Project3

December 7, 2017

```
In [1]: import numpy as np
        from math import sin, atan, tan, cos, log
        import matplotlib.pyplot as plt
```

0.1 Helper methods

```
In [2]: # First, a wrapper method that takes a function for finding yn+1
        # given yn and other pertinent data and gets the t and y vectors
        def get_method_results( method_f, yprime, a, b, y0, n ):
            h = (b-a)/n
            # Use the generator method below
            method = method_gen( method_f, yprime, a, y0, h )
            # And pull the ts and ys from it
            t0, y0 = next(method)
            ts = [t0]
            ys = [y0]
            while len(ts) <= n:
                tn, yn = next(method)
                ts.append(tn)
                ys.append(yn)
            return ts, ys

        # This is a method that takes a method for calculating a new y and returns a generator
        def method_gen( method, y_prime, t0, y0, h ):
            current_y = y0
            current_t = t0
            while True:
                yield (current_t, current_y)
                current_y = method(y_prime, current_y, current_t, h)
                current_t = current_t + h
```

0.2 Problem 1

```
In [3]: # Then, defining a method is as simple as creating a method with the right signature
        def forward_euler_method(y_prime, current_y, current_t, h):
            return current_y + h * y_prime( current_t, current_y )
```

```
# Wrap it with a convenience funtion, and we're done
def forward_euler(y_prime, a, b, y0, n = 100):
    return get_method_results( forward_euler_method, y_prime, a, b, y0, n )
```

0.3 Problem 2

```
In [4]: def RK4_method(y_prime, current_y, current_t, h):
            k1 = y_prime( current_t, current_y )
            k2 = y_prime( current_t + h/2, current_y + h*k1/2 )
            k3 = y_prime( current_t + h/2, current_y + h*k2/2 )
            k4 = y_prime( current_t + h, current_y + h*k3)
            return current_y + h / 6 * (k1 + 2*k2 + 2*k3 + k4)

        # Wrap it with a convenience funtion, like above
        def RK4(y_prime, a, b, y0, n = 100):
            return get_method_results( RK4_method, y_prime, a, b, y0, n )
```

0.4 Problem 3

```
In [5]: # For Backwards Euler, we need newtons method. Let's copy the code from project 2 here.
        def newtons_guesses( f, f_prime, x0 ):
            current_x = x0;
            while True:
                yield current_x
                next_x = current_x - (f(current_x) / f_prime(current_x))
                current_x = next_x

        def newtons_method( f, f_prime, x0, tol=1.0e-5, ittr_max=100 ):
            guesses = newtons_guesses( f, f_prime, x0 )
            ittr_count = 1
            current_guess = next(guesses) # it yields x0 first
            next_guess = next(guesses)
            while (np.abs(next_guess - current_guess) > tol
                   and ittr_count < ittr_max):
                current_guess = next_guess
                next_guess = next(guesses)
                ittr_count += 1
            return next_guess
```

```
In [6]: # This is a bit of a hack, but it alows us to use the above code by creating
        # the BE method at creation time, and passing in the y_prime_dy function. We
        # can then wrap it in a lambda to perserve the scope when this is evaluated.
        # Now if only python supported multi-line lambdas, this wouldn't look like
        # such a mess *rages*
        def build_BE_method( y_prime_dy ):
            return lambda y_prime, current_y, current_t, h : newtons_method(
                get_f( current_y, h, y_prime, current_t + h),
                get_f_prime( h, y_prime_dy, current_t + h ),
```

```
            current_y,
            tol = 10e-7,
            ittr_max = 100
        )

# This generates the non-linear function to solve in the BE method
def get_f(current_y, h, y_prime, next_t):
    return lambda x: current_y - x + h * y_prime(next_t, x)

# This is the derivative of that non-linear function, needed for Newtons method
def get_f_prime(h, y_prime_dy, next_t):
    return lambda x: h * y_prime_dy(next_t, x) - 1

# Wrap it with a convenience funtion, and we're done
def backward_euler(y_prime, y_prime_dy, a, b, y0, n = 100):
    return get_method_results( build_BE_method(y_prime_dy), y_prime, a, b, y0, n )
```

0.5 Problem 4

In [7]:
```
# First, we define a bunch of methods that we're going to need to plug in later:
def y_prime(t, y):
    return sin(y)/(1+t)

def y_true(t):
    cot_half = 1/tan(.5)
    return 2*atan((t + 1)/cot_half)

#This one is for backwards Euler only
def y_prime_dy(t, y):
    return cos(y)/(1+t)

#Short hand method for finding error
def max_error( ts, ys ):
    return max([ abs(y_true(t)-y) for t, y in zip(ts, ys)])

# as a little trick, we can define our methods as maps from names to functions
# that are already prepopulated with most of the data. This makes evaluation
# of every function easier to loop later.
def get_methods(a, b):
    return {
        "FE ": (lambda ittr_count:
                  forward_euler( y_prime, a, b, 1, ittr_count)),
        "RK4": (lambda ittr_count:
                  RK4( y_prime, a, b, 1, ittr_count)),
        "BE ": (lambda ittr_count:
                  backward_euler( y_prime, y_prime_dy, a, b, 1, ittr_count))
    }
```

