§2. **Equations Involving Differentials: Pfaffian Equations**

In Example 1.5 of the last section, we have derived from a functional relation between the variables x and y (namely $(x^2/a^2) + (y^2/b^2) = 1$) a relation between their differentials dx and dy (namely $(x/a^2)dx + (y/b^2)dy = 0$). In many practical problems, we are given relations between forms dx, dy, du etc., called **Pfaffian equations,** and we are asked to find functions which relate the variables x, y, u etc. The theory of Pfaffian equations is a rather difficult subject, as indicated below in Example 2.4, which gives us an unpleasant surprise. Let us begin with an easy example for encouragement.

Example 2.1. Solve the Pfaffian equation $xdy - ydx = 0$.

Solution: Recall the quotient rule $d\left(\dfrac{y}{x}\right) = \dfrac{xdy - ydx}{x^2} \equiv x^{-2}(xdy - ydx)$. This suggests us to multiply the equation $xdy - ydx = 0$ by x^{-2}, called an *integrating factor* of the equation. The equation becomes $x^{-2}(xdy - ydx) = 0$, where the left-hand side is an "exact differential", namely, $d(y/x)$. In view of the following new "rule of thumb", we see that $y/x =$constant gives a family of solutions to $xdy - ydx = 0$.

Rule DF4 (Constancy). If $du = 0$, then u remains constant.

In this example, another integrating factor is y^{-2}. Using this integrating factor, we get another family of solutions $x/y =$ constant, which is almost the same as the previous one, except that it has an extra solution $x = 0$ (y arbitrary), but it does not include $y = 0$. It is not hard to see that the general solution to $xdy - ydx = 0$ should be $ax - by = 0$, where a and b are constants, not simultaneously zero.

An explanation about Rule DF4 is in order. In the last section, we have derived that $dc = 0$ if c is a constant. This new rule says that the converse is also true. If u depends on a single variable x, according to Rule DF3 in the previous section, $du = 0$ means $\frac{du}{dx}dx = 0$ and hence $\frac{du}{dx}$ should be identically zero. It follows from a well-known theorem in calculus that u must be constant. However, as indicated in this example, Rule DF4 can be applied to situations where u is a function of more than one variable.

The general form of Pfaffian equations in two variables x and y is $Pdx + Qdy = 0$, where $P = P(x, y)$ and $Q = Q(x, y)$ are functions of x and y. Let us simply write this equation as $\omega = 0$, where $\omega = Pdx + Qdy$. If we can find functions $f = f(x, y)$ and $g = g(x, y)$ such that $\omega = gdf$, then $\omega = 0$ can be reduced to $df = 0$ with solutions $f(x, y) = c$ (c is any constant). The general solution $f(x, y) = c$ represents a family of curves with c as a parameter. In the above example, we have $xdy - ydx = x^2\, d(y/x)$ and hence $y/x = c$ is the general solution to $xdy - ydx = 0$, representing the family of straight lines through the origin.

The **Hodge star** of the differential form $\omega = Pdx + Qdy$ is defined by putting

$$*\omega = *(Pdx + Qdy) = Pdy - Qdx. \tag{2.1}$$

A systematic way to think of $*\omega$ is in two stages: first, $*(Pdx + Qdy) = P * dx + Q * dy$ and second, $*dx = dy$, $*dy = -dx$. It turns out that if $f = c$ and $g = c$ are solutions to Pfaffian equations $\omega = 0$ and $*\omega = 0$ respectively, then curves from the family $f = c$ are perpendicular to those from $g = c$. Thus, given a family of curves $f = c$, we can find an orthogonal family of curves by solving the Pfaffian equations $*df = 0$.

Example 2.2. Find the orthogonal family of straight lines $y = cx$ through O.

Solution. Rewrite $y = cx$ as $f = c$ with $f = y/x$. Then $*df = *x^{-2}(xdy - ydx) = x^{-2}(x(-dx) - ydy) = -\frac{1}{2}x^{-2}d(x^2 + y^2)$. So the solution to $*df = 0$ is $x^2 + y^2 = c$, the family of circles centered at the origin. Geometrically it is clear that this family of circles is orthogonal to rays from the origin.

