UNIVERSITY OF Southampton

Black Holes Orbital Mechanics

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Abstract

In this report, we derive the four types of orbits that appears in the Kepler problem. We start from the Two-body problem that we transform into a One-body problem with reduced mass. We then dig into orbits in the Schwarszchild metric where we derive the geodesics and equation of motion. We provide two proofs of the conservation laws in which we find values for the energy E and angular momentum L. We derive the relativistic potential as well as the location of stable and unstable circular orbits, the innermost stable circular orbit and the limit of bounded orbits. Finally, we explore briefly the radial infall motion and we describe in details the derivation of the perihelion advance/relativistic precession.

Acknowledgements

I would like to thank Leor Barack for his guidance and help during the project. I am also greatful to Nils Andersson for answering my questions about general relativity and letting me use his forthcoming book as reference.

Contents

Introduction

In the 1609 publication, *Astronomia Nova*, author Johannes Kepler investigated the parameters and physics dictating the motions of the planets within our solar system. Among these Keplerian laws of planetary motion, was the principle that all planets move along an ellipse with the Sun at one focus. In 1687, Isaac Newton published his *Philosophiae Naturalis Principia Mathematica*, presenting his own laws of motion. Under these laws, he demonstrated that the Keplerian relationships would apply in the solar system to a good approximation. [5] In 1907, only two years after the publication of his Special Theory of Relativity, Einstein wrote a paper attempting to modify Newton's theory of gravitation to fit special relativity [16] [6], a theory he developed over the following years. Many others contributed after 1915, like Karl Schwirzschild who published the first exact solution to the Einstein's fields equation used to describe the motion of particle around spherical bodies. [14]

This report aims to show how the understanding of particle motion around stars and black holes has evolved from the Newtonian theory to that of Einstein's general relativity. In Chapter 1, we introduce the different tools laid by Newton that we will use for calculations. We will then derive, from the statement of the 2-body problem, the different types of orbits that appears in Keplerian mechanics with illustrative examples. In Chapter 2, we will derive the equation of motion in the Schwarzschild metric. We will investigate the conserved quantities arising from the principles of General relativity. Finally we will discuss the behaviour of particle orbiting a static black hole and the phenomenom of perihelion advance.

Chapter 1

The Kepler Problem

1.1 Newtonian mechanics

In this chapter we will derive the orbits of planets as descibed by Kepler using the following properties of Newtonian mechanics:

$$\textbf{Newton's Second Law} \; : \quad \mathbf{F} = m\mathbf{a} \tag{1.1}$$

which means that the sum of all forces of a system is the mass m times the acceleration a, which is defined as $\ddot{\mathbf{r}} = \dot{\mathbf{v}} = \mathbf{a}$

$$\textbf{Newton's law of universal gravitation} \; : \quad F = -\frac{Gm_1m_2}{r^2} \tag{1.2}$$

which define gravity as a force. It is a conservative force such that

$$F = -\frac{dV}{dr}. \tag{1.3}$$

The total energy of a system is defined as

$$E = T + V, \tag{1.4}$$

where T is the inetic energy and V the potential energy. The angular momentum is given by

$$\mathbf{J} \equiv (\mathbf{r} \times \dot{\mathbf{r}})m.. \tag{1.5}$$

[9]

1.2 The 2-body problem

Consider a system of two particles of mass m_1 at position $\mathbf{r}_1$ and m_2 at $\mathbf{r}_2$ under the influence of a conservative central force and not influence by external forces. Our objective is to describe the trajectories of the two particles. From Newton's second law we get

$$\mathbf{F} = m_1\ddot{\mathbf{r}}_1, \qquad -\mathbf{F} = m_2\ddot{\mathbf{r}}_2. \tag{1.6}$$

So we have a system of coupled ordinary differential equations (ODE). To simplify the problem let us look into the central of the two bodies. Setting $M = m_1 + m_2$ we get

$$M\ddot{\mathbf{R}} = m_1\ddot{\mathbf{r}}_1 + m_2\ddot{\mathbf{r}}_2 = 0.. \tag{1.7}$$

Which means the center of mass moves with constant velocity. Now setting $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, the acceleration becomes

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \left(\frac{1}{m_1} - \frac{1}{m_2}\right)\mathbf{F} = \frac{m_1 + m_2}{m_1 m_2}\mathbf{F} \tag{1.8}$$

which we can write as

$$\mathbf{F} = \mu\ddot{\mathbf{r}} \tag{1.9}$$

where we define the *reduced mass* as

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{1.10}$$

[11] Now putting everything together using the total energy of the system we have

$$E = \frac{1}{2}M\ddot{\mathbf{R}}^2 + \frac{1}{2}\mu\ddot{\mathbf{r}}^2 + V(r). \tag{1.11}$$

Since the center of mass $\mathbf{R}$ moves with constant velocity we can set as inertial frame of reference with origin at $\mathbf{R}$ such that $\mathbf{R} = 0$. Using conservation of angular momentum, the orbit of our particle stays on the (x, y) plane, so we can reduce the problem to two dimensions. We will express $\ddot{\mathbf{r}}^2$ in polar coordinates to get the orbit in term of (r, θ) as it is easier to handle. The total energy of the system becomes

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V(r). \tag{1.12}$$

We want to express $\dot{\theta}$ in terms of r so using the definition of angular momentum we get

$$J = \mu r^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{J}{\mu r^2}. \tag{1.13}$$

Plugging back into (1.12) we get

$$T = \frac{1}{2}\mu(\dot{r}^2 + \frac{J^2}{\mu^2 r^2}). \tag{1.14}$$

Now, by definition, the kinetic energy is defined as

$$T = \frac{1}{2}m\dot{r}^2 \tag{1.15}$$

so the equation becomes

$$T = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}} \tag{1.16}$$

where

$$V_{\text{eff}} = \frac{J^2}{2mr^2} + V(r) \tag{1.17}$$

is the *effective potential*. Now, setting the force of the system as gravity defined by Newton's law of universal gravitation, using the reduced mass, we get

$$F = -\frac{G\mu}{r^2} \quad \Rightarrow \quad V = -\frac{G\mu}{r}. \tag{1.18}$$

Substitute in the effective potential we get:

$$V_{\text{eff}} = \frac{J^2}{2\mu r^2} - \frac{G\mu}{r}. \tag{1.19}$$

To simplify we set $k := G\mu$. [8] Looking back at (1.12) implementing the effective potential we get

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{J^2}{2\mu r^2} - \frac{k}{r}. \tag{1.20}$$

We effectively reduced the problem into a single ODE. We can investigate the properties of this equation to give us a hint about the nature of the trajectories. Figure 1.1 is the

