Definition of Squeezed States

Squeezed States are defined as the state obtained by the action of the operator $\hat{S}(\zeta)\hat{D}(\alpha)$ on the vaccum number state $|0\rangle$

$$|\alpha, \zeta\rangle = \hat{S}(\zeta)\hat{D}(\alpha)|0\rangle$$

Where the concerned operators are defined as,

$$\hat{D}(\alpha) = e^{-\frac{|\alpha|^2}{2}} e^{\alpha\hat{a}^\dagger} e^{-\alpha^* \hat{a}}$$
$$\hat{S}(\zeta) = e^{\frac{1}{2}\left(\zeta^* \hat{a}^2 - \zeta {\hat{a}^\dagger}^2\right)}$$

$\hat{D}(\alpha)$ and $\hat{S}(\zeta)$ are the translation and the squeeze operator respectively. It can be shown that,

$$\hat{D}(\alpha)^\dagger \hat{D}(\alpha) = \hat{S}(\zeta)^\dagger \hat{S}(\zeta) = 1$$

i.e., both the operators are unitary. This will help us a lot in the following section, where we will be looking at the expectations and variances of a few relevant quantities.

Variances in Squeezed States

δY_1 TODO

Mean Photon Number in a Squeezed State

We wish to compute $\langle n \rangle$ for the squeezed state $|\alpha, \zeta\rangle$

We know that

$$\hat{n} = \hat{a}^\dagger \hat{a}$$

Now, to compute the epectation of $\hat{n}$ in the Squeezed state we have to evaluate the following expression

$$\begin{aligned}
\langle n \rangle &= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{a} \hat{S}(\zeta) \hat{D}(\alpha) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) \hat{D}(\alpha)^\dagger \hat{a} \hat{S}(\zeta) \hat{D}(\alpha) \middle| 0 \right\rangle \qquad \left[\because \hat{D}(\alpha)^\dagger \hat{D}(\alpha) = 1\right] \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a} \hat{S}(\zeta) \hat{D}(\alpha) \middle| 0 \right\rangle \left[\because \hat{S}(\zeta)^\dagger \hat{S}(\zeta) = 1\right]
\end{aligned}$$

First, we evaluate the operator $\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha)$

$$\begin{aligned}
\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) &= e^{-|\alpha|^2} e^{-\alpha\hat{a}^\dagger} e^{-\alpha^* \hat{a}} \hat{a} e^{\alpha\hat{a}^\dagger} e^{-\alpha^* \hat{a}} \\
&= e^{-|\alpha|^2} e^{-\alpha\hat{a}^\dagger} e^{-\alpha^* \hat{a}} \left(e^{\alpha\hat{a}^\dagger} \hat{a} + \left[\hat{a}, e^{\alpha\hat{a}^\dagger}\right] \right) e^{-\alpha^* \hat{a}}
\end{aligned}$$

Now let us compute the comutator relation $\left[\hat{a}, e^{\alpha\hat{a}^\dagger}\right]$ which is given by,

$$\left[\hat{a}, e^{\alpha\hat{a}^\dagger}\right] = \sum_{n=1}^{\infty} \frac{\alpha^n \left[\hat{a}, {\hat{a}^\dagger}^n\right]}{n!}$$

We can easily show by induction that, $\left[\hat{a}, {\hat{a}^\dagger}^n\right] = n{\hat{a}^\dagger}^{n-1}$. Then the commutator evaluates to,

$$
\begin{aligned}
\left[\hat{a}, e^{\alpha\hat{a}^\dagger}\right] &= \sum_{n=1}^{\infty} \frac{n\alpha^n {\hat{a}^\dagger}^{n-1}}{n!} \\
&= \alpha \sum_{n=0}^{\infty} \frac{\left(\alpha\hat{a}^\dagger\right)^n}{n!} \\
&= \alpha e^{\alpha\hat{a}^\dagger}
\end{aligned}
$$

Substituting this commutator relation back into the expression for $\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha)$ we get,

$$
\begin{aligned}
\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) &= e^{-|\alpha|^2} e^{-\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} \left(e^{\alpha\hat{a}^\dagger}\hat{a} + \alpha e^{\alpha\hat{a}^\dagger}\right) e^{-\alpha^*\hat{a}} \\
&= e^{-|\alpha|^2} e^{-\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} (\alpha + \hat{a}) \\
&= \alpha + \hat{a}
\end{aligned}
$$

Taking the dagger of this equation on both sides we can also see that,

$$
\hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) = \alpha^* + \hat{a}^\dagger
$$