The Pfaffian equation $xdy - ydx = 0$ in Example 2.1 can be rewritten as an ordinary differential equation $dy/dx = y/x$, which can be solved by the method of "separation of variables". In general, given a Pfaffian equation in two variables $P(x, y)dx + Q(x, y)dy = 0$, we can rewrite it as a first order O.D.E. (ordinary differential equation) $dy/dx = -P(x, y)/Q(x, y)$. Conversely, a first order O.D.E. $dy/dx = F(x, y)$ can be converted to a Pfaffian equation $\omega = 0$ with $\omega = F(x, y)dx - dy$. Thus, for the case of two variables, Pfaffian equations are equivalent to first order O.D.E.s and hence they are in principle solvable. What about Pfaffian equations with three or more variables? This is a tough question. I challenge you to solve a simple equation, say $xdy - ydx + dz = 0$.

The general form of Pfaffian equations of n variables $x_1, x_2, \ldots, x_n$ is given by

$$\omega \equiv A_1 dx_1 + A_2 dx_2 + \cdots + A_n dx_n = 0, \tag{2.2}$$

where $A_1, A_2, \ldots, A_n$ are functions of $x_1, x_2, \ldots, x_n$. What do we mean by a solution to this equation? Well, it means a nontrivial functional relation among $x_1, x_2, \cdots, x_n$, say $f(x_1, x_2, \ldots, x_n) = \text{constant}$, so that $df = 0$ (obtained by differentiating both sides of this functional relation) is essentially the same as (2.2) above, that is, when we write df in the form $B_1dx_1 + B_2dx_2 + \cdots + B_ndx_n$, it is a nonzero multiple (which is a function called integrating factor) of the left-hand side of (2.2). Let us take a closer look of the identity

$$df = B_1dx_1 + B_2dx_2 + \cdots + B_ndx_n. \tag{2.3.}$$

Here B_kdx_k ($1 \leq k \leq n$) is the contribution of the change in x_k to the change of f (namely df) so that the sum $\sum_{k=1}^n B_kdx_k$ gives the total contribution to df from all variables. Now suppose only one variable changes, say x_1, and the rest remain constant, that is, $x_2 =$ constant c_2, $x_3 =$ constant c_3 etc., what will happen? Well, we have $dx_2 = 0$, $dx_3 = 0$ etc. and consequently (2.3) becomes $df = B_1dx_1$. According to Rule DF3 of the previous section, B_1 is just $\frac{d}{dx_1}f(x_1, c_2, c_3, \ldots, c_n)$, the derivative of the function f with respect to the variable x_1, keeping the other variables x_2, x_3 etc. fixed. As we know well, it is called the **partial derivative** of f with respect to x_1 and is denoted by

$$\frac{\partial f}{\partial x_1} \quad \text{or simply by} \quad f_{x_1}.$$

We conclude $B_1 = \partial f/\partial x_1$. Similarly, we have $B_2 = \partial f/\partial x_2$, etc. Thus we obtain the following generalization of Rule DF3 in the previous section:

Rule DF5 (Chain Rule). $df = \dfrac{\partial f}{\partial x_1}dx_1 + \dfrac{\partial f}{\partial x_2}dx_2 + \cdots + \dfrac{\partial f}{\partial x_n}dx_n,$

where f is a function of $x_1, x_2, \ldots, x_n$. In the above argument we have tacitly assumed:

Rule DF6 (Identification). *If we have*

$$A_1dx_1 + A_2dx_2 + \cdots + A_ndx_n = B_1dx_1 + B_2dx_2 + \cdots + B_ndx_n,$$

where $A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n$ are functions of variables $x_1, x_2, \ldots, x_n$, which can vary independently, then $A_1 = B_1, A_2 = B_2, \ldots, A_n = B_n$.

The following examples illustrate Rule DF5 and Rule DF6:

Example 2.3. In Example 1.1, from $r = \sqrt{x^2 + y^2 + z^2}$ we deduced

$$dr = \frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz.$$

Comparing this to $dr = r_xdx + r_ydy + r_zdz$, by Rule DF6, we get

$$r_x = \frac{x}{r}, \quad r_y = \frac{y}{r} \quad \text{and} \quad r_z = \frac{z}{r}.$$

Certainly they can be obtained by direct computation. (Reminder: $r_x = \partial r/\partial x$.)

