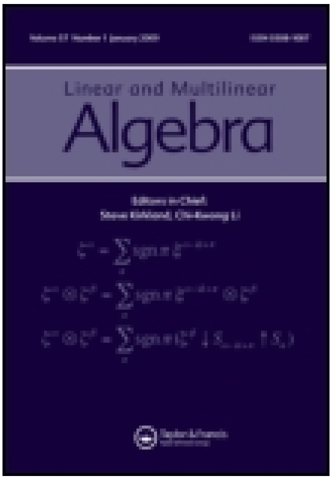

Linear and Multilinear Algebra
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/glma20

Determinants of matrices over semirings

Yi-Jia Tan[a]
[a] Department of mathematics, Fuzhou University, Fuzhou, China.
Published online: 15 Apr 2013.

To cite this article: Yi-Jia Tan (2014) Determinants of matrices over semirings, Linear and Multilinear Algebra, 62:4, 498-517, DOI: 10.1080/03081087.2013.784285

To link to this article: http://dx.doi.org/10.1080/03081087.2013.784285

Linear and Multilinear Algebra, 2014
Vol. 62, No. 4, 498–517, http://dx.doi.org/10.1080/03081087.2013.784285

Taylor & Francis
Taylor & Francis Group

Determinants of matrices over semirings

Yi-Jia Tan*

Department of mathematics, Fuzhou University, Fuzhou, China

Communicated by J. Y. Shao

(*Received 28 July 2012; final version received 2 March 2013*)

In this paper, the concept of determinants for the matrices over a commutative semiring is introduced, and a development of determinantal identities is presented. This includes a generalization of the Laplace and Binet–Cauchy Theorems, as well as on adjoint matrices. Also, the determinants and the adjoint matrices over a commutative difference-ordered semiring are discussed and some inequalities for the determinants and for the adjoint matrices are obtained. The main results in this paper generalize the corresponding results for matrices over commutative rings, for fuzzy matrices, for lattice matrices and for incline matrices.

Keywords: determinant; adjoint matrix; semiring; difference-ordered semiring

AMS Subject Classifications: 15A15; 16Y60

1. Introduction

A *semiring* [1] is an algebraic system $(R, +, \cdot)$ in which $(R, +)$ is an abelian monoid with identity element 0 and $(R, \cdot)$ is another monoid with identity element 1, connected by ring-like distributivity. Also, $0r = r0 = 0$ for all r in R and $0 \neq 1$. A semiring R is called *commutative* if $ab = ba$ for all a, b in R; R is called *zerosumfree* [1] if $a + b = 0$ implies that $a = b = 0$ for all a, b in R. Zerosumfree semirings were studied in [2,3] under the name of *antiring*. A semiring R is called an *additively idempotent semiring* [1] if $a + a = a$ for all $a \in R$. It is easy to verify that any additively idempotent semiring is zerosumfree.

Semirings are quite abundant, for example, any ring with identity is a semiring which is not zerosumfree; Boolean algebras, fuzzy algebras, bounded distributive lattices and inclines (see [4]) are commutative semirings which are additively idempotent. Also, the max-plus algebra $(\mathbb{R} \cup \{-\infty\}, \max, +)$ and the min-plus algebra $(\mathbb{R} \cup \{+\infty\}, \min, +)$ are commutative semirings which are additively idempotent (see [5,6]). In addition, the set $\mathbb{Z}^0$ of nonnegative integers with the usual operations of addition and multiplication of integers is a commutative semiring which is zerosumfree but not additively idempotent. The same is true for the set $\mathbb{Q}^0$ of all nonnegative rational numbers, for the set $\mathbb{R}^0$ of all nonnegative real numbers.

For a given $n \times n$ matrix $A = (a_{ij})_{n \times n}$ over a commutative ring, the *determinant* $|A|$ of A is defined by

$$|A| = \sum_{\sigma \in S_n} (-1)^{t(\sigma)} \prod_{i=1}^{n} a_{i\sigma(i)},$$

*Email: yjtan62@126.com

where S_n is the symmetric group on the set $\{1, 2, \ldots, n\}$ and $t(\sigma)$ is the number of inversions in the permutation σ. The determinant of matrices plays a fundamental role in linear algebra, it has many interesting properties, such as the Laplace and Binet–Cauchy Theorems (see [7]). In fact, the determinant, the matrix inverse and the solution to a system of linear equations are all closely related. By use of determinants, one can find a direct formula for the inverse of a invertible matrix (see [7]). Cramer's rule uses determinants to solve a system of linear equations.

The permanent of a square matrix is defined in a way similar to the determinant. For a given $n \times n$ matrix $A = (a_{ij})_{n\times n}$ over a commutative semiring, the *permanent* $per(A)$ of A is defined by

$$per(A) = \sum_{\sigma\in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

The concept of permanent was first introduced in 1812 by Binet [8] and Cauchy [9]. Since then, a large number of works on permanent theory have been published. In 1978, Minc [10] gave a complete account of the theory of permanents, their history and applications. Since the 1980s, many authors have studied permanents of matrices for some special cases of semirings (see e.g. [11–17]). In 1989, Kim et al. [14] studied permanent theory for fuzzy matrices and proved that $per(Apadj(A)) = per(A) = per((padj(A))A)$ for any square fuzzy matrix A, where $per(A)$ denotes the permanent of A and $padj(A)$ denotes the adjoint matrix of A with respect to permanent. This result was generalized to lattice matrices by Zhang [17] and to incline matrices by Huang and Tan [13].

To describe the invertible matrices over a commutative semiring, Tan [18] introduced the concept of determinants of the matrices over a commutative semiring. This concept contains determinants over commutative rings and permanents over commutative zerosumfree semirings. By using the determinants, Tan established Cramer's rule over a commutative semiring (see Theorem 4.1 in [18]).

In the present work, we continue to study the determinants of the matrices over general commutative semirings. We present a development of determinantal identities over commutative semirings. This includes a generalization of the Laplace and Binet–Cauchy Theorems, as well as on adjoint matrices (see Sections 3 and 4). Also, we discuss the determinants and the adjoint matrices over a commutative difference-ordered semiring and give some inequalities for the determinants and for the adjoint matrices (see Section 5). The main results in this paper generalize the corresponding results for matrices over a commutative ring in [7], for fuzzy matrices in [14,15], for lattice matrices in [17] and for incline matrices in [13].

2. Definitions and preliminaries

In this section, we give some definitions and preliminaries. For convenience, we use $\underline{n}$ to denote the set $\{1, 2, \ldots, n\}$.

Let R be a semiring and $a \in R$. We denote by a^k the k-th power of a and by ka the sum $a + a + \cdots + a$(k times) for any positive integer k. For $x \in R$, x is called *additively invertible* in R if $x + y = 0$ for some y in R. Such an element y is obviously unique and denoted by $-x$. Let $V(R)$ denote the set of all additively invertible elements in R. It is clear that $V(R) = \{0\}$ if and only if R is a zerosumfree semiring and that $V(R) = R$ if and only if R is a ring.

Let R be a commutative semiring. We denote by $M_{m\times n}(R)$ the set of all $m \times n$ matrices over R. Especially, we put $M_n(L) = M_{n\times n}(L)$. For $A \in M_{m\times n}(L)$, we denote by a_{ij} or A_{ij} the $(i, j)-$entry of A, and denote by A^T the *transpose* of A.

For any $A, B \in M_{m\times n}(R)$, $C \in M_{n\times l}(R)$ and $\lambda \in R$, we define:

$$A + B = (a_{ij} + b_{ij})_{m\times n}, \lambda A = (\lambda a_{ij})_{m\times n} \text{ and } AC = \left(\sum_{k=1}^{n} a_{ik}c_{kj}\right)_{m\times l}.$$

It is easy to verify that $M_n(R)$ forms a semiring with respect to the matrix addition and the matrix multiplication.

Let $A \in M_n(R)$ and $\mathcal{A}_n$ the alternating group on $\underline{n}$. The *positive determinant* $|A|^+$ and the *negative determinant* $|A|^-$ of A are defined as follows

$$|A|^+ = \sum_{\sigma\in\mathcal{A}_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

and

$$|A|^- = \sum_{\sigma\in S_n\setminus\mathcal{A}_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

It is clear that $per(A) = |A|^+ + |A|^-$ and that if R is a commutative ring then $|A| = |A|^+ - |A|^-$.

Let R be a semiring. A bijection ε on R is called an *ε-function* of R if $\varepsilon(\varepsilon(a)) = a$ and $\varepsilon(a + b) = \varepsilon(a) + \varepsilon(b)$ and $\varepsilon(ab) = a\varepsilon(b) = \varepsilon(a)b$ for all $a, b \in R$. It is easy to verify that $\varepsilon(a)\varepsilon(b) = ab$ and $\varepsilon(0) = 0$.

Remark 2.1 Any semiring R has at least an ε-function since the identical mapping of R is an ε-function of R. If R is a ring then the mapping: $a \mapsto -a$, $(a \in R)$, is an ε-function of R.

Remark 2.2 The concept of ε-function of a semiring R was also introduced in [18]. But the ε-function ε in [18] requires $\varepsilon(a) = -a$ for all $a \in V(R)$.

Example 2.1 Consider the set $R = \{(a, k) \mid a \in \mathbb{Z}_8, k \in \mathbb{Z}^0\}$, where $\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$ is the ring of residue classes of modulo 8 (Note that for any $l, m \in \mathbb{Z}, \bar{l} = \bar{m}$ if and only if $l \equiv m(mod 8)$) and $\mathbb{Z}^0$ is the semiring of nonnegative integers. Define operations of addition and multiplication on R by setting

$$(a, k_1) + (b, k_2) = (a + b, k_1 + k_2)$$

and

$$(a, k_1)(b, k_2) = (ab + k_2a + k_1b, k_1k_2)$$

for all $(a, k_1), (b, k_2) \in R$. Then R can be easily verified to be a commutative semiring with additive identity $(\bar{0}, 0)$ and multiplicative identity $(\bar{0}, 1)$(Note that R is called the *Dorroh extension of* $\mathbb{Z}_8$ *by* $\mathbb{Z}^0$ (see [1])). It is easy to verify that the mappings $\varepsilon_0 : (a, k) \mapsto (a, k)$, $\varepsilon_1 : (a, k) \mapsto (3a + \overline{2k}, k))$, $\varepsilon_2 : (a, k) \mapsto (5a + \overline{4k}, k)$ and $\varepsilon_3 : (a, k) \mapsto (7a + \overline{6k}, k)$ are ε-functions of R.

