Separation of variables and Frobenius solution for ODEs

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Abstract

Starting from the spherical wave equation, using separation of variables, we derive the spherical Bessel equation and the associated Legendre equation. Looking to find series solution for these two equations we introduce the power series solution theorem and the Fuchs theorem. We then classify their points of expansion i.e. regular and regular singular points. We obtain the first few term of the Bessel equation at $r = 0$ and $r = 1$ as well as the first few term for the associated Legendre equation at $x = 0$ and $x = 1$. We briefly explore the convergence of the solution for $r = 0$ and the value of the separation constant for regularity considerations.

Contents

1 Introduction

The wave equation is a hyperbolic partial differential equation (PDE) which contains a time variable t, one or more spatial variables $x_1, x_2, ..., x_n$ and a scalar function $u = (x_1, x_2, ..., x_n)$. In this report we will be demonstrating a brief calculation of series solutions for ODEs that arise when using the method of separation of variables, to solve a wave equation with three space variables, $\frac{\partial^2 u}{\partial t^2} = \Delta u$, where Δ is the Laplace operator. In our case we will be working with the Laplace equation in spherical polar coordinates, plus time, where $u = (t, r, \theta, \phi)$. This gives us the equation:

$$\Delta u = u_{,tt} = u_{,rr} + \frac{2}{r}u_{,r} + \frac{1}{r^2}(u_{,\theta\theta} + \cot\theta u_{,\theta} + \frac{1}{\sin^2\theta}u_{,\phi\phi}). \tag{1}$$

Which can be solved using the method of separation of variables and using the ansatz $u = T(t)R(r)X(x)\Phi(\phi)$, where $x = cos(\theta)$. This will give us a PDE which we will separate into four further ODEs, two of which are a spherical Bessel equation and an associated Legendre equation. We will then go into explaining the concepts of regular and singular points of linear ODE's, series solutions about regular points and Frobenius solutions about regular singular points.

2 Separation of variables

Start with the wave equation in spherical coordinates with $c = 1$ we have:

$$u_{,tt} = u_{,rr} + \frac{2}{r}u_{,r} + \frac{1}{r^2}(u_{,\theta\theta} + \cot\theta u_{,\theta} + \frac{1}{\sin^2\theta}u_{,\phi\phi}). \tag{2}$$

Using the ansatz $u(t, r, \theta, \phi) = T(t)R(r)X(x)\Phi(\phi)$ where $x = \cos(\theta)$, (2) becomes

$$\frac{T''}{T} = \frac{R''}{R} + \frac{2}{r}\frac{R'}{R} + \frac{1}{r^2}(\frac{X''}{X}(1-x^2) - 2x\frac{X'}{X} + \frac{1}{1-x^2}\frac{\Phi''}{\Phi}). \tag{3}$$

The LHS only depends on t so it differs only by a constant let's say K_1, we then get the ODE:

$$T'' = K_1 T. \tag{4}$$

We get the solution $T(t) = e^{i\omega t}$ setting $K_1 = -\omega^2$. Subbing back in (3) we have

$$-r^2\omega^2 - r^2\frac{R''}{R} - 2r\frac{R'}{R} = (\frac{X''}{X}(1-x^2) - 2x\frac{X'}{X} + \frac{1}{1-x^2}\frac{\Phi''}{\Phi}). \tag{5}$$

The LHS only depends on r so it differs only by a constant let's say λ, we then get the ODE:

$$r^2R'' + 2rR' + r^2\omega^2 R = \lambda R. \tag{6}$$

Plug back λ in (5) we have:

$$\lambda = \frac{X''}{X}(1-x^2) - 2x\frac{X'}{X} + \frac{1}{1-x^2}\frac{\Phi''}{\Phi}. \tag{7}$$

Multiplying by $(1-x^2)$ on both sides and bringing the expressions depending on x on the LHS we get

$$(1-x^2)\lambda - \frac{X''}{X}(1-x^2)^2 + 2x(1-x^2)\frac{X'}{X} = \frac{\Phi''}{\Phi}. \tag{8}$$

The LHS only depends on x so it differs only by a constant let's say K_2, we then get the ODE:

$$(1-x^2)\lambda - \frac{X''}{X}(1-x^2)^2 + 2x(1-x^2)\frac{X'}{X} = K_2. \tag{9}$$

Plug back K_2 in (8) we get the ODE

$$\Phi'' = K_2\Phi. \tag{10}$$

With solution $\Phi = e^{im\phi}$ with $K_2 = -m^2$ After this calculation, we have obtained two solution $\Phi(\phi)$ and $T(t)$ and two ODE

$$\Phi(\phi) = e^{im\phi} \tag{11}$$

$$T(t) = e^{i\omega t} \tag{12}$$

$$r^2R'' + 2rR' + (r^2\omega^2 - \lambda)R = 0 \tag{13}$$

$$(1 - x^2)X'' - 2xX' + (\lambda - \frac{m^2}{1 - x^2}) = 0. \tag{14}$$

Equation (13) is the **spherical Bessel equation** and (14) is the **associated Legendre equation**. These equation are more commonly used in the form where $\lambda = l(l + 1)$ where l and m are integers

$$r^2R'' + 2rR' + (r^2\omega^2 - l(l + 1))R = 0 \tag{15}$$

$$(1 - x^2)X'' - 2xX' + (l(l + 1) - \frac{m^2}{1 - x^2}) = 0 \tag{16}$$

In the last section of the report we will explore regularity considerations and explain the reasonning behind setting $\lambda = l(l + 1)$. Before moving on solving (13) and (14), we will need to state some definitions and theorems in order to contruct the method used to solve those two equations.

