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Introduction to Open Quantum Systems

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Course Outline

This is a collection of my notes on a series of five lectures delivered by Dr Manas Kulkarni for the Bangalore School on Statistical Physics-XII 2021.

- Lecture 1
 1. Motivation and General Construction of Open Quantum Systems
 2. System Reservoir Approach
- Lecture 2
 1. Quantum Master Equation(QME) : A general Setup
 2. Application to damped harmonic oscillator
- Lecture 3
 1. Quantum Langevin Equation(QLE)
 2. Comparisons between QME and QLE
 3. Transport through a system coupled to multiple reservoirs
- Lecture 4
 1. Open two-level/Multi-level systems: Perturbative Approach and exact results
 2. Jaynes-Cummings Model: Exact Solutions
- Lecture 5
 1. Dicke Model and Quantum Phase transitions
 2. Spectral Signatures in closed and open Dicke Model
 3. Connections to Hermitian and Non-Hermitian Random Matrix Theory.

There are also 3 tutorials accompanying the 5 lectures

- Tutorial 1
 1. Density matrix Approach to quantum Mechanics
 2. Quantum Mechanics of Composite Systems
 3. Partial Traces and Reduced Density Matrices of Subsystems
- Tutorial 2
 1. Numerical algorithm for the solution of the Driven Dissipative Jaynes-Cummings model.
 2. Numerical implementation using MATLAB.
- Tutorial 3 (July 9, Friday, 2021)
 1. Numerical algorithm for finding the spectrum of the Liouvillian of the dissipative Dicke model.
 2. Numerical implementation using MATLAB.

1 Introduction

We want to deal with problems where we have a quantum system interacting with some "Reservoir/ Bath/ Environment". Most of the systems that we have dealt with so far were closed systems which were isolated to the outer environment. We now relax the isolation condition and open up the system to the environment. This gives rise to Open Quantum Systems.

The subject aims to introduce the concept of dissipation and drive into quantum mechanics. If these elements are naively phenomenologically modeled and massaged into quantum mechanics, it might introduce inconsistencies in Heisenberg's Uncertainty Principle or the Commutation relations.

A helpful reference is found at

1.1 Examples of Such Systems

1.1.1 Damped Quantum Harmonic Oscillator

Most formal treatments of Quantum Mechanics use the quantum harmonic oscillator as the first toy model solved analytically. The damped harmonic oscillator is a toy example used here. This will be solved properly in Lecture 2.

This can be viewed as a single mode of an electromagnetic field in a lossy cavity (leaky cavity, cavity with imperfect mirrors). A nice solution can also be found at

1.1.2 Damped two level systems

These are two-level systems such as qubits that are subject to decay or dephasing due to its coupling to the environment

2 The system Reservoir approach

The aim here is to model the interaction between the system "S" and the environment "R" using a general Hamiltonian of the form

$$\begin{aligned} H &= H_S \otimes \mathbb{I}_R + \mathbb{I}_S \otimes H_R + H_{SR}(t) \\ &= H_S + H_R + H_{SR} \end{aligned} \quad (1)$$

where H_S and H_R is the Hamiltonian governing the evolution of the system and the reservoir respectively and H_{SR} is the Hamiltonian governing the coupling between the system and the reservoir. In most cases, the evolution of the reservoir doesn't concern us and it can be specified using its properties like its temperature, energy, or the density of states. Let us assume that the reservoir is large compared to the system in question. This means that the number of degrees of freedom associated with the reservoir is much larger than the number of degrees of freedom associated with the system. The evolution of the system is our main concern now, rather than the whole system+reservoir $S \otimes R$. Let $\chi(t)$ be the density operator for the whole system $S \otimes R$. Thus the reduced density matrix for the system is given by taking a partial trace over the states in the Hilbert space of the reservoir. The Density Matrix approach to Quantum Mechanics is discussed later under the Section "Tutorial 2". The reduced density matrix is given by

$$\rho(t) = \text{Tr}_R[\chi(t)] \quad (2)$$

where ρ is the reduced density matrix over the system of interest, $\text{Tr}_R[\]$ is the partial trace over the reservoir states and χ is the density operator for the system $S \otimes R$.