Let R be a semiring with an ε-function ε. For $A \in M_{m\times n}(R)$, we define $\tilde{\varepsilon}(A) = (\varepsilon(a_{ij}))_{m\times n}$. It is easy to verify that $\tilde{\varepsilon}$ is an ε-function of the semiring $M_n(R)$.

Definition 2.1 Let R be a commutative semiring with an ε-function ε and $A \in M_n(R)$. The *ε-determinant* of A, denoted by $det_\varepsilon(A)$, is defined by

$$det_\varepsilon(A) = \sum_{\sigma\in S_n} \varepsilon^{(t(\sigma))}(a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}), \tag{1}$$

where $t(\sigma)$ is the number of inversions in the permutation σ, and $\varepsilon^{(k)}$ is defined by $\varepsilon^{(0)}(a) = a$ and $\varepsilon^{(k)}(a) = \varepsilon^{(k-1)}(\varepsilon(a))$ for all positive integers k.

Since $\varepsilon^{(2)}(a) = a$, $det_\varepsilon(A)$ can be rewritten as follows:

$$det_\varepsilon(A) = |A|^+ + \varepsilon(|A|^-). \tag{2}$$

Remark 2.3 For any commutative semiring R, the identical mapping of R is an ε-function of R, and in this case

$$det_\varepsilon(A) = \sum_{\sigma\in S_n} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)} = per(A).$$

Remark 2.4 If R is a commutative ring, then the mapping: $a \mapsto -a$, $(a \in R)$, is an ε-function of R, and in this case

$$det_\varepsilon(A) = \sum_{\sigma\in S_n} (-1)^{t(\sigma)} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)} = |A|.$$

Remark 2.5 For any $A \in M_n(R)$, $det_\varepsilon(A)$ can be rewritten as follows:

$$det_\varepsilon(A) = \sum_{\sigma\in S_n} \varepsilon^{(t(\sigma))}(a_{\sigma(1)1}a_{\sigma(2)2}\cdots a_{\sigma(n)n}).$$

The following lemma is used.

LEMMA 2.1 *If ε is an ε-function of R, then*

(1) $\varepsilon^{(k+2l)} = \varepsilon^{(k)}$ *for any nonnegative integer k and any positive integer l;*
(2) $\varepsilon^{(k)}$ *is an ε-function of R for any nonnegative integer k;*
(3) $\varepsilon^{(k_1+k_2)}(xy) = \varepsilon^{(k_1)}(x)\varepsilon^{(k_2)}(y)$ *for any nonnegative integers k_1 and k_2 and $x, y \in R$.*

Proof

(1) By the definition of ε-function, we have $\varepsilon^{(2)}(x) = \varepsilon(\varepsilon(x)) = x$ for any $x \in R$. This implies that $\varepsilon^{(k+2l)}(x) = \varepsilon^{(k)}(x)$ for any nonnegative integer k and any positive integer l, i.e. $\varepsilon^{(k+2l)} = \varepsilon^{(k)}$.
(2) If k is an even number then $\varepsilon^{(k)} = \varepsilon^{(0)}$ is the identical mapping of R (by (1)), and if k is an odd number then $\varepsilon^{(k)} = \varepsilon$ (by (1)). Thus, $\varepsilon^{(k)}$ is an ε-function of R.
(3) By (2), $\varepsilon^{(k_1)}$ and $\varepsilon^{(k_2)}$ are ε-functions of R. Then $\varepsilon^{(k_1+k_2)}(xy) = \varepsilon^{(k_1)}(\varepsilon^{(k_2)}(xy)) = \varepsilon^{(k_1)}(x\varepsilon^{(k_2)}(y)) = \varepsilon^{(k_1)}(x)\varepsilon^{(k_2)}(y)$. □

3. Some identities of ε-determinant

In this section, we give some ε-determinantal identities over a commutative semiring R with an ε-function ε. These identities include generalizations of the Binet–Cauchy and Laplace Theorems.

THEOREM 3.1 *For any $A \in M_n(R)$, we have*

(1) *If B is a matrix obtained from A by multiplying all elements in a given row (or column) by a fixed element λ in R, then*
$$det_\varepsilon(B) = \lambda det_\varepsilon(A).$$

(2) *If the i-th row (resp. i-th column) of A is the sum of the i-th row (resp. i-th column) of a matrix B and the i-th row (resp. i-th column) of a matrix C, further, B and C different from A only in the i-th row (resp. i-th column), then*
$$det_\varepsilon(A) = det_\varepsilon(B) + det_\varepsilon(C).$$

(3) $det_\varepsilon(A) = det_\varepsilon(A^T)$.

(4) *If $B \in M_n(R)$ is obtained by interchanging two rows (or columns) of A, then*
$$det_\varepsilon(B) = \varepsilon(det_\varepsilon(A)).$$

(5) *If two rows (or two columns) of A are identical, then*
$$det_\varepsilon(A) = |A|^+ + \varepsilon(|A|^+).$$

(6) *If $B \in M_n(R)$ is obtained from A by adding the elements of its i-th row to the corresponding elements of its j-th row multiplied by $\lambda \in R$, then*
$$det_\varepsilon(B) = det_\varepsilon(A) + \lambda(|A_r(i \Rightarrow j)|^+ + \varepsilon(|A_r(i \Rightarrow j)|^+)).$$
where $A_r(i \Rightarrow j)$ denotes the matrix obtained from A by replacing the j-th row of A by the i-th row of A.

Proof (1), (2) and (3) are obvious.

(4) By Lemma 3.9 in [19], we have $|A|^+ = |B|^-$ and $|A|^- = |B|^+$, thus $det_\varepsilon(B) = |B|^+ + \varepsilon(|B|^-) = |A|^- + \varepsilon(|A|^+) = \varepsilon(|A|^+ + \varepsilon(|A|^-)) = \varepsilon(det_\varepsilon(A))$.

(5) By Lemma 3.11 in [19], we have $|A|^+ = |A|^-$, thus $det_\varepsilon(A) = |A|^+ + \varepsilon(|A|^+)$.

(6) It follows from (1), (2) and (5). □

In the following we give Laplace's expansion of ε-determinant. To do this, we need some notations.

For any positive integers n and r, let $\Gamma_{r,n}$ denote the set of all n^r sequences $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ of integers, $1 \le \alpha_i \le n, i \in \underline{n}$. Let

$$G_{r,n} = \{\alpha \in \Gamma_{r,n} | \alpha_i \neq \alpha_j \text{ for any } i, j \in \underline{r} \text{ with } i \neq j\} (r \le n) \text{ and}$$
$$\Omega_{r,n} = \{\alpha \in \Gamma_{r,n} \mid 1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_r \le n\} (r \le n).$$

Let $A \in M_{m\times n}(R), \alpha \in \Gamma_{p,m}$ and $\beta \in \Gamma_{p,n}$ $(1 \le p \le min\{m, n\})$, we denote by $A[\alpha|\beta]$ the $p \times p$ matrix whose $(u, v)-$entry is equal to $a_{\alpha_u \beta_v}$. If it happens that $\alpha \in \Omega_{p,m}$ and $\beta \in \Omega_{p,n}$, then $A[\alpha|\beta]$ is a $p \times p$ submatrix of A and in this case we denote by $A(\alpha|\beta)$ the $(m - p) \times (n - p)$ submatrix of A obtained from A by deleting rows α and columns β. The ε-determinant $det_\varepsilon(A[\alpha|\beta])$ of the submatrix $A[\alpha|\beta]$ is called an ε-*minor* *of order* p of A. If A is a square matrix, then the ε-determinant $det_\varepsilon(A(\alpha|\beta))$ is called the ε-*complementary minor* of the ε-minor $det_\varepsilon(A[\alpha|\beta])$, and $\varepsilon^{(|\alpha|+|\beta|)}(det_\varepsilon(A(\alpha|\beta)))$ is called the ε-*complementary cofactor* of the ε-minor $det_\varepsilon(A[\alpha|\beta])$, where $|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_p$ and $|\beta| = \beta_1+\beta_2+\cdots+\beta_p$. Especially, if $\alpha = (i)$ and $\beta = (j)$, then $\varepsilon^{(i+j)}(det_\varepsilon(A(i|j)))$ is called the *cofactor* of the element a_{ij}.

Laplace's Theorem states that

THEOREM 3.2 *Let R be a commutative ring and $A \in M_n(R)$. Then for any $\alpha, \beta \in \Omega_{p,n}$, we have*

$$|A| = \sum_{\gamma\in\Omega_{p,n}} (-1)^{|\alpha|+|\gamma|}|A[\alpha|\gamma]| \cdot |A(\alpha|\gamma)|$$

or, similarly

$$|A| = \sum_{\gamma\in\Omega_{p,n}} (-1)^{|\gamma|+|\beta|}|A[\gamma|\beta]| \cdot |A(\gamma|\beta)|.$$

We now give Laplace's theorem for semiring.

THEOREM 3.3 *(Laplace's theorem for semirings). Let $A \in M_n(R)$. Then for any $\alpha, \beta \in \Omega_{p,n}$, we have*

$$det_\varepsilon(A) = \sum_{\gamma\in\Omega_{p,n}} det_\varepsilon(A[\alpha|\gamma])\varepsilon^{(|\alpha|+|\gamma|)}(det_\varepsilon(A(\alpha|\gamma))) \tag{3}$$

or, similarly

$$det_\varepsilon(A) = \sum_{\gamma\in\Omega_{p,n}} det_\varepsilon(A[\gamma|\beta])\varepsilon^{(|\gamma|+|\beta|)}(det_\varepsilon(A(\gamma|\beta))). \tag{4}$$

Proof For any $\alpha \in \Omega_{p,n}$, let $\alpha' = (\alpha'_1, \alpha'_2, \ldots, \alpha'_{n-p})$ be the sequence complementary to α in $(1, 2, \ldots, n)$. Then $\alpha' \in \Omega_{n-p,n}$.