Now substituting these expressions for $\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha)$ and $\hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha)$ into our expression for $\langle n \rangle$ we get,

$$
\begin{aligned}
\langle n \rangle &= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \left(\hat{a}^\dagger + \alpha^*\right) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger (\hat{a} + \alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \left(\hat{S}(\zeta)^\dagger \hat{a}^\dagger \hat{S}(\zeta) + \alpha^*\right) \left(\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) + \alpha\right) \middle| 0 \right\rangle \left[\because \hat{S}(\zeta)^\dagger \hat{S}(\zeta) = 1\right]
\end{aligned}
$$

So, now we compute $\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta)$. Let us define $A = \frac{1}{2}\left(\zeta {\hat{a}^\dagger}^2 - \zeta^* \hat{a}^2\right)$. Then,

$$
\begin{aligned}
\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) &= e^A \hat{a} e^{-A} \\
&= \sum_{n=0}^{\infty} \frac{[A, \hat{a}]_n}{n!}
\end{aligned}
$$

where $[A, B]_1 = [A, B]$, $[A, B]_2 = [A, [A, B]]$ and so on. Lets compute $[A, \hat{a}]$

$$
\begin{aligned}
[A, \hat{a}] &= -\frac{\zeta}{2}\left[\hat{a}, {\hat{a}^\dagger}^2\right] \\
&= -\frac{\zeta}{2} 2\hat{a}^\dagger \\
&= -\zeta\hat{a}^\dagger
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
[A, \hat{a}]_2 &= [A, [A, \hat{a}]] \\
&= -\zeta\left[A, \hat{a}^\dagger\right] \\
&= \zeta\frac{\zeta^*}{2}\left[\hat{a}^2, \hat{a}^\dagger\right] \\
&= |\zeta|^2 \hat{a}
\end{aligned}
$$

We can see after this, that the results will be of a similar form when k has the same parity. It can be shown using induction that,

$$
[A, \hat{a}]_n = \begin{cases} -\zeta|\zeta|^{n-1}\hat{a}^\dagger & \text{if } n \text{ is odd} \\ |\zeta|^n \hat{a}^\dagger & \text{if } n \text{ is even} \end{cases}
$$

Then we can evaluate $\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta)$ to be,

$$
\begin{aligned}
\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) &= \sum_{n=0}^{\infty} \frac{1}{n!} [A, \hat{a}]_n \\
&= \hat{a} \sum_{k=0}^{\infty} \frac{|\zeta|^{2k}}{(2k)!} - \hat{a}^\dagger \sum_{k=0}^{\infty} \frac{\zeta |\zeta|^{2k}}{(2k+1)!} \\
&= \hat{a} \sinh(|\zeta|) - \hat{a}^\dagger \frac{\zeta}{|\zeta|} \sum_{k=0}^{\infty} \frac{|\zeta|^{2k+1}}{(2k+1)!} \\
&= \hat{a} \sinh(|\zeta|) - \hat{a}^\dagger \frac{\zeta}{|\zeta|} \sum_{k=0}^{\infty} \frac{|\zeta|^{2k+1}}{(2k+1)!} \\
&= \hat{a} \sinh(|\zeta|) - \hat{a}^\dagger \frac{\zeta}{|\zeta|} \cosh(|\zeta|) \\
&= \hat{a} \sinh(r) - \hat{a}^\dagger e^{i\theta} \cosh(r)
\end{aligned}
$$

Taking the dagger of this relation gives us

$$\hat{S}(\zeta)^\dagger \hat{a}^\dagger \hat{S}(\zeta) = \hat{a}^\dagger \sinh(r) - \hat{a} e^{-i\theta} \cosh(r)$$

Substituting these expressions into the expression for $\langle n \rangle$ we get,

$$
\begin{aligned}
\langle n \rangle &= \left\langle 0 \middle| \left(\hat{a}^\dagger \sinh(r) - \hat{a} e^{-i\theta} \cosh(r) + \alpha^* \right) \left(\hat{a} \sinh(r) - \hat{a}^\dagger e^{i\theta} \cosh(r) + \alpha \right) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| |\alpha|^2 + \hat{a} \hat{a}^\dagger \sinh^2(r) \middle| 0 \right\rangle \\
&= |\alpha|^2 + \sinh^2(r)
\end{aligned}
$$

Hence for a squeezed state, the mean photon number is given by

$$\langle n \rangle = |\alpha|^2 + \sinh^2(r)$$