begin{aligned} H(u+2I) &= -H(u) \\ H(u+2iI') &= -q^{-1}e^{-\pi iu/2I'}H(u) \end{aligned} \tag{10.7}$$

The other theta functions are related to $H(u)$ by the following relations,

$$\begin{aligned} H_1(u) &= H(u + I') \\ \Theta_1(u) &= \Theta(u + I) \\ \Theta(u) &= -iq^{\frac{1}{4}} e^{\pi i u/2I'} H(u + iI') \\ \Theta_1(u) &= q^{\frac{1}{4}} e^{\pi i u/2I'} H(u + I + iI') \end{aligned} \tag{10.8}$$

Using this, sn, cn and dn satisfy the following relations,

$$\begin{aligned} &\mathrm{sn}(-u) = -\mathrm{sn}(u), \quad \mathrm{cn}(-u) = \mathrm{cn}(u), \quad \mathrm{dn}(-u) = \mathrm{dn}(u) \\ &\mathrm{sn}(u + 2I) = -\mathrm{sn}(u) \\ &\mathrm{cn}(u + 2I) = -\mathrm{cn}(u) \\ &\mathrm{dn}(u + 2I) = \mathrm{dn}(u) \\ &\mathrm{sn}(u + 2iI') = \mathrm{sn}(u) \\ &\mathrm{cn}(u + 2iI') = -\ \mathrm{cn}(u) \\ &\mathrm{dn}(u + 2iI') = -\ \mathrm{dn}(u) \end{aligned} \tag{10.9}$$

and they satisfy the half period relations,

$$\begin{aligned} \mathrm{sn}(u + iI') &= (k\ \mathrm{sn}(u))^{-1} \\ \mathrm{cn}(u + iI') &= -i\ \mathrm{dn}(u)/k\ \mathrm{sn}(u) \\ \mathrm{dn}(u + iI') &= -i\ \mathrm{cn}(u)/\ \mathrm{sn}(u) \end{aligned} \tag{10.10}$$

Before we move on, we can see a visualization of the Jacobian elliptic functions in the complex plane. The Jacobian elliptic functions are periodic in the complex plane, and they can be visualized as a lattice in the complex plane. The period rectangle is defined by the half periods $2I$ and $2iI'$, and the functions repeat their values in this rectangle.

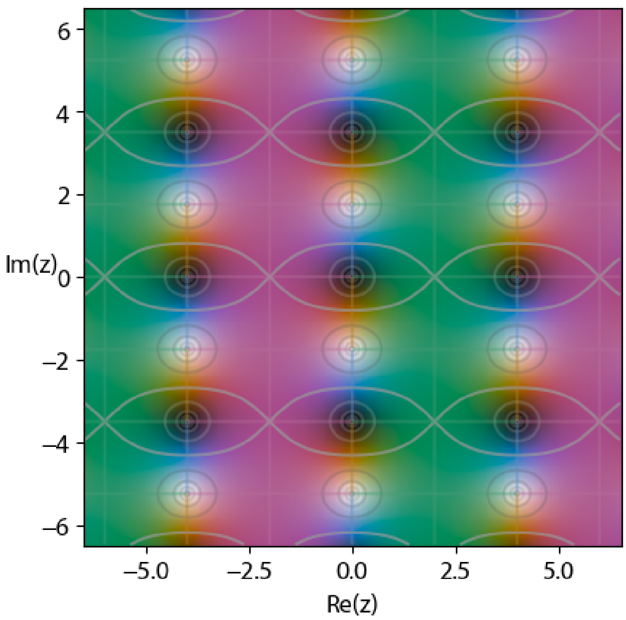


Figure 17: The Elliptic Sine Function with modulus $k = 0.8$. The half periods I and I' are given by 1.995 and 1.75. (Source: Wikipedia, Attribution: By Nschloe - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=111308860)

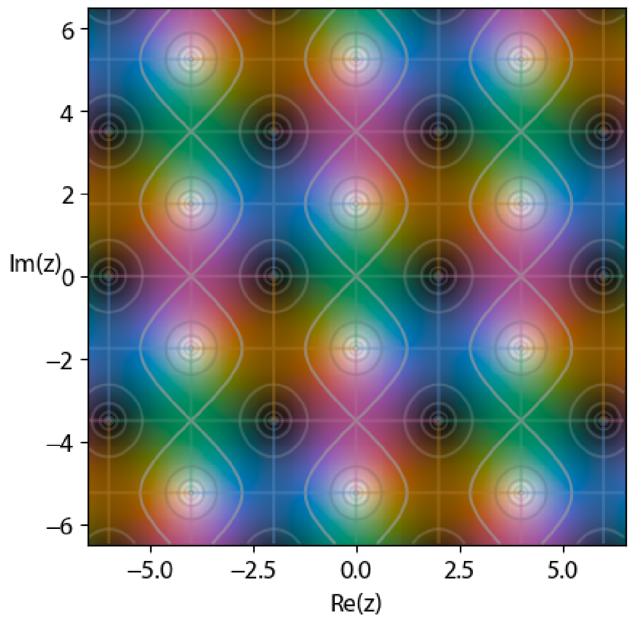


Figure 18: The Elliptic Cosine Function with modulus $k = 0.8$. The half periods I and I' are given by 1.995 and 1.75. (Source: Wikipedia, Attribution: By Nschloe - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=111308860)

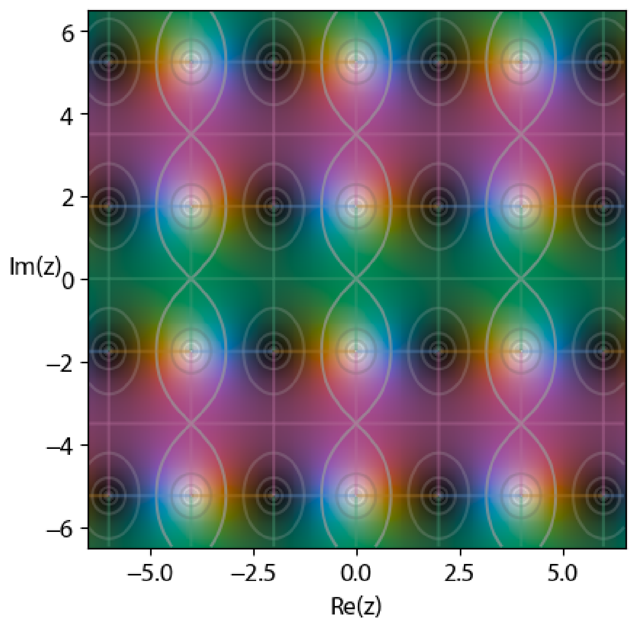


Figure 19: The Elliptic Function "dn" with modulus $k = 0.8$. The half periods I and I' are given by 1.995 and 1.75. (Source: Wikipedia, Attribution: By Nschloe - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=111308860)

Two important algebraic identities of Jacobian elliptic functions are,

$$\begin{aligned} \mathrm{sn}^2(u) + \mathrm{cn}^2(u) &= 1 \\ \mathrm{dn}^2(u) + k^2\mathrm{sn}^2(u) &= 1 \end{aligned} \tag{10.11}$$

The derivations of these identities can be found in Chapter 15 of [1]. The elliptic sine function sn satisfies the following addition theorem,

$$\mathrm{sn}(u-v) = \frac{\mathrm{sn}(u)\ \mathrm{cn}(v)\ \mathrm{dn}(v) - \mathrm{sn}(v)\ \mathrm{cn}(u)\ \mathrm{dn}(u)}{1 - k^2\mathrm{sn}^2(u)\mathrm{sn}^2(v)} \tag{10.12}$$

10.1. The modified Amplitude Function

The modified amplitude function is defined as,

$$\mathrm{Am}(u) = -i\ln\Big(ik^{\frac{1}{2}}\ \mathrm{sn}\Big(u - \frac{i}{2}I'\Big)\Big) \tag{10.13}$$

For real u, this function is real, odd and monotonic increasing. Later on we see that $\mathrm{Am}(u)$ goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ as u goes from $-I$ to I. This property is useful in showing that different choices of the argument u leads to different roots.

10.2. General Theorems

Theorem 10.2.1 (Liouville's Theorem): If a function is entire and bounded, then it is constant. A corollary of this theorem is that if a function is doubly periodic or anti-periodic and is analytic inside a period rectangle, then it is constant inside the rectangle.

Theorem 10.2.2 (Poles and Zeros): If a function $f(u)$ is doubly periodic or anti-periodic and meromorphic, and has n poles per period rectangle, then it also has n zeros per period rectangle. (Multiple poles or zeros of order r are counted as r poles or zeros.)

Theorem 10.2.3 (General Fundamental theorem of Algebra):

If a function $f(u)$ is meromorphic and satisfies the (anti-) periodic relations,

$$\begin{aligned} f(u+2I) &= (-1)^s f(u) \\ f(u+2iI') &= (-1)^r f(u) \end{aligned} \tag{10.14}$$

where r, s are integers, and if $f(u)$ has just n poles per period rectangle, at $u_1, ..., u_n$(counting a pole of order r as r coincident simple poles), then

$$f(u) = Ce^{i\lambda u} \prod_{j=1}^{n} \frac{H(u-v_j)}{H(u-u_j)} \tag{10.15}$$

where C, λ, v_j satisfy the constraint relations,

$$\begin{aligned} \sum_{j=1}^{n} v_j &= \sum_{j=1}^{n} u_j + (r+2m)I - i(s+2n)I' \\ \lambda &= \frac{1}{2}\pi \frac{s+2n}{I} \end{aligned} \tag{10.16}$$

where m, n are integers.

11. Eigenvalues for $T < T_c$

We now solve for $k \neq 1$. Thus, we have,

$$\begin{aligned} \sinh(2K) &= x \\ \sinh(2L) &= (kx)^{-1} \end{aligned} \tag{11.1}$$

Using this the elements of the matrix elements of $\mathbb{V}$ can be written as,

$$\begin{aligned} \exp(2K) &= x + \left(1+x^2\right)^{1/2} \\ \exp(2L) &= (kx)^{-1}\left[1 + \left(1+k^2x^2\right)^{1/2}\right] \end{aligned} \tag{11.2}$$

In general, there are no simple functions that can be used to parameterize (11.1). If $k = 1$ we can use the parameterization of $x = \tan(u)$. However, we can use the elliptic functions we introduced in the previous section if $k < 1$, to parameterize the eigenvalues of $\mathbb{V}$ and in turn express the elements of the matrix $\mathbb{V}$ in terms of meromorphic functions of the parameter u. These functions satisfy the properties,

$$\begin{aligned} \mathrm{cn}^2(u) &= 1 - \mathrm{sn}^2(u) \\ \mathrm{dn}^2(u) &= 1 - k^2\mathrm{sn}^2(u) \end{aligned} \tag{11.3}$$

Note that we abbreviate the notation $\mathrm{sn}(u,k) \iff \mathrm{sn}(u)$ and similarly for the other elliptic functions.

Let $x = -i\ \mathrm{sn}(iu)$. Using this, the parameterization of (11.1) can be written as,

$$\begin{aligned} \sinh(2K) &= -i\ \mathrm{sn}(iu) \\ \sinh(2L) &= -ik^{-1}\mathrm{sn}(iu)^{-1} \end{aligned} \tag{11.4}$$

Using this parameterization and (8.28), we can write the elements of the matrix $\mathbb{V}$ as,

$$\begin{aligned} \exp(\pm 2K) &= \mathrm{cn}(iu) \mp i\ \mathrm{sn}(iu) \\ \exp(\pm 2L) &= ik^{-1}\frac{\mathrm{dn}(iu) \pm 1}{\mathrm{sn}(iu)} \end{aligned} \tag{11.5}$$

Using the relations (10.5) between Jacobian elliptic functions and the theta functions,

$$\begin{aligned}\exp(\pm 2K) &= \frac{[k' H_1(iu) \mp iH(iu)]}{\left[k^{\frac{1}{2}}\Theta(iu)\right]} \\ \exp(\pm 2L) &= \frac{i[k'\Theta_1(iu) \pm \Theta(iu)]}{\left[k^{\frac{1}{2}}H(iu)\right]}\end{aligned} \tag{11.6}$$

The elliptic functions are only defined for $0 < k < 1$. This can be analytically continued to $k > 1$.

We outline the way we solve for the eigenvalues for the case of $k = 1$ and see the major steps involved.

1. For a given value of k, the transfer matrix elements $\exp(\pm 2K)$ and $\exp(\pm 2L)$ are expressed in terms of single-valued meromorphic functions of a single variable u. Meromorphic functions are functions that are holomorphic except at a discrete set of isolated points, which are called poles.
2. The eigenvalues of the transfer matrix $\mathbb{V}$ can be expressed as a linear combination of the matrix elements of the transfer matrix. Thus, the eigenvalue is also a single-valued meromorphic function of u. We can write a functional relation like (8.45) for the eigenvalues of the transfer matrix.
3. The zeros of the eigenvalue can be determined from the functional relation (8.45). The normalization of $\Lambda(u)$ can then be determined from that up to a sign.