We first consider the ε-minor $det_\varepsilon(A[\alpha|\alpha])$ for $\alpha = (1, 2, \ldots, p)$. In this case, $\alpha' = (p+1, p+2, \ldots, n)$ and $det_\varepsilon(A[\alpha|\alpha])\varepsilon^{(|\alpha|+|\alpha|)}(det_\varepsilon(A(\alpha|\alpha))) = det_\varepsilon(A[\alpha|\alpha])det_\varepsilon(A(\alpha|\alpha))$ (by Lemma 2.1(1)). Let T be any term of $det_\varepsilon(A[\alpha|\alpha])det_\varepsilon(A(\alpha|\alpha))$. Then

$$T = \varepsilon^{(t(\varphi))}(a_{1\varphi(1)}a_{2\varphi(2)}\cdots a_{p\varphi(p)})\varepsilon^{(t(\psi))}(a_{p+1,\psi(p+1)}a_{p+2,\psi(p+2)}\cdots a_{n\varphi(n)}),$$

where $\varepsilon^{(t(\varphi))}(a_{1\varphi(1)}a_{2\varphi(2)}\cdots a_{p\varphi(p)})$ and $\varepsilon^{(t(\psi))}(a_{p+1,\psi(p+1)}a_{p+2,\psi(p+2)}\cdots a_{n\varphi(n)})$ are terms of $det_\varepsilon(A[\alpha|\alpha])$ and $det_\varepsilon(A(\alpha|\alpha))$, respectively, $\varphi \in S_p$ and ψ is a permutation of the set $\{p+1, p+2, \ldots, n\}$. In the following we will show that T is a term of $det_\varepsilon(A)$.

Let σ be a mapping from $\underline{n}$ to $\underline{n}$ satisfying $\sigma(i) = \varphi(i)$ for $i \in \underline{p}$ and $\sigma(i) = \psi(i)$ for $i \in \{p+1, p+2, \ldots, n\}$. Then $\sigma \in S_n$ and $t(\sigma) = t(\varphi) + t(\psi)$ (because $\varphi(i) < \psi(j)$ for all $i \in \underline{p}$ and all $j \in \{p+1, p+2, \ldots, n\}$). Thus

$$\begin{aligned} T &= \varepsilon^{(t(\varphi))}(a_{1\varphi(1)}a_{2\varphi(2)}\cdots a_{p\varphi(p)})\varepsilon^{(t(\psi))}(a_{p+1,\psi(p+1)}a_{p+2,\psi(p+2)}\cdots a_{n\varphi(n)}) \\ &= \varepsilon^{(t(\varphi)+t(\psi))}(a_{1\varphi(1)}a_{2\varphi(2)}\cdots a_{p\varphi(p)}a_{p+1,\psi(p+1)}a_{p+2,\psi(p+2)}\cdots a_{n\varphi(n)}) \\ &\quad (\textit{by Lemma } 2.1(3)) \\ &= \varepsilon^{(t(\sigma))}(a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}), \end{aligned}$$

i.e. T is a term of $det_\varepsilon(A)$.

Now, we consider an arbitrary ε-minor $det_\varepsilon(A[\alpha|\gamma])$. We can exchange rows and columns of the matrix A until the submatrix $A[\alpha|\gamma])$ is in the top-left corner. This requires a total of $\sum_{1\le k\le p}(\alpha_k - k)$ exchanges of rows and $\sum_{1\le l\le p}(\gamma_l - l)$ exchanges of columns. Denote the rearranged matrix by B. Then, the ε-complementary minor of the ε-minor $det_\varepsilon(A[\alpha|\gamma])$ is the same in both A and B because the relative positions of rows and columns of the submatrix corresponding to this ε-minor $det_\varepsilon(A(\alpha|\gamma))$ are unchanged. Hence, using the special case already proved, we have that any term of $det_\varepsilon(A[\alpha|\gamma])det_\varepsilon(A(\alpha|\gamma))$ is a term of $det_\varepsilon(B)$. On the other hand, by Theorem 3.1(4), we have

$$
\begin{aligned}
det_\varepsilon(B) &= \varepsilon^{\left(\sum\limits_{1\le k\le p}(\alpha_k-k)\right)}\left(\varepsilon^{\left(\sum\limits_{1\le l\le p}(\gamma_l-l)\right)}(det_\varepsilon(A))\right)\\
&= \varepsilon^{\left(\sum\limits_{1\le k\le p}\alpha_k+\sum\limits_{1\le l\le p}\gamma_l-p(p+1)\right)}(det_\varepsilon(A))\\
&= \varepsilon^{(|\alpha|+|\gamma|)}(det_\varepsilon(A))(\text{because } p(p+1) \text{ is an even number}).
\end{aligned}
$$

Hence $det_\varepsilon(A) = \varepsilon^{(|\alpha|+|\gamma|)}(det_\varepsilon(B))$.

Let T be any term of $det_\varepsilon(A[\alpha|\gamma])\varepsilon^{(|\alpha|+|\gamma|)}(det_\varepsilon(A(\alpha|\gamma)))$. Since $det_\varepsilon(A[\alpha|\gamma])\varepsilon^{(|\alpha|+|\gamma|)}(det_\varepsilon(A(\alpha|\gamma))) = \varepsilon^{(|\alpha|+|\gamma|)}(det_\varepsilon(A[\alpha|\gamma])det_\varepsilon(A(\alpha|\gamma)))$ (by Lemma 2.1(3)), we have that $T = \varepsilon^{(|\alpha|+|\gamma|)}(T_1)$ for some term T_1 of $det_\varepsilon(A[\alpha|\gamma])det_\varepsilon(A(\alpha|\gamma))$. Since T_1 is a term of $det_\varepsilon(B)$, thus T is a term of $det_\varepsilon(A)$.

Since $det_\varepsilon(A[\alpha|\gamma])$ has $p!$ different terms and $\varepsilon^{(|\alpha|+|\gamma|)}(det_\varepsilon(A(\alpha|\gamma)))$ has $(n-p)!$ different terms, the product $det_\varepsilon(A[\alpha|\gamma])\varepsilon^{(|\alpha|+|\gamma|)}(det_\varepsilon(A(\alpha|\gamma)))$ has $p!(n-p)!$ different terms, thus $\sum_{\gamma\in\Omega_{p,n}} det_\varepsilon(A[\alpha|\gamma])\varepsilon^{(|\alpha|+|\gamma|)}(det_\varepsilon(A(\alpha|\gamma)))$ has $\binom{n}{p}p!(n-p)! = n!$ different terms. Since the number of the terms of $det_\varepsilon(A)$ is $n!$, Equation (3) holds. Equation (4) can be obtained by Equation (3) and Theorem 3.1(3). □

Remark 3.1 In Theorem 3.3, if R is a commutative ring and the ε-function ε of R is the mapping: $a \mapsto -a$, $(a \in R)$, then $det_\varepsilon(A) = |A|$ and $\varepsilon^{(|\alpha|+|\gamma|)}(det_\varepsilon(A(\alpha|\gamma))) = (-1)^{|\alpha|+|\gamma|}|A(\alpha|\gamma)|$. Thus, Theorem 3.3 generalizes Theorem 3.2.

In Theorem 3.3, if the ε-function ε is the identical mapping of R, then $det_\varepsilon(A) = per(A)$ and $\varepsilon^{(|\alpha|+|\gamma|)}(det_\varepsilon(A(\alpha|\gamma))) = per(A(\alpha|\gamma))$. By Theorem 3.3, we have

COROLLARY 3.1 *Let $A \in M_n(R)$. Then for any $\alpha, \beta \in \Omega_{p,n}$, we have*

$$per(A) = \sum_{\gamma\in\Omega_{p,n}} per(A[\alpha|\gamma])per(A(\alpha|\gamma))$$

or, similarly

$$per(A) = \sum_{\gamma\in\Omega_{p,n}} per(A[\gamma|\beta])per(A(\gamma|\beta)).$$

COROLLARY 3.2 *Let $A \in M_n(R)$. Then for any $i, j \in \underline{n}$, we have*

$$\sum_{k=1}^{n} a_{ik}\varepsilon^{(j+k)}(det_\varepsilon(A(j|k))) = \begin{cases} det_\varepsilon(A) & if\, i = j \\ |A_r(i \Rightarrow j)|^+ + \varepsilon(|A_r(i \Rightarrow j)|^+) & if\, i \neq j, \end{cases} \tag{5}$$

or, similarly,

$$\sum_{k=1}^{n} a_{ki}\varepsilon^{(k+j)}(det_\varepsilon(A(k|j))) = \begin{cases} det_\varepsilon(A) & if\, i = j \\ |A_c(i \Rightarrow j)|^+ + \varepsilon(|A_c(i \Rightarrow j)|^+) & if\, i \neq j, \end{cases} \tag{6}$$

where $A_c(i \Rightarrow j)$ denotes the matrix obtained from A by replacing the j-th column of A by the i-th column of A.

Proof Let $\alpha = (i)$. Then $\alpha \in \Omega_{1,n}$. By Theorem 3.3, we have $\sum_{k=1}^{n} a_{ik}\varepsilon^{(i+k)}$ $(det_\varepsilon(A(i|k))) = \sum_{\gamma\in\Omega_{1,n}} det_\varepsilon(A[\alpha|\gamma])\varepsilon^{(|\alpha|+|\gamma|)}(det_\varepsilon(A(\alpha|\gamma))) = det_\varepsilon(A)$.

