PH3203 Term Paper

Squeezed States of Light

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1 Introduction

Light is fundamentally a quantum field comprising an ensemble of quantized harmonic oscillators, each characterized by two non-commuting quadrature amplitudes whose uncertainties satisfy the Heisenberg bound. In 1983, Walls demonstrated that one may redistribute these fluctuations so as to suppress noise in a chosen quadrature below the coherent-state (standard quantum) limit—a phenomenon now known as "quadrature squeezing." Such squeezed states constitute a nonclassical resource capable of surpassing conventional quantum-noise constraints.

2 Literature Review

Hello this is a literature review

3 Normalized correlation function

PAM Dirac first put forth the quantization of the EM field into decoupled harmonic oscillators. A full derivation can be found in [1]. We use the same operators, just using different notation from his derivations. We have already seen that we quantize our electric field as,

$$\boldsymbol{E}(r,t) = \boldsymbol{E}^{+}(r,t) + \boldsymbol{E}^{-}(r,t) \tag{1}$$

where $\boldsymbol{E}^{+}(r,t)$ is the positive frequency part and $\boldsymbol{E}^{-}(r,t)$ is the negative frequency part. We will define some semblance of what coherence is in Quantum Optics. To do this we define the first order correlation function as

$$G^{(1)}(r_1,t_1;r_2,t_2) = \langle \boldsymbol{E}^{-}(r_1,t_1)\boldsymbol{E}^{+}(r_2,t_2)\rangle = \mathrm{Tr}[\rho \boldsymbol{E}^{-}(r_1,t_1)\boldsymbol{E}^{+}(r_2,t_2)] \tag{2}$$

In general we can define the nth order correlation functions. In order to write notation compactly, let us write $(\boldsymbol{r_j}, t_j) = x_j$. So our nth order correlation function is given to be

$$G^{(n)}(x_1,x_2,...,x_n,x_{n+1},...,x_{2n}) = \langle \boldsymbol{E}^{-}(x_1)...\boldsymbol{E}^{-}(x_n)\boldsymbol{E}^{+}(x_{n+1})...\boldsymbol{E}^{+}(x_{2n})\rangle \tag{3}$$

We use a certain normalization convention for the 1st order correlation function. We define the normalized correlation function as

$$g^{(1)}(x_1,x_2) = \frac{G^{(1)}(x_1,x_2)}{\sqrt{G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2)}} \tag{4}$$

Generalizing this, we can define the normalized nth order normalized correlation function as

Definition 3.1 (nth order normalized correlation function): The nth order normalized correlation function is defined as

$$g^{(n)}(x_1, x_2, ..., x_n, x_{n+1}, ..., x_{2n}) = \frac{G^{(n)}(x_1, x_2, ..., x_n, x_{n+1}, ..., x_{2n})}{\prod_{i=1}^{2n} G^{(1)}(x_i, x_i)} \tag{5}$$

where $G^{(n)}(x_1, x_2, ..., x_n, x_{n+1}, ..., x_{2n}) = \langle \boldsymbol{E}^-(x_1)...\boldsymbol{E}^-(x_n)\boldsymbol{E}^+(x_{n+1})...\boldsymbol{E}^+(x_{2n})\rangle$

Using some properties of the trace, we can show that, specifically that $\mathrm{tr}\left[\rho A^\dagger A\right]$ is positive definite, $|g^1(x_1, x_2)| \leq 1$. Note that this constraint is not for $n \geq 1$, but since we are generalizing from the first order correlation, we still call it a normalized correlation function.

The main point of this article to talk about squeezed states of light. To talk about these states, we need to talk about the second order normalized correlation function. The second order normalized correlation function is defined as

$$g^{(2)}(x_1, x_2, x_3, x_4) = \frac{G^{(2)}(x_1, x_2, x_3, x_4)}{\sqrt{G^{(1)}(x_1, x_1)G^{(1)}(x_2, x_2)G^{(1)}(x_3, x_3)G^{(1)}(x_4, x_4)}} \tag{6}$$

We need to use some properties of the correlation functions.

1. Permutation of the first half $(x_1, ..., x_n)$ and the second half $(x_{n+1}, ..., x_{2n})$ individually, of the correlation function does not change the value of the correlation function. This is because when we quantize the electric field, we end up with a bunch of decoupled harmonic oscillators, so for any two oscillators the commutation relation is $\left[a_i, a_j^\dagger\right] = \delta_{ij}$, so the correlation function is invariant under permutation of the first half and the second half.
2. **pls elaborate on this pt** If the field is nth order coherent it must satisfy the following condition $g^{(j)}\left(x_1, x_2, ..., x_j, x_j, ..., x_1\right) = 1 \quad \forall j \leq n$. Classically we only use first order coherence to mean coherence. If the field is nth order coherent, then we get

$$G^{(j)}\left(x_1, x_2, ..., x_j, x_j, ..., x_1\right) = \prod_{i=1}^{j} G^{(1)}(x_i, x_i) \quad \forall j \leq n \tag{7}$$

