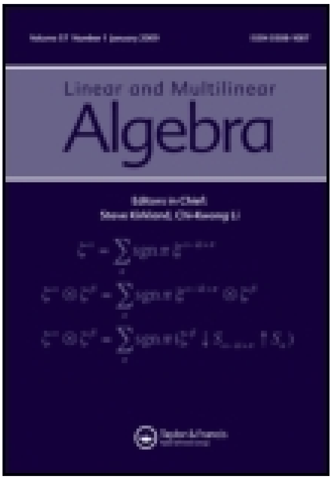

Linear and Multilinear Algebra
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/glma20

On invertible matrices over commutative semirings

Yi-Jia Tan [a]
[a] Department of Mathematics , Fuzhou University , Fuzhou 350108 , China
Published online: 18 Jul 2012.

To cite this article: Yi-Jia Tan (2013) On invertible matrices over commutative semirings, Linear and Multilinear Algebra, 61:6, 710-724, DOI: 10.1080/03081087.2012.703191

To link to this article: http://dx.doi.org/10.1080/03081087.2012.703191

Linear and Multilinear Algebra, 2013
Vol. 61, No. 6, 710–724, http://dx.doi.org/10.1080/03081087.2012.703191

On invertible matrices over commutative semirings

Yi-Jia Tan*

Department of Mathematics, Fuzhou University, Fuzhou 350108, China

Communicated by X. Zhan

(*Received 27 January 2012; final version received 7 June 2012*)

In this article, the invertible matrices over commutative semirings are studied. Some properties and equivalent descriptions of the invertible matrices are given and the inverse matrix of an invertible matrix is presented by analogues of the classic adjoint matrix. Also, Cramer's rule over a commutative semiring is established. The main results obtained in this article generalize the corresponding results for matrices over commutative rings, for lattice matrices, for incline matrices, for matrices over zerosumfree semirings and for matrices over additively regular semirings.

Keywords: invertible matrix; Cramer's rule; determinant; adjoint matrix; semiring

AMS Subject Classifications: 15A09; 15A15; 16Y60

1. Introduction

A *semiring* [14] is an algebraic system $(R, +, \cdot)$ in which $(R, +)$ is an abelian monoid with an identity element 0 and $(R, \cdot)$ is another monoid with an identity element 1, connected by ring-like distributivity. Also, $0r = r0 = 0$ for all r in R and $0 \neq 1$. A semiring R is called *commutative* if $ab = ba$ for all a, b in R; R is called *zerosumfree* [14] if $a + b = 0$ implies that $a = b = 0$ for all a, b in R. Zerosumfree semirings were studied in [25,26] under the name of *antiring*. A semiring R is called *additively regular* [14] if for every a in R, there exists an $a^\sharp$ in R such that $a + a^\sharp + a = a$ and $a^\sharp + a + a^\sharp = a^\sharp$. Additively regular semirings were studied in [16,24] under the name of *additively inversive semiring*. It is proved that for every element a in an additively regular semiring R, the element $a^\sharp$ is unique (see Proposition 13.1 in [14]).

Semirings are quite abundant; for example, any ring with identity is a semiring which is additively regular but not zerosumfree; every Boolean algebra, fuzzy algebra, every bounded distributive lattice and any incline are commutative semirings which are additively regular and zerosumfree. Also, the max-plus algebra $(\mathbb{R} \cup \{-\infty\}, \max, +)$ and the min-plus algebra $(\mathbb{R} \cup \{+\infty\}, \min, +)$ are commutative semirings which are additively regular and zerosumfree [7,28]. In addition, the set $\mathbb{Z}^0$ of nonnegative integers with the usual operations of addition and multiplication of integers is a commutative semiring which is zerosumfree but not additively regular.

*Email: yjtan62@126.com

The same is true for the set $\mathbb{Q}^0$ of all nonnegative rational numbers, for the set $\mathbb{R}^0$ of all nonnegative real numbers.

The study of matrices over general semirings has a long history. In 1964, Rutherford [22] gave a proof of the Cayley–Hamiltion theorem for a commutative semiring avoiding the use of determinants. Since then, a number of works on theory of matrices over semirings were published (see, e.g. [2–4,6,8,9,12,20,21]). In 1999, Golan described semirings and matrices over semirings in his work [14] comprehensively. The techniques of matrices over semirings have important applications in optimization theory, models of discrete event networks and graph theory. For further examples, see [1,11].

Invertible matrices are an important type of matrices. It is well known that a square matrix A over a commutative ring R is invertible if and only if the determinant $|A|$ of A is invertible in R and in this case $A^{-1}=|A|^{-1}\operatorname{adj}(A)$, where $\operatorname{adj}(A)$ denotes the adjoint matrix (see, e.g. [18, Theorem 1.7(b)]). If the ring R is a field, one can use elementary row operations of matrix to find the inverse of an invertible matrix over R. However, if R is an arbitrary commutative ring then the formula $A^{-1}=|A|^{-1}\operatorname{adj}(A)$ is necessary to compute the inverse of an invertible matrix over R. On the other hand, since the beginning of the 1950s, many authors have studied invertible matrices for some special cases of zerosumfree semirings (see, e.g. [5,10,13,15,17,23,27]). In 1952, Luce [17] showed that a matrix over a Boolean algebra of at least two elements is invertible if and only if it is an orthogonal matrix. Give'on [13] developed the theory of invertible lattice matrices, thus generalizing the result of Luce [17]. Zhao [27] discussed the conditions for the invertibility of matrices over a bounded distributive lattice. Skornyakov [23] gave an extensive description of the invertible lattice matrices. Cao et al. [5] were the first to study the condition for an incline matrix to be invertible and showed that the statement of Luce [17] holds for the incline matrices as well. Duan [10] studied the invertible matrices over an incline matrix and obtained some necessary and sufficient conditions for an incline matrix to be invertible. Moreover, Han and Li [15] gave the complete description of the invertible incline matrices and presented the Cramers rule over incline matrices. Tan [25] gave the complete description of the invertible matrices over an arbitrary commutative zerosumfree semiring and proved that if a square matrix A over a commutative zerosumfree semiring is invertible, then $A^{-1}=\operatorname{per}(A)^{-1}\operatorname{adj}(A)$, where $\operatorname{per}(A)$ denotes the permanent of A and $\operatorname{adj}(A)$ denotes the adjoint matrix of A with respect to permanent [25, Lemma 4.2], and presented the Cramers ruler over a commutative zerosumfree semiring [25, Theorem 4.1]. Recently, Sombatboriboon et al. [24] obtained an equivalent condition for a square matrix over a commutative additively regular semiring to be invertible (see Theorem 4 in [24]). It is clear that commutative rings, commutative zerosumfree semirings and commutative additively regular semirings are special kinds of commutative semirings.

A natural question to ask now is: Can this kind of approach be applied to the invertible matrices over a commutative semiring? The aim of this article is to discuss the invertible matrices over a commutative semiring and answer this question positively.

