# PH2101 : Waves and Optics

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1. **Question 1** Consider a beaded string of N beads each of mass m (approximated as a long chain of spring- mass system as shown in the figure). The beads are uniformly placed on the string and the string has a uniform tension T . The horizontal distance between any two beads in equilibrium is a. The unstretched lengths of the springs are negligible.
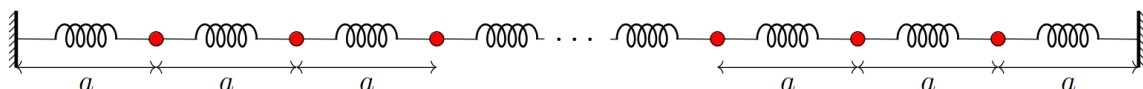
(a) Find the equation of the motion of nth bead for the longitudinal mode of vibration

> **Solution:**

(b) Assuming normal mode vibration, find the normal mode frequency ω_m for m^{th} mod

> **Solution:**

(c) Find the amplitudes of the beads in m^{th} mode ($A_n^{(m)}$ using the notation used in the class).

> **Solution:**

(d) Plot the dispersion relation ω versus k.

> **Solution:**

(e) Check whether we have $\omega_{N+2} = \omega_N$?

> **Solution:**

(f) Check whether we have $A_n^{(N+2)} = A_n^{(N)}$?

> **Solution:**

(g) Qualitatively plot $A_n^{(1)}$ and $A_n^{(N)}$ for all n.

> **Solution:**

2. **Question 2** Consider two pendulums, a and b, with the same string length L, but with different bob masses, M_a and M_b. They are coupled by a spring of spring constant K which is attached to the bobs. Assuming small angle oscillations,

(a) Find the equations of motion using angles of the pendulums (w.r.t. the vertical) as dynamical variables.

Solution: Let the coupled pendula be shifted by θ_a and θ_b for M_a and M_b respectively Then for the mass M_a we have the torque to be under small angles of displacement

$$\Gamma = -M_a g L \sin(\theta_a) - KL(L\sin(\theta_a) - L\sin(\theta_b))$$
$$\Rightarrow \Gamma = -M_a g L \theta_a - KL^2(\theta_a - \theta_b)$$

For mass M_b we have

$$\Gamma = -M_b g L \sin(\theta_b) - KL(L\sin(\theta_b) - L\sin(\theta_a))$$
$$\Rightarrow \Gamma = -M_b g L \theta_b - KL^2(\theta_b - \theta_a)$$

we know that $\Gamma = I\ddot{\theta}$ where I is the moment of Inertia and $\ddot{\theta}$ is the angular acceleration. Thus, we have

$$M_a L^2 2\ddot{\theta}_a = -M_a g L \theta_a - KL^2(\theta_a - \theta_b)$$
$$M_a L^2 2\ddot{\theta}_b = -M_b g L \theta_b - KL^2(\theta_b - \theta_a)$$

We thus have the equations of motion as follows

$$\boxed{\ddot{\theta}_a = -gL\theta_a - \frac{K}{M_a}(\theta_a - \theta_b)}$$

$$\boxed{\ddot{\theta}_b = -gL\theta_b - \frac{K}{M_b}(\theta_b - \theta_a)}$$

(b) Find the normal modes and the normal frequencies

Solution:

$$\ddot{\theta}_a = -gL\theta_a - \frac{K}{M_a}(\theta_a - \theta_b)$$
$$\ddot{\theta}_b = -gL\theta_b - \frac{K}{M_b}(\theta_b - \theta_a)$$

Since we are calculating a normal the frequencies of the oscillation of the the masses. Then the masses move at the same frequency, then we can use two guessed solutions i.e. $\theta_a = Ae^{i\omega t}$ and $\theta_b = Be^{i\omega t}$. Now we plus this in to the equations of motion and try to determine A and B by solving the system of linear equations. So we plug them in to the equation and eliminate $e^{i\omega t}$

$$-A\omega^2 = -AgL - A\frac{K}{M_a} + B\frac{K}{M_a} \qquad \text{(for mass } M_a\text{)}$$
$$\Rightarrow A\left(\omega^2 - gL - \frac{K}{M_a}\right) + B\left(\frac{K}{M_a}\right) = 0$$
$$-B\omega^2 = -gL - B\frac{K}{M_b} + A\frac{K}{M_b} \qquad \text{(for mass } M_b\text{)}$$
$$\Rightarrow B\left(\omega^2 - gL - \frac{K}{M_b}\right) + A\left(\frac{K}{M_b}\right) = 0$$