Physically this means

3.1 Coherent States

To check for squeezed states we are interested in the second order correlation function. The second order correlation function, with parameters x_1, x_2 is given by

$$\begin{aligned} g^2(x_1, x_2) &= \frac{G^{(2)}(x_1, x_2, x_1, x_2)}{G^{(1)}(x_1, x_1)G^{(1)}(x_2, x_2)} \\ &= \frac{\langle \boldsymbol{E}^-(x_1)\boldsymbol{E}^-(x_2)\boldsymbol{E}^+(x_1)\boldsymbol{E}^+(x_2)\rangle}{G^{(1)}(x_1, x_1)G^{(1)}(x_2, x_2)} \end{aligned} \tag{8}$$

We note that $\boldsymbol{E}^+$ is an anhilation operator which reduces photon number and $\boldsymbol{E}^-$ is a creation operator which increases photon number. So we can write $N_E = \boldsymbol{E}^-\boldsymbol{E}^+$ is a number operator which counts the number of photons. If the electric fields are classical , the number N_E is a representation of the intensity of the light. So we can write the second order correlation function as

$$g^2(x_1, x_2) = \frac{\left\langle : N_{E(x_1)} N_{E(x_2)} : \right\rangle}{\left\langle N_{E(x_1)} \right\rangle \left\langle N_{E(x_2)} \right\rangle} \tag{9}$$

where, $: X :$ represents the normal ordering of the operator X. The normal ordering of an operator is defined as the ordering of the operators such that all the creation operators are to the left of the annihilation operators.

For example, for an operator $M = a^\dagger a b^\dagger b$, the normally ordered operator is

$$: M := a^\dagger b^\dagger a b \tag{10}$$

.

Here, we consider time $t_1 = t$ and $t_2 = t + \tau$, and consider that we have stationary fields. So we can write the second order correlation function for $t_1 = 0, t_2 = \tau$ as

$$g^2(\tau) = g^2(0, \tau) = \frac{\left\langle : N_{E(0)} N_{E(\tau)} : \right\rangle}{\left\langle N_{E(0)} \right\rangle \left\langle N_{E(\tau)} \right\rangle} \tag{11}$$

Claim 3.1.1 (Correlation function for coherent states):

For coherent states of the electric field, which are the eigenstates of the $\hat{a} = \boldsymbol{E}^+$ operator.

$$g^{(2)}(0) = 1 \tag{12}$$

Proof: We see that the coherent states are defined as the eigenstates of the annihilation operator $a = E^+$, where fix the electric field in some polarization direction. So we can write

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{13}$$

where α is a complex number. We can now calculate

$$\begin{aligned} g^{(2)}(0) = \frac{\left\langle : N_{E(x,0)} : \right\rangle}{\left\langle N_{E(x,0)} \right\rangle^2} &= \frac{\left\langle : a^\dagger a a^\dagger a : \right\rangle}{\left\langle a^\dagger a \right\rangle^2} \\ = \frac{\left\langle a^\dagger a^\dagger a a \right\rangle}{\left\langle a^\dagger a \right\rangle^2} &= \frac{\langle \alpha | a^\dagger a^\dagger a a | \alpha \rangle}{|\alpha|^2} = \frac{|\alpha|^2}{|\alpha|^2} = 1 \end{aligned} \tag{14}$$

♣

For short counting times, the time delay in the second order correlation function is $\tau = 0$.

Claim 3.1.2 (Variance):

For sufficiently short counting times, the variance of the photon number distribution $V(n)$ is related to $g^{(2)}(0)$ by the relation

$$\frac{V(n) - \langle n \rangle}{\langle n \rangle^2} = g^{(2)}(0) - 1 \tag{15}$$

Proof: Note that $N_{E(x,0)} = n$, which the photon number operator. Then the variance is given by $V(n) = \langle n^2 \rangle - \langle n \rangle^2$ The second order correlation function is given by

$$g^{(2)}(0) = \frac{\langle : n^2 : \rangle}{\langle n \rangle^2} \tag{16}$$

Let us focus on the numerator, $\langle : n^2 : \rangle = \langle a^\dagger a^\dagger a a \rangle = \langle a^\dagger n a \rangle$. We use the commutator relation $[n, a] = -a$ to get $\langle : n^2 : \rangle = \langle a^\dagger n a \rangle = \langle a^\dagger a^\dagger a a \rangle - \langle a^\dagger a \rangle = \langle n^2 \rangle - \langle n \rangle$ Thus we get, from the definition of variance and the second order correlation function,

$$\begin{aligned} g^{(2)}(0) &= \frac{\langle n^2 \rangle - \langle n \rangle}{\langle n \rangle^2} = \frac{V(n) + \langle n \rangle^2 - \langle n \rangle}{\langle n \rangle^2} \\ \Rightarrow g^{(2)}(0) - 1 &= \frac{V(n) - \langle n \rangle}{\langle n \rangle^2} \end{aligned} \tag{17}$$