3 Power series solutions and the Frobenius Method

3.1 Power series solution of ODE

We will seek solution in the form of infinite series. Assuming we are working with infinitely differientiable functions, we can express them as an infinite series. More precisely:

Definition 3.1. (Taylor series) [4]

For a function f that has derivatives of all orders on $(x_0 - c, x_0 + c)$, we define the **Taylor series for f centered at x_0** to be the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \tag{17}$$

where $f^{(n)}(x_0)$ denotes the n^{th} derivative of $f(x)$ evaluated at x_0.

Not all points x_0 are created equal. Our function $f(x)$ is not necessarily define at x_0, therefore we need to classify different types of x_0 points. We are working with the following type of ODE

$$y'' + p(x)y' + q(x)y = 0. \tag{18}$$

Definition 3.2. (Analytic) [2]
A function f is **analytic at a point** x_0 if and only if it has a Taylor series in powers of $(x - x_0)$ which represents it on some open interval containing x_0. A function which is analytic at every point of an interval I is said to be **analytic** on I.

Definition 3.3. (Regular point) [1]
If both $p(x)$ and $q(x)$ can be expanded as a Taylor series in the neighbourhood of $x = x_0$ then (18) it is said to have an regular point at $x = x_0$.

Definition 3.4. (Singular point) [1]
If either $p(x)$ or $q(x)$ cannot be expanded as a taylor series about a point $x = x_0$ then x_0 is said to be a singular point for the differential equation.

Definition 3.5. (Regular Singular Point) [3, p. 194]
Equation (**??**) is said to have a **regular singular point** if $(x - x_0)p(x)$ and $(x - x_0)^2q(x)$ are analytic at x_0.

We have now the tools necessary to define the power series solution for our ODE.

Theorem 3.1. (Power series solution) [3, p. 182]
If p and q are analytic at x_0, then every solution of

$$y'' + p(x)y' + q(x)y = 0 \tag{19}$$

is too, and can therefore be found in the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n. \tag{20}$$

Further, the radius of convergence of every solution (20) is at least as large as the smaller radii of convergence of the Taylor series $p|_{x_0}$ and the Taylor series $q|_{x_0}$

Definition 3.6. (Interval of Convergence of Power Series) [3, p. 178]
The power series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{21}$$

converges at $x = x_0$. If it converges at other points as well, then those points necessarily comprise an interval $|x - x_0| < R$ centered at x_0 and possibly, one or both endpoints of that interval, where R can be determined from

$$R = \frac{1}{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|} \tag{22}$$

if the limit in the denominator exist and is nonzero. If the limit in (22) is zero then (21) converges for all x (for every finite x, no matter how large), and we say that "$R = \infty$". If the limit fail to exist by virtue of being infinite, then $R = 0$ and (21) converges only at x_0.

We call $|x - x_0| < R$ the **interval of convergence**, and R the **radius of convergence**. If a power series converges to a function f on some interval, we say that it **represents** f on that interval, and we call f its **sum function**.

We will use the ratio test to study the solutions of our ODEs.

3.2 Frobenius method

The following is a summary of [3, p. 195, 199]. Let's assume that (18) has a regular singular point at $x = 0$. There is no loss of generality as if $x_0 \neq 0$ we can make a substituion $\xi = (x - x_0)$ to bring the point x_0 to the origin in terms of ξ. Multiplying (18) by x^2 and rearanging we get :

$$x^2y'' + x[xp(x)y'] + [x^2q(x)] = 0. \tag{23}$$

Using Definition (3.5) and Definition (3.2) we can represent $xp(x)$ and $x^2q(x)$ as Taylor series to get :

$$x^2y'' + x(p_0 + xp_1 + x^2p_2 + ...) + (q_0 + xq_1 + x^2q_2 + ...) = 0. \tag{24}$$

Now looking at the neighborhood of $x = 0$ we can approximate (24) as

$$x^2y'' + xp_0y' + q_0y = 0. \tag{25}$$

Which is an Cauchy-Euler equation. It means that around the point $x = 0$ our equation behave like a Cauchy-Euler equation which has solution of the form x^r. It means that there is a least one solution of the form x^r and that we should look for solutions that behave like $y(x) \sim x^r$ as $x \to 0$. Then for a general solution we want to look for a solution that behave like a power series away from $x = 0$. We therefore look for a solution of the form:

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}. \tag{26}$$

This is the first step of the **Frobenius method**. If we put (26) into (24) we get:

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + (p_0 + xp_1 + x^2p_2 + ...)\sum_{n=0}^{\infty} a_n x^{n+r} + (q_0 + xq_1 + x^2q_2 + ...)\sum_{n=0}^{\infty} a_n x^{n+r} = 0. \tag{27}$$