Before we advance, we shall some constraints on the value of u for ferromagnetic Ising Models and that the calculations for the critical case is indeed a limiting case of the more general case of $k < 1$. Note that

$$\mathrm{sn}(iu) = -i\sinh(2K) = \sin(2iK)$$

Consider the equations 15.5.7 and 15.5.8 in [1]. Substituting $\alpha = i\beta$ in 15.5.8, we can write,

$$\begin{aligned}u &= \int_0^{2K} \frac{\mathrm{d}\beta}{(1 - i^2\sinh^2(\beta))^{\frac{1}{2}}} \\ &= \int_0^{2K} \frac{\mathrm{d}\beta}{(1 + \sinh^2(\beta))^{\frac{1}{2}}}\end{aligned} \tag{11.7}$$

If K, L are real and positive, then we have,

$$u > 0 \tag{11.8}$$

and note equation 15.5.10 in [1],

$$0 \leq u \leq \int_0^{\infty} \frac{\mathrm{d}\beta}{(1 + k^2\sinh^2(\beta))^{\frac{1}{2}}} = I' \tag{11.9}$$

We also need to show that the calculations for the critical case is indeed a limiting case of the more general case of $k < 1$. Again note equations in 15.5.9 and 15.5.10 which state that,

$$\begin{aligned}I &= \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\alpha}{(1 - k^2\sin^2(\alpha))^{\frac{1}{2}}} \\ I' &= \int_0^{\infty} \frac{\mathrm{d}\beta}{(1 + k^2\sinh^2(\beta))^{\frac{1}{2}}}\end{aligned} \tag{11.10}$$

For the case of $k = 1$, we have,

$$\begin{aligned}I_{k=1} &= \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\alpha}{(1 - \sin^2(\alpha))^{\frac{1}{2}}} \\ &= \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\alpha}{\cos(\alpha)} \longrightarrow \infty\end{aligned} \tag{11.11}$$

and,

$$\begin{aligned} I'_{k=1} &= \int_0^\infty \frac{\mathrm{d}\beta}{(1+\sinh^2(\beta))^{\frac{1}{2}}} \\ &= \int_0^\infty \frac{\mathrm{d}\beta}{\cosh(\beta)} = \frac{\pi}{2} \end{aligned} \tag{11.12}$$

At $k = 1$, we also get from solving the integral (11.7),

$$\begin{aligned} u &= \frac{\pi}{2} - \arctan(e^{-2K}) \\ \implies \sin(2iK) &= \frac{i}{2}\left[\tan\left(\frac{\pi}{4} - \frac{u}{2}\right) - \frac{1}{\tan(\frac{\pi}{4} - \frac{u}{2})}\right] \\ \implies \mathrm{sn}(iu) &= i\cot\left(\frac{\pi}{2} - u\right) = i\tan(u) \end{aligned} \tag{11.13}$$

The parameterization here is,

$$\sinh(2K) = -i\ \mathrm{sn}(iu) = \tan(u)$$

This is exactly our parameterization for the critical case of the 2D Ising model (9.1).

11.1. The form of $\Lambda(u)$

The solution to the eigenvalues can be derived in the same way as in the case of $k = 1$. The only difference is that we have to use the elliptic functions to parameterize the eigenvalues. Using this every element of the transfer matrix can be written as,

$$V_{\{\varphi,\varphi'\}}(u) = \frac{\Upsilon(u)}{(h(iu))^p} \tag{11.14}$$

where $\Upsilon(u)$ is an entire function of u and

$$h(u) = H(u)\Theta(u) \tag{11.15}$$

Again note that the eigenvalues of $\mathbb{V}$ can be expressed as a linear combination of the matrix elements of the transfer matrix. The coefficients of the linear combination of the matrix elements of the transfer matrix only depend on k. So the eigenvalue $\Lambda(u)$ can be expressed in terms of the elements of the transfer matrix as,

$$\Lambda(iu) = \frac{\Xi(u)}{(h(iu))^p} \tag{11.16}$$

where $\Xi(u)$ is another entire function of u.

We now try to find similar symmetry relations to those in the critical value of the 2D Ising model. Thus, consider incrementing u by $-2iI$ and $2I'$. First consider the increment by $2I'$. Then

$$\begin{aligned} \exp(\pm 2K') &= \mathrm{cn}(iu + 2iI') \mp i\ \mathrm{sn}(iu + 2iI') \\ &= -\ \mathrm{cn}(iu) \pm i\ \mathrm{sn}(iu) \\ &= -\exp(\mp 2K) \end{aligned} \tag{11.17}$$

Thus incrementing u by $2I'$ is the same as mapping K, L by $-K \pm \frac{1}{2}i\pi, -L \pm \frac{1}{2}i\pi$. Using (8.26) and (8.29), we see that this corresponds to multiplying the transfer matrix by the unlike spin matrix R (8.22). Thus the eigenvalue $\Lambda(u)$ satisfies the relation,

$$\Lambda(u + 2I') = -r\Lambda(u) \tag{11.18}$$

Again consider incrementing u by $-2iI$. Then,

$$\begin{aligned} \exp(\pm 2K') &= \mathrm{cn}(iu + 2I) \mp i\ \mathrm{sn}(iu + 2I) \\ &= -[\mathrm{cn}(u) \mp i\ \mathrm{sn}(u)] \\ &= -\exp(\pm 2K) \end{aligned} \tag{11.19}$$

Thus incrementing u by $-2iI$ is the same as mapping K, L by $K + \frac{1}{2}i\pi, L + \frac{1}{2}i\pi$. Using (8.29), we see that the eigenvalue $\Lambda(u)$ satisfies the relation,

$$\Lambda(u - 2iI) = \Lambda(u) \tag{11.20}$$

Thus, the eigenvalue $\Lambda(u)$ is doubly periodic with periods $2I'$ and $2iI$. The poles of $\Lambda(iu)$ are at $u = 0$, where $H(iu) = 0$ and $u = I'$ where $\Theta(iu) = 0$. Counting a pole of order p to p simple poles. Thus, there are $2p$ poles in one period rectangle. Thus, from Theorem 10.2.2, we obtain that there are $2p$ zeros in one period rectangle. Using the symmetry relations (11.18), (11.20) and thus the theorem Theorem 10.2.3, the eigenvalue $\Lambda(u)$ can be expressed as,

$$\Lambda(iu) = \rho e^{i\lambda u} \prod_{j=1}^{l} \frac{H(u - u_j)}{H(u - v_j)} \tag{11.21}$$

where v_j are the poles of $\Lambda(iu)$ and u_j are the zeros within a period rectangle. Note that $H(u + iI') = C\exp(-i\pi u/2I)\Theta(u)$. Using this, we can write the eigenvalue as,

$$\Lambda(u) = \rho e^{\lambda u}[h(iu)]^{-p} \prod_{\{j=1\}}^{l} H(iu - iu_j) \tag{11.22}$$

The constraint relations will be used later to properly count the eigenvalues of the transfer matrix. ρ and λ will be determined from the functional relation of the eigenvalues (8.45). The zeros u_j will be determined similarly to the case of $k = 1$.

11.2. The Zeros of $\Lambda(u)$

We can use the functional relation (8.45) to determine the zeros of $\Lambda(u)$. Note that setting $u' = u + I'$ in (11.5) is the same as setting $K \longrightarrow L + i\frac{\pi}{2}$ and $L \longrightarrow -K$

$$\begin{aligned}
\exp(\pm 2K') &= \text{cn}(iu + iI') \mp i\ \text{sn}(iu + iI') \\
&= -i\frac{\text{dn}(iu)}{k\ \text{sn}(iu)} \mp (k\text{sn}(iu)^{-1} \\
&= -ik^{-1}\frac{\text{dn}(iu) \mp 1}{\text{sn}(iu)} \\
&= -\exp(\pm 2L) = \exp\left(\pm 2\left(L + i\frac{\pi}{2}\right)\right) \\
\exp(\pm 2L') &= ik^{-1}\frac{\text{dn}(iu + iI') \pm 1}{\text{sn}(iu + iI')} \\
&= ik^{-1}\left(-i\frac{\text{cn}(iu)}{\text{sn}(iu)} \pm 1\right)k\ \text{sn}(iu) \\
&= \text{cn}(iu) \pm i\ \text{sn}(iu) \\
&= \exp(\mp 2K)
\end{aligned} \tag{11.23}$$

Using this and (11.4), we can write the functional relation (8.45) as,

$$\begin{aligned}
\Lambda(u)\Lambda(u + I') &= (2i\sinh(2L))^n + (-2i\sinh(2K))^n r \\
&= \left(-\frac{2}{k\ \text{sn}(iu)}\right)^n + (-2\ \text{sn}(iu))^n r
\end{aligned} \tag{11.24}$$

Using (11.24), (11.6), (11.22), we can write the functional relation as,

$$\Lambda(u)\Lambda(u + I') = \rho^2 e^{\lambda(2u+I')}[h(iu)h(iu + iI')]^{-p} \prod_{j=1}^{2p} H(iu - iu_j)H(iu - iu_j + iI')$$

$$\Longrightarrow \rho^2 e^{\lambda(2u+I')} \prod_{j=1}^{2p}[H(iu - iu_j)H(iu - iu_j + iI')] = \left(-\frac{2}{k\ \text{sn}(iu)}\right)^n + (-2\ \text{sn}(iu))^n r \tag{11.25}$$

$$\Longrightarrow \rho^2 e^{\lambda(2u+I')} \prod_{j=1}^{2p}[H(iu - iu_j)H(iu - iu_j + iI')] = \left(\frac{4}{k}\right)^p [H(iu)\Theta(iu)]^{-2p}[\Theta^{4p}(iu) + rH^{4p}(iu)]$$

Let us note the relations using the *Nome*, the *half period* I' and the *modulus* k and the Jacobian elliptic theta functions,

$$\begin{aligned} \Theta(u) &= -iq^{\frac{1}{4}} \exp(iu\pi/2I) H(u+iI') \\ H(u) &= -q \exp(i\pi u/I) H(u+2iI') \end{aligned} \tag{11.26}$$

Using this and (11.6), we can write the functional relation as,

$$\begin{aligned} &\rho^2 e^{\lambda(2u+I')} \prod_{j=1}^{2p} \left[H\left(iu - iu_j\right) H\left(iu - iu_j + iI'\right)\right] \\ &\qquad = [h(iu)h(iu+iI')]^p \left[\left(-\frac{2}{k\ \mathrm{sn}(iu)}\right)^n + (-2\ \mathrm{sn}(iu))^n r\right] \\ &\qquad = \left(\frac{4}{k}\right)^p \left[\Theta^{4p}(iu) + rH^{4p}(iu)\right] \end{aligned} \tag{11.27}$$

The zeros of H and Θ are exclusive. Thus, if u is a zero of the RHS, $H(u) \neq 0$. Note that the zeros of RHS of (11.27) are given by,

$$\left(k^2 \mathrm{sn}^2(iu)\right)^{2p} + r = 0 \tag{11.28}$$

This expression is a doubly periodic function of iu with periods $2I$ and $2iI'$. The function has a pole of order $4p$ where $\Theta(u) = 0$. Thus, from Theorem 10.2.2 we know the function has $4p$ zeros in the period rectangle. We now need to locate these zeros. We can use the amplitude function in (10.13) to relate the angles to this solution.

Set,

$$u = -\frac{1}{2}I' - i\phi \tag{11.29}$$

Then using (11.29) and (11.28), we can write the zeros of the RHS of (11.27) as,

$$\exp(i4p\ \mathrm{Am}(\phi)) + r = 0 \tag{11.30}$$

Let us define,

$$\mathrm{Am}(\phi) = \theta_j - \frac{\pi}{2}, \qquad j = 1, ..., 2p \tag{11.31}$$

where θ_j is defined similarly for $k = 1$ as in (9.20). Thus θ_j is defined as,

$$\theta_j = \begin{cases} \frac{\pi(j-1/2)}{2p} & j = \{1, ..., 2p\} \quad \text{if} \quad r = +1 \\ \frac{\pi j}{2p} & j = \{1, ..., 2p\} \quad \text{if} \quad r = -1 \end{cases} \tag{11.32}$$

The amplitude function $\mathrm{Am}(\phi)$ increases monotonically from $-\frac{\pi}{2} \longrightarrow \frac{\pi}{2}$ as ϕ goes from $-I \longrightarrow I$. Since $0 < \theta_j \leq \pi$, (11.31) takes unique values for $j = 1, ..., 2p$. Thus, if $j_1 \neq j_2$, they correspond to different roots of the eigenvalue $\Lambda(u)$.

Note that $r^2 = 1$, which implies that if u is a solution to (11.28), then so is $u + iI'$. Thus, (11.28) has $4p$ solutions in the period rectangle. The solutions are given by,

$$u_j = \mp\frac{1}{2}I' - i\phi_j, \quad j = 1, ..., 2p \tag{11.33}$$

These are unique zeros in one period rectangle. Thus, the zeros of (11.27) are given in pairs u_j and $u_j - I'$. In general the solution to (11.27) can be written as,

$$u_j = \frac{1}{2}\gamma_j I' - i\phi_j, \quad j = 1, ..., 2p \tag{11.34}$$

where γ_j takes the values ± 1.

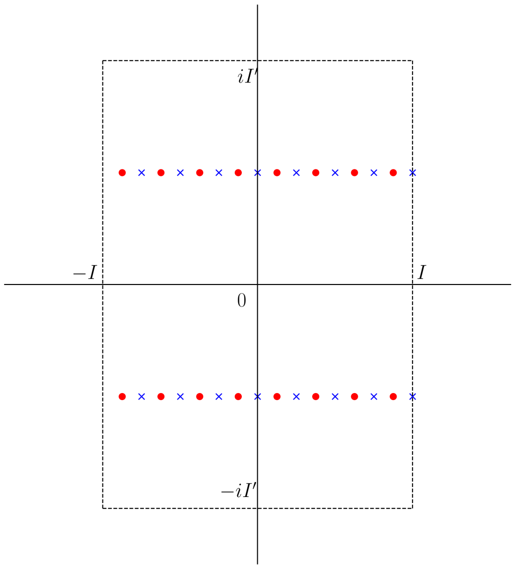


Figure 20: The zeros of the eigenvalue $\Lambda(u)$ in the period rectangle. The zeros are given by $u_j = \frac{1}{2}\gamma_j I' - i\phi_j$ where γ_j takes the values ± 1 and ϕ_j is defined in (9.20). The blue crosses are for the case of $r = -1$ and the red dots are for the case of $r = +1$. The zeros are given in pairs u_j and $u_j - iI'$.

