PH3203 Term Paper

Squeezed States of Light

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1 Introduction

Light is fundamentally a quantum field comprising an ensemble of quantized harmonic oscillators, each characterized by two non-commuting quadrature amplitudes whose uncertainties satisfy the Heisenberg bound. In 1983, Walls demonstrated that one may redistribute these fluctuations so as to suppress noise in a chosen quadrature below the coherent-state (standard quantum) limit—a phenomenon now known as "quadrature squeezing." Such squeezed states constitute a nonclassical resource capable of surpassing conventional quantum-noise constraints.

2 Literature Review

Hello this is a literature review

3 Normalized correlation function

PAM Dirac first put forth the quantization of the EM field into decoupled harmonic oscillators. A full derivation can be found in [1]. We use the same operators, just using different notation from his derivations. We have already seen that we quantize our electric field as,

$$\boldsymbol{E}(r,t) = \boldsymbol{E}^{+}(r,t) + \boldsymbol{E}^{-}(r,t) \tag{1}$$

where $\boldsymbol{E}^{+}(r,t)$ is the positive frequency part and $\boldsymbol{E}^{-}(r,t)$ is the negative frequency part. We will define some semblance of what coherence is in Quantum Optics. To do this we define the first order correlation function as

$$G^{(1)}(r_1,t_1;r_2,t_2) = \langle \boldsymbol{E}^{-}(r_1,t_1)\boldsymbol{E}^{+}(r_2,t_2)\rangle = \mathrm{Tr}[\rho \boldsymbol{E}^{-}(r_1,t_1)\boldsymbol{E}^{+}(r_2,t_2)] \tag{2}$$

In general we can define the nth order correlation functions. In order to write notation compactly, let us write $\left(\boldsymbol{r_j}, t_j\right) = x_j$. So our nth order correlation function is given to be

$$G^{(n)}(x_1,x_2,...,x_n,x_{n+1},...,x_{2n}) = \langle \boldsymbol{E}^{-}(x_1)...\boldsymbol{E}^{-}(x_n)\boldsymbol{E}^{+}(x_{n+1})...\boldsymbol{E}^{+}(x_{2n})\rangle \tag{3}$$

We use a certain normalization convention for the 1st order correlation function. We define the normalized correlation function as

$$g^{(1)}(x_1,x_2) = \frac{G^{(1)}(x_1,x_2)}{\sqrt{G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2)}} \tag{4}$$

Generalizing this, we can define the normalized nth order normalized correlation function as

Definition 3.1 (nth order normalized correlation function): The nth order normalized correlation function is defined as

$$g^{(n)}(x_1, x_2, ..., x_n, x_{n+1}, ..., x_{2n}) = \frac{G^{(n)}(x_1, x_2, ..., x_n, x_{n+1}, ..., x_{2n})}{\prod_{i=1}^{2n} G^{(1)}(x_i, x_i)} \quad (5)$$

where $G^{(n)}(x_1, x_2, ..., x_n, x_{n+1}, ..., x_{2n}) = \langle \boldsymbol{E}^-(x_1)...\boldsymbol{E}^-(x_n)\boldsymbol{E}^+(x_{n+1})...\boldsymbol{E}^+(x_{2n})\rangle$

Using some properties of the trace, we can show that, specifically that $\mathrm{tr}\left[\rho A^\dagger A\right]$ is positive definite, $|g^1(x_1, x_2)| \leq 1$. Note that this constraint is not for $n \geq 1$, but since we are generalizing from the first order correlation, we still call it a normalized correlation function.

The main point of this article to talk about squeezed states of light. To talk about these states, we need to talk about the second order normalized correlation function. The second order normalized correlation function is defined as

$$g^{(2)}(x_1, x_2, x_3, x_4) = \frac{G^{(2)}(x_1, x_2, x_3, x_4)}{\sqrt{G^{(1)}(x_1, x_1)G^{(1)}(x_2, x_2)G^{(1)}(x_3, x_3)G^{(1)}(x_4, x_4)}} \quad (6)$$

We need to use some properties of the correlation functions.

1. Permutation of the first half $(x_1, ..., x_n)$ and the second half $(x_{n+1}, ..., x_{2n})$ individually, of the correlation function does not change the value of the correlation function. This is because when we quantize the electric field, we end up with a bunch of decoupled harmonic oscillators, so for any two oscillators the commutation relation is $\left[a_i, a_j^\dagger\right] = \delta_{ij}$, so the correlation function is invariant under permutation of the first half and the second half.
2. If the field is nth order coherent it must satisfy the following condition $g^{(j)}(x_1, x_2, ..., x_j, x_j, ..., x_1) = 1 \quad \forall j \leq n$. Classically we only use first order coherence to mean coherence. If the field is nth order coherent, then we get

$$G^{(j)}(x_1, x_2, ..., x_j, x_j, ..., x_1) = \prod_{i=1}^{j} G^{(1)}(x_i, x_i) \quad \forall j \leq n \quad (7)$$

4 Coherent States

To check for squeezed states we are interested in the second order correlation function. The second order correlation function, with parameters x_1, x_2 is given by

$$\begin{aligned} g^2(x_1, x_2) &= \frac{G^{(2)}(x_1, x_2, x_1, x_2)}{G^{(1)}(x_1, x_1)G^{(1)}(x_2, x_2)} \\ &= \frac{\langle \boldsymbol{E}^-(x_1)\boldsymbol{E}^-(x_2)\boldsymbol{E}^+(x_1)\boldsymbol{E}^+(x_2)\rangle}{G^{(1)}(x_1, x_1)G^{(1)}(x_2, x_2)} \end{aligned} \quad (8)$$

We note that $\boldsymbol{E}^+$ is an anhilation operator which reduces photon number and $\boldsymbol{E}^-$ is a creation operator which increases photon number. So we can write $N_E = \boldsymbol{E}^-\boldsymbol{E}^+$ is a number operator which counts the number of photons. If the electric fields are classical , the number N_E is a representation of the intensity of the light. So we can write the second order correlation function as

$$g^2(x_1, x_2) = \frac{\left\langle : N_{E(x_1)} N_{E(x_2)} : \right\rangle}{\left\langle N_{E(x_1)} \right\rangle \left\langle N_{E(x_2)} \right\rangle} \tag{9}$$

where, $: X :$ represents the normal ordering of the operator X. The normal ordering of an operator is defined as the ordering of the operators such that all the creation operators are to the left of the annihilation operators.

For example, for an operator $M = a^\dagger a b^\dagger b$, the normally ordered operator is

$$: M := a^\dagger b^\dagger a b \tag{10}$$

.

