

Definition of Squeezed States

Squeezed States are defined as the state obtained by the action of the operator $\hat{S}(\zeta)\hat{D}(\alpha)$ on the vacuum number state $|0\rangle$

$$|\alpha, \zeta\rangle = \hat{D}(\alpha)\hat{S}(\zeta)|0\rangle$$

Where the concerned operators are defined as,

$$\begin{aligned}\hat{D}(\alpha) &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} \\ \hat{S}(\zeta) &= e^{\frac{1}{2}(\zeta^*\hat{a}^2 - \zeta\hat{a}^{\dagger 2})}\end{aligned}$$

where, $\zeta = re^{i\theta}$ with $r > 0$.

$\hat{D}(\alpha)$ and $\hat{S}(\zeta)$ are the translation and the squeeze operator respectively. It can be shown that,

$$\hat{D}(\alpha)^\dagger \hat{D}(\alpha) = \hat{S}(\zeta)^\dagger \hat{S}(\zeta) = 1$$

i.e., both the operators are unitary. This will help us a lot in the following section, where we will be looking at the expectations and variances of a few relevant quantities.

Mean Photon Number in a Squeezed State

We wish to compute $\langle n \rangle$ for the squeezed state $|\alpha, \zeta\rangle$

We know that

$$\hat{n} = \hat{a}^\dagger \hat{a}$$

Now, to compute the expectation of \hat{n} in the Squeezed state we have to evaluate the following expression

$$\begin{aligned}\langle n \rangle &= \langle 0 | \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) | 0 \rangle \\ &= \langle 0 | \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) \hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) | 0 \rangle \quad [\because \hat{D}(\alpha)^\dagger \hat{D}(\alpha) = 1] \\ &= \langle 0 | \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) | 0 \rangle \quad [\because \hat{S}(\zeta)^\dagger \hat{S}(\zeta) = 1]\end{aligned}$$

First, we evaluate the operator $\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha)$

$$\begin{aligned}\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) &= e^{-|\alpha|^2} e^{-\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} \hat{a} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} \\ &= e^{-|\alpha|^2} e^{-\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} (e^{\alpha\hat{a}^\dagger} \hat{a} + [\hat{a}, e^{\alpha\hat{a}^\dagger}]) e^{-\alpha^*\hat{a}}\end{aligned}$$

Now let us compute the commutator relation $[\hat{a}, e^{\alpha\hat{a}^\dagger}]$ which is given by,

$$[\hat{a}, e^{\alpha\hat{a}^\dagger}] = \sum_{n=1}^{\infty} \frac{\alpha^n [\hat{a}, \hat{a}^{\dagger n}]}{n!}$$

We can easily show by induction that, $[\hat{a}, \hat{a}^{\dagger n}] = n\hat{a}^{\dagger n-1}$. Then the commutator evaluates to,

$$\begin{aligned}[\hat{a}, e^{\alpha\hat{a}^\dagger}] &= \sum_{n=1}^{\infty} \frac{n\alpha^n \hat{a}^{\dagger n-1}}{n!} \\ &= \alpha \sum_{n=0}^{\infty} \frac{(\alpha\hat{a}^\dagger)^n}{n!} \\ &= \alpha e^{\alpha\hat{a}^\dagger}\end{aligned}$$

Substituting this commutator relation back into the expression for $\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha)$ we get,

$$\begin{aligned}\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) &= e^{-|\alpha|^2} e^{-\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \left(e^{\alpha \hat{a}^\dagger} \hat{a} + \alpha e^{\alpha \hat{a}^\dagger} \right) e^{-\alpha^* \hat{a}} \\ &= e^{-|\alpha|^2} e^{-\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} (\alpha + \hat{a}) \\ &= \alpha + \hat{a}\end{aligned}$$

Taking the dagger of this equation on both sides we can also see that,

$$\hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) = \alpha^* + \hat{a}^\dagger$$