♣

We know that photon statistics in the coherent state is poissonian. For poissonian statistics, the variance is given by $V(n) = \langle n \rangle$. So for poissonian statistics, we have $g^{(2)}(0) = 1$. When $g^{(0)} < 1$, we have sub-poissonian statistics and when $g^{(2)}(0) > 1$, we have super-poissonian statistics. Sub-poissonian statistics exhibit an phenomenon called photon antibunching. **Write more about photon antibunching.**

4 Squeezed States

4.1 Introduction to Squeezed States

Before we move onto the core topic of our term paper report, we must give an introduction to what we squeezed states are. The time dependent electric field operator is some specififc polarization direction for one single mode is given by [1]:

$$E(t) = \lambda\left(\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right) \tag{18}$$

where λ is a constant that contains information about the spatial wave functions. The operators $a, a^\dagger$ obey boson commutation relations ($\left[a, a^\dagger\right] = 1$). For more modes, we add up multiple different hilbert spaces of SHO, with different frequencies. We then write

$$a = X_1 + iX_2 \tag{19}$$

where we can see from the SHO equations, X_1 and X_2 are rescaled versions of the position and momentum operators, which obey the commutation relation $[X_1, X_2] = \frac{i}{2}$. We can then write the electric field operator as

$$E(t) = \frac{\lambda}{2}(X_1 \cos(\omega t) + X_2 \sin(\omega t)) \tag{20}$$

where X_1, X_2 are the amplitudes of the two quadratures of the field.

We use the generalized Heisenberg uncertainity principle to define the uncertainty in the two quadratures of the field. We can write the uncertainty in the two quadratures as

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4} \tag{21}$$

where $\Delta X_i = \sqrt{\langle X_i^2 \rangle - \langle X_i \rangle^2}$.

We are interested in a states with minimum uncertainty. To do that we must have, $\Delta X_1 \Delta X_2 = \frac{1}{4}$. A class of these states are the states with $\Delta X_1 = \Delta X_2 = \frac{1}{2}$ and $V(X_1) = V(X_2) = \frac{1}{4}$, where $V(X)$ is the variance of the operator X. These are our coherent states, which are the eigenstates of the anhilation operator in the quantized electric field description.

A larger class is by taking the variance in one quadrature to be lower than $\frac{1}{4}$ and the variance in the other quadrature to be more than that. These class of states are the called the squeezed states. We are interested in squeezed states which can be defined by the condition $V(X_i) < \frac{1}{4}$ where $i = 1$ or 2. We are dealing with a single mode of the field. For multiple modes, it is quite convinient to use a constrant on the normally ordered Variance of on the quadrature phases. We can write the variance of the quadrature phase as $V(X_i) = \langle X_i^2 \rangle - \langle X_i \rangle^2$. Lets define what our normally order variance is. We can write the normally ordered variance as

$$V(X_i) = \langle : X_i^2 : \rangle - \langle : X_i : \rangle^2 \tag{22}$$

where $: A :$ is the normally ordered operator. For our case, we can say

Claim 4.1.1 (Normally ordered variance of the quadrature phase): The normally ordered variance of the quadrature phase is given by

$$: V(X_i) : + \frac{1}{4} = V(X_i) \tag{23}$$

This implies that for squeezed states, $: V(X_i) :< 0$ for $i = 1$ or 2.

Proof: We can write the normally ordered variance as

$$\begin{aligned}
: V(X_i) :&= \langle : X_i^2 : \rangle - \langle : X_i : \rangle^2 \\
&= \frac{1}{4}\Big[\Big\langle : a^2 + \left(a^\dagger\right)^2 + aa^\dagger + a^\dagger a\Big\rangle : \Big) - \langle : a + a^\dagger : \rangle^2\Big] \\
&= \frac{1}{4}\Big[\Big\langle a^2 + \left(a^\dagger\right)^2 + 2a^\dagger a\Big\rangle - \langle : a + a^\dagger : \rangle^2\Big] \\
&= \frac{1}{4}\Big[\Big\langle a^2 + \left(a^\dagger\right)^2 + 2a^\dagger a + 1\Big\rangle - \langle a + a^\dagger \rangle^2 - 1\Big] \\
&= V(X_i) - \frac{1}{4}
\end{aligned} \tag{24}$$

Thus we have $: V(X_i) : + \frac{1}{4} = V(X_i)$, which implies that for squeezed states, $: V(X_i) :< 0$ for $i = 1$ or 2.