Here, we consider time $t_1 = t$ and $t_2 = t + \tau$, and consider that we have stationary fields. So we can write the second order correlation function for $t_1 = 0, t_2 = \tau$ as

$$g^2(\tau) = g^2(0, \tau) = \frac{\left\langle : N_{E(0)} N_{E(\tau)} : \right\rangle}{\left\langle N_{E(0)} \right\rangle \left\langle N_{E(\tau)} \right\rangle} \tag{11}$$

Claim 4.1 (Correlation function for coherent states):

For coherent states of the electric field, which are the eigenstates of the $\hat{a} = \boldsymbol{E}^+$ operator.

$$g^{(2)}(0) = 1 \tag{12}$$

Proof: We see that the coherent states are defined as the eigenstates of the annihilation operator $a = E^+$, where fix the electric field in some polarization direction. So we can write

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{13}$$

where α is a complex number. We can now calculate

$$\begin{aligned} g^{(2)}(0) = \frac{\left\langle : N_{E(x,0)} : \right\rangle}{\left\langle N_{E(x,0)} \right\rangle^2} &= \frac{\langle : a^\dagger a a^\dagger a : \rangle}{\langle a^\dagger a \rangle^2} \\ = \frac{\langle a^\dagger a^\dagger a a \rangle}{\langle a^\dagger a \rangle^2} &= \frac{\langle \alpha | a^\dagger a^\dagger a a | \alpha \rangle}{|\alpha|^2} = \frac{|\alpha|^2}{|\alpha|^2} = 1 \end{aligned} \tag{14}$$

♣

For short counting times, the time delay in the second order correlation function is $\tau = 0$.

Claim 4.2 (Variance):

For sufficiently short counting times, the variance of the photon number distribution $V(n)$ is related to $g^{(2)}(0)$ by the relation

$$\frac{V(n) - \langle n \rangle}{\langle n \rangle^2} = g^{(2)}(0) - 1 \tag{15}$$

Proof: Note that $N_{E(x,0)} = n$, which the photon number operator. Then the variance is given by $V(n) = \langle n^2 \rangle - \langle n \rangle^2$ The second order correlation function is given by

$$g^{(2)}(0) = \frac{\langle : n^2 : \rangle}{\langle n \rangle^2} \tag{16}$$

Let us focus on the numerator, $\langle : n^2 : \rangle = \langle a^\dagger a^\dagger a a \rangle = \langle a^\dagger n a \rangle$. We use the commutator relation $[n, a] = -a$ to get $\langle : n^2 : \rangle = \langle a^\dagger n a \rangle = \langle a^\dagger a^\dagger a a \rangle - \langle a^\dagger a \rangle = \langle n^2 \rangle - \langle n \rangle$ Thus we get, from the definition of variance and the second order correlation function,

$$\begin{aligned} g^{(2)}(0) &= \frac{\langle n^2 \rangle - \langle n \rangle}{\langle n \rangle^2} = \frac{V(n) + \langle n \rangle^2 - \langle n \rangle}{\langle n \rangle^2} \\ \Rightarrow g^{(2)}(0) - 1 &= \frac{V(n) - \langle n \rangle}{\langle n \rangle^2} \end{aligned} \tag{17}$$

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We know that photon statistics in the coherent state is poissonian. For poissonian statistics, the variance is given by $V(n) = \langle n \rangle$. So for poissonian statistics, we have $g^{(2)}(0) = 1$. When $g^{(0)} < 1$, we have sub-poissonian statistics and when $g^{(2)}(0) > 1$, we have super-poissonian statistics. Sub-poissonian statistics exhibit an phenomenon called photon antibunching. **Write more about photon antibunching.**

5 Squeezed States

5.1 Introduction to Squeezed States

Before we move onto the core topic of our term paper report, we must give an introduction to what we squeezed states are. The time dependent electric field operator is some specififc polarization direction for one single mode is given by [1]:

$$E(t) = \lambda\left(\hat{a} e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right) \tag{18}$$

where λ is a constant that contains information about the spatial wave functions. The operators $a, a^\dagger$ obey boson commutation relations ($\left[a, a^\dagger\right] = 1$). For more modes, we add up multiple different hilbert spaces of SHO, with different frequencies. We then write

$$a = X_1 + iX_2 \tag{19}$$

where we can see from the SHO equations, X_1 and X_2 are rescaled versions of the position and momentum operators, which obey the commutation relation $[X_1, X_2] = \frac{i}{2}$. We can then write the electric field operator as

$$E(t) = \frac{\lambda}{2}(X_1 \cos(\omega t) + X_2 \sin(\omega t)) \tag{20}$$

where X_1, X_2 are the amplitudes of the two quadratures of the field.

We use the generalized Heisenberg uncertainity principle to define the uncertainty in the two quadratures of the field. We can write the uncertainty in the two quadratures as

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4} \tag{21}$$

where $\Delta X_i = \sqrt{\langle X_i^2 \rangle - \langle X_i \rangle^2}$.

We are interested in a states with minimum uncertainty. To do that we must have, $\Delta X_1 \Delta X_2 = \frac{1}{4}$. A class of these states are the states with $\Delta X_1 = \Delta X_2 = \frac{1}{2}$ and $V(X_1) = V(X_2) = \frac{1}{4}$, where $V(X)$ is the variance of the operator X. These are our coherent states, which are the eigenstates of the anhilation operator in the quantized electric field description.

A larger class is by taking the variance in one quadrature to be lower than $\frac{1}{4}$ and the variance in the other quadrature to be more than that. These class of states are the called the squeezed states. We are interested in squeezed states which can be defined by the condition $V(X_i) < \frac{1}{4}$ where $i = 1$ or 2. We are dealing with a single mode of the field. For multiple modes, it is quite convinient to use a constrant on the normally ordered Variance of on the quadrature phases. We can write the variance of the quadrature phase as $V(X_i) = \langle X_i^2 \rangle - \langle X_i \rangle^2$. Lets define what our normally order variance is. We can write the normally ordered variance as

$$V(X_i) = \langle : X_i^2 : \rangle - \langle : X_i : \rangle^2 \tag{22}$$

where $: A :$ is the normally ordered operator. For our case, we can say

Claim 5.1.1 (Normally ordered variance of the quadrature phase): The normally ordered variance of the quadrature phase is given by

$$: V(X_i) : + \frac{1}{4} = V(X_i) \tag{23}$$

This implies that for squeezed states, $: V(X_i) :< 0$ for $i = 1$ or 2.

Proof: We can write the normally ordered variance as

$$\begin{aligned} : V(X_i) :&= \langle : X_i^2 : \rangle - \langle : X_i : \rangle^2 \\ &= \frac{1}{4}\Big[\Big\langle : a^2 + \left(a^\dagger\right)^2 + aa^\dagger + a^\dagger a \Big\rangle : \Big) - \langle : a + a^\dagger : \rangle^2] \\ &= \frac{1}{4}\Big[\Big\langle a^2 + \left(a^\dagger\right)^2 + 2a^\dagger a \Big\rangle - \langle : a + a^\dagger : \rangle^2 \Big] \\ &= \frac{1}{4}\Big[\Big\langle a^2 + \left(a^\dagger\right)^2 + 2a^\dagger a + 1 \Big\rangle - \langle a + a^\dagger \rangle^2 - 1 \Big] \\ &= V(X_i) - \frac{1}{4} \end{aligned} \tag{24}$$

Thus we have $: V(X_i) : + \frac{1}{4} = V(X_i)$, which implies that for squeezed states, $: V(X_i) :< 0$ for $i = 1$ or 2.