This article is organized as follows. In Section 2, we introduce the concept of ε-determinant of the matrices over a commutative semiring and some lemmas. In Section 3, we give some equivalent descriptions of invertible matrices over a commutative semiring in term of ε-determinant and present the inverse matrix of an invertible matrix by analogues of the classic adjoint matrix. Finally, in Section 4, we establish Cramer's rule over a commutative semiring. The main results in this article generalize the corresponding results for matrices over commutative rings in [18], for lattice matrices in [27], for incline matrices in [10], for matrices over zerosumfree semirings in [25] and for matrices over additively regular semirings in [24].

2. Definitions and preliminaries

In this section, we will give some definitions and preliminaries. For convenience, we use $\underline{n}$ to denote the set $\{1, 2, \ldots, n\}$.

Let R be a semiring. An element x in R is called *additively invertible* in R if $x+y=y+x=0$ for some y in R. Such an element y is obviously unique and denoted by $-x$. Let $V(R)$ denote the set of all additively invertible elements in R. It is clear that $V(R)=\{0\}$ if and only if R is a zerosumfree semiring and that $V(R)=R$ if and only if R is a ring. An element x in R is called *multiplicatively invertible* in R if $xy=yx=1$ for some y in R. Such an element y is obviously unique and denoted by x^{-1}. Let $U(R)$ denote the set of all multiplicatively invertible elements in R. Then $U(R)$ is a subgroup of the semigroup $(R, \cdot)$.

Let R be a commutative semiring. We denote by $M_{m\times n}(R)$ and $V_n(R)$ the set of all $m\times n$ matrices and the set of all column vectors of order n over R, respectively. Especially, we put $M_n(R)=M_{n\times n}(R)$. For $A\in M_{m\times n}(R)$, we denote by a_{ij} the (i, j)-entry of A, and denote by A^T the *transpose* of A.

For any $A, B\in M_{m\times n}(R)$, $C\in M_{n\times l}(R)$ and $\lambda\in R$, we define:

$$A+B=(a_{ij}+b_{ij})_{m\times n}, \quad \lambda A=(\lambda a_{ij})_{m\times n} \quad \text{and} \quad AC=\left(\sum_{k=1}^{n} a_{ik}c_{kj}\right)_{m\times l}.$$

It is easy to see that $M_n(R)$ forms a semiring with respect to the matrix addition and the matrix multiplication. Especially, $M_n(R)$ is a ring if R is a commutative ring.

Let $A\in M_n(R)$. A is called *left invertible* (*right invertible*) in $M_n(R)$ if $BA=I_n$ ($AB=I_n$) for some $B\in M_n(R)$, where I_n denotes the identity matrix of order n over R. Such a matrix B is called a *left inverse* (*right inverse*) of A. If A is both left and right invertible, then A is called *invertible* in $M_n(R)$. Obviously, if A is invertible then its left inverse coincides with its right inverse, which is called its *inverse*. The inverse of A is obviously unique and denoted by A^{-1}. Let $GL_n(R)$ denote the set of all invertible matrices in $M_n(R)$. Then $GL_n(R)$ forms a group under the matrix multiplication, which is called the *linear group* over R.

Let $A\in M_n(R)$. The *positive determinant* $|A|^+$ and the *negative determinant* $|A|^-$ of A are defined as follows:

$$|A|^+=\sum_{\sigma\in\mathcal{A}_n} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}, \quad |A|^-=\sum_{\sigma\in S_n\setminus\mathcal{A}_n} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)},$$

where S_n and $\mathcal{A}_n$ denote the symmetric group and the alternating group on the set $\underline{n}$, respectively [14]. It is clear that $\mathrm{per}(A)=|A|^+ + |A|^-$ and that if R is a commutative ring then $|A|=|A|^+ - |A|^-$.

Let R be a semiring. A bijection ε on R is called an *ε-function* of R if $\varepsilon(\varepsilon(a))=a$ and $\varepsilon(a+b)=\varepsilon(a)+\varepsilon(b)$ and $\varepsilon(ab)=a\varepsilon(b)=\varepsilon(a)b$ for all $a,b\in R$ and $\varepsilon(a)=-a$ for all $a\in V(R)$. It is easy to verify that $\varepsilon(a)\varepsilon(b)=ab$ and $\varepsilon(0)=0$.

Definition 2.1 Let R be a commutative semiring with an ε-function ε and $A\in M_n(R)$. The *ε-determinant* of A, denoted by $\det_\varepsilon(A)$, is defined by

$$\det_\varepsilon(A) = \sum_{\sigma\in S_n} \varepsilon^{(t(\sigma))}(a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}), \tag{1}$$

where $t(\sigma)$ is the number of inversions in the permutation σ, and $\varepsilon^{(k)}$ is defined by $\varepsilon^{(0)}(a)=a$ and $\varepsilon^{(k)}(a)=\varepsilon^{(k-1)}(\varepsilon(a))$ for all positive integers k.

Since $\varepsilon^{(2)}(a)=a$, $\det_\varepsilon(A)$ can be rewritten as follows:

$$\det_\varepsilon(A) = |A|^+ + \varepsilon(|A|^-). \tag{2}$$

Remark 2.1 If R is a commutative ring, then $V(R)=R$ and the mapping ε: $a\longmapsto(-1)a$, $(a\in R)$, is an ε-function of R, and in this case

$$\det_\varepsilon(A) = \sum_{\sigma\in S_n}(-1)^{t(\sigma)}a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)} = |A|.$$

If R is a commutative zerosumfree semiring, then $V(R)=\{0\}$ and the identical mapping ε: $a\longmapsto a$, $(a\in R)$, is an ε-function of R, and in this case

$$\det_\varepsilon(A) = \sum_{\sigma\in S_n} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)} = \mathrm{per}(A).$$

Remark 2.2 If R is a commutative additively regular semiring, then the mapping ε: $a\longmapsto a^\sharp$, $(a\in R)$, is an ε-function of R (see Theorem 3(2) in [16]), and in this case

$$\det_\varepsilon(A) = |A|^+ + (|A|^-)^\sharp.$$

Let $A\in M_n(R)$. An *ε-minor* of order $n-1$ of A is defined to be the ε-determinant of a submatrix of A obtained by striking out one row and one column from A. The ε-minor obtained by striking out the i-th row and the j-th column is written as M_{ij} $(i,j\in \underline{n})$. $\varepsilon^{(i+j)}(M_{ij})$ is called the *ε-cofactor* of the element a_{ij} and denoted by A_{ij}. The *ε-adjoint matrix* of A, denoted by $\mathrm{adj}_\varepsilon(A)$, is defined to be the transposed matrix of ε-cofactors of A, i.e.

$$\mathrm{adj}_\varepsilon(A) = ((A_{ij})_{n\times n})^T.$$

It is clear that if R is a commutative ring and $A\in M_n(R)$ then $\mathrm{adj}_\varepsilon(A)=\mathrm{adj}(A)$.

The following lemmas are used.

Lemma 2.1 *Let R be a semiring. Then*

(1) *for any $a,b\in R$, $a+b\in V(R)$ if and only if $a,b\in V(R)$;*
(2) *for any $a\in V(R)$ and any $r\in R$, we have ra, $ar\in V(R)$.*

Proof

(1) For any $a,b\in R$, if $a+b\in V(R)$ then $a+b+x=0$ for some x in R and so a, $b\in V(R)$. Conversely, if $a,b\in V(R)$ then $a+x=0$ and $b+y=0$ for some x and y in R and so $(a+b)+(x+y)=0$, i.e. $a+b\in V(R)$.

