Inference for MVN

meanvector, μ ← unknown

- hypothesis testing
- confidence regions/interval

Model $X_1 \dots X_n$ iid (vector)

$X_i \sim N_p(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ (unknown)

modelcheck → Scatter matrix
should be → mahalonobis QQ plot

Hypothesis

$H_0: \underset{(p\times 1)}{\mu} \overset{?}{=} \underset{(p\times 1)}{\mu_\alpha}$ ← residual vector

used to check if our postulate is true. e.g. check if the robot work correctly over many test.

$H_1: \mu \overset{!}{=} \mu_0$

2-Sided

Start by estimating the unknown variables

$$\underset{p\times 1}{\hat{\mu}} = \bar{X} = \frac{1}{n}\sum_{j=1}^{n} X_j \quad \text{matlab mean}$$

$$\underset{p\times p}{\hat{\Sigma}} = S \begin{bmatrix} S_1^2 & S_{12} & S_{13} & S_{1p} \\ & S_2^2 & & \\ -11- & & \ddots & \\ & & & S_p^2 \end{bmatrix} = \sum (X_j - \bar{X})(X_j - \bar{X})^T$$

matlab cov

Test statistic

1D: $t = \dfrac{\bar{x} - \mu_0}{\frac{S}{\sqrt{n}}} \sim t_{n-1}$

$$t_2 = (\bar{x} - \mu_0) \frac{1}{S^2/n} (\bar{x} - \mu_0) \sim F_{1,n-1}$$

p number of dimensions

pD: Hotelling T^2

$$T^2 = \underbrace{(\bar{x} - \mu_0)^T}_{1 \times p} \underbrace{\left(\frac{S}{n}\right)^{-1}}_{p \times p} \cdot \underbrace{(\bar{x} - \mu_0)}_{p \times 1} \sim$$