♣

We now see how these states are generated and what some of their mathematical properties are,

A light field with sub-poissonian statistics will exhibit an effect known as photon antibunching. The Glauber Sudarshan P representation of the density operator for a light field is given by

$$\rho = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha| \tag{25}$$

where $P(\alpha)$ is the Glauber-Sudarshan P function, and $d^2\alpha = d\,\mathrm{Re}(\alpha)d\,\mathrm{Im}(\alpha)$. More about this is given in Appendix A.

The quantum mechanical average resemble classical averaging procedures when the P function is positive non singular function. We show that photon antibunching is a quantum effect by showing there is no positive non singular P function. Note that we generate squeezed states by reducing the variance in one quadrature, which we choose here to be X_1.

Claim 5.1.2 (Variance of Squeezed States(P representation)):

The Variance of the X_1 operator. the glauber sudarshan P representation of the squeezed state is given by

$$V(X_1) = \frac{1}{4}\left[1 + \int d^2\alpha P(\alpha)\big\{(\alpha+\alpha^*) - \big(\langle a\rangle + \langle a^\dagger\rangle\big)\big\}^2\right] \tag{26}$$

Proof: We can write the variance of the X_1 operator as $V(X_1) = \langle X_1^2\rangle - \langle X_1\rangle^2$ where $\langle A\rangle = \mathrm{Tr}(\rho A)$. We can write the density operator in the Glauber-Sudarshan P representation as

$$\rho = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha| \tag{27}$$

Recall the completeness relation $\sum_n |n\rangle\langle n| = \mathbb{I}$. We can write the expectation value of the X_1 operator as

$$\begin{aligned}
\langle X_1\rangle = \mathrm{Tr}(\rho X_1) &= \sum_n \int d^2\alpha P(\alpha)\langle n|\alpha\rangle\langle\alpha|X_1|n\rangle \\
&= \sum_n \int d^2\alpha P(\alpha)\langle\alpha|X_1|n\rangle\langle n|\alpha\rangle = \int d^2\alpha P(\alpha)\langle\alpha|X_1|\alpha\rangle \\
&= \frac{1}{2}\int d^2\alpha P(\alpha)(\alpha+\alpha^*) \\
&= \frac{1}{2}\left[\langle a\rangle + \langle a^\dagger\rangle\right]
\end{aligned} \tag{28}$$

where the last equality is obtained by taking $\langle a\rangle = \mathrm{Tr}(\rho a) = \int d^2\alpha P(\alpha)\langle\alpha|a|\alpha\rangle = \int d^\alpha P(\alpha)\alpha$. We now find the expectation value of the X_1^2 operator. We can write the expectation value of the X_1^2 operator as

$$\begin{aligned}
\langle X_1^2\rangle = \mathrm{Tr}(\rho X_1^2) &= \sum_n \int d^2\alpha P(\alpha)\langle n|\alpha\rangle\langle\alpha|X_1^2|n\rangle \\
&= \sum_n \int d^2\alpha P(\alpha)\langle n|X_1^2|\alpha\rangle\langle\alpha|n\rangle \\
&= \int d^2\alpha P(\alpha)\langle\alpha|X_1^2|\alpha\rangle \\
&= \frac{1}{4}\int d^2\alpha P(\alpha)\Big\langle\alpha\Big|a^2 + aa^\dagger + a^\dagger a + \left(a^\dagger\right)^2\Big|\alpha\Big\rangle \\
&= \frac{1}{4}\int d^2\alpha P(\alpha)\left[\alpha^2 + \left(\alpha^*\right)^2 + 2\alpha\alpha^* + 1\right]
\end{aligned} \tag{29}$$

We note that the P-function is a quasiprobability distribution, which doesnt satisfy all the Kolmogorov axioms, but still satifies $\int P(\alpha)d^2\alpha = 1$. Then the variance is given by

$$\begin{aligned}
V(X_1) = \langle X_1^2\rangle - \langle X_1\rangle^2 &= \frac{1}{4}\left[1 + \int d^2\alpha P(\alpha)\left(\alpha^2 + (\alpha^*)^2 + 2\ |\alpha|^2\right) - \left(\langle a\rangle + \langle a^\dagger\rangle\right)^2\right] \\
&= \frac{1}{4}\left[1 + \int d^2\alpha P(\alpha)(\alpha + \alpha^*)^2\right] + \left(\langle a\rangle + \langle a^\dagger\rangle\right)^2 - 2\left(\langle a\rangle + \langle a^\dagger\rangle\right)\left(\langle a\rangle + \langle a^\dagger\rangle\right) \\
&= \frac{1}{4}[1 + \int d^2\alpha P(\alpha)\left[(\alpha + \alpha^*)^2 + \left(\langle a\rangle + \langle a^\dagger\rangle\right)^2 - 2(\alpha + \alpha^*)\left(\langle a\rangle + \langle a^\dagger\rangle\right]\right) \\
&= \frac{1}{4}\left[1 + \int d^2\alpha P(\alpha)\left\{(\alpha + \alpha^*) - \left(\langle a\rangle + \langle a^\dagger\rangle\right)\right\}^2\right]
\end{aligned} \tag{30}$$

We know that a squeezed has $V(X_1) < \frac{1}{4}$ in one quadrature, and $V(X_2) > \frac{1}{4}$ in the other quadrature. We can see that the squeezed state has a negative P function, which is a signature of nonclassicality.

5.2 Definition of Squeezed States

Squeezed States are defined as the state obtained by the action of the operator $\hat{D}(\alpha)\hat{S}(\zeta)$ on the vaccum number state $|0\rangle$

$$|\alpha,\zeta\rangle = \hat{D}(\alpha)\hat{S}(\zeta)|0\rangle \tag{31}$$

Where the concerned operators are defined as,

$$\begin{aligned}
\hat{D}(\alpha) &= e^{-\frac{|\alpha|^2}{2}}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}} \\
\hat{S}(\zeta) &= e^{\frac{1}{2}\left(\zeta^*\hat{a}^2 - \zeta\hat{a}^{\dagger 2}\right)}
\end{aligned} \tag{32}$$

where, $\zeta = re^{i\theta}$ with $r > 0$, and $\alpha = |\alpha|e^{i\varphi}$.

$\hat{D}(\alpha)$ and $\hat{S}(\zeta)$ are the translation and the squeeze operator respectively. It can be shown that(we show that in Appendix B.),

$$\hat{D}(\alpha)^\dagger\hat{D}(\alpha) = \hat{S}(\zeta)^\dagger\hat{S}(\zeta) = 1 \tag{33}$$

i.e., both the operators are unitary. This will help us a lot in the following section, where we will be looking at the expectations and variances of a few relevant quantities.