Theorem 10.2.3 poses constraints on the signs of γ_j. Note that the poles of (11.16) are given at $u_j = 0$ for the $H(iu)$ in the denominator and $u_j = iI'$ for the $\Theta(iu)$ in the denominator. The constraints are given by,

$$\begin{aligned}\sum_{j=1}^{2p} v_j &= \left(\frac{1}{i}\right)\left[\sum_{j=1}^{2p} u_j + \left(\frac{1}{2}(1-r) + 2l\right)I + 2il'I'\right] \\ &= (p + 2l')I' + i\left[\frac{1}{2}(1-r) + 2l\right]I\end{aligned} \tag{11.35}$$

where r takes the values ± 1 and l, l' are any integers. For $r = 1$, each root ϕ occurs in pairs $(\phi, -\phi)$. When $r = -1$, all roots occur in pairs $(-\phi, -\phi)$ other than $r_0 = 0$ and $r_{2p} = I$. This exact statement was shown when we calculated the constraint relations for the Ising model at the critical point. Thus plugging in (11.34), we obtain,

$$\sum_{j=1}^{2p} \gamma_j = 2p - 4 \times \text{integer} \tag{11.36}$$

Thus r and all but one γ_j can be chosen independently, giving us 2^{2p} eigenvalues. The value of λ in (11.22) can be determined from the functional relation (11.24) and the constraint in Theorem 10.2.3,

$$\lambda = \pi[p + 2l']/2I' = -\pi\left[\sum_{j=1}^{2p} \gamma_j\right]/4I' \tag{11.37}$$

Then the eigenvalue $\Lambda(u)$ is given by

$$\Lambda(u) = \rho[h(iu)]^{-p} \prod_{j=1}^{2p} \exp(-\pi\gamma_j u/4I) H\left(iu - \phi_j + \frac{1}{2}i\gamma_j I'\right) \tag{11.38}$$

Squaring, we obtain,

$$\Lambda^2(u) = \rho^2 \prod_{j=1}^{2p} \exp(-\pi\gamma_j u/2I) \frac{H^2(iu - \phi_j + \frac{1}{2}i\gamma_j I')}{H(iu)\Theta(iu)} \tag{11.39}$$

and using relations between $H(u + iI')$ and $\Theta(u)$,

$$iq^{-\frac{1}{4}}\frac{\Theta\left(iu-\phi_j-\frac{1}{2}i\gamma_j I'\right)}{H\left(iu-\phi_j+\frac{1}{2}i\gamma_j I'\right)}=\exp\left(-\pi u\gamma_j/2I'\right) \tag{11.40}$$

we obtain,

$$\Lambda^2(u)=\rho'\prod_{j=1}^{2p}\frac{H\left(iu-\phi_j+\frac{1}{2}i\gamma_j I'\right)\Theta\left(iu-\phi_j+\frac{1}{2}i\gamma_j I'\right)}{H(iu)\Theta(iu)} \tag{11.41}$$

Again using the relations between $H(u+iI')$ and $\Theta(u)$, we can write the eigenvalue as,

$$\Lambda^2(u)=D\prod_{j=1}^{2p}k^{\frac{1}{2}}\operatorname{sn}\left(iu-\phi_j+\frac{1}{2}i\gamma_j I'\right) \tag{11.42}$$

where the gamma independent function D of u is given by,

$$D=\rho'\prod_{j=1}^{2p}\frac{\Theta\left(iu-\phi_j-\frac{1}{2}iI'\right)\Theta\left(iu-\phi_j+\frac{1}{2}iI'\right)}{H(iu)\Theta(iu)} \tag{11.43}$$

Note that there are poles in $D(u)$ of order $2p$ at $u=0$ and $u=I'$ and $4p$ zeros at $u=\phi_j\pm\frac{1}{2}iI'$. If we can come up with some function $g(u)$ that is easily evaluated and has the same poles and zeros as $D(u)$, the ration of $D(u)$ and $g(u)$ is an entire doubly periodic function of u. Thus, by Liouville's theorem Theorem 10.2.1, the ratio is a constant.

One such function $g(u)$ is

$$\frac{\left(k\operatorname{sn}^2(iu)\right)^{2p}+r}{\left(k^{\frac{1}{2}}\operatorname{sn}(iu)\right)^{2p}} \tag{11.44}$$

Then we can write D as,

$$D=\xi\frac{\left(k\operatorname{sn}^2(iu)\right)^{2p}+r}{\left(k^{\frac{1}{2}}\operatorname{sn}(iu)\right)^{2p}} \tag{11.45}$$

where ξ is a constant. Using (10.10), (11.45) and (11.42) and $r^2=1$, we obtain,

$$\begin{aligned}\Lambda^2(u)&=\xi\left[\frac{\left(k\operatorname{sn}^2(iu)\right)^{2p}+r}{\left(k^{\frac{1}{2}}\operatorname{sn}(iu)\right)^{2p}}\right]\prod_{j=1}^{2p}k^{\frac{1}{2}}\operatorname{sn}\left(iu-\phi_j+\frac{1}{2}i\gamma_j I'\right)\\ \Lambda^2(u+iI')&=\xi\left[\frac{\left(k\operatorname{sn}^2(iu)\right)^{-2p}+r}{\left(k^{\frac{1}{2}}\operatorname{sn}(iu)\right)^{-2p}}\right]\prod_{j=1}^{2p}k^{-\frac{1}{2}}\left(\operatorname{sn}\left(iu-\phi_j+\frac{1}{2}i\gamma_j I'\right)\right)^{-1}\end{aligned} \tag{11.46}$$

Then, we obtain,

$$\Lambda^u\Lambda^2(u+I')=\xi^2\left[2+r\left[\left(\left(k\operatorname{sn}^2(iu)\right)^{2p}+\left(k\operatorname{sn}^2(iu)\right)^{-2p}\right)\right]\right] \tag{11.47}$$

Squaring the functional relation (11.24), we obtain,

$$\Lambda^2(u)\Lambda^2(u+I')=\left(\frac{4}{k}\right)^{2p}r\left[2+r\left[\left(\left(k\operatorname{sn}^2(iu)\right)^{2p}+\left(k\operatorname{sn}^2(iu)\right)^{-2p}\right)\right]\right] \tag{11.48}$$

Comparing the above two relations, we obtain ξ to be,

$$\xi=\left(\frac{4}{k}\right)^{p}r^{\frac{1}{2}} \tag{11.49}$$

Finally, we can write the eigenvalue as,

$$\Lambda(u)=\zeta\left[\left(\frac{2}{(k\operatorname{sn}(iu))^{2p}}\right)+r(2\operatorname{sn}(iu))^{2p}\right]\times\prod_{j=1}^{2p}k^{\frac{1}{2}}\operatorname{sn}\left(iu-\phi_j+\frac{1}{2}i\gamma_j I'\right) \tag{11.50}$$

where

$$\zeta = \begin{cases} 1 & \text{if } r = +1 \\ -i & \text{if } r = -1 \end{cases} \tag{11.51}$$

After this whole ordeal we have obtained the eigenvalue of the transfer matrix $\mathbb{V}$ in terms of elliptic functions. In the next section, we revert to using L, K instead of the parameter u.

Result 10 (Eigenvalues for $T < T_c$): The eigenvalues of the transfer matrix $\mathbb{V}$ for $T < T_c$ can be expressed in terms of elliptic functions. The eigenvalue is given by,

$$\Lambda(u) = \zeta\left[\left(\frac{2}{(k\ \mathrm{sn}(iu))^{2p}}\right) + r(2\ \mathrm{sn}(iu))^{2p}\right] \times \prod_{j=1}^{2p} k^{\frac{1}{2}}\ \mathrm{sn}\left(iu - \phi_j + \frac{1}{2} i\gamma_j I'\right)$$

where

$$\zeta = \begin{cases} 1 & \text{if } r = +1 \\ -i & \text{if } r = -1 \end{cases}$$

r takes the values ± 1 and γ_j also takes the values ± 1. ϕ_j is given by,

$$\mathrm{Am}(\phi_j) = \theta_j - \frac{\pi}{2}, \qquad j = 1, ..., 2p$$

where θ_j is defined as,

$$\theta_j = \begin{cases} \frac{\pi(j-1/2)}{2p} & j = \{1, ..., 2p\} \quad \text{if} \quad r = +1 \\ \frac{\pi j}{2p} & j = \{1, ..., 2p\} \quad \text{if} \quad r = -1 \end{cases}$$

and $r = +1$, γ_j can only assume values subject to the constraint,

$$\sum_{j=1}^{2p} \gamma_j = 2p - 4 \times \text{integer}$$

$k^{-1} = \sinh(2K)\sinh(2L)$ is the modulus of the elliptic functions, $\mathrm{sn}(iu)$ is the elliptic sine function, and ϕ_j are the zeros of the eigenvalue.

Our task of finding the eigenvalues at a more general setting is done. However, k is only restricted to values less than unity. We shall analytically continue the form of the eigenvalue to the more general case, after removing the parameterizations on elliptic functions in the next section.

12. The Eigenvalues of the Transfer Matrix

Let us recollect the parameterizations that we used to obtain the elliptic functions in the previous section.

From the definition of the amplitude of the elliptic function (10.13) and (11.31), we have,

$$\begin{aligned} \mathrm{Am}(\phi_j) &= -i \ln\left[ik^{\frac{1}{2}}\ \mathrm{sn}\left(\phi_j - \frac{i}{2}I'\right)\right] \\ \implies k^{\frac{1}{2}}\ \mathrm{sn}\left(\phi_j - \frac{i}{2}I'\right) &= \exp(i\theta_j) \end{aligned} \tag{12.1}$$

The elliptic trigonometric identities are given as (10.11),

$$\mathrm{sn}^2(u) + \mathrm{cn}^2(u) = 1$$
$$\mathrm{dn}^2(u) + k^2\mathrm{sn}^2(u) = 1$$

Let $t = \phi_j - \frac{i}{2}I'$. Using these, we obtain,

$$
\begin{aligned}
\mathrm{cn}^2(t)\mathrm{dn}^2(t) &= (1-\mathrm{sn}^2(t))(1-k^2\mathrm{sn}^2(t)) \\
&= \left(1-k^{-1}e^{i\theta_j}\right)\left(1-ke^{i\theta_j}\right) \\
&= 1-ke^{i\theta_j}-k^{-1}e^{i\theta_j}+e^{-4i\theta_j} \\
&= \exp\left(2i\theta_j\right)\left[2\cosh\left(2\theta_j\right)-k-\frac{1}{k}\right]
\end{aligned}
\tag{12.2}
$$

Using (12.2), we can write,

$$
\mathrm{cn}(t)\ \mathrm{dn}(t) = -ik^{\frac{1}{2}}\exp\left(2i\theta_j\right)c_j \tag{12.3}
$$

where

$$
c_j = k^{-1}\left[1+k^2-2k\cosh\left(2\theta_j\right)\right]^{\frac{1}{2}} \tag{12.4}
$$

The addition formula for the elliptic functions is given in (10.12). Using (12.1) and (12.3) and the addition theorem, we obtain,

$$
\begin{aligned}
\mathrm{sn}(iu-t) &= \frac{\mathrm{sn}(iu)\ \mathrm{cn}(t)\ \mathrm{dn}(t)-\mathrm{cn}(iu)\ \mathrm{dn}(iu)\ \mathrm{sn}(t)}{1-k^2\mathrm{sn}^2(iu)\mathrm{sn}^2(t)} \\
&= \frac{k^{-\frac{1}{2}}\ \mathrm{cn}(iu)\ \mathrm{dn}(iu)e^{i\theta_j}-ik^{\frac{1}{2}}e^{i\theta_j}\ \mathrm{sn}(iu)c_j}{1-ke^{2i\theta_j}\mathrm{sn}^2(iu)} \\
&= k^{-\frac{1}{2}}\frac{\mathrm{cn}(iu)\ \mathrm{dn}(iu)-ikc_j\ \mathrm{sn}(iu)}{\exp\left(-i\theta_j\right)-k\exp\left(i\theta_j\right)\mathrm{sn}^2(iu)}
\end{aligned}
\tag{12.5}
$$

From (11.4), we obtain,

$$
\begin{aligned}
\sinh(2K) &= -i\ \mathrm{sn}(iu) \\
\sinh(2L) &= \frac{i}{k\ \mathrm{sn}(iu)} \\
\cosh(2K) &= \sqrt{1-\mathrm{sn}^2(iu)} = \mathrm{cn}(iu) \\
\cosh(2L) &= \sqrt{1-k^{-2}\mathrm{sn}^{-2}(iu)} = \frac{i\ \mathrm{dn}(iu)}{k\ \mathrm{sn}(iu)}
\end{aligned}
\tag{12.6}
$$

Using these, we can remove the elliptic functions from (12.5),

$$
\begin{aligned}
k^{\frac{1}{2}}\ \mathrm{sn}(iu-t) &= \frac{\mathrm{cn}(iu)\ \mathrm{dn}(iu)-ikc_j\ \mathrm{sn}(iu)}{\exp\left(-i\theta_j\right)-k\exp\left(i\theta_j\right)\mathrm{sn}^2(iu)} \\
&= \frac{\cosh(2K)\cosh(2L)+c_j}{\exp\left(-i\theta_j\right)\sinh(2L)+k\exp\left(i\theta_j\right)\sinh(2K)}
\end{aligned}
\tag{12.7}
$$

Thus, we have reverted the parameterization the factors in the product in (11.50). Also note that in (11.50), γ_j can take the values ± 1. Using this we can write that ,

$$
k^{\frac{1}{2}}\ \mathrm{sn}\left(iu-\phi_j+\frac{i}{2}I'\right) = \left(k^{\frac{1}{2}}\ \mathrm{sn}(iu-t)\exp\left(i\theta_j\right)\right)^{-1} \tag{12.8}
$$

Using this we can write ν_j as,

$$
\begin{aligned}
\nu_j &= k^{\frac{1}{2}}\ \mathrm{sn}\left(iu-\phi_j+\frac{i}{2}I'\right) \\
\implies k^{\frac{1}{2}}\ \mathrm{sn}&\left(iu-\phi_j+\frac{i}{2}\gamma_j I'\right) = \nu_j^{\gamma_j}
\end{aligned}
\tag{12.9}
$$

The factors preceding the product in (11.50) can also be reverted in the same way,

$$
\begin{aligned}
\frac{2}{k\ \mathrm{sn}(iu)} &= -2i\sinh(2L) \\
2\ \mathrm{sn}(iu) &= 2i\sinh(2K)
\end{aligned}
\tag{12.10}
$$

Using these relations we can completely eliminate the dependence of the eigenvalue on the elliptic functions. The eigenvalue can then be written as,

$$\Lambda^2(u) = \zeta(-4)^p\left[\sinh(2L)^{2p} + r(\sinh(2K))^{2p}\right]\prod_{j=1}^{2p}\nu_j^{\gamma_j} \tag{12.11}$$

where γ_j takes the values ±1 and ν_j is given as,

$$\nu_j = \frac{\cosh(2K)\cosh(2L) + c_j}{\exp\left(-i\theta_j\right)\sinh(2L) + \exp\left(i\theta_j\right)\sinh(2K)} \tag{12.12}$$

and c_j is given by,

$$c_j = k^{-1}\left[1 + k^2 - 2k\cos\left(2\theta_j\right)\right]^{\frac{1}{2}} \tag{12.13}$$

Here we have only obtained the general forms to the eigenvalue if $k < 1$. We can analytically continue the solution to $k > 1$. The constraint on $k < 1$ was imposed by our use of elliptic functions. Since, we have removed the dependence to elliptic functions in the eigenvalue, we can analytically continue the solution to $k > 1$ as well.