Again like we did in the previous assignment write this in matrix and the determinant of coefficients have to be 0 in order to have a non-trivial solution.

$$\begin{pmatrix} \omega^2 - \dfrac{g}{L} - \dfrac{K}{M_a} & \dfrac{K}{M_a} \\ \dfrac{K}{M_b} & \omega^2 - \dfrac{g}{L} - \dfrac{K}{M_b} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} \omega^2 - \dfrac{g}{L} - \dfrac{K}{M_a} & \dfrac{K}{M_a} \\ \dfrac{K}{M_b} & \omega^2 - \dfrac{g}{L} - \dfrac{K}{M_b} \end{vmatrix} = 0$$

$$\Rightarrow \left(\omega^2 - \frac{g}{L} \right)^2 - \left(\omega^2 - \frac{g}{L} \right) \left(\frac{K}{M_a} + \frac{K}{M_b} \right) + \frac{\cancel{K^2}}{\cancel{M_a M_b}} - \frac{\cancel{k^2}}{\cancel{M_a M_b}} = 0$$

$$\Rightarrow \omega^4 - \omega^2 \left(2gL + K\mu \right) + \frac{g}{L} \left(\frac{g}{L} + K\mu \right) = 0 \qquad \text{(Let } \mu = \frac{1}{M_a} + \frac{1}{M_b} \text{)}$$

$$\Rightarrow \omega^2 = \frac{\left(\dfrac{2g}{L} + K\mu \right) \pm \sqrt{\left(\dfrac{2g}{L} + K\mu \right)^2 - 4\dfrac{g}{L} \left(\dfrac{g}{L} + K\mu \right)}}{2}$$

$$\Rightarrow \omega^2 = \frac{\left(\dfrac{2g}{L} + K\mu \right) \pm \left(K \left(\dfrac{M_a + M_b}{M_a M_b} \right) \right)}{2}$$

Thus the normal modes ω_1(for in phase motion) and ω_2(for out of phase motion) are,

$$\boxed{\omega_1^2 = \frac{g}{L}} \boxed{\omega_2^2 = \frac{g}{L} + K \left(\frac{M_a + M_b}{M_a M_b} \right)}$$

(c) For $M_a = M_b = M$, does this reduce to the case considered in class?

Solution: For $M_a = M_b = M$ we have $\omega_1^2 = \dfrac{g}{L}$, $\omega_2^2 = \dfrac{g}{L} + \dfrac{2K}{M}$, which does reduce to the case considered to the one done in the class i.e.

3. **Question 3** Consider the following string with the given configuration.
(a)Find the Fourier series representation of the string. You should use the sine representation for the string and not the full representation.

$$L/4 \qquad\qquad 3L/4$$
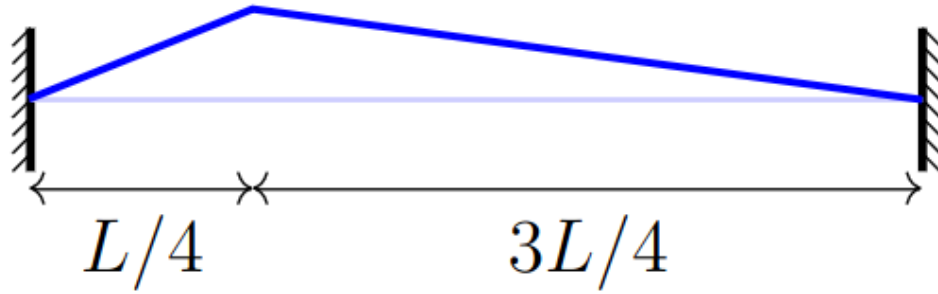
Solution: We can see that this is a piecewise function woth two linear parts, one from $0 < x < L4$ and the other from $\frac{L}{4} < x < L$. Let the height to which the string is plucked some value say h. So for the first part the function passes through the coordinates $(L4, h)$ and $(0, 0)$, thus we have the function.

