PH4104 Notes

Non Linear Dynamics Notes

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Abstract

This is a collection of my Notes for the course Non Linear Dynamics (PH4104) given at IISER Kolkata. We start with 1D systems and move on to higher dimensions. The main topics covered are Fixed Points, Stability Analysis, Bifurcations, Limit Cycles and Chaos. Then we move onto Maps and Fractals. After, this we move onto Hamiltonian Systems and the KAM theorem. We then cover some nonlinear PDEs and Solitons. Finally we end with some topics in nonlinear PDEs.

Contents

1. Introduction

We divide the whole notes into six main sections:

1. Continuous flows: 1D flows, 2D flows, chaotic systems.
2. Discrete maps: fixed points, stability, bifurcations, chaos in maps, coupled maps and synchronization.
3. Fractals.(very brief)
4. Hamiltonian Dynamics and KAM theory.
5. Nonlinear PDEs : Method of characteristics, shocks and Solitons.
6. Misc Topics: Hopf and turing bifurcations, pattern formation.

These notes cover the fundamentals of one-dimensional flows in nonlinear dynamics, including fixed points, linear stability analysis, bifurcations, and applications such as the logistic model for population growth. Then we move on to 2D flows and their unique bifurcations. Then we explore chaotic systems, strange attractors, and fractals, only for the Lorenz system. Then we move onto discrete dynamical systems, covering fixed points, stability, bifurcations, and chaos in maps like the logistic map and Henon map.

The references are:

Reference Book: Nonlinear Dynamics and Chaos by Steven Strogatz [1](for 1,2,3).

Reference Book: Lecture notes on Nonlinear Dynamics by Daniel Arovas [2] (for parts of 5 and 6).

2. Introduction to 1-D Flows

The general form of a first-order differential equation is:

$$\frac{dx}{dt} = f(t, x(t)) \tag{2.1}$$

For **autonomous systems**, generally there is no explicit time dependence:

$$\frac{dx}{dt} = f(x(t)) \tag{2.2}$$

2.1. Existence and Uniqueness

These differential equations, along with initial conditions, have a unique solution provided that the function f is continuous. **Picard-Lindeloef Theorem** guarantees this existence and uniqueness.

Theorem 2.1.1 (Picard-Lindeloef Theorem):

Let $D \subset \mathbb{R} \times \mathbb{R}^n$. Let $f : D \to \mathbb{R}^n$ be a function continuous in t and Lipschitz continuous in x, with Lipschitz constant independent of t. Then for any initial condition $(t_0, x_0) \in D$, there exists $\varepsilon > 0$, such that the initial value problem

$$y'(t) = f(t, y(t)), y(t_0) = x_0 \tag{2.3}$$

has a unique solution on the interval $|t - t_0| < \varepsilon$.

2.2. Example: Finite Escape Time

Consider the system:

$$\frac{dx}{dt} = 1 + x^2 \tag{2.4}$$

With initial condition $x(t = 0) = x_0$.

The solution is:

$$x(t) = \tan(t + \arctan(x_0)) \tag{2.5}$$

For times $t = \frac{\pi}{2} - \arctan(x_0) + k\pi$, the solution goes to infinity ($x(t) \to \infty$). This phenomenon is called **finite escape time**.

Vector fields (quiver plots) can be used to visualize the flow, where arrows indicate the slope $\frac{\mathrm{d}x}{\mathrm{d}t}$.

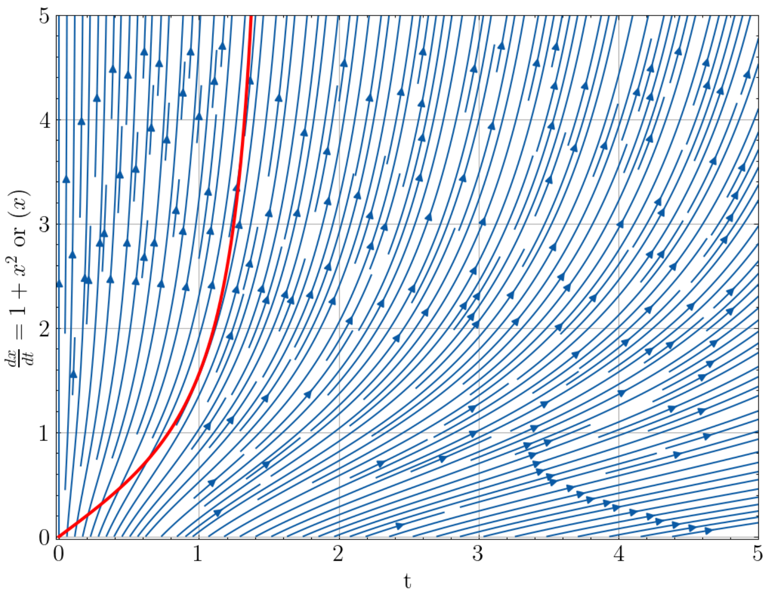


Figure 1: Vector field (quiver plot) for $\frac{dx}{dt} = 1 + x^2$. The red line represents the trajectory starting from $x(0) = 0$.

2.3. Fixed Points and Stability

Fixed points x^* are points such that the flow stops:

$$\dot{x} = 0 \Longrightarrow f(x^*) = 0 \tag{2.6}$$

where $x \in \mathbb{R}^n$. These represent the real solutions (equilibrium states) of the system.

Definition 2.3.1 (Stability):

- **Stable Fixed Point:** Flows tend to approach this point.
- **Unstable Fixed Point:** Flows move away from this point.
- The **basin of attraction** is the set of initial conditions that eventually flow toward a specific stable fixed point.

2.4. Linear Stability Analysis

Nonlinear systems are hard to solve exactly. However, we can analyze the stability of fixed points using **linear stability analysis**. We linearize the system around the fixed points.

To determine stability mathematically, we consider a small perturbation $\varepsilon(t)$ away from a fixed point x^*:

$$x(t) = x^* + \varepsilon(t) \tag{2.7}$$

Substituting this into the differential equation:

$$\frac{dx}{dt} = \frac{dx^*}{dt} + \frac{d\varepsilon}{dt} = f(x^* + \varepsilon(t)) \tag{2.8}$$

Using Taylor expansion around x^*:

$$f(x^* + \varepsilon) \approx f(x^*) + f'(x^*)\varepsilon + \mathcal{O}(\varepsilon^2) \tag{2.9}$$

Since $f(x^*) = 0$ (it is a fixed point) and $\frac{\mathrm{d}x^*}{\mathrm{d}t} = 0$, we get the linearized equation:

$$\frac{d\varepsilon}{dt} \approx f'(x^*)\varepsilon \tag{2.10}$$

The solution behaves as

$$\varepsilon(t) = \varepsilon(0) \exp\left[\int f'(x^*) dt\right] \tag{2.11}$$

- If $f'(x^*) > 0$: The perturbation grows → **Unstable**.
- If $f'(x^*) < 0$: The perturbation decays → **Stable**.

Note: Linear stability analysis only analyzes local behavior near fixed points.

2.5. Bifurcations

We study how the qualitative behavior of a system changes as parameters vary.

Definition 2.5.1 (Bifurcation): A bifurcation occurs when a change in a parameter causes a change in the qualitative behavior of the system (e.g., number or stability of fixed points).

Generally to study bifurcations, we use the concept of normal forms, which are simplified equations that capture the essential behavior near the bifurcation point.

Definition 2.5.2 (Normal Form): Normal forms are simplified equations that capture the essential behavior of a system near a bifurcation point. All systems exhibiting the same type of bifurcation are topologically equivalent to the normal form of the bifurcation.

2.5.1. Saddle-Node Bifurcation

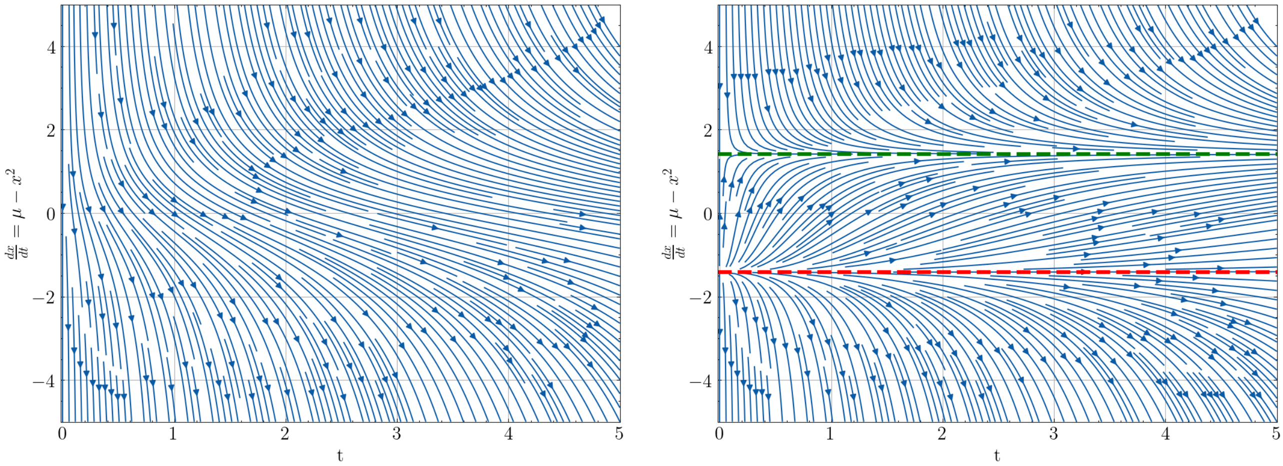
Normal Form:

$$\dot{x} = \mu - x^2 \tag{2.12}$$

Fixed points occur where $f(x) = 0$:

$$x_\pm = \pm\sqrt{\mu} \tag{2.13}$$

- If $\boldsymbol{\mu < 0}$**:** No real fixed points.
- If $\boldsymbol{\mu = 0}$**:** One semi-stable fixed point at $x = 0$.
- If $\boldsymbol{\mu > 0}$**:** Two fixed points created, $x = +\sqrt{\mu}$ (stable) and $x = -\sqrt{\mu}$ (unstable).



(a) Before the bifurcation. (b) After the bifurcation.

Figure 2: Before the bifurcation (a) we do not have any fixed points. After the bifurcation (b) two fixed points are created: one stable μ and one unstable $-\mu$.

The bifurcation diagram shows a parabolic curve $x^2 = \mu$ representing the creation of fixed points as μ increases.

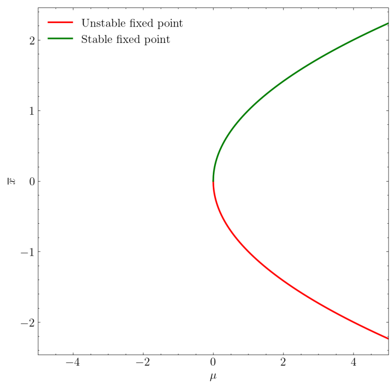


Figure 3: The saddle node bifurcation occurs at $\mu = 0$

Definition 2.5.1.1 (Saddle-Node Bifurcation):
A saddle-node bifurcation occurs when two fixed points (one stable and one unstable) collide and annihilate each other as a parameter is varied, or vice versa.