12.1. Analytic Continuation of the Eigenvalue

The eigenvalue, for finite p, must be an algebraic function of $\exp(2K)$ and $\exp(2L)$, since these constitute the elements of the transfer matrix. We know that the eigenvalue is a linear combination of the matrix elements of the transfer matrix. Thus, the elliptic function removed version of the eigenvalue can be analytically continued to the case of $k \geq 1$.

The only dependence on k in the eigenvalue is in c_j. The sign of c_j should concern us. If $\gamma = -1$, then the numerator of c_j introduces potential singularities into the eigenvalue, where

$$\cosh(2K)\cosh(2L) + c_j = 0 \tag{12.14}$$

The form of c_j is given by,

$$\begin{aligned} c_j &= k^{-1}\left[1 + k^2 - 2k\cos\left(2\theta_j\right)\right]^{\frac{1}{2}} \\ &= k^{-1}\left[(1-k)^2 + 2k\left(1 - \cos\left(2\theta_j\right)\right)\right]^{\frac{1}{2}} \end{aligned}$$

Thus for any value of k, c_j is always real and positive, given $0 < \theta_j < \pi$. For all roots $j \in \{1, ..., 2p\}$, this approaches a positive limit when $k \longrightarrow 1$, so we choose the positive root for c_j.

The problem arises when $r = -1$ and $j = 2p$, then $\theta_j = \pi$ and for $k < 1$,

$$c_{2p} = \frac{1-k}{k} \longrightarrow 0 \tag{12.15}$$

Thus, the analytic continuation of the eigenvalue to $k > 1$ for this case needs to be the negative square root. A clearer idea of the analytic continuation can be viewed in the following plot.

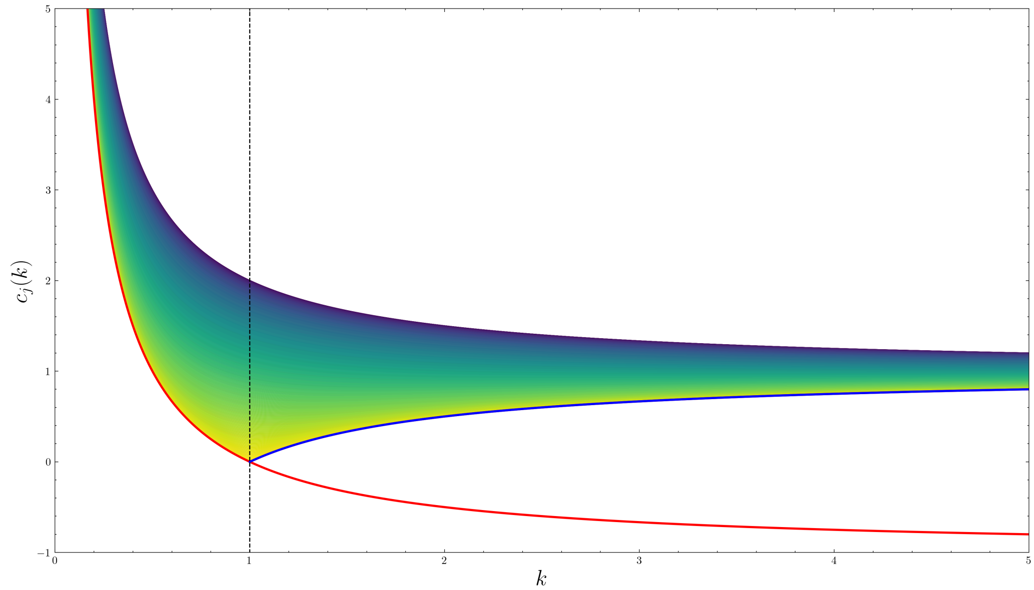


Figure 21: The plot of c_j as a function of k. The purple line corresponds to $\frac{\pi}{2}$ and the yellow line corresponds to π. The solution is obtained from the interval of $k \in (0, 1)$ which is the region where the elliptic functions are defined. The blue line is the positive root and the red line is the negative root.

Observe Figure 21. The colors in the color map go from purple to yellow, where the purple line corresponds to $\theta_j = \frac{\pi}{2}$ and the yellow line corresponds to $\theta_j = \pi$. The blue line is the positive root of c_j at $\theta_j = \pi$ while the red line is the negative root of c_j at $\theta_j = \pi$. The analytic continuation of the eigenvalue to $k > 1$ is given by the negative root of c_j at $\theta_j = \pi$. Thus, from the graph we can clearly see that the negative root is the analytic continuation of the eigenvalue to $k > 1$ for the case $r = -1$ and $j = 2p$. This type of non-analytic behavior also arises at $\theta_j = 0$, since these graphs obtained will be symmetric about $\theta_j = \frac{\pi}{2}$. However, the $\theta = 0$ is not a root of the eigenvalue, since $\theta_j^{\min} = \pi/2p$ for $r = -1$ and $\theta_j^{\min} = \pi/4p$ for $r = 1$.

Thus, the analytically continued form of c_j is given by,

$$c_j = k^{-1}\left[1 + k^2 - 2k\cos(2\theta_j)\right]^{\frac{1}{2}} \tag{12.16}$$

unless $r = -1$ and $j = 2p$, then

$$c_j = -k^{-1}\left[1 + k^2 - 2k\cos(2\theta_j)\right]^{\frac{1}{2}} \tag{12.17}$$

Result 11 (General expression of the eigenvalues): The eigenvalue of the transfer matrix $\Lambda^2(u)$ is given by,

$$\Lambda^2(u) = \zeta(-4)^p\left[\sinh(2L)^{2p} + r(\sinh(2K))^{2p}\right]\prod_{j=1}^{2p}\nu_j^{\gamma_j}$$

where γ_j takes the values ± 1 and ν_j is given as,

$$\nu_j = \frac{\cosh(2K)\cosh(2L) + c_j}{\exp(-i\theta_j)\sinh(2L) + \exp(i\theta_j)\sinh(2K)}$$

and c_j is given by,

$$c_j = \begin{cases} -k^{-1}\left[1 + k^2 - 2k\cos(2\theta_j)\right]^{\frac{1}{2}} & \text{if } k > 1 \text{ and } j = 2p \text{ and } r = -1 \\ k^{-1}\left[1 + k^2 - 2k\cos(2\theta_j)\right]^{\frac{1}{2}} & \text{Otherwise} \end{cases}$$

where

$$\theta_j = \begin{cases} \frac{\pi j}{2p} & j \in \{1, ..., 2p\} \quad \text{if } r = -1 \\ \frac{\pi(j-\frac{1}{2})}{2p} & j \in \{1, ..., 2p\} \quad \text{if } r = +1 \end{cases}$$

and the normalization factor ζ is given by,

$$\zeta = \begin{cases} -i & \text{if } r = -1 \\ 1 & \text{if } r = +1 \end{cases}$$

13. The maximum eigenvalue

The partition function for the 2D Ising model is given by (8.4). There are m rows in the 2D Ising model and each row has $2p$ sites. The number of sites N is then given by $2mp$. The free energy per site is given by,

$$\begin{aligned} -\beta f &= \lim_{N\longrightarrow\infty}\left(\frac{1}{N}\right)\ln Z_N \\ &= \lim_{m\longrightarrow\infty}\left(\frac{m}{2mp}\right)\frac{1}{m}\ln(Z_N) \\ &= (2p)^{-1}\Lambda_{\max}(u) \end{aligned} \tag{13.1}$$

The transfer matrix has all positive entries, and hence the matrix is irreducible. The Perron Frobenius theorem [10] guarantees that the maximum eigenvalue is positive and unique and the corresponding eigenvector has all positive entries. The matrix of unlike entries R (8.22) is a matrix of all positive entries. Note that the eigenvector of the dominant eigenvalue is given by $\boldsymbol{x}(k)$, which is a vector of all positive entries. Thus, the eigenvalue r has to be positive, which means $r = +1$.

After this we need to maximize the expression,

$$\Lambda^2(u) = (-4)^p\left[\sinh(2L)^{2p} + (\sinh(2K))^{2p}\right]\prod_{j=1}^{2p}\nu_j^{\gamma_j} \tag{13.2}$$

The value of $|\nu_j|$ determines the choices for γ_j for the maximum eigenvalue. Using (8.38), (12.13) and (12.11) we can find $|\nu|$. From (12.12), we can express $\nu_j = \frac{f}{g}$ where f and g are the numerator and denominator respectively of ν_j, such that f is real and g is in general complex. Then,

$$
\begin{aligned}
\nu_j \nu_j^* &= \left[\frac{fg^*}{|g|^2}\right]\left[\frac{fg}{|g|^2}\right] \\
&= \frac{f^2}{|g|^2} \\
&= \frac{\left(\cosh(2K)\cosh(2L)+c_j\right)^2}{\sinh^2(2L)+\sinh^2(2K)+2\sinh(2L)\sinh(2K)\cos(2\theta_j)} \\
&= \frac{\left(\cosh(2K)\cosh(2L)+c_j\right)^2}{\cosh^2(2L)+\cosh^2(2K)-2+1+\sinh^2(2L)\sinh^2(2K)-c_j^2} \\
&= \frac{\left(\cosh(2K)\cosh(2L)+c_j\right)^2}{\cosh^2(2L)\cosh^2(2K)-c_j^2} \\
&= \frac{\cosh(2K)\cosh(2L)+c_j}{\cosh(2K)\cosh(2L)-c_j}
\end{aligned}
\tag{13.3}
$$

Note the following inequality,

$$
\begin{aligned}
&(\sinh(2L)-\sinh(2K))^2 \geq 0 \\
\Longrightarrow\ &\sinh^2(2L)+\sinh^2(2K) \geq 2\sinh(2L)\sinh(2K) \geq -2\sinh(2L)\sinh(2K)\cos(2\theta_j)
\end{aligned}
\tag{13.4}
$$

As a result of the previous inequality, c_j also satisfies the constraint relation,

$$
\begin{aligned}
0 \leq c_j \leq \left(1+k^{-2}-\frac{2}{k}\cos(2\theta_j)\right)^{\frac{1}{2}} &\leq \left(1+k^{-2}+\sinh^2(2L)+\sinh^2(2K)\right)^{\frac{1}{2}} \\
&\leq \cosh(2L)\cosh(2K)
\end{aligned}
\tag{13.5}
$$

Using this inequality, $|v_j| \geq 1$. Thus, the eigenvalue is clearly maximized when all $\gamma_j = +1$. Using this and the fact that for the maximum eigenvalue $r = 1$, we can reduce the expression further.

