# Area to Volume Mapping

To volume of urine can be computed from a measured spot area, for a given paper type. We model the mapping as a 2nd order polynomial with a zero constant term.

$$V = k_2 A^2 + k_1 A$$

The program uses a least-squares method to compute the coefficients, $k_1$ and $k_2$, based on user selected spots of known volume. The error function to minimize is the sum of the squares of the difference between the known volumes and the modeled volumes, in terms of the unknown coefficients. Given N data points ($x_i$, $y_i$), where the first coordinate is measured area and the second coordinate is known volume, the error function is given by the following.

$$E = \sum_{i=1}^{N} (k_2 x_i^2 + k_1 x_1 - y_i)^2$$

To find the minimum set the partial derivatives with respect to the coefficients to zero and solve.

$$\frac{\partial E}{\partial k_1} = \sum_{i=1}^{N} 2(k_2 x_i^2 + k_1 x_1 - y_i)\, x_i = 0$$
$$\frac{\partial E}{\partial k_2} = \sum_{i=1}^{N} 2(k_2 x_i^2 + k_1 x_1 - y_i) x_i^2 = 0$$

Simplifying:

$$\sum_{i=1}^{N} (k_2 x_i^3 + k_1 x_i^2 - x_i y_i) = 0$$
$$\sum_{i=1}^{N} (k_2 x_i^4 + k_1 x_i^3 - x_i^2 y_i) = 0$$

This pair of equations can be written as two linear equations:

$$\left(\sum_{i=1}^{N} x_i^3\right) k_2 + \left(\sum_{i=1}^{N} x_i^2\right) k_1 - \left(\sum_{i=1}^{N} x_i y_i\right) = 0$$
$$\left(\sum_{i=1}^{N} x_i^4\right) k_2 + \left(\sum_{i=1}^{N} x_i^3\right) k_1 - \left(\sum_{i=1}^{N} x_i^2 y_i\right) = 0$$

Any pair of non-colinear equations of the form $a_1x+b_1y+c_1 = 0$, and $a_2x+b_2y+c_2 = 0$ have a solution given by:

$$x = (b_1c_2 - b_2c_1)/(a_1b_2 - a_2b_1)$$
$$y = (c_1a_2 - c_2a_1)/(a_1b_2 - a_2b_1)$$

Applying this to our problem, the solution is given by the following:

Let:

$$a_1 = \sum_{i=1}^{N} x_i^3 \qquad a_2 = \sum_{i=1}^{N} x_i^4$$
$$b_1 = \sum_{i=1}^{N} x_i^2 \qquad b_2 = \sum_{i=1}^{N} x_i^3$$
$$c_1 = -\sum_{i=1}^{N} x_i y_i \qquad c_2 = -\sum_{i=1}^{N} x_i^{32} y_i$$

Then:

$$k_2 = (b_1c_2 - b_2c_1)/(a_1b_2 - a_2b_1)$$
$$k_1 = (c_1a_2 - c_2a_1)/(a_1b_2 - a_2b_1)$$