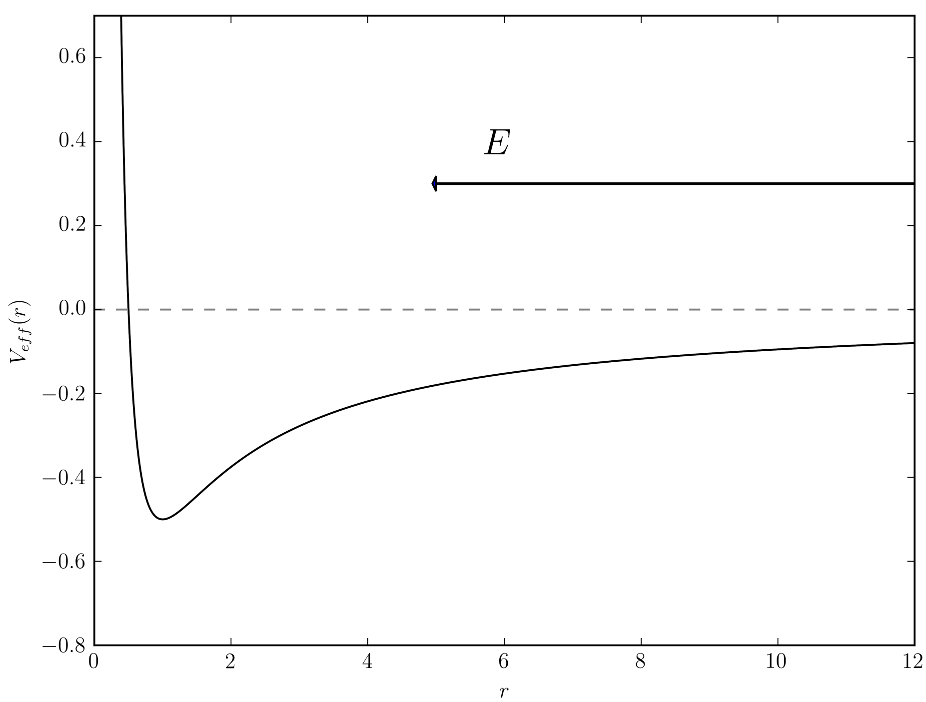

Figure 1.1: Plot of the effective potential V_{eff}. The energy $E = \text{const.}$ is represented by an arrow.

plot of the effective potential. It describes the motion of the particle orbiting. In setting the energy of the system to a certain value, we will determine the orbit of the particle.

We can see on the plot that the energy E will cross the effective potential at one point. This point is call *turning point.* The particle travels from infinity in the direction $r = 0^+$, once it hits the point $E = V_{\text{eff}}$ with $\frac{dV_{\text{eff}}}{dr} > 0$, the particle travels in the opposite direction until it hits $E = V_{\text{eff}}$ again or goes to $r = \infty$. We can also see that there exist a point where $\frac{dV_{\text{eff}}}{dr} = 0$, which, from Figure 1.1, is the minimum value of the potential V_{min}. What would happen if $E = V_{\text{min}}$? To answer this question, we would like to know how we can describe the orbit as function of $r(\theta)$, so we can investigate particular points of the potential. Solving for $\dot{r}$ we get

$$\dot{r} = \sqrt{\frac{2}{\mu}(E - V_{\text{eff}})}. \tag{1.21}$$

Looking for solution of the form $r(\theta)$. Using the chain rule we get

$$\dot{\theta} = \frac{d\theta}{dr}\dot{r}. \tag{1.22}$$

Combining $\dot{\theta} = \frac{J}{\mu r^2}$ and equation (1.21), considering $dr = \dot{r}$ and $d\theta = \dot{\theta}$ we get the following

$$\frac{\dot{\theta}}{\dot{r}} = \frac{d\theta}{dr} = \frac{J/\mu r^2}{\sqrt{\frac{2}{\mu}(E - V_{\text{eff}})}}. \tag{1.23}$$

Integrating we get:

$$\theta = \int \frac{J/\mu r^2}{\sqrt{\frac{2}{\mu}(E - V_{\text{eff}})}} dr + \theta_0 \tag{1.24}$$

where θ_0 is the constant of integration. Putting the value of V_{eff}

$$\theta = \int \frac{J/\mu r^2}{\sqrt{\frac{2}{\mu}(E - \frac{J^2}{2\mu r^2} - \frac{k}{r})}} dr + \theta_0. \tag{1.25}$$

We get a rather messy integral. However we can use the substitution $x = 1/r$ to get a nicer form

$$\int \frac{-J/\mu}{\sqrt{\frac{2}{\mu}(E - x^2\frac{J^2}{2\mu} - xk)}} dx + \theta_0 \tag{1.26}$$

$$= \int \frac{-dx}{\sqrt{\frac{2\mu}{J^2}(E - x^2\frac{J^2}{2\mu} - xk)}} + \theta_0 \tag{1.27}$$

$$= \int \frac{-dx}{\sqrt{-x^2 - x\frac{2\mu k}{J^2} + \frac{2\mu E}{J^2}}} + \theta_0. \tag{1.28}$$

Now we have a polynomial in the denominator so we can complete the square to get

$$= \int \frac{-dx}{\sqrt{-(x + \frac{\mu k}{J^2})^2 + \frac{2\mu E}{J^2} - \frac{\mu^2 k^2}{J^4}}} + \theta_0. \tag{1.29}$$

Which is in the form of an inverse trigonometric function. We use the fact that $\frac{d}{dx}\arccos(x/a) = \frac{-1}{\sqrt{a^2-x^2}}$ to get

$$= \int \frac{d}{dx}\left(\arccos\left(\frac{x+\frac{\mu k}{J^2}}{\sqrt{\frac{2\mu E}{J^2}-\frac{\mu^2 k^2}{J^4}}}\right)\right) + \theta_0 \tag{1.30}$$

$$= \arccos\left(\frac{\frac{J}{\mu r}-\frac{k}{J}}{\sqrt{\frac{2E}{\mu}+\frac{k^2}{J}}}\right) + \theta_0. \tag{1.31}$$

[4]

Setting

$$p = \frac{J^2}{\mu} \quad \text{and} \quad e = \sqrt{1+\frac{2EJ^2}{\mu^2}} \tag{1.32}$$

we get

$$\theta - \theta_0 = \arccos\left(\frac{p/r-1}{e}\right). \tag{1.33}$$

Which in terms of r gives

$$r = \frac{p}{1+e\cos(\theta-\theta_0)}. \tag{1.34}$$

Which describe the motion of the particle. We can set $\theta_0 = 0$ so that the orbit starts at the aphelion as the denominator is minimized, hence r maximized.

1.3 Types of orbits

The motion of the particle depends on the value of e. To illustrate this, let us rewrite the equation of r in terms of cartesian coordinates. From (r,θ) to (x,y) we get

$$\sqrt{x^2+y^2} = \frac{p}{1+e\cos(\arctan(y/x))} \tag{1.35}$$

$$\Rightarrow \sqrt{x^2+y^2} = \frac{p}{1+e/\sqrt{1+(y^2/x^2)}} \tag{1.36}$$

$$\Rightarrow \left(1+\frac{xe}{\cancel{\sqrt{x^2+y^2}}}\right)\cancel{\sqrt{x^2+y^2}} = p. \tag{1.37}$$

Squaring everything we get

$$\sqrt{x^2+y^2} = p - xe \tag{1.38}$$

$$x^2+y^2 = p^2 - 2pxe + x^2e^2. \tag{1.39}$$

This equation represent the shape of the orbit depending on the value of e.

