

# Root locus Technique

In the root locus diagram, we can observe the path of the closed loop poles. Hence, we can identify the nature of the control system. In this technique, we will use an open loop transfer function to know the stability of the closed loop control system.

## Basics of Root Locus

The Root locus is the locus of the roots of the characteristic equation by varying system gain  $K$  from zero to infinity.

We know that, the characteristic equation of the closed loop control system is

$$1 + G(s)H(s) = 0$$

We can represent  $G(s)H(s)$  as

$$G(s)H(s) = K \frac{N(s)}{D(s)}$$

Where,

- $K$  represents the multiplying factor
- $N(s)$  represents the numerator term having (factored)  $n^{\text{th}}$  order polynomial of 's'.
- $D(s)$  represents the denominator term having (factored)  $m^{\text{th}}$  order polynomial of 's'.

Substitute,  $G(s)H(s)$  value in the characteristic equation.

$$1 + k \frac{N(s)}{D(s)} = 0$$

$$\Rightarrow D(s) + KN(s) = 0$$

**Case 1 –  $K = 0$** 

If  $K = 0$ , then  $D(s) = 0$ .

That means, the closed loop poles are equal to open loop poles when  $K$  is zero.

**Case 2 –  $K = \infty$** 

Re-write the above characteristic equation as

$$K \left( \frac{1}{K} + \frac{N(s)}{D(s)} \right) = 0 \Rightarrow \frac{1}{K} + \frac{N(s)}{D(s)} = 0$$

Substitute,  $K = \infty$  in the above equation.

$$\frac{1}{\infty} + \frac{N(s)}{D(s)} = 0 \Rightarrow \frac{N(s)}{D(s)} = 0 \Rightarrow N(s) = 0$$

If  $K = \infty$ , then  $N(s) = 0$ . It means the closed loop poles are equal to the open loop zeros when  $K$  is infinity.

From above two cases, we can conclude that the root locus branches start at open loop poles and end at open loop zeros.

**Angle Condition and Magnitude Condition**

The points on the root locus branches satisfy the angle condition. So, the angle condition is used to know whether the point exist on root locus branch or not. We can find the value of  $K$  for the points on the root locus branches by using magnitude condition. So, we can use the magnitude condition for the points, and this satisfies the angle condition.

Characteristic equation of closed loop control system is

$$1 + G(s)H(s) = 0 \\ \Rightarrow G(s)H(s) = -1 + j0$$

The **phase angle** of  $G(s)H(s)$  is

$$\angle G(s)H(s) = \tan^{-1} \left( \frac{0}{-1} \right) = (2n + 1)\pi$$

The **angle condition** is the point at which the angle of the open loop transfer function is an odd multiple of  $180^\circ$ .

Magnitude of  $G(s)H(s)$  is –

$$|G(s)H(s)| = \sqrt{(-1)^2 + 0^2} = 1$$

The magnitude condition is that the point (which satisfied the angle condition) at which the magnitude of the open loop transfer function is one.

The **root locus** is a graphical representation in s-domain and it is symmetrical about the real axis. Because the open loop poles and zeros exist in the s-domain having the values either as real or as complex conjugate pairs. In this chapter, let us discuss how to construct (draw) the root locus.

#### Rules for Construction of Root Locus

Follow these rules for constructing a root locus.

**Rule 1** – Locate the open loop poles and zeros in the 's' plane.

**Rule 2** – Find the number of root locus branches.

We know that the root locus branches start at the open loop poles and end at open loop zeros. So, the number of root locus branches **N** is equal to the number of finite open loop poles **P** or the number of finite open loop zeros **Z**, whichever is greater.

Mathematically, we can write the number of root locus branches **N** as

$N=P$  if  $P \geq Z$

$N=Z$  if  $P < Z$

**Rule 3** – Identify and draw the **real axis root locus branches**.

If the angle of the open loop transfer function at a point is an odd multiple of  $180^\circ$ , then that point is on the root locus. If odd number of the open loop poles and zeros exist to the left side of a point on the real axis, then that point is on the root locus branch. Therefore, the branch of points which satisfies this condition is the real axis of the root locus branch.

**Rule 4** – Find the centroid and the angle of asymptotes.

- If  $P=Z$ , then all the root locus branches start at finite open loop poles and end at finite open loop zeros.
- If  $P > Z$ , then  $Z$  number of root locus branches start at finite open loop poles and end at finite open loop zeros and  $P-Z$  number of root locus branches start at finite open loop poles and end at infinite open loop zeros.
- If  $P < Z$ , then  $P$  number of root locus branches start at finite open loop poles and end at finite open loop zeros and  $Z-P$  number of root locus branches start at infinite open loop poles and end at finite open loop zeros.

So, some of the root locus branches approach infinity, when  $P \neq Z$ . Asymptotes give the direction of these root locus branches. The intersection point of asymptotes on the real axis is known as **centroid**.

We can calculate the **centroid**  $\alpha$  by using this formula,

$$\alpha = \frac{\sum \text{Real part of finite open loop poles} - \sum \text{Real part of finite open loop zeros}}{P - Z}$$

The formula for the angle of **asymptotes**  $\theta$  is

$$\theta = \frac{(2q + 1)180^\circ}{P - Z}$$

Where,

$$q = 0, 1, 2, \dots, (P - Z) - 1$$

**Rule 5** – Find the intersection points of root locus branches with an imaginary axis.

We can calculate the point at which the root locus branch intersects the imaginary axis and the value of **K** at that point by using the Routh array method and special **case (ii)**.

- If all elements of any row of the Routh array are zero, then the root locus branch intersects the imaginary axis and vice-versa.
- Identify the row in such a way that if we make the first element as zero, then the elements of the entire row are zero. Find the value of **K** for this combination.
- Substitute this **K** value in the auxiliary equation. You will get the intersection point of the root locus branch with an imaginary axis.

**Rule 6** – Find Break-away and Break-in points.

- If there exists a real axis root locus branch between two open loop poles, then there will be a **break-away point** in between these two open loop poles.
- If there exists a real axis root locus branch between two open loop zeros, then there will be a **break-in point** in between these two open loop zeros.

**Note** – Break-away and break-in points exist only on the real axis root locus branches. Follow these steps to find break-away and break-in points.

- Write K in terms of s from the characteristic equation  $1+G(s)H(s)=0$ .
- Differentiate K with respect to s and make it equal to zero. Substitute these values of ss in the above equation.
- The values of ss for which the K value is positive are the **break points**.

**Rule 7** – Find the angle of departure and the angle of arrival.

The Angle of departure and the angle of arrival can be calculated at complex conjugate open loop poles and complex conjugate open loop zeros respectively.

The formula for the **angle of departure**  $\phi_d$  is

$$\phi_d = 180^\circ - \phi$$

The formula for the **angle of arrival**  $\phi_a$  is

$$\phi_a = 180^\circ + \phi$$

Where,

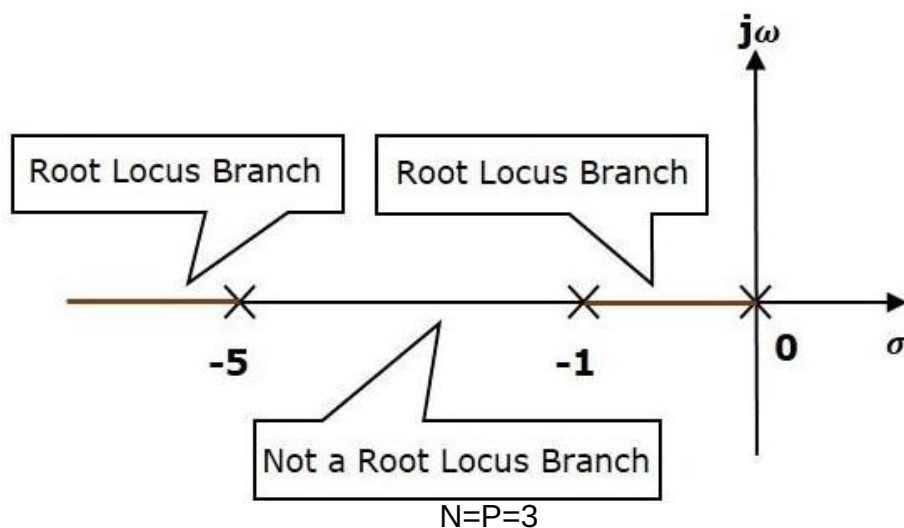
$$\phi = \sum \phi_P - \sum \phi_Z$$

Example

Let us now draw the root locus of the control system having open loop transfer

Function,  $G(s)H(s) = \frac{K}{s(s+1)(s+5)}$

**Step 1** – The given open loop transfer function has three poles at  $s = 0$ ,  $s = -1$ ,  $s = -5$ . It doesn't have any zero. Therefore, the number of root locus branches is equal to the number of poles of the open loop transfer function.



