Homework 1 solution

2019/09/12

1. (Page 8, Q 0.1. Only (f), (i), (l), (m) and (n) required.)

   a. By limit, we have

   $$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n-100}{n-200} = 1$$

   leading to $f(n) = \theta(g(n))$

   b. By limit, we have

   $$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n^{1/2}}{n^{2/3}} = \lim_{n\to\infty}\frac{1}{n^{1/6}} = 0$$

   leading to $f(n) = O(g(n))$

   c. By limit, we have

   $$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{100n+\log n}{n+(\log n)^2} = \lim_{n\to\infty}\frac{100n}{n} = 100$$

   leading to $f(n) = \theta(g(n))$

   d. By limit, we have

   $$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{\mathrm{n}\log n}{10\mathrm{n}\log 10n} = \lim_{n\to\infty}\frac{\log n}{10\ \log 10n} = \lim_{n\to\infty}\frac{\log n}{10\ \log n + 10\ \log 10}$$
   $$= \frac{1}{10}$$

   leading to $f(n) = \theta(g(n))$

   e. By limit, we have

   $$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{\log 2n}{\log 3n} = \lim_{n\to\infty}\frac{\log n+\log 2}{\log n+\log 3} = 1$$

leading to $f(n) = \theta(g(n))$

f. By limit, we have

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{10\log n}{\log n^2} = \lim_{n\to\infty}\frac{10\log n}{2\log n} = 5$$

leading to $f(n) = \theta\big(g(n)\big)$

g. By limit, we have

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n^{1.01}}{n\,(\log n)^2} = \lim_{n\to\infty}(\frac{n^{0.005}}{\log n})^2$$

Since polynomial function always dominates logarithmic function, which is because that polynomial function always has larger growth than logarithmic function when n goes infinity, so

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n^{1.01}}{n\,(\log n)^2} = \lim_{n\to\infty}(\frac{n^{0.005}}{\log n})^2 = \infty$$

leading to $f(n) = \Omega\big(g(n)\big)$

h. By limit, we have

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n^2/\log n}{n\,(\log n)^2} = \lim_{n\to\infty}\frac{n}{(\log n)^3} = \lim_{n\to\infty}\frac{n}{3(\log n)^2} = \lim_{n\to\infty}\frac{n}{6(\log n)^1}$$
$$= \lim_{n\to\infty}\frac{n}{6} = \infty$$

leading to $f(n) = \Omega\big(g(n)\big)$

i. By limit, we have

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n^{0.1}}{(\log n)^{10}} = \lim_{n\to\infty}(\frac{n^{0.01}}{\log n})^{10}$$

Since polynomial function always dominates logarithmic function, which is because that polynomial function always has larger growth than logarithmic function when n goes infinity, so

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n^{0.1}}{(\log n)^{10}} = \lim_{n\to\infty}(\frac{n^{0.01}}{\log n})^{10} = \infty$$

leading to $f(n) = \Omega\big(g(n)\big)$

j. Since

$$\log(a^{\log\ }) = \log a \log b = \log(b^{\log a})$$

we know

$$a^{\log b} = b^{\log a}$$

so

$$f(n) = (\log n)^{\log n} = n^{\log\log n}$$

By limit, we have

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n^{\log\log n}}{n/\log n} = \lim_{n\to\infty}\left(n^{\log\log n-1}\log n\right) = \infty$$

leading to $f(n) = \Omega\big(g(n)\big)$

k. By limit, we have

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{\sqrt{n}}{\log n^3} = \lim_{n\to\infty}(\frac{n^{1/6}}{\log n})^3 = \infty$$

leading to $f(n) = \Omega\big(g(n)\big)$

l. Since

$$\log(a^{\log b}) = \log a \log b = \log(b^{\log a})$$

we know

$$a^{\log b} = b^{\log a}$$

so

$$f(n) = 5^{\log_2 n} = n^{\log_2 5}$$

By limit, we have

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n^{1/2}}{5^{\log_2 n}} = \lim_{n\to\infty}\frac{n^{1/2}}{n^{\log_2 5}} = 0$$

leading to $f(n) = O\big(g(n)\big)$

m. By limit, we have

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n2^n}{3^n} = \lim_{n\to\infty} n(\frac{2}{3})^n = 0$$

leading to $f(n) = O\big(g(n)\big)$

n. By limit, we have

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{2^n}{2^{n+1}} = \frac{1}{2}$$

leading to $f(n) = \theta\big(g(n)\big)$

o. Since

$$n! > 2^n$$

So by limit, we have

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{n!}{2^n} = \infty$$

leading to $f(n) = \Omega\big(g(n)\big)$

p. Since

$$\log(a^{\log b}) = \log a \log b = \log(b^{\log a})$$

we know

$$a^{\log b} = b^{\log a}$$

so

$$f(n) = (\log n)^{\log n} = n^{\log\log n}$$

and

$$g(n) = 2^{(\log_2 n)^2} = 2^{\log_2 n * \log_2 n} = (2^{\log_2 n})^{\log_2 n} = (n^{\log_2 2})^{\log_2 n} = n^{\log_2 n}$$