When $e = 0$ we get

$$x^2+y^2 = p^2, \tag{1.40}$$

which is the equation of a circle.
When $0 < e < 1$ we get

$$\frac{\left(x + \frac{pe}{(1-e^2)}\right)^2}{\left(\frac{p}{1-e^2}\right)} + \frac{y^2}{\left(\frac{p}{\sqrt{e^2-1}}\right)^2} = 1 \tag{1.41}$$

which is the equation of a shifted ellipse. [3]
When $e = 1$ we get

$$x = -\frac{y^2}{2p} + \frac{p}{2}. \tag{1.42}$$

Which is the equation of a parabola. [3]
When $e > 1$ we get

$$\frac{\left(x + \frac{pe}{(1-e^2)}\right)^2}{\left(\frac{p}{1-e^2}\right)} - \frac{y^2}{\left(\frac{p}{\sqrt{e^2-1}}\right)^2} = 1. \tag{1.43}$$

Which is the equation of a hyperbola. [3]
Setting those value of e in (1.32), we can solve for E to get the following table classifying the orbits.

Type of orbit	Energy
Elliptic	$-mk^2/2j^2 < E < 0$
Circular	$E = -mk^2/2j^2$
Parabolic	$E = 0$
Hyperbolic	$E > 0$

[2]

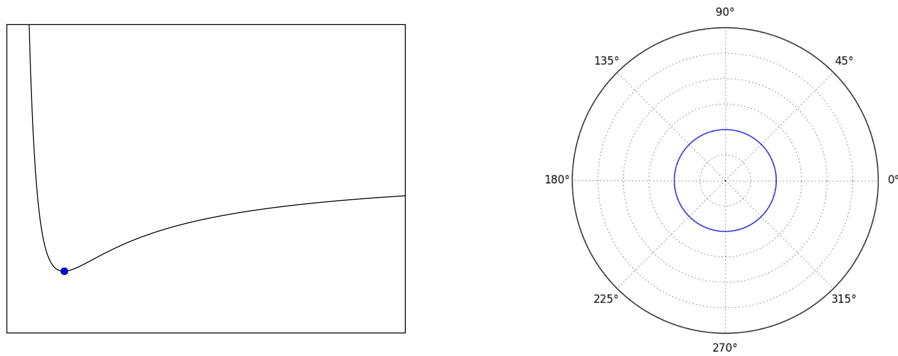

Figure 1.2: The unique bounded circular orbit with, $e = 0$, $p = 10$, $E = -\mu k^2/2j^2$

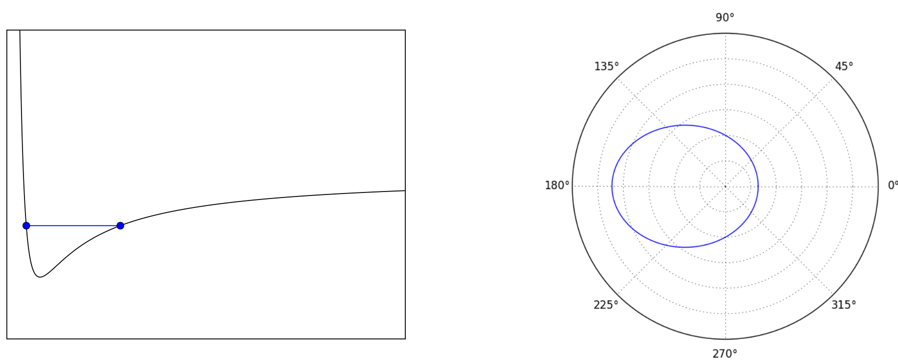

Figure 1.3: Elliptic orbit with $e = 0.55$, $p = 10$, $-mk^2/2j^2 < E < 0$. The point closest to the center of gravity is the perihelion, the furthest excursion is the aphelion.

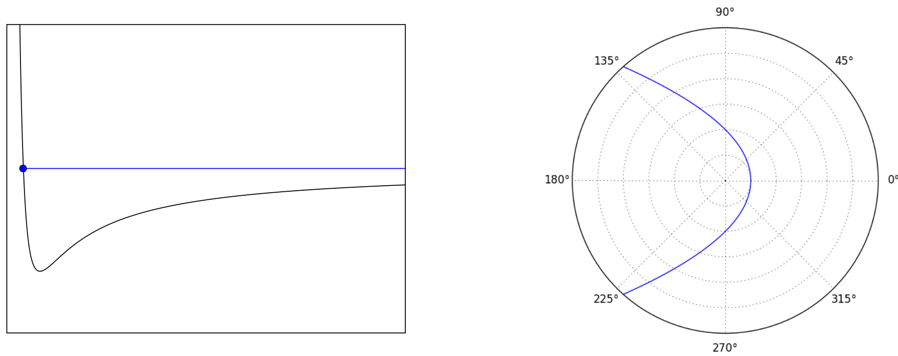

Figure 1.4: Parabolic orbit with $e = 1, \;\; p = 10, \;\; E = 0$. This is the point where orbits becomes unbounded as when $r \to \infty$, $V_{\text{eff}} \to 0$.

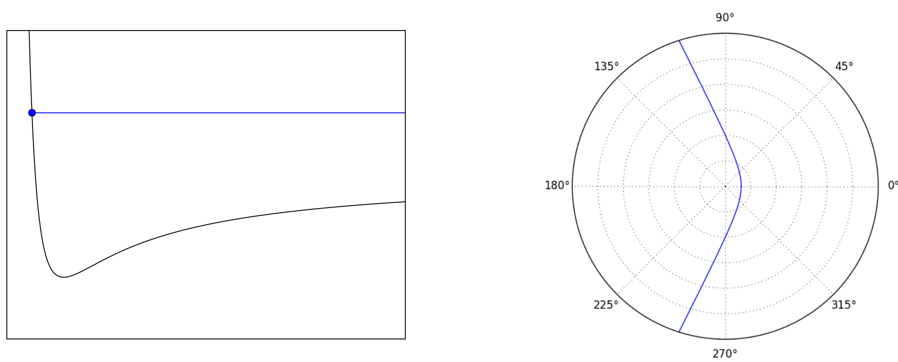

Figure 1.5: Hyperbolic orbit with $e = 2.22, \;\; p = 10, \;\; E > 0$

Chapter 2

Orbits in the Schwarzschild metric

2.1 Equation of motion

In this chapter we will be looking at the motion of particles in terms of the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.1}$$

It represent a non-rotating blackhole with mass m in a vacuum. It can be used to study the behaviour around non-rotating stars.
The usual way to derive the equations of motion is to use the Euler-Lagrange equations. We can use a "shortcut" to get the equation starting with the line element

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta. \tag{2.2}$$

Taking the derivative with proper time we get

$$\frac{ds^2}{d\tau^2} = g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \tag{2.3}$$

hence for massive particle we have

$$\Rightarrow -1 = g_{\alpha\beta} u^\alpha u^\beta. \tag{2.4}$$

Plugging the Schwarzschild metric into the metric tensor with u^α components we get the equation of motion

$$-1 = -\left(1 - \frac{2m}{r}\right) \dot{t}^2 + \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2). \tag{2.5}$$

This is essentially the Euler-Lagrange method where $2\mathcal{L} = ds^2$. However to simplify the expression we need to take a look at conserved quantities.