Example 2.4. In Example 1.3, we derived the relation between the differentials dr, $d\theta$ for polar coordinates and the differentials dx, dy for rectangular coordinates:

$$dx = \cos\theta\ dr - r\sin\theta\ d\theta, \qquad dy = \sin\theta\ dr + r\cos\theta\ d\theta.$$

It is easy to solve for dr, $d\theta$ in terms of dx and dy. For example, multiplying the first identity by $\cos\theta$ and the second by $\sin\theta$, and then adding the resulting identities, we obtain the first one of the following identities:

$$dr = \cos\theta\ dx + \sin\theta\ dy, \qquad d\theta = -\frac{\sin\theta}{r}dx + \frac{\cos\theta}{r}dy.$$

It follows from Rule DF5 and Rule DF6 that

$$\frac{\partial r}{\partial x} = \cos\theta, \qquad \frac{\partial r}{\partial y} = \sin\theta, \qquad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}, \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}.$$

Replacing $\cos\theta$ by x/r and $\sin\theta$ by y/r, we have

$$r_x = \frac{x}{r}, \quad r_y = \frac{y}{r}, \quad \theta_x = -\frac{y}{r^2}, \quad \theta_y = \frac{x}{r^2}.$$

It is possible but quite clumsy to derive these identities by direct computation.

The following two "interchange rules" for taking partial derivatives are crucial. Here we assume that F is a function of several variables, among which u, v are two of them.

Rule PD1. $\dfrac{\partial}{\partial u}\dfrac{\partial}{\partial v}F = \dfrac{\partial}{\partial v}\dfrac{\partial}{\partial u}F$, or simply $F_{uv} = F_{vu}$.

Rule PD2. $\dfrac{\partial}{\partial u}\displaystyle\int_a^b F(u,v)\ dv = \int_a^b \frac{\partial}{\partial u}F(u,v)\ dv.$

Here is some sort of phony philosophy to look at these two identities: taking partial derivatives and/or integrals with respect to different variables are considered to be independent actions upon functions. The order of these actions does not affect the final result because they are independent of each other. To make a very crude analogue of this, consider kicking and punching as independent actions. No matter which comes first, the result is the same: a black eye and a bruised knee. Let us dub the exchangeability of order for independent actions as "kicking and punching principle"; (despite of the nickname here, mathematical operations are nonviolent.)

Example 2.5. Now let us try to solve the "challenging" Pfaffian equation

$$xdy - ydx + dz = 0. \tag{2.4}$$

Let $g(x, y, z) = 0$ be a solution. Rule DF5 tells us $g_x dx + g_y dy + g_z dz = 0$, which is essentially the same as (2.4). This means that there is some nonzero function F such that $g_x dx + g_y dy + g_z dz = F(xdy - ydx + dz) \equiv Fxdy - Fydx + Fdz$, i.e.

$$g_x = -Fy, \qquad g_y = Fx, \qquad g_z = F.$$

It follows that

$$F_x = (g_z)_x = g_{zx} = g_{xz} \text{ (by Rule PD1)} = (-Fy)_z = -F_z y \quad \text{ and}$$
$$F_y = g_{zy} = g_{yz} = (Fx)_z = F_z x.$$

Therefore $xF_x + yF_y = -xF_z y + yF_z x = 0$.

On the other hand, from $g_x = -Fy$ we have $g_{xy} = -(Fy)_y = F_y y - F$ and from $g_y = Fx$ we have $g_{yx} = (Fx)_x = F_x x + F$. By Rule PD1, $g_{xy} = g_{yx}$, that is, $-F_y y - F = F_x x + F$, or $2F = xF_x + yF_y$, which is zero by the previous paragraph. But F is not allowed to be zero! What goes wrong? Well, this contradictory conclusion tells us that solution to $xdy - ydx + dz = 0$ *does not exist*. That is, there is no genuine functional relation among the variables x, y and z. Or, in technical terms, $xdy - ydx + dz = 0$ is *not integrable*. My previous challenge to solve this equation is a trap!

Example 2.6. Solve the Pfaffian system $xdy - ydx + dz = 0$ and $ydx + dz = 0$.