Gathering the powers of x for the first few n and equating them to zero we get:

$$n = 0 : x^r[r(r-1) + p_0 r + q_0]a_0 = 0 \tag{28}$$
$$n = 1 : x^{r+1}[(r+1)r + p_0(r+1) + q_0]a_1 + (p_1 r + q_1)a_0 = 0 \tag{29}$$
$$n = 2 : x^{r+2}[(r+2)(r+1) + p_0(r+2) + q_0]a_2 + ... = 0 \tag{30}$$
$$\vdots$$

Assuming $a_0 \neq 0$ we get :

$$r^2 + (p_0 - 1)r + q_0 = 0. \tag{31}$$

This polynomial equation is called the **indicial equation** and has roots r_1 and r_2. Putting $r = r_1$ in (28)-(30) gives a system of linear equation to find the values $a_1, a_2, a_3, ...$ in terms of a_0. This relation between the coefficients a_n is called the **recurrence relation**. We apply the same process for $r = r_2$ into (28)-(30) and try to find the recurrence relation. After these steps we should obtain two linearly independent solutions of (18). Similarly to the Cauchy-Euler equation, we need to be careful when there is a *repeated root* i.e. $r_1 = r_2$. We would only get one solution. For the Cauchy-Euler equation, in case of repeated root, the second solution has the form of $\ln(x)x^r$, found using reduction of order. We will get a similar solution containing an $\ln(x)$ term that we will describe in the theorem below. There exist a case, unlike Cauchy-Euler, where the two roots *differ by an integer*. The following theorem summurize the different types of solutions.

Theorem 3.2. (Fuchs' theorem) [3, p. 201]
Let $x = 0$ be a regular singular point of the differential equation

$$y'' + p(x)y' + q(x)y = 0. \tag{32}$$

with $xp(x) = p_0 + p_1 x + ...$ and $x^2 q(x) = q_0 + q_1 x + ...$ having radii of convergence R_1, R_2 respectively. Let r_1,r_2 be the roots of the indicial equation

$$r^2 + (p_0 - 1)r + q_0 = 0. \tag{33}$$

where $r_1 \geq r_2$ if the roots are real. (Otherwise they are complex conjugates.) Seeking y(x) in the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}. \tag{34}$$

with $r = r_1$ inevitably leads to a solution

$$y_1(x) = x^{r_1} \sum_{n=0}^{\infty} a_n x^n. \tag{35}$$

where $a_1, a_2, ...$ are known multiples of a_0, which remains arbitrary. For definiteness, we choose $a_0 = 1$ in (35). The form of the second linearly independent solution, $y_2(x)$, depends on r_1 and r_2 as follows:
(i) **r_1 and r_2 distinct and not differing by an integer**. (Complex conjugate roots belong to this case.) Then with r_2, (34) yields

$$y_2(x) = x^{r_2} \sum_{n=0}^{\infty} b_n x^n. \tag{36}$$

where the b_n's are generated by the same recursion relation as the a_n's, but with $r = r_2$ instead of $r = r_1$;b_1,b_2,... are known multiple of b_0, which is arbitrary. For definiteness, we choose $b_0 = 1$ in (36).
(ii) **Repeated roots, $r_1 = r_2 \equiv r$**. Then $y_2(x)$ can be found in the form

$$y_2(x) = y_1(x)\ln(x) + x^{r} \sum_{n=0}^{\infty} c_n x^n. \tag{37}$$

(iii) **$r_1 - r_2$ equal to an integer**. Then the smaller root r_2 leads to both solutions, $y_1(x)$ and $y_2(x)$, or to neither. In either case, the larger root r_1 gives the single solution (35). In the latter case, $y_2(x)$ can be found in the form

$$y_2(x) = \kappa y_1(x)\ln(x) + x^{r_2} \sum_{n=0}^{\infty} d_n x^n. \tag{38}$$

where the constant κ may turn out to be zero, in which case there is no logarithmic term in (38).

4 Spherical Bessel equation

4.1 Series solutions about $r = 1$

$$r^2 R'' + 2rR' + (r^2\omega^2 - l(l+1))R = 0. \tag{39}$$

Here $r = 1$ is a regular point as, by Definition 3.3, the Taylor series of $p(r) = \frac{2}{r}$ and $q(r) = \frac{r^2\omega^2 - l(l+1)}{r^2}$ are analytic at $r = 1$. We want to find a power series solution about the point $r_0 = 1$. We set $r = y + 1$ and plug back into (39).

$$[y^2 + 2y + 1]R'' + 2yR' + 2R' + [y^2 + 2y + 1]\omega^2 R - l(l+1)R = 0. \tag{40}$$

Looking for power series solution of the form $R = \sum_{n=0}^{\infty} a_n y^n$, with derivatives $R' = \sum_{n=0}^{\infty} a_n n y^{n-1}$,$R'' = \sum_{n=0}^{\infty} a_n n(n-1) y^{n-2}$ the last equation becomes,

$$\begin{aligned} &\sum_{n=0}^{\infty} a_n n(n-1)y^n + \sum_{n=0}^{\infty} 2a_n(n-1)y^{n-1} + \sum_{n=0}^{\infty} a_n n(n-1)y^{n-2} + \sum_{n=0}^{\infty} 2a_n n y^{n-1} + \\ &+ \sum_{n=0}^{\infty} 2a_n n y^n + \sum_{n=0}^{\infty} a_n \omega^2 y^{n+2} + \sum_{n=0}^{\infty} a_n 2\omega^2 y^{n+1} + \sum_{n=0}^{\infty} a_n(\omega^2 - l(l+1))y^n = 0 \end{aligned}. \tag{41}$$