5.3 Mean Photon Number in a Squeezed State

Claim 5.3.1 (Mean Photon Number in a Squeezed State): Hence for a squeezed state, the mean photon number is given by

$$\langle \hat{n} \rangle = |\alpha|^2 + \sinh^2(r) \tag{34}$$

We wish to compute $\langle \hat{n} \rangle$ for the squeezed state $|\alpha, \zeta\rangle$

We know that

$$\hat{n} = \hat{a}^\dagger \hat{a} \tag{35}$$

Now, to compute the epectation of $\hat{n}$ in the Squeezed state we have to evaluate the following expression

$$\begin{aligned} \langle \hat{n} \rangle &= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\ &= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) \hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \qquad \left[\because \hat{D}(\alpha)^\dagger \hat{D}(\alpha) = 1\right] \\ &= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \left[\because \hat{S}(\zeta)^\dagger \hat{S}(\zeta) = 1\right] \end{aligned} \tag{36}$$

First, we evaluate the operator $\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha)$

$$\begin{aligned} \hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) &= e^{-|\alpha|^2} e^{-\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \hat{a} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \\ &= e^{-|\alpha|^2} e^{-\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \left(e^{\alpha \hat{a}^\dagger} \hat{a} + \left[\hat{a}, e^{\alpha \hat{a}^\dagger}\right] \right) e^{-\alpha^* \hat{a}} \end{aligned} \tag{37}$$

Now let us compute the comutator relation $\left[\hat{a}, e^{\alpha \hat{a}^\dagger}\right]$ which is given by,

$$\left[\hat{a}, e^{\alpha \hat{a}^\dagger}\right] = \sum_{n=1}^{\infty} \frac{\alpha^n \left[\hat{a}, \hat{a}^{\dagger n}\right]}{n!} \tag{38}$$

We can easily show by induction that, $\left[\hat{a}, \hat{a}^{\dagger n}\right] = n \hat{a}^{\dagger n-1}$. Then the commutator evaluates to,

$$\begin{aligned} \left[\hat{a}, e^{\alpha \hat{a}^\dagger}\right] &= \sum_{n=1}^{\infty} \frac{n \alpha^n \hat{a}^{\dagger n-1}}{n!} \\ &= \alpha \sum_{n=0}^{\infty} \frac{\left(\alpha \hat{a}^\dagger\right)^n}{n!} \\ &= \alpha e^{\alpha \hat{a}^\dagger} \end{aligned} \tag{39}$$

Substituting this commutator relation back into the expression for $\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha)$ we get,

$$\begin{aligned} \hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) &= e^{-|\alpha|^2} e^{-\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \left(e^{\alpha \hat{a}^\dagger} \hat{a} + \alpha e^{\alpha \hat{a}^\dagger} \right) e^{-\alpha^* \hat{a}} \\ &= e^{-|\alpha|^2} e^{-\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} (\alpha + \hat{a}) \\ &= \alpha + \hat{a} \end{aligned} \tag{40}$$

Taking the dagger of this equation on both sides we can also see that,

$$\hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) = \alpha^* + \hat{a}^\dagger \tag{41}$$

Now substituting these expressions for $\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha)$ and $\hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha)$ into our expression for $\langle \hat{n} \rangle$ we get,

$$
\begin{aligned}
\langle \hat{n} \rangle &= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger (\hat{a}^\dagger + \alpha^*) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger (\hat{a} + \alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \left(\hat{S}(\zeta)^\dagger \hat{a}^\dagger \hat{S}(\zeta) + \alpha^* \right) \left(\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) + \alpha \right) \middle| 0 \right\rangle \left[\because \hat{S}(\zeta)^\dagger \hat{S}(\zeta) = 1 \right]
\end{aligned} \tag{42}
$$

So, now we compute $\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta)$. Let us define $A = \frac{1}{2}\left(\zeta {\hat{a}^\dagger}^2 - \zeta^* \hat{a}^2\right)$. Then,

$$
\begin{aligned}
\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) &= e^A \hat{a} e^{-A} \\
&= \sum_{n=0}^{\infty} \frac{[A, \hat{a}]_n}{n!}
\end{aligned} \tag{43}
$$

where $[A, B]_1 = [A, B]$, $[A, B]_2 = [A, [A, B]]$ and so on. Lets compute $[A, \hat{a}]$

$$
\begin{aligned}
[A, \hat{a}] &= -\frac{\zeta}{2} \left[\hat{a}, {\hat{a}^\dagger}^2 \right] \\
&= -\frac{\zeta}{2} 2 \hat{a}^\dagger \\
&= -\zeta \hat{a}^\dagger
\end{aligned} \tag{44}
$$

Similarly,

$$
\begin{aligned}
[A, \hat{a}]_2 &= [A, [A, \hat{a}]] \\
&= -\zeta \left[A, \hat{a}^\dagger \right] \\
&= \zeta \frac{\zeta^*}{2} \left[\hat{a}^2, \hat{a}^\dagger \right] \\
&= |\zeta|^2 \hat{a}
\end{aligned} \tag{45}
$$

We can see after this, that the results will be of a similar form when k has the same parity. It can be shown using induction that,

$$
[A, \hat{a}]_n = \begin{cases} -\zeta |\zeta|^{n-1} \hat{a}^\dagger & \text{if } n \text{ is odd} \\ |\zeta|^n \hat{a}^\dagger & \text{if } n \text{ is even} \end{cases} \tag{46}
$$

Then we can evaluate $\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta)$ to be,

$$
\begin{aligned}
\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) &= \sum_{n=0}^{\infty} \frac{1}{n!} [A, \hat{a}]_n \\
&= \hat{a} \sum_{k=0}^{\infty} \frac{|\zeta|^{2k}}{(2k)!} - \hat{a}^\dagger \sum_{k=0}^{\infty} \frac{\zeta |\zeta|^{2k}}{(2k+1)!} \\
&= \hat{a} \cosh(|\zeta|) - \hat{a}^\dagger \frac{\zeta}{|\zeta|} \sum_{k=0}^{\infty} \frac{|\zeta|^{2k+1}}{(2k+1)!} \\
&= \hat{a} \cosh(|\zeta|) - \hat{a}^\dagger \frac{\zeta}{|\zeta|} \sum_{k=0}^{\infty} \frac{|\zeta|^{2k+1}}{(2k+1)!} \\
&= \hat{a} \cosh(|\zeta|) - \hat{a}^\dagger \frac{\zeta}{|\zeta|} \sinh(|\zeta|) \\
&= \hat{a} \cosh(r) - \hat{a}^\dagger e^{i\theta} \sinh(r)
\end{aligned} \tag{47}
$$