You would probably say: this system has no solutions because the first equation is the same one as the previous example, which is known for nonexistence of solutions. This conclusion is mistakenly drawn from your past experience with systems of linear equations, by ignoring the fact that the meanings of solutions to these two types of systems are completely different. This and the previous examples indicate some difficulty working with Pfaffian systems. Let us try to solve this system. Cancelling dz, we have $xdy - 2ydx = 0$, which can be solved by separation of variables to obtain $y = cx^2$ (c is a constant). Rewrite $ydx + dz = 0$ as $dz = -ydx$ and substituting $y = cx^2$ in it, we have $dz = -cx^2 dx$ and hence $z = \int -cx^2 dx = -(c/3)x^3 + b$, where b is another constant. Letting $a = c/3$, we can put the general solution to the given Pfaffian system as $y = 3ax^2$, $z = -ax^3 + b$, which represents a family of curves depending on two parameters a and b.

Given functions u and v of two variables x and y, we have

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy, \qquad dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy.$$

Using matrix notation, this can be rewritten as:

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}.$$

The 2×2 matrix in the above identity is called the Jacobian matrix for the transformation φ which sends (x, y) to (u, v) (or, interpreted in another way, the change of coordinates from (x, y) to (u, v).) The determinant of this matrix is simply called **Jacobian**, and is denoted by J_φ or $\partial(u, v)/\partial(x, y)$.

Example 2.7. Find the Jacobian $\partial(x, y)/\partial(r, \theta)$, where x, y are the Cartesian coordinates and r, θ are the polar coordinates of the Euclidean plane.

Solution: We know $dx = \cos\theta dr - r\sin\theta d\theta$ and $dy = \sin\theta dr + r\cos\theta d\theta$. So

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

(This simple answer is useful for computing double integrals by using polar coordinates.)

Example 2.8. Find the Jacobian $\partial(u, v)/\partial(x, y)$ for the planar transform given by $u = x^2 - y^2$, $v = 2xy$. (Here u and v are the real and the imaginary parts of the complex function $f(x + iy) = (x + iy)^2$.)

Solution: We have $du = 2xdx - 2ydy$ and $dv = 2ydx + 2xdy$. So

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2 + y^2).$$

Notice that this transform is a two-to-one (except at the origin) map: if (x, y) is mapped to (u, v), then so is $(-x, -y)$. (This corresponds to a fact about the solutions to the algebraic equation $z^2 = a$ for a given complex number a.)

Using the summation sign, we put Rule DF5 as $df = \sum_{k=1}^n \frac{\partial f}{\partial x_k} dx_k$. Suppose that we are given m functions $u_1, u_2, \ldots, u_m$ of n variables $v_1, v_2, \ldots, v_n$. Then

$$du_j = \sum\nolimits_{k=1}^n \frac{\partial u_j}{\partial v_k} dv_k, \qquad j = 1, 2 \ldots, m. \tag{2.5}$$

(We may rewrite the relations (2.5) between differentials $du_1, \ldots, du_m$ and $dv_1, \ldots, dv_n$ in matrix form as $d\mathbf{u} = J(\mathbf{u}, \mathbf{v})d\mathbf{v}$, where $J(\mathbf{u}, \mathbf{v})$ is the Jacobian matrix.) Now assume further that $v_1, v_2, \ldots, v_n$ are functions of variables $w_1, w_2, \ldots, w_p$. Then

$$dv_k = \sum\nolimits_{\ell=1}^p \frac{\partial v_k}{\partial w_\ell} dw_\ell; \qquad k = 1, \ldots, n. \tag{2.6}$$

Substituting (2.5) into (2.6), we have

$$
\begin{aligned}
du_j &= \sum_{k=1}^{n} \frac{\partial u_j}{\partial v_k} \sum_{\ell=1}^{p} \frac{\partial v_k}{\partial w_\ell}\, dw_\ell = \sum_{k=1}^{n} \sum_{\ell=1}^{p} \frac{\partial u_j}{\partial v_k} \frac{\partial v_k}{\partial w_\ell}\, dw_\ell \\
&= \sum_{\ell=1}^{p} \sum_{k=1}^{n} \frac{\partial u_j}{\partial v_k} \frac{\partial v_k}{\partial w_\ell}\, dw_\ell = \sum_{\ell=1}^{p} \left(\sum_{k=1}^{n} \frac{\partial u_j}{\partial v_k} \frac{\partial v_k}{\partial w_\ell} \right) dw_\ell.
\end{aligned}
\tag{2.7}
$$