The denominator in (12.12) is,

$$
\begin{aligned}
e^{-i\theta_j}\sinh(2L)+e^{i\theta_j}\sinh(2K) &= e^{i\theta_j}\left[\sinh(2K)+e^{-2i\theta_j}\sinh(2L)\right]] \\
&= e^{i\pi(j-\frac{1}{2})/2p}\left[\sinh(2K)+e^{-i\pi(2j-1)/2p}\sinh(2L)\right]
\end{aligned}
\tag{13.6}
$$

Consider a general expression $(t+\alpha t_j)$. Take the product of this expression from $j=1$ to $2p$. Evaluated,

$$
\prod_{j=1}^{2p}(t+\alpha t_j) = t^{2p} + \left[\sum_{j=1}^{2p} t_j\right]\alpha t^{2p-1} + \alpha^2\left[\sum_{1\leq i<j\leq 2p} t_i t_j\right] t^{2p-2} + \dots + \prod_{j=1}^{2p} t_j \tag{13.7}
$$

The factor before $\sinh(2L)$ in (12.12) corresponds to the solutions of $(2p)$th roots of -1. The equation with solutions as the $2p$th roots of -1 is given by,

$$
z^{2p} = -1 \tag{13.8}
$$

Let the j^{th} root of this equation be η_j. By Vieta's Theorem [13], the following relations hold,

$$
\begin{aligned}
\sum_{j=1}^{2p}\eta_j &= 0 \\
\sum_{1\leq i<j\leq 2p}\eta_i\eta_j &= 0
\end{aligned}
\tag{13.9}
$$

This holds for all summations of combinations of roots, other than the product of all roots. Using this fact, all terms in (13.7) vanish except the first and the last term. Using these relations, we can write the denominator in (12.12) as,

$$\prod_{j=1}^{2p} e^{\frac{i\pi\left(j-\frac{1}{2}\right)}{2p}} \left[\sinh(2K) + e^{-\frac{i\pi(2j-1)}{2p}} \sinh(2L)\right] = \left[\sinh^{2p}(2K) + \sinh^{2p}(2L)\right] \prod_{j=1}^{2p} e^{i\pi\left(j-\frac{1}{2}\right)/2p} = (-1)^p \left[\sinh^{2p}(2K) + \sinh^{2p}(2L)\right] \tag{13.10}$$

Thus using this in (12.11), we can write the maximum eigenvalue as,

$$\Lambda^2(u) = (-1)^p 4^p \left[\sinh^{2p}(2K) + \sinh^{2p}(2L)\right] \prod_{j=1}^{2p} \nu_j = \prod_{j=1}^{2p} 2\left[\cosh(2K)\cosh(2L) + c_j\right] \tag{13.11}$$

We are finally ready to take the thermodynamic limit. Define the function $F(\theta)$,

$$F(\theta) = \ln\left[2\left\{\cosh(2K)\cosh(2L) + k^{-1}\left[1 + k^2 - 2k\cos(2\theta)\right]^{\frac{1}{2}}\right\}\right] \tag{13.12}$$

Thus, the maximum eigenvalue is given from (13.11), (13.12) and (9.20) and the fact that the Perron Frobenius theorem asserts that $r = 1$ for the maximum eigenvalue as,

$$\ln(\Lambda_{\max}(u)) = \frac{1}{2}\sum_{j=1}^{2p} F\left(\pi\left(j - \frac{1}{2}\right)/2p\right) \tag{13.13}$$

In the thermodynamic limit this sum over uniformly distributed values of θ becomes an integral, which runs from 0 to π. The interval of length between roots is given by $\Delta\theta = \pi/2p$. Thus, the free energy per site is given by,

$$-\beta f = (2\pi)^{-1} \int_0^{\pi} F(\theta)\, \mathrm{d}\theta \tag{13.14}$$

This is the major result of this article.

Result 12 (Free energy of the 2D Ising model): The free energy of the 2D Ising model is given by,

$$-\beta f = (2\pi)^{-1} \int_0^{\pi} F(\theta)\, \mathrm{d}\theta$$

where $F(\theta)$ is given by,

$$F(\theta) = \ln\left[2\left\{\cosh(2K)\cosh(2L) + k^{-1}\left[1 + k^2 - 2k\cos(2\theta)\right]^{\frac{1}{2}}\right\}\right]$$

and $k^{-1} = \sinh(2K)\sinh(2L)$.

14. The Criticality of the 2D Ising Model

The free energy of the 2D Ising model is given by (13.14) and (13.12). Note the form of $F(\theta)$, which, for positive k and real θ, is an analytic function in θ, k and $\cosh(2L), \cosh(2K)$ except possibly in one case. Consider the case when $\theta = 0$, we then obtain from (13.12),

$$F(0) = \ln\left[2\left\{\cosh(2K)\cosh(2L) + k^{-1}\sqrt{1 + k^2 - 2k}\right\}\right] = \ln\left[2\{\cosh(2K)\cosh(2L) \pm k^{-1}(1-k)\}\right] \tag{14.1}$$

Thus, $F(\theta)$ might be non-analytic when $k = 1$ and $\theta = 0$. So let us examine this further. Near $k = 1$ and $\theta = 0$, the square root becomes zero. So we can write,

$$\begin{aligned}F(\theta) &= \ln[2\{\cosh(2K)\cosh(2L)\}] + \ln\left(1 + \frac{k^{-1}\sqrt{1+k^2-2k\cos(2\theta)}}{\cosh(2K)\cosh(2L)}\right)\\ &= \ln[2\{\cosh(2K)\cosh(2L)\}] + \frac{k^{-1}\sqrt{1+k^2-2k\cos(2\theta)}}{\cosh(2K)\cosh(2L)} + \dots\end{aligned} \tag{14.2}$$

where we used the Taylor expansion of $\ln(1+x)$ around $x = 0$, which is $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + O(x^4)$. Plugging this form of $F(\theta)$ into the integral in (13.14), we obtain,

$$-2\pi\beta f = \pi\ln(2\cosh(2K)\cosh(2L)) + \int_0^\pi \mathrm{d}\theta\, \frac{k^{-1}\sqrt{1+k^2-2k\cos(2\theta)}}{\cosh(2K)\cosh(2L)} \tag{14.3}$$

The first term in the above equation cannot give us singularities. All nuisance points must arise from the second term f_n,

$$-2\pi\beta f_n = \int_0^\pi \mathrm{d}\theta\, \frac{k^{-1}\sqrt{1+k^2-2k\cos(2\theta)}}{\cosh(2K)\cosh(2L)} \tag{14.4}$$

Changing the variable of integration from θ to $\frac{\pi}{2} - \theta$ and the trigonometric formula, we obtain $\cos(2\theta) = 2\cos^2(\theta) - 1$,

$$-\beta f_n = (2\pi)^{-1}\frac{[2(1+k)]}{\cosh(2K)\cosh(2L)}\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \left(1 - \left[\frac{2k^{\frac{1}{2}}}{1+k}\right]^2 \sin^2(\theta)\right)^{\frac{1}{2}} \tag{14.5}$$

This integral has a special name and is called the complete elliptic integral of the second kind, $E(k)$. Thus, we can write,

$$-\beta f_n = (2\pi)^{-1}\frac{[2(1+k)]}{\cosh(2K)\cosh(2L)}E(k_1) \tag{14.6}$$

where

$$E(k) = \int_0^{\frac{\pi}{2}} \mathrm{d}\theta\,(1 - k^2\sin^2(\theta))^{\frac{1}{2}} \tag{14.7}$$

$$k_1 = \frac{2k^{\frac{1}{2}}}{1+k} \tag{14.8}$$

Define the complex or complementary modulus like $k' = \sqrt{1-k^2}$. The Table of Integrals Series and products mentions in Sec. 8.114 equation 3 [19] that when k' is near 0,

$$\begin{aligned}E(k) &= 1 + \frac{1}{2}\left(\ln\left(\frac{4}{k'}\right) - \frac{1}{2}\right)k'^2 + O(k'^3)\\ &= 1 + \frac{1}{4}\ln\left(\frac{16}{1-k^2}\right)(1-k^2) - \frac{1}{4}(1-k^2)\\ &\approx 1 + \frac{1}{4}\ln\left(\frac{16}{1-k^2}\right)(1-k^2)\end{aligned} \tag{14.9}$$

Thus, we obtain from this and (14.6), neglecting analytic contributions,

$$-\beta f_n = \frac{[(1+k)(1-k)^2]}{[2\pi k\cosh(2K)\cosh(2L)]}\ln\left|\frac{1+k}{1-k}\right| \tag{14.10}$$

Thus, we can clearly observe from this that the free energy is non-analytic at $k = 1$. This matches our result from Kramers Wannier Duality, where we showed that if there is only one critical point, then it must be at $k = 1$ (5.21). Thus, we conclude that the 2D Ising model has a critical point at $k = 1$.

15. Numerical Simulation of 2D Ising Model

After our arduous journey through the theoretical landscape of the 2D Ising model, we now relax and see how numerical simulations relate to the theory. We will use the Metropolis algorithm to simulate the 2D Ising model on a square lattice and compute various thermodynamic quantities.

15.1. Metropolis Algorithm

The Metropolis algorithm is a Monte Carlo method used to simulate the behavior of systems in statistical mechanics. For the 2D Ising model, the algorithm proceeds as follows:

1. Initialize a square lattice of size $L \times L$ with random spin configurations (each spin can be either $+1$ or -1).
2. For a given temperature T, repeat the following steps for a large number of Monte Carlo steps: a. Randomly select a spin σ_i on the lattice. b. Calculate the change in energy ΔE that would result from flipping the spin. c. If $\Delta E \leq 0$, flip the spin. If $\Delta E > 0$, flip the spin with probability $\exp\left(-\frac{\Delta E}{k_B T}\right)$.
3. After a sufficient number of Monte Carlo steps, measure the desired thermodynamic quantities.

Here we will see how the lattice evolves with time at different temperatures. We will observe the formation of domains and how they grow as the system approaches equilibrium. Before we need to define what our time scale is. We will define one Monte Carlo Step (MCS) as one complete sweep through the lattice, where each spin has been considered for a possible flip once. So one MCS consists of L^2 spin flip attempts for a lattice of size $L \times L$.

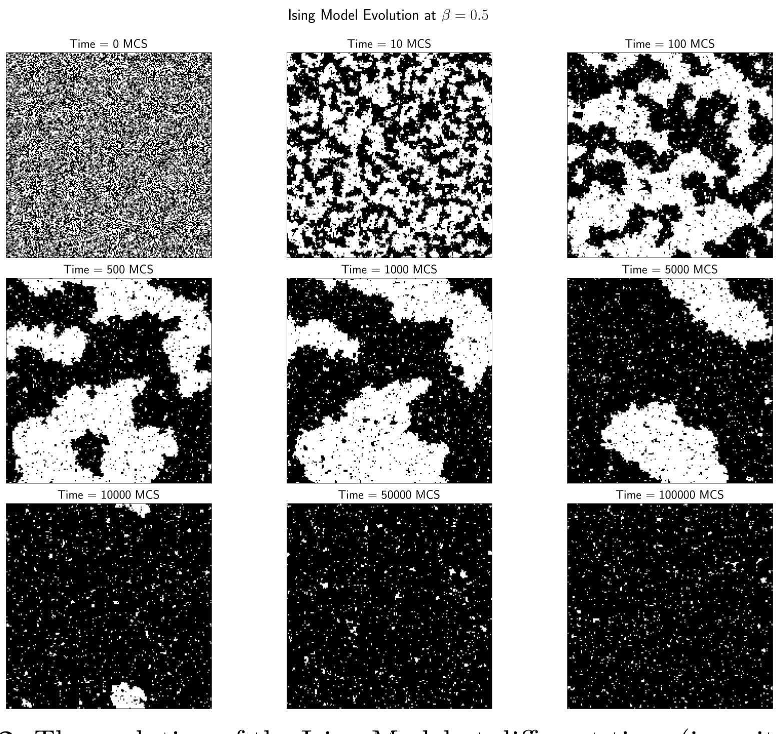


Figure 22: The evolution of the Ising Model at different times(in units of MCS) at $T = 2$.

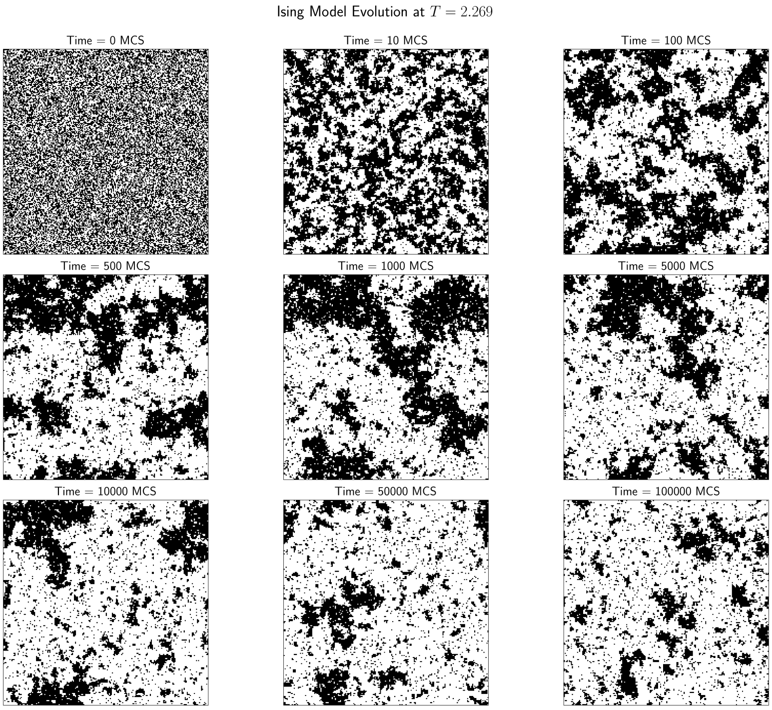


Figure 23: The evolution of the Ising Model at different times(in units of MCS) at $T \approx 2.269$.

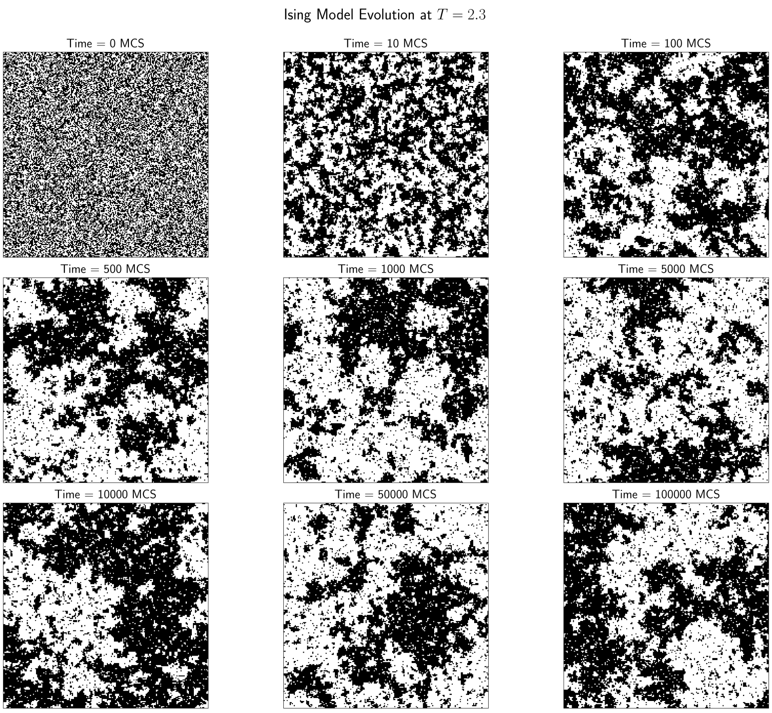


Figure 24: The evolution of the Ising Model at different times(in units of MCS) at $T = 2.3$.