(2) Let $a \in V(R)$ and $r \in R$. Then there exists an x in R such that $a+x=0$ and so $ra+rx=r(a+x)=0$ and $ar+xr=(a+x)r=0$, i.e. ra, $ar \in V(R)$. ■

Lemma 2.2 *Let R be a semiring and ε be an ε-function of R. Then*

$$\varepsilon^{(k)}(x+y)=\varepsilon^{(k)}(x)+\varepsilon^{(k)}(y), \quad \varepsilon^{(k)}(xy)=\varepsilon^{(k)}(x)\,y=x\varepsilon^{(k)}(y)$$

for any x and y in R, where k is a positive integer.

Proof The proof is obvious. ■

Lemma 2.3 [21] *Let $A, B \in M_n(R)$. If $AB=I_n$ then $BA=I_n$.*

Lemma 2.4 *Let R be a commutative semiring with an ε-function and $A \in M_n(R)$. Then*

(1) $\det_\varepsilon(A)=\det_\varepsilon(A^T)$;
(2) *if two rows (or columns) of A are identical, then* $\det_\varepsilon(A)=|A|^+ + \varepsilon(|A|^+)$;
(3) *if $B \in M_n(R)$ is obtained by interchanging two rows (or columns) of A, then* $\det_\varepsilon(B)=\varepsilon(\det_\varepsilon(A))$.

Proof

(1) is obvious.
(2) By Lemma 3.11 in [19], we have $|A|^+=|A|^-$ and so $\det_\varepsilon(A)=|A|^+ + \varepsilon(|A|^+)$.
(3) By Lemma 3.9 in [19], we have $|A|^+=|B|^-$ and $|A|^-=|B|^+$, and so $\det_\varepsilon(B)=|B|^+ + \varepsilon(|B|^-)=|A|^- + \varepsilon(|A|^+)=\varepsilon(|A|^+ + \varepsilon(|A|^-))=\varepsilon(\det_\varepsilon(A))$.

■

Lemma 2.5 *Let R be a commutative semiring with an ε-function and $A \in M_n(R)$. Then for any $i, j \in \underline{n}$, we have*

$$\sum_{k=1}^{n} a_{ik}A_{jk} = \begin{cases} \det_\varepsilon(A) & \text{if } i=j \\ |A_r(i \Rightarrow j)|^+ + \varepsilon(|A_r(i \Rightarrow j)|^+) & \text{if } i \neq j, \end{cases} \tag{3}$$

or, similarly,

$$\sum_{k=1}^{n} a_{ki}A_{kj} = \begin{cases} \det_\varepsilon(A) & \text{if } i=j \\ |A_c(i \Rightarrow j)|^+ + \varepsilon(|A_c(i \Rightarrow j)|^+) & \text{if } i \neq j, \end{cases} \tag{4}$$

where $A_r(i \Rightarrow j)$ (resp. $A_c(i \Rightarrow j)$) denotes the matrix obtained from A by replacing row j (resp. column j) of A by row i (resp. column i) of A.

Proof First, we know that every term of $\det_\varepsilon(A)$ contains an element of the i-th row. If the term $\varepsilon^{(t(\sigma))}(a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)})$ contains the element a_{ik} (note that in this case $k=\sigma(i)$) then, by Lemma 2.2, we have

$$\varepsilon^{(t(\sigma))}(a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}) = a_{ik}\varepsilon^{(t(\sigma))}(a_{1\sigma(1)}\cdots a_{i-1,\sigma(i-1)}a_{i+1,\sigma(i+1)}\cdots a_{n\sigma(n)}).$$

Therefore, collecting together the terms containing a_{ik} ($k \in \underline{n}$), we obtain

$$\sum_{k=1}^{n} a_{ik}A'_{ik} = \det_\varepsilon(A). \tag{5}$$

Hence, to establish $\sum_{k=1}^{n} a_{ik}A_{ik} = \det_\varepsilon(A)$, it suffices to show that

$$A'_{ik} = A_{ik} \tag{6}$$

for all i, $k \in \underline{n}$.

First, consider the case $i=k=1$, we show that $A'_{11} = A_{11}$. Indeed, from Equation (1) and the construction of A'_{11} in Equation (5), it follows that

$$A'_{11} = \sum_{\sigma \in S_n, \sigma(1)=1} \varepsilon^{(t(\sigma))}(a_{2\sigma(2)} \cdots a_{n\sigma(n)}), \tag{7}$$

where $t(\sigma)$ is the number of inversions of the permutation $(1, \sigma(2), \sigma(3), \ldots, \sigma(n))$ or, equivalently, of the permutation $(\sigma(2), \sigma(3), \ldots, \sigma(n))$ of the numbers $2, 3, \ldots, n$. Thus $A'_{11} = M_{11} = \varepsilon^{(1+1)}(M_{11}) = A_{11}$.

To prove Equation (6) in the general case, we first shift a_{ik} to the $(1, 1)$ position by means of $i-1$ successive interchanges of adjacent rows followed by $k-1$ interchanges of adjacent columns. Call the rearranged matrix B. The ε-minor associated with a_{ik} is the same in both A and B because the relative positions of rows and columns of the submatrix corresponding to this ε-minor M_{ik} are unchanged. Hence, using the special case already proved, $\det_\varepsilon(B) = a_{ik}M_{ik} +$ (terms not involving a_{ik}). But Lemma 2.4(3) implies that

$$\det_\varepsilon(B) = \varepsilon^{(i-1)}(\varepsilon^{(k-1)}(\det_\varepsilon(A))) = \varepsilon^{(i+k-2)}(\det_\varepsilon(A)) = \varepsilon^{(i+k)}(\det_\varepsilon(A)).$$

Hence

$$\begin{aligned}\det_\varepsilon(A) &= \varepsilon^{(i+k)}(a_{ik}M_{ik}) + (\text{terms not involving } a_{ik})\\ &= a_{ik}\varepsilon^{(i+k)}(M_{ik}) + (\text{terms not involving } a_{ik}).\end{aligned}$$

Then $A'_{ik} = \varepsilon^{(i+k)}(M_{ik}) = A_{ik}$ and so $\sum_{k=1}^{n} a_{ik}A_{ik} = \det_\varepsilon(A)$.

If $i \neq j$, then, by Lemma 2.4(2), we have

$$\sum_{k=1}^{n} a_{ik}A_{jk} = \det_\varepsilon(A_r(i \Rightarrow j)) = |A_r(i \Rightarrow j)|^+ + \varepsilon(|A_r(i \Rightarrow j)|^+).$$

This completes the proof of Equation (3). Equation (4) can be obtained using Equation (3) and Lemma 2.4(1). ■

3. Some conditions for invertible matrices over a commutative semiring

In this section, we give some properties and equivalent descriptions of invertible matrices over a commutative semiring R in terms of ε-determinant and present the inverse matrix of an invertible matrix by the ε-adjoint matrix. In this section and Section 4, R is supposed to be a commutative semiring with an ε-function.