♣

We now see how these states are generated and what some of their mathematical properties are,

A light field with sub-poissonian statistics will exhibit an effect known as photon antibunching. The Glauber Sudarshan P representation of the density operator for a light field is given by

$$\rho = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha| \tag{25}$$

where $P(\alpha)$ is the Glauber-Sudarshan P function, and $d^2\alpha = d\,\mathrm{Re}(\alpha)d\,\mathrm{Im}(\alpha)$. More about this is given in Appendix A.

The quantum mechanical average resemble classical averaging procedures when the P function is positive non singular function. We show that photon antibunching is a quantum effect by showing there is no positive non singular P function. Note that we generate squeezed states by reducing the variance in one quadrature, which we choose here to be X_1.

Claim 4.1.2 (Variance of Squeezed States(P representation)):

The Variance of the X_1 operator. the glauber sudarshan P representation of the squeezed state is given by

$$V(X_1) = \frac{1}{4}\left[1 + \int d^2\alpha P(\alpha)\left\{(\alpha+\alpha^*) - \left(\langle a\rangle + \langle a^\dagger\rangle\right)\right\}^2\right] \tag{26}$$

Proof: We can write the variance of the X_1 operator as $V(X_1) = \langle X_1^2\rangle - \langle X_1\rangle^2$ where $\langle A\rangle = \mathrm{Tr}(\rho A)$. We can write the density operator in the Glauber-Sudarshan P representation as

$$\rho = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha| \tag{27}$$

Recall the completeness relation $\sum_n |n\rangle\langle n| = \mathbb{I}$. We can write the expectation value of the X_1 operator as

$$\begin{aligned}
\langle X_1\rangle = \mathrm{Tr}(\rho X_1) &= \sum_n \int d^2\alpha P(\alpha)\langle n|\alpha\rangle\langle\alpha|X_1|n\rangle \\
&= \sum_n \int d^2\alpha P(\alpha)\langle\alpha|X_1|n\rangle\langle n|\alpha\rangle = \int d^2\alpha P(\alpha)\langle\alpha|X_1|\alpha\rangle \\
&= \frac{1}{2}\int d^2\alpha P(\alpha)(\alpha+\alpha^*) \\
&= \frac{1}{2}\left[\langle a\rangle + \langle a^\dagger\rangle\right]
\end{aligned} \tag{28}$$

where the last equality is obtained by taking $\langle a\rangle = \mathrm{Tr}(\rho a) = \int d^2\alpha P(\alpha)\langle\alpha|a|\alpha\rangle = \int d^\alpha P(\alpha)\alpha$. We now find the expectation value of the X_1^2 operator. We can write the expectation value of the X_1^2 operator as

$$\begin{aligned}
\langle X_1^2\rangle = \mathrm{Tr}(\rho X_1^2) &= \sum_n \int d^2\alpha P(\alpha)\langle n|\alpha\rangle\langle\alpha|X_1^2|n\rangle \\
&= \sum_n \int d^2\alpha P(\alpha)\langle n|X_1^2|\alpha\rangle\langle\alpha|n\rangle \\
&= \int d^2\alpha P(\alpha)\langle\alpha|X_1^2|\alpha\rangle \\
&= \frac{1}{4}\int d^2\alpha P(\alpha)\left\langle\alpha\middle|a^2 + aa^\dagger + a^\dagger a + \left(a^\dagger\right)^2\middle|\alpha\right\rangle \\
&= \frac{1}{4}\int d^2\alpha P(\alpha)\left[\alpha^2 + \left(\alpha^*\right)^2 + 2\alpha\alpha^* + 1\right]
\end{aligned} \tag{29}$$

We note that the P-function is a quasiprobability distribution, which doesnt satisfy all the Kolmogorov axioms, but still satifies $\int P(\alpha)d^2\alpha = 1$. Then the variance is given by

$$\begin{aligned}
V(X_1) = \langle X_1^2\rangle - \langle X_1\rangle^2 &= \frac{1}{4}\left[1 + \int d^2\alpha P(\alpha)\left(\alpha^2 + \left(\alpha^*\right)^2 + 2\ |\alpha|^2\right) - \left(\langle a\rangle + \left\langle a^\dagger\right\rangle\right)^2\right] \\
&= \frac{1}{4}\left[1 + \int d^2\alpha P(\alpha)\left(\alpha + \alpha^*\right)^2\right] + \left(\langle a\rangle + \left\langle a^\dagger\right\rangle\right)^2 - 2\left(\langle a\rangle + \left\langle a^\dagger\right\rangle\right)\left(\langle a\rangle + \left\langle a^\dagger\right\rangle\right) \\
&= \frac{1}{4}[1 + \int d^2\alpha P(\alpha)\left[\left(\alpha + \alpha^*\right)^2 + \left(\langle a\rangle + \left\langle a^\dagger\right\rangle\right)^2 - 2\left(\alpha + \alpha^*\right)\left(\langle a\rangle + \left\langle a^\dagger\right\rangle\right]\right) \\
&= \frac{1}{4}\left[1 + \int d^2\alpha P(\alpha)\left\{\left(\alpha + \alpha^*\right) - \left(\langle a\rangle + \left\langle a^\dagger\right\rangle\right)\right\}^2\right]
\end{aligned} \tag{30}$$

We know that a squeezed has $V(X_1) < \frac{1}{4}$ in one quadrature, and $V(X_2) > \frac{1}{4}$ in the other quadrature. We can see that the squeezed state has a negative P function, which is a signature of nonclassicality.