If $i \neq j$, then, by Theorem 3.1(5), we have $\sum_{k=1}^{n} a_{ik}\varepsilon^{(k+j)}(det_\varepsilon(A(j|k))) = det_\varepsilon(A_r(i \Rightarrow j)) = |A_r(i \Rightarrow j)|^+ + \varepsilon(|A_r(i \Rightarrow j)|^+)$.

This completes the proof of (5). The formula in (6) follows from (5) by Theorem 3.1(3). □

The Cauchy–Binet Theorem is well known and it takes the form

THEOREM 3.4 (Binet–Cauchy) [7, Theorem 1.5] *Let R be a commutative ring, and let $A \in M_{m\times n}(R)$, $B \in M_{n\times p}(R)$ and $M = AB$. Then for any $\alpha \in \Omega_{t,m}$ and $\beta \in \Omega_{t,p}$, where $1 \leq t \leq min\{m, n, p\}$,*

$$|M[\alpha|\beta]| = \sum_{\gamma\in\Omega_{t,n}} |A[\alpha|\gamma]||B[\gamma|\beta]|.$$

For a commutative semiring R with an $\varepsilon-$ function ε, we have

THEOREM 3.5 *Let $A \in M_{m\times n}(R)$, $B \in M_{n\times p}(R)$ and $M = AB$. Then for any $\alpha \in \Omega_{t,m}$ and $\beta \in \Omega_{t,p}$, where $1 \leq t \leq min\{m, n, p\}$, there exists $\delta \in R$ such that*

$$det_\varepsilon(M[\alpha|\beta]) = \sum_{\gamma\in\Omega_{t,n}} det_\varepsilon(A[\alpha|\gamma])det_\varepsilon(B[\gamma|\beta]) + (\delta + \varepsilon(\delta)).$$

To prove Theorem 3.5, we need a lemma.

LEMMA 3.1 *Let $A \in M_n(R)$ and $\sigma \in S_n$. If $B \in M_n(R)$ satisfies $b_{ij} = a_{\sigma(i)j}$ for all $i, j \in \underline{n}$, then $det_\varepsilon(B) = \varepsilon^{(t(\sigma))}(det_\varepsilon(A))$.*

Proof If $\sigma \in \mathcal{A}_n$ then $2|t(\sigma)$, thus $\varepsilon^{(t(\sigma))}(det_\varepsilon(A)) = det_\varepsilon(A)$ (by Lemma 2.1(1)). On the other hand, since $\sigma \in \mathcal{A}_n$, σ can be expressed as a product of k transpositions for some even number k. By Theorem 3.1(4), we have $det_\varepsilon(B) = \varepsilon^{(k)}(det_\varepsilon(A)) = det_\varepsilon(A)$. Then $det_\varepsilon(B) = \varepsilon^{(t(\sigma))}(det_\varepsilon(A))$. Similarly, we can prove $det_\varepsilon(B) = \varepsilon^{(t(\sigma))}(det_\varepsilon(A))$ for $\sigma \in S_n\backslash\mathcal{A}_n$. □

Proof of Theorem 3.5 By $M = AB = \left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)_{m\times p}$, we have

$$
\begin{aligned}
det_\varepsilon(M[\alpha|\beta]) &= \sum_{\sigma\in S_t} \varepsilon^{(t(\sigma))}\left(\left(\sum_{i_1=1}^{n} a_{\alpha_1 i_1} b_{i_1\beta_{\sigma(1)}}\right)\cdots\left(\sum_{i_t=1}^{n} a_{\alpha_t i_t} b_{i_t\beta_{\sigma(t)}}\right)\right)\\
&= \sum_{\sigma\in S_t} \varepsilon^{(t(\sigma))}\left(\sum_{1\le i_1,i_2,\ldots,i_t\le n} a_{\alpha_1 i_1}\cdots a_{\alpha_t i_t}\left(b_{i_1\beta_{\sigma(1)}}\cdots b_{i_t\beta_{\sigma(t)}}\right)\right)\\
&= \sum_{\sigma\in S_t}\left(\sum_{1\le i_1,i_2,\ldots,i_t\le n} a_{\alpha_1 i_1}\cdots a_{\alpha_t i_t}\varepsilon^{(t(\sigma))}\left(b_{i_1\beta_{\sigma(1)}}\cdots b_{i_t\beta_{\sigma(t)}}\right)\right) \text{ (by Lemma 2.1(2))}\\
&= \sum_{1\le i_1,i_2,\ldots,i_t\le n} a_{\alpha_1 i_1}\cdots a_{\alpha_t i_t}\left(\sum_{\sigma\in S_t}\varepsilon^{(t(\sigma))}(b_{i_1\beta_{\sigma(1)}}\cdots b_{i_t\beta_{\sigma(t)}})\right)\\
&= \sum_{\gamma\in\Gamma_{t,n}} a_{\alpha_1\gamma_1}\cdots a_{\alpha_t\gamma_t}\left(\sum_{\sigma\in S_t}\varepsilon^{(t(\sigma))}(b_{\gamma_1\beta_{\sigma(1)}}\cdots b_{\gamma_t\beta_{\sigma(t)}})\right)\\
&= \sum_{\gamma\in\Gamma_{t,n}} a_{\alpha_1\gamma_1}\cdots a_{\alpha_t\gamma_t} det_\varepsilon(B[\gamma|\beta])\\
&= \sum_{\gamma\in G_{t,n}} a_{\alpha_1\gamma_1}\cdots a_{\alpha_t\gamma_t} det_\varepsilon(B[\gamma|\beta]) + \sum_{\gamma\in\Gamma_{t,n}\setminus G_{t,n}} a_{\alpha_1\gamma_1}\cdots a_{\alpha_t\gamma_t} det_\varepsilon(B[\gamma|\beta]).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\sum_{\gamma\in G_{t,n}} a_{\alpha_1\gamma_1}\cdots a_{\alpha_t\gamma_t} det_\varepsilon(B[\gamma|\beta])\\
&\quad= \sum_{\gamma\in\Omega_{t,n}}\sum_{\rho\in S_t} a_{\alpha_1\gamma_{\rho(1)}}\cdots a_{\alpha_t\gamma_{\rho(t)}} det_\varepsilon(B[\gamma_{\rho(1)},\gamma_{\rho(2)},\ldots,\gamma_{\rho(t)}|\beta])\\
&\quad= \sum_{\gamma\in\Omega_{t,n}}\sum_{\rho\in S_t} a_{\alpha_1\gamma_{\rho(1)}}\cdots a_{\alpha_t\gamma_{\rho(t)}}\varepsilon^{(t(\rho))}(det_\varepsilon(B[\gamma_1,\gamma_2,\ldots,\gamma_t|\beta]))\text{(by Lemma 3.1)}\\
&\quad= \sum_{\gamma\in\Omega_{t,n}}\sum_{\rho\in S_t}\varepsilon^{(t(\rho))}(a_{\alpha_1\gamma_{\rho(1)}}\cdots a_{\alpha_t\gamma_{\rho(t)}}) det_\varepsilon(B[\gamma|\beta]) \text{ (by Lemma 2.1(3))}\\
&\quad= \sum_{\gamma\in\Omega_{t,n}}\Big(\sum_{\rho\in S_t}\varepsilon^{(t(\rho))}(a_{\alpha_1\gamma_{\rho(1)}}\cdots a_{\alpha_t\gamma_{\rho(t)}})\Big) det_\varepsilon(B[\gamma|\beta])\\
&\quad= \sum_{\gamma\in\Omega_{t,n}} det_\varepsilon(A[\alpha|\gamma]) det_\varepsilon(B[\gamma|\beta])
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{\gamma\in\Gamma_{t,n}\setminus G_{t,n}} a_{\alpha_1\gamma_1}\cdots a_{\alpha_t\gamma_t} det_\varepsilon(B[\gamma|\beta])\\
&\quad= \sum_{\gamma\in\Gamma_{t,n}\setminus G_{t,n}} a_{\alpha_1\gamma_1}\cdots a_{\alpha_t\gamma_t}(|B[\gamma|\beta]|^{+} + \varepsilon(|B[\gamma|\beta]|^{+}) \text{ (by Theorem 3.1(5)))}
\end{aligned}
$$

$$= \sum_{\gamma \in \Gamma_{t,n} \setminus G_{t,n}} a_{\alpha_1\gamma_1} \cdots a_{\alpha_t\gamma_t} |B[\gamma|\beta]|^+ + \varepsilon \left(\sum_{\gamma \in \Gamma_{t,n} \setminus G_{t,n}} a_{\alpha_1\gamma_1} \cdots a_{\alpha_t\gamma_t} |B[\gamma|\beta]|^+ \right)$$
$$= \delta + \varepsilon(\delta) \text{ (where } \delta = \sum_{\gamma \in \Gamma_{t,n} \setminus G_{t,n}} a_{\alpha_1\gamma_1} \cdots a_{\alpha_t\gamma_t} |B[\gamma|\beta]|^+),$$

we have $det_\varepsilon(M[\alpha|\beta]) = \sum_{\gamma \in \Omega_{t,n}} det_\varepsilon(A[\alpha|\gamma])det_\varepsilon(B[\gamma|\beta]) + (\delta + \varepsilon(\delta)).$ □

Remark 3.2 In Theorem 3.5, if R is a commutative ring and the ε-function ε of R is the mapping: $a \mapsto -a$ $(a \in R)$, then $\delta + \varepsilon(\delta) = 0$ for any $\delta \in R$ and $det_\varepsilon(A) = |A|$ for any $A \in M_n(R)$. Thus, Theorem 3.5 generalizes Theorem 3.4.

By Theorem 3.5, we have

Corollary 3.3 *For any $A, B \in M_n(R)$, there exists $\delta \in R$ such that*

$$det_\varepsilon(AB) = det_\varepsilon(A)det_\varepsilon(B) + (\delta + \varepsilon(\delta)).$$

At the end of this section, we give an equivalent description for a commutative semiring R with an ε-function ε to have the property that $det_\varepsilon(AB) = det_\varepsilon(A)det_\varepsilon(B)$ for any $A, B \in M_n(R)$ $(n \geq 2)$.