$$\sim \frac{p(n-1)}{n-p} F_{p,n-p}$$

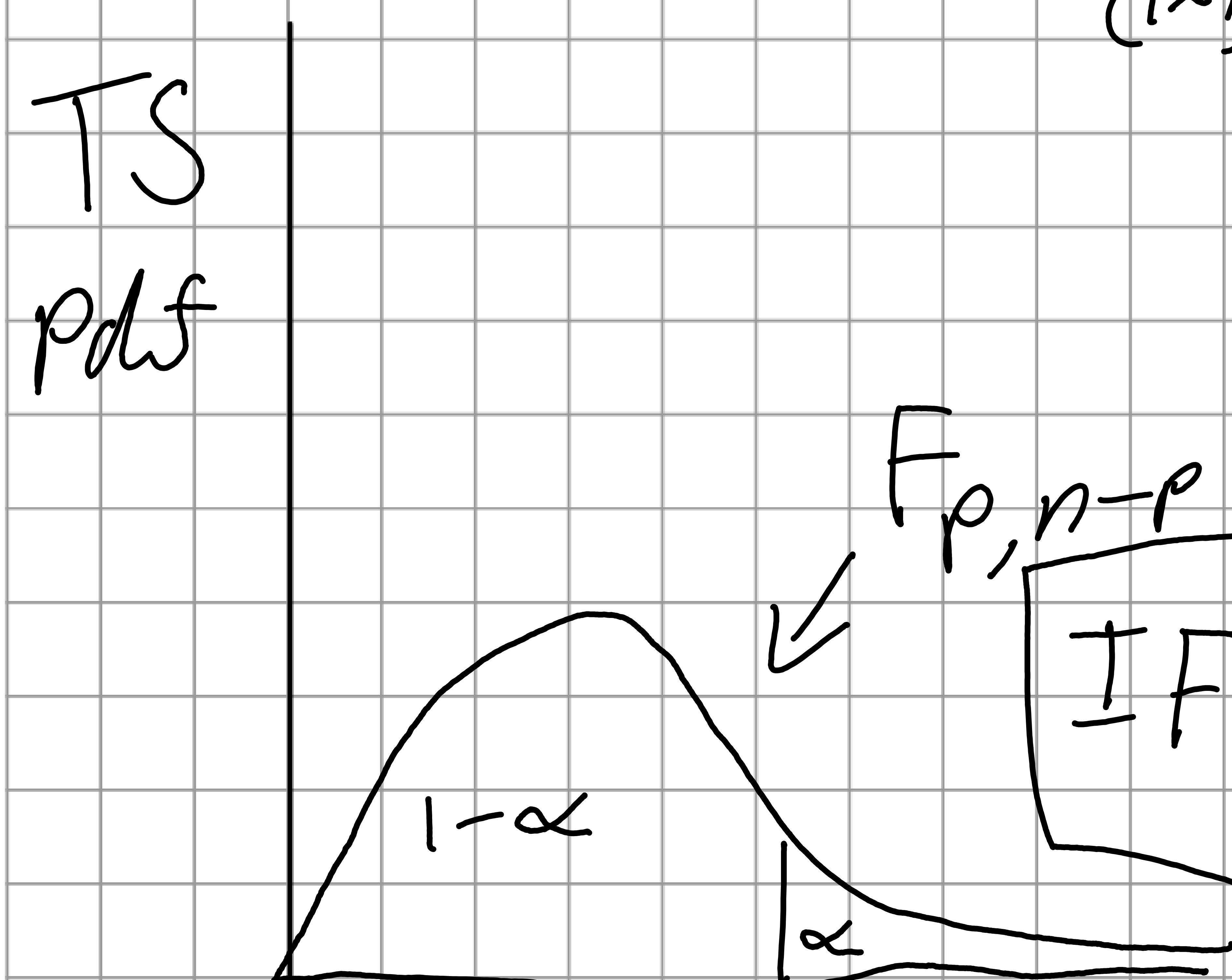
T^2 = big value → reject H_0

$T^2 \approx 0$ → accept H_0

Where H is our hypothesis

Test $T_0^2 \xrightarrow{\text{data}} t_0^2$

(1×1) value (1×1)



