opg 4

$x = \rho \cdot \sin\phi \cdot \cos\theta$
$y = \rho \cdot \sin\phi \cdot \sin\theta$
$z = \rho \cdot \cos\phi$

$$\iiint x^2+y^2+z^2 \, dV$$

$z = c \cdot \sqrt{x^2+y^2}$ and $x^2+y^2+z^2 = a^2$

convert to spherical coordinates.

$\rho^2 = x^2+y^2+z^2 \quad \wedge \quad r^2 = x^2+y^2$

and then the integral.

$$\iiint \rho^2 \cdot \rho^2 \cdot \sin(\varphi) \, d\rho \, d\theta \, d\varphi$$

We then find the limits.

$x^2+y^2+z^2 = \rho^2 = a^2$

$\rho = a$

therefore $0 \leq \rho \leq a$

the limit for θ i $0 \leq \theta \leq 2\pi$ since we are integrating the volume between a sphere and a cone around t z-axis.

$z = c\sqrt{x^2+y^2} = c \cdot \sqrt{(\rho \cdot \sin\varphi \cdot \cos\theta)^2 + (\rho \cdot \sin\varphi \cdot \sin\theta)^2}$

$z = \rho \cdot \cos(\varphi)$

We put them next to each other

$\rho \cdot \cos(\varphi) = c \cdot \sqrt{\rho^2 \cdot \sin^2\varphi \cdot \cos^2\theta + \rho^2 \cdot \sin^2\varphi \cdot \sin\theta^2}$

$\rho \cdot \cos(\varphi) = c \cdot \sqrt{\rho^2 \sin^2\varphi \underbrace{(\cos^2\theta + \sin^2\theta)}_{=1}}$

$\rho \cdot \cos(\varphi) = c \cdot \rho \cdot \sin(\varphi) \cdot 1$

$\dfrac{\rho \cdot \cos(\varphi)}{c} = \rho \cdot \sin(\varphi)$

$\dfrac{1}{c} = \dfrac{\sin(\varphi)}{\cos(\varphi)}$

$\dfrac{1}{c} = \tan(\varphi)$

$\varphi = \tan^{-1}\left(\dfrac{1}{c}\right)$

therefore the limit is $0 \leq \varphi \leq \tan^{-1}\left(\frac{1}{c}\right)$

We can now solve the integral.

$$= \int_0^{2\pi} \int_0^{\tan^{-1}(\frac{1}{c})} \int_0^a \rho^4 \sin(\varphi) \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\tan^{-1}(\frac{1}{c})} \left[\sin(\varphi) \cdot \frac{1}{5} \cdot (\rho^5) \right]_0^a d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\tan^{-1}(\frac{1}{c})} \sin(\varphi) \cdot \frac{a^5}{5} \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \left[\frac{a^5}{5} \cdot (-\cos\varphi) \right]_0^{\tan^{-1}(\frac{1}{c})} d\theta$$

$$= \frac{a^5}{5} \int_0^{2\pi} -\cos\left(\tan^{-1}\left(\frac{1}{c}\right)\right) - \cos(0) \, d\theta$$

$$= \frac{a^5}{5} \int_0^{2\pi} -\cos\left(\tan^{-1}\left(\frac{1}{c}\right)\right) - 1 \, d\theta$$

$$= -\frac{a^5}{5} \cdot \left(-\cos\left(\tan^{-1}\left(\frac{1}{c}\right)\right) + 1\right) \cdot \left[\theta\right]_0^{2\pi}$$

$$= -\frac{a^5}{5} \cdot \left(-\cos\left(\tan^{-1}\left(\frac{1}{c}\right)\right) + 1\right) \cdot 2\pi$$

$$= \frac{2\pi a^5}{5} \cdot \left(-\cos\left(\tan^{-1}\left(\frac{1}{c}\right)\right) + 1\right)$$