Theorem 3.6 *Let R be a commutative semiring and ε an ε-function of R and $n \geq 2$. Then $det_\varepsilon(AB) = det_\varepsilon(A)det_\varepsilon(B)$ for any $A, B \in M_n(R)$ if and only if R is a commutative ring and $det_\varepsilon(A) = |A|$ for all $A \in M_n(R)$.*

Proof If R is a commutative ring and $det_\varepsilon(A) = |A|$ for all $A \in M_n(R)$, then, clearly, $det_\varepsilon(AB) = det_\varepsilon(A)det_\varepsilon(B)$ for any $A, B \in M_n(R)$.

Conversely, suppose that $det_\varepsilon(AB) = det_\varepsilon(A)det_\varepsilon(B)$ for any $A, B \in M_n(R)$. For any $a \in R$, let

$$A = \begin{pmatrix} 1 & 0 & \\ 1 & 0 & \\ & & I_{n-2} \end{pmatrix}, \quad B = \begin{pmatrix} a & 1 & \\ 0 & 0 & \\ & & I_{n-2} \end{pmatrix} \in M_n(R). \text{ Then}$$

$$AB = \begin{pmatrix} a & 1 & \\ a & 1 & \\ & & I_{n-2} \end{pmatrix},$$ and $det_\varepsilon(A) = det_\varepsilon(B) = 0$ and $det_\varepsilon(AB) = a + \varepsilon(a)$. Since $det_\varepsilon(AB) = det_\varepsilon(A)det_\varepsilon(B)$, we have $a + \varepsilon(a) = 0$, i.e. a is additively invertible in R. Then R is a commutative ring and $\varepsilon(a) = -a$ for all $a \in R$, thus $det_\varepsilon(A) = |A|$ for all $A \in M_n(R)$. □

4. ε-djoint matrices

Let R be a commutative semiring with an ε- function ε and $A \in M_n(R)$. The *ε-adjont matrix* of A, written as $adj_\varepsilon(A)$, is defined to be the transposed matrix of ε-cofactors of A, i.e.

$$adj_\varepsilon(A) = ((\varepsilon^{(i+j)}(A(i|j)))_{n\times n})^T.$$

It is clear that if R is a commutative ring and ε is the mapping: $a \mapsto -a$, $(a \in R)$, and $A \in M_n(R)$, then $adj_\varepsilon(A) = adj A$.

In this section, we discuss some properties of ε-adjoin matrices over a commutative semiring R with an ε-function ε. Partial results obtained in this section generalize the corresponding results for matrices over commutative rings.

THEOREM 4.1 *For any $A \in M_n(R)$ and $\lambda \in R$, we have*

(1) $adj_\varepsilon(\lambda A) = \lambda^{n-1} adj_\varepsilon(A)$.
(2) $adj_\varepsilon(A^T) = (adj_\varepsilon(A))^T$.

Proof It is obvious. □

THEOREM 4.2 *For any $A, B \in M_n(R)$, there exists $\Delta \in M_n(R)$ such that*

$$adj_\varepsilon(AB) = adj_\varepsilon(B) adj_\varepsilon(A) + (\Delta + \tilde{\varepsilon}(\Delta)).$$

Proof Let $AB = M$ and $adjM = C$. Then for any $i, j \in \underline{n}$, we have $c_{ij} = \varepsilon^{(i+j)} (det_\varepsilon(M(j|i))) = \varepsilon^{(i+j)}(det_\varepsilon(M[\underline{n} - \{j\}|\underline{n} - \{i\}]))$. By Theorem 3.5, there exists $\delta_{ij} \in R$ such that

$$det_\varepsilon(M[\underline{n}-\{j\}|\underline{n}-\{i\}]) = \sum_{\gamma \in \Omega_{n-1,n}} det_\varepsilon(A[\underline{n}-\{j\}|\gamma]) det_\varepsilon(B[\gamma|\underline{n}-\{i\}]) + (\delta_{ij} + \varepsilon(\delta_{ij})).$$

Thus, we have

$$\begin{aligned}
c_{ij} &= \varepsilon^{(i+j)}(det_\varepsilon(M[\underline{n} - \{j\}|\underline{n} - \{i\}])) \\
&= \varepsilon^{(i+j)} \left(\sum_{\gamma \in \Omega_{n-1,n}} det_\varepsilon(A[\underline{n} - \{j\}|\gamma]) det_\varepsilon(B[\gamma|\underline{n} - \{i\}]) + (\delta_{ij} + \varepsilon(\delta_{ij})) \right) \\
&= \sum_{k=1}^{n} \varepsilon^{(i+j)}(det_\varepsilon(A(j|k)) det_\varepsilon(B(k|i))) + (\varepsilon^{(i+j)}(\delta_{ij}) + \varepsilon^{(i+j+1)}(\delta_{ij})) \text{ (by Lemma 2.1(2))} \\
&= \sum_{k=1}^{n} \varepsilon^{(i+j+2k)}(det_\varepsilon(A(j|k)) det_\varepsilon(B(k|i))) + (\delta_{ij} + \varepsilon(\delta_{ij})) \text{ (by Lemma 2.1(1))} \\
&= \sum_{k=1}^{n} \varepsilon^{(j+k)}(det_\varepsilon(A(j|k)))\varepsilon^{(i+k)}(det_\varepsilon(B(k|i))) + (\delta_{ij} + \varepsilon(\delta_{ij})) \text{ (by Lemma 2.1(3))} \\
&= \sum_{k=1}^{n} (adj_\varepsilon(B))_{ik}(adj_\varepsilon(A))_{kj} + (\delta_{ij} + \varepsilon(\delta_{ij})) \\
&= (adj_\varepsilon(B) adj_\varepsilon(A))_{ij} + (\delta_{ij} + \varepsilon(\delta_{ij})),
\end{aligned}$$

i.e. $adj_\varepsilon(AB) = adj_\varepsilon(B) adj_\varepsilon(A) + (\Delta + \tilde{\varepsilon}(\Delta))$, where $\Delta = (\delta_{ij}) \in M_n(R)$. This completes the proof. □

If R is a commutative ring and ε is the mapping: $a \mapsto -a$, $(a \in R)$, then $\Delta + \tilde{\varepsilon}(\Delta) = O$ for any $\Delta \in M_n(R)$ and $adj_\varepsilon(A) = adj(A)$ for any $A \in M_n(R)$. By Theorem 4.2, we have

COROLLARY 4.1 *If R is a commutative ring and $A, B \in M_n(R)$, then*

$$adj(AB) = adj(B) adj(A).$$

THEOREM 4.3 *For any $A \in M_n(R)$, we have*

(1) $Aadj_\varepsilon(A) = (det_\varepsilon(A_r(i \Rightarrow j)))_{n\times n}$ *and* $adj_\varepsilon(A)A = (det_\varepsilon(A_c(i \Rightarrow j)))_{n\times n}$, *where* $A_r(i \Rightarrow i) = A_c(i \Rightarrow i) = A$ *for all* $i \in \underline{n}$;

(2) *there exist* $\delta_1, \delta_2 \in R$ *such that*

$$det_\varepsilon(Aadj_\varepsilon(A)) = (det_\varepsilon(A))^n + (\delta_1 + \varepsilon(\delta_1))$$

and

$$det_\varepsilon(adj_\varepsilon(A)A) = (det_\varepsilon(A))^n + (\delta_2 + \varepsilon(\delta_2)).$$

Proof (1) Let $B = Aadj_\varepsilon(A)$. Then, by Corollary 3.2,

$$b_{ij} = \sum_{k=1}^{n} a_{ik}\varepsilon^{(j+k)}(det_\varepsilon(A(j|k))) = det_\varepsilon(A_r(i \Rightarrow j)),$$

i.e. $Aadj_\varepsilon A = (det_\varepsilon(A_r(i \Rightarrow j)))_{n\times n}$.

Similarly, we can prove $(adj_\varepsilon(A))A = (det_\varepsilon(A_c(i \Rightarrow j)))_{n\times n}$.

(2) By (1), we have $Aadj_\varepsilon(A) = (det_\varepsilon(A_r(i \Rightarrow j)))_{n\times n}$. Then

$$det_\varepsilon(Aadj_\varepsilon(A)) = \sum_{\sigma\in S_n} \varepsilon^{(t(\sigma))}\left(\prod_{1\le i\le n} det_\varepsilon(A_r(i \Rightarrow \sigma(i)))\right).$$

For any $\sigma \in S_n$. If $\sigma = 1_n$, where 1_n is the identity in the group S_n, then $\sigma(i) = i$ for all $i \in \underline{n}$, thus

$$\varepsilon^{(t(1_n))}\left(\prod_{1\le i\le n} det_\varepsilon(A_r(i \Rightarrow \sigma(i)))\right) = \prod_{1\le i\le n} det_\varepsilon(A_r(i \Rightarrow i)) = (det_\varepsilon(A))^n.$$

If $\sigma \neq 1_n$, then $k \neq \sigma(k)$ for some $k \in \underline{n}$. In this case

$$\begin{aligned}
&\prod_{1\le i\le n} det_\varepsilon(A_r(i \Rightarrow \sigma(i)))\\
&= det_\varepsilon(A_r(k \Rightarrow \sigma(k))) \prod_{\substack{1\le i\le n\\ i\neq k}} det_\varepsilon(A_r(i \Rightarrow \sigma(i)))\\
&= (|A_r(k \Rightarrow \sigma(k))|^+ + \varepsilon(|A_r(k \Rightarrow \sigma(k))|^+))\\
&\quad\times \prod_{\substack{1\le i\le n\\ i\neq k}} det_\varepsilon(A_r(i \Rightarrow \sigma(i))) \text{ (by Theorem 3.1(5))}\\
&= |A_r(k \Rightarrow \sigma(k))|^+ \prod_{\substack{1\le i\le n\\ i\neq k}} det_\varepsilon(A_r(i \Rightarrow \sigma(i))) +\\
&\quad + \varepsilon(|A_r(k \Rightarrow \sigma(k))|^+ \prod_{\substack{1\le i\le n\\ i\neq k}} det_\varepsilon(A_r(i \Rightarrow \sigma(i))))\\
&= \delta_\sigma + \varepsilon(\delta_\sigma) \text{ (where } \delta_\sigma = |A_r(k \Rightarrow \sigma(k))|^+ \prod_{\substack{1\le i\le n\\ i\neq k}} det_\varepsilon(A_r(i \Rightarrow \sigma(i))))
\end{aligned}$$