Taking the dagger of this relation gives us

$$\hat{S}(\zeta)^\dagger \hat{a}^\dagger \hat{S}(\zeta) = \hat{a}^\dagger \cosh(r) - \hat{a}e^{-i\theta}\sinh(r) \tag{48}$$

Substituting these expressions into the expression for $\langle \hat{n} \rangle$ we get,

$$\begin{aligned}\langle \hat{n} \rangle &= \langle 0 \big| \big(\hat{a}^\dagger \cosh(r) - \hat{a}e^{-i\theta}\sinh(r) + \alpha^*\big)\big(\hat{a}\cosh(r) - \hat{a}^\dagger e^{i\theta}\sinh(r) + \alpha\big)\big|0\rangle \\ &= \langle 0 \big| |\alpha|^2 + \hat{a}\hat{a}^\dagger \sinh^2(r) \big| 0 \rangle \\ &= |\alpha|^2 + \sinh^2(r)\end{aligned} \tag{49}$$

Hence for a squeezed state, the mean photon number is given by

$$\langle \hat{n} \rangle = |\alpha|^2 + \sinh^2(r) \tag{50}$$

♣

5.4 Variances in Squeezed States

Let us define Y_1 and Y_2 such that, $Y_1 + iY_2 = (X_1 + X_2)e^{-i\frac{\theta}{2}} := \hat{b}$. Then, we have $\hat{b} = \hat{a}e^{-i\frac{\theta}{2}}$. And we also have,

$$\hat{S}(\zeta) = e^{\frac{1}{2}\left(\hat{b}^2 - \hat{b}^{\dagger 2}\right)} \tag{51}$$

Observe that, $\hat{b}^\dagger \hat{b} = \hat{a}^\dagger \hat{a} = \hat{n}$ and $\left[\hat{b}^\dagger, \hat{b}\right] = \left[\hat{a}^\dagger, \hat{a}\right] = 1$ Also, lets define $\beta = \alpha e^{-\frac{\theta}{2}}$

Claim 5.4.1 (Variance of the quadrature phase): The variance of the quadrature phase is given by

$$Var(Y_1) = \delta Y_1 = \frac{1}{4}e^{-2r} \tag{52}$$

$$Var(Y_2) = \delta Y_2 = \frac{1}{4}e^{2r} \tag{53}$$

Proof: Let's compute δY_1 and δY_2 for the squeezed state $|\alpha, \zeta\rangle$

$$\begin{aligned}\delta Y_1 &= \langle Y_1^2 \rangle - \langle Y_1 \rangle^2 \\ &= \frac{1}{4}\left\langle \left(\hat{b} + \hat{b}^\dagger\right)^2 \right\rangle - \left\langle \hat{b} + \hat{b}^\dagger \right\rangle^2 \\ &= \frac{1}{4}\left[\left\langle \left(\hat{b}^2 + \hat{b}^{\dagger 2} + 2\hat{b}^\dagger \hat{b} + 1\right)^2 \right\rangle - \left(\left\langle \hat{b} \right\rangle + \left\langle \hat{b}^\dagger \right\rangle\right)^2\right] \\ &= \frac{1}{4}\left[\left\langle \hat{b}^2 \right\rangle + \left\langle \hat{b}^{\dagger 2} \right\rangle + 2\langle \hat{n} \rangle + 1 - \left(\left\langle \hat{b} \right\rangle + \left\langle \hat{b}^\dagger \right\rangle\right)^2\right]\end{aligned} \tag{54}$$

First, we'll compute $\left\langle \hat{b} \right\rangle$

$$
\begin{aligned}
\left\langle \hat{b} \right\rangle &= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{b} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger (\hat{a} + \alpha) \middle| 0 \right\rangle e^{-i\frac{\theta}{2}} \\
&= \left\langle 0 \middle| \left(\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) + \alpha \right) \middle| 0 \right\rangle e^{-\frac{\theta}{2}} \\
&= \left\langle 0 \middle| \hat{a}\cosh(r) - e^{i\theta}\sinh(r) + \alpha \middle| 0 \right\rangle e^{-\frac{\theta}{2}} \\
&= \alpha e^{-i\frac{\theta}{2}}
\end{aligned}
\tag{55}
$$

Similarly,

$$
\left\langle \hat{b}^\dagger \right\rangle = \alpha^* e^{i\frac{\theta}{2}} \tag{56}
$$

Now, we compute $\left\langle \hat{b}^2 \right\rangle$

$$
\begin{aligned}
\left\langle \hat{b}^2 \right\rangle &= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{b}^2 \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{b} \hat{D}(\alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{b} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger (\hat{a} + \alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger (\hat{a} + \alpha) \hat{S}(\zeta) \middle| 0 \right\rangle e^{-i\theta} \\
&= \left\langle 0 \middle| \left(\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) + \alpha \right)^2 \middle| 0 \right\rangle e^{-i\theta} \\
&= \left\langle 0 \middle| \left(\hat{a}\cosh(r) - e^{i\theta} \hat{a}^\dagger \sinh(r) + \alpha \right)^2 \middle| 0 \right\rangle e^{-i\theta} \\
&= \left\langle 0 \middle| -\hat{a}\hat{a}^\dagger e^{i\theta} \cosh(r)\sinh(r) + \alpha^2 \middle| 0 \right\rangle e^{-i\theta} \\
&= \alpha^2 e^{-i\theta} - \sinh(r)\cosh(r)
\end{aligned}
\tag{57}
$$

Similarly,

$$
\left\langle \hat{b}^{\dagger 2} \right\rangle = (\alpha^*)^2 e^{i\theta} - \sinh(r)\cosh(r) \tag{58}
$$

Substituting these expressions in the expression for δY_1 we get,

$$
\begin{aligned}
\delta Y_1 &= \frac{1}{4}\left(\alpha^2 e^{-i\theta} + (\alpha^*)^2 e^{i\theta} - 2\sinh(r)\cosh(r) + 2|\alpha|^2 + 2\sinh^2(r) + 1 - \left(\alpha e^{-i\frac{\theta}{2}} + \alpha e^{i\frac{\theta}{2}} \right)^2 \right) \\
&= \frac{1}{4}\left(\left(\alpha e^{-i\frac{\theta}{2}} + \alpha e^{i\frac{\theta}{2}} \right)^2 + 1 + 2\sinh^2(r) - 2\cosh(r)\sinh(r) - \left(\alpha e^{-i\frac{\theta}{2}} + \alpha e^{i\frac{\theta}{2}} \right)^2 \right) \\
&= \frac{1}{4}(1 + 2\sinh^2(r) - 2\cosh(r)\sinh(r)) \\
&= \frac{1}{4}(\cosh(2r) - \sin(2r)) \\
&= \frac{1}{4} e^{-2r}
\end{aligned}
\tag{59}
$$

With a very similar computation we can show that,

$$
\delta Y_2 = \frac{1}{4} e^{2r} \tag{60}
$$

From these fluctuations we can clearly see that the amplitude is decaying exponentially in one of the quadratures while, increasing in the other one at the same rate, with r. Also, the fluctuations are independent of α.