4.2 Generating Squeezed States

We denote a coherent state as $|\alpha\rangle$. We know that a coherent state can be generated from the vaccuum state using the displacement operator $D(\alpha)$,

$$|\alpha\rangle = D(\alpha)|0\rangle = \exp\left(\alpha a^\dagger - \alpha^* a\right)|0\rangle \tag{31}$$

A squeezed state $|\alpha,\zeta\rangle$ can be generated by applying the squeezed operator and then the displacement operator on the ground state of the Simple Harmonic Oscillator.

4.2.1 Definition of Squeezed States

Squeezed States are defined as the state obtained by the action of the operator $\hat{D}(\alpha)\hat{S}(\zeta)$ on the vaccum number state $|0\rangle$

$$|\alpha, \zeta\rangle = \hat{D}(\alpha)\hat{S}(\zeta)|0\rangle \tag{32}$$

Where the concerned operators are defined as,

$$\begin{aligned}
\hat{D}(\alpha) &= e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} \\
\hat{S}(\zeta) &= e^{\frac{1}{2}\left(\zeta^*\hat{a}^2 - \zeta\hat{a}^{\dagger 2}\right)}
\end{aligned} \tag{33}$$

where, $\zeta = re^{i\theta}$ with $r > 0$.

$\hat{D}(\alpha)$ and $\hat{S}(\zeta)$ are the translation and the squeeze operator respectively. It can be shown that,

$$\hat{D}(\alpha)^\dagger \hat{D}(\alpha) = \hat{S}(\zeta)^\dagger \hat{S}(\zeta) = 1 \tag{34}$$

i.e., both the operators are unitary. This will help us a lot in the following section, where we will be looking at the expectations and variances of a few relevant quantities.

4.2.2 Mean Photon Number in a Squeezed State

We wish to compute $\langle \hat{n} \rangle$ for the squeezed state $|\alpha, \zeta\rangle$

We know that

$$\hat{n} = \hat{a}^\dagger \hat{a} \tag{35}$$

Now, to compute the epectation of $\hat{n}$ in the Squeezed state we have to evaluate the following expression

$$\begin{aligned}
\langle \hat{n} \rangle &= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) \hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \qquad \left[\because \hat{D}(\alpha)^\dagger \hat{D}(\alpha) = 1\right] \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \left[\because \hat{S}(\zeta)^\dagger \hat{S}(\zeta) = 1\right]
\end{aligned} \tag{36}$$

First, we evaluate the operator $\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha)$

$$\begin{aligned}
\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) &= e^{-|\alpha|^2} e^{-\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \hat{a} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \\
&= e^{-|\alpha|^2} e^{-\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \left(e^{\alpha \hat{a}^\dagger} \hat{a} + \left[\hat{a}, e^{\alpha \hat{a}^\dagger}\right] \right) e^{-\alpha^* \hat{a}}
\end{aligned} \tag{37}$$

Now let us compute the comutator relation $\left[\hat{a}, e^{\alpha \hat{a}^\dagger}\right]$ which is given by,

$$\left[\hat{a}, e^{\alpha \hat{a}^\dagger}\right] = \sum_{n=1}^{\infty} \frac{\alpha^n \left[\hat{a}, \hat{a}^{\dagger n}\right]}{n!} \tag{38}$$

We can easily show by induction that, $\left[\hat{a}, \hat{a}^{\dagger n}\right] = n \hat{a}^{\dagger n-1}$. Then the commutator evaluates to,

$$\begin{aligned}
\left[\hat{a}, e^{\alpha \hat{a}^\dagger}\right] &= \sum_{n=1}^{\infty} \frac{n \alpha^n \hat{a}^{\dagger n-1}}{n!} \\
&= \alpha \sum_{n=0}^{\infty} \frac{\left(\alpha \hat{a}^\dagger\right)^n}{n!} \\
&= \alpha e^{\alpha \hat{a}^\dagger}
\end{aligned} \tag{39}$$

Substituting this commutator relation back into the expression for $\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha)$ we get,

$$\begin{aligned}
\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) &= e^{-|\alpha|^2} e^{-\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \left(e^{\alpha \hat{a}^\dagger} \hat{a} + \alpha e^{\alpha \hat{a}^\dagger} \right) e^{-\alpha^* \hat{a}} \\
&= e^{-|\alpha|^2} e^{-\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} (\alpha + \hat{a}) \\
&= \alpha + \hat{a}
\end{aligned} \tag{40}$$