Comparing (2.7) with $du_j = \sum_{\ell=1}^{p} \frac{\partial u_j}{\partial w_\ell}\, dw_\ell$ (from Rule DF5 again) and applying Rule DF6, we obtain the following

Rule PD3 (Chain rule for partial derivatives). $\dfrac{\partial u_j}{\partial w_\ell} = \sum_{k=1}^{n} \dfrac{\partial u_j}{\partial v_k} \dfrac{\partial v_k}{\partial w_\ell}.$

For example, if f is a function of u, v, w, and u, v, w are functions of r, s, then

$$
\frac{\partial f}{\partial r} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial r} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial r}, \qquad \frac{\partial f}{\partial s} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial s}.
$$

The chain rule (Rule PD3) is very handy for handling functions with variables at odd places, by introducing new variables, as shown in the following example.

Example 2.9. Let $J \equiv J(x) = \int_{f(x)}^{g(x)} F(x, h(x), p)\, dp$. Derive a formula for computing the derivative dJ/dx.

Solution: Let us put $r = f(x)$, $s = g(x)$, $t = x$ and $u = h(x)$ so that we can write $J = \int_s^r F(t, u, p)\, dp$. Notice that $\partial J/\partial r = F(t, u, r)$, in view of the fundamental theorem of calculus, as well as $\partial J/\partial s = -F(t, u, s)$, (why?). Also,

$$
\frac{\partial J}{\partial t} = \frac{\partial}{\partial t} \int_s^r F(t, u, p)\, dp = \int_s^r \frac{\partial}{\partial t} F(t, u, p)\, dp,
$$

using the "punching and kicking principle" to exchange the order of taking derivative and taking integral, which is possible because the relevant variables t and p are independent of each other. Similarly we have $\partial J/\partial u = \int_s^r \partial F/\partial u\, du$. Thus

$$
\begin{aligned}
\frac{dJ}{dx} &= \frac{\partial J}{\partial r}\frac{dr}{dx} + \frac{\partial J}{\partial s}\frac{ds}{dx} + \frac{\partial J}{\partial t}\frac{dt}{dx} + \frac{\partial J}{\partial u}\frac{du}{dx} \\
&= F(t, u, r)\frac{dr}{dx} - F(t, u, s)\frac{ds}{dx} + \frac{dt}{dx} \int_s^r \frac{\partial F}{\partial t}\, dp + \frac{du}{dx} \int_s^r \frac{\partial F}{\partial u}\, dp \\
&= F(x, h(x), f(x)) f'(x) - F(x, h(x), g(x)) g'(x) \\
&\qquad + \int_{g(x)}^{f(x)} \left(\frac{\partial F}{\partial t}(x, h(x), p) + h'(x) \frac{\partial F}{\partial u}(x, h(x), p) \right) dp.
\end{aligned}
$$

(Notice that from $t = x$ we have $dt/dx = 1$.)

To give another illustration, we show how to derive the product rule from the chain rule. Consider the product of two functions, say $y = uv$ with $u = u(x)$ and $v = v(x)$. Then

$$\frac{dy}{dx} = \frac{\partial y}{\partial u}\frac{du}{dx} + \frac{\partial y}{\partial v}\frac{dv}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

Done. As you can see, in applying the chain rule it is of utmost importance to keep track of variables and handle them with great care. Differential forms, on the other hand, are often allowed to be casual about their variables in manipulation. This is one of many great advantages of differentials over derivatives.

Another typical application of the chain rule is to derive **Euler's equation for homogeneous functions**. A function $f(x_1, x_2, \ldots, x_n)$ defined everywhere except at the origin is said to be homogeneous of degree r if the identity

$$f(\lambda u_1, \lambda u_2, \ldots, u_n) = \lambda^r f(u_1, u_2, \ldots, u_n) \tag{2.8}$$

holds for all $\lambda > 0$ and all $(u_1, u_2, \ldots, u_n) \neq (0, 0, \ldots, 0)$. For example, the function $f(x, y) = (2x - 3y)/(x^2 + xy + y^2)$ is homogeneous of degree -1. The following so called Euler's equation is obtained by differentiating both sides of (2.8) above with respect to λ and then evaluating at $\lambda = 1$, keeping $u_1, u_2, \ldots, u_n$ fixed throughout the process. Differentiating the RHS of (2.8) is easy:

$$\frac{d}{d\lambda}\,\lambda^r f(u_1, u_2, \ldots, u_n) = r\lambda^{r-1} f(u_1, u_2, \ldots, u_n).$$