Now, we want to gather the powers of y. We first need to shift the starting index each of the summations to match with the highest power of y i.e y^{n+2}.

$$\sum_{n=-4}^{\infty} a_{n+4}(n+4)(n+3)y^{n+2} + \sum_{n=-3}^{\infty} 2a_{n+3}(n+3)(n+2)y^{n+2} + \sum_{n=-2}^{\infty} a_{n+2}(n+2)(n+1)y^{n+2} + \sum_{n=-3}^{\infty} 2a_{n+3}(n+3)y^{n+2} + \\ + \sum_{n=-2}^{\infty} 2a_{n+2}(n+2)y^{n+2} + \sum_{n=0}^{\infty} a_n\omega^2 y^{n+2} + \sum_{n=-1}^{\infty} a_{n+1}2\omega^2 y^{n+2} + \sum_{n=-2}^{\infty} a_{n+2}(\omega^2 - l(l+1))y^{n+2} = 0 . \tag{42}$$

We can also remove the terms that equal zero for some of the summations.

$$\sum_{n=-2}^{\infty} a_{n+4}(n+4)(n+3)y^{n+2} + \sum_{n=-1}^{\infty} 2a_{n+3}(n+3)(n+2)y^{n+2} + \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)y^{n+2} + \sum_{n=-2}^{\infty} 2a_{n+3}(n+3)y^{n+2} + \\ + \sum_{n=-1}^{\infty} 2a_{n+2}(n+2)y^{n+2} + \sum_{n=0}^{\infty} a_n\omega^2 y^{n+2} + \sum_{n=-1}^{\infty} a_{n+1}2\omega^2 y^{n+2} + \sum_{n=-2}^{\infty} a_{n+2}(\omega^2 - l(l+1))y^{n+2} = 0 . \tag{43}$$

Gathering each sums starting from the same index we get

$$\begin{aligned} &\sum_{n=-2}^{\infty} [a_{n+4}(n+4)(n+3) + a_{n+3}2(n+3) + a_{n+2}\omega^2 - l(l+1)]y^{n+2} \\ &+ \sum_{n=-1}^{\infty} [a_{n+3}2(n+3)(n+2) + a_{n+2}2(n+2) + a_{n+1}2\omega^2]y^{n+2} \\ &+ \sum_{n=0}^{\infty} [a_{n+2}(n+2)(n+1) + a_n(\omega^2 - l(l+1))]y^{n+2} = 0 \end{aligned} . \tag{44}$$

Putting all the sums into one by plugging $n = -2, -1$ we get

$$\begin{aligned} &[2a_2 + 2a_1 + (\omega^2 - l(l+1))a_0] + [6a_3 + 6a_2 + (2 + \omega^2 - l(l+1))a_1 + 2\omega^2 a_0]y + \\ &\sum_{n=0}^{\infty} [a_{n+4}(n+4)(n+3) + a_{n+3}2(n+3)^2 + a_{n+2}((n+3)(n+2) + (\omega^2 - l(l+1))) + a_{n+1}2\omega^2 + a_n\omega^2]y^{n+2} = 0 \end{aligned} . \tag{45}$$

We get the following recurrence formula:

$$a_{n+4} = -\frac{a_{n+3}2(n+3)^2 + a_{n+2}((n+3)(n+2) + (\omega^2 - l(l+1))) + a_{n+1}2\omega^2 + a_n\omega^2}{(n+4)(n+3)} . \tag{46}$$

Hence, putting $n = 0, 1, 2, ...$ we get the solution

$$\begin{aligned}
R(r) = a_0 &\Big[(\omega^2 - l(l+1)) + 2\omega^2(r-1) + \frac{\omega^2}{12}(r-1)^2 + \frac{32\omega^2}{240}(r-1)^3 + ...\Big] \\
&+ a_1\Big[2 + (2 + \omega^2 - l(l+1))(r-1) + \frac{\omega^2}{6}(r-1)^2 + \frac{38\omega^2}{6}(r-1)^3 + ...\Big] \\
&+ a_2\Big[2 + 6(r-1) + \frac{(6+\omega^2 - l(l+1))}{12}(r-1)^2 + \frac{(5 + 2(\omega^2 - l(l+1)))}{20}(r-1)^3 + ...\Big] \\
&+ a_3\Big[6(r-1) + \frac{3}{2}(r-1)^2 + \frac{(52 + \omega^2 - l(l+1))}{20}(r-1)^3 + ...\Big]
\end{aligned} \tag{47}$$

with arbitrary a_3, a_2, a_1, a_0.