Taking the dagger of this equation on both sides we can also see that,

$$\hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) = \alpha^* + \hat{a}^\dagger \tag{41}$$

Now substituting these expressions for $\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha)$ and $\hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha)$ into our expression for $\langle \hat{n} \rangle$ we get,

$$\begin{aligned}\langle\hat{n}\rangle &= \left\langle 0\middle|\hat{S}(\zeta)^\dagger(\hat{a}^\dagger+\alpha^*)\hat{S}(\zeta)\hat{S}(\zeta)^\dagger(\hat{a}+\alpha)\hat{S}(\zeta)\middle|0\right\rangle \\ &= \left\langle 0\middle|\left(\hat{S}(\zeta)^\dagger\hat{a}^\dagger\hat{S}(\zeta)+\alpha^*\right)\left(\hat{S}(\zeta)^\dagger\hat{a}\hat{S}(\zeta)+\alpha\right)\middle|0\right\rangle \left[\because \hat{S}(\zeta)^\dagger\hat{S}(\zeta)=1\right]\end{aligned} \tag{42}$$

So, now we compute $\hat{S}(\zeta)^\dagger\hat{a}\hat{S}(\zeta)$. Let us define $A=\frac{1}{2}\left(\zeta{\hat{a}^\dagger}^2-\zeta^*\hat{a}^2\right)$. Then,

$$\begin{aligned}\hat{S}(\zeta)^\dagger\hat{a}\hat{S}(\zeta) &= e^A\hat{a}e^{-A} \\ &= \sum_{n=0}^{\infty}\frac{[A,\hat{a}]_n}{n!}\end{aligned} \tag{43}$$

where $[A,B]_1=[A,B]$, $[A,B]_2=[A,[A,B]]$ and so on. Lets compute $[A,\hat{a}]$

$$\begin{aligned}[A,\hat{a}] &= -\frac{\zeta}{2}\left[\hat{a},{\hat{a}^\dagger}^2\right] \\ &= -\frac{\zeta}{2}2\hat{a}^\dagger \\ &= -\zeta\hat{a}^\dagger\end{aligned} \tag{44}$$

Similarly,

$$\begin{aligned}[A,\hat{a}]_2 &= [A,[A,\hat{a}]] \\ &= -\zeta\left[A,\hat{a}^\dagger\right] \\ &= \zeta\frac{\zeta^*}{2}\left[\hat{a}^2,\hat{a}^\dagger\right] \\ &= |\zeta|^2\hat{a}\end{aligned} \tag{45}$$

We can see after this, that the results will be of a similar form when k has the same parity. It can be shown using induction that,

$$[A,\hat{a}]_n = \begin{cases} -\zeta|\zeta|^{n-1}\hat{a}^\dagger & \text{if } n \text{ is odd} \\ |\zeta|^n\hat{a}^\dagger & \text{if } n \text{ is even}\end{cases} \tag{46}$$

Then we can evaluate $\hat{S}(\zeta)^\dagger\hat{a}\hat{S}(\zeta)$ to be,

$$\begin{aligned}\hat{S}(\zeta)^\dagger\hat{a}\hat{S}(\zeta) &= \sum_{n=0}^{\infty}\frac{1}{n!}[A,\hat{a}]_n \\ &= \hat{a}\sum_{k=0}^{\infty}\frac{|\zeta|^{2k}}{(2k)!} - \hat{a}^\dagger\sum_{k=0}^{\infty}\frac{\zeta|\zeta|^{2k}}{(2k+1)!} \\ &= \hat{a}\cosh(|\zeta|) - \hat{a}^\dagger\frac{\zeta}{|\zeta|}\sum_{k=0}^{\infty}\frac{|\zeta|^{2k+1}}{(2k+1)!} \\ &= \hat{a}\cosh(|\zeta|) - \hat{a}^\dagger\frac{\zeta}{|\zeta|}\sum_{k=0}^{\infty}\frac{|\zeta|^{2k+1}}{(2k+1)!} \\ &= \hat{a}\cosh(|\zeta|) - \hat{a}^\dagger\frac{\zeta}{|\zeta|}\sinh(|\zeta|) \\ &= \hat{a}\cosh(r) - \hat{a}^\dagger e^{i\theta}\sinh(r)\end{aligned} \tag{47}$$

Taking the dagger of this relation gives us

$$\hat{S}(\zeta)^\dagger \hat{a}^\dagger \hat{S}(\zeta) = \hat{a}^\dagger \cosh(r) - \hat{a}e^{-i\theta}\sinh(r) \tag{48}$$

