Bayesian Inference and Information based model check of Langevin Systems

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Introduction

1. Stochastic Thermodynamics
    - Stochastic Processes
    - Brownian Motion
    - Stochastic Differential equation
    - Stochastic Integrals
    - Langevin Equation
    - Fokker-Planck Equation
    - Euler-Maruyuma Integrator
2. Bayesian Inference
    - Bayes Theorem
    - Prior Assignment
    - Example

Introduction(Contd.)

4. Information Theory
    - Shannon Information
    - Fisher Information
    - Kullback Liebler Divergence
5. Nested Sampling
    - Likelihood Function
    - MCMC
    - Evidence Calculation
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    - p-value check

Definition
A stochastic process is a sequence of random variables where the indexing of the variables often carries the notion of time.

For example, we have Brownian motion, which is represented using the Wiener process (a stochastic process).

Definition
A Wiener process $\hat{W}$ is a stochastic process which has the following conditional pdf

$$P(\hat{W}(t+\Delta t) = x | \hat{W}(t) = x') = \frac{1}{\sqrt{2\pi\Delta t}} \exp\left(-\frac{(x-x')^2}{2\Delta t}\right)$$

The initial conditions are $P(\hat{W}(0) = x) = \delta(x)$ where $\delta(x)$ is the Dirac Delta function.

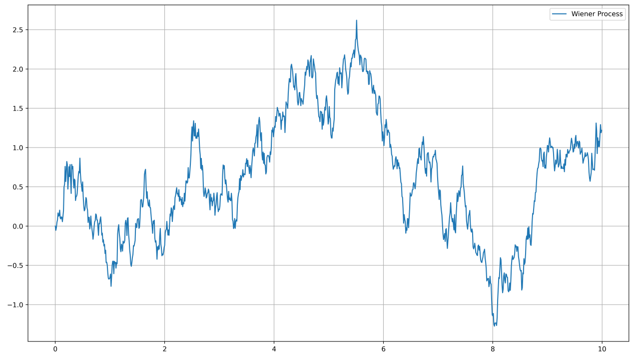

Figure 1: Wiener Process

Definition

The White Gaussian noise $\hat{\xi}(x)$ is defined as

$$\lim_{\Delta t \to 0} \frac{\hat{W}(t+\Delta t) - \hat{W}(t)}{\Delta t}$$

Definition

The White Gaussian noise $\hat{\xi}(x)$ is also defined as any random variable that has the following properties

- $\mathbb{E}_t[\hat{\xi}(t)] = 0$
- $\mathbb{E}_t[\hat{\xi}(t')\hat{\xi}(t)] \propto \delta(t - t')$

We can have other types of noise as well like telegraphic noise, which we'll see briefly at the very end.

Definition
A stochastic differential equation is an equation of the form

$$\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = a(\hat{x}, t) + b(\hat{x}, t) \circ \hat{\zeta}(x)$$

where $\hat{\zeta}(t)$ is some form of stochastic noise and $\circ$ is the type of product rule being used.

The two most common type of product rules are the Ito and the Stratonovich product rules. We are not going to discuss on this further.

This is type of equation provides a nice way to model SDE, named after Paul Langevin(also known for his contributions towards the twin paradox). The Langevin equation is basically Newton's second law of motion but with an added noise term.

Definition
A Langevin Equation is an equation of the form

$$m\frac{\mathrm{d}\hat{v}}{\mathrm{d}t} = a(\hat{x}, \hat{v}, t) + b(\hat{x}, \hat{v}, t) \circ \hat{\xi}(t)$$

where a and b are known functions.

Definition (Fokker Planck Equation)
Suppose we have the following langevin equation

$$m\frac{\mathrm{d}\hat{x}}{\mathrm{d}t} = a(\hat{x}, t) + b(\hat{x}, t)\hat{\zeta}(t) \tag{1}$$

where $\zeta(t)$ satisfies the following properties.

- $\mathbb{E}_t[\zeta(t)] = 0$
- $\mathbb{E}_t[\zeta(t)^2] = 1$

The corresponding Fokker-Planck is given as

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}(a(x,t)P(x,t)) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(b(x,t)^2 P(x,t)\right) \tag{2}$$

Definition (EM Integrator)
To simulate Stochastic processes, we use the Euler-Maruyama Integrator which is given as, which is just the discretized SDE,

$$\hat{x}(t+dt) = \hat{x}(t) + a(\hat{x},t)dt + b(\hat{x},t)\sqrt{dt}\hat{N} \tag{3}$$

where $\hat{N} \sim Normal(0,1)$

Bayesian Inference : Prior Assignment

Bayesian Inference : Example

Information Theory : Fisher Information

Information Theory : Typical set of models

Information Theory : Jeffrey's prior

Nested Sampling : Likelihood

Nested Sampling : MCMC

Nested Sampling : Evidence

Nested Sampling : Error

Model Check

Model Check