Now substituting these expressions for $\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha)$ and $\hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha)$ into our expression for $\langle n \rangle$ we get,

$$\begin{aligned}\langle n \rangle &= \langle 0 | \hat{S}(\zeta)^\dagger (\hat{a}^\dagger + \alpha^*) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger (\hat{a} + \alpha) \hat{S}(\zeta) | 0 \rangle \\ &= \langle 0 | (\hat{S}(\zeta)^\dagger \hat{a}^\dagger \hat{S}(\zeta) + \alpha^*) (\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) + \alpha) | 0 \rangle [\because \hat{S}(\zeta)^\dagger \hat{S}(\zeta) = 1]\end{aligned}$$

So, now we compute $\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta)$. Let us define $A = \frac{1}{2}(\zeta \hat{a}^{\dagger 2} - \zeta^* \hat{a}^2)$. Then,

$$\begin{aligned}\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) &= e^A \hat{a} e^{-A} \\ &= \sum_{n=0}^{\infty} \frac{[A, \hat{a}]_n}{n!}\end{aligned}$$

where $[A, B]_1 = [A, B]$, $[A, B]_2 = [A, [A, B]]$ and so on. Lets compute $[A, \hat{a}]$

$$\begin{aligned}[A, \hat{a}] &= -\frac{\zeta}{2} [\hat{a}, \hat{a}^{\dagger 2}] \\ &= -\frac{\zeta}{2} 2\hat{a}^\dagger \\ &= -\zeta \hat{a}^\dagger\end{aligned}$$

Similarly,

$$\begin{aligned}[A, \hat{a}]_2 &= [A, [A, \hat{a}]] \\ &= -\zeta [A, \hat{a}^\dagger] \\ &= \zeta \frac{\zeta^*}{2} [\hat{a}^2, \hat{a}^\dagger] \\ &= |\zeta|^2 \hat{a}\end{aligned}$$

We can see after this, that the results will be of a similar form when k has the same parity. It can be shown using induction that,

$$[A, \hat{a}]_n = \begin{cases} -\zeta |\zeta|^{n-1} \hat{a}^\dagger & \text{if } n \text{ is odd} \\ |\zeta|^n \hat{a}^\dagger & \text{if } n \text{ is even} \end{cases}$$

Then we can evaluate $\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta)$ to be,

$$\begin{aligned}
\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) &= \sum_{n=0}^{\infty} \frac{1}{n!} [A, \hat{a}]_n \\
&= \hat{a} \sum_{k=0}^{\infty} \frac{|\zeta|^{2k}}{(2k)!} - \hat{a}^\dagger \sum_{k=0}^{\infty} \frac{\zeta |\zeta|^{2k}}{(2k+1)!} \\
&= \hat{a} \cosh(|\zeta|) - \hat{a}^\dagger \frac{\zeta}{|\zeta|} \sum_{k=0}^{\infty} \frac{|\zeta|^{2k+1}}{(2k+1)!} \\
&= \hat{a} \cosh(|\zeta|) - \hat{a}^\dagger \frac{\zeta}{|\zeta|} \sum_{k=0}^{\infty} \frac{|\zeta|^{2k+1}}{(2k+1)!} \\
&= \hat{a} \cosh(|\zeta|) - \hat{a}^\dagger \frac{\zeta}{|\zeta|} \sinh(|\zeta|) \\
&= \hat{a} \cosh(r) - \hat{a}^\dagger e^{i\theta} \sinh(r)
\end{aligned}$$

Taking the dagger of this relation gives us

$$\hat{S}(\zeta)^\dagger \hat{a}^\dagger \hat{S}(\zeta) = \hat{a}^\dagger \cosh(r) - \hat{a} e^{-i\theta} \sinh(r)$$

Substituting these expressions into the expression for $\langle n \rangle$ we get,

$$\begin{aligned}
\langle n \rangle &= \langle 0 | (\hat{a}^\dagger \cosh(r) - \hat{a} e^{-i\theta} \sinh(r) + \alpha^*) (\hat{a} \cosh(r) - \hat{a}^\dagger e^{i\theta} \sinh(r) + \alpha) | 0 \rangle \\
&= \langle 0 | |\alpha|^2 + \hat{a} \hat{a}^\dagger \sinh^2(r) | 0 \rangle \\
&= |\alpha|^2 + \sinh^2(r)
\end{aligned}$$