Substituting these expressions into the expression for $\langle \hat{n} \rangle$ we get,

$$\begin{aligned}\langle \hat{n} \rangle &= \big\langle 0 \big| \big(\hat{a}^\dagger \cosh(r) - \hat{a}e^{-i\theta}\sinh(r) + \alpha^*\big)\big(\hat{a}\cosh(r) - \hat{a}^\dagger e^{i\theta}\sinh(r) + \alpha\big) \big| 0 \big\rangle \\ &= \big\langle 0 \big| |\alpha|^2 + \hat{a}\hat{a}^\dagger \sinh^2(r) \big| 0 \big\rangle \\ &= |\alpha|^2 + \sinh^2(r)\end{aligned} \tag{49}$$

Hence for a squeezed state, the mean photon number is given by

$$\langle \hat{n} \rangle = |\alpha|^2 + \sinh^2(r) \tag{50}$$

4.2.3 Variances in Squeezed States

Let us define Y_1 and Y_2 such that, $Y_1 + iY_2 = (X_1 + X_2)e^{-i\frac{\theta}{2}} := \hat{b}$. Then, we have $\hat{b} = \hat{a}e^{-i\frac{\theta}{2}}$. And we also have,

$$\hat{S}(\zeta) = e^{\frac{1}{2}\left(\hat{b}^2 - \hat{b}^{\dagger 2}\right)} \tag{51}$$

Observe that, $\hat{b}^\dagger \hat{b} = \hat{a}^\dagger \hat{a} = \hat{n}$ and $\left[\hat{b}^\dagger, \hat{b}\right] = \left[\hat{a}^\dagger, \hat{a}\right] = 1$ Also, lets define $\beta = \alpha e^{-\frac{\theta}{2}}$

Let's compute δY_1 and δY_2 for the squeezed state $|\alpha, \zeta\rangle$

$$\begin{aligned}\delta Y_1 &= \langle Y_1^2 \rangle - \langle Y_1 \rangle^2 \\ &= \frac{1}{4}\left\langle \left(\hat{b} + \hat{b}^\dagger\right)^2 \right\rangle - \left\langle \hat{b} + \hat{b}^\dagger \right\rangle^2 \\ &= \frac{1}{4}\left[\left\langle \left(\hat{b}^2 + \hat{b}^{\dagger 2} + 2\hat{b}^\dagger \hat{b} + 1\right)^2 \right\rangle - \left(\left\langle \hat{b} \right\rangle + \left\langle \hat{b}^\dagger \right\rangle\right)^2\right] \\ &= \frac{1}{4}\left[\left\langle \hat{b}^2 \right\rangle + \left\langle \hat{b}^{\dagger 2} \right\rangle + 2\langle \hat{n} \rangle + 1 - \left(\left\langle \hat{b} \right\rangle + \left\langle \hat{b}^\dagger \right\rangle\right)^2\right]\end{aligned} \tag{52}$$

First, we'll compute $\left\langle \hat{b} \right\rangle$

$$\begin{aligned}\left\langle \hat{b} \right\rangle &= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{b} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\ &= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger (\hat{a} + \alpha) \middle| 0 \right\rangle e^{-i\frac{\theta}{2}} \\ &= \left\langle 0 \middle| \left(\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) + \alpha\right) \middle| 0 \right\rangle e^{-\frac{\theta}{2}} \\ &= \left\langle 0 \middle| \hat{a}\cosh(r) - e^{i\theta}\sinh(r) + \alpha \middle| 0 \right\rangle e^{-\frac{\theta}{2}} \\ &= \alpha e^{-i\frac{\theta}{2}}\end{aligned} \tag{53}$$

Similarly,

$$\left\langle \hat{b}^\dagger \right\rangle = \alpha^* e^{i\frac{\theta}{2}} \tag{54}$$

Now, we compute $\left\langle \hat{b}^2 \right\rangle$

$$\begin{aligned}
\left\langle \hat{b}^2 \right\rangle &= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{b}^2 \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{b} \hat{D}(\alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{b} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger (\hat{a}+\alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger (\hat{a}+\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle e^{-i\theta} \\
&= \left\langle 0 \middle| \left(\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) + \alpha \right)^2 \middle| 0 \right\rangle e^{-i\theta} \\
&= \left\langle 0 \middle| \left(\hat{a}\cosh(r) - e^{i\theta} \hat{a}^\dagger \sinh(r) + \alpha \right)^2 \middle| 0 \right\rangle e^{-i\theta} \\
&= \left\langle 0 \middle| -\hat{a}\hat{a}^\dagger e^{i\theta} \cosh(r)\sinh(r) + \alpha^2 \middle| 0 \right\rangle e^{-i\theta} \\
&= \alpha^2 e^{-i\theta} - \sinh(r)\cosh(r)
\end{aligned} \tag{55}$$