$$y = \frac{h}{L4}(x)$$
$$\Rightarrow y = \frac{4h}{L}x$$

For the second part we have the line passing through the coordinates $(L/4, h)$ and $(L, 0)$. Thus we have the function,

$$y = \frac{h}{L/4 - L}(x - L)$$
$$\Rightarrow y = -\frac{4h}{3L}(x - L)$$

Thus we have the piecewise function definition

$$f(x) = \begin{cases} \dfrac{4h}{L}x & \text{if } 0 < x < \dfrac{L}{4} \\ -\dfrac{4h}{3L}(x - L) & \text{if } \dfrac{L}{4} < x < L \end{cases}$$

We need to show a fourier series representation of the above function.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

$a_n = 0 \;\; \forall n$ as the cosine terms do not become 0 and thus do not satisfy the boundary conditions as the ends of the string are fixed and at $y = 0$. we can take a_0 to be any value we want as that value in th physical scenario doesn't matter as the string is tied between two and ends and we can take one of the ends to be the origin without consequence. The a_0 is just a term that translates the whole fourier function graph upward or downward. We only need to find the coefficients of the sine

terms i.e. b_n.

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

$$\Rightarrow \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = f(x)\sin\left(\frac{m\pi x}{L}\right)$$

$$\Rightarrow \int_0^L \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

As we know that the integral picks out the nth value from the sum, thus giving us the coefficient b_n

$$\Rightarrow \int_0^L b_n \sin^2\left(\frac{n\pi x}{L}\right)dx = \int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$\Rightarrow b_n\frac{L}{2} = \int_0^{L/4} \frac{4h}{3L}x\sin\left(\frac{n\pi x}{L}\right)dx - \int_{L/4}^L \frac{4h}{3L}(x-L)\sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow b_n = \frac{8h}{L^2}\left[\int_0^{L/4} x\sin\left(\frac{n\pi x}{L}\right)dx - \frac{1}{3}\int_{L/4}^L (x-L)\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$\Rightarrow b_n = \frac{8h}{L^2}\left[\frac{L^2}{\pi^2 n^2}\sin\left(\frac{\pi n x}{L}\right) - \frac{L}{\pi n}x\cos\left(\frac{\pi n x}{L}\right)\Big|_0^{L/4} - 13\left(\frac{L^2}{\pi^2 n^2}\sin\left(\frac{\pi n x}{L}\right) + \frac{L}{\pi n}(L-x)\cos\left(\frac{\pi n x}{L}\right)\right)\Big|_{L/4}^L\right]$$

Let us solve the intgrals piecewise :

$$\int_0^{L/4} x\sin\left(\frac{n\pi x}{L}\right)dx = \left[\left(L^2\pi^2 n^2\sin\left(\frac{\pi n x}{L}\right) - \frac{L}{\pi n}x\cos\left(\pi n x L\right)\right)\Big|_0^{L/4}\right] \qquad \text{(Part 1)}$$

$$= \frac{L^2}{\pi^2 n^2}\sin\left(\frac{\pi n}{4}\right) - \frac{L^2}{4\pi n}\cos\left(\frac{\pi n}{4}\right)$$

$$13\int_{L/4}^L (x-L)\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{3}\left(\frac{L^2}{\pi^2 n^2}\sin\left(\frac{\pi n x}{L}\right) + \frac{L}{\pi n}(L-x)\cos\left(\frac{\pi n x}{L}\right)\right)\Big|_{L/4}^L \qquad \text{(Part 2)}$$

$$= \frac{L^2}{3\pi^2 n^2}\cdot\sin\left(\pi n\right) - \frac{L^2}{3\pi^2 n^2}\sin\left(\pi n 4\right) - \frac{L^2}{4\pi n}\cos\left(\frac{\pi n}{4}\right)$$

$$= -\frac{L^2}{3\pi^2 n^2}\sin\left(\pi n 4\right) - \frac{L^2}{4\pi n}\cos\left(\frac{\pi n}{4}\right) \qquad \text{(since n} \in \mathbb{N}, \sin(n\pi)=0)$$