2.5.2. Transcritical Bifurcation

Normal Form:

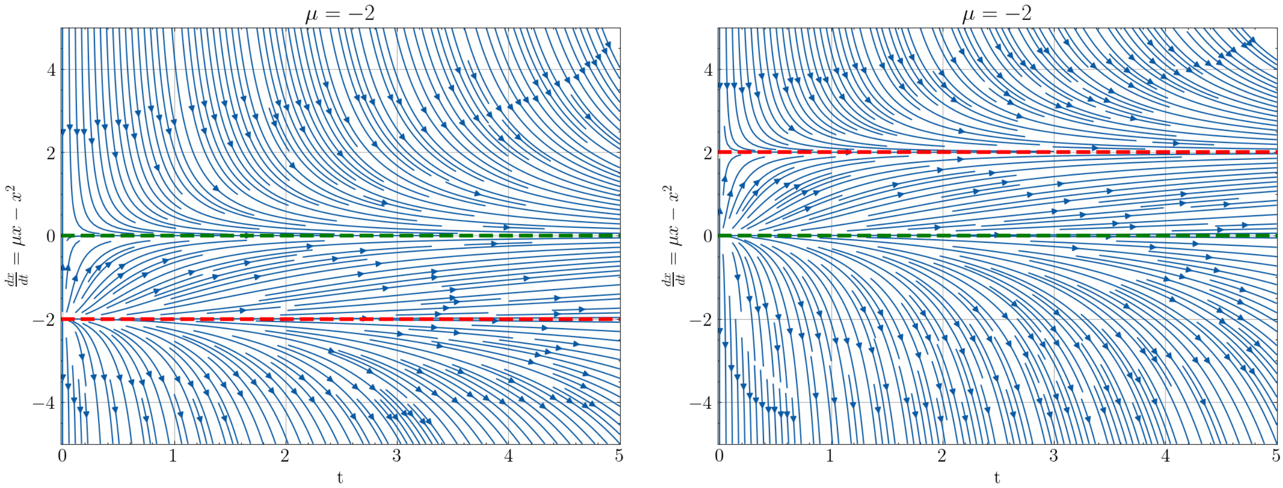
$$\dot{x} = \mu x - x^2 \tag{2.14}$$

Fixed points:

$$
\begin{aligned}
f(x) = x(\mu - x) = 0 \\
\implies x^* = 0 \quad \text{or} \quad x^* = \mu
\end{aligned}
\tag{2.15}
$$

Stability depends on $f'(x) = \mu - 2x$:

1. For $x^* = 0$: $f'(0) = \mu$.
    1. **Stable** if $\mu < 0$.
    2. **Unstable** if $\mu > 0$.
2. For $x^* = \mu$: $f'(\mu) = -\mu$.
    1. **Stable** if $\mu > 0$.
    2. **Unstable** if $\mu < 0$.



(a) Before the bifurcation. (b) After the bifurcation.

Figure 4: Before the bifurcation (a) there is one stable fixed point at x $= \mu$ and one unstable $x = 0$. After the bifurcation (b) there is one stable fixed point at $x = 0$ and one unstable $x = \mu$. In the first figure the colors of the unstable and stable fixed points are swapped.

There is an exchange of stability between the two fixed points at $\mu = 0$.

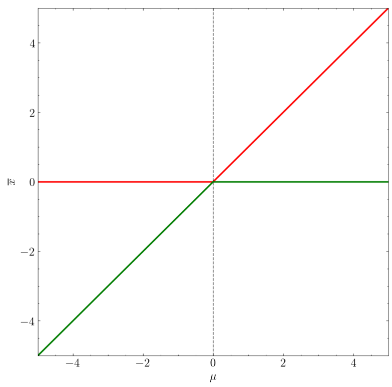


Figure 5: There is an exchange of stability at the point of the bifurcation.

Definition 2.5.2.1 (Transcritical Bifurcation):
A transcritical bifurcation occurs when two fixed points exchange their stability as a parameter is varied.

2.5.3. Pitchfork Bifurcation

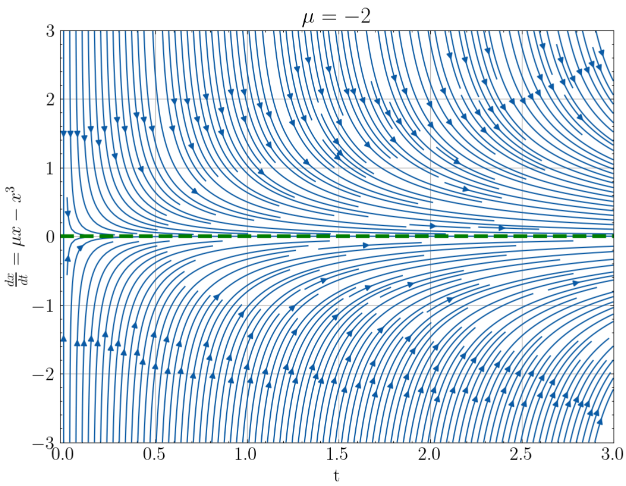
Normal Form:

$$\dot{x} = \mu x - x^3 \tag{2.16}$$

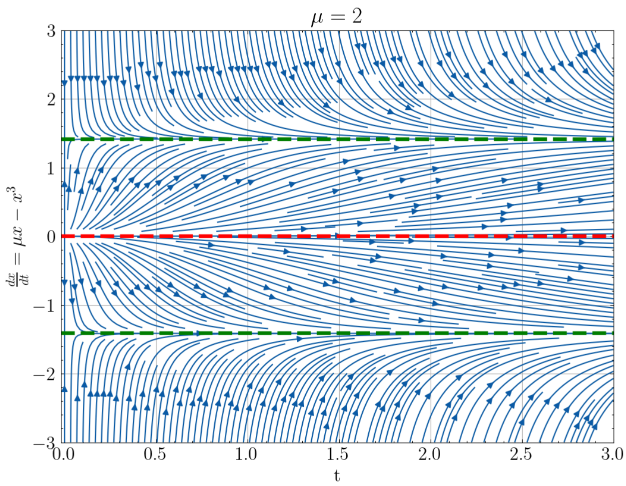
Fixed points:
Fixed points occur where $f(x) = 0$:

$$x_{\pm} = \pm\sqrt{\mu}, 0 \tag{2.17}$$

- If $\boldsymbol{\mu < 0}$**:** One stable fixed point at $x = 0$.
- If $\boldsymbol{\mu > 0}$**:** Two fixed points created, $x = +\sqrt{\mu}$ (stable) and $x = -\sqrt{\mu}$ (stable), with $x = 0$ becoming unstable.



(a) Before the bifurcation.



(b) After the bifurcation.

Figure 6: Before the bifurcation (a) there is only one stable fixed point at x = 0. After (b) there is one unstable fixed point at $x = 0$ and two unstable $x = \pm\mu$.

One stable fixed point becomes unstable, and two new stable fixed points emerge symmetrically. There are concepts of supercritical and subcritical bifurcations, which are discussed further in [1]

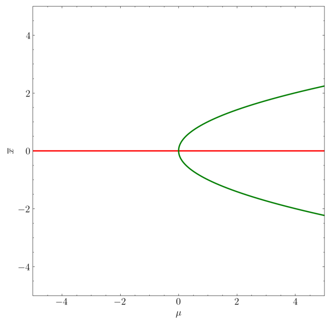


Figure 7: Stable fixed point becomes unstable and two new stable ones emerge.

Definition 2.5.3.1 (Pitchfork Bifurcation):
A pitchfork bifurcation occurs when a single stable fixed point becomes unstable and gives rise to two new stable fixed points as a parameter is varied.

2.6. Logistic Model (Population Growth)

Let N be the population and K be the carrying capacity and $r > 0$ be the rate of growth.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \tag{2.18}$$

Population does not grow indefinitely; it saturates. To simplify, we let $x = \frac{N}{K}$:

$$\frac{dx}{dt} = rx(1 - x) \tag{2.19}$$

Analysis of fixed points:

- $x^* = 0$: Unstable (growth starts here, $f'(0) > 0$).
- $x^* = 1$: Stable (population reaches carrying capacity, $f'(1) < 0$).

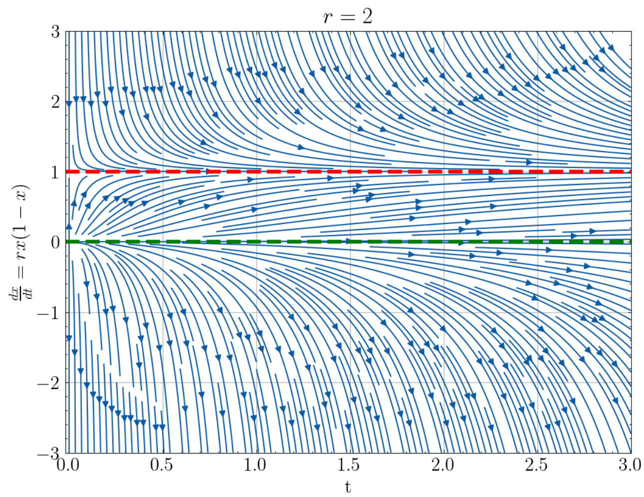


Figure 8: The phase plot of the logistic model. One unstable fixed point at zero population and a stable fixed point at the carrying capacity.

3. 2D Systems

The general form of a linear 2D system is:

$$\frac{dx_1}{dt} = ax_1 + bx_2 \tag{3.1}$$

$$\frac{dx_2}{dt} = cx_1 + dx_2 \tag{3.2}$$

In matrix notation:

$$\frac{d(x)}{dt} = M(x) \tag{3.3}$$

The solution is given by:

$$(x)(t) = e^{At}(x)(0) \tag{3.4}$$

Let η_1 and η_2 be the eigenvectors of A with eigenvalues λ_1 and λ_2.

$$e^{At}\eta_1 = e^{\lambda_1 t}\eta_1 \tag{3.5}$$

The general solution is:

$$(x)(t) = C_1 e^{\lambda_1 t}\eta_1 + C_2 e^{\lambda_2 t}\eta_2 \tag{3.6}$$

3.1. Eigenvalues and Eigenvectors

The behavior depends on the nature of the eigenvalues λ.

3.1.1. 1. Complex Eigenvalues

If $\lambda = \alpha \pm i\beta$ and eigenvectors $(v) = (a) \pm i(b)$, the solution involves oscillations:

$$(x)(t) = C_1 e^{\alpha t}[(a)\cos(\beta t) - (b)\sin(\beta t)] + C_2 e^{\alpha t}[(a)\sin(\beta t) + (b)\cos(\beta t)] \tag{3.7}$$

3.1.2. 2. Repeated Eigenvalues (Degenerate Case)

If the matrix A is not diagonalizable, it can be brought to **Jordan Normal Form**:

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \tag{3.8}$$

This occurs when there is a repeated eigenvalue and only one independent eigenvector. The solution takes the form:

$$(x)(t) = C_1 e^{\lambda t}(v_1) + C_2 e^{\lambda t}(t(v_1) + (v_2)) \tag{3.9}$$

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4. Linear Stability Analysis in 2D

Consider a nonlinear system:

$$\dot{(x)} = f((x)) \tag{4.1}$$

Let $(x)^*$ be a fixed point such that $f((x)^*) = 0$. We consider a small perturbation $(\varepsilon)(t)$:

$$(x)(t) = (x)^* + (\varepsilon)(t) \tag{4.2}$$

Linearizing around the fixed point using the Jacobian matrix J:

$$\frac{d(\varepsilon)}{dt} \approx Df|_{x^*}\,(\varepsilon) \tag{4.3}$$

where Df is the Jacobian.

The stability is determined by the eigenvalues λ_1, λ_2 of the Jacobian.