Then

$$\varepsilon^{(t(\sigma))}\left(\prod_{1\le i\le n} det_\varepsilon(A_r(i \Rightarrow \sigma(i)))\right)$$
$$= \varepsilon^{(t(\sigma))}(\delta_\sigma) + \varepsilon^{(t(\sigma)+1)}(\delta_\sigma)$$
$$= \delta_\sigma + \varepsilon(\delta_\sigma) \text{ (by Lemma 2.1(1))}$$

Taking $\delta_1 = \sum_{\sigma\in S_n \sigma\neq 1_n} \delta_\sigma$, we have

$$det_\varepsilon(A adj_\varepsilon(A))$$
$$= (det_\varepsilon(A))^n + \sum_{\sigma\in S_n\sigma\neq 1_n} (\delta_\sigma + \varepsilon(\delta_\sigma))$$
$$= (det_\varepsilon(A))^n + \sum_{\sigma\in S_n\sigma\neq 1_n} \delta_\sigma + \varepsilon\left(\sum_{\sigma\in S_n\sigma\neq 1_n} \delta_\sigma\right)$$
$$= (det_\varepsilon(A))^n + \delta_1 + \varepsilon(\delta_1).$$

Similarly, we can prove $det_\varepsilon(adj_\varepsilon(A)A) = (det_\varepsilon(A))^n + (\delta_2 + \varepsilon(\delta_2))$ for some δ_2 in R. □

If R is a commutative ring and ε is the mapping: $a \mapsto -a$, $(a \in R)$, then $\delta+\varepsilon(\delta) = 0$ for any δ in R and $det_\varepsilon(A) = |A|$ and $adj_\varepsilon(A) = adj(A)$ for any $A \in M_n(R)$, and moreover, $det_\varepsilon(A_r(i \Rightarrow j)) = det_\varepsilon(A_c(i \Rightarrow j)) = 0$ for any $i, j \in \underline{n}$ with $i \neq j$. By Theorem 4.3, we have

Corollary 4.2 *If R is a commutative ring and $A \in M_n(R)$, then*

(1) $A adj A = (adj A)A = |A|I_n$, *where I_n is the $n \times n$ identity matrix over R;*
(2) $|A adj A| = |(adj A)A| = |A|^n$.

5. ε-determinants over difference-ordered semirings

A semiring $(R, +, \cdot)$ is called *partially ordered* (see [1]) if there exists a partial order relation $\le$ on R satisfying the following conditions for all elements $a, b, c \in R$:

(1) If $a \le b$, then $a + c \le b + c$;
(2) If $a \le b$ and $c \ge 0$, then $ac \le bc$ and $ca \le cb$.

If the relation $\le$ is a total order, then R is called *totally ordered*.

A partially ordered semiring R is called *difference ordered* (see [1]) if $a \le b$ in R if and only if there exists an element c in R such that $a + c = b$. Difference-ordered semirings are clearly zerosumfree.

Note that if R is an additively idempotent semiring, then the order '$\le$' on R (defined by $a \le b$ if and only if $a + b = b$ for $a, b \in R$) is just the difference order on R (see Proposition 20.19 and Example 20.26 in [1]). This means that any additively idempotent semiring is a difference-ordered semiring.

Boolean algebras, fuzzy algebras, bounded distributive lattices and inclines are difference-ordered semirings (In fact, they are additively idempotent semirings). In addition, the semirings $(\mathbb{Z}^0, +, \cdot)$, $(\mathbb{Q}^0, +, \cdot)$ and $(\mathbb{R}^0, +, \cdot)$ are difference-ordered semirings which are not additively idempotent semirings.

Let R be a commutative partially ordered semiring. For any $A, B \in M_n(R)$, we define $A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all $i, j \in \underline{n}$. Then the semiring $M_n(R)$ is a partially ordered semiring. Especially, $M_n(R)$ is a difference- ordered semiring if R is a commutative difference ordered semiring.

In this section, we discuss ε-determinants and ε-adjoint matrices over a commutative difference ordered semiring R with an ε-function ε, and give some inequalities for ε-determinants and for ε-adjoint matrices. Partial results in this section generalize corresponding results for fuzzy matrices in [14,15], for lattice matrices in [17] and for incline matrices in [13].

Lemma 5.1 *Let R be a difference ordered semiring and $a, b, c, d \in R$. Then*

(1) *$a \leq b$ and $c \leq d$ imply that $a + c \leq b + d$ and $ac \leq bd$;*
(2) *if ε is an ε-function on R, then $a \leq b$ implies $\varepsilon^k(a) \leq \varepsilon^k(b)$ for any nonnegative integer k.*

Proof It is obvious. □

Theorem 5.1 *For any $A, B \in M_n(R)$, we have*

(1) *$A \leq B$ implies $det_\varepsilon(A) \leq det_\varepsilon(B)$;*
(2) *$det_\varepsilon(A) + det_\varepsilon(B) \leq det_\varepsilon(A + B)$.*

Proof (1) If $A \leq B$, then $a_{ij} \leq b_{ij}$ for all $i, j \in \underline{n}$ and so $a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)} \leq b_{1\sigma(1)}b_{2\sigma(2)} \cdots b_{n\sigma(n)}$ for any $\sigma \in S_n$ (by Lemma 5.1(1)). Thus $|A|^+ \leq |B|^+$ and $|A|^- \leq |B|^-$ (by Lemma 5.1(1)). Then $det_\varepsilon(A) = |A|^+ + \varepsilon(|A|^-) \leq |B|^+ + \varepsilon(|B|^-) = det_\varepsilon(B)$ (by Lemma 5.1).

(2) We have

$$\begin{aligned} det_\varepsilon(A+B) &= \sum_{\sigma \in S_n} \varepsilon^{(t(\sigma))}((a_{1\sigma(1)} + b_{1\sigma(1)}) \cdots (a_{n\sigma(n)} + b_{n\sigma(n)})) \\ &\geq \sum_{\sigma \in S_n} \varepsilon^{(t(\sigma))}(a_{1\sigma(1)} \cdots a_{n\sigma(n)} + b_{1\sigma(1)} \cdots b_{n\sigma(n)}) \text{ (by Lemma 5.1)} \\ &= \sum_{\sigma \in S_n} \varepsilon^{(t(\sigma))}(a_{1\sigma(1)} \cdots a_{n\sigma(n)}) + \sum_{\sigma \in S_n} \varepsilon^{(t(\sigma))}(b_{1\sigma(1)} \cdots b_{n\sigma(n)}) \text{ (Lemma 2.1(2))} \\ &= det_\varepsilon(A) + det_\varepsilon(B). \end{aligned}$$

□

By Theorem 5.1, we have

Corollary 5.1 *For any $A, B \in M_n(R)$, we have*

(1) *$A \leq B$ implies $per(A) \leq per(B)$;*
(2) *$per(A) + per(B) \leq per(A + B)$.*

THEOREM 5.2 *Let $A \in M_{m\times n}(R)$, $B \in M_{n\times p}(R)$ and $M = AB$. Then for any $\alpha \in \Omega_{t,m}$ and $\beta \in \Omega_{t,p}$, where $1 \le t \le min\{m, n, p\}$, we have*

$$det_\varepsilon(M[\alpha|\beta]) \ge \sum_{\gamma\in\Omega_{t,n}} det_\varepsilon(A[\alpha|\gamma])det_\varepsilon(B[\gamma|\beta]).$$

Especially, we have

$$per(M[\alpha|\beta]) \ge \sum_{\gamma\in\Omega_{t,n}} per(A[\alpha|\gamma])per(B[\gamma|\beta]).$$

Proof By Theorem 3.5, we have

$$\begin{aligned} det_\varepsilon(M[\alpha|\beta]) &= \sum_{\gamma\in\Omega_{t,n}} det_\varepsilon(A[\alpha|\gamma])det_\varepsilon(B[\gamma|\beta]) + (\delta + \varepsilon(\delta)) \text{ (where } \delta \in R) \\ &\ge \sum_{\gamma\in\Omega_{t,n}} det_\varepsilon(A[\alpha|\gamma])det_\varepsilon(B[\gamma|\beta]). \end{aligned}$$

This completes the proof. □

By Theorem 5.2, we have

COROLLARY 5.2 *For any $A, B \in M_n(R)$, we have*

$$det_\varepsilon(AB) \ge det_\varepsilon(A)det_\varepsilon(B).$$

Especially, we have

$$per(AB) \ge per(A)per(B).$$

Remark 5.1 Since any fuzzy algebra and any bounded distributive lattice are difference-ordered semirings, Corollary 5.2 generalizes Theorem 3.4 in [15] and Theorem 3(1) in [17].

THEOREM 5.3 *For any $A, B \in M_n(R)$ $(n \ge 2)$, we have*

(1) $A \le B$ *implies* $adj_\varepsilon(A) \le adj_\varepsilon(B)$;
(2) $adj_\varepsilon(A) + adj_\varepsilon(B) \le adj_\varepsilon(A + B)$;
(3) $adj_\varepsilon(A^T) = (adj_\varepsilon(A))^T$;
(4) $adj_\varepsilon(B)adj_\varepsilon(A) \le adj_\varepsilon(AB)$;
(5) $A^2 \le A$ *implies* $(adj_\varepsilon(A))^2 \le adj_\varepsilon(A)$.