4.2 Frobenius solutions about $r = 0$

$$r^2R'' + 2rR' + (r^2\omega^2 - l(l+1))R = 0. \tag{48}$$

Here $r = 0$ is a regular singular point as, by Definition 3.5, $rp(r) = 2$ and $r^2q(r) = -l(l+1)$ are analytic at $r = 0$. We want to find a Frobenius solution about the point $r_0 = 0$. Using the ansatz $R = \sum_{n=0}^{\infty} a_n r^{n+k} n$, with derivatives $R' = \sum_{n=0}^{\infty} a_n(n+k)r^{n+k-1}$,$R'' = \sum_{n=0}^{\infty} a_n(n+k)(n+k-1)r^{n+k-2}$ the last equation becomes,

$$\sum_{n=0}^{\infty} a_n(n+k)(n+k-1)r^{n+k} + \sum_{n=0}^{\infty} 2a_n(n+k)r^{n+k} + \sum_{n=0}^{\infty} a_n\omega^2 r^{n+k+2} - \sum_{n=0}^{\infty} a_n l(l+1)r^{n+k} = 0. \tag{49}$$

We first need to shift the starting index each of the summations to match with the highest power of r i.e r^{n+2}.

$$\sum_{n=-2}^{\infty} a_{n+2}(n+k+2)(n+k+1)r^{n+k+2} + \sum_{n=-2}^{\infty} 2a_{n+2}(n+k+2)r^{n+k+2} + \sum_{n=0}^{\infty} a_n\omega^2 r^{n+k+2} - \sum_{n=-2}^{\infty} a_{n+2}l(l+1)r^{n+k+2} = 0. \tag{50}$$

Putting all the sums into one by plugging $n = -2, -1$ we get

$$a_0[k(k+1) - l(l+1)]r^k + a_1[(k+1)(k+2) - l(l+1)]r^{k+1} + \sum_{n=0}^{\infty} a_{n+2}[((n+k+2)(n+k+3) - l(l+1)) + a_n\omega^2]r^{n+k+2}. \tag{51}$$

Proceeding similarly as (28) to (31), we get the indicial equation

$$\begin{gathered} k(k+1) - l(l+1) = 0 \quad \text{which has roots} \\ k = l, -l-1 \end{gathered} \quad . \tag{52}$$

Our recursive formula is

$$a_{n+2} = \frac{-a_n\omega^2}{(n+k+2)(n+k+3) - l(l+1)}. \tag{53}$$

Solving for $k = l$ our recursive formula becomes

$$a_{n+2} = \frac{-a_n\omega^2}{(n+l+2)(n+l+3) - l(l+1)}. \tag{54}$$

$$R_1(r) = r^l(a_0\Big[\frac{-\omega^2}{4l+6} + \frac{\omega^4}{(4l+6)(8l+20)}r - \frac{\omega^6}{(4l+6)(8l+20)(12l+42)}r^2 + ...\Big] \\ +a_1\Big[\frac{-\omega^2}{6l+12} + \frac{\omega^4}{(6l+12)(10l+30)}r - \frac{\omega^6}{(6l+12)(10l+30)(14l+56)}r^2 + ...\Big]). \tag{55}$$

Now solving for $k = -l-1$ our recursive formula becomes

$$a_{n+2} = \frac{-a_n\omega^2}{(n+1-l)(n+2-l) - l(l+1)}. \tag{56}$$

We get a solution R_2 similar to (55)

$$R_2(r) = r^{-l-1}(a_0\Big[\frac{-\omega^2}{2-4l} + \frac{\omega^4}{(2-4l)(12-8l)}r - \frac{\omega^6}{(2-4l)(12-8l)(30-12l)}r^2 + ...\Big] \\ +a_1\Big[\frac{-\omega^2}{6-6l} + \frac{\omega^4}{(6-6l)(20-10l)}r - \frac{\omega^6}{(6-6l)(20-10l)(42-14l)}r^2 + ...\Big]). \tag{57}$$

$$R_2(r) = a_0y_3 + a_1y_4. \tag{58}$$

The final solution for $R(r)$ is the sum of the two series solution

$$R(r) = R_1(r) + R_2(r). \tag{59}$$

4.3 Convergence analysis

As the a_{n+1} term in (56) is missing, we can use Definition 3.6 to find the radius of convergence C of $R(r)$

$$C = \lim_{n\to\infty}\left|\frac{a_{n+2}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{-\omega^2}{(n+k+2)(n+k+3) - l(l+1)}\right| = 0. \tag{60}$$

By Definition 3.6, our solution converges for all r.

5 Legendre equation

5.1 Series solutions about $x = 0$

$$(1-x^2)X'' - 2xX' + (l(l+1) - \frac{m^2}{1-x^2})X = 0. \tag{61}$$

Here $x = 0$ is a regular point as, by Definition 3.3, the Taylor series of $p(x) = \frac{-2x}{(1-x^2)}$ and $q(x) = \frac{l(l+1)}{(1-x^2)} - \frac{m^2}{(1-x^2)^2}$ are analytic at $x = 0$. We want to find a power series solution about the point $x_0 = 0$. Multiplying by $(1-x^2)^2$ we get

$$(x^4 - 2x + 1)X'' - (2x^3 + 2x)X' + l(l+1)X - l(l+1)x^2X - m^2X = 0. \tag{62}$$