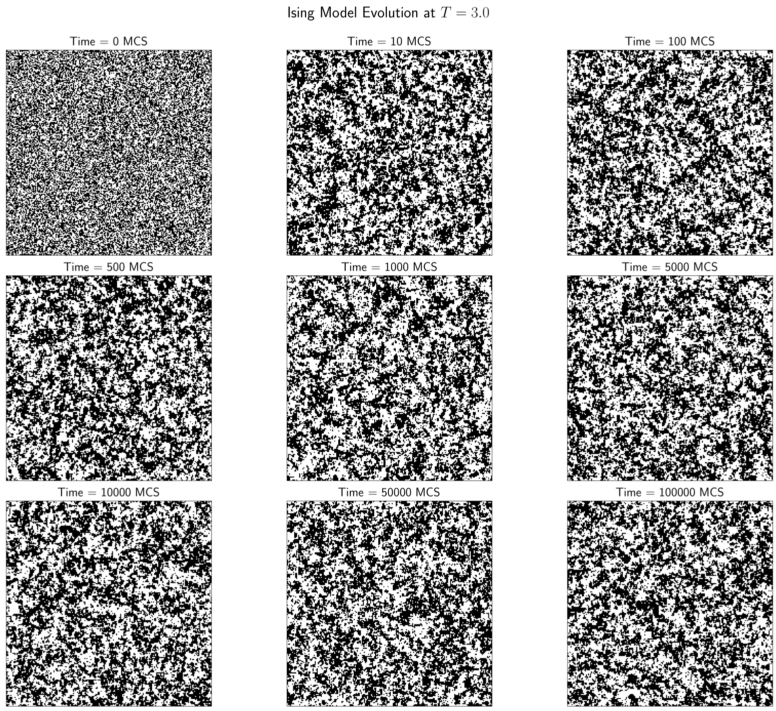


Figure 25: The evolution of the Ising Model at different times(in units of MCS) at $T = 3$.

15.2. Thermodynamic Quantities

The quantitites of interest to us are the average magnetization per spin $\langle|m|\rangle$, the average energy per spin $\langle E\rangle$, the specific heat C, and the magnetic susceptibility χ. These quantities can be computed as follows:

$$\langle|m|\rangle = \frac{1}{N}\left\langle |\sum_{\{i\}}\sigma_i| \right\rangle \tag{15.1}$$

$$\langle E\rangle = \frac{1}{N}\left\langle -J\sum_{\{ij\}}\sigma_i\sigma_j \right\rangle \tag{15.2}$$

$$C = \frac{\langle E^2\rangle - \langle E\rangle^2}{k_B T^2} \tag{15.3}$$

$$\chi = \frac{\langle|m|^2\rangle - \langle|m|\rangle^2}{k_B T} \tag{15.4}$$

The analytical solution Energy can be found from the free energy per spin f as:

$$\langle E\rangle = -\frac{\partial(\beta f)}{\partial\beta} \tag{15.5}$$

where $\beta = \frac{1}{k_B T}$. The exact solution for magnetization per spin is given by:

$$\langle|m|\rangle = \left(1 - \sinh^{-4}\left(\frac{2J}{k_B T}\right)\right)^{\frac{1}{8}} \quad \text{for} \quad T < T_c \tag{15.6}$$

$$\langle|m|\rangle = 0 \quad \text{for} \quad T \geq T_c \tag{15.7}$$

We can compare the numerical results obtained from the Metropolis algorithm with the exact solutions for these quantities. Below are the plots showing the comparison.

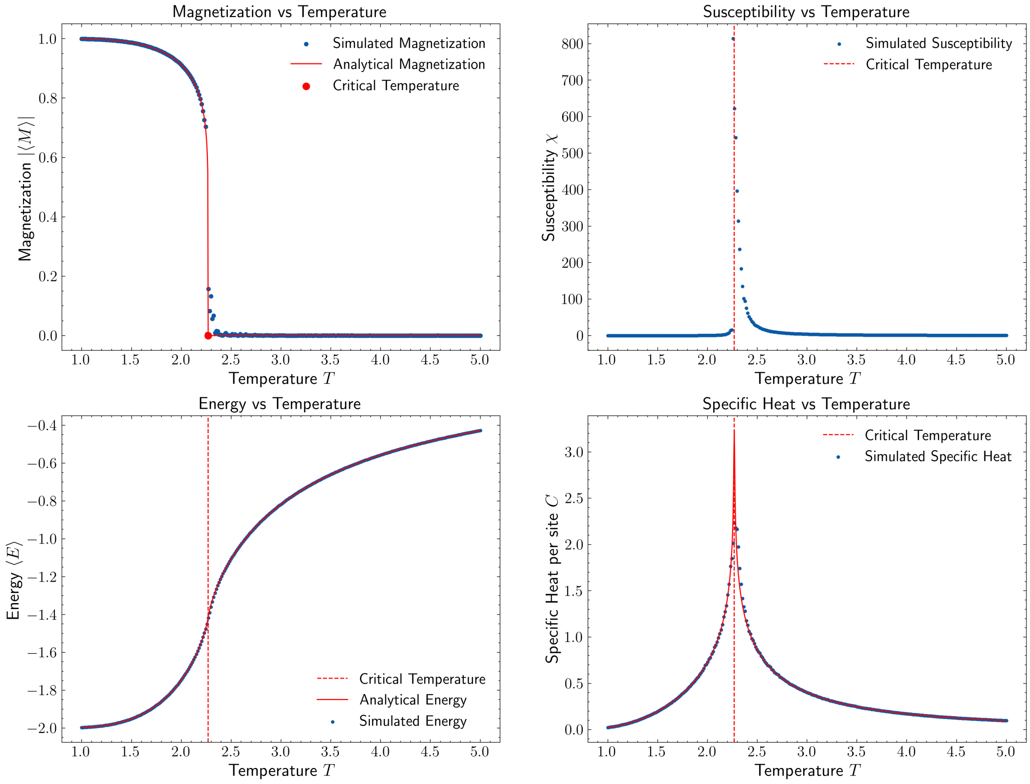


Figure 26: Magnetization per spin $\langle |m| \rangle$, Magnetic Susceptibility χ, Energy per spin $\langle E \rangle$, and Specific Heat C as functions of temperature T obtained using the Metropolis algorithm compared against the exact solutions.

15.3. Time Delayed Autocorrelation function

The time-delayed autocorrelation function is a measure of how the state of the system at one time is correlated with its state at a later time. This function is particularly useful for understanding the dynamics of the system and how well the numerical algorithm samples the configuration space. The autocorrelation function for a quantity A is defined as:

$$C_{A(\tau)} = \frac{\langle A(t)A(t+\tau)\rangle - \langle A\rangle^2}{\langle A^2\rangle - \langle A\rangle^2} \tag{15.8}$$

where τ is the time delay, and $\langle ... \rangle$ denotes the ensemble average. The integral of the autocorrelation function gives the correlation time τ_c:

$$\tau_c = \int_0^\infty C_A(\tau)d\tau \tag{15.9}$$

In practice, the correlation time can be estimated by fitting the autocorrelation function to an exponential decay and extracting the characteristic timescale. The integrated correlation time is generally not used in simulations, since we require quite a large amount of data to compute it accurately. Instead, we can estimate the correlation time by finding the time lag τ at which the autocorrelation function decays to a small value.

When computing thermodynamic quantities, it is essential to ensure that the measurements are taken from uncorrelated configurations to obtain accurate results. The autocorrelation function helps in determining the correlation time τ_c, which indicates how many Monte Carlo steps are needed before the configurations become effectively uncorrelated.

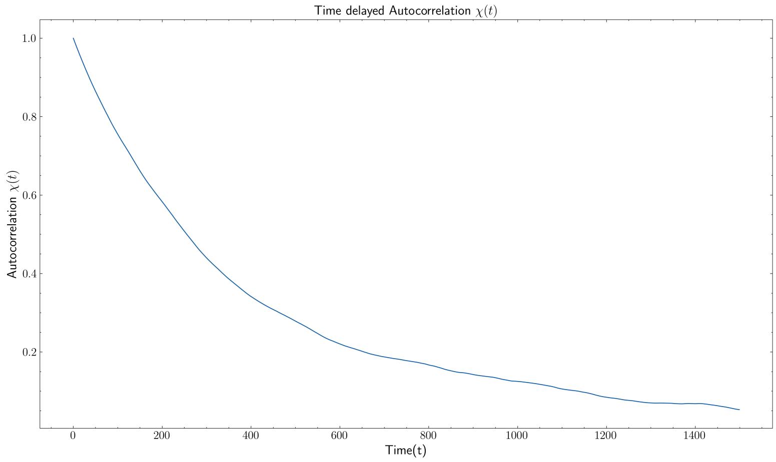


Figure 27: The time-delayed autocorrelation function for magnetization at $T \approx 2.269$.

This plot shows the autocorrelation function for magnetization at a temperature near the critical temperature, for small timelags where the lattice configurations(or magnetization in our case). Afte r a certain time lag, the autocorrelation function decays to a noise level which is completely random. The correlation time τ_c can be estimated from this plot.

Before we go on with that, let us understand something about critical slowing down.

15.4. Critical Slowing Down

Critical slowing down refers to the phenomenon where the relaxation time of a system near its critical point diverges. In the context of the Ising model, as the temperature approaches the critical temperature T_c, the time it takes for the system to reach equilibrium increases significantly. Let's now understand why this happens.

The two point spin-spin correlation function $G(r)$ describes how the spin at one site is correlated with the spin at another site a distance r away. Now this correlation function has a characteristic length scale called the correlation length ξ, which diverges as the system approaches the critical temperature. This means that spins become correlated over larger distances, leading to the formation of large domains of aligned spins. When the system is near the critical temperature, the correlation length diverges and essentially similar domains of spins form throughout the lattice, over all scales of the lattice. This makes it difficult for the Metropolis algorithm to efficiently sample the configuration space, as flipping a single spin may not significantly change the overall configuration of the system. As a result, the system takes longer to be uncorrelated, leading to an increase in the autocorrelation time τ_c. To fix this problem, cluster algorithms like the Wolff algorithm are used, which flip clusters of spins instead of individual spins, thereby reducing the autocorrelation time near the critical point.

Before we move on to a quantification of critical slowing down, let us examine one more consequence of diverging correlation length.

15.5. Scale Invariance at Criticality

At the critical temperature, the system exhibits scale invariance, meaning that the properties of the system look similar at different length scales. This is a direct consequence of the diverging correlation length ξ, which implies that there is no characteristic length scale in the system. As a result, the system displays self-similar structures, where clusters of aligned spins can be found at all scales.

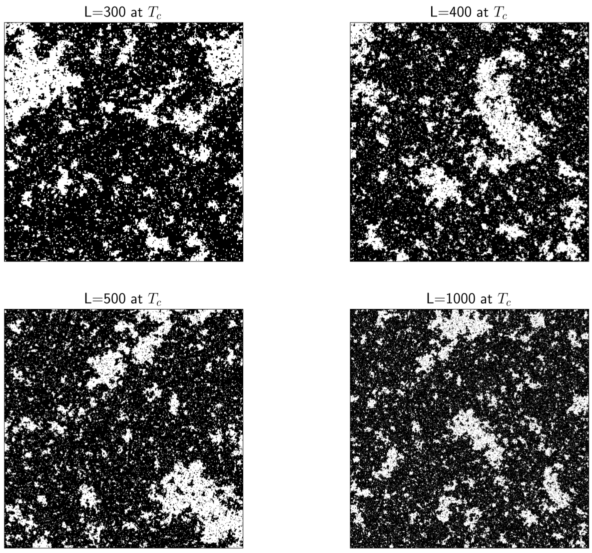


Figure 28: Scale Invariance at Criticality. These images show the same configuration of the Ising model at T_c zoomed in at different scales. Notice how the patterns look similar at different scales.

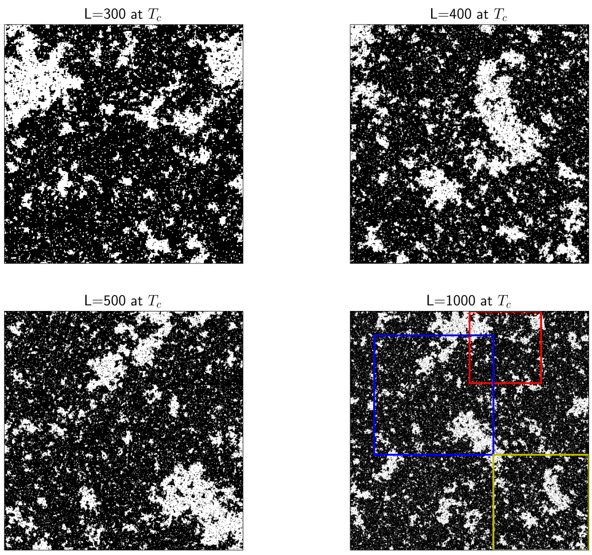


Figure 29: Scale Invariance at Criticality.The red box is the 300x300 patch shown top left, the yellow box is the 400x400 patch shown top right, and the yellow box is the 500x500 patch shown bottom left.