4.1. Classification of Fixed Points

1. **Unstable Node:** $\lambda_1 > \lambda_2 > 0$. All trajectories move away.
2. **Stable Node:** $\lambda_1 < \lambda_2 < 0$. All trajectories approach the fixed point.
3. **Saddle Point:** $\lambda_1 > 0 > \lambda_2$. Unstable in one direction, stable in the other.
4. **Spirals (Focus):** Eigenvalues are complex conjugate $\lambda = \alpha \pm i\beta$. Unstable Spiral: $\boldsymbol{\alpha > 0}$ **(Real part positive). Stable Spiral:** $\alpha < 0$ (Real part negative).
5. **Center:** $\lambda = \pm i\beta$ (Purely imaginary, $\alpha = 0$). Trajectories are periodic orbits (closed loops).
6. **Star/Degenerate Node:** Repeated eigenvalues.

4.1.1. Trace-Determinant Classification

Let $\tau = \mathrm{Tr}(J)$ be the trace and $\Delta = \det(J)$ be the determinant. The characteristic equation is:

$$\lambda^2 - \tau\lambda + \Delta = 0 \tag{4.4}$$

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} \tag{4.5}$$

- **Saddle Points:** $\Delta < 0$.
- **Nodes:** $\tau^2 > 4\Delta$ and $\Delta > 0$. **Stable if** $\boldsymbol{\tau < 0}$**.** Unstable if $\tau > 0$.
- **Spirals:** $\tau^2 < 4\Delta$ and $\Delta > 0$. **Stable if** $\boldsymbol{\tau < 0}$**.** Unstable if $\tau > 0$.
- **Centers:** $\tau = 0, \Delta > 0$.
- **Degenerate/Star Nodes:** On the parabola $\tau^2 = 4\Delta$.

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5. Application Example: Competitive Species

Consider the system:

$$\dot{x} = x(3 - x) - 2xy \tag{5.1}$$

$$\dot{y} = y(2 - y) - xy \tag{5.2}$$

Here x and y represent populations (e.g., rabbits and sheep).

5.1. Fixed Points

We solve $f(x, y) = 0$:

1. $(0, 0)$: Extinction of both.
2. $(0, 2)$: Sheep only.
3. $(3, 0)$: Rabbits only.
4. Intersection of nullclines: $3 - x - 2y = 0$ and $2 - y - x = 0$ (Check if valid in first quadrant).

5.2. Stability Analysis

The Jacobian is:

$$J = \begin{pmatrix} 3 - 2x - 2y & -2x \\ -y & 2 - x - 2y \end{pmatrix} \tag{5.3}$$

1. **At (0,0):**

$$J = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \tag{5.4}$$

$\lambda_1 = 3, \lambda_2 = 2$. Both positive $\to$ **Unstable Node**.

2. **At (0,2):**

$$J = \begin{pmatrix} 3-4 & 0 \\ -2 & 2-2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2 & 0 \end{pmatrix} \tag{5.5}$$

(Recalculate Jacobian evaluation carefully based on notes) Actually, notes indicate saddle behavior or stable node depending on parameters. For $(0,2)$: $x = 0, y = 2$. $J_{\{11\}} = 3 - 4 = -1$. $J_{\{22\}} = 2 - 0 - 4 = -2$. $J = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$. Eigenvalues are $-1, -2$. Both negative → **Stable Node**.

3. **At (3,0):** $x = 3, y = 0$. $J_{\{11\}} = 3 - 6 = -3$. $J_{\{22\}} = 2 - 3 = -1$. $J = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$. Eigenvalues are $-3, -1$. Both negative → **Stable Node**.

Interpretation: The system exhibits **bistability**. Depending on initial conditions, one species drives the other to extinction. They cannot coexist stably in this specific configuration (Competitive Exclusion).

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6. Bifurcations and Periodic Orbits

6.1. Hopf Bifurcation

A Hopf bifurcation occurs when a stable fixed point changes stability and gives birth to a periodic orbit (limit cycle) as a pair of complex eigenvalues crosses the imaginary axis ($\mu = 0$).

Normal Form (in polar coordinates):

$$\dot{r} = \mu r - r^3 \tag{6.1}$$

$$\dot{\theta} = \omega \tag{6.2}$$

If $\mu < 0$: Stable spiral at origin. If $\mu > 0$: Unstable spiral at origin, stable limit cycle at $r = \sqrt{\mu}$.

This is a **Supercritical Hopf Bifurcation**.

6.2. SNIC / SNIPER Bifurcation

(Saddle-Node Infinite Period) A bifurcation on a periodic orbit where a saddle-node collision occurs on the cycle. As the parameter approaches the bifurcation, the period of oscillation diverges:

$$T \propto \frac{1}{\sqrt{\mu - \mu_c}} \tag{6.3}$$

Unlike Hopf, where the period is roughly constant close to bifurcation, here the period goes to infinity.

6.3. Homoclinic Bifurcation

A periodic orbit collides with a saddle point, creating a homoclinic orbit (a trajectory connecting a saddle point to itself) effectively breaking the cycle.

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7. Topological Equivalence and Hartman-Grobman

Two systems defined on sets A and B are **topologically equivalent** if there exists a homeomorphism (continuous, invertible map) $H : A \to B$ that maps trajectories of one system onto the other, preserving direction of time.

7.1. Hartman-Grobman Theorem

If x^* is a **hyperbolic fixed point** (no eigenvalues with zero real part) of a nonlinear system $\dot{x} = f(x)$, then the system is locally topologically equivalent to its linearization $\dot{x} = Jx$ near x^*. This means the local phase portrait of the nonlinear system looks like the linear system (distorted but qualitatively the same).

Example:

$$\dot{y} = -y \tag{7.1}$$

$$\dot{z} = z + y^2 \tag{7.2}$$

Linearization at $(0,0)$ gives $\dot{y} = -y, \dot{z} = z$, which is a saddle. The nonlinear term y^2 bends the trajectories, but the local saddle structure remains.

7.2. Stable and Unstable Manifolds

For a fixed point p: Stable Manifold (W^s): **Set of initial conditions that approach p as $t \to \infty$. Unstable Manifold (W^u):** Set of initial conditions that approach p as $t \to -\infty$.

Example: Nonlinear Pendulum

$$\frac{d^2\theta}{dt^2} = -\sin(\theta) \tag{7.3}$$

Fixed points at $(k\pi, 0)$. **Even k (downward): Centers (marginally stable in conservative case).** Odd k (upward): Saddles. With dissipation (friction), centers become stable spirals.

8. Conservative Systems

Consider the second-order system:

$$\frac{d^2x}{dt^2} = F(t, x, \dot{x}) \tag{8.1}$$

Let $x_1 = x$ and $x_2 = \dot{x}$. We define some kind of **Energy functional**:

$$\mathcal{E}(x_1, x_2) = \frac{x_2^2}{2} + V(x_1) \tag{8.2}$$

The time derivative is:

$$\dot{x}_2 = F(t, x_1, x_2) \tag{8.3}$$

$$\frac{d\mathcal{E}}{dt} = \frac{\partial \mathcal{E}}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \mathcal{E}}{\partial x_2}\frac{dx_2}{dt} \tag{8.4}$$

If the system is conservative, $\frac{d\mathcal{E}}{dt} = 0$.

8.1. Theorem

A conservative system cannot have a stable fixed point (attractor). This can be shown by contradiction using a Hamiltonian H.

Consider a system described by:

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} = f_1(x_1, x_2) \tag{8.5}$$

$$\dot{x}_2 = -\frac{\partial H}{\partial x_1} = f_2(x_1, x_2) \tag{8.6}$$

The equations of motion satisfy the divergence-free condition (Liouville's Theorem):

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \frac{\partial^2 H}{\partial x_1 \partial x_2} - \frac{\partial^2 H}{\partial x_2 \partial x_1} = 0 \tag{8.7}$$

8.1.1. Example

Consider the system derived from the Hamiltonian:

$$H(x_1, x_2) = -\frac{x_1^3}{3} + x_1 x_2 - \frac{x_2^2}{2} \tag{8.8}$$

The equations of motion are:

$$\dot{x}_1 = \frac{\partial H}{\partial x_2} = x_1 - x_2 \tag{8.9}$$

$$\dot{x}_2 = -\frac{\partial H}{\partial x_1} = -(-x_1^2 + x_2) = x_1^2 - x_2 \tag{8.10}$$

Fixed Points: Solving $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$:

1. $(0, 0)$: Saddle point.
2. $(1, 1)$: Center.

Level Curves: The trajectories lie on curves of constant energy:

$$-\frac{x_1^3}{3} + x_1 x_2 - \frac{x_2^2}{2} = \text{const} \tag{8.11}$$

We can analyze the curvature using the Hessian matrix of H:

$$\hat{H} = \begin{pmatrix} -2x_1 & 1 \\ 1 & -1 \end{pmatrix} \tag{8.12}$$

At $(0, 0)$: $\widehat{H} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$. Determinant is $-1 < 0 \longrightarrow$ Saddle. At $(1, 1)$: $\hat{H} = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$. Determinant is $2 - 1 = 1 > 0$ and Trace is $-3 \longrightarrow$ Local Maximum. Near $(1, 1)$, the level curves form closed loops (Centres).

9. Gradient Systems

A gradient system is defined by a potential function $V(x)$:

$$\dot{(x)} = -\nabla V \tag{9.1}$$

Properties:

1. The potential V decreases along trajectories:

$$\frac{dV}{dt} = (\nabla V) \cdot \dot{(x)} = (\nabla V) \cdot (-\nabla V) = -\left\|\nabla V\right\|^2 \leq 0 \tag{9.2}$$

2. Fixed points occur at extrema of V.
3. No periodic orbits are possible (since V must strictly decrease).

10. Reversible Systems

A system is reversible if it is invariant under the transformation $t \longrightarrow -t$ and $y \longrightarrow -y$ (time reversal and reflection in phase space). If a fixed point $(x^*, 0)$ is a linear center in the reversible system, then it is a nonlinear center (surrounded by closed orbits).

10.0.1. Example

$$\dot{x} = y - y^3 \tag{10.1}$$

$$\dot{y} = -x - y^2 \tag{10.2}$$

Fixed points:

1. $(0, 0)$: Linear center.
2. $(-1, 1)$ and $(-1, -1)$: Saddle points.

The system exhibits a **Heteroclinic Orbit** connecting the saddle points.

Figure 9: Phase portrait showing heteroclinic orbits connecting saddle points.

11. Stability Definitions

Consider the system $\dot{(x)} = f((x))$ with a fixed point $(x)^* = 0$.

1. **Stable (Lyapunov Stable):** The fixed point is stable if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that:

$$\|(x)(t_0)\| < \delta \Longrightarrow \|(x)(t)\| < \varepsilon, \quad \forall t > t_0 \tag{11.1}$$

2. **Asymptotically Stable:** The fixed point is stable AND attractive:

$$\|(x)(t_0)\| < \delta \Longrightarrow \lim_{t \longrightarrow \infty} \|(x)(t)\| = 0 \tag{11.2}$$

12. Lyapunov's Direct Method

Let $V(x_1, x_2)$ be a scalar function (Lyapunov function) in a region U. $\boldsymbol{V((x)) = 0}$ **at the fixed point.** $V((x)) > 0$ for all $(x) \neq 0$ (Positive Definite).

We examine the derivative of V along trajectories:

$$\dot{V}((x)) = \nabla V \cdot f((x)) \tag{12.1}$$

- If $\dot{V} \leq 0$: The fixed point is **Stable**.
- If $\dot{V} < 0$ (for $x \neq 0$): The fixed point is **Asymptotically Stable**.
- If $\dot{V} > 0$: The fixed point is **Unstable**.