For differentiating the LHS, we introduce variables

$$x_1 = \lambda u_1,\ x_2 = \lambda u_2,\ \ldots,\ x_n = \lambda u_n.$$

Let y be the LHS of (4.4) so that $y = f(x_1, x_2, \ldots, x_n) = f(\lambda u_1, \lambda u_2, \ldots, \lambda u_n)$. Then

$$\frac{dy}{d\lambda} = \frac{\partial y}{\partial x_1}\frac{dx_1}{d\lambda} + \frac{\partial y}{\partial x_2}\frac{dx_2}{d\lambda} + \cdots + \frac{\partial y}{\partial x_n}\frac{dx_n}{d\lambda} = \frac{\partial y}{\partial x_1}u_1 + \frac{\partial y}{\partial x_1}u_2 + \cdots + \frac{\partial y}{\partial x_n}u_n,$$

where the partial derivatives $\partial y/\partial x_j$ $(1 \leq j \leq n)$ are evaluated at $(\lambda u_1, \lambda u_2, \ldots, \lambda u_n)$. Equating the derivatives of the both sides of (2.8) with respect to λ and then putting $\lambda = 1$, we have $(\partial y/\partial x_1)u_1 + (\partial y/\partial x_2)u_2 + \cdots + (\partial y/\partial x_n)u_n = f(u_1, u_2, \ldots u_n)$, where the partial derivatives $\partial y/\partial x_j$ are evaluated at $(u_1, u_2, \ldots, u_n)$. Thus we have

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \cdots + x_n\frac{\partial f}{\partial x_n} = rf. \tag{2.9}$$

because two sides evaluated at $(u_1, u_2, \ldots, u_n)$ are equal and $(u_1, u_2, \ldots, u_n)$ is arbitrary. To confirm (2.9), let us check a special case given in the following example.

Example 2.10. Verify Euler's equation for $f(x, y) = Ax^2 + 2Bxy + Cy^2$.

Solution. We have $f_x = 2Ax + 2By$ and $f_y = 2Bx + 2Cy$. Hence

$$xf_x + yf_y = x(2Ax + 2By) + y(2Bx + 2Cy) = 2(Ax^2 + 2Bxy + Cy^2),$$

which is just $2f(x, y)$.

Exercises

1. Solve the following Pfaffian equations: (a) $x^3dx + 2ydy = 0$, (b) $xdy + 2ydx = 0$, (c) $ydx + dy = 0$, (d) $\cos x \cos y \; dx - \sin x \sin y \; dy = 0$.
2. Solve the following Pfaffian equations:
   (a) $e^x dx + ydy - z^2dz = 0$, (b) $xdy + ydx + zdz = 0$, (c) $xdy - ydx + x^2zdz = 0$.
3. Solve the Pfaffian system: $xdy + ydx + dz = 0$, $xdy - ydx + 2zdz = 0$.
4. Show that the Pfaffian equation $xdy + ydx + ydz = 0$ is not solvable.
5. Verify that, for polar coordinates r and θ, we have $*dr = rd\theta$ and $*rd\theta = -dr$. (Hint: express dr and $rd\theta$ in the form $fdx + gdy$.)
6. In each of the following cases, find an family of curves orthogonal to the given one: (a) $xy = c$, (b) $y = -\frac{1}{2}x^2 + c$, (c) $x^3 - 3xy^2 = c$, (d) $y = \sin x + c$, (e) $e^x \cos y = c$, (f) the family of parallel lines $2x - 3y + c$, (g) the family of circles through the origin and tangent to the y-axis, namely $(x - c)^2 + y^2 = c^2$.
7. In each of the following parts, find the Jacobian $\partial(u, v)/\partial(x, y)$ of the given mapping which sends (x, y) to (u, v) as described: (a) $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$; (b) $u = e^x \cos y$, $v = e^x \sin y$; (c) $u = \log \sqrt{x^2 + y^2}$, $v = \arctan(y/x)$, $(x, y > 0)$.
8. Let (r, θ, ϕ) be the spherical coordinates and let (x, y, z) be the rectangular coordinates of $\mathbf{R}^3$. Find the Jacobian $\partial(x, y, z)/\partial(r, \theta, \phi)$.
9. Suppose that u is a function of r and s with $\partial u/\partial r = 2r$ and $\partial u/\partial s = 2s$, and suppose $r = x^2 - y^2$ and $s = 2xy$. Find an expression for du in variables x and y. Use the chain rule (Rule PD3) to compute the partial derivatives $\partial u/\partial x$ and $\partial u/\partial y$. Check that $(\partial u/\partial x)dx + (\partial u/\partial y)dy$ agrees with du.