2.2 Conservation laws

2.2.1 Using Geodesics equation

When looking at the motion of an object, it is important to consider symmetry. The motion of a particle is descibed by the geodesic equation

$$\frac{du^\beta}{d\tau} + \Gamma^\beta_{\gamma\delta} u^\gamma u^\delta = 0. \tag{2.6}$$

We can derive an expression of the geodesic equation for the 'lowered' component u_α. Taking its derivative with respect to proper time we have

$$\begin{aligned}
\frac{du_\alpha}{d\tau} &= \frac{d(g_{\alpha\beta}u^\beta)}{d\tau} \\
&= g_{\alpha\beta}\frac{du^\beta}{d\tau} + u^\beta \frac{dg_{\alpha\beta}}{d\tau} \\
&= u^\beta \frac{dg_{\alpha\beta}}{d\tau} - g_{\alpha\beta}\Gamma^\beta_{\gamma\delta}u^\gamma u^\delta \qquad \text{(plugging the geodesic equation in } \tfrac{du^\beta}{d\tau}\text{)} \\
&= \frac{dx^\gamma}{d\tau}\frac{dg_{\alpha\beta}}{dx^\gamma}u^\beta - g_{\alpha\beta}\Gamma^\beta_{\gamma\delta}u^\gamma u^\delta \\
&= u^\gamma u^\beta g_{\alpha\beta,\gamma} - g_{\alpha\beta}\frac{1}{2}g^{\beta\epsilon}(g_{\epsilon\delta,\gamma} + g_{\epsilon\gamma,\delta} - g_{\gamma\delta,\epsilon})u^\gamma u^\delta \\
&= u^\gamma u^\beta g_{\alpha\beta,\gamma} - \frac{1}{2}\delta^\epsilon_\alpha(g_{\epsilon\delta,\gamma} + g_{\epsilon\gamma,\delta} - g_{\gamma\delta,\epsilon})u^\gamma u^\delta \\
&= \cancel{u^\gamma u^\beta g_{\alpha\beta,\gamma}} - (\cancel{\tfrac{1}{2}g_{\alpha\delta,\gamma}} + \cancel{\tfrac{1}{2}g_{\alpha\gamma,\delta}} - \frac{1}{2}g_{\gamma\delta,\alpha})u^\gamma u^\delta \\
&\qquad \text{(Cancel by renaming the free indices } \delta \Leftrightarrow \beta \text{ and } \delta \Leftrightarrow \gamma,\ \gamma \Leftrightarrow \beta\text{)} \\
&= \frac{1}{2}g_{\gamma\delta,\alpha}u^\delta u^\gamma.
\end{aligned}$$

Hence, we get the general results

$$\frac{du_\alpha}{d\tau} = \frac{1}{2}g_{\gamma\delta,\alpha}u^\delta u^\gamma. \tag{2.7}$$

Now, using this results with the Schwirszchild metric, there are only non-zero components when $\gamma = \delta$. Taking the derivative with respect to time we get

$$\frac{du_0}{dt} = \frac{1}{2}\frac{\partial}{\partial t}\left(1 - \frac{2m}{r}\right)\dot{t}^2 = 0. \tag{2.8}$$

The component of the metric tensor does not depend on time so the equation equal zero. It means that u_0 is constant along the particle's time coordinate trajectory. Setting the constant as energy we have

$$u_0 = g_{00}u^0 = \left(1 - \frac{2m}{r}\right)\dot{t} = \text{constant} = E. \tag{2.9}$$

We get a similar situation in the case of $\alpha = \phi$.

$$\frac{du_4}{d\phi} = \frac{1}{2}\frac{\partial}{\partial\phi}\left(r^2\sin^2(\theta)\right)\dot{\phi} = 0. \tag{2.10}$$

The component of the metric does not depend on ϕ hence u_4 is constant along the ϕ coordinate trajectory. Without loss of generality, we can set $\theta = \pi/2$ so that we get

$$u_4 = g_{44}u^4 = r^2\dot{\phi} = \text{const} = L. \tag{2.11}$$

Where we recognize $r^2\dot{\phi}$ as the angular momentum from the Newton laws of motion.

2.2.2 Using Killing vectors

A killing vector is an object that satisfies the *Killing's equation*

$$\nabla_\alpha k_\beta + \nabla_\beta k_\alpha = 0. \tag{2.12}$$

Now, looking at the following

$$k_\alpha u^\alpha. \tag{2.13}$$

Taking the derivative with respect to proper time we have

$$\begin{aligned} \frac{d}{d\tau}(k_\alpha u^\alpha) &= \nabla_\beta(k_\alpha u^\alpha) && (\text{as } u^\alpha \text{ is a scalar } \partial_\alpha = \nabla_\alpha) \\ &= u^\alpha \nabla_\beta(k_\alpha) + k_\alpha \nabla_\beta(u^\alpha) \end{aligned}$$

From the geodesic equation we know that the four velocity satisfy

$$\nabla_\beta u^\alpha = 0. \tag{2.14}$$

[1, p. 47] From equation (**??**), we know $\nabla_\beta k_\alpha$ is antisymmetric, renaming the indices $\alpha \to \beta$ we get

$$u^\alpha \nabla_\beta(k_\alpha) = -u^\beta \nabla_\alpha(k_\beta) = -u^\alpha \nabla_\beta(k_\alpha) = 0. \tag{2.15}$$

Therefore, all the terms are zero so

$$\frac{d}{d\tau}(k_\alpha u^\alpha) = 0. \tag{2.16}$$

Hence, $k_\alpha u^\alpha$ is constant along a geodesics.

We can associate this conservation law with spacetime symmetries. Indeed, if a metric does not depend on one or more of its coordinate component we can associate a killing vector to it and the property (2.16) will hold.