12.0.1. Example 1

$$V(x_1, x_2) = x_1^2 + x_2^2 \tag{12.2}$$

$$\dot{V} = -2x_1^4 - 2x_2^4 < 0 \tag{12.3}$$

This implies asymptotic stability for the associated system.

12.0.2. Example 2: Duffing-like Oscillator

$$\dot{x} = y \tag{12.4}$$

$$\dot{y} = x - x^3 \tag{12.5}$$

Construct Energy/Lyapunov function:

$$V(x, y) = \frac{1}{2}y^2 - \int (x - x^3)dx = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 \tag{12.6}$$

To make it positive definite around equilibrium, we can adjust constants.

$$H(\pm 1, 0) = -\frac{1}{4} \tag{12.7}$$

. This represents a conservative system with periodic orbits around the centers.

13. Van der Pol Oscillator

Equation:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \tag{13.1}$$

Liénard form:

$$\dot{x} = y \tag{13.2}$$

$$\dot{y} = -\mu(x^2 - 1)y - x \tag{13.3}$$

Using Energy function $\alpha(x, y) = \frac{1}{2}(x^2 + y^2)$:

$$\frac{d\alpha}{dt} = x\dot{x} + y\dot{y} = -\mu y^2(x^2 - 1) \tag{13.4}$$

If $|x| < 1$: $x^2 - 1 < 0 \implies \dot{\alpha} > 0$ (Energy grows, damping is negative). If $|x| > 1$: $x^2 - 1 > 0 \implies \dot{\alpha} < 0$ (Energy decays, damping is positive).

This balance leads to a stable **Limit Cycle**.

Figure 10: Limit cycle for the Van der Pol oscillator.

13.0.1. Example: Positive System Stability

$$\dot{x}_1 = 1 - 2x_1 + x_2 \tag{13.5}$$

$$\dot{x}_2 = x_1 - x_2 \tag{13.6}$$

Jacobian J:

$$J = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \tag{13.7}$$

Trace $= -3$, Determinant $= 1$. Since $\Delta > 0$ and $\tau < 0$, the fixed point is a **Stable Node**.

Using a logarithmic Lyapunov function candidate:

$$\dot{\alpha} = (1 - 2x_1 + x_2)\ln(x_1) + (x_1 - x_2)\ln(x_2) \tag{13.8}$$

This relates to the property $-(\alpha - \beta)(\ln \alpha - \ln \beta) < 0$.

14. Limit Cycles

Consider the system:

$$\frac{dx_1}{dt} = x_1 - x_2 - x_1(x_1^2 + x_2^2) \tag{14.1}$$

$$\frac{dx_2}{dt} = x_1 + x_2 - x_2(x_1^2 + x_2^2) \tag{14.2}$$

14.1. Linear Stability Analysis

At the fixed point $(\overline{x}_1, \overline{x}_2) = (0, 0)$: The Jacobian is:

$$J|_{0,0} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{14.3}$$

The eigenvalues are $\lambda = 1 \pm i$. Since the real part is positive $(1 > 0)$, the origin is an **Unstable Spiral**.

14.2. Lyapunov Direct Method

Consider the Lyapunov function candidate:

$$V(x_1, x_2) = x_1^2 + x_2^2 \tag{14.4}$$

Taking the time derivative along trajectories:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 \tag{14.5}$$

$$\frac{dV}{dt} = 2(x_1^2 + x_2^2)[1 - (x_1^2 + x_2^2)] \tag{14.6}$$

Analysis:
1. If $x_1^2 + x_2^2 < 1$: $\frac{dV}{dt} > 0$. Trajectories move away from the origin (Unstable).
2. If $x_1^2 + x_2^2 = 1$: $\frac{dV}{dt} = 0$. This implies a periodic orbit (Limit Cycle).
3. If $x_1^2 + x_2^2 > 1$: $\frac{dV}{dt} < 0$. Trajectories move inwards (Stable).

Thus, we obtain a stable **Limit Cycle** at $r = 1$.

—

15. Poincaré-Bendixson Theorem

Let R be a closed, bounded region in the plane for a 2D autonomous system.
1. If all vector fields on the boundary of R point inwards (making R a **trapping region**).
2. And if there are no fixed points in R.
3. Then, any trajectory starting in R must eventually approach a periodic orbit inside R.

15.1. Example: Perturbed System

Consider the system in polar coordinates:

$$\dot{r} = r(1 - r^2) + \mu r \cos(\theta) \tag{15.1}$$

$$\dot{\theta} = 1 \tag{15.2}$$

We want to find a trapping region (an annulus $r_{\min} < r < r_{\max}$). We look for $r_{\min}$ such that $\dot{r} > 0$:

$$r(1 - r^2) + \mu r \cos(\theta) > 0 \tag{15.3}$$

$$1 - r^2 > -\mu \cos(\theta) \tag{15.4}$$

Since $-\mu \cos(\theta) \le \mu$, it suffices to have $1 - r^2 > \mu$.

$$r_{\min} < \sqrt{1 - \mu} \tag{15.5}$$

Similarly, for $r_{\max}$ such that $\dot{r} < 0$:

$$r_{\max} > \sqrt{1+\mu} \tag{15.6}$$

For $0 \leq \mu < 1$, no fixed points exist in the annulus, so a limit cycle exists.

—

16. Lotka-Volterra Systems (Predator-Prey)

Equations:

$$\frac{dx}{dt} = \alpha x - \beta xy \tag{16.1}$$

$$\frac{dy}{dt} = -\gamma y + \delta xy \tag{16.2}$$

where parameters $\alpha, \beta, \gamma, \delta > 0$.

16.1. Fixed Points

1. $(0, 0)$: Saddle point.
2. $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$: Non-trivial fixed point.

16.2. Jacobian Analysis

At the non-trivial fixed point:

$$J = \begin{pmatrix} 0 & -\beta\left(\frac{\gamma}{\delta}\right) \\ \delta\left(\frac{\alpha}{\beta}\right) & 0 \end{pmatrix} \tag{16.3}$$

Trace $\tau = 0$, Determinant $\Delta = \alpha\gamma > 0$. Linear stability predicts a **Center** (neutral stability with purely imaginary eigenvalues).

16.3. Conserved Quantity

We can separate variables:

$$\frac{dy}{dx} = \frac{-y(\gamma - \delta x)}{x(\alpha - \beta y)} \tag{16.4}$$

$$\frac{\alpha - \beta y}{y} dy + \frac{\gamma - \delta x}{x} dx = 0 \tag{16.5}$$

Integrating gives the conserved quantity (Lyapunov function):

$$V(x, y) = \delta x - \gamma \ln(x) + \beta y - \alpha \ln(y) = C \tag{16.6}$$

The level sets of V are closed orbits (periodic solutions).

This system can be written as a Hamiltonian system using the transformation $x = e^p, y = e^q$.

—

17. Selkov Model (Glycolysis)

Equations:

$$\frac{dx_1}{dt} = -x_1 + ax_2 + x_1^2 x_2 \tag{17.1}$$

$$\frac{dx_2}{dt} = b - ax_2 - x_1^2 x_2 \tag{17.2}$$

17.1. Nullclines

1. $\dot{x}_1 = 0 \Longrightarrow x_2 = \frac{x_1}{a+x_1^2}$
2. $\dot{x}_2 = 0 \Longrightarrow x_2 = \frac{b}{a+x_1^2}$

17.2. Fixed Point

Intersection occurs at:

$$x_1 = b \tag{17.3}$$

$$x_2 = \frac{b}{a+b^2} \tag{17.4}$$

Fixed point: $\left(b, \frac{b}{a+b^2}\right)$.

17.3. Stability

Jacobian J at the fixed point determines stability.

1. $\Delta = a + b^2 > 0$ (Always positive).
2. Trace τ determines Hopf bifurcation.

If $\tau = 0$, a **Hopf Bifurcation** occurs, leading to the creation of limit cycles.

—

18. Josephson Junction

In the overdamped limit, the equation for the phase difference φ is:

$$\dot{\varphi} + \alpha\dot{\varphi} + \sin(\varphi) = I \tag{18.1}$$

This can be written as a system ($\varphi = y$):

$$\dot{\varphi} = y \tag{18.2}$$

$$\dot{y} = I - \sin(\varphi) - \alpha y \tag{18.3}$$

18.1. Fixed Points

Fixed points satisfy $y = 0$ and $\sin(\varphi) = I$.

1. If $I < 1$: Two fixed points exist (Stable Node/Spiral and Saddle).
2. If $I > 1$: No fixed points exist.

18.2. Dynamics on a Cylinder

The phase space is a cylinder (since φ is periodic mod 2π). For $I > 1$, there are no fixed points. The flow goes around the cylinder. We can define a **Poincaré Map** $P(y)$ from $\varphi = 0$ to $\varphi = 2\pi$.

1. If $P(y^*) = y^*$, we have a periodic orbit (limit cycle of the second kind, corresponding to continuous rotation).

—

19. Coupled Oscillators

Equations for two coupled phase oscillators:

$$\dot{\theta}_1 = \omega_1 + k_1 \sin(\theta_2 - \theta_1) \tag{19.1}$$

$$\dot{\theta}_2 = \omega_2 + k_2 \sin(\theta_1 - \theta_2) \tag{19.2}$$

Define phase difference $\varphi = \theta_1 - \theta_2$.

$$\dot{\varphi} = (\omega_1 - \omega_2) - (k_1 + k_2)\sin(\varphi) \tag{19.3}$$

19.1. Synchronization

1. If $|\omega_1 - \omega_2| < k_1 + k_2$: Fixed points exist for φ. The oscillators **synchronize** (phase lock).
2. If $|\omega_1 - \omega_2| > k_1 + k_2$: No fixed points. φ drifts indefinitely (**Phase Drift** / Beats).

20. Lorenz Equations

The Lorenz system is given by the following set of differential equations:

$$\dot{x} = \sigma(y - x) \tag{20.1}$$

$$\dot{y} = rx - y - xz \tag{20.2}$$

$$\dot{z} = xy - bz \tag{20.3}$$

Parameters: $\sigma, r, b > 0$.

20.1. Properties

1. **Symmetry:** The system is invariant under the transformation $(x, y, z) \to (-x, -y, z)$. This implies that if $(x(t), y(t), z(t))$ is a solution, then $(-x(t), -y(t), z(t))$ is also a solution.
2. **Dissipativity (Volume Contraction):** Consider a volume V in phase space. The rate of change of volume is given by the divergence of the flow field (f):

$$\frac{dV}{dt} = \int_V (\nabla \cdot (f))\, \mathrm{d}V \tag{20.4}$$

Calculating the divergence:

$$\nabla \cdot (f) = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} \tag{20.5}$$

$$\nabla \cdot (f) = -\sigma - 1 - b \tag{20.6}$$

Since $\sigma, b > 0$, the divergence is always negative:

$$\frac{dV}{dt} = -(\sigma + 1 + b)V < 0 \tag{20.7}$$

The system is **dissipative**, meaning phase space volumes contract exponentially fast.