Looking for power series solution of the form $X = \sum_{n=0}^{\infty} a_nx^n$, with derivatives $X' = \sum_{n=0}^{\infty} a_nnx^{n-1}$,$X'' = \sum_{n=0}^{\infty} a_nn(n-1)x^{n-2}$ the last equation becomes,

$$\sum_{n=0}^{\infty} a_nn(n-1)x^{n+2} - \sum_{n=0}^{\infty} 2a_nn(n-1)x^n + \sum_{n=0}^{\infty} a_nn(n-1)x^{n-2} - \sum_{n=0}^{\infty} 2a_nnx^{n+2} - \\ \sum_{n=0}^{\infty} 2a_nnx^n + \sum_{n=0}^{\infty} a_nl(l+1)x^n - \sum_{n=0}^{\infty} a_nl(l+1)x^{n+2} - \sum_{n=0}^{\infty} a_nm^2x^n = 0. \tag{63}$$

We shift the starting index for each of the summations to match with the highest power of x i.e x^{n+2}. Gathering each sums starting from the same index we get

$$\sum_{n=-4}^{\infty} a_n[n(n-3)-l(l+1)]x^{n+2}+\sum_{n=-2}^{\infty} a_{n+2}[-2(n+2)^2+l(l+1)-m^2]x^{n+2}+\sum_{n=0}^{\infty} a_n[n(n-3)-l(l+1)]x^{n+2}=0. \quad (64)$$

Putting all the sums into one by plugging $n=-2,-1$ we get

$$\begin{aligned}&a_0(l(l+1)-m^2)+2a_2+a_1(l(l+1)-m^2-2)x+6a_3x\\&+\sum_{n=0}^{\infty}[a_{n+4}(n+4)(n+3)+a_{n+2}(-2(n+2)^2+l(l+1)-m^2)+a_n(n(n-3)-l(l+1))]x^{n+2}=0\end{aligned}. \quad (65)$$

The recursive formula is

$$a_{n+4}=\frac{-a_{n+2}[-2(n+2)^2+l(l+1)-m^2]-a_n[n(n-3)-l(l+1)]}{(n+4)(n+3)}. \quad (66)$$

Putting for $n=0,1,2,...$ we get the solution

$$\begin{aligned}X(x)=a_0&\Big[(l(l+1)-m^2)-\frac{l(l+1)}{12}x^2+\frac{l(l+1)(l(l+1)-m^2-32)}{12\cdot 30}x^4-...\Big]\\+a_1&\Big[(l(l+1)-m^2-2)x+\frac{(2+l(l+1))}{20}x^3+\frac{(l(l+1)+2)(l(l+1)-m^2-50)}{20\cdot 42}x^5+...\Big]\\+a_2&\Big[2-\frac{(l(l+1)-m^2-8)}{12}x^2+\frac{(l(l+1)-m^2-8)(l(l+1)-m^2-32)+24+12l(l+1)}{12\cdot 30}x^4-...\Big]\\+a_3&\Big[6x-\frac{(l(l+1)-m^2-18)}{20}x^3+\frac{(l(l+1)-m^2-50)(l(l+1)-m^2-18)+20l(l+1)}{20\cdot 42}-...\Big]\end{aligned}. \quad (67)$$

5.2 Frobenius solutions about $x = 1$

We start by rewriting () as

$$X''-\frac{2x}{1-x^2}X'+(\frac{l(l+1)}{1-x^2}-\frac{m^2}{(1-x^2)^2})X=0. \quad (68)$$

Here $x=1$ is a regular singular point as, by Definition 3.5, $xp(x)$ and $x^2q(x)$ are analytic at $x=1$. Looking to find the series solution around $x=1$. Setting $y:=x+1$, multiply by $(1-x^2)$ and expanding we get

$$y^4X''+4y^3X''+4y^2X''+2y^3X'+6y^2X'+4yX'+y^2l(l+1)X-2yl(l+1)X-m^2X=0. \quad (69)$$

We want to find a Frobenius solution about the point $x_0 = 0$. Using the ansatz $X = \sum_{n=0}^{\infty} a_n x^{n+k} n$, with derivatives $X' = \sum_{n=0}^{\infty} a_n (n+k) x^{n+k-1}$,$X'' = \sum_{n=0}^{\infty} a_n (n+k)(n+k-1) x^{n+k-2}$ the last equation becomes,

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n (k+n-1)(k+n) y^{n+k+2} + \sum_{n=0}^{\infty} a_n 4(k+n-1)(k+n) y^{n+k+1} \\
&+ \sum_{n=0}^{\infty} a_n 4(k+n-1)(k+n) y^{n+k} + \sum_{n=0}^{\infty} a_n 2(k+n) y^{n+k+2} + \sum_{n=0}^{\infty} a_n 6(k+n) y^{n+k+1} \\
&+ \sum_{n=0}^{\infty} a_n 4(k+n) y^{n+k} - \sum_{n=0}^{\infty} a_n l(l+1) y^{n+k+2} - \sum_{n=0}^{\infty} a_n 2l(l+1) y^{n+k+1} - \sum_{n=0}^{\infty} a_n m^2 y^{n+k} = 0
\end{aligned} \quad . \tag{70}$$

Shifingt the starting index for each of the summations to match with the highest power of x i.e x^{n+2}.