Proposition 3.1 *Let $A \in M_n(R)$. If A is invertible, then*

(1) $\sum_{1 \leq i \leq n} a_{ij}a_{ik} \in V(R)$ *for all j, $k \in \underline{n}$ with $j \neq k$;*
(2) $\sum_{1 \leq j \leq n} a_{ij}a_{kj} \in V(R)$ *for all i, $k \in \underline{n}$ with $i \neq k$.*

Proof

(1) Since A is invertible in $M_n(R)$, $AB = BA = I_n$ for some B in $M_n(R)$. By $AB = I_n$, we have $\sum_{k=1}^{n} a_{ik}b_{kj} = 0$ for all $i, j \in \underline{n}$ with $i \neq j$ and $\sum_{k=1}^{n} a_{ik}b_{ki} = 1$ for all $i \in \underline{n}$, and so $a_{ik}b_{kj} \in V(R)$ for all $i, j, k \in \underline{n}$ with $i \neq j$ (by Lemma 2.1(1)) and $\prod_{i=1}^{n}(\sum_{s=1}^{n} a_{is}b_{si}) = 1$.
For any $i, j, k \in \underline{n}$ with $j \neq k$, we have

$$\begin{aligned} a_{ij}a_{ik} &= a_{ij}a_{ik}\prod_{t=1}^{n}\left(\sum_{s=1}^{n} a_{ts}b_{st}\right) \\ &= a_{ij}a_{ik}\left(\sum_{1\le s_1,s_2,\ldots,s_n\le n} a_{1s_1}b_{s_11}a_{2s_2}b_{s_22}\cdots a_{ns_n}b_{s_nn}\right) \\ &= \sum_{1\le s_1,s_2,\ldots,s_n\le n} a_{ij}a_{ik}a_{1s_1}b_{s_11}a_{2s_2}b_{s_22}\cdots a_{ns_n}b_{s_nn}. \end{aligned}$$

For any $s_1, \ldots, s_n \in \underline{n}$, if $s_u = s_v$ for some $u \neq v$ then $a_{us_u}b_{s_vv} = a_{us_u}b_{s_uv} \in V(R)$ and so $a_{ij}a_{ik}a_{1s_1}b_{s_11}a_{2s_2}b_{s_22}\cdots a_{ns_n}b_{s_nn} \in V(R)$ (by Lemma 2.1(2)); if $s_1, \ldots, s_n$ are mutually different, then there must be $u, v \in \underline{n}$ with $u \neq v$ such that $j = s_u$ and $k = s_v$ and so $a_{ij}b_{s_uu}a_{ik}b_{s_vv} = a_{ij}b_{ju}a_{ik}b_{kv} \in V(R)$. By Lemma 2.1(2), we have $a_{ij}a_{ik}a_{1s_1}b_{s_11}a_{2s_2}b_{s_22}\cdots a_{ns_n}b_{s_nn} \in V(R)$. Consequently, we have $a_{ij}a_{ik} = \sum_{1\le s_1,s_2,\ldots,s_n\le n} a_{ij}a_{ik}a_{1s_1}b_{s_11}a_{2s_2}b_{s_22}\cdots a_{ns_n}b_{s_nn} \in V(R)$ (by Lemma 2.1(1)). Then $\sum_{1\le i\le n} a_{ij}a_{ik} \in V(R)$ for all $j, k \in \underline{n}$ with $j \neq k$ (by Lemma 2.1(1)). This proves (1).

(2) Similar to that of (1). ∎

Proposition 3.2 *Let $A \in M_n(R)$. If A is invertible, then*

$$\det_\varepsilon(AB) = \det_\varepsilon(BA) = \det_\varepsilon(A)\det_\varepsilon(B)$$

for any $B \in M_n(R)$.

Proof We have

$$\begin{aligned} \det_\varepsilon(AB) &= \sum_{\sigma\in S_n} \varepsilon^{(t(\sigma))}\left(\left(\sum_{i_1=1}^{n} a_{1i_1}b_{i_1\sigma(1)}\right)\cdots\left(\sum_{i_n=1}^{n} a_{ni_n}b_{i_n\sigma(n)}\right)\right) \\ &= \sum_{\sigma\in S_n} \varepsilon^{(t(\sigma))}\left(\sum_{1\le i_1,i_2,\ldots,i_n\le n} a_{1i_1}\cdots a_{ni_n}(b_{i_1\sigma(1)}\cdots b_{i_n\sigma(n)})\right) \\ &= \sum_{\sigma\in S_n}\left(\sum_{1\le i_1,i_2,\ldots,i_n\le n} a_{1i_1}\cdots a_{ni_n}\varepsilon^{(t(\sigma))}(b_{i_1\sigma(1)}\cdots b_{i_n\sigma(n)})\right) \text{ (by Lemma 2.2)} \\ &= \sum_{1\le i_1,i_2,\ldots,i_n\le n} a_{1i_1}\cdots a_{ni_n}\left(\sum_{\sigma\in S_n}\varepsilon^{(t(\sigma))}(b_{i_1\sigma(1)}\cdots b_{i_n\sigma(n)})\right). \end{aligned}$$

For any $i_1, i_2, \ldots, i_n \in \underline{n}$, if $i_s = i_t$ for some $s, t \in \underline{n}$ with $s \neq t$, then $a_{si_s}a_{ti_t} \in V(R)$ (by Proposition 3.1(2) and Lemma 2.1(1)). Let $c_{uv} = b_{i_uv}$ for all $u, v \in \underline{n}$ and $C = (c_{uv})_{n\times n}$. Then the s-th row and the t-th row of C are identical. By Lemma 2.4(2) we have

$$\sum_{\sigma\in S_n}\varepsilon^{(t(\sigma))}(b_{i_1\sigma(1)}\cdots b_{i_n\sigma(n)}) = \det_\varepsilon(C) = |C|^+ + \varepsilon(|C|^+)$$

and so

$$\begin{aligned} a_{1i_1}\cdots a_{ni_n}\left(\sum_{\sigma\in S_n}\varepsilon^{(t(\sigma))}(b_{i_1\sigma(1)}\cdots b_{i_n\sigma(n)})\right) &= a_{1i_1}\cdots a_{ni_n}|C|^+ + a_{1i_1}\cdots a_{ni_n}\varepsilon(|C|^+) \\ &= a_{1i_1}\cdots a_{ni_n}|C|^+ + \varepsilon(a_{1i_1}\cdots a_{ni_n}|C|^+). \end{aligned}$$

Since $a_{si_s}a_{ti_t}\in V(R)$, we have $a_{1i_1}\cdots a_{ni_n}|C|^+\in V(R)$ (by Lemma 2.1(2)) and so

$$\varepsilon(a_{1i_1}\cdots a_{ni_n}|C|^+) = -a_{1i_1}\cdots a_{ni_n}|C|^+.$$

Therefore

$$a_{1i_1}\cdots a_{ni_n}\left(\sum_{\sigma\in S_n}\varepsilon^{(t(\sigma))}(b_{i_1\sigma(1)}\cdots b_{i_n\sigma(n)})\right)=0.$$