$F_{p,n-p}$ pdf

IFF $t^2 > \frac{p(n-1)}{n-p} F_{p,n-p,\alpha}$ reject H_0

either choos α: 0,01 / 0,05 / 0,10

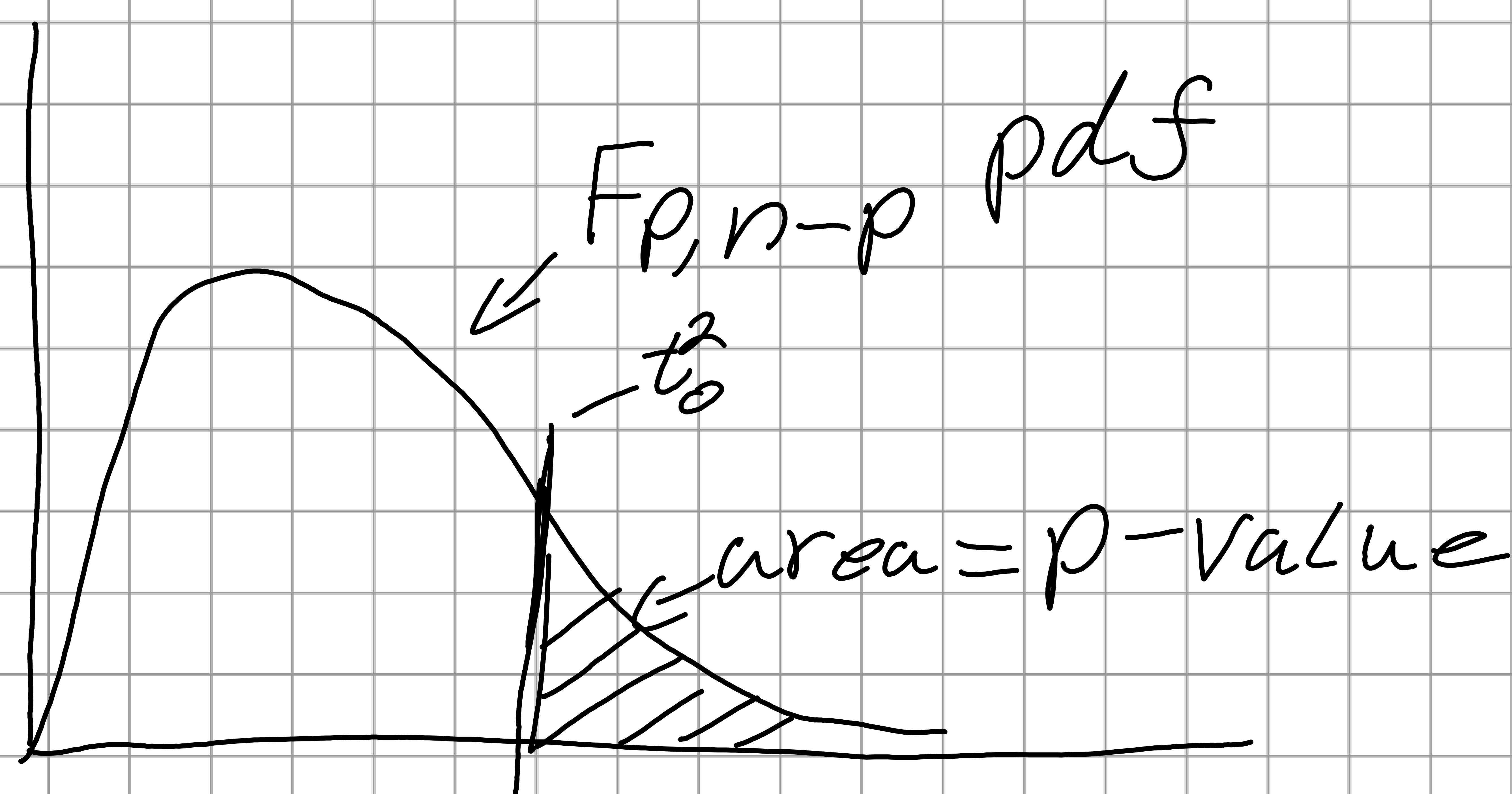
$F_{p,n-p}$ - quantile

matlab finv(1-α, p, n-p)

Lower α means Lower acceptance.

$\alpha = 0,1 \rightarrow$ im 90% confident in my data.

<u>or</u> calculate p-value



$$p\text{-value} = P\left(F_{p,n-p} > \frac{n-p}{p(n-1)} t^2\right)$$

$$= 1 - P\left(F_{p,n-p} < \frac{n-p}{p(n-1)} t^2\right)$$

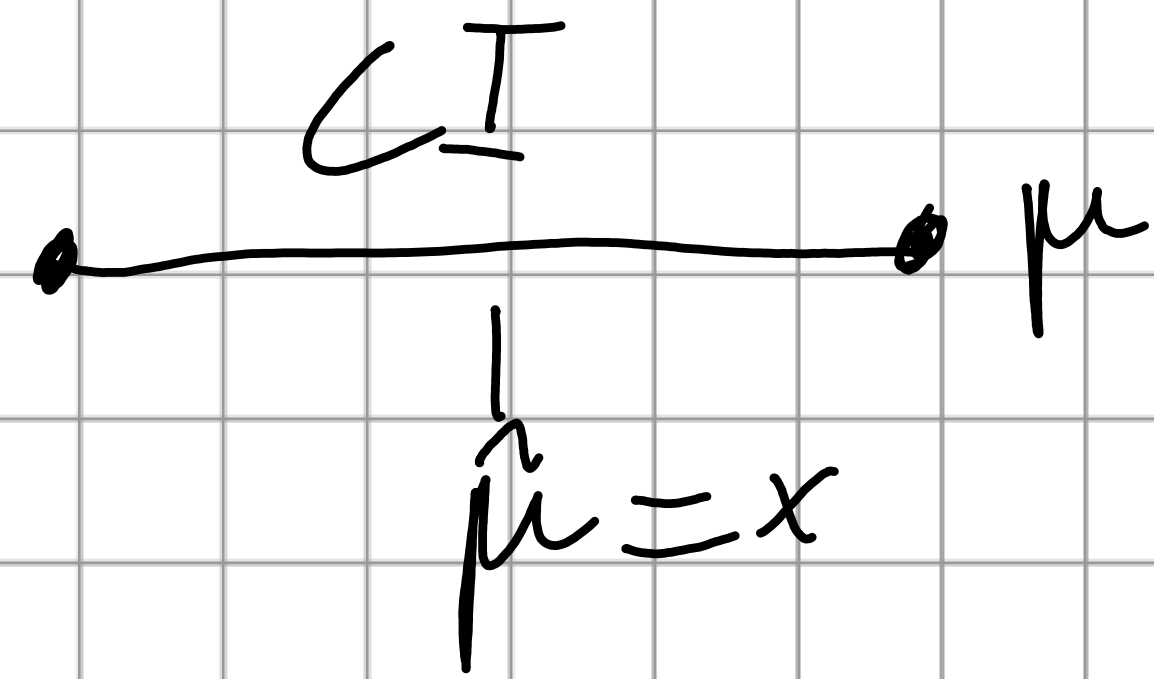
fcdf(, p, n-p)

p-value = "low" → reject H_0

$\alpha = 0,05 \rightarrow p < \alpha \rightarrow$ reject H_0

$\alpha = 0,01 \rightarrow p > \alpha \rightarrow$ accept H_0

1D Confidence interval for μ



pD confidence region in $\mathbb{R}^p$

$100(1-\alpha)\%$ CR:

$$\boxed{\forall \mu \in \mathbb{R}^p : (\bar{x}-\mu)^T \left(\frac{S}{n}\right)^{-1} (\bar{x}-\mu) \le \frac{p(n-1)}{n-p} F_{p,n-p,\alpha}}$$

quadratic form

$\hookrightarrow$ pD hyperellipsoid

CI (confidenc intervals for μ variables, μ_i, $i=1 \ldots, p$)

Marginal confidence. one variable at a time.