5.5 Second Order Correlation Function for the Squeezed State

Claim 5.5.1 (Second Order Correlation Function for the Squeezed State):

The second order correlation function for the squeezed state is given by

$$g^{(2)}(0) = \frac{|\alpha|^4 + 3\sinh^4(r) + \left(1 + 2|\alpha|^2\right)\sinh^2(r) - \mathrm{Re}\left(\alpha^2 e^{-i\theta}\right)\sinh(2r)}{\left(|\alpha|^2 + \sinh^2(r)\right)^2} \tag{61}$$

where $g^{(2)}(0) = \frac{\langle : \hat{n}^2 : \rangle}{\langle \hat{n} \rangle^2}$ and α is the parameter of displacement operator and $\zeta = re^{i\theta}$, where $r > 0$ is the amplitude of squeezing.

Now let us compute the second order correlation function for this state $|\alpha, \zeta\rangle$ We know that,

$$g^{(2)}(0) = \frac{\langle : \hat{n}^2 : \rangle}{\langle \hat{n} \rangle^2} \tag{62}$$

So, we must compute $\langle : \hat{n}^2 : \rangle$

$$\begin{aligned}
\langle : \hat{n}^2 : \rangle &= \left\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) \hat{S}(\zeta)^\dagger \hat{S}(\zeta) \hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \left(\alpha * + \hat{a}^\dagger\right) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger \left(\alpha * + \hat{a}^\dagger\right) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger (\alpha + \hat{a}) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger (\alpha + \hat{a}) \hat{S}(\zeta) \right\rangle \\
&= \left\langle 0 \middle| \left(\hat{a}^\dagger \cosh(r) - e^{-i\theta} \hat{a} \sinh(r) + \alpha^*\right)^2 \left(\hat{a} \cosh(r) - e^{i\theta} \hat{a}^\dagger \sinh(r) + \alpha\right)^2 \middle| 0 \right\rangle \\
&= \Big\langle 0 \Big| \hat{a} \hat{a}^\dagger \hat{a} \hat{a}^\dagger \sinh^2(r) \cosh^2(r) + \hat{a} \hat{a} \hat{a}^\dagger \hat{a}^\dagger \sinh^4(r) \\
&\quad + \hat{a} \hat{a}^\dagger \left(2|\alpha|^2 \sinh^2(r) - \left(\alpha^2 e^{-i\theta} + (\alpha^*)^2 e^{i\theta}\right) \sinh(r) \cosh(r)\right) + |\alpha|^4 \Big| 0 \Big\rangle \\
&= 2\sinh^4(r) + \sinh^2(r) \cosh^2(r) + 2|\alpha|^2 \sinh^2(r) - \left(\alpha^2 e^{-i\theta} + (\alpha^*)^2 e^{i\theta}\right) \sinh(r) \cosh(r) \\
&= 3\sinh^4(r) + \left(1 + 2|\alpha|^2\right) \sinh^2(r) - \mathrm{Re}\left(\alpha^2 e^{-i\theta}\right) \sinh(2r) + |\alpha|^4
\end{aligned} \tag{63}$$

Hence, the correlation function turns out to be

$$g^{(2)}(0) = \frac{|\alpha|^4 + 3\sinh^4(r) + \left(1 + 2|\alpha|^2\right)\sinh^2(r) - \mathrm{Re}\left(\alpha^2 e^{-i\theta}\right)\sinh(2r)}{\left(|\alpha|^2 + \sinh^2(r)\right)^2} \tag{64}$$

Now, we will discuss two limiting conditions for this expression.

First we will look at the limit when $|\alpha| \ll 1$, i.e., we are considering the squeezed state $|0, \zeta\rangle$. Then we get the following form of the correlation function.

$$
\begin{aligned}
g^{(2)}(0) &= \frac{3\sinh^4(r) + \sinh^2(r)}{\sinh^4(r)} \\
&= 3 + \frac{1}{\sinh^2(r)} \\
&= 3 + \frac{1}{\langle n \rangle} \\
&= 1 + \frac{\cosh(2r)}{\sinh^2(r)}
\end{aligned}
\tag{65}
$$

As we can see in the small α limit, no matter what value of ζ we choose, we always have $g^{(2)}(0) > 1$ So, in this case we will always have photon bunching.

Now let's look at the second limiting case, when $|\alpha|^2 \gg \sinh^2(r)$

$$
\begin{aligned}
g^{(2)}(0) = & \frac{|\alpha^4| + 3\sinh^4(r) + 2|\alpha|^2\sinh^2(r) - \mathrm{Re}\big(\alpha^2 e^{-i\theta}\big)\sinh(2r)}{|\alpha|^4} \\
& = 1 + 3\left(\frac{\sinh^2(r)}{|\alpha|^2}\right)^2 + 2\frac{\sinh^2(r)}{|\alpha|^2} - \cos(2\varphi - \theta)\frac{\sinh(2r)}{|\alpha|^2} \\
& \approx 1 - \cos(2\varphi - \theta)\frac{\sinh(2r)}{|\alpha|^2}
\end{aligned}
\tag{66}
$$

From this expression we can see that in this limit, if $r > 0$, we eventually get $g^{(2)}(0) < 1$ i.e. photon anti-bunching and sub poisonnian photon statistics, and for $r < 0$ we also get $g^{(2)}(0) > 1$ i.e. photon bunching and super poissonian photon statistics.

6 Production of Squeezed States

The production of squeezed states of light essentially requires the generation of a mixing of a particular mode of the field with its conjugate mode. This ican not be achieved by transformations offered by linear optical devices (mirror, beam splitter, phase shifter). Th only way to achieve this is through the use of nonlinear optical devices.

In general, what we desire is a canonical Bogoliubov transformation of the form:

$$
\hat{b} = \mu\hat{a} + \nu\hat{a}^\dagger \tag{67}
$$

Where controlling μ and ν allows us to control the extent of squeezing.

Phrased in other terms, what we require is a Hamiltonian that contains quadratic terms in the creation and annihilation operators of that mode. This is given in the general form:

$$
H = i\frac{h}{2\pi}\kappa\Big(\big(\hat{a}^\dagger\big)^2 - \hat{a}^2\Big) \tag{68}
$$

This can be achieved through two main methods:

1. Degenerate Parametric Down-Conversion ($\mathrm{X}^{(2)}$): Here, a strong classical photon pump is used to drive a $\mathrm{X}^{(2)}$ crystal at ssome frequency 2ω. this results in the creation of two photons of almost perfectly correlated

phases of ω, and the process gives us an interaction Hamiltonian of the form mentioned above. The extent of squeezing is controlled by the non-linear susceptibility of the crystal used.