20.2. Fixed Points

Fixed points satisfy $\dot{x} = \dot{y} = \dot{z} = 0$:

1. From $\dot{x} = 0 \Longrightarrow y = x$.
2. Substitute into $\dot{y} = 0$: $rx - x - xz = 0 \Longrightarrow x(r - 1 - z) = 0$.
    1. **Case A :** $x = 0 \Longrightarrow y = 0$. From $\dot{z} = 0$, $-bz = 0 \Longrightarrow z = 0$.
        1. **Fixed Point 1:** $C_0 = (0, 0, 0)$ (Origin).
    2. **Case B:** $z = r - 1$. Substitute into $\dot{z} = 0$: $x^2 - b(r - 1) = 0$.
        1. $x = \pm\sqrt{b(r - 1)}$.
        2. Fixed Points C_+ and C_- exist only if $r > 1$:

$$C_\pm = \left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r - 1\right) \tag{20.8}$$

20.3. Linear Stability Analysis

The Jacobian matrix is:

$$J(x, y, z) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix} \tag{20.9}$$

20.3.1. Stability of the Origin $(0, 0, 0)$

Evaluating J at $(0, 0, 0)$:

$$J_0 = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \tag{20.10}$$

The z-direction is decoupled with eigenvalue $\lambda_3 = -b$ (Stable). The xy-block $A_{xy} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}$ determines the other eigenvalues. **Determinant** $\boldsymbol{\det(A_{xy}) = \sigma - \sigma r = \sigma(1 - r).}$ Trace $\mathrm{Tr}(A_{xy}) = -(\sigma + 1) < 0$.

Classification based on r:

1. **If $\boldsymbol{r < 1}$: det $\boldsymbol{> 0}$ and Tr $\boldsymbol{< 0}$.** The origin is a **Stable Node** (or Sink). It is globally attracting.
2. **If $\boldsymbol{r > 1}$: det $\boldsymbol{< 0}$.** The origin is a **Saddle Point** (Unstable).

20.3.2. Stability of C_+ and C_-

For these fixed points, the characteristic equation is cubic. **For $\boldsymbol{1 < r < r_H}$ (where $\boldsymbol{r_H = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}}$): $\boldsymbol{C_+}$ and $\boldsymbol{C_-}$ are stable.** At $r = r_H$: A **Subcritical Hopf Bifurcation** occurs. The fixed points lose stability, and unstable periodic orbits (saddle cycles) exist for $r < r_H$.

20.4. Global Stability of Origin (for $r < 1$)

Construct a Lyapunov function:

$$V(x, y, z) = \frac{1}{\sigma}x^2 + y^2 + z^2 \tag{20.11}$$

$V > 0$ everywhere except at the origin. Taking the time derivative:

$$\frac{dV}{dt} = \frac{2}{\sigma}x\dot{x} + 2y\dot{y} + 2z\dot{z} \tag{20.12}$$

$$\frac{dV}{dt} = \frac{2}{\sigma}x(\sigma(y - x)) + 2y(rx - y - xz) + 2z(xy - bz) \tag{20.13}$$

$$\frac{dV}{dt} = 2xy - 2x^2 + 2rxy - 2y^2 - 2xyz + 2xyz - 2bz^2 \tag{20.14}$$

The term $2xyz$ cancels out. For $r < 1$, it can be shown that $\frac{dV}{dt} < 0$ everywhere (except origin). Thus, the origin is **Globally Stable** for $r < 1$.

20.5. Trapping Region

Since the system is dissipative, solutions must eventually enter and remain in a bounded region (Trapping Region). Consider the surface defined by:

$$rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \leq C \tag{20.15}$$

We examine the derivative of this distance function along trajectories to show that for large enough C, the flow points inward on the boundary. This proves that trajectories are bounded and do not escape to infinity.

20.6. Chaos

When r is large (standard value $r = 28$), the system exhibits **Chaos**. Characteristics:

1. **Aperiodic long-time behavior:** Trajectories do not settle to fixed points or limit cycles.
2. **Deterministic:** The equations are not random.
3. **Sensitive Dependence on Initial Conditions:** Nearby trajectories diverge exponentially (Butterfly Effect).
4. **Strange Attractor:** The trajectories settle onto a fractal set (the Lorenz Attractor).

20.6.1. The Lorenz Map

Since analyzing the 3D trajectory is difficult, Lorenz analyzed the local maxima of the $z(t)$ variable. Let z_n be the n-th local maximum. He plotted z_{n+1} vs z_n. The resulting plot resembles a "Tent Map".

$$z_{n+1} = f(z_n) \tag{20.16}$$

This reduction to a 1D map explains the chaotic behavior and sensitivity to initial conditions (slope $|f'| > 1$ leads to expansion/divergence).

21. Maps

We consider discrete time dynamical systems:

$$y_{n+1} = g(y_n) \tag{21.1}$$

Fixed Points: A fixed point $\overline{y}$ satisfies:

$$\overline{y} = g(\overline{y}) \tag{21.2}$$

21.1. Stability Analysis

Consider a small perturbation ε_n around the fixed point:

$$y_n = \overline{y} + \varepsilon_n \tag{21.3}$$

Expanding $g(y_n)$ using Taylor series:

$$g(y_n) = g(\overline{y} + \varepsilon_n) = g(\overline{y}) + g'(\overline{y})\varepsilon_n + \dots \tag{21.4}$$

Since $y_{n+1} = \overline{y} + \varepsilon_{n+1}$, and $g(\overline{y}) = \overline{y}$:

$$\overline{y} + \varepsilon_{n+1} \approx \overline{y} + g'(\overline{y})\varepsilon_n \tag{21.5}$$

$$\varepsilon_{n+1} \approx g'(\overline{y})\varepsilon_n \tag{21.6}$$

Let $\lambda = g'(\overline{y})$ be the multiplier. **If $|g'| > 1$: The perturbation grows →** Unstable**.** If $|g'| < 1$: The perturbation decays → **Stable**.

21.2. Higher Dimensions

For $y \in \mathbb{R}^n$, the derivative g' becomes the **Jacobian Matrix** $Dg|_{\overline{y}}$. Stability is determined by the eigenvalues μ_i of the Jacobian: **Stable if $|\mu_i| < 1$ for all $i = 1, \dots, n$.** Unstable if at least one $|\mu_i| > 1$.

Figure 11: Cobweb plot illustrating stability.

22. Bifurcations in Maps

Bifurcations occur when eigenvalues cross the unit circle in the Argand Plane (complex plane).

1. **Flip Bifurcation (Period Doubling):** An eigenvalue crosses at $\lambda = -1$.
2. **Fold / Saddle-Node Bifurcation:** An eigenvalue crosses at $\lambda = +1$.
3. **Neimark-Sacker Bifurcation:** Complex conjugate eigenvalues cross the unit circle ($|\lambda| = 1$) away from the real axis. This is the discrete-time analog of the Hopf bifurcation.

22.1. Example: Logistic Map

$$x_{n+1} = ax_n(1 - x_n) \tag{22.1}$$

Fixed Points: Solve $f(x) = x$:

$$x = ax(1 - x) \tag{22.2}$$

$$x - ax(1 - x) = 0 \implies x(1 - a(1 - x)) = 0 \tag{22.3}$$

Two fixed points:

1. $\overline{x}_1 = 0$
2. $\overline{x}_2 = 1 - \frac{1}{a}$

Stability:

$$f'(x) = a(1 - 2x) \tag{22.4}$$

1. For $\overline{x}_1 = 0$:

$$f'(0) = a \tag{22.5}$$

. **Stable if $|a| < 1$.** Unstable if $a > 1$.

2. For $\overline{x}_2 = 1 - \frac{1}{a}$:

$$f'\left(1 - \frac{1}{a}\right) = a\left(1 - 2\left(1 - \frac{1}{a}\right)\right) = a\left(1 - 2 + \frac{2}{a}\right) = a\left(-1 + \frac{2}{a}\right) = 2 - a \tag{22.6}$$

. **Stable if $|2 - a| < 1 \implies -1 < 2 - a < 1 \implies 1 < a < 3$.** At $a = 3$, $f'(\overline{x}) = -1$, leading to a **Flip Bifurcation** (Period Doubling).

Figure 12: Bifurcation diagram of the Logistic Map.

22.2. Period Doubling to Chaos

As a increases beyond 3, the stable fixed point becomes unstable and gives birth to a stable period-2 orbit.

$$f(x_1) = x_2, \quad f(x_2) = x_1 \tag{22.7}$$

This corresponds to fixed points of the second iterate map $f^{(2)}(x) = f(f(x))$.

The stability of the 2-cycle is determined by the chain rule:

$$\frac{d}{dx}f^{(2)}(x) = f'(x_2)f'(x_1) \tag{22.8}$$

Sharkovsky's Theorem implies: Period 3 $\implies$ Chaos. If a continuous map on an interval has a period-3 orbit, it has orbits of all other periods. For the logistic map, the period-3 window appears near $a \approx 1 + \sqrt{8} \approx 3.83$ (approx). (Note: Notes mention $3 < a < 1 + \sqrt{6}$ in context of pitchfork/flip analysis).

—

23. Lyapunov Exponent

The Lyapunov exponent λ measures the average exponential rate of divergence of nearby trajectories. Consider a small perturbation δ_0. After n iterations:

$$|\delta_n| \approx |\delta_0|\ e^{n\lambda} \tag{23.1}$$

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \ln|\ \frac{\delta_n}{\delta_0}\ | \tag{23.2}$$

Using the chain rule $\delta_n = \prod_{i=0}^{n-1} f'(x_i)\delta_0$:

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \ln|\prod_{i=0}^{n-1} f'(x_i)\ | = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|\ f'(x_i)\ | \tag{23.3}$$

Interpretation: $\lambda < 0$: Stable fixed point or periodic orbit. $\boldsymbol{\lambda > 0}$**:** Chaotic**.** $\lambda = 0$: Marginally stable (bifurcation point).

To calculate this, we need the distribution of x (Invariant measure / PDF). We use the Ergodic Hypothesis to replace the time average with an ensemble average.

—

24. Tent Map

$$x_{n+1} = f(x_n) = \begin{cases} rx_n & \text{if } 0 \le x_n < \frac{1}{2} \\ r(1-x_n) & \text{if } \frac{1}{2} \le x_n \le 1 \end{cases} \tag{24.1}$$

For $r = 2$:

$$f'(x_n) = \begin{cases} 2 & \text{if } 0 \le x_n < \frac{1}{2} \\ -2 & \text{if } \frac{1}{2} \le x_n \le 1 \end{cases} \tag{24.2}$$

Lyapunov Exponent for Tent Map ($\boldsymbol{r = 2}$):

$$|f'(x)| = 2 \tag{24.3}$$

everywhere (except at $\frac{1}{2}$).

$$\lambda = \int_0^1 \rho(x) \ln|f'(x)| \, \mathrm{d}x = \int_0^1 \rho(x) \ln(2) \, \mathrm{d}x = \ln(2) \tag{24.4}$$

Since $\ln(2) > 0$, the map is chaotic.

24.1. Relation to Logistic Map ($a = 4$)

There is a coordinate transformation between the Logistic Map ($a = 4$) and the Tent Map ($r = 2$). Let y_n be the Logistic variable and x_n be the Tent variable.

$$x_n = \frac{2}{\pi} \arcsin(\sqrt{y_n}) \tag{24.5}$$

This homeomorphism proves they are topologically conjugate.

Invariant Density (PDF) for Logistic Map: The Tent map has a uniform density $\rho(x) = 1$. Using the transformation:

$$\rho(y) = \rho(x) \mid \frac{dx}{dy} \mid \tag{24.6}$$

$$\rho(y) = 1 \cdot \frac{d}{dy}\left(\frac{2}{\pi}\arcsin(\sqrt{y})\right) = \frac{1}{\pi\sqrt{y(1-y)}} \tag{24.7}$$

This is the invariant measure for the fully chaotic logistic map ($a = 4$).

