Definition of Squeezed States

Squeezed States are defined as the state obtained by the action of the operator $\hat{D}(\alpha)\hat{S}(\zeta)$ on the vaccum number state $|0\rangle$

$$|\alpha, \zeta\rangle = \hat{D}(\alpha)\hat{S}(\zeta)|0\rangle$$

Where the concerned operators are defined as,

$$\hat{D}(\alpha) = e^{-\frac{|\alpha|^2}{2}} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}}$$
$$\hat{S}(\zeta) = e^{\frac{1}{2}\left(\zeta^* \hat{a}^2 - \zeta \hat{a}^{\dagger 2}\right)}$$

where, $\zeta = re^{i\theta}$ with $r > 0$.

$\hat{D}(\alpha)$ and $\hat{S}(\zeta)$ are the translation and the squeeze operator respectively. It can be shown that,

$$\hat{D}(\alpha)^\dagger \hat{D}(\alpha) = \hat{S}(\zeta)^\dagger \hat{S}(\zeta) = 1$$

i.e., both the operators are unitary. This will help us a lot in the following section, where we will be looking at the expectations and variances of a few relevant quantities.

Mean Photon Number in a Squeezed State

We wish to compute $\langle \hat{n} \rangle$ for the squeezed state $|\alpha, \zeta\rangle$

We know that

$$\hat{n} = \hat{a}^\dagger \hat{a}$$

Now, to compute the epectation of $\hat{n}$ in the Squeezed state we have to evaluate the following expression

$$\begin{aligned}
\langle \hat{n} \rangle &= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) \hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \qquad \left[\because \hat{D}(\alpha)^\dagger \hat{D}(\alpha) = 1\right] \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \left[\because \hat{S}(\zeta)^\dagger \hat{S}(\zeta) = 1\right]
\end{aligned}$$

First, we evaluate the operator $\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha)$

$$\begin{aligned}
\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) &= e^{-|\alpha|^2} e^{-\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \hat{a} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \\
&= e^{-|\alpha|^2} e^{-\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \left(e^{\alpha \hat{a}^\dagger} \hat{a} + \left[\hat{a}, e^{\alpha \hat{a}^\dagger}\right] \right) e^{-\alpha^* \hat{a}}
\end{aligned}$$

Now let us compute the comutator relation $\left[\hat{a}, e^{\alpha \hat{a}^\dagger}\right]$ which is given by,

$$\left[\hat{a}, e^{\alpha \hat{a}^\dagger}\right] = \sum_{n=1}^{\infty} \frac{\alpha^n \left[\hat{a}, \hat{a}^{\dagger n}\right]}{n!}$$

We can easily show by induction that, $\left[\hat{a}, \hat{a}^{\dagger n}\right] = n \hat{a}^{\dagger n-1}$. Then the commutator evaluates to,

$$\begin{aligned}
\left[\hat{a}, e^{\alpha \hat{a}^\dagger}\right] &= \sum_{n=1}^{\infty} \frac{n \alpha^n \hat{a}^{\dagger n-1}}{n!} \\
&= \alpha \sum_{n=0}^{\infty} \frac{\left(\alpha \hat{a}^\dagger\right)^n}{n!} \\
&= \alpha e^{\alpha \hat{a}^\dagger}
\end{aligned}$$

Substituting this commutator relation back into the expression for $\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha)$ we get,

$$\begin{aligned}\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) &= e^{-|\alpha|^2} e^{-\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} \left(e^{\alpha \hat{a}^\dagger} \hat{a} + \alpha e^{\alpha \hat{a}^\dagger}\right) e^{-\alpha^* \hat{a}} \\ &= e^{-|\alpha|^2} e^{-\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}} (\alpha + \hat{a}) \\ &= \alpha + \hat{a}\end{aligned}$$

Taking the dagger of this equation on both sides we can also see that,

$$\hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) = \alpha^* + \hat{a}^\dagger$$

Now substituting these expressions for $\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha)$ and $\hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha)$ into our expression for $\langle \hat{n} \rangle$ we get,

$$\begin{aligned}\langle \hat{n} \rangle &= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger (\hat{a}^\dagger + \alpha^*) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger (\hat{a} + \alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\ &= \left\langle 0 \middle| \left(\hat{S}(\zeta)^\dagger \hat{a}^\dagger \hat{S}(\zeta) + \alpha^*\right) \left(\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) + \alpha\right) \middle| 0 \right\rangle \left[\because \hat{S}(\zeta)^\dagger \hat{S}(\zeta) = 1\right]\end{aligned}$$

