Normal form of a PDE

See video.

1. Compare with the standard PDE

$$Au_{xx} + Bu_{xy} + C_{yy} + Du_x + Eu_y + Fu = G$$

Obtain constant values

2. Classify with Discriminant

Classify PDE:

Discriminant	PDE type
$B^2 - 4AC > 0$	Hyperbolic
$B^2 - 4AC = 0$	Parabolic
$B^2 - 4AC < 0$	Elliptic

3. Find Characteristic Equation (normal form)

Hyperbolic:

$$\begin{cases} \frac{dy}{dx} & = & \frac{\xi(y)}{\xi(x)} & = & \frac{-B+\sqrt{B^2-4AC}}{2A} \\ \frac{dy}{dx} & = & \frac{\eta(y)}{\eta(x)} & = & \frac{-B-\sqrt{B^2-4AC}}{2A} \end{cases}$$

Parabolic:

$$\frac{dy}{dx} = \frac{B}{2A}$$

Hyperbolic:

$$\begin{cases} \frac{dy}{dx} & = & \frac{B+\sqrt{B^2-4AC}}{2A} \\ \frac{dy}{dx} & = & \frac{B-\sqrt{B^2-4AC}}{2A} \end{cases}$$

4. Integrate Characteristic Equations

We are trying to obtain the following values:

$$\xi(x,y) = c_1, \qquad \eta(x,y) = c_2$$

This is done differently depending on the type of PDE.

Hyperbolic:

$$\xi(x,y) = c_1, \qquad \eta(x,y) = c_2$$

Parabolic:
$\xi(x,y) = c_1$ and η will be chosen such that is will not be parallel to the ξ coordinate; η is chosen such that *the jacobian of the transformation is not zero*.

Elliptic:
A *second transformation is done* after finding $\xi(x,y) = c_1$ and $\eta(x,y) = c_2$:

$$\begin{cases} \alpha = \frac{\xi+\eta}{2} \\ \beta = \frac{\xi-\eta}{2i} \end{cases}$$

5. Write the canonical equation (normal form)!

$$\bar{A}u_{\xi\xi} + \bar{B}u_{\xi\eta} + \bar{C}_{\eta\eta} + \bar{D}u_{\xi} + \bar{E}u_{\eta} + \bar{F}u = \bar{G}$$

where

$$
\begin{cases}
\bar{A} &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
\bar{B} &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_x \\
\bar{C} &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
\bar{D} &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y \\
\bar{E} &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y \\
\bar{F} &= F \\
\bar{G} &= G
\end{cases}
$$

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