Homework 9 Solutions

December 7, 2019

10.1

$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ is one of the famous "Bell states", a highly entangled state of its two qubits. In this question we examine some of its strange properties.

(a) Suppose this Bell state could be decomposed as the (tensor) product of two qubits (recall the box on page 314), the first in state $\alpha_0|0\rangle + \alpha_1|1\rangle$ and the second in state $\beta_0|0\rangle + \beta_1|1\rangle$. Write four equations that the amplitudes $\alpha_0, \alpha_1, \beta_0$ and β_1 must satisfy. Conclude that the Bell state cannot be so decomposed.

(b) What is the result of measuring the first qubit of $|\psi\rangle$?

(c) What is the result of measuring the second qubit after measuring the first qubit?

(d) If the two qubits in state $|\psi\rangle$ are very far from each other, can you see why the answer to (c) is surprising?

Solution

(a) (By contradiction) The tensor product of two qubits (left) must be equal to the given Bell state:

$$\alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

which implies

$$\begin{cases} \alpha_0\beta_0 = \frac{1}{\sqrt{2}} \\ \alpha_0\beta_1 = 0 \\ \alpha_1\beta_0 = 0 \\ \alpha_1\beta_1 = \frac{1}{\sqrt{2}} \end{cases}$$

Based on $\alpha_0\beta_0 = \frac{1}{\sqrt{2}}$ and $\alpha_1\beta_1 = \frac{1}{\sqrt{2}}$, we must have $\alpha_0, \alpha_1, \beta_0, \beta_1 \neq 0$.

However $\alpha_0\beta_1 = 0$ requires α_0 or β_1 be 0, which contradicts $\alpha_0, \alpha_1, \beta_0, \beta_1 \neq 0$.

So, the Bell state cannot be decomposed into the tensor product of two single qubits.

(b) Measuring the first qubit will have two possible outcomes of either 0 with probability 1/2 or 1 with probability 1/2:

Pr $\{$1st bit$=0\} = |\frac{1}{\sqrt{2}}|^2 = 1/2$

Pr $\{$1st bit $= 1\} = |\frac{1}{\sqrt{2}}|^2 = 1/2$

(c) **If the first qubit is measured to zero,** the new state of the two qubits is

$$\frac{1/\sqrt{2}}{\sqrt{(1/\sqrt{2})^2}}\,|00\rangle = |00\rangle$$

Now if the second qubit is measured, it can only be zero with probability 1:

$$\Pr\{\text{2nd qubit} = 0\} = |1|^2 = 1$$

If the first qubit is measured to one, the new state of the two qubits is

$$\frac{1/\sqrt{2}}{\sqrt{(1/\sqrt{2})^2}}\,|11\rangle = |11\rangle$$

Now if the second qubit is measured, it can only be one with probability 1:

$$\Pr\{\text{2nd qubit} = 1\} = |1|^2 = 1$$

This implies that measurements of the two qubits will always be identical for this Bell state.

(d) That means, no matter how far the distance between the two qubits, the measurement of the second qubit is always the same as that of the first qubit.