Part 1 + Part 2 = $b_n\dfrac{L^2}{8h}$

$$b_n = \frac{8h}{L^2}\left[\frac{L^2}{\pi^2 n^2}\sin\left(\frac{\pi n}{4}\right) - \frac{L^2}{4\pi n}\cancel{\cos\left(\frac{\pi n}{4}\right)} + \frac{L^2}{3\pi^2 n^2}\sin\left(\frac{\pi n}{4}\right) + \frac{L^2}{4\pi n}\cancel{\cos\left(\frac{\pi n}{4}\right)}\right]$$

$$= \frac{32h}{3L^2\pi^2 n^2}\sin\left(\frac{n\pi}{4}\right)$$

Thus we have the Fourier series as $\boxed{f(x, t = 0) = \sum_n^\infty \dfrac{32h}{3L^2\pi^2 n^2} \sin\left(\dfrac{n\pi}{4}\right) \sin\left(\dfrac{n\pi x}{L}\right)}$

(b) Show that normal modes having nodes at L/4 are absent.

Solution: The normal modes with a node at $L4$ are absent because b_n vanishes. Physically it also makes sense as the amplitude of the string at $t = 0$ is at $\frac{L}{4}$. We have a node at $L4$ meaning $\sin\left(\frac{n\pi x}{L}\right) = \sin\left(\frac{n\pi L}{4L}\right) = \sin\left(\frac{n\pi}{4}\right) = 0 \Rightarrow n = 4m$ for some $m \in \mathbb{N}$. So the whole normal mode

$$b_n = \frac{32h}{3L^2\pi^2 n^2} \sin\left(\frac{4m\pi}{4}\right)$$

$$= \frac{32h}{3L^2\pi^2 n^2} \sin(m\pi)$$

$$= 0 \qquad\qquad (\sin(m\pi) = 0 \ \forall \ m \in \mathbb{N})$$

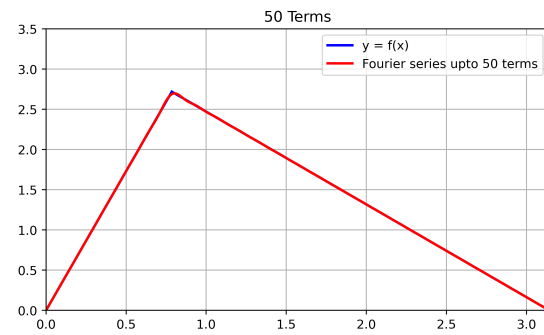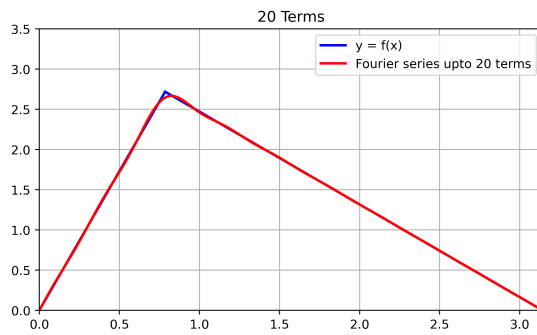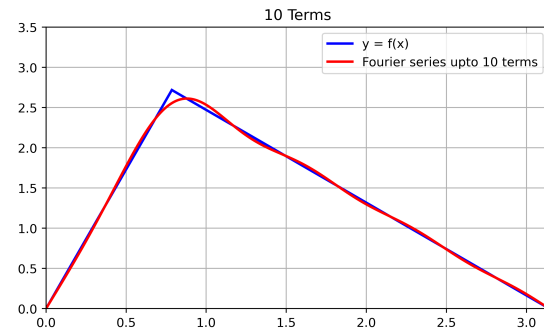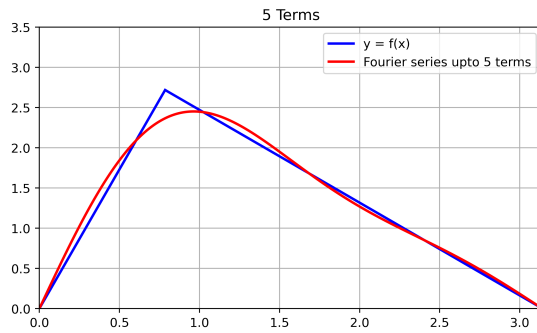Thus the normal modes with nodes at $x = \dfrac{L}{4}$ vanish to 0.