Hence for a squeezed state, the mean photon number is given by

$$\langle n \rangle = |\alpha|^2 + \sinh^2(r)$$

Variances in Squeezed States

Let us define Y_1 and Y_2 such that, $Y_1 + iY_2 = (X_1 + X_2)e^{-i\frac{\theta}{2}} := \hat{b}$. Then, we have $\hat{b} = \hat{a}e^{-i\frac{\theta}{2}}$. And we also have,

$$\hat{S}(\zeta) = e^{\frac{1}{2}(\hat{b}^2 - \hat{b}^{\dagger 2})}$$

Observe that, $\hat{b}^\dagger \hat{b} = \hat{a}^\dagger \hat{a} = \hat{n}$ and $[\hat{b}^\dagger, \hat{b}] = [\hat{a}, \hat{a}] = 1$ Also, let's define $\beta = \alpha e^{-\frac{\theta}{2}}$

Let's compute δY_1 and δY_2 for the squeezed state $|\alpha, \zeta\rangle$

$$\begin{aligned}
\delta Y_1 &= \langle Y_1^2 \rangle - \langle Y_1 \rangle^2 \\
&= \frac{1}{4} \langle (\hat{b} + \hat{b}^\dagger)^2 \rangle - \langle \hat{b} + \hat{b}^\dagger \rangle^2 \\
&= \frac{1}{4} \left[\langle (\hat{b}^2 + \hat{b}^{\dagger 2} + 2\hat{b}^\dagger \hat{b} + 1)^2 \rangle - (\langle \hat{b} \rangle + \langle \hat{b}^\dagger \rangle)^2 \right] \\
&= \frac{1}{4} \left[\langle \hat{b}^2 \rangle + \langle \hat{b}^{\dagger 2} \rangle + 2\langle \hat{n} \rangle + 1 - (\langle \hat{b} \rangle + \langle \hat{b}^\dagger \rangle)^2 \right]
\end{aligned}$$

First, we'll compute $\langle \hat{b} \rangle$

$$\begin{aligned}
\langle \hat{b} \rangle &= \langle 0 | \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{b} \hat{D}(\alpha) \hat{S}(\zeta) | 0 \rangle \\
&= \langle 0 | \hat{S}(\zeta)^\dagger (\hat{a} + \alpha) | 0 \rangle e^{-i\frac{\theta}{2}} \\
&= \langle 0 | (\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) + \alpha) | 0 \rangle e^{-\frac{\theta}{2}} \\
&= \langle 0 | \hat{a} \cosh(r) - e^{i\theta} \sinh(r) + \alpha | 0 \rangle e^{-\frac{\theta}{2}} \\
&= \alpha e^{-i\frac{\theta}{2}}
\end{aligned}$$

Similarly,

$$\langle \hat{b}^\dagger \rangle = \alpha^* e^{i\frac{\theta}{2}}$$

Now, we compute $\langle \hat{b}^2 \rangle$

$$\begin{aligned}
\langle \hat{b}^2 \rangle &= \langle 0 | \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{b}^2 \hat{D}(\alpha) \hat{S}(\zeta) | 0 \rangle \\
&= \langle 0 | \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{b} \hat{D}(\alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{b} \hat{D}(\alpha) \hat{S}(\zeta) | 0 \rangle \\
&= \langle 0 | \hat{S}(\zeta)^\dagger (\hat{a} + \alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger (\hat{a} + \alpha) \hat{S}(\zeta) | 0 \rangle e^{-i\theta} \\
&= \langle 0 | (\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) + \alpha)^2 | 0 \rangle e^{-i\theta} \\
&= \langle 0 | (\hat{a} \cosh(r) - e^{i\theta} \hat{a}^\dagger \sinh(r) + \alpha)^2 | 0 \rangle e^{-i\theta} \\
&= \langle 0 | -\hat{a} \hat{a}^\dagger e^{i\theta} \cosh(r) \sinh(r) + \alpha^2 | 0 \rangle e^{-i\theta} \\
&= \alpha^2 e^{-i\theta} - \sinh(r) \cosh(r)
\end{aligned}$$