Then

$$\begin{aligned} \det_\varepsilon(AB) &= \sum_{\substack{1\le i_1,i_2,\ldots,i_n\le n\\ i_s\neq i_t(s\neq t)}} a_{1i_1}\cdots a_{ni_n}\left(\sum_{\sigma\in S_n}\varepsilon^{(t(\sigma))}(b_{i_1\sigma(1)}\cdots b_{i_n\sigma(n)})\right)\\
&= \sum_{\gamma\in S_n} a_{1\gamma(1)}\cdots a_{n\gamma(n)}\left(\sum_{\sigma\in S_n}\varepsilon^{(t(\sigma))}(b_{\gamma(1)\sigma(1)}\cdots b_{\gamma(n)\sigma(n)})\right)\\
&= \sum_{\gamma\in S_n} a_{1\gamma(1)}\cdots a_{n\gamma(n)}\left(\sum_{\sigma\in \mathcal{A}_n} b_{\gamma(1)\sigma(1)}\cdots b_{\gamma(n)\sigma(n)}+\varepsilon\left(\sum_{\sigma\in S_n\setminus\mathcal{A}_n} b_{\gamma(1)\sigma(1)}\cdots b_{\gamma(n)\sigma(n)}\right)\right)\\
&= \sum_{\gamma\in \mathcal{A}_n} a_{1\gamma(1)}\cdots a_{n\gamma(n)}\left(\sum_{\sigma\in \mathcal{A}_n} b_{\gamma(1)\sigma(1)}\cdots b_{\gamma(n)\sigma(n)}+\varepsilon\left(\sum_{\sigma\in S_n\setminus\mathcal{A}_n} b_{\gamma(1)\sigma(1)}\cdots b_{\gamma(n)\sigma(n)}\right)\right)\\
&\quad + \sum_{\gamma\in S_n\setminus\mathcal{A}_n} a_{1\gamma(1)}\cdots a_{n\gamma(n)}\left(\sum_{\sigma\in \mathcal{A}_n} b_{\gamma(1)\sigma(1)}\cdots b_{\gamma(n)\sigma(n)}\right.\\
&\quad \left. +\varepsilon\left(\sum_{\sigma\in S_n\setminus\mathcal{A}_n} b_{\gamma(1)\sigma(1)}\cdots b_{\gamma(n)\sigma(n)}\right)\right)\\
&= \sum_{\gamma\in \mathcal{A}_n} a_{1\gamma(1)}\cdots a_{n\gamma(n)}\left(\sum_{\sigma\gamma^{-1}\in \mathcal{A}_n} b_{1\sigma\gamma^{-1}(1)}\cdots b_{n\sigma\gamma^{-1}(n)}\right.\\
&\quad \left. +\varepsilon\left(\sum_{\sigma\gamma^{-1}\in S_n\setminus\mathcal{A}_n} b_{1\sigma\gamma^{-1}(1)}\cdots b_{n\sigma\gamma^{-1}(n)}\right)\right)\\
&\quad + \sum_{\gamma\in S_n\setminus\mathcal{A}_n} a_{1\gamma(1)}\cdots a_{n\gamma(n)}\left(\sum_{\sigma\gamma^{-1}\in S_n\setminus\mathcal{A}_n} b_{1\sigma\gamma^{-1}(1)}\cdots b_{n\sigma\gamma^{-1}(n)}\right.\\
&\quad \left. +\varepsilon\left(\sum_{\sigma\gamma^{-1}\in \mathcal{A}_n} b_{1\sigma\gamma^{-1}(1)}\cdots b_{n\sigma\gamma^{-1}(n)}\right)\right)\\
&= \sum_{\gamma\in \mathcal{A}_n} a_{1\gamma(1)}\cdots a_{n\gamma(n)}(|B|^+ +\varepsilon(|B|^-)) + \sum_{\gamma\in S_n\setminus\mathcal{A}_n} a_{1\gamma(1)}\cdots a_{n\gamma(n)}(|B|^- +\varepsilon(|B|^+))\\
&= \left(\sum_{\gamma\in \mathcal{A}_n} a_{1\gamma(1)}\cdots a_{n\gamma(n)}\right)(|B|^+ +\varepsilon(|B|^-)) + \left(\sum_{\gamma\in S_n\setminus\mathcal{A}_n} a_{1\gamma(1)}\cdots a_{n\gamma(n)}\right)(|B|^- +\varepsilon(|B|^+))
\end{aligned}$$

$$\begin{aligned}
&= |A|^+(|B|^+ + \varepsilon(|B|^-)) + |A|^-(|B|^- + \varepsilon(|B|^+))\\
&= |A|^+|B|^+ + |A|^+\varepsilon(|B|^-) + |A|^-|B|^- + |A|^-\varepsilon(|B|^+)\\
&= |A|^+|B|^+ + |A|^+\varepsilon(|B|^-) + \varepsilon(|A|^-)|B|^+ + \varepsilon(|A|^-)\varepsilon(|B|^-)\\
&= (|A|^+ + \varepsilon(|A|^-))(|B|^+ + \varepsilon(|B|^-)) = \det_\varepsilon(A)\det_\varepsilon(B).
\end{aligned}$$

Similarly, we can prove $\det_\varepsilon(BA) = \det_\varepsilon(A)\det_\varepsilon(B)$. ■

The following theorem is one of our main results.

THEOREM 3.1 *For any $A \in M_n(R)$, the following statements are equivalent.*

(1) *A is invertible in $M_n(R)$.*
(2) *$\det_\varepsilon(A) \in U(R)$ and $\sum_{1\le i\le n} a_{ij}a_{ik} \in V(R)$ for all j, $k \in \underline{n}$ with $j \neq k$.*
(3) *$\det_\varepsilon(A) \in U(R)$ and $\sum_{1\le j\le n} a_{ij}a_{kj} \in V(R)$ for all i, $k \in \underline{n}$ with $i \neq k$.*

And in this case

$$A^{-1} = (\det_\varepsilon(A))^{-1}\mathrm{adj}_\varepsilon(A).$$

Proof (1) $\Rightarrow$ (2). If A is invertible in $M_n(R)$ then $AB = BA = I_n$ for some $B \in M_n(R)$ and so $1 = \det_\varepsilon(I_n) = \det_\varepsilon(AB) = \det_\varepsilon(A)\det_\varepsilon(B)$ (by Proposition 3.2), i.e. $\det_\varepsilon(A) \in U(R)$. By Proposition 3.1(1), $\sum_{1\le i\le n} a_{ij}a_{ik} \in V(R)$ for all j, $k \in \underline{n}$ with $j \neq k$.

(2) $\Rightarrow$ (1). Suppose that $\det_\varepsilon(A) \in U(R)$ and $\sum_{1\le i\le n} a_{ij}a_{ik} \in V(R)$ for all $j, k \in \underline{n}$ with $j \neq k$. Then $a_{ij}a_{ik} \in V(R)$ for all i, j, $k \in \underline{n}$ with $j \neq k$ (by Lemma 2.1(1)) and so $a_{ij}a_{ik} + \varepsilon(a_{ij})a_{ik} = a_{ij}a_{ik} + \varepsilon(a_{ij}a_{ik}) = a_{ij}a_{ik} + (-(a_{ij}a_{ik})) = 0$.