—

25. Synchronization of Coupled Maps

Consider two coupled chaotic maps:

$$x_{n+1} = f(x_n) \tag{25.1}$$

$$y_{n+1} = f(y_n) \tag{25.2}$$

With symmetric coupling (ε):

$$x_{n+1} = (1-\varepsilon)f(x_n) + \varepsilon f(y_n) \tag{25.3}$$

$$y_{n+1} = \varepsilon f(x_n) + (1-\varepsilon)f(y_n) \tag{25.4}$$

Change of variables to average (u) and difference (v):

$$u_n = \frac{x_n + y_n}{2} \tag{25.5}$$

$$v_n = \frac{x_n - y_n}{2} \tag{25.6}$$

For synchronization, we need $v_n \to 0$. Linearizing the difference dynamics for small v_n:

$$v_{n+1} \approx (1-2\varepsilon)f'(u_n)v_n \tag{25.7}$$

The **Transverse Lyapunov Exponent** λ_v governs stability:

$$\lambda_v = \lim_{n\to\infty} \frac{1}{n} \sum \ln\mid (1-2\varepsilon)f'(x_i) \mid \tag{25.8}$$

$$\lambda_v = \ln\mid 1-2\varepsilon \mid + \lambda_f \tag{25.9}$$

where λ_f is the Lyapunov exponent of the single uncoupled map. Synchronization occurs if $\lambda_v < 0$.

—

26. Other Maps

26.1. Bernoulli Shift Map

$$x_{n+1} = 2x_n \bmod 1 \tag{26.1}$$

$$x_{n+1} = \begin{cases} 2x_n & \text{if } 0 \le x_n < \frac{1}{2} \\ 2x_n - 1 & \text{if } \frac{1}{2} \le x_n \le 1 \end{cases} \tag{26.2}$$

This is related to bit-shifting operations on the binary representation of x.

$$x_n = 0.a_1a_2a_3... = \sum \frac{a_i}{2^i} \tag{26.3}$$

Multiplying by 2 shifts the bits to the left and drops the integer part. It exemplifies sensitivity to initial conditions.

26.2. Baker's Map (2D)

A generalization of the Bernoulli shift to 2D, involving stretching and folding.

$$x_{n+1} = 2x_n \bmod 1 \tag{26.4}$$

$$y_{n+1} = \begin{cases} ay_n & \text{if } 0 \le x_n < \frac{1}{2} \\ \frac{1}{2} + ay_n & \text{if } \frac{1}{2} \le x_n \le 1 \end{cases} \tag{26.5}$$

Jacobian:

$$J = \begin{pmatrix} 2 & 0 \\ 0 & a \end{pmatrix} \tag{26.6}$$

Determinant $\det(J) = 2a$. **If $a = \frac{1}{2}$: Area preserving (Conservative).** If $a < \frac{1}{2}$: Dissipative (Fractal attractor).

26.3. Hénon Map

Inspired by the stretching and folding of the Baker's map but using polynomial functions.

$$x_{n+1} = 1 - ax_n^2 + y_n \tag{26.7}$$

$$y_{n+1} = bx_n \tag{26.8}$$

Jacobian:

$$J = \begin{pmatrix} -2ax_n & 1 \\ b & 0 \end{pmatrix} \tag{26.9}$$

Determinant $\det(J) = -b$. Since the determinant is constant (b), area contracts at a constant rate if $|b| < 1$. Typical parameters for chaos: $a = 1.4, b = 0.3$. This map exhibits a **Strange Attractor**.

27. Hamilton's Variational Principles

The action S is defined as the integral of the Lagrangian L over time:

$$S = \int_{t_1}^{t_2} L \, \mathrm{d}t \tag{27.1}$$

The principle of least action states that the variation of the action is zero ($\delta S = 0$) along the true path:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \tag{27.2}$$

This is the Euler-Lagrange equation.

Define the Hamiltonian H via the Legendre transform:

$$H(q_k, p_k, t) = \sum p_k \dot{q}_k - L(q_k, \dot{q}_k, t) \tag{27.3}$$

Taking the differential dH:

$$dH = d\Big(\sum p_k \dot{q}_k - L\Big) \tag{27.4}$$

$$dH = \sum(\dot{q}_k dp_k + p_k d\dot{q}_k) - \sum\left(\frac{\partial L}{\partial q_k} dq_k + \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k\right) - \frac{\partial L}{\partial t} dt \tag{27.5}$$

Using the definition of canonical momentum $p_k = \frac{\partial L}{\partial \dot{q}_k}$, the terms involving $d\dot{q}_k$ cancel out:

$$dH = \sum \dot{q}_k dp_k - \sum \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial t} dt \tag{27.6}$$

Comparing this to the total differential of $H(q, p, t)$:

$$dH = \sum \frac{\partial H}{\partial q_k} dq_k + \sum \frac{\partial H}{\partial p_k} dp_k + \frac{\partial H}{\partial t} dt \tag{27.7}$$

We obtain Hamilton's Canonical Equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{27.8}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{27.9}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{27.10}$$

If H does not explicitly depend on time, H (energy) is conserved. If q_i is cyclic (doesn't appear in L), then p_i is conserved.

—

28. Canonical Transformations

A transformation from coordinates (q, p) to (Q, P) is canonical if it preserves the form of Hamilton's equations.

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i} \tag{28.1}$$

where $K(Q, P, t)$ is the new Hamiltonian.

This implies that the phase space volume is conserved (Liouville's Theorem).

$$V(t + dt) - V(t) = 0 \tag{28.2}$$

The transformation satisfies the principle of least action in both coordinate systems:

$$\delta \int \left(\sum p_i \dot{q}_i - H \right) \mathrm{d}t = 0 \tag{28.3}$$

$$\delta \int \left(\sum P_i \dot{Q}_i - K \right) \mathrm{d}t = 0 \tag{28.4}$$

The integrands can differ by the total time derivative of a generating function F:

$$\sum p_i \dot{q}_i - H = \sum P_i \dot{Q}_i - K + \frac{dF}{dt} \tag{28.5}$$

28.1. Generating Functions

There are four basic types of generating functions depending on the variables they depend on.

1. **Type 1:** $F_1(q, Q, t)$

$$p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i} \tag{28.6}$$

$$K = H + \frac{\partial F_1}{\partial t} \tag{28.7}$$

2. **Type 2:** $F_2(q, P, t)$ Using Legendre transform: $F_2 = F_1 + \sum P_i Q_i$

$$p_i = \frac{\partial F_2}{\partial q_i}, \quad Q_i = \frac{\partial F_2}{\partial P_i} \tag{28.8}$$

$$K = H + \frac{\partial F_2}{\partial t} \tag{28.9}$$

Identity Transformation: $F_2 = \sum q_i P_i$ leads to $Q_i = q_i, P_i = p_i$.

3. **Type 3:** $F_3(p, Q, t)$

$$F_3 = F_1 - \sum p_i q_i \tag{28.10}$$

$$q_i = -\frac{\partial F_3}{\partial p_i}, \quad P_i = -\frac{\partial F_3}{\partial Q_i} \tag{28.11}$$

4. **Type 4:** $F_4(p, P, t)$

$$F_4 = F_2 - \sum p_i q_i \tag{28.12}$$

$$q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = \frac{\partial F_4}{\partial P_i} \tag{28.13}$$

28.1.1. Example: Harmonic Oscillator

Hamiltonian: $H = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$ Generating function of Type 1:

$$F_1(q, Q) = \frac{m\omega q^2}{2} \cot Q \tag{28.14}$$

This transformation leads to:

$$p = \frac{\partial F_1}{\partial q} = m\omega q \cot Q \tag{28.15}$$

$$P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2 \sin^2 Q} \tag{28.16}$$

Solving for q and p:

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q \tag{28.17}$$

$$p = \sqrt{2m\omega P} \cos Q \tag{28.18}$$

The new Hamiltonian K becomes:

$$K(Q, P) = \omega P \tag{28.19}$$

Equations of motion:

$$\dot{Q} = \frac{\partial K}{\partial P} = \omega \to Q(t) = \omega t + \alpha \tag{28.20}$$

$$\dot{P} = -\frac{\partial K}{\partial Q} = 0 \to P(t) = \text{constant} \tag{28.21}$$

—

29. Hamilton-Jacobi Theory

We seek a canonical transformation to variables (Q, P) such that the new Hamiltonian $K = 0$. This requires:

$$K = H + \frac{\partial F_2}{\partial t} = 0 \tag{29.1}$$

Let the generating function be $S(q, P, t) = F_2(q, P, t)$, called Hamilton's Principal Function. The Hamilton-Jacobi Equation (HJE) is:

$$H\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{29.2}$$

If H is conserved (independent of time), we use the ansatz:

$$S(q, P, t) = W(q, P) - Et \tag{29.3}$$

where W is Hamilton's Characteristic Function. The HJE becomes:

$$H\left(q, \frac{\partial W}{\partial q}\right) = E \tag{29.4}$$

29.0.1. Application to Harmonic Oscillator

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 + \frac{\partial S}{\partial t} = 0 \tag{29.5}$$

Using $S(q, t) = W(q) - Et$:

$$\frac{1}{2m}(W')^2 + \frac{1}{2}m\omega^2 q^2 = E \tag{29.6}$$

$$W(q) = \int \sqrt{2mE - m^2\omega^2 q^2}\,\mathrm{d}q \tag{29.7}$$

Integration leads to the solution for motion $q(t)$.

—

30. Action-Angle Variables and Perturbation Theory

For integrable systems with conserved quantities I_i, we can transform to Action-Angle variables (I, θ).

$$H(q, p) \to H_0(I) \tag{30.1}$$

$$\dot{I}_i = -\frac{\partial H_0}{\partial \theta_i} = 0 \to I_i = \text{const} \tag{30.2}$$

$$\dot{\theta}_i = \frac{\partial H_0}{\partial I_i} = \omega_{i(I)} \to \theta_i = \omega_i t + \delta_i \tag{30.3}$$

Consider a perturbed Hamiltonian:

$$H(J, \theta) = H_0(J) + \varepsilon H_1(J, \theta) \tag{30.4}$$

We seek a canonical transformation $(J, \theta) \to (I, \varphi)$ using a generating function $S(I, \theta)$:

$$S(I, \theta) = I \cdot \theta + \varepsilon S_1(I, \theta) \tag{30.5}$$

The new Hamiltonian $K(I)$ should depend only on I. Expanding to first order in ε:

$$H_0(I) + \varepsilon \frac{\partial H_0}{\partial I}\frac{\partial S_1}{\partial \theta} + \varepsilon H_1(I, \theta) = K(I) \tag{30.6}$$

To eliminate angle dependence, we choose S_1 such that it cancels the oscillating part of H_1. Expanding H_1 and S_1 in Fourier series:

$$H_1 = \sum_k H_1^{(k)}(I) e^{ik\cdot\theta} \tag{30.7}$$

$$S_1 = \sum_k S_1^{(k)}(I) e^{ik\cdot\theta} \tag{30.8}$$

The condition becomes:

$$i(k\cdot\omega) S_1^{(k)} = -H_1^{(k)} \tag{30.9}$$

$$S_1^{(k)} = \frac{iH_1^{(k)}}{k\cdot\omega} \tag{30.10}$$

Small Divisor Problem: If the frequencies are commensurate ($k \cdot \omega \approx 0$), the denominator vanishes, causing the series to diverge. This is a central problem in KAM theory.