10. Assume that variables P, V, T are functionally related, say $f(P, V, T) = 0$. Assume that each variable can be explicitly "solved" from this functional relation in terms of two other variables, which are allowed to vary freely. For example, we can solve for P to obtain an expression of the form $P = g(V, T)$, where V and T are chosen as free variables. Any function of P, V, T can be expressed as a function of any pair of free variables of your choice, e.g. $F(P, V, T) = F(g(V, T), V, T)$ is expressed as a function of a pair of free variables V and T. The symbol $(\partial F/\partial V)_T$ means the partial derivative of F with respect to V, keeping T fixed, when F is regarded as a function of free variables V and T. The expressions $(\partial F/\partial T)_V$, $(\partial F/\partial P)_V$, $(\partial F/\partial V)_P$, $(\partial F/\partial T)_P$ and $(\partial F/\partial P)_T$ have the similar meanings. Verify the following identities which are important in equilibrium thermodynamics: (a) $(\partial F/\partial T)_V = (\partial F/\partial P)_T(\partial P/\partial T)_V + (\partial F/\partial T)_P$, (b) $(\partial P/\partial V)_T(\partial V/\partial T)_P(\partial T/\partial P)_V = -1$, (c) $(\partial P/\partial T)_V = \partial(P, V)/\partial(T, V)$. (The correct interpretation of the partial derivatives and the Jacobian here is essential for these rather mysterious identities.)

11. Let $L = L(q, \dot{q})$ be a function of $2n$ variables $(q, \dot{q}) \equiv (q^1, \dots, q^n, \dot{q}^1, \dots, \dot{q}^n)$, (called Lagrangian). Introduce new variables $(q, p) = (q^1, \dots, q^n, p_1, \dots, p_n)$ which are related to $(q, \dot{q})$ by $q^j = q^j$ and $p_j = \partial L/\partial \dot{q}^j$ $(1 \leq j \leq n)$. (These identities tell us how to change the variables $(q, \dot{q})$ into (q, p). Here we tacitly assume that this change is reversible. Thus $(q, \dot{q})$ can be considered as functions of (q, p), if necessary.) Let $H = H(q, p)$ be $\sum_{j=1}^n p_j \dot{q}^j - L(q, \dot{q})$ written in variables (q, p), (called Hamiltonian). Verify the identities $\partial H/\partial q^j = -\partial L/\partial q^j$ and $\partial H/\partial p_j = \dot{q}^j$. Hence derive that if a trajectory $(q(t), \dot{q}(t))$ with $\dot{q}(t) = dq/dt$ satisfies the **Euler-Lagrange equations**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^j} - \frac{\partial L}{\partial q^j} = 0, \quad j = 1, \dots, n,$$

then, using (q, p) as variables, this trajectory satisfies **Hamilton's equations**

$$\frac{dq^j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q^j}, \quad n = 1, \dots, n.$$

(Hint: compute dH.)

12. Suppose that u is a function of r, s, which are functions of x, y. Verify the identity

$$u_{xx} = u_{rr} r_x^2 + 2u_{rs} r_x s_x + u_{ss} s_x^2 + u_r r_{xx} + u_s s_{xx}.$$

Use this identity to deduce that the Laplacian $\Delta u \equiv u_{xx} + u_{yy}$ of u in polar coordinates is given by

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}.$$

(The explicit expressions for r_{xx}, r_{xy}, r_{yy}, θ_{xx}, θ_{xy} and θ_{yy} used in this exercise can be read off from the answer to Exercise 7 in §1.)