2.1 Liouville von-Neuman Equation

Let \hat{O} be an operator acting on the Hilbert Space associated with the System S. Then the average of the operator can be computed its average in the Schrodinger Picture using the already known density matrix $\rho(t)$. Note that since \hat{O} acts only on the Hilbert Space of S, $\hat{O} = \hat{O}_S \otimes \mathbb{I}_R$

$$\begin{aligned} \langle \hat{O} \rangle &= \text{Tr}_{S \otimes R}[\hat{O}\chi(t)] = \text{Tr}_S[\hat{O}_S \text{Tr}_R[\chi(t)]] \\ &= \text{Tr}_S[\hat{O}\rho(t)] \end{aligned} \quad (3)$$

The goal is to obtain the evolution of the density matrix $\rho(t)$ with time. This is given by the Liouville von-Neumann equation

$$\frac{d\chi(t)}{dt} = \dot{\chi}(t) = \frac{1}{i\hbar} [H, \chi(t)] \quad (4)$$

where H is the full Hamiltonian $H = H_S + H_R + H_{SR}$.

The problem is taken to the interaction picture where the operators evolve with time using the time evolution operator U_0 corresponding to uncoupled Hamiltonian $H_0 = H_S + H_R$. This separates the rapid motion due to $H_S + H_R$ from the slow motion due to H_{SR} . Note that we have an assumption that the uncoupled Hamiltonian is independent of time. Then the form of the evolution of the full density matrix is then given as

$$\tilde{\chi}(t) = e^{i/\hbar(H_S+H_R)t} \chi(t) e^{-i/\hbar(H_S+H_R)t} \quad (5)$$

Using eq. (5) and the Liouville von-Neumann equation, we get

$$\begin{aligned} \dot{\tilde{\chi}}(t) &= \frac{i}{\hbar} (H_S + H_R) \tilde{\chi}(t) - \frac{i}{\hbar} \tilde{\chi}(t) (H_S + H_R) + e^{i/\hbar(H_S+H_R)t} \dot{\chi}(t) e^{-i/\hbar(H_S+H_R)t} \\ &= \frac{1}{i\hbar} [\tilde{H}_{SR}(t), \tilde{\chi}(t)] \end{aligned} \quad (6)$$

Also note that the operator $H_{SR}(t) = e^{i/\hbar(H_S+H_R)t} H_{SR} e^{-i/\hbar(H_S+H_R)t}$. Using eq. (7), we can write the evolution of the reduced density matrix as

$$\tilde{\chi}(t) = \chi(0) + \frac{1}{i\hbar} \int_0^t dt' [\tilde{H}_{SR}(t'), \tilde{\chi}(t')]$$

Using this we get,

$$\dot{\tilde{\chi}}(t) = \frac{1}{i\hbar} [\tilde{H}_{SR}(t), \chi(0)] - \frac{1}{\hbar^2} \int_0^t dt' [\tilde{H}_{SR}(t), [\tilde{H}_{SR}(t'), \tilde{\chi}(t')]] \quad (7)$$

2.2 Born and Markov Approximation

To simplify the previous equation to actually understand stuff from it, we use some approximations. So let us set the stage for the application of these approximations.

Let the interaction be turned on at $t = 0$. The systems are assumed to be initially uncorrelated. This means that the initial density matrix $\chi(0) = \tilde{\chi}(0)$ factorizes as

$$\chi(0) = \rho(0) R_0 \quad (8)$$

where R_0 is the initial thermal density operator for the reservoir. More on this later. Note that

$$Tr_R[\tilde{\chi}(t)] = e^{i/\hbar(H_S+H_R)t} Tr_R[\chi(t)] e^{-i/\hbar(H_S+H_R)t} = \tilde{\rho}(t)$$

Tracing over the Hilbert space of the reservoir,

$$\dot{\tilde{\rho}}(t) = -\frac{1}{\hbar^2} \int_0^t dt' Tr_R[\tilde{H}_{SR}(t), [\tilde{H}_{SR}(t'), \tilde{\chi}(t')]]$$

with the assumption of $\frac{1}{i\hbar} Tr_R([\tilde{H}_{SR}(0), \chi(0)]) = 0$. A note on this assumption is provided later.