31. Action-Angle Variables

Consider a Hamiltonian system in action-angle variables $H_0(J_1, J_2)$. The equations of motion are:

$$\frac{dJ_i}{dt} = -\frac{\partial H_0}{\partial \theta_i} = 0 \Longrightarrow J_i = \text{const} \tag{31.1}$$

$$\frac{d\theta_i}{dt} = \frac{\partial H_0}{\partial J_i} = \omega_{i((J))} \tag{31.2}$$

The non-degeneracy condition is required:

$$\det\left(\frac{\partial \omega_i}{\partial J_j}\right) \neq 0 \tag{31.3}$$

If the frequency ratio $\frac{\omega_1}{\omega_2}$ is irrational, the trajectory covers the torus densely. If rational, the orbits are closed.

32. KAM Theorem (Kolmogorov-Arnold-Moser)

The theorem addresses the stability of quasi-periodic motions (tori) under small perturbations. Books on classical mechanics often refer to this.

1. **Statement:** If the frequencies are sufficiently irrational, the invariant tori are stable under small perturbations ($\varepsilon \ll 1$).
2. **Diophantine Condition:** A torus with frequency ratio $\frac{\omega_1}{\omega_2}$ survives if there exist constants k, τ such that for all integers m, s:

$$| \frac{\omega_1}{\omega_2} - \frac{m}{s} | > \frac{k(\varepsilon)}{s^\tau} \tag{32.1}$$

   (Notes mention $s^{\frac{5}{2}}$).

As perturbation ε increases, the resonant regions (where ratios are rational like $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$) grow, destroying the nearby tori.

33. Mechanism of Tori-Breaking (Poincaré-Birkhoff Theorem)

Consider a rational torus where resonant condition holds:

$$a(r) = \frac{\omega_1}{\omega_2} = \frac{n}{s} \tag{33.1}$$

The unperturbed map T is a twist map:

$$r_{i+1} = r_i \tag{33.2}$$

$$\theta_{i+1} = \theta_i + 2\pi a(r_i) \tag{33.3}$$

For the s-th iterate of the map T^s:

$$T^s \begin{pmatrix} r_0 \\ \theta_0 \end{pmatrix} = \begin{pmatrix} r_0 \\ \theta_0 + 2\pi(\frac{n}{s})s \end{pmatrix} = \begin{pmatrix} r_0 \\ \theta_0 + 2\pi n \end{pmatrix} \equiv \begin{pmatrix} r_0 \\ \theta_0 \end{pmatrix} \tag{33.4}$$

Thus, every point on the rational torus is a fixed point of the s-th iterate.

33.1. Perturbation

Now add a perturbation ε:

$$r_{i+1} = r_i + \varepsilon f(r_i, \theta_i) \tag{33.5}$$

$$\theta_{i+1} = \theta_i + 2\pi a(r_i) + \varepsilon g(r_i, \theta_i) \tag{33.6}$$

The map must remain area-preserving (conservative).

According to the **Poincaré-Birkhoff Theorem**:

1. Not all fixed points survive the perturbation.
2. An even number of fixed points survive for the s-th iterate.
3. They alternate between **Elliptic** (stable) and **Hyperbolic/Saddle** (unstable) fixed points.
4. This creates a chain of islands (stable regions around elliptic points) separated by saddle points.

34. Homoclinic Bifurcations and Tangles

Consider the saddle fixed point P formed from the broken torus.

1. $W^{s(P)}$: Stable manifold (set of points approaching P as $n \to \infty$).
2. $W^{u(P)}$: Unstable manifold (set of points approaching P as $n \to -\infty$).

34.1. Homoclinic Point

If the stable and unstable manifolds intersect at a point P' (transversal intersection), this point is called a **Homoclinic Point**.

1. Since P' is on W^s, $f^{n(P')} \to P$ as $n \to \infty$.
2. Since P' is on W^u, $f^{n(P')} \to P$ as $n \to -\infty$.

Because the map is continuous and invertible, the existence of one homoclinic point implies the existence of infinitely many. The manifolds must oscillate wildly to accommodate this, creating a structure known as the **Homoclinic Tangle**.

This complex geometry is a signature of **Chaos**.

35. Smale Horseshoe Map

This map provides a geometric model for chaos, involving stretching and folding dynamics.

1. **Process:**
    1. Start with a unit square region.
    2. Stretch it into a long thin strip.
    3. Fold it into a horseshoe shape.
    4. Map it back onto the original square.
2. **Invariant Set:** The set of points that remain in the square for all forward and backward iterations constitutes a fractal set (Cantor set cross Cantor set).
3. **Symbolic Dynamics:** The dynamics on this invariant set are topologically conjugate to a shift map on a space of symbol sequences, demonstrating sensitivity to initial conditions and dense periodic orbits.

36. Non-linear systems with Partial Differential Equations (PDEs)

Consider a density $\rho((x), t)$ and flux $(q)((x), t)$. The conservation law is given by:

$$\int \left[\frac{\partial \rho}{\partial t} + \text{div }(q)\right] \mathrm{d}V = 0 \quad (36.1)$$

$$\frac{\partial \rho}{\partial t} + \text{div }(q) = 0 \quad (36.2)$$

In 1-D systems:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad (36.3)$$

Assuming $q = Q(\rho)$:

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \quad (36.4)$$

where $c(\rho) = Q'(\rho)$. This is the Kinematic Wave Equation.

If c is constant, we get the **Linear Transport Equation**:

$$u_t + cu_x = 0 \quad (36.5)$$

36.1. Method of Characteristics

For the equation $u_t + cu_x = 0$, we consider curves $x(t)$ such that:

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} \quad (36.6)$$

Comparing with the PDE, if we choose curves where $\frac{dx}{dt} = c$, then $\frac{du}{dt} = 0$.

Characteristic Curves: $x - ct = \text{const} = \xi$ **Solution:** $u(x,t) = f(\xi) = f(x - ct)$ where $f(x)$ is determined by the initial condition $u(x, 0) = f(x)$.

The solution represents a traveling wave with velocity c.

36.1.1. Non-uniform Linear Transport

Equation: $u_t + c(x)u_x = 0$ Characteristics are defined by:

$$\frac{dx}{dt} = c(x) \quad (36.7)$$

$$\int \frac{dx}{c(x)} = \int dt = t + \text{const} \quad (36.8)$$

Define $b(x) = \int^x \frac{ds}{c(s)}$. Then $b(x) - t = \xi$. The general solution is:

$$u(x,t) = f\big(b^{-1}(b(x) - t)\big) \quad (36.9)$$

Example: $u_t - xu_x = 0$ with $u(x, 0) = \frac{1}{1+x^2}$ Here $c(x) = -x$.

$$\frac{dx}{dt} = -x \to \ln|x| = -t + C \to xe^t = \text{const} = \xi \quad (36.10)$$

Solution:

$$u(x,t) = f(xe^t) = \frac{1}{1 + (xe^t)^2} \quad (36.11)$$

—

37. General Method of Characteristics (Quasilinear PDE)

Consider the PDE:

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u) \quad (37.1)$$

This can be viewed as the dot product of a vector field (a, b, c) with the surface normal vector $\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, -1\right)$ being zero. Thus, the vector field is tangent to the solution surface.

Parameterize the characteristics by s:

$$\frac{dx}{ds} = a, \quad \frac{dy}{ds} = b, \quad \frac{du}{ds} = c \tag{37.2}$$

37.1. Non-linear Transport Equation (Riemann Equation)

Equation:

$$u_t + uu_x = 0 \tag{37.3}$$

Characteristic equations:

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = u, \quad \frac{du}{ds} = 0 \tag{37.4}$$

From $\frac{du}{ds} = 0$, u is constant along characteristics. Since u is constant, $\frac{dx}{dt} = u$ implies characteristic lines are straight lines with slope $\frac{1}{u}$.

$$x = ut + \xi \to \xi = x - ut \tag{37.5}$$

Implicit solution:

$$u(x,t) = f(x - ut) \tag{37.6}$$

If f is increasing ($\alpha > 0$), characteristics spread out (**Rarefaction Wave**). If f is decreasing ($\alpha < 0$), characteristics intersect.

37.1.1. Shock Waves

When characteristics intersect, the solution becomes multi-valued, leading to a shock (discontinuity). The time of shock formation t^* occurs when $\frac{\partial u}{\partial x} \to \infty$.

$$\frac{\partial u}{\partial x} = \frac{f'(\xi)}{1 + tf'(\xi)} \tag{37.7}$$

Blow up occurs when $1 + tf'(\xi) = 0$.

Rankine-Hugoniot Condition: The speed of the shock $s'(t)$ is related to the jump in u:

$$\frac{ds}{dt} = \frac{J}{[u]} = \frac{\left[\frac{u^2}{2}\right]}{[u]} = \frac{\left(\frac{u_-^2}{2}\right) - \left(\frac{u_+^2}{2}\right)}{u_- - u_+} = \frac{1}{2}(u_- + u_+) \tag{37.8}$$

where u_- is the value behind the shock and u_+ is ahead.

This is derived from the integral form of the conservation law:

$$\frac{d}{dt}\int_a^b u \,\mathrm{d}x = -\left[\frac{u^2}{2}\right]_a^b \tag{37.9}$$

—

38. Viscous Burgers' Equation

Equation:

$$u_t + uu_x = \nu u_{xx} \tag{38.1}$$

This combines non-linear transport (steepening) with diffusion (smoothing). Using the Hopf-Cole transformation, this can be linearized to the Heat Equation.

$$u = -2\nu \frac{\varphi_x}{\varphi} \tag{38.2}$$

Then $\varphi_t = \nu \varphi_{xx}$.

Solution involves convolution with the heat kernel:

$$u(x,t) = \frac{\int \left(-\frac{x-y}{t}\right) e^{-\frac{(x-y)^2}{4\nu t}} e^{-\frac{1}{2\nu}\int_0^y f(\eta)\,\mathrm{d}\eta} \,\mathrm{d}y}{\int e^{-\frac{(x-y)^2}{4\nu t}} e^{-\frac{1}{2\nu}\int_0^y f(\eta)\,\mathrm{d}\eta} \,\mathrm{d}y} \tag{38.3}$$

For traveling wave solutions $u(x,t) = v(x - ct)$, we get a profile connecting two states (e.g., u_- to u_+), representing a smooth shock structure.

39. KdV Equation and Solitons

The Korteweg-de Vries (KdV) equation is given by:

$$u_t + u_{\{xxx\}} + uu_x = 0 \tag{39.1}$$

We apply the **Travelling Wave ansatz**:

$$\xi = x - ct \tag{39.2}$$

$$u(x,t) = v(\xi) \tag{39.3}$$

This leads to the **Soliton** solution, which has a sech^2 profile. This is useful in equilibration problems.

39.1. FPUT Paradox and Hamiltonian Systems

The Fermi-Pasta-Ulam-Tsingou (FPUT) paradox involves a lattice with a nonlinear potential:

$$V(\delta) = \frac{1}{2}k\delta^2 + \beta\delta^3 \tag{39.4}$$

Kruskal proposed that a continuous limit can be invoked. The system has conserved quantities (Integrals of Motion), often denoted as T_n or I_n.

- **Mass / Momentum:** $T_1 = u$
- **Energy:** $T_2 = \frac{u^2}{2} + u_{\{xx\}}$ (Note: Standard form is usually $\frac{u^2}{2}$)
- **Hamiltonian:** $T_3 = \frac{u^3}{3} - \frac{u_x^2}{2}$

39.2. Derivation of Travelling Wave ODE

Substituting $u(x,t) = v(x - ct)$ into the KdV equation:

$$-cv' + v''' + vv' = 0 \tag{39.5}$$

Integrating once with respect to ξ:

$$-cv + v'' + \frac{v^2}{2} = A \tag{39.6}$$

where A is an integration constant. Assuming boundary conditions where $v, v', v'' \to 0$ as $\xi \to \infty$, we set $A = 0$.