Let $C = A\,\mathrm{adj}_\varepsilon(A)$. For any $i, j \in \underline{n}$, if $i = j$ then $c_{ii} = \sum_{k=1}^n a_{ik}A_{ik} = \det_\varepsilon(A)$ (by Lemma 2.5); if $i \neq j$ then

$$\begin{aligned}
c_{ij} &= \sum_{k=1}^n a_{ik}A_{jk} = |A_r(i \Rightarrow j)|^+ + \varepsilon(|A_r(i \Rightarrow j)|^+) \text{ (by Lemma 2.5)}\\
&= \sum_{\sigma \in \mathcal{A}_n} a_{1\sigma(1)}\cdots a_{i\sigma(i)}\cdots a_{i\sigma(j)}\cdots a_{n\sigma(n)} + \varepsilon\left(\sum_{\sigma \in \mathcal{A}_n} a_{1\sigma(1)}\cdots a_{i\sigma(i)}\cdots a_{i\sigma(j)}\cdots a_{n\sigma(n)}\right)\\
&= \sum_{\sigma \in \mathcal{A}_n} a_{1\sigma(1)}\cdots a_{i\sigma(i)}\cdots a_{i\sigma(j)}\cdots a_{n\sigma(n)} + \sum_{\sigma \in \mathcal{A}_n} a_{1\sigma(1)}\cdots \varepsilon(a_{i\sigma(i)})\cdots a_{i\sigma(j)}\cdots a_{n\sigma(n)}\\
&= \sum_{\sigma \in \mathcal{A}_n} (a_{i\sigma(i)}a_{i\sigma(j)} + \varepsilon(a_{i\sigma(i)})a_{i\sigma(j)}) \prod_{k\neq i,j} a_{k\sigma(k)}\\
&= 0 \text{ (because } \sigma(i) \neq \sigma(j)).
\end{aligned}$$

Consequently, $A\,\mathrm{adj}_\varepsilon(A) = \det_\varepsilon(A)I_n$. Taking $B = (\det_\varepsilon(A))^{-1}\mathrm{adj}_\varepsilon(A)$, we have $AB = I_n$. By Lemma 2.3, we have $BA = I_n$. Then A is invertible in $M_n(R)$.

Similarly, we can prove that the statements (1) and (3) are equivalent.

By the proof of (2) $\Rightarrow$ (1), we have $A^{-1} = (\det_\varepsilon(A))^{-1}\mathrm{adj}_\varepsilon(A)$. ■

If R is a commutative ring then $V(R) = R$ and the mapping ε: $a \longmapsto -a$, $(a \in R)$, is an ε-function of R, and in this case, $\det_\varepsilon(A) = |A|$ and $\mathrm{adj}_\varepsilon(A) = \mathrm{adj}(A)$. By Theorem 3.1, we have the following corollary.

Corollary 3.1 [18, Theorem 1.7(b)] *If R is a commutative ring and $A \in M_n(R)$, then A is invertible in $M_n(R)$ if and only if $|A| \in U(R)$, and in this case*

$$A^{-1} = |A|^{-1} \operatorname{adj}(A).$$

■

If R is a commutative additively regular semiring then the mapping ε: $a \longmapsto a^{\sharp}$, $(a \in R)$, is an ε-function of R, and in this case, $\det_\varepsilon(A) = |A|^+ + (|A|^-)^{\sharp}$. By Theorem 3.1 and Lemma 2.1(1), we have the following corollary.

Corollary 3.2 [24, Theorem 4] *If R is a commutative additively regular semiring and $A \in M_n(R)$, then A is invertible in $M_n(R)$ if and only if $|A|^+ + (|A|^-)^{\sharp} \in U(R)$ and $a_{ij}a_{ik} \in V(R)$ for all $i, j, k \in \underline{n}$ with $j \neq k$.* ■

Corollary 3.3 *If R is a commutative zerosumfree semiring and $A \in M_n(R)$, then the following statements are equivalent.*

(1) *A is invertible in $M_n(R)$.*
(2) $\operatorname{per}(A) \in U(R)$ *and $a_{ij}a_{ik} = 0$ for all $i, j, k \in \underline{n}$ with $j \neq k$.*
(3) $\operatorname{per}(A) \in U(R)$ *and $a_{ij}a_{kj} = 0$ for all $i, j, k \in \underline{n}$ with $i \neq k$.*
(4) *$\sum_{1 \leq u \leq n} a_{uj} \in U(R)$ for all $j \in \underline{n}$ and $a_{ij}a_{ik} = 0$ for all $i, j, k \in \underline{n}$ with $j \neq k$.*
(5) *$\sum_{1 \leq u \leq n} a_{iu} \in U(R)$ for all $i \in \underline{n}$ and $a_{ij}a_{kj} = 0$ for all $i, j, k \in \underline{n}$ with $i \neq k$.*

And in this case

$$A^{-1} = (\operatorname{per}(A))^{-1} \operatorname{adj}(A) = D^{-1} A^T,$$

where adj(A) *denotes the adjoint matrix of A with respect to permanent,* $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ *and $d_j = \sum_{1 \leq u \leq n} a_{uj}^2 \in U(R)$ for all $j \in \underline{n}$.*

Proof Let R be a commutative zerosumfree semiring. Then $V(R) = \{0\}$ and the identical mapping ε: $a \mapsto a$, $(a \in R)$, is an ε- function of R and $\det_\varepsilon(A) = \operatorname{per}(A)$. By Theorem 3.1, the statements (1)–(3) are equivalent and $A^{-1} = (\operatorname{per}(A))^{-1}\operatorname{adj}(A)$.

Suppose that $a_{ij}a_{kj} = 0$ for all $i, j, k \in \underline{n}$ with $i \neq k$. Then

$$\prod_{1 \leq i \leq n} \left(\sum_{1 \leq u \leq n} a_{iu} \right) = \sum_{1 \leq u_1, u_2, \ldots, u_n \leq n} a_{1u_1} a_{2u_2} \cdots a_{nu_n}.$$

For any $u_1, u_2, \ldots, u_n \in \underline{n}$, if $u_s = u_t$ for some $s, t \in \underline{n}$ with $s \neq t$, then $a_{su_s}a_{tu_t} = 0$ and so $a_{1u_1}a_{2u_2} \cdots a_{nu_n} = 0$. Therefore

$$\begin{aligned} \prod_{1 \leq i \leq n} \left(\sum_{1 \leq u \leq n} a_{iu} \right) &= \sum_{\substack{1 \leq u_1, u_2, \ldots, u_n \leq n \\ u_s \neq u_t (s \neq t)}} a_{1u_1} a_{2u_2} \cdots a_{nu_n} \\ &= \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \\ &= \operatorname{per}(A). \end{aligned}$$

This implies that $\operatorname{per}(A) \in U(R)$ if and only if $\sum_{1 \leq u \leq n} a_{iu} \in U(R)$ for all $i \in \underline{n}$. Hence the statements (3) and (5) are equivalent. Similarly, we can prove that the statements (2) and (4) are equivalent. In the following we prove $A^{-1} = D^{-1}A^T$.