To illustrate, let us have a killing vector associated with x^0 component in some coordinate system such that

$$k^\alpha = \delta^\alpha_0 \quad \Rightarrow \quad k_\alpha = g_{\alpha\beta}k^\beta = g_{\alpha 0}. \tag{2.17}$$

Now plugging this in the expression (2.12) we get

$$\nabla_\alpha k_\beta + \nabla_\beta k_\alpha = \partial_0 g_{\alpha 0} = 0. \tag{2.18}$$

Therefore the existence this specific illing vector implies that the metric does not depend on the x^0 component. [1, p. 48]

Now, we can use this result on the Schwarschild metric. We can see that the metric does not depend on $x^0 = t$ or $x^3 = \phi$. We can set two killing vector such that $k^\alpha = \delta_0^\alpha$ and $d^\alpha = \delta_3^\alpha$. Applying equation (2.16) for t, with the only non zero term of the metric tensor we get

$$\frac{d}{d\tau}(k_\alpha u^\alpha) = \frac{d}{d\tau}(g_{00}u^0) = -\frac{\partial}{\partial\tau}\left[\left(1 - \frac{2m}{r}\right)\dot{t}\right] = 0 \tag{2.19}$$

$$\Rightarrow \quad -\left(1 - \frac{2m}{r}\right)\dot{t} = \text{const} = E. \tag{2.20}$$

From Noether's theorem, if the symmetry is time invariance then the conserved quantity E is energy. Similarly for ϕ we get

$$\frac{d}{d\tau}(k_\alpha u^\alpha) = \frac{d}{d\tau}(g_{44}u^4) = \frac{\partial}{\partial\tau}\left(r^2\sin^2(\theta)\dot{\phi}\right) = 0 \tag{2.21}$$

$$\Rightarrow \quad r^2\sin^2(\theta)\dot{\phi} = \text{const} = L. \tag{2.22}$$

From Noether's theorem, if the symmetry is rotational invariance then the conserved quantity L is the angular momentum.

2.3 Schwarzschild orbits

We can use the conservation laws that we derived and apply them on the equation of motion. Substituing (2.20) and (2.22) into (2.5) we get

$$-\frac{E^2}{(1-\frac{2m}{r})} + \frac{\dot{r}^2}{(1-\frac{2m}{r})} + \frac{L^2}{r^2} = -1. \tag{2.23}$$

Multiplying by $(1 - \frac{2m}{r})$ on both sides we have

$$E^2 = \dot{r}^2 + \left(1 - \frac{2m}{r}\right)\left(1 + \frac{L^2}{r^2}\right). \tag{2.24}$$

Similarly to the Keplerian case, let us set the effective potential V_{eff} as

$$V_{\text{eff}}^2 = \left(1 - \frac{2m}{r}\right)\left(1 + \frac{L^2}{r^2}\right). \tag{2.25}$$

Hence we reduced the equation of motion to

$$\dot{r}^2 = E^2 - V_{\text{eff}}^2. \tag{2.26}$$

Using V^2 or V is a matter of preference, as we will be looking at the points where $V = E$ or $V^2 = E^2$.[10]

Comparing this potential with the keplerian one, in geometrized units with $V_{\text{GR}} = V_{\text{eff}} - 1$ we can see that

$$V_{\text{GR}} = \frac{1}{2}V_{\text{Kepler}} - \frac{2mL^2}{r^3}. \tag{2.27}$$

For large values of r, the Keplerian potential is similar to the Schwarzschild potential. It is only when r get closer to zero that the corrective term $2mL^2/r^3$ from general relativity makes significant changes. In the Keplerian potential there is an infinite centrifugal barrier that becomes a barrier of finite height in General relativity [7, p. 195]. We can see on figure (2.1) when the two potential diverge from each other.

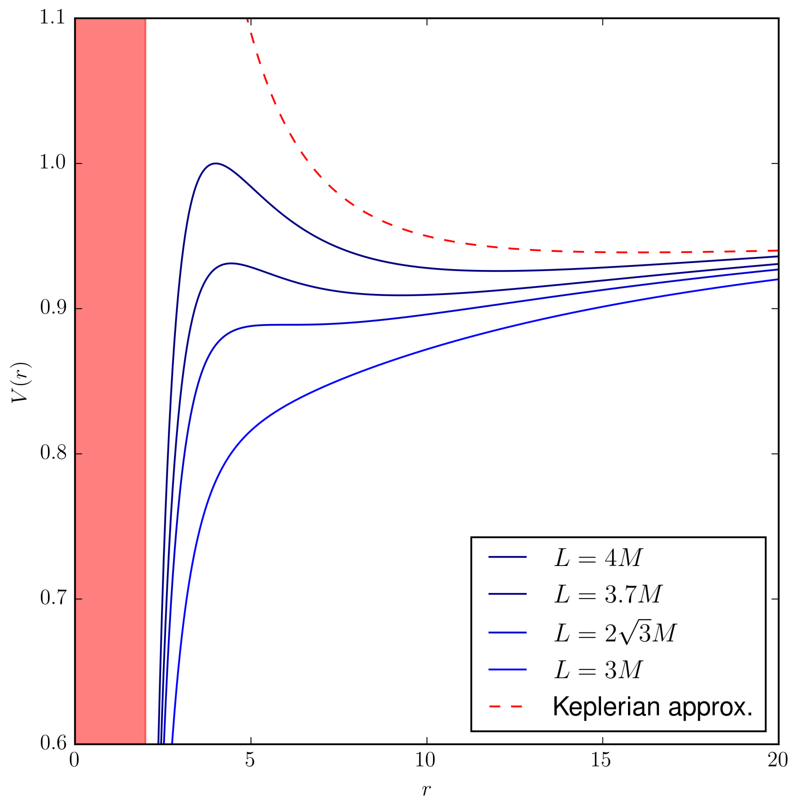

Figure 2.1: Plot of the effective potential V_{eff} for the Schwarszchild metric. The red area is the event horizon.

The first property of interest that we will look at, in the following section, are the *turning points* where $V = E$ or $V^2 = E^2$.

2.3.1 Stable and Unstable Orbits

In order to have a more precise description of the different types of orbit, we look at the equilibrium points of the potential.

The equilibrium points are located where the derivative of V_{eff} is zero. So we have

$$V_{\text{eff}} = 1 + \frac{L^2}{r^2} - \frac{2m}{r} - \frac{2mL^2}{r^3}.. \tag{2.28}$$

Taking the derivative and looking at equilibrium point we have

$$\frac{dV_{\text{eff}}}{dr} = -\frac{2L^2}{r^3} + \frac{2m}{r^2} + \frac{6mL^2}{r^4} = 0. \tag{2.29}$$

$$\Rightarrow mr^2 - L^2 r + 3mL^2 = 0. \tag{2.30}$$

Which is a quadratic equation with roots

$$r_{\pm} = \frac{L^2 \pm L^2\sqrt{1 - \frac{12m^2}{L^2}}}{2m}. \tag{2.31}$$

The two roots describe two different types of circular orbits. The smallest root r_- is an *unstable* circular orbit. A small perturbation δE will cause the particle either to fall in the black hole ($\delta E < 0$) or travel far away to $r = \infty$ ($\delta E > 0$). The biggest root r_+ is a *stable* circular orbit. A small perturbation would put the partical in an elliptical orbit with very small eccentricity, and would not diverge away. [10, p. 662]

Now, looking at the equation (2.31) we can see that there is a situation where there is only one root, namely when $L^2 = 12m^2 \Rightarrow L = 2\sqrt{3}$. In that case the radius become $r = 6m$. This is called the *innermost stable circular orbit* (ISCO). It is the minimum value of r for which circular orbits exists. We can confirm looking at Figure 2.1 that the potential at $L = 2\sqrt{3}$ has only one stable point. [1, p. 209]

What is the radial limit of the unstable circular orbits ? Clearly, they cannot exist inside the horizon as the potential tends to $-\infty$ as $r \to 2m$. We cannot find the value of r using equation (2.31) as there is an L^2 term outside that prevent us looing at the limit $L \to \infty$. However we can solve equation (2.30) with respect to L^2 to get

$$L^2 = \frac{mr^2}{(r - 3m)}. \tag{2.32}$$

Now we can see that as $r \to 3m$, $L^2 \to \infty$. So the maximum unstable circular orbit has limit $r = 3m$.