The lower figure shows different patches of the same configuration at T_c. The three images look statistically similar, demonstrating the scale invariance of the system at criticality. One can see that the scale invariance is broken when we move away from the critical temperature, as shown below.

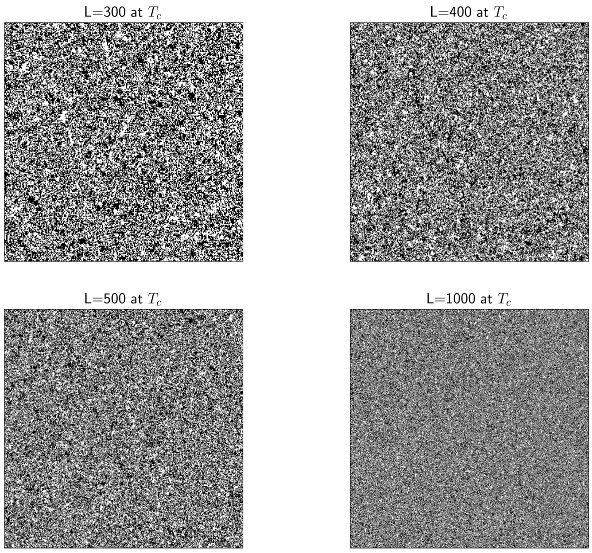


Figure 30: Breaking of Scale Invariance above T_c. These images show the same configuration of the Ising model at $T > T_c$ zoomed in at different scales. Notice how the patterns do not look similar at different scales. Top left image shows larges clusters of aligned spins, while the bottom right image shows mostly random spins.

To drive the point about the slowness of Metropolis Hastings home, we will see a measure of how the Metropolis algorithm compares at the critical temperature from other algorithms with respect to autocorrelation time.

15.6. Dynamical exponent

The dynamical exponent z characterizes how the autocorrelation time τ_c scales with the system size L near the critical point. It is defined by the relation:

$$\tau_c \sim \xi^z \sim L^z \tag{15.10}$$

To physically motivate this, consider that for one time step, to completely decorrelate the system, information must propagate across the largest clusters of correlated spins, which are of size ξ. Now since the correlation length diverges at the critical point, the time taken to decorrelate also diverges. However, in a finite system of size L, the correlation length is limited by the system size, leading to the scaling relation above. The value of the dynamical exponent depends on the specific algorithm used for the simulation. For the Metropolis algorithm, the dynamical exponent is approximately $z \approx 2.17$ for the 2D Ising model. This means that as the system size increases, the autocorrelation time increases significantly, making simulations near the critical point computationally expensive. The plot below shows the dynamical exponent for the Metropolis algorithm.

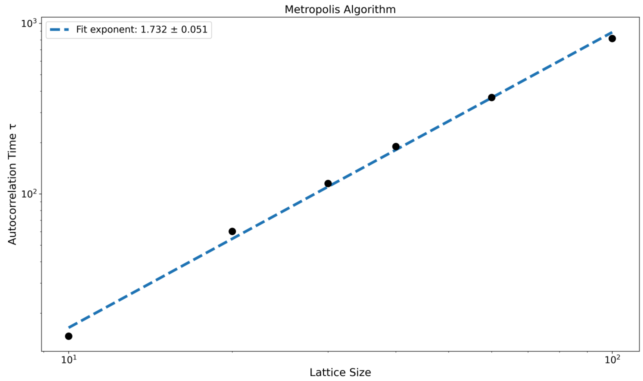


Figure 31: The time-delayed autocorrelation function for magnetization at $T \approx 2.269$.

15.7. The Wolff Algorithm

The Ising model simulation done using the Metropolis algorithm, which suffers from critical slowing down near the critical temperature. To overcome this, we can use cluster algorithms like the Wolff algorithm, which flips clusters of spins instead of individual spins. This significantly reduces the autocorrelation time near the critical point.

The Wolff algorithm works by randomly selecting a seed spin and then recursively adding neighboring spins of the same orientation to the cluster with a probability $p = 1 - \exp\left(-\frac{2J}{k_B T}\right)$. Once the cluster is formed, all spins in the cluster are flipped simultaneously. Below are two examples of clusters formed and flipped using the Wolff algorithm.

In short, the Wolff algorithm works:

1. Randomly select a seed spin on the lattice.
2. Initialize an empty cluster and add the seed spin to it.
3. For each spin in the cluster, check its neighboring spins. If a neighboring spin has the same orientation as the seed spin, add it to the cluster with probability $p = 1 - \exp\left(-\frac{2J}{k_B T}\right)$.
4. Repeat the previous step until no more spins can be added to the cluster.
5. Flip all spins in the cluster simultaneously.

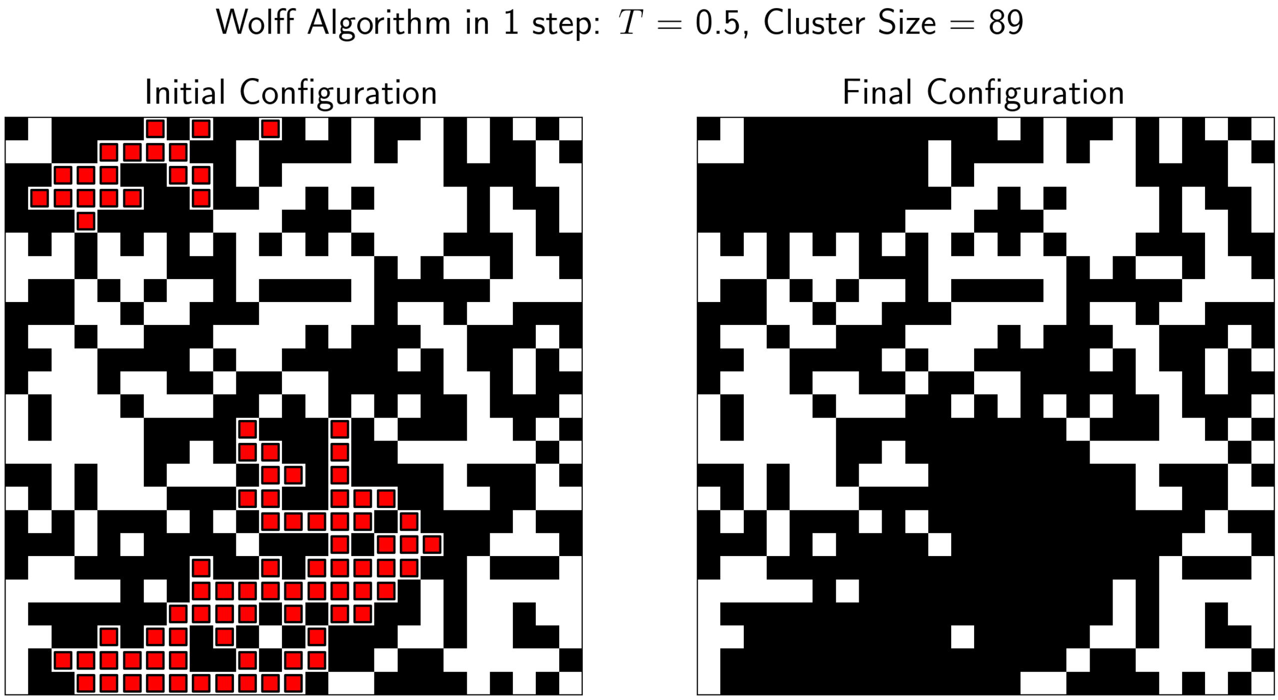


Figure 32: Flipping a Cluster of spins using the Wolff algorithm.

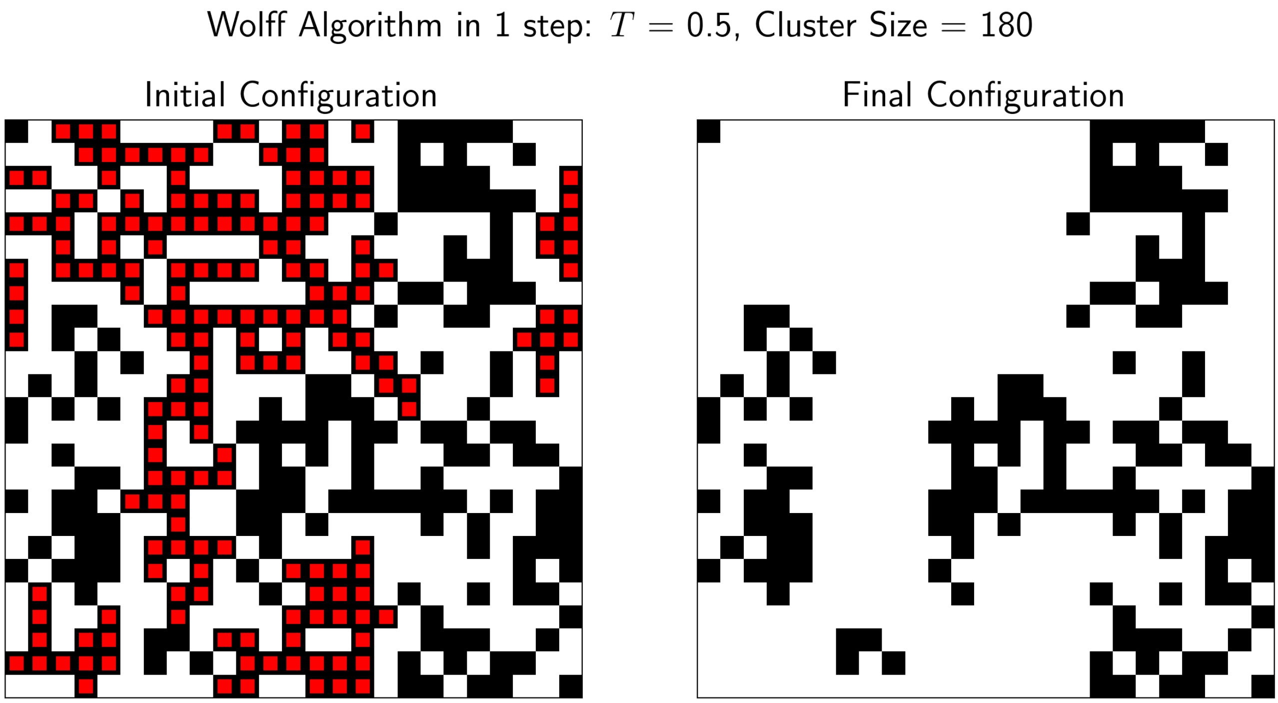


Figure 33: Flipping a Cluster of spins using the Wolff algorithm.

We check the magnetization curve obtained using the Wolff algorithm against the exact solution.

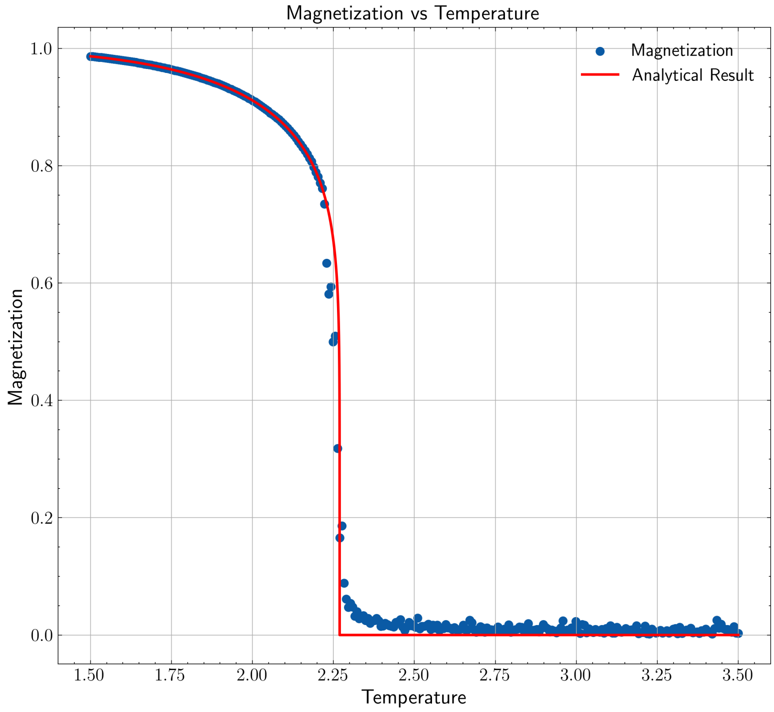


Figure 34: Magnetization per spin $\langle|m|\rangle$ as a function of temperature T obtained using the Wolff algorithm compared against the exact solution.

Similarly, we can check the dynamical exponent for the Wolff algorithm.

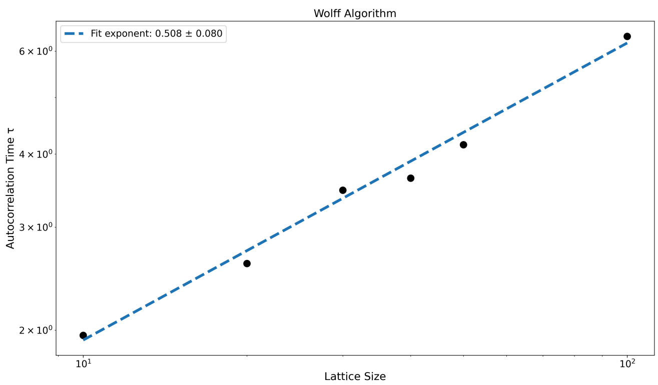


Figure 35: The time-delayed autocorrelation function for magnetization at $T \approx 2.269$ using the Wolff algorithm.

The wolff algorithm does not follow the scaling relation $\tau_c \sim L^z$ as closely as the Metropolis algorithm. This is because the Wolff algorithm effectively reduces the correlation time by flipping clusters of spins, leading to a much smaller dynamical exponent for larger system sizes. The dynamical exponent for the Wolff algorithm is approximately $z \approx 0.25$, indicating that the autocorrelation time increases much more slowly with system size compared to the Metropolis algorithm. This makes the Wolff algorithm much more efficient for simulating the Ising model near the critical point.