Proof

(1) Since $A \le B$, we have $A(j|i) \le B(j|i)$ for any $i, j \in \underline{n}$. This implies $det_\varepsilon(A(j|i)) \le det_\varepsilon(B(j|i))$ for any $i, j \in \underline{n}$ (by Theorem 5.1(1)), thus $\varepsilon^{(i+j)}(det_\varepsilon(A(j|i))) \le \varepsilon^{(i+j)}(det_\varepsilon(B(j|i)))$ (by Lemma 5.1(2)), i.e. $(adj_\varepsilon(A))_{ij} \le (adj_\varepsilon(B))_{ij}$ for all $i, j \in \underline{n}$. Then $adj_\varepsilon(A) \le adj_\varepsilon(B)$.

(2) For any $i, j \in \underline{n}$, we have

$$\begin{aligned}
&(adj_\varepsilon(A) + adj_\varepsilon(B))_{ij} \\
&\quad = (adj_\varepsilon(A))_{ij} + (adj_\varepsilon(B))_{ij} \\
&\quad = \varepsilon^{(i+j)}(det_\varepsilon(A(j|i))) + \varepsilon^{(i+j)}(det_\varepsilon(B(j|i))) \\
&\quad = \varepsilon^{(i+j)}(det_\varepsilon(A(j|i)) + det_\varepsilon(B(j|i))) \text{ (by Lemma 2.1(2))} \\
&\quad \leq \varepsilon^{(i+j)}(det_\varepsilon(A(j|i) + B(j|i))) \text{ (by Theorem 5.1(2) and Lemma 5.1(2))} \\
&\quad = (adj_\varepsilon(A+B))_{ij}.
\end{aligned}$$

Then $adj_\varepsilon(A) + adj_\varepsilon(B) \leq adj_\varepsilon(A+B)$.

(3) It is trivial.

(4) Let $AB = M$ and $adjM = C$. Then for any $i, j \in \underline{n}$,

$$\begin{aligned}
c_{ij} &= \varepsilon^{(i+j)}(det_\varepsilon(M(j|i))) \\
&= \varepsilon^{(i+j)}(det_\varepsilon(M[\underline{n} - \{j\}|\underline{n} - \{i\}])) \\
&\geq \varepsilon^{(i+j)}\left(\sum_{\delta \in \Omega_{n-1,n}} det_\varepsilon(A[\underline{n} - \{j\}|\delta])det_\varepsilon(B[\delta|\underline{n} - \{i\}])\right) \\
&\quad \text{(by Theorem 5.2 and Lemma 5.1(2))} \\
&= \varepsilon^{(i+j)}(\sum_{k=1}^{n} det_\varepsilon(A(j|k))det_\varepsilon(B(k|i))) \\
&= \sum_{k=1}^{n} \varepsilon^{(i+j)}(det_\varepsilon(A(j|k))det_\varepsilon(B(k|i))) \text{ (by Lemma 2.1(2))} \\
&= \sum_{k=1}^{n} \varepsilon^{(i+j+2k)}(det_\varepsilon(A(j|k))det_\varepsilon(B(k|i))) \text{ (by Lemma 2.1(1))} \\
&= \sum_{k=1}^{n} \varepsilon^{(j+k)}(det_\varepsilon(A(j|k)))\varepsilon^{(i+k)}(det_\varepsilon(B(k|i))) \text{ (by Lemma 2.1(3))} \\
&= \sum_{k=1}^{n} (adj_\varepsilon(B))_{ik}(adj_\varepsilon(A))_{kj} \\
&= (adj_\varepsilon(B)adj_\varepsilon(A))_{ij},
\end{aligned}$$

i.e. $adj_\varepsilon(B)adj_\varepsilon(A) \leq adj_\varepsilon(AB)$.

(5) It follows from (1) and (4). □

If the ε- function ε is the identical mapping of the semiring R and $A \in M_n(R)$ then $adj_\varepsilon(A) = padjA$, where $padj(A)$ denotes the adjoint matrix of A with respect to permanent. By Theorem 5.3, we have

Corollary 5.3 *For any $A, B \in M_n(R)$ $(n \geq 2)$, we have*

(1) *$A \leq B$ implies $padj(A) \leq padj(B)$;*

(2) *$padj(A) + padj(B) \leq padj(A+B)$;*

(3) $padj(A^T) = (padj(A))^T$;
(4) $padj(B)padj(A) \leq padj(AB)$;
(5) $A^2 \leq A$ *implies* $(padj(A))^2 \leq padj(A)$. □

THEOREM 5.4 *For any* $A \in M_n(R)$ $(n \geq 2)$, *we have*

$$(det_\varepsilon(A))^n \leq det_\varepsilon(Aadj_\varepsilon(A)) \leq n!(n!det_\varepsilon(A))^n + \varepsilon(n!(n!det_\varepsilon(A))^n)$$

and

$$(det_\varepsilon(A))^n \leq det_\varepsilon((adjA)A) \leq n!(n!det_\varepsilon(A))^n + \varepsilon(n!(n!det_\varepsilon(A))^n)$$

To prove Theorem 5.4, we need a lemma.

LEMMA 5.2 [13, Lemma 2] *Let*

$$\begin{array}{cccc} (1, a_{11}) & (2, a_{12}) & \cdots & (n, a_{1n}) \\ (1, a_{21}) & (2, a_{22}) & \cdots & (n, a_{2n}) \\ \vdots & \vdots & \vdots & \vdots \\ (1, a_{k1}) & (2, a_{k2}) & \cdots & (n, a_{kn}) \end{array} \tag{7}$$

be kn ordered pairs with $a_{ij} \in \underline{n}$ *for all* $i \in \underline{k}$ *and all* $j \in \underline{n}$ *and* $|\{(i, j) \in \underline{k} \times \underline{n} \mid a_{ij} = l\}| = k$ *for all* $l \in \underline{n}$. *Then there exist* $\sigma_1, \sigma_2, \ldots, \sigma_k \in S_n$ *such that (7) can be rearranged as follows:*

$$\begin{array}{cccc} (1, \sigma_1(1)) & (2, \sigma_1(2)) & \cdots & (n, \sigma_1(n)) \\ (1, \sigma_2(1)) & (2, \sigma_2(2)) & \cdots & (n, \sigma_2(n)) \\ \vdots & \vdots & \vdots & \vdots \\ (1, \sigma_k(1)) & (2, \sigma_k(2)) & \cdots & (n, \sigma_k(n)) \end{array} \tag{8}$$

□

Proof of Theorem 5.4 By Theorem 4.3(2), we have $(det_\varepsilon(A))^n \leq det_\varepsilon(Aadj_\varepsilon(A))$. In the following we prove $det_\varepsilon(Aadj_\varepsilon(A)) \leq n!(n!det_\varepsilon(A))^n + \varepsilon(n!(n!det_\varepsilon(A))^n)$.

Since $Aadj_\varepsilon A = (det_\varepsilon(A_r(i \Rightarrow j)))_{n\times n}$ (by Theorem 4.3(1)), we have

$$\begin{aligned} &det_\varepsilon(Aadj_\varepsilon(A)) \\ &= \sum_{\sigma\in S_n} \varepsilon^{(t(\sigma))} \left(\prod_{i=1}^{n} det_\varepsilon(A_r(i \Rightarrow \sigma(i)))\right) \\ &= \sum_{\sigma\in S_n} \varepsilon^{(t(\sigma))} \left(\prod_{i=1}^{n} det_\varepsilon(A_r(\sigma(i) \Rightarrow i))\right) \text{ (by Remark 2.5)} \\ &= \sum_{\sigma\in S_n} \varepsilon^{(t(\sigma))} \left(\prod_{i=1}^{n} \left(\sum_{\sigma_i\in S_n} \varepsilon^{(t(\sigma_i))}(a_{1\sigma_i(1)} \cdots a_{\sigma(i)\sigma_i(i)} \cdots a_{n\sigma_i(n)})\right)\right) \\ &= \sum_{\sigma\in S_n} \varepsilon^{(t(\sigma))} \left(\sum_{\sigma_1,\ldots,\sigma_n\in S_n} \varepsilon^{(t(\sigma_1))}(a_{\sigma(1)\sigma_1(1)}a_{2\sigma_1(2)} \cdots a_{n\sigma_1(n)}) \cdots \right. \\ &\qquad \left. \varepsilon^{(t(\sigma_n))}(a_{1\sigma_n(1)}a_{2\sigma_n(2)} \cdots a_{\sigma(n)\sigma_i(n)})\right) \\ &= \sum_{\sigma,\sigma_1,\ldots,\sigma_n\in S_n} \varepsilon^{\left(t(\sigma)+\sum_{1\leq i\leq n} t(\sigma_i)\right)}((a_{\sigma(1)\sigma_1(1)}a_{2\sigma_1(2)} \cdots a_{n\sigma_1(n)}) \\ &\qquad \cdots (a_{1\sigma_n(1)}a_{2\sigma_n(2)} \cdots a_{\sigma(n)\sigma_n(n)})). \end{aligned}$$

For any $\sigma, \sigma_1, \dots, \sigma_n \in S_n$, let

$$\Delta(\sigma, \sigma_1, \dots, \sigma_n) = (a_{\sigma(1)\sigma_1(1)} a_{2\sigma_1(2)} \cdots a_{n\sigma_1(n)}) \cdots (a_{1\sigma_n(1)} a_{2\sigma_n(2)} \cdots a_{\sigma(n)\sigma_n(n)})$$

and

$$\tilde{\Delta}(\sigma, \sigma_1, \dots, \sigma_n) = \varepsilon^{\left(t(\sigma)+\sum\limits_{1\le i\le n} t(\sigma_i)\right)}(\Delta(\sigma, \sigma_1, \dots, \sigma_n)).$$

Denoting By (i, j) the subscript of a_{ij}, we can get n^2 ordered pairs from the expression of $\Delta(\sigma, \sigma_1, \dots, \sigma_n)$ as follows.

$$\begin{array}{cccc} (\sigma(1), \sigma_1(1)) & (2, \sigma_1(2)) & \cdots & (n, \sigma_1(n)) \\ (1, \sigma_2(1)) & (\sigma(2), \sigma_2(2)) & \cdots & (n, \sigma_2(n)) \\ \vdots & \vdots & \vdots & \vdots \\ (1, \sigma_n(1)) & (2, \sigma_n(2)) & \cdots & (\sigma(n), \sigma_n(n)) \end{array} \tag{9}$$