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n (k+n-1)(k+n) y^{n+k+2} + \sum_{n=-1}^{\infty} a_{n+1} 4(k+n+1)(k+n) y^{n+k+2} \\
&+ \sum_{n=-2}^{\infty} a_{n+2} 4(k+n+2)(k+n+1) y^{n+k+2} + \sum_{n=0}^{\infty} a_n 2(k+n) y^{n+k+2} + \sum_{n=-1}^{\infty} a_{n+1} 6(k+n+1) y^{n+k+2} \\
&+ \sum_{n=-2}^{\infty} a_{n+2} 4(k+n+2) y^{n+k+2} - \sum_{n=0}^{\infty} a_n l(l+1) y^{n+k+2} - \sum_{n=-1}^{\infty} a_{n+1} 2l(l+1) y^{n+k+2} - \sum_{n=-2}^{\infty} a_{n+2} m^2 y^{n+k+2} = 0
\end{aligned} \quad . \tag{71}$$

Putting all the sums into one by plugging $n = -2, -1$ we get

$$\begin{aligned}
&a_0[4k^2 - m^2] y^k + a_0[k(k+1) - 2l(l+1)] y^{k+1} + a_1[4(k+1)^2 - m^2] y^{k+1} + \\
&\sum_{n=0}^{\infty} a_n[(k+n+1)(k+n) - l(l+1)] + a_{n+1}[(k+n+1)(4(k+n)+6) - 2l(l+1)]. \\
&+ a_{n+2}[(k+n+1)(k+n) - m^2] y^{n+k+2} = 0
\end{aligned} \tag{72}$$

Proceeding similarly as (28) to (31), we get the indicial equation

$$\begin{gathered}
4k^2 - m^2 = 0 \quad \text{which has roots} \\
k = \frac{m}{2}, \frac{-m}{2}
\end{gathered} \quad . \tag{73}$$

Our recursive formula is

$$a_{n+2} = \frac{a_{n+1}[(k+n+1)(4(k+n)+6) - 2l(l+1)] + a_n[(k+n+1)(k+n) - l(l+1)]}{(k+n+1)(k+n) - m^2}. \tag{74}$$

Finding for $k_1 = \frac{m}{2}$ we get

$$a_{n+2} = \frac{a_{n+1}[(\frac{m}{2}+n+1)(4(\frac{m}{2}+n)+6) - 2l(l+1)] + a_n[(\frac{m}{2}+n+1)(\frac{m}{2}+n) - l(l+1)]}{(\frac{m}{2}+n+1)(\frac{m}{2}+n) - m^2}. \tag{75}$$

Hence, the first few terms of the solution for $k_1 = \frac{m}{2}$ are

$$a_2 = \frac{a_1[(\frac{m}{2}+1)(2m+6) - 2l(l+1)] + a_0[(\frac{m}{2}+1)\frac{m}{2} - l(l+1)]}{(\frac{m}{2}+1)\frac{m}{2} - m^2}. \tag{76}$$

$$a_3 = \frac{a_2[(\frac{m}{2}+2)(2m+10) - 2l(l+1)] + a_1[(\frac{m}{2}+2)(\frac{m}{2}+1) - l(l+1)]}{(\frac{m}{2}+2)(\frac{m}{2}+1) - m^2}. \tag{77}$$

$$a_4 = \frac{a_3[(\frac{m}{2}+3)(2m+14) - 2l(l+1)] + a_2[(\frac{m}{2}+3)(\frac{m}{2}+2) - l(l+1)]}{(\frac{m}{2}+3)(\frac{m}{2}+2) - m^2}. \tag{78}$$

We set m as an integer in Section 2. By Theorem 3.2, the second solution takes the form

$$X_2(x) = \kappa X_1(x)\ln(x-1) + (x-1)^{-\frac{m}{2}} \sum_{n=0}^{\infty} d_n (x-1)^n. \tag{79}$$

Taking the derivatives of y_2 and plugging back in (14) we get

$$\kappa \ln(x-1)\left[\underbrace{(1-x^2)X_1'' - 2xX_1' + (l(l+1) - \frac{m^2}{(1-x^2)})X_1}_{=0}\right] + F(X_1', X_1, x)\sum_{n=0}^{\infty} b_n (x-1)^{n-\frac{m}{2}} = 0. \tag{80}$$

We can compare the coefficent b_n to the coefficient of $F(X_1', X_1, x)$ to get the second solution X_2.

5.3 Value of λ

The following argument is a summary of [3, p. 212, 213]. Starting from equation (14) with the special case $m = 0$ we get the Legendre equation:

$$X'' - \frac{2x}{1-x^2}X' + \frac{\lambda}{1-x^2}X = 0. \tag{81}$$

Taking the power series solution at the regular point $x = 0$ we get the recursive formula:

$$a_{k+2} = \frac{k(k+1) - \lambda}{(k+1)(k+2)} a_k. \tag{82}$$

Setting $k = 0, 1, 2, ...$ we get the general solution:

$$\begin{aligned} X(x) =& a_0\left[1 - \frac{\lambda}{2}x^2 - \frac{(6-\lambda)\lambda}{4!}x^4 - \frac{(20-\lambda)(6-\lambda)\lambda}{6!} - ...\right] \\ &+ a_1\left[x + \frac{2-\lambda}{3!}x^3 + \frac{(12-\lambda)(2-\lambda)}{5!}x^5 + ...\right] \\ =& a_0 y_1(x) + a_1 y_2(x) \end{aligned}. \tag{83}$$

Looking at the radius of convergence of the general solution (83), we notice that a_{k+2} is really the next coefficient after a_k. Using Definition 3.6, we have

$$C = \lim_{n\to\infty}\left|\frac{a_{n+2}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{k(k+1)-\lambda}{(k+1)(k+2)}\right| = 1. \tag{84}$$

Hence the radius of convergence is 1, so the series converges in $-1 < x < 1$.