Definition 1 (Quantum Master Equation):

The Quantum Master Equation is given by

$$\dot{\tilde{\rho}}(t) = -\frac{1}{\hbar^2} \int_0^t dt' Tr_R[\tilde{H}_{SR}(t), [\tilde{H}_{SR}(t'), \tilde{\chi}(t')]] \quad (9)$$

with the assumption of $\frac{1}{i\hbar} Tr_R([\tilde{H}_{SR}(0), \chi(0)]) = 0$.

Let us come to the Born approximation. Let us assume that the coupling between the System and the Reservoir is weak. This means that the interaction Hamiltonian is small compared to the uncoupled Hamiltonian. Note that the total density matrix at time $t = 0$ is factorizes as $\chi(0) = \rho(0)R_0$, which means there is no correlation between the system and the reservoir at $t = 0$. The weak coupling assumption says that $\tilde{\chi}(t)$ shows deviations from the uncorrelated state only upto an order $O(H_{SR})$. Also note that we have assumed that the reservoir is quite large. This means that the weak coupling does not appreciably affect the reservoir. A note on this relating to reservoir correlation times, is given later. This weak coupling assumption gives us

$$\tilde{\chi}(t) = \tilde{\rho}(t)R_0 + O(H_{SR})$$

This is called the Born Approximation.

Definition 2 (Born Approximation):

The Born Approximation is the weak coupling assumption which says that the interaction Hamiltonian causes the density matrix to deviate from the uncorrelated state only upto an order $O(H_{SR})$. Mathematically it is given by

$$\tilde{\chi}(t) = \tilde{\rho}(t)R_0 + O(H_{SR}) \quad (10)$$

Ignoring terms of order $O(H_{SR}^3)$ in the Quantum Master Equation, we get the Quantum Master Equation in the Born Approximation.

$$\dot{\tilde{\rho}}(t) = -\frac{1}{\hbar^2} \int_0^t dt' \text{Tr}_R[\tilde{H}_{SR}(t), [\tilde{H}_{SR}(t'), \tilde{\rho}(t')R_0]] \quad (11)$$

Note that the equation is still non Markovian. The integrand in the above equation, depends on $\tilde{\rho}(t)$ at all times upto t . This non markovian nature is manifested in the fact that the system at a previous time might affect the reservoir, which can then affect the system at a later time. This causes the system to have a memory of its past, which relates to its current non Markovian nature. A note on this will be given after this, which relates to the reservoir correlation times.

Definition 3 (Markov Approximation):

The Markov Approximation is the assumption that the system does not have a memory of its past. This means that the system does not remember its past states. This is mathematically given by

$$\tilde{\rho}(t') = \tilde{\rho}(t) \quad \forall t' < t \quad (12)$$

Definition 4 (Born-Markov Master Equation):

The Born-Markov Master Equation is given by

$$\dot{\tilde{\rho}}(t) = -\frac{1}{\hbar^2} \int_0^t dt' \text{Tr}_R[\tilde{H}_{SR}(t), [\tilde{H}_{SR}(t'), \tilde{\rho}(t)R_0]] \quad (13)$$

with the assumption of $\frac{1}{i\hbar} \text{Tr}_R\left([\tilde{H}_{SR}(0), \chi(0)]\right) = 0$. This takes into the account the Born and Markov Approximations.

The above equation can be expanded to give the Redfield Equation. A derivation is given at ??

Remark 5. Why does $\tilde{H}_{SR}(t')$ not cause an issue?

2.3 Note on Markov Approximation and Reservoir correlations

There are physical grounds on which we can justify the Markovian Approximation. The non-Markovian variant manifests in the fact that the system can affect the reservoir at some time and the reservoir might affect the system at a later time. This results in the system being able to remember its past states. Now, here is our assumption of the reservoir being large and having low correlation times. The reservoir evolves rapidly and cannot remember how it is affected by the system at a past time. Thus, the reservoir doesn't allow the retention of memory of past states. This can happen say when the reservoir

is held at thermal equilibrium. The markov approximation relies on the fact that there are two separate time scales for the evolution of the composite system+reservoir. There is a slow time scale over which the system evolves and a fast time scale over which the reservoir correlation functions decay.

The determination of these time scales are quite involved. A small remark will be included in ?? for the determination of the system time scale in the case of the Damped Harmonic Oscillator.

3 The Damped Harmonic Oscillator

3.1 The Lindblad Master Equation

3.2 Detailed balance Condition

3.3 Expectation values for Operators

4 Tutorial 1