0.5.1 4 a, b, and c

In [8]:
```
methods = get_methods(0,20)

for ittr_count in [500, 1000, 2000]:
    for m in methods:
        ts, ys = methods[m](ittr_count)
        print( "Error at {} ittrs for method {} is {}".
              format(ittr_count, m, max_error(ts, ys)))
```

```
Error at 500 ittrs for method FE  is 0.010659661546484678
Error at 500 ittrs for method RK4 is 5.279037207372994e-10
Error at 500 ittrs for method BE  is 0.010593098515223165
Error at 1000 ittrs for method FE  is 0.005321602711385598
Error at 1000 ittrs for method RK4 is 3.455902231053187e-11
Error at 1000 ittrs for method BE  is 0.005305015170636196
Error at 2000 ittrs for method FE  is 0.0026587485869593586
Error at 2000 ittrs for method RK4 is 2.2086776851892864e-12
Error at 2000 ittrs for method BE  is 0.0026545956748784683
```

0.5.2 4 d

In [9]:
```
methods = get_methods(0,20)

for m in methods:
    ts1, ys1 = methods[m](1000)
    e1000 = max_error( ts1, ys1 )
    ts2, ys2 = methods[m](2000)
    e2000 = max_error( ts2, ys2 )
    cvgnc = log(e1000/ e2000) / log( (20/1000) / (20/2000) )
    print( "Rate of convergence for method {} is {}".
            format( m, cvgnc))
```

```
Rate of convergence for method FE  is 1.001113447359553
Rate of convergence for method RK4 is 3.9678075985598333
Rate of convergence for method BE  is 0.9988647378950698
```

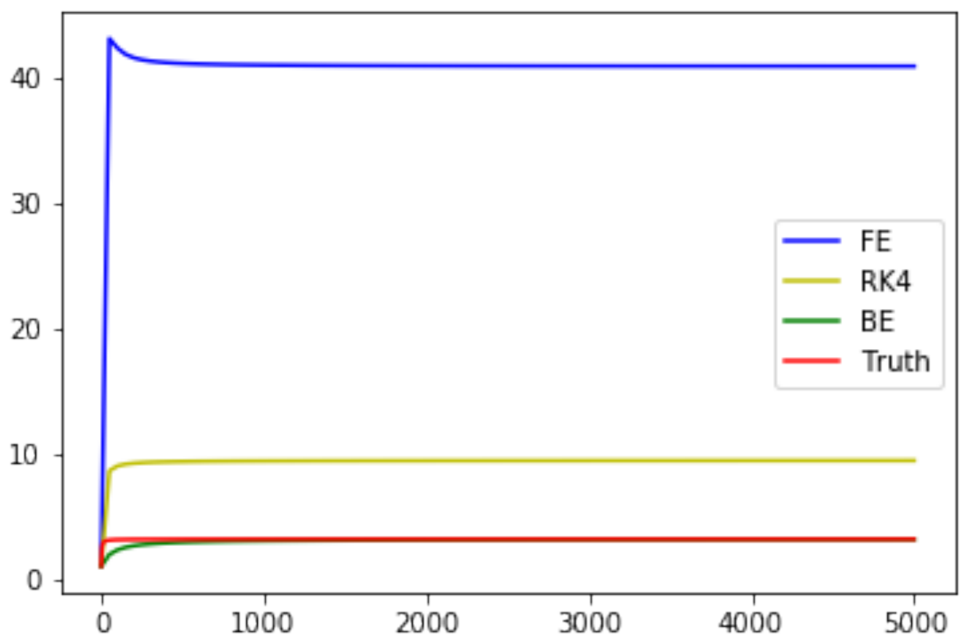
0.5.3 4 e

Yes. These rates of convergence confirm our theoretical analysis. For both FE and BE, the rate is very close to 1, as expected, while for RK4, it is almost exactly 4.

0.6 Problem 5

In [10]:
```
long_term_methods = get_methods(0, 5000)

colors = {
```

```
        "FE ": 'b-',
        "RK4": 'y-',
        "BE ": 'g-'
}


for m in methods:
    ts, ys = long_term_methods[m](100)
    plt.plot(ts, ys, colors[m], label = m)

ts = range(0, 5000)
ys = [y_true(t) for t in ts]
plt.plot(ts, ys, 'r-', label = "Truth")
plt.legend()

plt.show()
```



In this case, BE is the best approximation. This is because BE is an A-stable method, unlike RK4 or FE. For those methods, the relativly large h means that we no longer fall within the zone of stability for the function.