The Legendre equation is used a lot in physical application (electromagnetism, temperature distribution,...). In these topics, one would use a bounded solution of (83).

Definition 5.1. (Bounded)
Let I be an interval. If there exists a finite constant M such that

$$|F(x)| \leq M \qquad \text{for all } x \text{ in } I. \tag{85}$$

then $F(x)$ is **bounded**. If that is not the case $F(x)$ is **unbounded**.

Our solution (83), however, increases and decreases unboundedly as $x \to \pm 1$. As seen in Figure 1, our solution goes to $\pm\infty$ as $x \to +1$ and to $-\infty$ as $x \to -1$. There is no M that satisfies Definition 5.1.
Now looking at the regular singular points $x = \pm 1$, we look at the Frobenius solution about $x = 1$. As seen in the previous section, a Frobenius solution about $x = 1$ contain an $\ln(x-1)$ term. This term reveal the singular nature of the solution as $x \to 1$. In our Frobenius solution about $x = 0$, the $\ln(x-1)$ term is "hidden", so one cannot see explicitely that the series diverges as $x \to 1$.

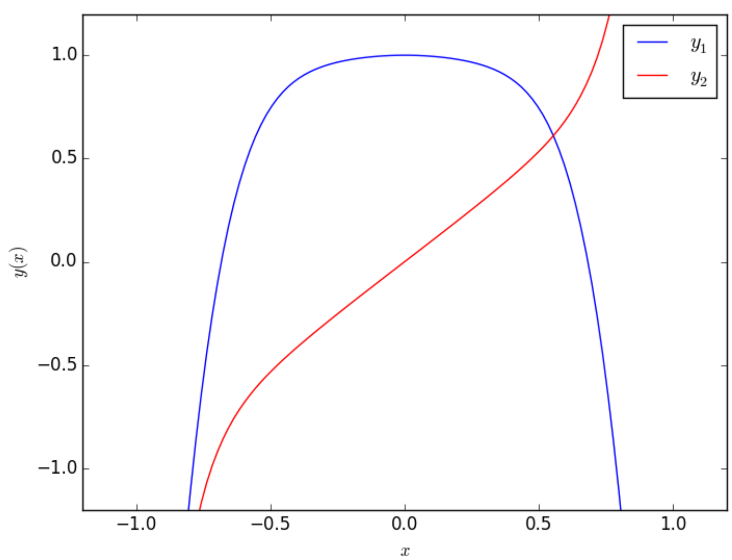

Figure 1: y_1 and y_2 from (83) for $\lambda = 1$

To avoid this, we want our solution to be bounded on $-1 < x < 1$ so the infinite series needs to terminate. For certain value of λ the series will be bounded. Setting $\lambda = l(l+1)$ for $l = 0, 1, 2, ...$ will terminate the series (82) at $k = l$. For example, at $l = 2$ our λ equal 6 and the series $y_1(x)$ stops at $1 - 3x^2$ as all the other terms of the series become zero, i.e. $(6 - \lambda)$ cancels the rest of the series. Hence, if l is even, then the even powered series terminates at $k = l$ as $a_{n+2} = a_{n+4} = ... = 0$ and if l is odd, the odd powered series terminates.
Equation (13) and (14) now share the same separation constant $\lambda = l(l+1)$ when regularity of the solutions is considered. [3]

6 Conclusion

The Frobenius method is a powerful tool to find solutions to a broad type of ODE. Behaving like Euler equations, the Frobenius solutions have different forms directed by the nature of the roots of the indicial equation. We used this method to find solutions of the spherical Bessel equation at $r = 0, 1$ and solutions of the associated Legendre equation at $x = 0, 1$. The Bessel and Legendre equation arise when using separation of variables for the wave equation in spherical coordinates. They are used in many application in physics, area where using bounded solutions can be required. Hence, we showed that setting $\lambda = l(l+1)$ for $l = 0, 1, 2, ..$ satisfy the bounded condition.

References

[1] A.C. Bajpai, L.R. Mustoe, and D. Walker. *Advanced engineering mathematics*. Wiley, 1990, p. 170. ISBN: 9780471925958.

[2] Wylie C. Ray and Barrett Louis C. *Advanced Engineering Mathematics 6th ed.* Ed. by McGraw Hill. 1995, p. 128. ISBN: 0-07-072206-4.

[3] Michael D. Greenberg. *Advanced Engineering Mathematics 2nd ed.* Ed. by Prentice Hall. 1998, pp. 178, 182, 194, 195, 199, 201, 212, 213. ISBN: 0-13-321431.

[4] Vesna Perisic. *MATH2039 Analysis*. University of Southampton, 2018.