2. Degenerate 4-wave mixing ($X^{(3)}$): Same as the previous method, but in this case we use two photon pumps to drive the crystal with two photons of some frequency ω. This generates two photons that are again nearly perfectly correlated in phase. This gives us a nearly identical interaction Hamiltonian and the same Bogoliubov transformation as before.

7 Detection of Squeezed States

7.1 Basic idea of the detection

The measure and characterise the quadrature squeezing of the light field, we must be able to perform phase-sensitive measurements of the field operators:

$$\widehat{X_\theta} = \frac{1}{2}\left(e^{-i\theta}\hat{a} + e^{i\theta}\widehat{a^\dagger}\right) \tag{69}$$

Where $\hat{a}$ is the signal mode's annihilation operator, and θ is a tunable phase that decides the quadrature that is to be measured.

Two of the basic techniques that are used for the detection of the squeezed states that are discussed in the paper involve using a strong local oscillator (hereby referred to as LO) to procude a coherent state $\left|\beta e^{i\theta}\right\rangle$ with $\beta \gg 1$, and using it to perform a homodyne or a heterodyne measurement of the signal. this process involves the mixing of the singal moed with the LO mode using a 50/50 beam splitter, the action of which is represented by the following transformation:

$$\begin{pmatrix} \hat{c} \\ \hat{d} \end{pmatrix} = \frac{1}{\sqrt{a}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} \tag{70}$$

This yields the modes:

$$\hat{c} = \frac{1}{\sqrt{2}}\left(\hat{a} + \hat{b}\right) \tag{71}$$

$$\hat{d} = \frac{1}{\sqrt{2}}\left(\hat{a} - \hat{b}\right) \tag{72}$$

Where $\hat{a}$ is the signal mode, and $\hat{b}$ is the LO mode. The output modes are then detected using photodiodes, which essentially carry out a photon counting operation, i.e. $\hat{c}^\dagger\hat{c}$ and $\hat{d}^\dagger d$.

7.2 Homodyne detection

In this mode, the LO mode is prepared in the same frequency as the signal mode ω, and the phase θ is varied. The difference in the photocurrent produced by the two photodiodes is given by:

$$\widehat{N_-} = \hat{c}^\dagger\hat{c} - \hat{d}^\dagger\hat{d} = \hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a} \tag{73}$$

In the strong LO approximation limit, $\left\langle\hat{b}\right\rangle = \beta e^{-\theta}$, and one can replace the $\hat{b}$ operators with their expectation values. This yields:

$$\widehat{N_-} \approx 2\ |\beta|\ \widehat{X_\theta} \tag{74}$$

Now, the variance in the photocurrent is given by:

$$Var\left(\widehat{N_-}\right) = 4\ |\beta|^2\ \left\langle\left(\Delta\widehat{X_\theta}\right)^2\right\rangle \tag{75}$$

If we observe a variance in the different photocurrent that is less than what is expected from a coherent state, i.e. $\mathrm{mod}(\beta)^2$, we can be sure that we are observing a squeezed state. The main benefit of this method is the direct access to all quadratures by modifying the phase of the LO mode, which we can alter from $\theta = 0$ to $\theta = \frac{\pi}{2}$ to access the two orthogonal quadratures. The disadvantage of this method is that the LO mode must be prepared in the same frequency as the signal mode, which is not always possible. This can be a problem in cases where the signal mode is at a different frequency, such as in the case of squeezed light generated by four-wave mixing or parametric down-conversion.

7.3 Heterodyne detection

In this method, the LO mode is prepared in a slightly different frequency of $\omega + \delta\omega$, and the phase θ is varied. One of the output modes from the beamsplitter is given by:

$$\hat{c}(t) = \frac{1}{\sqrt{2}}\left(\hat{a}e^{-\omega t} + \beta e^{-i(\omega+\delta\omega)t}\right) \tag{76}$$

and the intensity of the photocurrent is given by:

$$\hat{I}(t) = \frac{1}{2}\left[\hat{a}^\dagger\hat{a} + |\beta|^2 + \beta^*\hat{a}e^{-i\delta\omega t} + \beta\hat{a}^\dagger e^{i\delta\omega t}\right] \tag{77}$$

Electronic filtering of the photocurrent is then done to remove the high frequency terms, and the resulting signal is given by:

$$\widehat{I_{beat}}(t) = \frac{1}{2}\left[\beta^*\hat{a}e^{-i\delta\omega t} + \beta\hat{a}^\dagger e^{i\delta\omega t}\right] \tag{78}$$

This contains information about both the quadrature amplitudes, as can be seen by substituting $\beta = |\beta|\ e^{i\theta}$ to get;

$$\widehat{I_{beat}}(t) = |\beta|\ \left[\widehat{X_1}\cos(\delta\omega t) + \widehat{X_2}\sin(\delta\omega t)\right] \tag{79}$$

Demmodulation of this signal gives us the quadrature amplitudes, and thus the variances - and if we detect that any of them are less than the coherent state limit we can be sure that we have observed a squeezed state. The main advantage of this method is that the LO mode does not need to be prepared in the same frequency as the signal mode, which makes it more versatile. However, the disadvantage is that we do not have direct access to all quadratures, and we need to demodulate the signal to obtain the quadrature amplitudes.

8 Applications of Squeezed States

8.1 Quantum-Noise–Limited Interferometry

High-precision interferometric measurements (e.g. gravitational-wave detectors, cavity-length stabilization) are fundamentally limited by quantum shot noise and radiation-pressure back-action. Injecting squeezed vacuum into the interferometer's unused port reduces fluctuations in the pertinent quadrature, thereby lowering the noise floor. Caves (1981) showed that a squeeze parameter r yields a noise reduction factor of e^{-r} in the shot-noise-dominated regime.

8.2 Phase-Sensitive Amplification

Conventional, phase-insensitive amplifiers must add at least half a quantum of noise per quadrature. In contrast, a **phase-sensitive parametric amplifier** based on a $\chi^{(2)}$ or $\chi^{(3)}$ nonlinearity can amplify one quadrature without excess added noise by locking to the squeezed quadrature.

8.3 Optical Communication Channels

Yuen and Shapiro [2] first proposed encoding information in a squeezed-quadrature field to surpass the classical Shannon limit imposed by coherent-state (shot-noise) channels. By modulating the squeezed quadrature and homodyning at the receiver, one can achieve a signal-to-noise ratio increased by a factor e^{2r} over a coherent-state channel for the same mean photon number . Thus we can lock onto the squeezed quadrature to achieve a higher signal to noise ratio, and the noise can be put into the other quadrature.

8.4 Optical Waveguide Taps and Quantum-Limited Routing

Shapiro [3] showed that tapping an optical waveguide with a squeezed-state probe can extract a portion of the signal with arbitrarily low back-action when the probe is squeezed in the appropriate quadrature. This **optical waveguide tap** enables non-invasive monitoring of guided signals over multikilometer networks without intermediate optical amplifiers, offering a route to quantum-enhanced data buses and sensing arrays .