So, now we compute $\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta)$. Let us define $A = \frac{1}{2}\left(\zeta {\hat{a}^\dagger}^2 - \zeta^* \hat{a}^2\right)$. Then,

$$\begin{aligned}\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) &= e^A \hat{a} e^{-A} \\ &= \sum_{n=0}^{\infty} \frac{[A, \hat{a}]_n}{n!}\end{aligned}$$

where $[A, B]_1 = [A, B]$, $[A, B]_2 = [A, [A, B]]$ and so on. Lets compute $[A, \hat{a}]$

$$\begin{aligned}[A, \hat{a}] &= -\frac{\zeta}{2}\left[\hat{a}, {\hat{a}^\dagger}^2\right] \\ &= -\frac{\zeta}{2} 2\hat{a}^\dagger \\ &= -\zeta \hat{a}^\dagger\end{aligned}$$

Similarly,

$$\begin{aligned}[A, \hat{a}]_2 &= [A, [A, \hat{a}]] \\ &= -\zeta \left[A, \hat{a}^\dagger\right] \\ &= \zeta \frac{\zeta^*}{2} \left[\hat{a}^2, \hat{a}^\dagger\right] \\ &= |\zeta|^2 \hat{a}\end{aligned}$$

We can see after this, that the results will be of a similar form when k has the same parity. It can be shown using induction that,

$$[A, \hat{a}]_n = \begin{cases} -\zeta |\zeta|^{n-1} \hat{a}^\dagger & \text{if } n \text{ is odd} \\ |\zeta|^n \hat{a}^\dagger & \text{if } n \text{ is even} \end{cases}$$

Then we can evaluate $\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta)$ to be,

$$
\begin{aligned}
\hat{S}(\zeta)^\dagger \hat{a}\hat{S}(\zeta) &= \sum_{n=0}^{\infty} \frac{1}{n!}[A, \hat{a}]_n \\
&= \hat{a}\sum_{k=0}^{\infty} \frac{|\zeta|^{2k}}{(2k)!} - \hat{a}^\dagger \sum_{k=0}^{\infty} \frac{\zeta|\zeta|^{2k}}{(2k+1)!} \\
&= \hat{a}\cosh(|\zeta|) - \hat{a}^\dagger \frac{\zeta}{|\zeta|} \sum_{k=0}^{\infty} \frac{|\zeta|^{2k+1}}{(2k+1)!} \\
&= \hat{a}\cosh(|\zeta|) - \hat{a}^\dagger \frac{\zeta}{|\zeta|} \sum_{k=0}^{\infty} \frac{|\zeta|^{2k+1}}{(2k+1)!} \\
&= \hat{a}\cosh(|\zeta|) - \hat{a}^\dagger \frac{\zeta}{|\zeta|} \sinh(|\zeta|) \\
&= \hat{a}\cosh(r) - \hat{a}^\dagger e^{i\theta} \sinh(r)
\end{aligned}
$$

Taking the dagger of this relation gives us

$$
\hat{S}(\zeta)^\dagger \hat{a}^\dagger \hat{S}(\zeta) = \hat{a}^\dagger \cosh(r) - \hat{a}e^{-i\theta}\sinh(r)
$$

Substituting these expressions into the expression for $\langle \hat{n} \rangle$ we get,

$$
\begin{aligned}
\langle \hat{n} \rangle &= \langle 0 | \left(\hat{a}^\dagger \cosh(r) - \hat{a}e^{-i\theta}\sinh(r) + \alpha^*\right)\left(\hat{a}\cosh(r) - \hat{a}^\dagger e^{i\theta}\sinh(r) + \alpha\right)|0\rangle \\
&= \langle 0 | |\alpha|^2 + \hat{a}\hat{a}^\dagger \sinh^2(r) | 0 \rangle \\
&= |\alpha|^2 + \sinh^2(r)
\end{aligned}
$$

Hence for a squeezed state, the mean photon number is given by

$$
\langle \hat{n} \rangle = |\alpha|^2 + \sinh^2(r)
$$

Variances in Squeezed States

Let us define Y_1 and Y_2 such that, $Y_1 + iY_2 = (X_1 + X_2)e^{-i\frac{\theta}{2}} := \hat{b}$. Then, we have $\hat{b} = \hat{a}e^{-i\frac{\theta}{2}}$. And we also have,

$$
\hat{S}(\zeta) = e^{\frac{1}{2}\left(\hat{b}^2 - \hat{b}^{\dagger 2}\right)}
$$

Observe that, $\hat{b}^\dagger \hat{b} = \hat{a}^\dagger \hat{a} = \hat{n}$ and $\left[\hat{b}^\dagger, \hat{b}\right] = \left[\hat{a}^\dagger, \hat{a}\right] = 1$ Also, lets define $\beta = \alpha e^{-\frac{\theta}{2}}$

