Bayesian Inference and Information based model check of Langevin Systems

Sabarno Saha [1]   Dr. Rajesh Singh [2]

[1]Department of Physical Sciences IISERK

[2]Department of Physical Sciences IIT Madras

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Introduction

1. Stochastic Thermodynamics
   - Stochastic Processes
   - Brownian Motion
   - Stochastic Differential equation
   - Stochastic Integrals
   - Langevin Equation
   - Fokker-Planck Equation
   - Euler-Maruyuma Integrator
2. Bayesian Inference
   - Bayes Theorem
   - Prior Assignment
   - Example

Introduction(Contd.)

4. Information Theory
   - Shannon Information
   - Fisher Information
   - Kullback Liebler Divergence
5. Nested Sampling
   - Likelihood Function
   - MCMC
   - Evidence Calculation
6. Model Check
   - Information check
   - Scaling of Steps
   - p-value check

Stochastic Thermodynamics

Definition

A stochastic process is a sequence of random variables where the indexing of the variables often carries the notion of time.

For example, we have Brownian motion, which is represented using the Wiener process (a stochastic process).

Definition

A Wiener process $\hat{W}$ is a stochastic process which has the following conditional pdf

$$P(\hat{W}(t + \Delta t) = x | \hat{W}(t) = x') = \frac{1}{\sqrt{2\pi\Delta t}} \exp\left(-\frac{(x - x')^2}{2\Delta t}\right)$$

The initial conditions are $P(\hat{W}(0) = x) = \delta(x)$ where $\delta(x)$ is the Dirac Delta function.

Stochastic Thermodyamics : Noise

Definition

The White Gaussian noise $\hat{\xi}(x)$ is defined as

$$\lim_{\Delta t \to 0} \frac{\hat{W}(t + \Delta t) - \hat{W}(t)}{\Delta t}$$

Definition

The White Gaussian noise $\hat{\xi}(x)$ is also defined as any random variable that has the following properties

- $\mathbb{E}_t[\hat{\xi}(t)] = 0$
- $\mathbb{E}_t[\hat{\xi}(t')\hat{\xi}(t)] \propto \delta(t - t')$

Stochastic Thermodynamics : SDE

Definition

A stochastic differential equation is an equation of the form

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = a(\hat{x}, t) + b(\hat{x}, t) \circ \hat{\zeta}(x)$$

where $\hat{\zeta}(t)$ is some form of stochastic noise and $\circ$ is the type of product rule being used.

The two most common type of product rules are the Ito and the Stratonovich product rules. We are not going to discuss on this further.

Stochastic Thermodynamics : Langevin Equation

This is type of equation provides a nice way to model SDE, named after Paul Langevin(also known for his contributions towards the twin paradox). The Langevin equation is basically Newton's second law of motion but with an added noise term.

Definition

A Langevin Equation is an equation of the form

$$m\frac{\mathrm{d}\hat{v}}{\mathrm{d}t} = a(\hat{x}, \hat{v}, t) + b(\hat{x}, \hat{v}, t) \circ \hat{\xi}(t)$$

where a and b are known functions.

Definition (Fokker Planck Equation)

Suppose we have the following langevin equation

$$m\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = a(\hat{x}, t) + b(\hat{x}, t)\hat{\zeta}(t) \tag{1}$$

where $\zeta(t)$ satisfies the following properties.

- $\mathbb{E}_t[\zeta(t)] = 0$
- $\mathbb{E}_t[\zeta(t)^2] = 1$

The corresponding Fokker-Planck is given as

$$\frac{\partial P(x, t)}{\partial t} = -\frac{\partial}{\partial x}(a(x, t)P(x, t)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(b(x, t)^2 P(x, t)) \tag{2}$$

Definition (EM Integrator)

To simulate Stochastic processes, we use the Euler-Maruyama Integrator which is given as, which is just the discretized SDE,

$$\hat{x}(t + dt) = \hat{x}(t) + a(\hat{x}, t)dt + b(\hat{x}, t)\sqrt{dt}\hat{N} \tag{3}$$

where $\hat{N} \sim Normal(0, 1)$

Nested Sampling : Evidence

Model Check