Similarly,

$$\left\langle \hat{b}^{\dagger 2} \right\rangle = (\alpha^*)^2 e^{i\theta} - \sinh(r)\cosh(r) \tag{56}$$

Substituting these expressions in the expression for δY_1 we get,

$$\begin{aligned}
\delta Y_1 &= \frac{1}{4}\left(\alpha^2 e^{-i\theta} + (\alpha^*)^2 e^{i\theta} - 2\sinh(r)\cosh(r) + 2|\alpha|^2 + 2\sinh^2(r) + 1 - \left(\alpha e^{-i\frac{\theta}{2}} + \alpha e^{i\frac{\theta}{2}} \right)^2 \right) \\
&= \frac{1}{4}\left(\left(\alpha e^{-i\frac{\theta}{2}} + \alpha e^{i\frac{\theta}{2}} \right)^2 + 1 + 2\sinh^2(r) - 2\cosh(r)\sinh(r) - \left(\alpha e^{-i\frac{\theta}{2}} + \alpha e^{i\frac{\theta}{2}} \right)^2 \right) \\
&= \frac{1}{4}(1 + 2\sinh^2(r) - 2\cosh(r)\sinh(r)) \\
&= \frac{1}{4}(\cosh(2r) - \sin(2r)) \\
&= \frac{1}{4}e^{-2r}
\end{aligned} \tag{57}$$

With a very similar computation we can show that,

$$\delta Y_2 = \frac{1}{4}e^{2r} \tag{58}$$

4.2.4 Second Order Correlation Function for the Squeezed State

Now let us compute the second order correlation function for this state $|\alpha, \zeta\rangle$ We know that,

$$g^{(2)}(0) = \frac{\langle : \hat{n}^2 : \rangle}{\langle \hat{n} \rangle^2} \tag{59}$$

So, we must compute $\langle : \hat{n}^2 : \rangle$

$$
\begin{aligned}
\langle :\hat{n}^2: \rangle &= \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a} \rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a} \hat{D}(\alpha)\hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha)\hat{S}(\zeta)\hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha)\hat{S}(\zeta)^\dagger \hat{S}(\zeta)\hat{D}(\alpha)^\dagger \hat{a}\hat{D}(\alpha)\hat{S}(\zeta)\hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}\hat{D}(\alpha)\hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger (\alpha * + \hat{a}^\dagger)\hat{S}(\zeta)\hat{S}(\zeta)^\dagger (\alpha * + \hat{a}^\dagger)\hat{S}(\zeta)\hat{S}(\zeta)^\dagger (\alpha + \hat{a})\hat{S}(\zeta)\hat{S}(\zeta)^\dagger (\alpha + \hat{a})\hat{S}(\zeta) \right\rangle \\
&= \left\langle 0 \middle| \left(\hat{a}^\dagger \cosh(r) - e^{-i\theta}\hat{a}\sinh(r) + \alpha^*\right)^2 \left(\hat{a}\cosh(r) - e^{i\theta}\hat{a}^\dagger \sinh(r) + \alpha\right)^2 \middle| 0 \right\rangle \qquad (60) \\
&= \left\langle 0 \middle| \hat{a}\hat{a}^\dagger \hat{a}\hat{a}^\dagger \sinh^2(r)\cosh^2(r) + \hat{a}\hat{a}\hat{a}^\dagger \hat{a}^\dagger \sinh^4(r) \right. \\
&\quad \left. + \hat{a}\hat{a}^\dagger \left(2|\alpha|^2 \sinh^2(r) - \left(\alpha^2 e^{-i\theta} + (\alpha^*)^2 e^{i\theta}\right)\sinh(r)\cosh(r)\right) \middle| 0 \right\rangle \\
&= 2\sinh^4(r) + \sinh^2(r)\cosh^2(r) + 2|\alpha|^2 \sinh^2(r) - \left(\alpha^2 e^{-i\theta} + (\alpha^*)^2 e^{i\theta}\right)\sinh(r)\cosh(r) \\
&= 3\sinh^4(r) + (1 + 2|\alpha|^2)\sinh^2(r) - \mathrm{Re}\left(\alpha^2 e^{-i\theta}\right)\sinh(2r)
\end{aligned}
$$

Hence, the correlation function turns out to be

$$
g^{(2)}(0) = \frac{3\sinh^4(r) + (1 + 2|\alpha|^2)\sinh^2(r) - \mathrm{Re}\left(\alpha^2 e^{-i\theta}\right)\sinh(2r)}{\left(|\alpha|^2 + \sinh^2(r)\right)^2} \qquad (61)
$$

5 Production of Squeezed States

6 Detection of Squeezed States

7 Applications of Squeezed States

Bibliography

[1] P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th ed. Oxford University Press, 1958.

A : Glauber Sudarshan P function

B : Correlation functions

C : Photon antibunching