Bibliography

[1] P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th ed. Oxford University Press, 1958.

[2] H. P. Yuen and J. H. Shapiro, "Quantum Statistics of Homodyne and Heterodyne Detection," in *Coherence and Quantum Optics IV*, Springer US, 1978, pp. 719–727. doi: 10.1007/978-1-4757-0665-9_75.

[3] J. H. Shapiro, "Quantum Noise and Excess Noise in Optical Communications," *Optics Letters*, vol. 5, no. 8, pp. 351–353, Aug. 1980, doi: 10.1364/OL.5.000351.

[4] H. J. Carmichael, *Statistical Methods in Quantum Optics 1: Master Equations and Fokker-Planck Equations*, 2nd ed. in Theoretical and Mathematical Physics. Springer, 2002. doi: 10.1007/978-3-662-04103-1.

A : Glauber Sudarshan P function

The Glauber Sudarshan P representation of the density operator for a light field is given by [4]

$$\rho = \int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha| \tag{80}$$

where $P(\alpha)$ is the Glauber-Sudarshan P function, and $d^2\alpha = d\,\mathrm{Re}(\alpha)d\,\mathrm{Im}(\alpha)$

If we find a function that is positive and non singular, we can write the density operator in the form of a classical statistical ensemble. The quantum mechanical average resemble classical averaging procedures when the P function is positive non singular function. Then for normal ordered operators the classical statistical averaging is same as the quantum mechanical averaging.

Note that the function P has the property that the density matrix ρ is diagonal in the coherent state basis$\{|\alpha\rangle\}$.

This representation of the P function is quite nice since we can interpret P as a quasi-probability distribution. Lets start by observing that the P function is indeed normalized.

$$\begin{aligned}
\int P(\alpha)d^2\alpha &= \int d^2\alpha P(\alpha)\langle\alpha|\alpha\rangle \\
&= \sum_n \int d^2\alpha P(\alpha)\langle\alpha|n\rangle\langle n|\alpha\rangle \\
&= \sum_n \int d^2\alpha P(\alpha)\langle n|\alpha\rangle\langle\alpha|n\rangle \\
&= \mathrm{tr}\left(\int d^2\alpha P(\alpha)|\alpha\rangle\langle\alpha|\right) \\
&= \mathrm{tr}(\rho) = 1
\end{aligned} \tag{81}$$

Thus we see that the P function is normalized. The beauty of this representation lies in the fact than normal ordered averages can be calculated the same way we do in classical probability theory, using P as the probability distribution. Normal ordering refers to writing the creation operators to the left and the anhilation operators to the right.

$$\begin{aligned}
\left\langle a^{\dagger p}a^q\right\rangle &= \mathrm{tr}\left(\rho a^{\dagger p}a^q\right) \\
&= \sum_n \int d^2\alpha P(\alpha)\langle n|\alpha\rangle\left\langle\alpha\middle|a^{\dagger p}a^q n\right\rangle \\
&= \int d^2\alpha P(\alpha)\left\langle\alpha\middle|a^{\dagger p}a^q\middle|\alpha\right\rangle \\
&= \int d^2\alpha P(\alpha)(\alpha^*)^p\alpha^q \\
&= \left(\overline{\alpha^{*p}\alpha^q}\right)_P
\end{aligned} \tag{82}$$

To interpret the P function as a probability distribution we need P to be positive and non singular. Thus we can intuitively say that we do not find a positive definite P function , we can say that the state is not in a classically possible statistical ensemble.

B : Unitarity of Squeeze and Displacement operators

Definition 2.1 (Displacement Operator):

The displacement operator is defined as

$$D(\alpha) = \exp\left(\left[\alpha a^\dagger - \alpha^* a\right]\right)$$

where α is a complex number.

We need to show that $D(\alpha)$ is unitary. Note that $D^\dagger(\alpha) = D(-\alpha) = \exp\left(\left[\alpha^* a - \alpha a^\dagger\right]\right)$. Note that $\left[\alpha^* a - \alpha a^\dagger, \alpha^* a + \alpha a^\dagger\right] = 0$.Thus we get,

$$\begin{aligned} D^\dagger(\alpha)D(\alpha) &= D^{-\alpha}D(\alpha) \\ &= \exp\left(\left[\alpha^* a - \alpha a^\dagger\right]\right)\exp\left(\left[\alpha a^\dagger - \alpha^* a\right]\right) \\ &= \exp\left(\left[\alpha^* a - \alpha a^\dagger + \alpha a^\dagger - \alpha^* a\right]\right) \\ &= \exp([0]) = \mathbb{I} \end{aligned} \tag{83}$$

Thus we say that $D(\alpha)$ is unitary. There is a simpler the argument. The exponent of the exponential is anti-hermitian, so the operator is unitary. Now we move onto the squeeze operator.

Definition 2.2 (Squeeze Operator): The squeeze operator is defined as

$$S(\zeta) = \exp\left(\frac{1}{2}\left[\zeta^* a^2 - \zeta a^{\dagger 2}\right]\right)$$

where $\zeta = re^{i\theta}$ where $r > 0$.

Note again that $S^\dagger(\zeta) = S(-\zeta)$. Using the same steps again, we note that the squeeze operator is unitary. (This proof uses $\left[\zeta^* a^2 - \zeta a^{\dagger 2}, \zeta a^{\dagger 2} - \zeta^* a^2\right]$)

C : Photon antibunching

Photon Antibunching refers to a light field where the photon distribution is sub-poissonian. And, we know that the variance and mean of the photon number operator $\hat{n}$ are related to the second order correlation function according to the following relation

$$\frac{\delta\hat{n} - \langle\hat{n}\rangle}{\langle\hat{n}\rangle^2} = g^{(2)}(0) - 1 \tag{84}$$

When we say the distribution is sub-poissonian or super-poissonian, what we mean is that $\delta\hat{n} < \langle\hat{n}\rangle$ and $\delta\hat{n} > \langle\hat{n}\rangle$ respectively. So, in case of antibunched photons, we have sub-poissonian photon statistics, implying that

$$g^{(2)}(0) < 1 \tag{85}$$

The exact opposite scenario to this is called Photon Bunching, where we observe super-poissonian photon statistics, i.e. $\delta\hat{n} > \langle\hat{n}\rangle$ or, in terms of the second order correlation function,

$$g^{(2)}(0) > 1 \tag{86}$$

We also know, that the correlation function $g^{(2)}(0)$ is defined as,

$$g^{(2)}(0) = \frac{\left\langle\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\right\rangle}{\langle\hat{n}\rangle^2} \tag{87}$$

Which essentially corresponds to observing two photons at the detector simulatinously. For a coherent photon state, we have $g^{(2)}(0) = 1$. Hence, for a bunched state there's a higher probability of a detector observing two photons at once than that for a coherent state. The opposite is true for a antibunched state.