$e I's$ $\bar{X}_i \pm t_{n-1, \alpha/2} \sqrt{\frac{S_i^2}{n}}$

for μ_i

$i=1$

$1 \ldots p$

Naive!

we get too low confidence overall. I 1D the $\hat{\mu}$ belongs to its own confidence intval. I pD we have to do it simultaneous.

$1D \to P(\mu \in CI_\mu) = 1-\alpha$ $\quad pD \to (1-\alpha)^p$

Bonferroni CI for μ_i

$\alpha \ll 1$: $(1-\alpha)^p \cong 1-\alpha p$

choose $\alpha_B \stackrel{def}{=} \frac{\alpha}{p}$

$(1-\alpha_B)^p \cong 1-\alpha_B \cdot p = 1-\frac{\alpha}{p}\cdot p = 1\,\alpha$

$$\left[\bar{X}_i \pm t_{n-1,\alpha/2p}\sqrt{\frac{S_i^2}{n}}\right] \quad i=1,\dots,p$$

Simultaneus CI's for μ_i

$$\left[\bar{x}_i \pm \sqrt{\frac{p(n-1)}{n-p} F_{p,n-p,\alpha}} \sqrt{\frac{S_i^2}{n}}\right]_{i=1\ldots p}$$

too wide a square

Large sample ($n \gg p$) approximate

CR for μ (based on CLT
central limit theory)

$X \sim N_p(\mu, \Sigma) \Rightarrow d^2(x,\mu) = (x-\mu)^T \Sigma^{-1} (x-\mu) \sim$

$\sim \chi^2_p$

distributed as

$\bar{X} \sim N_p\left(\mu, \frac{\Sigma}{n}\right) \Rightarrow (\bar{x}-\mu)^T \left(\frac{\Sigma}{n}\right)^{-1} (x-\mu) \sim \chi^2_p$

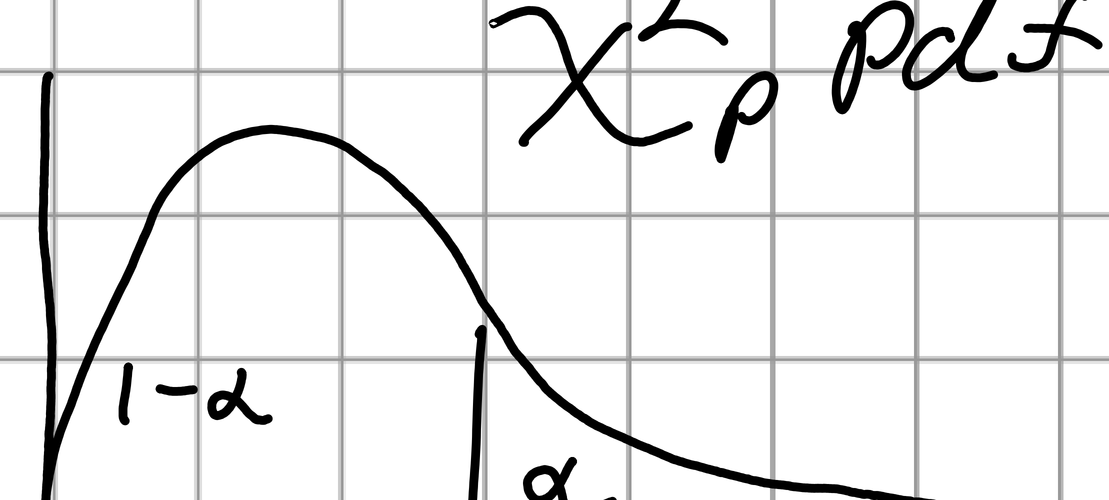
$\Downarrow$

$n \gg p$, due CLT $(\bar{x}-\mu)^T \left(\frac{S}{n}\right)^{-1} (\bar{x}-\mu) \overset{approx}{\sim} \chi^2_p$

Approximate large sample

$100(1-\alpha)\%$ CR for μ

χ^2_p pdf



$\chi^2_{p,\alpha}$

chi2inv(1-α, p)

The likelyhood Ratio test principle. applied to inference for MVN μ

Given $X_1 \dots X_n$ iid $X_j \sim N_p(\mu, \Sigma)$

$H_0: \mu \stackrel{?}{=} \mu_0$ Full model (unconstrained)

$H_1: \mu \neq \mu_0$

$L(\hat{\mu}, \hat{\Sigma}) = P(\text{given obs} \mid \mu, \Sigma)$

F-dist

χ^2

normal

student^T