For any $i, j \in \underline{n}$, if $i = j$ then $(A^TA)_{jj} = \sum_{1 \le u \le n} a_{uj}^2 = d_j$. Since $a_{uj}a_{vj} = 0$ for $u \neq v$, we have $\sum_{1 \le u \le n} a_{uj}^2 = (\sum_{1 \le u \le n} a_{uj})^2 \in U(R)$ and so $d_j \in U(R)$; if $i \neq j$ then $(A^TA)_{ij} = \sum_{1 \le u \le n} a_{ui}a_{uj} = 0$. Hence $A^TA = D$ and so $A^{-1} = D^{-1}A^T$. ■

If R is a bounded distributive lattice or an incline then R is a commutative zerosumfree semiring and $U(R) = \{1\}$. By Corollary 3.3 and Lemma 2.3, we have the following corollary.

Corollary 3.4 [10, Theorem 3] *If R is an incline and $A, B \in M_n(R)$, then the following statements are equivalent.*

(1) $AB = BA = I_n$.
(2) $AB = I_n$.
(3) $\sum_{1 \le u \le n} a_{uj} = 1$ *for all* $j \in \underline{n}$ *and* $a_{ij}a_{ik} = 0$ *for all* $i, j, k \in \underline{n}$ *with* $j \neq k$.
(4) $\sum_{1 \le u \le n} a_{iu} = 1$ *for all* $i \in \underline{n}$ *and* $a_{ij}a_{kj} = 0$ *for all* $i, j, k \in \underline{n}$ *with* $i \neq k$.

And in this case

$$A^{-1} = A^T.$$

Corollary 3.5 [27, Theorem 1] *If $L = (L, \vee, \wedge, 0, 1)$ is a distributive lattice and $A \in M_n(R)$, then a necessary and sufficient condition for the invertibility of A is that the equalities*

$$\bigvee_{1 \le u \le n} a_{iu} = 1, \quad i \in \underline{n}$$

$$\bigvee_{1 \le u \le n} a_{uj} = 1, \quad j \in \underline{n}$$

$$\bigvee_{1 \le j \le n} a_{ij}a_{kj} = 0, \quad i \neq k,$$

$$\bigvee_{1 \le i \le n} a_{ij}a_{ik} = 0, \quad j \neq k,$$

hold simultaneously. In this case, A^T is the unique inverse of A.

Example 3.1 Consider the set $R = \{(a, b) \mid a \in \mathbb{R}, b \in \mathbb{R}^0\}$, where $\mathbb{R}$ and $\mathbb{R}^0$ denote the real number field and the semiring of nonnegative real numbers, respectively. Define operations of addition and multiplication on R by setting

$$(a, b) + (c, d) = (a + c, b + d)$$

and

$$(a, b)(c, d) = (ac + ad + bc, bd)$$

for all $(a, b), (c, d) \in R$. Then R can be easily verified to be a commutative semiring with an additive identity $(0, 0)$ and a multiplicative identity $(0, 1)$, and $V(R) = \{(a, 0) \mid a \in \mathbb{R}\}$ and $U(R) = \{(a, b) \in R \mid b \neq 0, a + b \neq 0\}$ (note that R is called the *Dorroh extension of* $\mathbb{R}$ *by* $\mathbb{R}^0$ (see [14])). It is easy to verify that the function

$$\varepsilon\colon (a, b) \longmapsto (-a - 2b, b), ((a, b) \in R),$$

is an ε-function of R.

Now let

$$A = \begin{pmatrix} (2,0) & (-1,1) & (1,0) \\ (-1,0) & (0,0) & (-1,1) \\ (0,1) & (1,0) & (1,0) \end{pmatrix} \in M_3(R).$$

Then

$$\begin{aligned} \det_\varepsilon(A) &= |A|^+ + \varepsilon(|A|^-) \\ &= (2,0)(0,0)(1,0) + (-1,1)(-1,1)(0,1) + (1,0)(-1,0)(1,0) \\ &\quad + \varepsilon((1,0)(0,0)(0,1) + (-1,1)(-1,0)(1,0) + (2,0)(-1,1)(1,0)) \\ &= (0,0) + (-1,1) + (-1,0) + \varepsilon((0,0) + (0,0) + (0,0)) \\ &= (-2,1) \end{aligned}$$

and

$$\begin{aligned} a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} &= (2,0)(-1,1) + (-1,0)(0,0) + (0,1)(1,0) = (1,0), \\ a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} &= (2,0)(1,0) + (-1,0)(-1,1) + (0,1)(1,0) = (3,0), \\ a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} &= (-1,1)(1,0) + (0,0)(-1,1) + (1,0)(1,0) = (1,0). \end{aligned}$$

Since $(-2,1) \in U(R)$ and $(1,0),(3,0) \in V(R)$ (note that $(-2,1)^{-1} = (-2,1)$ and $-(1,0) = (-1,0)$ and $-(3,0) = (-3,0)$), we have that A is invertible in $M_3(R)$ (by Theorem 3.1). Moreover, by a computation, we have

$$A^{-1} = (\det_\varepsilon(A))^{-1}\mathrm{adj}_\varepsilon(A) = \begin{pmatrix} (0,0) & (-1,0) & (-1,1) \\ (-2,1) & (-1,0) & (1,0) \\ (1,0) & (1,1) & (0,0) \end{pmatrix}.$$

Finally, by Proposition 3.2 and Theorem 3.1, we have the following theorem.

THEOREM 3.2 *The mapping* $\varphi: A \longmapsto \det_\varepsilon(A)$, $(A \in GL_n(R))$, *is a homomorphism from the group* $GL_n(R)$ *to the group* $U(R)$.

4. Cramer's rule over a commutative semiring

In this section, we establish Cramer's rule for a matrix equation over a commutative semiring R.

THEOREM 4.1 *Let* $A \in M_n(R)$ *and* $b = (b_1, b_2, \ldots, b_n)^T \in V_n(R)$. *If* A *is invertible in* $M_n(R)$ *then the matrix equation* $Ax = b$ *has a unique solution*

$$x = (d^{-1}d_1, d^{-1}d_2, \ldots, d^{-1}d_n)^T,$$

where $d = \det_\varepsilon(A)$ *and* $d_j = \det_\varepsilon(A_j)$ *for* $j \in \underline{n}$ *and* A_j *is the matrix formed by replacing the* j*-th column of* A *by the column vector* b.

Proof It is clear that the matrix equation $Ax = b$ has a solution $x = A^{-1}b$. Let $y \in V_n(R)$ be any solution of this equation, then $Ay = b$ and so $y = A^{-1}b$, which means that the equation $Ax = b$ has a unique solution.