This reduces to a second-order ordinary differential equation (ODE):

$$v'' = cv - \frac{v^2}{2} \tag{39.7}$$

This can be analyzed as a particle in a potential well.

—

40. Fisher's Equation

The Fisher equation describes a reaction-diffusion system. We start with the reaction term (Logistic growth):

$$u_t = \mu u(1 - u) \tag{40.1}$$

This has fixed points at $u = 0$ and $u = 1$. For $\mu > 0$, the growth curve is a parabola starting at 0, peaking, and returning to 0 at 1.

Adding the diffusion term:

$$u_t = \mu u(1 - u) + Du_{\{xx\}} \tag{40.2}$$

40.1. Steady State Analysis

For steady states ($u_t = 0$):

$$Du_{\{xx\}} + \mu u(1 - u) = 0 \tag{40.3}$$

$$u_{\{xx\}} = -\frac{\mu}{D}u(1 - u) \tag{40.4}$$

We convert this to a system of first-order equations:

$$u_x = v \tag{40.5}$$

$$v_x = -\frac{\mu}{D}u(1-u) \tag{40.6}$$

Fixed Points: Solving $v = 0$ and $-\frac{\mu}{D}u(1-u) = 0$ gives two fixed points in phase space:

1. $(u, v) = (0, 0)$
2. $(u, v) = (1, 0)$

Jacobian Analysis: The Jacobian matrix is:

$$J = \begin{pmatrix} 0 & 1 \\ -\frac{\mu}{D}(1-2u) & 0 \end{pmatrix} \tag{40.7}$$

1. **At (0,0):**

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -\frac{\mu}{D} & 0 \end{pmatrix} \tag{40.8}$$

$$\Delta = \frac{\mu}{D} > 0 \tag{40.9}$$

,

$$\mathrm{Tr} = 0 \tag{40.10}$$

. Since the trace is zero and determinant is positive, this is a **Center** (Linear stability). However, due to nonlinear terms, it might differ, but locally trajectories circle the origin.

2. **At (1,0):**

$$J(1,0) = \begin{pmatrix} 0 & 1 \\ \frac{\mu}{D} & 0 \end{pmatrix} \tag{40.11}$$

$$\Delta = -\frac{\mu}{D} < 0 \tag{40.12}$$

. Since the determinant is negative, this is a **Saddle Point**.

40.2. Boundary Conditions

Dirichlet Boundary Conditions (DBC): $\boldsymbol{u(0) = u(L) = 0}$**. Neumann Boundary Conditions (NBC):** $u'(0) = u'(L) = 0$.

Due to reversibility considerations in the steady state equation, NBC might not be feasible for non-trivial solutions connecting states. For DBC, solutions like $u \approx C \sin\left(\sqrt{\frac{\mu}{D}}x\right)$ exist for small amplitudes.

This leads to a **Critical Length** L^* below which no non-trivial solution can exist (population goes extinct):

$$L^* = \pi\sqrt{\frac{D}{\mu}} \tag{40.13}$$

—

41. Travelling Waves in Fisher's Equation

We look for solutions of the form $u(x,t) = \varphi(x - ct) = \varphi(\xi)$. Boundary conditions:

$$\varphi(-\infty) = 1 \tag{41.1}$$

(Populated past)

$$\varphi(+\infty) = 0 \tag{41.2}$$

(Empty future)

Substituting into the PDE:

$$-c\varphi' = D\varphi'' + \mu\varphi(1-\varphi) \tag{41.3}$$

$$\varphi'' + \frac{c}{D}\varphi' + \frac{\mu}{D}\varphi(1-\varphi) = 0 \tag{41.4}$$

Let $\varphi' = \psi$. The system becomes:

$$\varphi' = \psi \tag{41.5}$$

$$\psi' = -\frac{c}{D}\psi - \frac{\mu}{D}\varphi(1-\varphi) \tag{41.6}$$

Fixed Points Analysis:

1. **At (0,0) (The "Empty" state ahead):**

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -\frac{\mu}{D} & -\frac{c}{D} \end{pmatrix} \tag{41.7}$$

Eigenvalues are determined by $\lambda^2 - \text{Tr}\ \lambda + \Delta = 0$.

$$\text{Tr} = -\frac{c}{D} \tag{41.8}$$

,

$$\Delta = \frac{\mu}{D} \tag{41.9}$$

.

$$\lambda = \frac{-\frac{c}{D} \pm \sqrt{\left(\frac{c}{D}\right)^2 - 4\frac{\mu}{D}}}{2} \tag{41.10}$$

For the solution to be physically stable (non-oscillatory approach to 0, since population u cannot be negative), we require real eigenvalues:

$$\left(\frac{c}{D}\right)^2 - 4\frac{\mu}{D} \geq 0 \tag{41.11}$$

$$c^2 \geq 4D\mu \tag{41.12}$$

$$c \geq 2\sqrt{D\mu} \tag{41.13}$$

2. **At (1,0) (The "Populated" state behind):**

$$J(1,0) = \begin{pmatrix} 0 & 1 \\ \frac{\mu}{D} & -\frac{c}{D} \end{pmatrix} \tag{41.14}$$

$$\Delta = -\frac{\mu}{D} < 0 \tag{41.15}$$

. This is a **Saddle Point**.

41.1. Critical Velocity

The minimum velocity for a stable travelling wave front is:

$$c^* = 2\sqrt{D\mu} \tag{41.16}$$

If $c < 2\sqrt{D\mu}$: The fixed point at $(0,0)$ is a stable spiral. This implies φ would oscillate around 0, leading to negative population values (unphysical). If $c \geq 2\sqrt{D\mu}$: The fixed point is a stable node. The wavefront decays monotonically to zero.

Thus, the travelling wave solution exists only for speeds $c \geq c^*$.

42. KdV Equation

The Korteweg-de Vries (KdV) equation is given by:

$$u_t + u_{\{xx\}} + uu_x = 0 \tag{42.1}$$

We look for **Solitonic solutions** using the travelling wave ansatz:

$$\xi = x - ct \tag{42.2}$$

$$u(x,t) = v(\xi) \tag{42.3}$$

42.1. Hamiltonian Formulation (FPUT Context)

Related to the Fermi-Pasta-Ulam-Tsingou (FPUT) problem with potential:

$$V(s) = \frac{1}{2}ks^2 + \beta s^3 \tag{42.4}$$

Hamiltonian:

$$H = \sum \frac{1}{2}m\dot{u}_n^2 + \sum \varphi\left(u_{\{n+1\}} - u_n\right) \tag{42.5}$$

If the system is integrable, we have conserved quantities (Integrals of Motion) $\{T_n, I_n\}$ which are infinite in number.

42.2. Derivation of Soliton ODE

Substituting $u(x,t) = v(x - ct)$ into the KdV equation:

$$-cv' + v''' + vv' = 0 \tag{42.6}$$

Integrating once with respect to ξ:

$$-cv + v'' + \frac{1}{2}v^2 = k \tag{42.7}$$

Assuming boundary conditions where $v, v', v'' \to 0$ at infinity, we set the integration constant $k = 0$.

This reduces to a system of first-order ODEs:

$$\frac{dv}{d\xi} = w \tag{42.8}$$

$$\frac{dw}{d\xi} = cv - \frac{v^2}{2} \tag{42.9}$$

Fixed Points: Setting derivatives to zero:

1. $w = 0$
2. $cv - \frac{v^2}{2} = 0 \Longrightarrow v\left(c - \frac{v}{2}\right) = 0$

Fixed points are $(0,0)$ and $(2c, 0)$.

Jacobian Analysis:

$$J = \begin{pmatrix} 0 & 1 \\ c - v & 0 \end{pmatrix} \tag{42.10}$$

1. **At $(0,0)$:**

$$J = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix} \tag{42.11}$$

   Eigenvalues: $\lambda^2 - c = 0 \Longrightarrow \lambda = \pm\sqrt{c}$. Since $c > 0$ (velocity), eigenvalues are real and opposite sign. This is a **Saddle Point**.

2. **At $(2c, 0)$:**

$$J = \begin{pmatrix} 0 & 1 \\ c - 2c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -c & 0 \end{pmatrix} \tag{42.12}$$

   Eigenvalues: $\lambda^2 + c = 0 \Longrightarrow \lambda = \pm i\sqrt{c}$. This is a **Center** (linear analysis).

The soliton solution corresponds to a **homoclinic orbit** connecting the saddle point at the origin to itself.

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43. Turing Patterns (Reaction-Diffusion Systems)

Consider a reaction-diffusion system:

$$u_t = f(u, v) + D_u \nabla^2 u \tag{43.1}$$

$$v_t = g(u, v) + D_v \nabla^2 v \tag{43.2}$$

We look for a Turing instability, where a stable homogeneous equilibrium becomes unstable due to diffusion, leading to spatial patterns.

43.1. Linear Stability Analysis

Let (u_0, v_0) be a stable fixed point of the reaction kinetics (without diffusion). Perturbation:

$$(w) = \begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix} \sim e^{\lambda t} e^{ikx} \tag{43.3}$$

The Jacobian with diffusion included is:

$$J_k = J_{\text{reaction}} - k^2 D \tag{43.4}$$

$$J_k = \begin{pmatrix} f_u - D_u k^2 & f_v \\ g_u & g_v - D_v k^2 \end{pmatrix} \tag{43.5}$$

For Turing instability, we need:

1. Trace $\mathrm{Tr} < 0$ (for stability at $k = 0$).
2. Determinant $\Delta(k) < 0$ for some $k \neq 0$ (to induce instability).

43.2. The Brusselator Model

Equations:

$$u_t = A - (B + 1)u + u^2 v + D_u u_{\{xx\}} \tag{43.6}$$

$$v_t = Bu - u^2 v + D_v v_{\{xx\}} \tag{43.7}$$

Fixed point: $(u_0, v_0) = \left(A, \frac{B}{A}\right)$.

Jacobian Matrix:

$$J = \begin{pmatrix} B - 1 & A^2 \\ -B & -A^2 \end{pmatrix} \tag{43.8}$$

(Note: Standard Brusselator Jacobian usually involves these terms).

Applying the perturbation analysis to find the critical parameter B_c for instability. We solve for eigenvalues λ of the matrix:

$$M = \begin{pmatrix} \lambda + k^2 D_u - (B - 1) & -A^2 \\ B & \lambda + k^2 D_v + A^2 \end{pmatrix} \tag{43.9}$$

The characteristic equation is:

$$\lambda^2 - \mathrm{Tr} \cdot \lambda + \mathrm{Det} = 0 \tag{43.10}$$

We analyze the determinant condition for instability. The threshold occurs when the minimum of the determinant curve touches zero.

Hopf Bifurcation: Occurs when $\mathrm{Tr} = 0$.

$$b^H = k^2(1 + D) + 1 + a^2 \tag{43.11}$$

Note: If $k = 0$, the perturbation has no spatial dependence, effectively reducing to the ODE case.

Turing Bifurcation: Occurs when $\mathrm{Det} = 0$.

$$b^T(k) = k^2 + 1 + \frac{a^2}{k^2 D} + \ldots \tag{43.12}$$

We minimize this function with respect to k^2 to find the critical wavenumber k_c and critical parameter b_c.

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