Since $\sigma \in S_n$, for any $i \in \underline{n}$, there is a unique $j \in \underline{n}$ such that $i = \sigma(j)$, thus $j = \sigma^{-1}(i)$. Then the n ordered pairs $(\sigma(1), \sigma_1(1)), (\sigma(2), \sigma_2(2)), \cdots, (\sigma(n), \sigma_n(n))$ can be rearranged as follows:

$$(1, \sigma_{\sigma^{-1}(1)}(\sigma^{-1}(1))), (2, \sigma_{\sigma^{-1}(2)}(\sigma^{-1}(2))), \cdots, (n, \sigma_{\sigma^{-1}(n)}(\sigma^{-1}(n))).$$

Thus, The n^2 ordered pairs in (9) can be rearranged as follows:

$$\begin{array}{cccc} (1, \sigma_{\sigma^{-1}(1)}(\sigma^{-1}(1))) & (2, \sigma_1(2)) & \cdots & (n, \sigma_1(n)) \\ (1, \sigma_2(1)) & (2, \sigma_{\sigma^{-1}(2)}(\sigma^{-1}(2))) & \cdots & (n, \sigma_2(n)) \\ \vdots & \vdots & \vdots & \vdots \\ (1, \sigma_n(1)) & (2, \sigma_n(2)) & \cdots & (n, \sigma_{\sigma^{-1}(n)}(\sigma^{-1}(n))) \end{array} \tag{10}$$

It is clear that $|\{(i, j) \in \underline{n} \times \underline{n} \mid \sigma_i(j) = l\}| = n$ for all $l \in \underline{n}$. By Lemma 5.2, there exist $\pi_1, \pi_2, \dots, \pi_n \in S_n$ such that (10) can be rearranged as follows:

$$\begin{array}{cccc} (1, \pi_1(1)) & (2, \pi_1(2)) & \cdots & (n, \pi_1(n)) \\ (1, \pi_2(1)) & (2, \pi_2(2)) & \cdots & (n, \pi_2(n)) \\ \vdots & \vdots & \vdots & \vdots \\ (1, \pi_n(1)) & (2, \pi_n(2)) & \cdots & (n, \pi_n(n)). \end{array} \tag{11}$$

Then, we have

$$\Delta(\sigma, \sigma_1, \dots, \sigma_n) = \prod_{1\le i\le n} (a_{1\pi_i(1)} a_{2\pi_i(2)} \cdots a_{n\pi_i(n)}).$$

It is clear that $\varepsilon^{(\sum_{1\le i\le n} t(\pi_i))}(\Delta(\sigma, \sigma_1, \dots, \sigma_n)) = \prod_{1\le i\le n}(\varepsilon^{(t(\pi_i))}(a_{\pi_i(1)} a_{2\pi_i(2)} \cdots a_{n\pi_i(n)}))$ is a term of $(det_\varepsilon(A))^n$. If $2|(t(\sigma) + \sum_{1\le i\le n} t(\sigma_i) - \sum_{1\le i\le n} t(\pi_i))$, then $\tilde{\Delta}(\sigma, \sigma_1, \dots, \sigma_n) = \varepsilon^{(t(\sigma)+\sum_{1\le i\le n} t(\sigma_i))}(\Delta(\sigma, \sigma_1, \dots, \sigma_n)) = \varepsilon^{(\sum_{1\le i\le n} t(\pi_i))}(\Delta(\sigma, \sigma_1, \dots, \sigma_n))$, thus $\tilde{\Delta}(\sigma, \sigma_1, \dots, \sigma_n)$ is a term of $(det_\varepsilon(A))^n$. If $2 \nmid (t(\sigma) + \sum_{1\le i\le n} t(\sigma_i) - \sum_{1\le i\le n} t(\pi_i))$, then $t(\sigma)+\sum_{1\le i\le n} t(\sigma_i) = \sum_{1\le i\le n} t(\pi_i)+2k+1$ for some integer k, thus

$\tilde{\Delta}(\sigma, \sigma_1, \ldots, \sigma_n) = \varepsilon^{(t(\sigma)+\sum_{1\le i\le n} t(\sigma_i))}(\Delta(\sigma, \sigma_1, \ldots, \sigma_n)) = \varepsilon(\varepsilon^{(\sum_{1\le i\le n} t(\pi_i))}(\Delta(\sigma, \sigma_1, \ldots, \sigma_n)))$ is a term of $\varepsilon((det_\varepsilon(A))^n)$. Consequently, we have

$$\tilde{\Delta}(\sigma, \sigma_1, \ldots, \sigma_n) \le (det_\varepsilon(A))^n + \varepsilon((det_\varepsilon(A))^n).$$

Since the number of the terms of $det_\varepsilon(Aadj_\varepsilon(A))$ is $(n!)^{n+1}$, we have

$$det_\varepsilon(Aadj_\varepsilon(A)) \le n!(n!det_\varepsilon(A))^n + \varepsilon(n!(n!det_\varepsilon(A))^n).$$

Similarly, we can prove

$$(det_\varepsilon(A))^n \le det_\varepsilon((adjA)A) \le n!(n!det_\varepsilon(A))^n + \varepsilon(n!(n!det_\varepsilon(A))^n). \qquad \square$$

Since any additively idempotent semiring is a difference-ordered semiring, by Theorem 5.4, we have

COROLLARY 5.4 *If R is a commutative additively idempotent semiring and $A \in M_n(R)$ $(n \ge 2)$, then*

$$(det_\varepsilon(A))^n \le det_\varepsilon(Aadj_\varepsilon(A)) \le (det_\varepsilon(A))^n + \varepsilon((det_\varepsilon(A))^n)$$

and

$$(det_\varepsilon(A))^n \le det_\varepsilon((adjA)A) \le (det_\varepsilon(A))^n + \varepsilon((det_\varepsilon(A))^n).$$

By Corollary 5.4, we have

COROLLARY 5.5 *If R is a commutative additively idempotent semiring and $A \in M_n(R)$ $(n \ge 2)$, then*

$$per(Apadj(A)) = per((padj(A))A) = (per(A))^n.$$

Remark 5.2 Since fuzzy algebras, bounded distributive lattices and commutative inclines are commutative additively idempotent semirings, Corollary 5.5 generalizes Theorem 1 in [13], Theorem 6 in [17] and Theorem 4 in [14].

6. Conclusions

This paper studied the determinants for the matrices over commutative semirings and presented a development of determinantal identities. This includes a generalization of the Laplace and Binet–Cauchy Theorems, as well as on adjoint matrices. Also, the paper discussed the determinants and the adjoint matrices over commutative difference-ordered semirings and obtained some inequalities for the determinants and for the adjoint matrices. The main results in this paper generalize the corresponding results for matrices over commutative rings, for fuzzy matrices, for lattice matrices and for incline matrices.

Acknowledgements

The author would like to thank the referees for a number of constructive comments and valuable suggestions. This work was supported by the Natural Science Foundation of Fujian Province (2012J01008), China.

References

[1] Golan JS. Semirings and their applications. Dordrecht: Kluwer Academic; 1999.

[2] Tan YJ. On invertible matrices over antirings. Linear Algebra Appl. 2007;423:428–444.

[3] Vechtomov EM. Two general structure theorems on submodules, Abelian Groups and Modules (in Russian). Tomsk State University, Tomsk. 2000;No. 15:17–23.

[4] Cao ZQ, Kim KH, Roush FW. Incline algebra and applications. New York (NY): John Wiley; 1984.

[5] Cuninghame-Green RA. Minimax algebra, Lecture Notes in Economics and Mathematical Systems 166. Berlin: Springer-Verlag; 1979.

[6] Zimmermann U. Linear and combinatorial optimization in ordered algebraic structures. vol. 10. Annals of Discrete Mathematics. Amsterdam: North-Holland Publishing Company; 1981.

[7] Mcdonald BR. Linear algebra over commutative rings. New York (NY): Marcel Dekker Inc.; 1984.

[8] Binet JPM. Mémoire sur un système de formules analytiques, et leur application à des considérations géometriques [Memory on a system of analytical formulas and their application to geometrical considerations]. J. Éc. Polyt. 1812;9:280–302.

[9] Cauchy AL. Mémoire sur les fonctions qui ne peuvent obtenir que deux valeurs égales et de signes contraires par suite des traspositions opérées entre les variables qu'elles renferment [Memory functions that can only get two equal values and opposite signs as a result of transpositions between the variables they contain]. J. Éc. Polyt. 1812;10:29–112.

[10] Minc H. Permanents. Reading (MA): Addison-Wesley; 1978.

[11] Duan JS. The transitive closure, convergence of powers and adjoint of generalized fuzzy matrices. Fuzzy Sets Syst. 2004;145:301–311.

[12] Han SC, Li HX. Invertible incline matrices and Cramer's rule over inclines. Linear Algebra Appl. 2004;389:121–138.

[13] Huang Y, Tan YJ. A problem on incline matrices (in Chinese). J. Fuzhou Univ. 2009;37:12–18.

[14] Kim JB, Baartmans A, Sahadin NS. Determinant theory for fuzzy matrices. Fuzzy Sets Syst. 1989;29:349–356.

[15] Ragab MZ, Emam EG. The determinant and adjoint of a square fuzzy matrix. Fuzzy Sets Syst. 1994;61:297–307.

[16] Tian ZJ, Yan KM, Li DG, Zhao H. Determinant of matrices over completely distributive lattices (in Chinese). J. Gansu Univ. Technol. 2002;28(4):115–118.

[17] Zhang KL. Determinant theory for D_{01}-lattices. Fuzzy Sets Syst. 1994;62:347–353.

[18] Tan YJ. On invertible matrices over commutative semirings. Linear Multilinear Algebra. 2013;61:710–714.

[19] Poplin PL, Hartwig RE. Determinantal identities over commutative semirings. Linear Algebra Appl. 2004;387:99–132.