Similarly,

$$\langle \hat{b}^{\dagger 2} \rangle = (\alpha^*)^2 e^{i\theta} - \sinh(r) \cosh(r)$$

Substituting these expressions in the expression for δY_1 we get,

$$\begin{aligned}
\delta Y_1 &= \frac{1}{4} \left(\alpha^2 e^{-i\theta} + (\alpha^*)^2 e^{i\theta} - 2 \sinh(r) \cosh(r) + 2|\alpha|^2 + 2 \sinh^2(r) + 1 - (\alpha e^{-i\frac{\theta}{2}} + \alpha e^{i\frac{\theta}{2}})^2 \right) \\
&= \frac{1}{4} \left((\alpha e^{-i\frac{\theta}{2}} + \alpha e^{i\frac{\theta}{2}})^2 + 1 + 2 \sinh^2(r) - 2 \cosh(r) \sinh(r) - (\alpha e^{-i\frac{\theta}{2}} + \alpha e^{i\frac{\theta}{2}})^2 \right) \\
&= \frac{1}{4} (1 + 2 \sinh^2(r) - 2 \cosh(r) \sinh(r)) \\
&= \frac{1}{4} (\cosh(2r) - \sinh(2r)) \\
&= \frac{1}{4} e^{-2r}
\end{aligned}$$

With a very similar computation we can show that,

$$\delta Y_2 = \frac{1}{4} e^{2r}$$

Second Order Correlation Function for the Squeezed State

Now let us compute the second order correlation function for this state $|\alpha, \zeta\rangle$. We know that,

$$g^{(2)}(0) = \frac{\langle : \hat{n}^2 : \rangle}{\langle \hat{n} \rangle^2}$$

So, we must compute $\langle : \hat{n}^2 : \rangle$

$$\begin{aligned}
\langle : \hat{n}^2 : \rangle &= \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle \\
&= \langle 0 | \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) | 0 \rangle \\
&= \langle 0 | \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) \hat{S}(\zeta)^\dagger \hat{S}(\zeta) \hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) | 0 \rangle \\
&= \langle 0 | \hat{S}(\zeta)^\dagger (\alpha * + \hat{a}^\dagger) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger (\alpha * + \hat{a}^\dagger) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger (\alpha + \hat{a}) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger (\alpha + \hat{a}) \hat{S}(\zeta) \rangle \\
&= \langle 0 | (\hat{a}^\dagger \cosh(r) - e^{-i\theta} \hat{a} \sinh(r) + \alpha^*)^2 (\hat{a} \cosh(r) - e^{i\theta} \hat{a}^\dagger \sinh(r) + \alpha)^2 | 0 \rangle \\
&= \langle 0 | \hat{a} \hat{a}^\dagger \hat{a} \hat{a}^\dagger \sinh^2(r) \cosh^2(r) + \hat{a} \hat{a} \hat{a}^\dagger \hat{a}^\dagger \sinh^4(r) \\
&\quad + \hat{a} \hat{a}^\dagger (2|\alpha|^2 \sinh^2(r) - (\alpha^2 e^{-i\theta} + (\alpha^*)^2 e^{i\theta}) \sinh(r) \cosh(r)) | 0 \rangle \\
&= 2 \sinh^4(r) + \sinh^2(r) \cosh^2(r) + 2|\alpha|^2 \sinh^2(r) - (\alpha^2 e^{-i\theta} + (\alpha^*)^2 e^{i\theta}) \sinh(r) \cosh(r) \\
&= 3 \sinh^4(r) + (1 + 2|\alpha|^2) \sinh^2(r) - \text{Re}(\alpha^2 e^{-i\theta}) \sinh(2r)
\end{aligned}$$

Hence, the correlation function turns out to be

$$g^{(2)}(0) = \frac{3 \sinh^4(r) + (1 + 2|\alpha|^2) \sinh^2(r) - \text{Re}(\alpha^2 e^{-i\theta}) \sinh(2r)}{(|\alpha|^2 + \sinh^2(r))^2}$$