MA2202 :Probability 1

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1. **Question 1**
   There are n boxes numbered $1, 2, \ldots, n$, among which the r^{th} box contains $r - 1$ white cubes and $n - r$ red cubes. Suppose, we choose a box at random and we remove two cubes from it, one after another, without replacement.

   (a) Find the probability of the second cube being red.

   (b) Find the probability of the second cube being red, given that the first cube is red.

   Solution:

2. **Question 2**
Let $(\Omega, \mathcal{E}, P)$ be a probability space and let $A_1, A_2, \ldots, A_n \in \mathcal{E}$ and $P(\bigcap_{i=1}^{n} A_i) \neq 0$. Show that

$$P(\bigcap_{i=1}^{n} A_i) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)\ldots P(A_n|\bigcap_{i=1}^{n-1} A_i) \tag{1}$$

Solution:
Note: $P(\bigcap_{i=1}^{m} A_i) \neq 0$ as $\bigcap_{i=1}^{n} A_i \subseteq \bigcap_{i=1}^{m} A_i$, thus $P(\bigcap_{i=1}^{n} A_i \leq P(\bigcap_{i=1}^{m} A_i)) \quad \forall m \in 1, 2, \ldots, n$.

Method 1: We solve this by induction.

Case n=1
$P(A_1) = P(A_1)$ which is true.
Case n = 2
By definition of conditional probability, we have $P(A_1|A_2) = \dfrac{P(A_1 \cap A_2)}{P(A_2)}$.

Induction Hypothesis for n=k

$$P(\bigcap_{i=1}^{k} A_i) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)\ldots P(A_k|\bigcap_{i=1}^{k-1} A_i) \tag{2}$$

We now prove this for n = k+1. Let $\mathcal{A} = \bigcap_{i=1}^{k} A_i$. By using the definition of conditional probability or the case n=2, $P(A_{k+1}|\mathcal{A}) = \dfrac{P(A_{k+1} \cap \mathcal{A})}{P(\mathcal{A})}$.

$$\begin{aligned}
P(\bigcap_{i=1}^{k+1} A_i) &= P(A_{k+1} \cap \mathcal{A}) \\
&= P(A_{k+1}|\mathcal{A})P(\mathcal{A}) \\
&= P(A_{k+1}|\bigcap_{i=1}^{k} A_i)P(\bigcap_{i=1}^{k} A_i) \\
&= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)\ldots P(A_k|\bigcap_{i=1}^{k-1} A_i)P(A_{k+1}|\bigcap_{i=1}^{k} A_i) \\
&\qquad \text{(Using Induction Hypothesis)} \\
&= P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)\ldots P(A_{k+1}|\bigcap_{i=1}^{k} A_i)
\end{aligned}$$

Thus we have proved the equation (1) $\forall\, n \in \mathbb{N}$.

Method 2: Let $B_k = \bigcap_{i=1}^{k} A_i$. Then we need to prove that

$$P(B_n) = P(A_1)P(A_2|A_1)P(A_3|B_2)\ldots P(A_n|B_{n-1})$$

We just use our definition of conditional probability i.e. $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

$$\begin{aligned}
P(A_1)P(A_2|A_1)P(A_3|B_2)\dots P(A_n|B_{n-1}) &= P(A_1)\frac{P(A_2 \cap A_1)}{P(A_1)}\frac{P(A_3 \cap B_2)}{P(B_2)}\cdots\frac{P(B_n)}{P(B_{n-1})} \\
&= \cancel{P(A_1)}\frac{\cancel{P(A_2 \cap A_1)}}{\cancel{P(A_1)}}\frac{\cancel{P(A_3 \cap B_2)}}{\cancel{P(B_2)}}\cdots\frac{P(B_n)}{\cancel{P(B_{n-1})}} \\
&= P(B_n) \\
&= P(\bigcap_{i=1}^{n} A_i)
\end{aligned}$$

3. **Question 3**

Let $(\Omega, \mathcal{E}, P)$ be a probability space and let $A_1, A_2, \dots, A_n \in \mathcal{E}$ be pairwise mutually exclusive. Let $A = \bigcup_{i=1}^{\infty} A_i$ and $B \in \mathcal{E}$ and $P(B) \neq 0$. Show that

$$P(A|B) = \sum_{i=1}^{\infty} P(A_i|B) \tag{3}$$

Solution:

4. **Question 4**

We are familiar with the famous Monty Hall problem. Now suppose, instead of 3 doors, there are n doors, only one among which has a prize behind it.

(a) Find the probability of winning upon switching given that Monty opens k doors. Will switching benefit you?

(b) Find the probability of winning upon switching given that Monty opens maximum num- ber of doors. Will switching benefit you?

(c) Find the probability of winning upon switching given that Monty opens no doors. Will switching benefit you?

Solution:

5. **Question 5**

Dropping two points uniformly at random on $[0, 1]$, the unit interval is divided into three segments. Find the probability that the three segments obtained in this way form a triangle.

Solution: