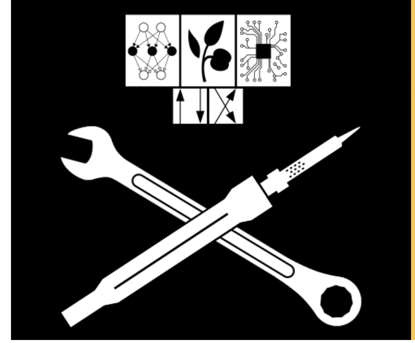

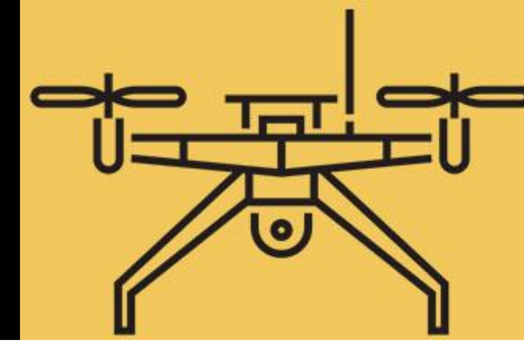

State estimation and observers

Lecture 7

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Today's lecture

- Observability concept and Kalman criterion
- Main idea behind observers
- Observers with feedback measurement
- The observer as a filter
- The Kalman Filter as an observer

What is observability?

We now assume that not the whole state is measured

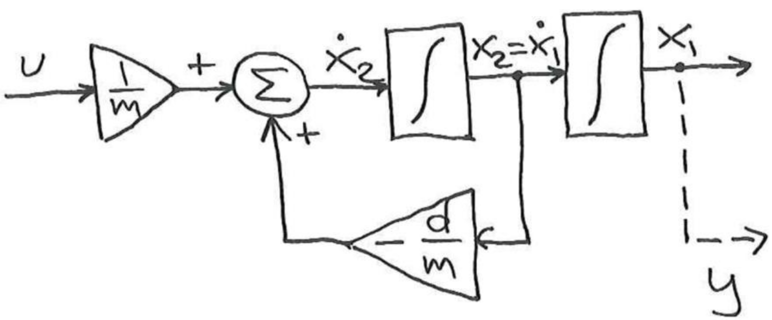
Use the measurements y(t) to deduce/give an estimate of x(t)

Is that always possible? Notion of observability

Rough intuition on a simplified example:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -\frac{d}{m}x_2(t) + \frac{1}{m}u(t) \end{cases}$$

GPS sensor only: $y(t) = x_1(t)$



All the state components have an influence on y(t) $\approx$ observable

Speedometer only: $y(t) = x_2(t)$

not observable

The Kalman criterion for observability (1/2)

Definition: The system $\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases}$

is said to be observable if the knowledge of y(t) on [0,T] is sufficient to determine the whole state x(t).

Criterion to work systematically on matrices A and C: Kalman criterion

Basic idea: in the SISO case, dim(y)=1 while dim(x)=n (difficult)

BUT we have y(t) on [0,T] $\Longrightarrow$ we can compute the derivatives of y(t)

$$\begin{cases} y(t) = \mathbf{C}\mathbf{x}(t) \\ \dot{y}(t) = \mathbf{C}\dot{\mathbf{x}}(t) = \mathbf{C}\mathbf{A}\mathbf{x}(t) \\ \ddot{y}(t) = \mathbf{C}\ddot{\mathbf{x}}(t) = \mathbf{C}\mathbf{A}^2\mathbf{x}(t) \\ \vdots \\ y^{(n-1)}(t) = \mathbf{C}\mathbf{x}^{(n-1)}(t) = \mathbf{C}\mathbf{A}^{n-1}\mathbf{x}(t) \end{cases} \Rightarrow \bar{\mathbf{y}}(t) = \mathbf{W}_o\mathbf{x}(t)$$

with $\mathbf{W}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}$

Observability matrix

and

$$\bar{\mathbf{y}}(t) = [y(t), \dot{y}(t), \ddot{y}(t), \cdots, y^{(n-1)}(t)]^T$$

The Kalman criterion for observability (2/2)

Then we can obtain an estimate of x(t): $$\hat{\mathbf{x}}(t) = \mathbf{W}_o^{-1}\bar{\mathbf{y}}(t)$$

which is possible only if W_O is invertible

Theorem: The system $\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases}$ is observable if and only if matrix W_O is invertible.

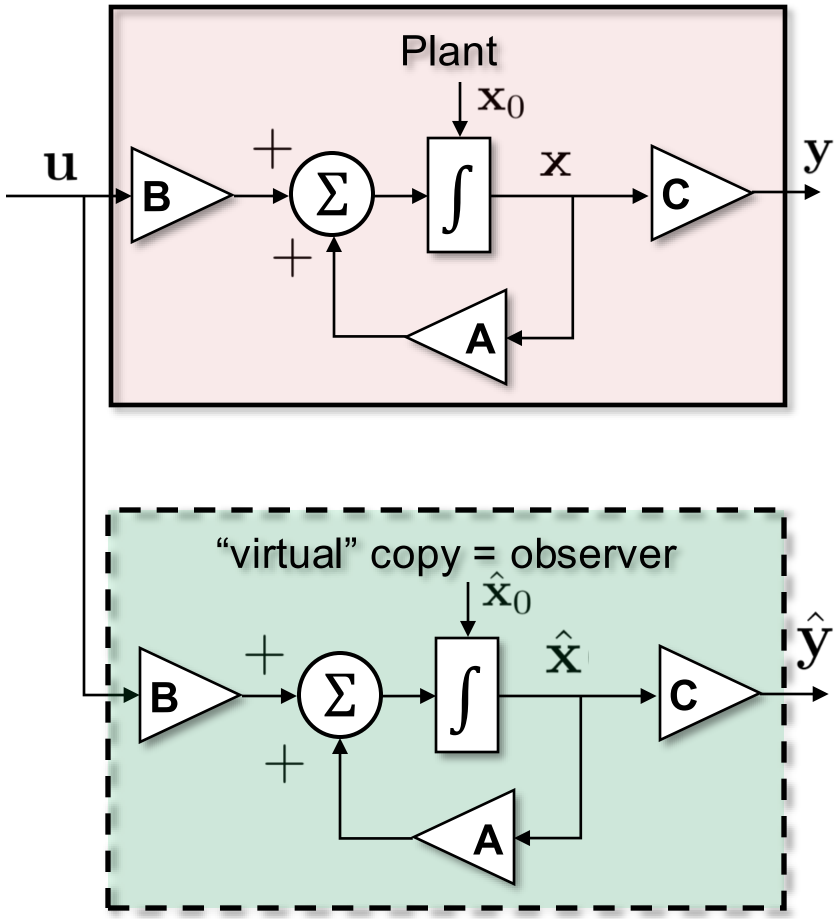
We could use the above expression to compute an estimate of x(t) BUT…
differentiating y(t) (corrupted by noise) amplifies the noise!

Discrete-time case: same result except that $$\bar{\mathbf{y}}(k) = [y(k), y(k+1), \cdots, y(k+n-1)]^T$$

(we could use the above expression for estimate but not much to tune/adjust)

Systems with inputs: exactly the same result (does not depend on inputs for linear systems)

The observer as a virtual plant (1/2)



<u>Basic idea behind an observer</u>: if we feed 2 identical systems with the same input, they will have the same behavior…

…in one of them (the virtual copy), we have access to <u>everything</u>, including the state!

The observer (the virtual copy) gives us an estimate of the real state x(t) of the plant!

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The observer as a virtual plant (2/2)

More mathematically:

Plant
$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), & \mathbf{x}(0) = \mathbf{x}_0 \\ y(t) = \mathbf{C}\mathbf{x}(t) \end{cases}$$

Observer
$$\begin{cases} \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t), & \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0 \\ \hat{y}(t) = \mathbf{C}\hat{\mathbf{x}}(t) \end{cases}$$

Important: generally, $\mathbf{x}_0 \neq \hat{\mathbf{x}}_0$.

If the observer is doing a good job, then there should be no difference between the state and its estimate... ➡ Define $\tilde{\mathbf{x}}(t) := \hat{\mathbf{x}}(t) - \mathbf{x}(t)$

Estimation error

Evolution of estimation error:

$$\begin{aligned} \dot{\tilde{\mathbf{x}}}(t) = \dot{\hat{\mathbf{x}}}(t) - \dot{\mathbf{x}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) - \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{u}(t) \\ &= \mathbf{A}(\hat{\mathbf{x}}(t) - \mathbf{x}(t)) \end{aligned}$$

Estimation error dynamics

$$\dot{\tilde{\mathbf{x}}}(t) = \mathbf{A}\tilde{\mathbf{x}}(t)$$

If this system is stable, then the estimation error will eventually go to 0...

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➡ The state estimate will converge to the true state of the plant!

Adding feedback on the measurements

Wait a sec: what if the plant we want to observe is not stable?

Use measurements y(t) to improve the estimate gradually

We have the observer equations
$$\begin{cases} \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(y(t) - \hat{y}(t)) \\ \hat{y}(t) = \mathbf{C}\hat{\mathbf{x}}(t) \end{cases}$$

Trivial but important fact: if $\hat{\mathbf{x}}(t) = \mathbf{x}(t)$ then $\hat{y}(t) = y(t)$

and therefore $\mathbf{L}(y - \hat{y}) = 0$

(ie the state estimate does not need any improvement)

Estimation error dynamics?

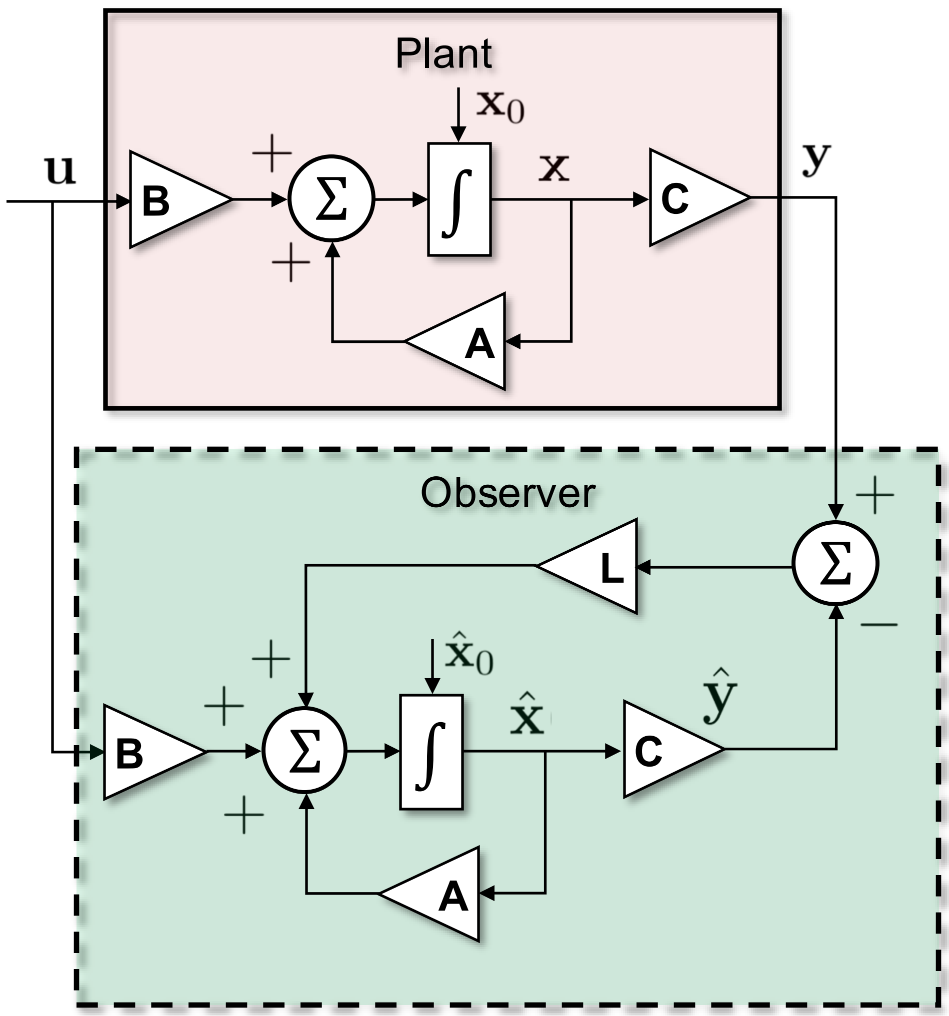
$$\begin{aligned} \dot{\tilde{\mathbf{x}}}(t) = \dot{\hat{\mathbf{x}}}(t) - \dot{\mathbf{x}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(y(t) - \hat{y}(t)) - \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{u}(t) \\ &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{L}\mathbf{C}(\mathbf{x}(t) - \hat{\mathbf{x}}(t)) - \mathbf{A}\mathbf{x}(t) \\ &= (\mathbf{A} - \mathbf{L}\mathbf{C})(\hat{\mathbf{x}}(t) - \mathbf{x}(t)) \end{aligned}$$

$$\dot{\tilde{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\tilde{\mathbf{x}}(t)$$

Tuning possibilities

Estimate converges to true state if this system is stable

Implementation (block diagram)



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The observer as a filter (1/2)

An observer can also have a filtering effect on noisy measurements.

Consider the example of a constant signal corrupted by noise:

$$\dot{x}(t) = 0, \quad x(0) = x_0 \qquad \text{with} \qquad y(t) = x + \cos \omega t$$

(high-frequency noise)

Define the observer
$$\begin{cases} \dot{\hat{x}}(t) = 0 + l(y(t) - \hat{y}(t)) \\ \hat{y}(t) = \hat{x}(t) \end{cases}$$

which can be rewritten as
$$\boxed{\dot{\hat{x}}(t) = -l\hat{x}(t) + ly(t)} \qquad (1)$$

(first-order LP filter)

asymptotic behavior?

$$\hat{x}(t) = e^{-lt}\hat{x}(0) + \int_0^t e^{-l(t-\tau)} l y(\tau) d\tau$$ is a solution of ODE (1)

assume (for simplicity) that $\hat{x}(0) = 0$ and $l = 1$

$$\Rightarrow \quad \hat{x}(t) = \int_0^t e^{-(t-\tau)} \left(x + \cos \omega \tau\right) d\tau$$

The observer as a filter (2/2)

Rewrite
$$\begin{aligned}\hat{x}(t) &= \int_0^t e^{-(t-\tau)}\,(x+\cos\omega\tau)\,d\tau \\ &= x\,e^{-t}\int_0^t e^{\tau}d\tau + e^{-t}\int_0^t e^{\tau}\cos\omega\tau d\tau \\ &= x\,e^{-t}\left[e^{\tau}\right]_0^t + e^{-t}\left[e^{\tau}\frac{\omega\sin\omega\tau+\cos\omega\tau}{\omega^2+1}\right]_0^t\end{aligned}$$

After a transient, we have
$$\hat{x}(t) \approx x + \frac{\omega\sin\omega t+\cos\omega t}{\omega^2+1}$$

and since $\omega \gg 1$, we also have $\dfrac{\omega\sin\omega t+\cos\omega t}{\omega^2+1} \approx \dfrac{1}{\omega}\sin\omega t$

so that
$$\boxed{\hat{x}(t) \approx x + \frac{1}{\omega}\sin\omega t}$$
to be compared with
$$\boxed{y(t) = x + \cos\omega t}$$

→ The impact of the noise was reduced by a factor ω!

Tuning of observer gain L

Different algorithms/methods exist that put the eigenvalues/poles of A-LC in the left-half plane

Can we use the methods we have seen for state-feedback?

For state-feedback, we used the cmd 'place' so that we could have eig(A-BK)=p

we entered 'K=place(A,B,p)'

Equation unknown

For observer tuning, we want eig(A-LC)=p_{obs}

use the place cmd again?

Notice that $(\mathbf{A}-\mathbf{LC})^T = \mathbf{A}^T - \mathbf{C}^T\mathbf{L}^T$ and since transposition does not change eigenvalues:

$$\mathbf{P}_{obs} = \mathrm{eig}(\mathbf{A}^T - \mathbf{C}^T\mathbf{L}^T)$$

same spot in the equation!

L = (place(A^T,C^T,p_{obs}))^T (and similarly using 'acker')

Tuning remark: generally, p_{obs} around 3-10 times faster than p

The Kalman Filter as an observer

The KF can be applied for Linear Time-Varying systems:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), & \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \end{cases}$$

Kalman Filter:

$$\begin{cases} \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}(t)\hat{\mathbf{x}}(t) + \mathbf{B}(t)\mathbf{u}(t) + \mathbf{L}(t)\left(\mathbf{y}(t) - \hat{\mathbf{y}}(t)\right) \\ \hat{\mathbf{y}}(t) = \mathbf{C}(t)\hat{\mathbf{x}}(t) \end{cases}$$

it IS an observer!

How is observer gain L calculated?

$$\mathbf{L}(t) = \mathbf{P}(t)\mathbf{C}^T(t)\mathbf{R}^{-1}(t)$$

with $\dot{\mathbf{P}}(t) = \mathbf{A}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^T(t) - \mathbf{P}(t)\mathbf{C}^T(t)\mathbf{R}^{-1}(t)\mathbf{C}(t)\mathbf{P}(t) + \mathbf{Q}(t)$

where $\mathbf{P}(0) = \mathbf{P}^T(0) > 0$ (Riccati matrix differential equation)

and where Q(t) and R(t) are positive tuning matrices

Remark: in case the plant is time invariant, so is L and the Riccati equation is now an algebraic equation

Discrete-time version of the Kalman Filter

Consider the DT system

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}(k)\mathbf{x}(k) + \mathbf{B}(k)\mathbf{u}(k) + \mathbf{v}(k) \\ \mathbf{y}(k) = \mathbf{C}(k)\mathbf{x}(k) + \mathbf{w}(k) \end{cases}$$

where v(k) and w(k) are Gaussian white noise signals with

$$\mathbb{E}[\mathbf{v}(k)\mathbf{v}(j)] = \begin{cases} 0 & k \neq j \\ \mathbf{Q} & k = j \end{cases}$$

$$\mathbb{E}[\mathbf{w}(k)\mathbf{w}(j)] = \begin{cases} 0 & k \neq j \\ \mathbf{R} & k = j \end{cases}$$

Observer form

$$\boxed{\hat{\mathbf{x}}(k+1) = \mathbf{A}(k)\hat{\mathbf{x}}(k) + \mathbf{B}(k)\mathbf{u}(k) + \mathbf{L}(k)(\mathbf{y}(k) - \mathbf{C}\hat{\mathbf{x}}(k))}$$

with (dropping the (k)'s) $\boxed{\mathbf{L} = \mathbf{APC}^T(\mathbf{CPC}^T + \mathbf{R})^{-1}}$

and $\boxed{\mathbf{P}^+ = \mathbf{APA}^T + \mathbf{Q} - \mathbf{APC}^T[\mathbf{CPC}^T + \mathbf{R}]^{-1}\mathbf{CPA}^T}$

Kalman Filter

where $\mathbf{P}^+ := \mathbf{P}(k+1)$

Remark: one can also describe the above in the well-known predict/update form often used in the literature