15.8. Numerical Scaling Exponents

Using the Wolff algorithm, we can also estimate the critical exponents of the 2D Ising model numerically. We could take the magnetization data near the critical temperature and fit it to the scaling relation:

$$\langle |m| \rangle \sim (T_c - T)^\beta \tag{15.11}$$

where $\beta = \frac{1}{8}$ is the critical exponent for magnetization. By fitting the numerical data to this relation, we can extract the value of β and compare it to the exact value.

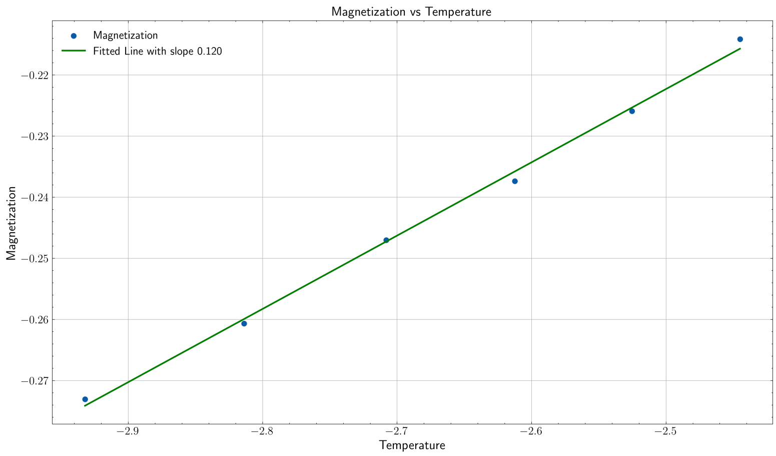


Figure 36: Fitting the magnetization data near the critical temperature to extract the critical exponent β. The fitted value is close to the exact value of $\frac{1}{8}$.

Similarly, we can estimate γ for susceptibility, which diverges as:

$$\chi \sim |T - T_c|^{-\gamma} \tag{15.12}$$

where $\gamma = \frac{7}{4}$ is the critical exponent for susceptibility.

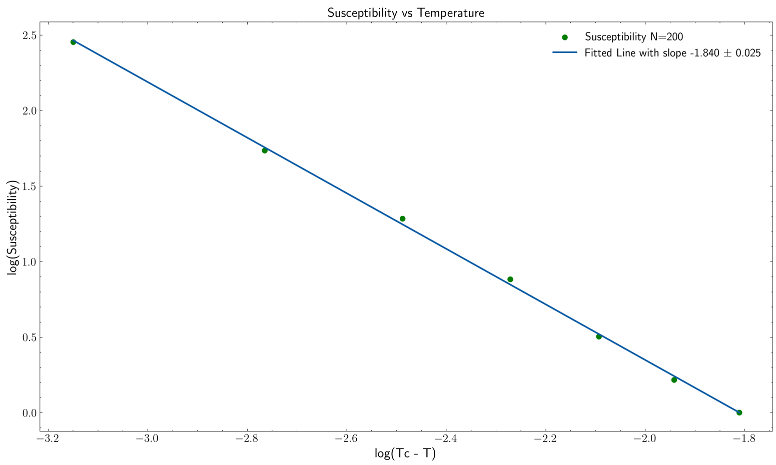


Figure 37: Fitting the specific heat data near the critical temperature to extract the critical exponent γ. The fitted value is close to the exact value of $\frac{7}{4}$.

15.9. Finite Size Effects

Phase transitions cannot take place in finite systems, as the partition function is a sum of analytic functions, and hence is itself analytic. However, in finite systems, we can observe signatures of phase transitions. As the system size increases, the thermodynamic quantities exhibit sharper features near the critical temperature, resembling the behavior of an infinite system. For example, the magnetization curve becomes steeper, and the peaks in specific heat and susceptibility become more pronounced as the system size increases. We see the finite size effects in the magnetization curve below.

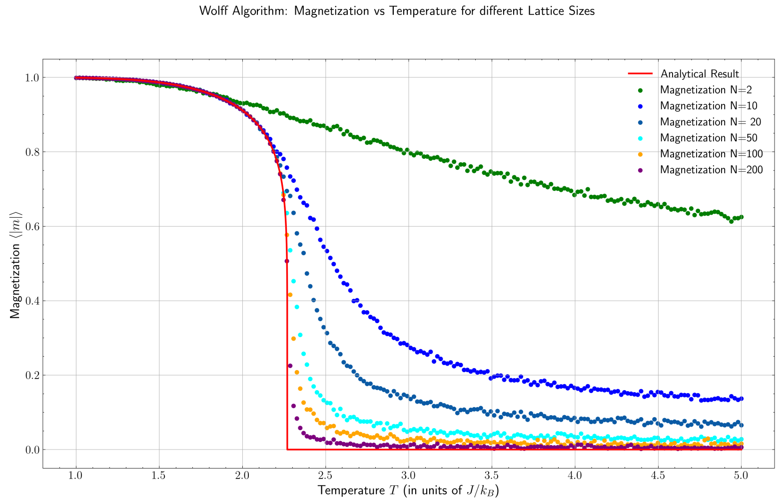


Figure 38: Magnetization per spin $\langle|m|\rangle$ as a function of temperature T for different system sizes using the Wolff algorithm. Notice how the curves become sharper near the critical temperature as the system size increases, indicating finite size effects.

This happens because for small systems, any outlier can significantly affect the overall behavior of the system. As the system size increases, these outliers have a diminishing effect, and the system's behavior approaches that of an infinite system.

15.10. Spin-Spin Correlations

The spin-spin correlation function $G(r)$ measures how the spin at one site is correlated with the spin at another site a distance r away. It is defined as:

$$G(r) = \langle\sigma_i\sigma_{i+r}\rangle - \langle\sigma_i\rangle\langle\sigma_{i+r}\rangle \tag{15.13}$$

The two point spin covariance function is plotted below for different temperatures. This function is important, this the correlation length ξ can be extracted from this function, which diverges as the system approaches the critical temperature, a signature of continuous phase transitions. Ising's original argument for why the 1D model does not have a phase transition was based on the fact that the correlation length remains finite and decays exponentially, for all non-zero temperatures in 1D.

The plot below shows the spin-spin correlation function for different temperatures using the Wolff algorithm. The amount of yellow indicates the strength of correlation between spins at different distances. As the temperature approaches the critical temperature T_c, the correlation length increases, indicating that spins become correlated over larger distances. We can see that at $T = 1J/k_B$ the graph becomes completely yellow, indicating that all spins are aligned and perfectly correlated, atleast for the lattice size used in the simulation.

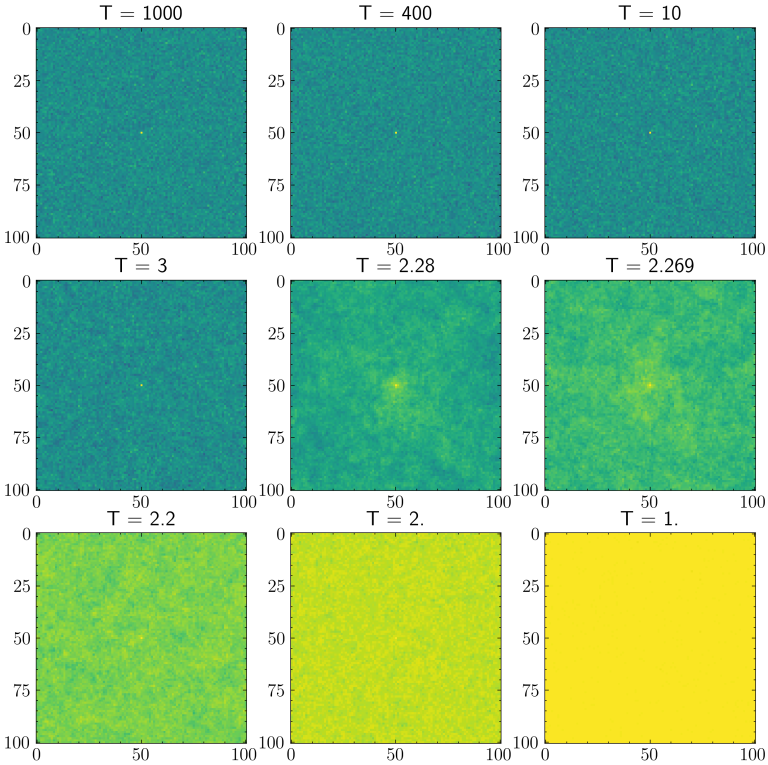


Figure 39: The spin-spin correlation function $G(r)$ for different temperatures. Notice how the correlation length increases as the temperature approaches the critical temperature T_c.

16. Conclusion

After this extensive analytical and numerical exploration of the 2D Ising model, we have understood atleast one solution to a non-trivial model in statistical mechanics. To summarize,

1. We saw some models before the Ising model, and how they led to the formulation of the Ising model.
2. We formulated the Ising model and understood its Hamiltonian.
3. We saw Peierls argument for the existence of a phase transition in higher dimensions.
4. We saw some series expansion methods which can give many useful results to many models.
5. We found the critical temperature of the 2D Ising model using duality arguments.
6. We construct the lattice diagonally and find the transfer matrix for the 2D Ising model.
7. We diagonalized the transfer matrix using symmetry properites and Jacobian elliptic functions.
8. We found the analytical solution for free energy.
9. We simulated the 2D Ising model using the Metropolis algorithm and Wolff algorithm.
10. We saw critical slowing down and scale invariance at criticality.
11. We found the dynamical exponent for both algorithms.
12. We found the critical exponents numerically using the Wolff algorithm.
13. We saw finite size effects in the magnetization curve.
14. We saw the spin-spin correlation functions at different temperatures.

The 2D Ising model serves as a cornerstone in the study of phase transitions and critical phenomena, providing deep insights into the behavior of complex systems. The techniques and concepts learned here can be extended to more complex models and higher dimensions, paving the way for further exploration in statistical mechanics and condensed matter physics.

References

[1] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*. London: Academic Press, 1982. [Online]. Available: https://books.google.com/books/about/Exactly_Solved_Models_in_Statistical_Mec.html?id=dbHvAAAAMAAJ

[2] R. Mills, E. Ascher, and R. Jaffee, *Critical phenomena in alloys, magnets, and superconductors. Battelle Institute materials science colloquia, Geneva and Gstaad, Switzerland, September 7–12, 1970*. McGraw-Hill Book Company., 1971.

[3] E. Ising, "Beitrag zur Theorie des Ferro- und Paramagnetismus," 1924.

[4] R. Peierls, "On Ising's model of ferromagnetism," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 32, no. 3, pp. 477–481, 1936, doi: 10.1017/S0305004100019174.

[5] L. Onsager, "Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition," *Physical Review*, vol. 65, no. 3–4, pp. 117–149, Feb. 1944, doi: 10.1103/PhysRev.65.117.

[6] S. G. BRUSH, "History of the Lenz-Ising Model," *Reviews of Modern Physics*, vol. 39, no. 4, pp. 883–893, Oct. 1967, doi: 10.1103/revmodphys.39.883.

[7] A. Taroni, "90 years of the Ising model," *Nature Physics*, vol. 11, no. 12, p. 997, Dec. 2015, doi: 10.1038/nphys3595.

[8] J. S. Walker, *A Student's Guide to the Ising Model*. in Student's Guides. Cambridge University Press, 2023. doi: 10.1017/9781009089579.

[9] K. Hoffman and R. Kunze, *Linear Algebra*, 2nd ed. Englewood Cliffs, NJ: Prentice Hall, 1971.

[10] C. D. Meyer, *Applied Linear Algebra and Matrix Analysis*. Philadelphia, PA: Society for Industrial, Applied Mathematics, 2000.

[11] J. P. Sethna, *Statistical Mechanics: Entropy, Order Parameters, and Complexity*. Oxford, UK: Oxford University Press, 2006.

[12] D. A. R. Sakthivadivel, "Magnetisation and Mean Field Theory in the Ising Model," *SciPost Physics Lecture Notes*, p. 35, Jan. 2022, doi: 10.21468/SciPostPhysLectNotes.35.

[13] E. of Mathematics, "Viète theorem," *Encyclopedia of Mathematics*, 2001, Accessed: Oct. 15, 2023. [Online]. Available: https://encyclopediaofmath.org/wiki/Vi%C3%A8te_theorem

[14] B. Kaufman and L. Onsager, "Crystal Statistics. II. Partition Function Evaluated by Spinor Analysis," *Physical Review*, vol. 76, no. 8, pp. 1232–1243, 1949, doi: 10.1103/PhysRev.76.1232.

[15] M. E. Fisher, "On the Dimer Solution of Planar Ising Models," *Journal of Mathematical Physics*, vol. 7, no. 10, pp. 1776–1781, Oct. 1966, doi: 10.1063/1.1704825.

[16] P. W. Kasteleyn, "The statistics of dimers on a lattice: I. The number of dimer arrangements on a quadratic lattice," *Physica*, vol. 27, no. 12, pp. 1209–1225, 1961, doi: 10.1016/0031-8914(61)90063-5.

[17] H. A. Kramers and G. H. Wannier, "Statistics of the Two-Dimensional Ferromagnet. Part I," *Physical Review*, vol. 60, no. 3, pp. 252–262, Aug. 1941, doi: 10.1103/physrev.60.252.

[18] G. Wolschin, "Statistical Mechanics, Summer Semester 2021," 2021. Accessed: Jul. 08, 2024. [Online]. Available: https://www.thphys.uni-heidelberg.de/~wolschin/statsem21_3s.pdf

[19] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 8th ed. Academic Press, 2015.