Now let $(\mathrm{adj}_\varepsilon(A))b=(d_1,d_2,\ldots,d_n)^T$. Then for any $j\in \underline{n}$, we have $d_j=\sum_{i=1}^n b_iA_{ij}$. On the other hand, we have

$$\det_\varepsilon(A_j)=\det_\varepsilon\begin{pmatrix} a_{11} & \ldots & a_{1,j-1} & b_1 & a_{1,j+1} & \ldots & a_{1n}\\ a_{21} & \ldots & a_{2,j-1} & b_2 & a_{2,j+1} & \ldots & a_{2n}\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ a_{n1} & \ldots & a_{n,j-1} & b_n & a_{n,j+1} & \ldots & a_{nn}\end{pmatrix}$$
$$=\sum_{i=1}^n b_iA_{ij}\ \text{(by Lemma 2.5)}.$$

Therefore

$$x=A^{-1}b=(\det_\varepsilon(A))^{-1}(\mathrm{adj}_\varepsilon(A))b\ \text{(by Theorem 3.1)}$$
$$=d^{-1}(\mathrm{adj}_\varepsilon(A))b=(d^{-1}d_1,d^{-1}d_2,\ldots,d^{-1}d_n)^T.$$

This completes the proof. ■

By Theorem 4.1, we have the following corollary.

COROLLARY 4.1 *If R is a commutative ring and $A\in M_n(R)$ is invertible, then for any $b=(b_1,b_2,\ldots,b_n)^T\in V_n(R)$ the matrix equation $Ax=b$ has a unique solution*

$$x=(d^{-1}d_1,d^{-1}d_2,\ldots,d^{-1}d_n)^T,$$

where $d=|A|$ and $d_j=|A_j|$ for $j\in\underline{n}$ and A_j is the matrix formed by replacing the j-th column of A by the column vector b.

COROLLARY 4.2 [25, THEOREM 4.1] *If R is a commutative zerosumfree semiring and $A\in M_n(R)$ is invertible, then for any $b=(b_1,b_2,\ldots,b_n)^T\in V_n(R)$ the matrix equation $Ax=b$ has a unique solution*

$$x=(d^{-1}d_1,d^{-1}d_2,\ldots,d^{-1}d_n)^T,$$

where $d=\mathrm{per}(A)$ and $d_j=\mathrm{per}(A_j)$ for $j\in\underline{n}$ and A_j is the matrix formed by replacing the j-th column of A by the column vector b.

5. Conclusions

This article studied the invertible matrices over commutative semirings and obtained some properties and equivalent descriptions of the invertible matrices and present the inverse matrix of an invertible matrix by analogues of the classic adjoint matrix. Also, this article established Cramer's rule over a commutative semiring. The main results obtained in this article generalize the corresponding results for matrices over commutative rings, for lattice matrices, for incline matrices, for matrices over zerosumfree semirings and for matrices over additively regular semirings.

Acknowledgements

The author would like to thank the referees for a number of constructive comments and valuable suggestions. This work was supported by the Natural Science Foundation of Fujian Province (2012J01008), China.

References

[1] F.L. Baccelli and I. Mairesse, *Ergodic theorems for stochastic operators and discrete event networks*, in *Idempotency*, J. Gunawardena, ed., Cambridge University Press, Cambridge, 1998, pp. 171–208.

[2] L.B. Beasley and S.G. Lee, *Linear operators strongly preserving r-potent matrices over semirings*, Linear Algebra Appl. 162–164 (1992), pp. 589–599.

[3] L.B. Beasley and S.G. Lee, *Linear operators strongly preserving r-cyclic matrices over semirings*, Linear Multilinear Algebra 35 (1993), pp. 325–337.

[4] L. Beasley and N.J. Pullman, *Linear operators strongly preserving idempotent matrices over semirings*, Linear Algebra Appl. 160 (1992), pp. 217–229.

[5] Z.Q. Cao, K.H. Kim, and F.W. Roush, *Incline Algebra and Applications*, John Wiley, New York, 1984.

[6] H.H. Cho and S.R. Kim, *Factorizations of matrices over semirings*, Linear Algebra Appl. 373 (2003), pp. 289–296.

[7] R.A. Cuninghame-Green, *Minimax Algebra*, Lecture Notes in Economics and Mathematical Systems, Vol. 166, Springer-Verlag, Berlin, 1979.

[8] D. Dolžan and P. Oblak, *Noncommuting graphs of matrices over semirings*, Linear Algebra Appl. 435 (2011), pp. 1649–1656.

[9] D. Dolžan and P. Oblak, *Commuting graphs of matrices over semirings*, Linear Algebra Appl. 435 (2011), pp. 1657–1665.

[10] J.S. Duan, *Invertible conditions for matrices over an incline*, Adv. Math. 35 (2006), pp. 285–288.

[11] B. Gaujal and A. Jean-Marie, *Computational issues in recursive stochastic systems*, in *Idempotency*, J. Gunawardena, ed., Cambridge University Press, Cambridge, 1998, pp. 209–230.

[12] S. Ghosh, *Matrices over semirings*, Inform. Sci. 90 (1996), pp. 221–230.

[13] Y. Give'on, *Lattice matrices*, Inform. Control 7 (1964), pp. 477–484.

[14] J.S. Golan, *Semirings and Their Applications*, Kluwer Academic Publishers, Dordrecht, 1999.

[15] S.C. Han and H.X. Li, *Invertible incline matrices and Cramer's rule over inclines*, Linear Algebra Appl. 389 (2004), pp. 121–138.

[16] P.H. Karvellas, *Inversive semirings*, J. Australian Math. Soc. 18 (1974), pp. 277–288.

[17] R.D. Luce, *A note on Boolean matrix theory*, Proc. Amer. Math. Soc. 3 (1952), pp. 382–388.

[18] B.R. McDonald, *Linear Algebra Over Commutative Rings*, Marcel Dekker, New York, 1984.

[19] P.L. Poplin and R.E. Hartwig, *Determinantal identities over commutative semirings*, Linear Algebra Appl. 387 (2004), pp. 99–132.

[20] O.A. Pshenitsyna, *Mappings that preserve the invertibility of matrices over semirings*, Russ. Math. Surveys 64 (2009), pp. 162–164.

[21] C. Reutenauer and H. Straubing, *Inversion of matrices over a commutative semiring*, J. Algebra 88 (1984), pp. 350–360.

[22] D.E. Rutherford, *The Cayley–Hamiltion theorem for semiring*, Proc. Roy. Soc. Edinburgh. Sec. A 66 (1964), pp. 211–215.

[23] L.A. Skornyakov, *Invertible matrices over distributive structures*, Sibirsk. Mat. Zh. 27 (1986), pp. 182–185 (in Russian).

[24] S. Sombatboriboon, W. Mora, and Y. Kemprasit, *Some results concerning invertible matrices over semirings*, Sci. Asia 37 (2011), pp. 130–135.

[25] Y.J. Tan, *On invertible matrices over antirings*, Linear Algebra Appl. 423 (2007), pp. 428–444.

[26] E.M. Vechtomov, *Two General Structure Theorems on Submodules, Abelian Groups and Modules*, Tomsk State University, Tomsk, 2000 (in Russian).
[27] C.K. Zhao, *Inverses of L-fuzzy matrices*, Fuzzy Sets Syst. 34 (1990), pp. 103–116.
[28] U. Zimmermann, *Linear and Combinatorial Optimization in Ordered Algebraic Structures*, Annals of Discrete Mathematics, Vol. 10, North Holland, Amsterdam, 1981.