Let's compute δY_1 and δY_2 for the squeezed state $|\alpha, \zeta\rangle$

$$
\begin{aligned}
\delta Y_1 &= \langle Y_1^2 \rangle - \langle Y_1 \rangle^2 \\
&= \frac{1}{4}\left\langle \left(\hat{b} + \hat{b}^\dagger\right)^2 \right\rangle - \left\langle \hat{b} + \hat{b}^\dagger \right\rangle^2 \\
&= \frac{1}{4}\left[\left\langle \left(\hat{b}^2 + \hat{b}^{\dagger 2} + 2\hat{b}^\dagger \hat{b} + 1\right)^2 \right\rangle - \left(\left\langle \hat{b} \right\rangle + \left\langle \hat{b}^\dagger \right\rangle\right)^2\right] \\
&= \frac{1}{4}\left[\left\langle \hat{b}^2 \right\rangle + \left\langle \hat{b}^{\dagger 2} \right\rangle + 2\langle \hat{n} \rangle + 1 - \left(\left\langle \hat{b} \right\rangle + \left\langle \hat{b}^\dagger \right\rangle\right)^2\right]
\end{aligned}
$$

First, we'll compute $\left\langle \hat{b} \right\rangle$

$$
\begin{aligned}
\left\langle \hat{b} \right\rangle &= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{b} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger (\hat{a} + \alpha) \middle| 0 \right\rangle e^{-i\frac{\theta}{2}} \\
&= \left\langle 0 \middle| \left(\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) + \alpha \right) \middle| 0 \right\rangle e^{-\frac{\theta}{2}} \\
&= \left\langle 0 \middle| \hat{a} \cosh(r) - e^{i\theta} \sinh(r) + \alpha \middle| 0 \right\rangle e^{-\frac{\theta}{2}} \\
&= \alpha e^{-i\frac{\theta}{2}}
\end{aligned}
$$

Similarly,

$$
\left\langle \hat{b}^\dagger \right\rangle = \alpha^* e^{i\frac{\theta}{2}}
$$

Now, we compute $\left\langle \hat{b}^2 \right\rangle$

$$
\begin{aligned}
\left\langle \hat{b}^2 \right\rangle &= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{b}^2 \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{b} \hat{D}(\alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{b} \hat{D}(\alpha) \hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger (\hat{a} + \alpha) \hat{S}(\zeta) \hat{S}(\zeta)^\dagger (\hat{a} + \alpha) \hat{S}(\zeta) \middle| 0 \right\rangle e^{-i\theta} \\
&= \left\langle 0 \middle| \left(\hat{S}(\zeta)^\dagger \hat{a} \hat{S}(\zeta) + \alpha \right)^2 \middle| 0 \right\rangle e^{-i\theta} \\
&= \left\langle 0 \middle| \left(\hat{a} \cosh(r) - e^{i\theta} \hat{a}^\dagger \sinh(r) + \alpha \right)^2 \middle| 0 \right\rangle e^{-i\theta} \\
&= \left\langle 0 \middle| -\hat{a} \hat{a}^\dagger e^{i\theta} \cosh(r) \sinh(r) + \alpha^2 \middle| 0 \right\rangle e^{-i\theta} \\
&= \alpha^2 e^{-i\theta} - \sinh(r) \cosh(r)
\end{aligned}
$$

Similarly,

$$
\left\langle {\hat{b}^\dagger}^2 \right\rangle = (\alpha^*)^2 e^{i\theta} - \sinh(r) \cosh(r)
$$

Substituting these expressions in the expression for δY_1 we get,

$$
\begin{aligned}
\delta Y_1 &= \frac{1}{4} \left(\alpha^2 e^{-i\theta} + (\alpha^*)^2 e^{i\theta} - 2 \sinh(r) \cosh(r) + 2|\alpha|^2 + 2 \sinh^2(r) + 1 - \left(\alpha e^{-i\frac{\theta}{2}} + \alpha e^{i\frac{\theta}{2}} \right)^2 \right) \\
&= \frac{1}{4} \left(\left(\alpha e^{-i\frac{\theta}{2}} + \alpha e^{i\frac{\theta}{2}} \right)^2 + 1 + 2 \sinh^2(r) - 2 \cosh(r) \sinh(r) - \left(\alpha e^{-i\frac{\theta}{2}} + \alpha e^{i\frac{\theta}{2}} \right)^2 \right) \\
&= \frac{1}{4} (1 + 2 \sinh^2(r) - 2 \cosh(r) \sinh(r)) \\
&= \frac{1}{4} (\cosh(2r) - \sin(2r)) \\
&= \frac{1}{4} e^{-2r}
\end{aligned}
$$

With a very similar computation we can show that,

$$
\delta Y_2 = \frac{1}{4} e^{2r}
$$

From these fluctuations we can clearly see that the amplitude is decaying exponentially in one of the quadratures while, increasing in the other one at the same rate, with r. Also, the fluctuations are independent of α.