By limit, we have

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{(\log n)^{\log n}}{2^{(\log_2 n)^2}} = \lim_{n\to\infty} \frac{n^{\log\log n}}{n^{\log_2 n}} = 0$$

Since $\log_2 n$ domains $\log\log n$

leading to $f(n) = O(g(n))$

q. By limit, we have

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} i^k}{n^{k+1}} < \lim_{n\to\infty} \frac{\sum_{i=1}^{n} n^k}{n^{k+1}} = \lim_{n\to\infty} \frac{n \cdot n^k}{n^{k+1}} = 1$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} i^k}{n^{k+1}} \geq \lim_{n\to\infty} \frac{\sum_{i=\frac{n}{2}}^{n} \left(\frac{n}{2}\right)^k}{n^{k+1}} = \lim_{n\to\infty} \frac{\left(\frac{n}{2}+1\right)\left(\frac{n}{2}\right)^k}{n^{k+1}}$$
$$= \lim_{n\to\infty} \left(\frac{1}{2}\right)^{k+1} + \frac{1}{n2^k} = \left(\frac{1}{2}\right)^{k+1} > 0$$

Finally, get $0 < \lim_{n\to\infty} \frac{f(n)}{g(n)} < 1$

leading to $f(n) = \Theta(g(n))$.

2. (Page 9, Q 0.2.)
Based on

$$\mathrm{g(n)} = 1 + c + c^2 + \cdots + c^n = \frac{1-c^{n+1}}{1-c}$$

a. If c < 1,

$$\lim_{n\to\infty} g(n) = \frac{1}{1-c} = \theta(1)$$

b. If c = 1,

$$g(n) = n + 1 = \theta(n)$$

c. If c > 1,

$$\lim_{n\to\infty} g(n) = \frac{c^{n+1}-1}{c-1} = \theta(c^n)$$

3. (Page 9, Q 0.3)
   a. Base cases:
   When n = 6,
   $$F_6 = 13 \geq 8 = 2^{0.5*6}$$
   When n = 7,
   $$F_7 = 21 \geq 8\sqrt{2} = 2^{0.5*7}$$
   When n = k, assume
   $$F_k \geq 2^{0.5k}$$
   For all k >= 7
   By induction, when n = k + 2,
   $$F_{k+2} = F_{k+1} + F_k = F_{k-1} + F_k + F_k \geq 2F_k \geq 2 * 2^{0.5k} = 2^{0.5(k+2)}$$
   Proved.

   b. When $\mathrm{n} = 0$,
   $$F_0 = 0 \leq 2^{c*0} (c \in \mathbb{R})$$
   When $\mathrm{n} = 1$,
   $$F_1 = 1 \leq 2^{c*1} (c \geq 0)$$
   When $\mathrm{n} = \mathrm{k}\ (k \geq 2)$, assume there is a constant c, which
   $$F_k \leq 2^{ck}$$
   When $\mathrm{n} = \mathrm{k}+1\ (k \geq 2)$,
   $$F_{k+1} = F_k + F_{k-1}$$
   $$= 2F_{k-1} + F_{k-2}$$
   $$\leq 3F_{k-1}$$
   $$\leq 3 * 2^{c(k-1)}$$
   $$= 2^{c(k-1)+\log_2 3}$$
   $$= 2^{c(k-1+\frac{\log_2 3}{c})}$$
   So when

$$-1 + \frac{\log_2 3}{c} \leq 1$$

$$c \geq \frac{\log_2 3}{2}$$

Finally, when

$$\frac{\log_2 3}{2} \leq c < 1$$

the induction will work, which means $F_n = 0 \leq 2^{cn}$.

c. Assume there is a constant m>0, which makes $F_n = \Omega(2^{cn})$.
When $\mathrm{n} = 0$,

$$F_0 = 0 \geq m * 2^{c*0}$$

When $\mathrm{n} = 1$,

$$F_1 = 1 \geq m * 2^{c*1}$$

When $\mathrm{n} = \mathrm{k}\ (k \geq 2)$, assume there is a constant c, which

$$F_k \geq m * 2^{ck}$$

and

$$F_{k+1} \geq m * 2^{c(k+1)}$$

When $\mathrm{n} = \mathrm{k} + 2\ (k \geq 2)$,

$$F_{k+2} = F_{k+1} + F_k$$

$$\geq m * 2^{c(k+1)} + m * 2^{ck}$$

$$= m * 2^{ck} * (2^c + 1)$$

So set

$$2^{ck} * (2^c + 1) = 2^{c(k+2)}$$

$$2^c + 1 = 2^{2c}$$

$$(2^c)^2 - 2^c - 1 = 0$$

$$2^c = \frac{1 \pm \sqrt{5}}{2}$$

So finally, the largest c can be found is

$$c = \log_2 \frac{\sqrt{5}+1}{2}$$

the induction will work, which means $F_n = 0 \leq 2^{cn}$