To summarize, we were able to find the location of stable and unstable circular orbits such that

$$\text{Stable} : \quad 6m \leq r_+ < \infty. \tag{2.33}$$

$$\text{Unstable} : \quad 3m \leq r_- < 6m. \tag{2.34}$$

[12, p. 37]

2.3.2 Bounded and Unbounded Orbits

Similarly to the Keplerian case. We can distinguish another two different types of orbits which are, recalling section 1.3, bounded and unbounded orbits.

We know that when $E = V_{\text{eff}}$, the particle hit a turning point and change direction on the potential. When E is always greater than V_{eff}, the particle never hits a turning point, so depending on the starting point and the direction, it will fall inward or outward. Looking at equation 2.1 we notice that $V_{\text{eff}} \to 1$ as $r \to \infty$. So for our orbit to be bounded the maxima of V_{eff} cannot exceed 1. Therefore one of the condition for an orbit to be bounded is that $E < V_{\max}$, where the maximum value of $V_{\max} = 1$.

In the previous section we introduce the ISCO, that exist at $L = 2\sqrt{3}$. Any orbit with $L < 2\sqrt{3}$ will be pulled in the black hole or more precisely $r = 2m$. There is no point E intersecting V_{eff} as $V'_{\text{eff}} > 0$. [10, p.662] Hence we derive the following condition for bounded orbits

$$E < 1 \quad \text{and} \quad L > 2\sqrt{3}. \tag{2.35}$$

The orbits that have $L < 2\sqrt{3}$ or $E > V_{\max}$ will be pulled in the black hole. However, when we have $E < V_{\max}$, with $E > 1$, the particle will reach a turning point and then diverges to $r = \infty$.

A simple case of an unbounded orbit is when we release a particle at rest from infinity, i.e. $E = 1$, with $L = 0$. From equation (2.24) we get the equation of motion for radial infall

$$\frac{dr}{d\tau} = -\sqrt{\frac{2m}{r}}. \tag{2.36}$$

Integrating we get

$$\int d\tau = -\frac{1}{\sqrt{2m}} \int r^{1/2} dr. \tag{2.37}$$

Which gives

$$\tau - \tau_0 = -\frac{1}{\sqrt{2m}} \frac{2}{3} \left(r^{3/2} - r_0^{3/2} \right). \tag{2.38}$$

where τ_0 and r_0 are integration constant. Setting $r = 2m$ we see that the particle reaches the horizon at a finite proper time. So nothing special happens we a particle crosses the horizon. This is in contrast to the well known fact that it takes an infinite coordinate time t to reach the horizon [1, p. 209]. Which is just one indication that the Schwarzschild coordinate are flawed near $r = 2m$ [7, p. 199].

2.3.3 Perihelion advance

In the previous section, we classified different type of orbits derived from the potential. We are now interested about the shape of bounded orbits. The following argument is a summary of [15, p. 159]. Similarly to the Keplerian case, we are interested in the orbit $r(\phi)$. Hence $\dot{r} = r'\dot{\phi}$, we know from (2.11) that $\dot{\phi} = L/r^2$. Subbing this in equation (2.26) we get

$$r' \frac{L^2}{r^4} = E^2 - V(r). \tag{2.39}$$

As in the Kepler problem we work with $u = 1/r$ and thus $r' = -u'/u^2$. In terms of this variable, we have

$$L^2 u'^2 = E^2 - (1 - 2mu)(1 + L^2 u^2) \tag{2.40}$$

or

$$u'^2 + u^2 = \frac{E^2 - 1}{L^2} + \frac{2m}{L^2} u + 2mu^3. \tag{2.41}$$

Differentiating with respect to ϕ gives

$$2u'u'' + 2uu' = \frac{2m}{L^2} u' + 6mu'u^2. \tag{2.42}$$

Either $u' = 0$ so we have a circular motion or u satisfy the following ODE

$$u'' + u = \frac{m}{L^2} + 3mu^2. \tag{2.43}$$

Now we can approximate the solution u using the Kepler orbit found in 1.2 such that

$$u(\phi) = \frac{m}{L^2}(1 + e\cos(\phi)). \tag{2.44}$$

The $3mu^2$ term is considered as a small perturbation from the Kepler orbit. Putting u in (2.43) we get

$$u'' + u = \frac{m}{L^2} + \frac{3m^3}{L^4}\left(1 + 2e\cos(\phi) + e^2\cos^2(\phi)\right). \tag{2.45}$$

Now the right hand side is a sum of constant, cos and $\cos^2$, which can be considered as the sum of solutions of a particular integral such that

$$u_{\mathrm{P}}'' + u_{\mathrm{P}} = \begin{cases} A \\ B\cos(\phi) \\ C\cos^2(\phi) \end{cases} \tag{2.46}$$

where A,B,C are constant. This have solution

$$u_{\mathrm{P}} = \begin{cases} A \\ \frac{1}{2}B\phi\sin(\phi) \\ \frac{1}{2}C - \frac{1}{6}C\cos(2\phi) \end{cases}. \tag{2.47}$$

We know the value of the constant from (2.45): $A = 1$, $B = 2e$, $C = e^2$. By superposition, we add the keplerian solution (2.43) and the particular solution to get

$$u(\phi) = \frac{m}{L^2}(1 + e\cos(\phi)) + \frac{3m^3}{L^4}\left(1 + \frac{e^2}{2} - \frac{e^2}{6}\cos(2\phi) + e\phi\sin(\phi)\right).. \tag{2.48}$$

This should be the solution of the orbit, however, we see that after each rotation, $\phi = 2n\pi$ with $n = 0, 1, 2, ...$, that the term $\phi\sin\phi$ is getting bigger after each revolution. When $\sin(\phi) > 0$, u increases after each revolution as $\phi\sin\phi$ is unbounded, $r \to 0$ as

$u \to \infty$. When $\sin(\phi) < 0$, u decreases after each revolution with no lower bound, so it means that as $u \to \infty$ there will be one point where $u = 0$ and so $r \to \infty$. These results are very different from observations and appear to break conservation laws. The term causing for the unphysical behaviour is $\frac{3m^3}{L^4} e\phi \sin(\phi)$, which becomes larger as ϕ increases. We saw in section 2.3.2 that there exists bounded solution in the case $E < 1$ and $L = 2\sqrt{3}$. Therefore, our solution must be bounded, at least for the first few term of the approximation. Rearranging (2.48) such that

$$u(\phi) = \frac{m}{L^2}\left(1 + e\cos(\phi) + \frac{3m^2}{L^2} e\phi\sin(\phi) + \frac{3m^2}{L^2}(1 + \frac{e^2}{2} - \frac{e^2}{6}\cos(2\phi))\right).. \tag{2.49}$$