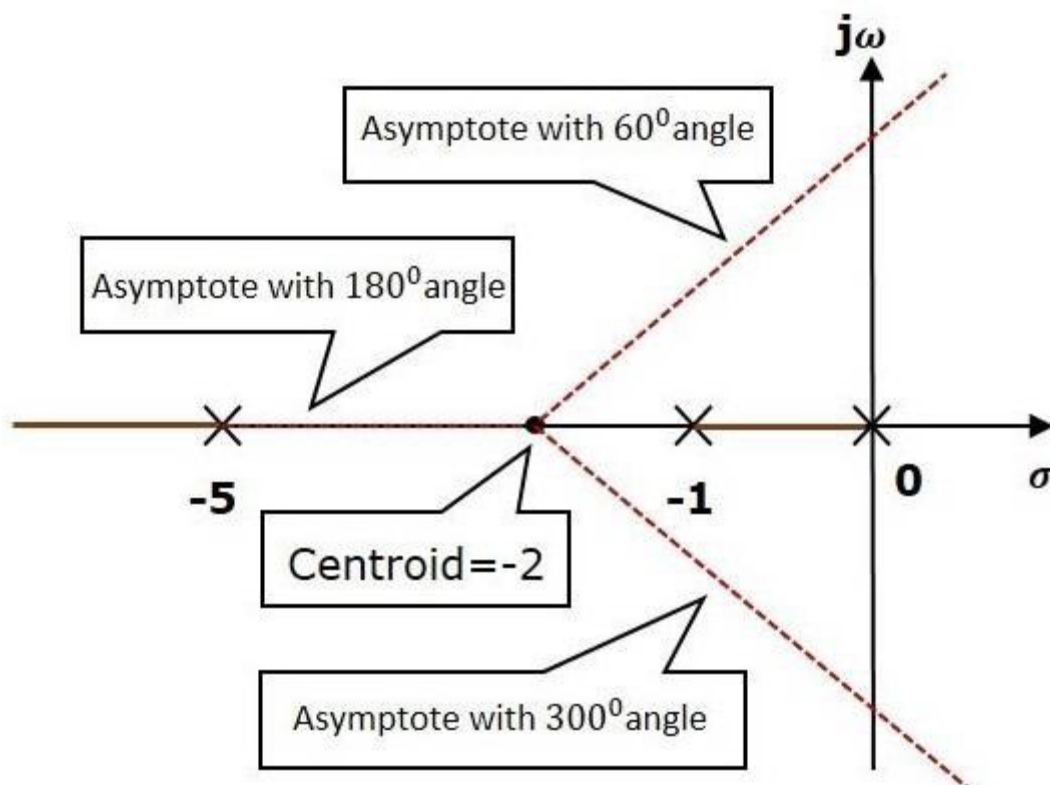
The three poles are located are shown in the above figure. The line segment between  $s=-1$ , and  $s=0$  is one branch of root locus on real axis. And the other branch of the root locus on the real axis is the line segment to the left of  $s=-5$ .

**Step 2** – We will get the values of the centroid and the angle of asymptotes by using the given formulae.

Centroid

The angle of asymptotes  $\theta = 60^\circ, 180^\circ$  and  $300^\circ$ .

The centroid and three asymptotes are shown in the following figure.



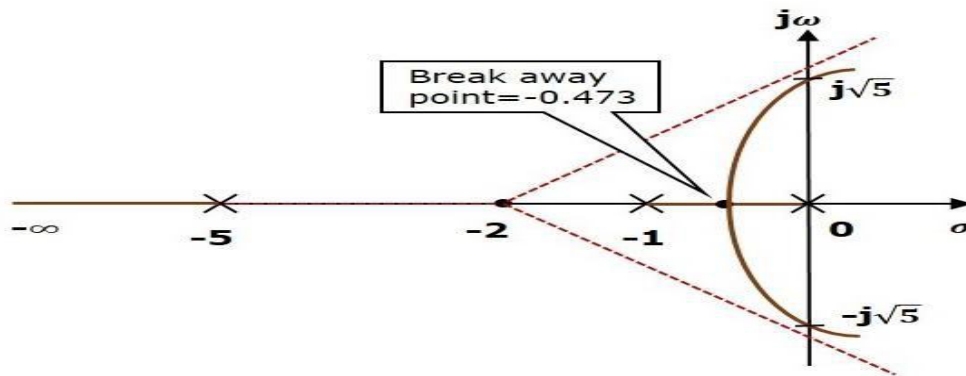
**Step 3** – Since two asymptotes have the angles of  $60^\circ$  and  $300^\circ$ , two root locus branches intersect the imaginary axis. By using the Routh array method and special case(ii),  $j\sqrt{5}$  and  $-j\sqrt{5}$ .

the root locus branches intersect the imaginary axis at

There will be one break-away point on the real axis root locus branch between the poles  $s$

$s=-1$  and  $s=0$ . By following the procedure given for the calculation of break-away point, we will get it as  $s = -0.473$ .

The root locus diagram for the given control system is shown in the following figure.



In this way, you can draw the root locus diagram of any control system and observe the movement of poles of the closed loop transfer function.

From the root locus diagrams, we can know the range of  $K$  values for different types of damping.

#### Effects of Adding Open Loop Poles and Zeros on Root Locus

The root locus can be shifted in '**s**' plane by adding the open loop poles and the open loop zeros.

- If we include a pole in the open loop transfer function, then some of root locus branches will move towards right half of '**s**' plane. Because of this, the damping ratio  $\delta$  decreases. Which implies, damped frequency  $\omega_d$  increases and the time domain specifications like delay time  $t_d$ , rise time  $t_r$  and peak time  $t_p$  decrease. But, it effects the system stability.
- If we include a zero in the open loop transfer function, then some of root locus branches will move towards left half of '**s**' plane. So, it will increase the control system stability. In this case, the damping ratio  $\delta$  increases. Which implies, damped frequency  $\omega_d$  decreases and the time domain specifications like delay time  $t_d$ , rise time  $t_r$  and peak time  $t_p$  increase.

So, based on the requirement, we can include (add) the open loop poles or zeros to the transfer function.