Second Order Correlation Function for the Squeezed State

Now let us compute the second order correlation function for this state $|\alpha, \zeta\rangle$ We know that,

$$g^{(2)}(0) = \frac{\langle : \hat{n}^2 : \rangle}{\langle \hat{n} \rangle^2}$$

So, we must compute $\langle : \hat{n}^2 : \rangle$

$$\begin{aligned}
\langle : \hat{n}^2 : \rangle &= \left\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a} \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a} \hat{D}(\alpha)\hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha)\hat{S}(\zeta)\hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha)\hat{S}(\zeta)^\dagger \hat{S}(\zeta)\hat{D}(\alpha)^\dagger \hat{a}\hat{D}(\alpha)\hat{S}(\zeta)\hat{S}(\zeta)^\dagger \hat{D}(\alpha)^\dagger \hat{a}\hat{D}(\alpha)\hat{S}(\zeta) \middle| 0 \right\rangle \\
&= \left\langle 0 \middle| \hat{S}(\zeta)^\dagger (\alpha * + \hat{a}^\dagger) \hat{S}(\zeta)\hat{S}(\zeta)^\dagger (\alpha * + \hat{a}^\dagger) \hat{S}(\zeta)\hat{S}(\zeta)^\dagger (\alpha + \hat{a})\hat{S}(\zeta)\hat{S}(\zeta)^\dagger (\alpha + \hat{a})\hat{S}(\zeta) \right\rangle \\
&= \left\langle 0 \middle| \left(\hat{a}^\dagger \cosh(r) - e^{-i\theta}\hat{a}\sinh(r) + \alpha^*\right)^2 \left(\hat{a}\cosh(r) - e^{i\theta}\hat{a}^\dagger \sinh(r) + \alpha\right)^2 \middle| 0 \right\rangle \\
&= \Big\langle 0 \Big| \hat{a}\hat{a}^\dagger \hat{a}\hat{a}^\dagger \sinh^2(r)\cosh^2(r) + \hat{a}\hat{a}\hat{a}^\dagger \hat{a}^\dagger \sinh^4(r) \\
&\quad + \hat{a}\hat{a}^\dagger \left(2|\alpha|^2 \sinh^2(r) - \left(\alpha^2 e^{-i\theta} + (\alpha^*)^2 e^{i\theta}\right)\sinh(r)\cosh(r)\right) \Big| 0 \Big\rangle \\
&= 2\sinh^4(r) + \sinh^2(r)\cosh^2(r) + 2|\alpha|^2\sinh^2(r) - \left(\alpha^2 e^{-i\theta} + (\alpha^*)^2 e^{i\theta}\right)\sinh(r)\cosh(r) \\
&= 3\sinh^4(r) + (1 + 2|\alpha|^2)\sinh^2(r) - \mathrm{Re}\left(\alpha^2 e^{-i\theta}\right)\sinh(2r)
\end{aligned}$$

Hence, the correlation function turns out to be

$$g^{(2)}(0) = \frac{3\sinh^4(r) + (1 + 2|\alpha|^2)\sinh^2(r) - \mathrm{Re}\left(\alpha^2 e^{-i\theta}\right)\sinh(2r)}{\left(|\alpha|^2 + \sinh^2(r)\right)^2}$$

Now, we will discuss two limiting conditions for this expression.

First we will look at the limit when $|\alpha| \ll 1$, i.e., we are considering the squeezed state $|0, \zeta\rangle$. Then we get the following form of the correlation function.

$$\begin{aligned}
g^{(2)}(0) &= \frac{3\sinh^4(r) + \sinh^2(r)}{\sinh^4(r)} \\
&= 3 + \frac{1}{\sinh^2(r)} \\
&= 3 + \frac{1}{\langle n \rangle}
\end{aligned}$$

As we can see in the small α limit, no matter what value of ζ we choose, we always have $g^{(2)}(0) > 1$ So, in this case we will always have photon bunching.

Now let's look at the second limiting case, when $|\alpha|^2 \gg \sinh^2(r)$

$$\begin{aligned}
g^{(2)}(0) &= \frac{3\sinh^4(r) + 2|\alpha|^2\sinh^2(r) - \mathrm{Re}\left(\alpha^2 e^{-i\theta}\right)\sinh(2r)}{|\alpha|^4} \\
&= 3\left(\frac{\sinh^2(r)}{|\alpha|^2}\right)^2 + 2\frac{\sinh^2(r)}{|\alpha|^2} - \cos(2\varphi - \theta)\frac{\sinh(2r)}{|\alpha|^2} \\
&\approx -\cos(2\varphi - \theta)\frac{\sinh(2r)}{|\alpha|^2}
\end{aligned}$$

From this expression we can see that in this limit, if $r > 0$, we eventually get $g^{(2)}(0) < 1$ i.e. photon anti-bunching and sub poisonnian photon statistics, and for $r < 0$ we also get $g^{(2)}(0) > 1$ i.e. photon bunching and super poissonian photon statistics.