Now introducing the value

$$\varepsilon := \frac{3m^2}{L^2} \ll 1 \tag{2.50}$$

such that ε is very small [1, p. 211] .

$$u(\phi) = \frac{m}{L^2}\left(1 + e\left(\cos(\phi) + \varepsilon\phi\sin(\phi)\right) + \varepsilon(1 + \frac{e^2}{2} - \frac{e^2}{6}\cos(2\phi))\right).. \tag{2.51}$$

Now using the small angle formula we know that $\sin(\varepsilon\phi) \approx \varepsilon\phi$ and $\cos(\varepsilon\phi) \approx 1$, so we can introduce in (2.51)

$$u(\phi) = \frac{m}{L^2}\left(1 + e\left(\cos(\phi)\cos(\varepsilon\phi) + \sin(\varepsilon\phi)\sin(\phi)\right) + \varepsilon(1 + \frac{e^2}{2} - \frac{e^2}{6}\cos(2\phi))\right).. \tag{2.52}$$

We can now use the following trigonometric identity

$$\cos((1-\varepsilon)\phi) = \cos(\phi)\cos(\varepsilon\phi) + \sin(\varepsilon\phi)\sin(\phi). \tag{2.53}$$

Hence to have a bounded solution and get rid of the $\phi\sin(\phi)$ term, we set our particular solution to be

$$u(\phi) = \frac{m}{L^2}(1 + e\cos((1-\varepsilon)\phi)). \tag{2.54}$$

The general solution becomes

$$u(\phi) = \frac{m}{L^2}(1 + e\cos((1-\varepsilon)\phi)) + \frac{3m^3}{L^4}\left(1 + \frac{e^2}{2} - \frac{e^2}{6}\cos(2\phi(1-\varepsilon)) + e\cos((1-\varepsilon)\phi)\right).. \tag{2.55}$$

We now see in our solution that the period in not 2π anymore but

$$\frac{2\pi}{(1-\varepsilon)} > 2\pi \tag{2.56}$$

which means that after two rotation the perihelion advance with a angle of

$$\Delta\phi = \frac{2\pi}{(1-\varepsilon)} - 2\pi \approx 2\pi\varepsilon. \tag{2.57}$$

Describing ε in terms of semi-minor, semi-major axis and eccentricity we get

$$\frac{6\pi m}{a(1-e^2)}. \tag{2.58}$$

Which is the famous equation of the relativistic precession. It was one of the argument that consolidated Einstein with his theory as, when calculated for Mercury, the precession rate was 43 arcseconds. [7, p. 204] In accordance with physical observation.

Figure 2.2 and 2.3 shows how bounded orbits are different from the Keplerian one. Of course, the effect of the precession is exaggerated as, for a planet like Mercury, it is extremely small.

We manage to see that the equation (2.45) is wrong because we had knowledge of physical observations to compare with our solution. One has to be care ful using perturbation method to draw conclusion without a way to check if the result is physical.

We stress the fact that the solution (2.55) was obtained with the assumption that the radius of the orbit stays far away from the radius of the central body. It does not apply for a particle that come close to the central body. To get the orbit closer to the center of gravity one has to use numerical methods or use elliptical functions. However, numerical approximation become necessary anyway as the orbit get perturbed in so many ways. It gets influenced by the rotation of the star/black hole, perturbation from other planets orbit or loss of angular momentum caused by friction from the interplanetary medium. [13]

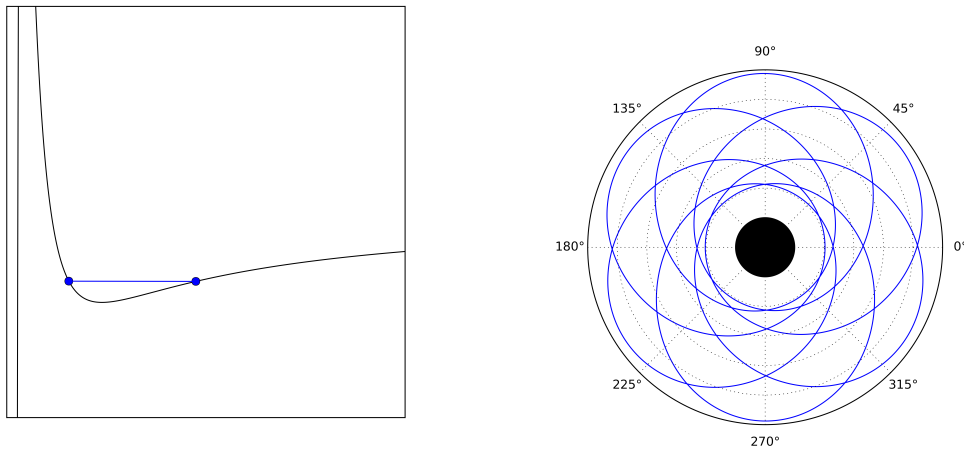

Figure 2.2: The two points are when $E = V_{\text{eff}}$. Relativistic perihelion shift with eccentricity $e = 0.55$, $m = 1$. The black circle is the event horizon.

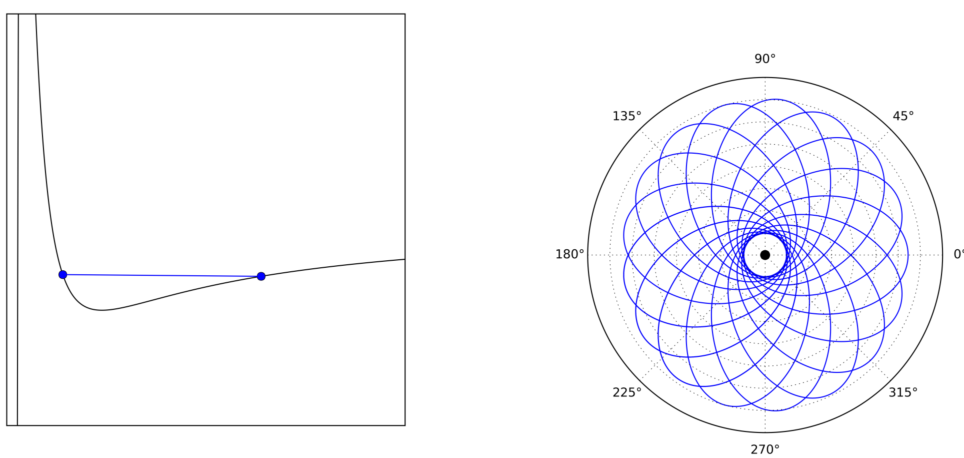

Figure 2.3: Relativistic perihelion shift with eccentricity $e = 0.8$, $m = 5$. The black circle is the event horizon.