(c) Check numerically that your solution matches with the given shape. You may submit your codes.
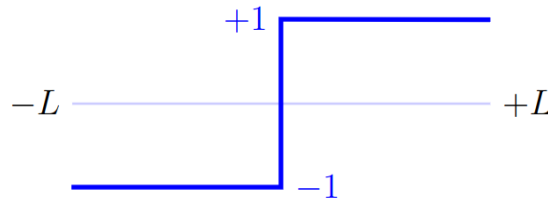
Solution:

```
import matplotlib.pyplot as plt
import numpy as np
L=np.pi
h=np.e
x = np.linspace(0,L,20000)
x1 = np.linspace(0,L/4,5000)
x2 = np.linspace(L/4,L,5000)
y1 = (4*h/L)*x1
y2 = -(4*h/(3*L))*(x2-L)
def fourierfunc(x,n):
    y=0
    for i in range(1,n+1):
        y+= ((32*h/(3*i*i*np.pi*np.pi))*np.sin((np.pi*i)/4) * np.sin(i*np.pi
            ↪ *x/L))
    return y
fig,axes = plt.subplots(figsize=(16,9),dpi=400,nrows = 2, ncols = 2)
turns = [1,5,10,50]
fig.suptitle("Fourier␣Series")
for i in range(0,2):
    for j in range(0,2):
        axes[i][j].set_title(str(turns[2*i+j])+"␣Terms")
        axes[i][j].plot(x1,y1,lw = '2',color = "blue",label= "y␣=␣f(x)␣")
        axes[i][j].plot(x2,y2,lw = '2',color = "blue")
        axes[i][j].plot(x, fourierfunc(x,turns[2*i+j]),color = "red",lw =2,
            ↪ label = "Fourier␣series␣upto␣"+str(turns[2*i+j])+"␣terms")
        axes[i][j].set_ylim([0,3.5])
```

```
            axes[i][j].set_xlim([0,np.pi])
            axes[i][j].legend(loc = "best")
            axes[i][j].grid()
```


Fourier Series

4. **Question 4** Consider the following pattern. Find the Fourier representation of this pattern. Use the complete representation (using sine and cosine). Also, check numerically that your solution matches with the given shape. You may submit your codes.

Solution:

$$f(x) = \begin{cases} 1 & \text{if } -L < x < 0 \\ -1 & \text{if } 0 < x < L \end{cases}$$

The graphs plotted by the software are shown below.

```python
import matplotlib.pyplot as plt
import numpy as np
L=np.pi*np.e
x = np.linspace(-L,L,10000)
x1 = [-L,0,L]
y1 = [-1,-1,1]

def fourierfunc(x,n):
    y=0
    for i in range(1,n+1,2):
        y = y + ((4/(i*np.pi))*np.sin(i*np.pi*x/L))
    return y
fig,axes = plt.subplots(figsize=(16,16),dpi=400,nrows = 3, ncols = 2)
turns = [5,10,20,50,100,200]
fig.suptitle("Fourier Series")
for i in range(0,3):
    for j in range(0,2):
        axes[i][j].set_title(str(turns[2*i+j])+" Terms")
        axes[i][j].axvline(0, color = 'black', lw = 2)
        axes[i][j].axhline(0, color = 'black', lw = 2)
        axes[i][j].step(x1,y1,lw = 3,color = "blue",label= "y = f(x) ")
        axes[i][j].plot(x, fourierfunc(x,turns[2*i+j]),color = "red",lw =2,
            ↪ label = "Fourier series upto "+str(turns[2*i+j])+" terms")
        axes[i][j].set_ylim([-1.5,1.5])
        labels = [item.get_text() for item in axes[i][j].get_xticklabels()]
        labels[1] = "L"
        labels[-2] = "-L"
        axes[i][j].set_xticklabels(labels)
        axes[i][j].legend(loc = "best")
        axes[i][j].grid()
```

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