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List of Figures

Schwarszchild Potential plot

```python
# -*- coding: utf-8 -*-
"""
Created on Thu May 02 21:35:48 2019

@author: Jonah
"""

import numpy as np
import matplotlib.pyplot as plt



##SCHARWSCHILD METRIC
def Vs(r,M,L):
    F = (1.0-2.0*M/r)*(1.0+L**2.0/r**2.0)
    return F

#Kepler Potential
def V(r,M,L):
    F = -(r**-1.0)+((L**2)*0.5*(r**-2.0))+0.97
    return F

x0=np.linspace(0.0,0.0,1000)
fig, ax = plt.subplots(figsize=(7,7))
fig.set_figwidth(7)
fig.set_figheight(7)
x = np.linspace(-1.0,50.0,1000)
K = V(x,1,4)

F=np.zeros([4,1000])
L=np.array([4.0,3.7,3.0,np.sqrt(12)])
for i in range(4):
    F[i]=Vs(x,1,L[i])

l1, = ax.plot(x, F[0],color='navy', label='$L=4M$')
l2, = ax.plot(x, F[1],color='darkblue', label='$L=3.7M$')
l4, = ax.plot(x, F[3],color='mediumblue', label='$L=2\sqrt{3}M$')
l3, = ax.plot(x, F[2],color='b', label='$L=3M$')
l5, = ax.plot(x, K,'r',linestyle='--', label='Keplerian approx.')

ax.axvspan(0, 2, alpha=0.5, color='red')
#ax.fill_between(x,x0,F[3] ,facecolor='grey', alpha=0.3)
ax.set_xlabel("$r$")
ax.set_ylabel("$V(r)$")
ax.set_xlim([0, 20])
ax.set_ylim([0.6, 1.1])
plt.legend(handles=[l1,l2,l4,l3,l5],loc='lower right')

plt.show(fig)
plt.savefig('SCHwarplot.png', dpi=600)
```

Perihelion Advance plot

```python
# -*- coding: utf-8 -*-
"""
Created on Wed Mar 27 16:27:50 2019

@author: Jonah
"""

import numpy as np
import matplotlib.pyplot as plt

cos = np.cos
sin=np.sin
pi = np.pi

a = 10
e = 0.8
m = 5
L = m*a*(1.0-e**2)


theta = np.linspace(0,42*pi, 21*360)

r =(1/L)+(1/L**2)*(1+(e**2)/3)+(e/L)*cos((1-1/L)*theta)+((e**2)/(3*L**2))*(cos(2*theta))

fig = plt.figure()
ax = fig.add_subplot(111, polar=True)
ax.set_yticklabels([])
ax.plot(theta,r**-1)
ax.fill_between(np.linspace(0.0, 2*pi,200), 2.0*np.ones(200),facecolor='black')
#ax.set_rmax(15)

plt.show()
plt.savefig('periadv_2', dpi=600)
```

Polar Keplerian orbits plot

```python
# -*- coding: utf-8 -*-
"""
Created on Wed Mar 27 23:03:38 2019

@author: Jonah
"""

import numpy as np
import matplotlib.pyplot as plt
cos = np.cos
sin=np.sin
pi = np.pi
theta = np.linspace(0,2*pi, 360)

e =np.array([0.0 , 0.55, 1.0, 2.2])
r = np.zeros((4,360))
for i in range(0,4):
    p=10
    r[i]=p/(1.0+e[i]*cos(theta))


fig = plt.figure()
ax = fig.add_subplot(111, polar=True)
ax.set_yticklabels([])
ax.set_ylim(30, 0)

#ax.plot(theta,r[1])
#ax.plot(theta,r[2])
#   ax.plot(theta,r[3])
ax.plot(theta,r[0])

plt.show()
```

Keplerian Potential plot

```python
# -*- coding: utf-8 -*-
"""
Created on Sat Feb 16 19:45:31 2019

@author: Jonah
"""

import numpy as np
import matplotlib.pyplot as plt
from matplotlib import rc
rc('text', usetex=True)


x = np.linspace(-1.0,15.0,1000)
x1 = np.linspace(0,0,1000)


def V(r,J):
    F = -(r**-1.0)+0.5*(J**2)*(r**-2.0)
    return F

v1 =V(x,1)
minn  = np.min(v1)
x2 = np.linspace(4.0,100,1000)
x3 = np.linspace(0,100,1000)
v199=np.linspace(-0.25,-0.25,177)
#v180=np.linspace(0.3,0.3,810)
vpara=np.linspace(0.0,0.0,807)
plt.plot(x,v1, 'k',  label='$y_1$')
#plt.plot(x[99:276],v199, 'b',  label='$y_1$')
#plt.plot( x[99],-0.25, 'bo',markersize=8)

#plt.plot( x[276],-0.25, 'bo',markersize=8)
#plt.plot(x[93:900],vpara, 'b',  label='$y_1$')
#plt.plot( x[93],0.0, 'bo',markersize=8)
#
plt.plot(x,x1, color='grey',linestyle='--')
#plt.plot( 1.01,minn, 'bo',markersize=8)
#plt.plot(x,x2, color='lightblue',linestyle='--')
plt.arrow(20.0,0.3,-15.0,0.0,width=0.0018)
plt.text(5.7,0.37,'$E$',size=20)

"""


plt.text(5.88, 0.5, r'\textac{Hyperbolic}', fontsize= 18)
plt.text(5.88, 0.02, r'\textac{Parabolic}', fontsize= 18, color='red')
plt.text(5.88, -0.33, r'\textac{Circular}', fontsize= 18, color='blue')
plt.text(7.5, -0.2, r'\textac{Elliptic}', fontsize= 18)
"""
#plt.plot(x,x3, 'r--')


plt.xlabel('$r$')
plt.ylabel('$V_{eff}(r)$')
#plt.legend(loc='upper right')
```

```
"""
plt.fill_between(x,0,x3, facecolor='grey', alpha=0.3)
plt.fill_between(x,0,minn, facecolor='red', alpha=0.3)
"""
plt.axis([0,12,-0.8,0.7])
#plt.gca().axes.get_yaxis().set_visible(False)

#plt.gca().axes.get_xaxis().set_visible(False)

plt.show()
plt.savefig('potential1.png', dpi=600)
```

Small Potential Sketch plot

```
# -*- coding: utf-8 -*-
"""
Created on Mon Mar 18 18:30:17 2019

@author: Jonah
"""

import numpy as np
import matplotlib.pyplot as plt

##SCHARWSCHILD METRIC
x = np.linspace(-1.0,90.0,1000)
x1 = np.linspace(0,0,1000)
def Vs(r,M,L):
     F = (1.0-2.0*M/r)*(1.0+L**2.0/r**2.0)
     return F
M =1

inters = np.linspace(v1[165],v1[480])
plt.figure(figsize=(7,7))
v1 =Vs(x,M,5*M)
inters = np.linspace(v1[150],v1[642],492)

minn = np.min(v1)
plt.plot(x,v1, 'k', label='$L=3.7M$')
plt.plot(x[150],v1[150], 'bo',markersize=8)
plt.plot(x[642],v1[642], 'bo',markersize=8)
plt.plot(x[150:642],inters, 'b')
#492
plt.axis([0,90,0.9,1.1])

plt.gca().axes.get_yaxis().set_visible(False)

plt.gca().axes.get_xaxis().set_visible(False)

plt.show()
plt.savefig('smallsch2', dpi=600)
```