Markov Melodies

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Abstract

We assess whether finite-state Markov chains can effectively simulate Western music by creating Markov models and simulating their outputs.

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1 Introduction

Music is a central element to human culture and society. Different societies have developed different styles and types of music [1]. The style most prevalent in the history of Western civilization is based on twelve fundamental notes. Anyone habituated to this style of music is also habituated to the underlying theory behind the music, which is the first part of the paper.

In the second part of the paper, we investigate how perception of the music changes when randomness is heavily involved in its composition. Generally, Western music is created by composers who follow certain rules when composing, and although they have many degrees of freedom, their choices are deliberate, based on the mood and style they want the piece to achieve. Consequently, detailed analysis of a musical composition created in this way is possible, and examples abound (see [2]). We use these rules to establish the random pattern (i.e. a Markov chain), but do nothing further to the composition.

We do this both directly and indirectly. The direct approach involves estimating the transition probabilities based on the rules mentioned above; the indirect approach involves computing the transition probabilities from songs (which, of course, are written using these rules).

2 The Western Theory of Music

Though styles within Western music (e.g. classical, rock, jazz) are diverse, they all share a common footing in *music theory*, the study of the twelve-note system of music from which most pieces composed now, and historically, have been constructed. The notes are: A, A♯ (equivalent to B♭), B, C, C♯ (equivalent to D♭), D, D♯ (equivalent to E♭), E, F, F♯ (equivalent to G♭), G, and G♯ (equivalent to A♭); these notes form the "chromatic scale". The "natural notes" (the white keys on a piano) are A, B, C, D, E, F, G. The ♯ symbol is "sharp", which represents a note one semitone higher (to be explained shortly) than the natural note it is written next to, and the ♭ symbol is "flat", which represents a note one semitone lower than the natural note it is written next to.

The *interval* is a crucial component of Western music. The most basic type of interval is a *tone*; it represents the musical distance between two adjacent white keys on a piano[1], or equivalently, the distance between two adjacent natural notes (e.g. A, B and G, A). A *semitone* is half a tone; it is the musical distance between adjacent white and black keys (as well as the musical distance between the natural notes E and F, and B and C).

[1] Except for the interval between E and F, and B and C.

Of course, on a piano, there are more than twelve keys. There is another important interval, the *octave*, which allows for the expansion of the basic twelve notes. At this point, it is important to explain exactly what the notes mean. Music is sound, and sound is composed of waves; these waves oscillate at certain frequencies. One can think of frequencies as the central invariant in music. The frequency a musical note is oscillating at is called its *pitch*. An oscillation of 440 Hz produced by a guitar, by definition, has the same pitch as an oscillation of 440 Hz produced by a piano. The reason they sound different to humans is beyond the scope of this explanation; it has to do with other qualities, the most important of which are timbre and intensity [1].

The octave is the interval between two notes whose ratio of frequency is 2. For example, a note at 440 Hz is one octave below a note at 880 Hz. Symbolically, these notes are equivalent. In particular, a note at 440 Hz is an A, and a note at 880 Hz is also an A. However, the A (880 Hz) sounds different to the human ear, and even an untrained ear could recognize it sounds higher than the 440 Hz note. The octave enables musicians to work with many more notes than the twelve that were presented initially. In principle, this means the notes can go from the lower range of human hearing (around 20 Hz, depending on age and hearing conditions) to the upper range (around 20 kHz, again depending on age and hearing conditions) [1]. Because the notes repeat themselves periodically, the chromatic scale is often represented as a circle instead of a line (Figure 1).

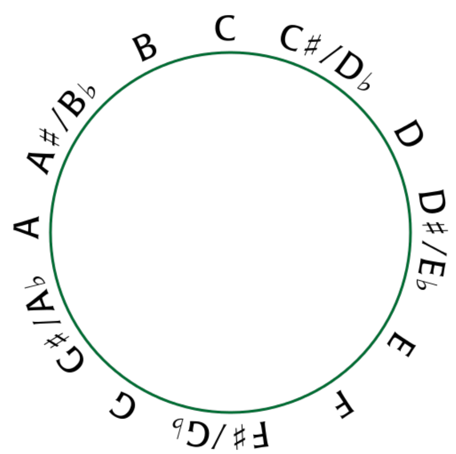

Figure 1: The chromatic scale.

In general, a scale is just a collection of notes; the chromatic scale is notable because it contains all the notes in Western 12-tone equal temperament. But suppose you wanted to create a composition using just the notes of the chromatic scale (within one octave), and you did this by sampling the notes randomly. You would produce a rather chaotic-sounding piece, much like a child striking notes for fun.[2] It turns out that certain arrangements of notes sound much better than others, such that playing notes randomly within these arrangements sounds much more pleasant. Moreover, certain patterns within these arrangements produce very organized sounds; musicians use these patterns to compose.

There are two such arrangements most relevant to this project: the *major scale* and the *minor scale*. Each of the twelve notes in the chromatic scale has a major and minor scale. The major scale is produced as follows: whole step, whole step, half step, whole step, whole step, whole step, half step [4]. (Notation: a whole step is the musical distance equivalent to a tone, and a half step is the musical distance equivalent to the semitone). In other words, we take the root note, or *tonic*, and apply this pattern. As an example, the C major scale is constructed as follows: C + whole step = D; D + whole step = E; E + half step = F; F + whole step = G; G + whole step = A; A + whole step = B; B + half step = C. Therefore, the C major scale is C, D, E, F, G, A, B, C. It is important to note that the order of the scale is important; although A, B, C, D, E, F, G, A is equivalent in the sense that it is a permutation of the original scale, with a second A added as the octave, it is *not* the C major scale. (In fact, as we will soon see, the scale we have just constructed is the A minor scale).

The reason why the order is important lies in the distance each note is from the tonic. Again, in the C major scale, C is the tonic note. Each note is defined by this distance; for instance, B is referred to as the *seventh* (the major seventh, to be exact), because if you numbered the notes C-B from 1-7, B would be 7. Similarly, A is the sixth, G is the fifth, and so on. Each interval has a particular "feel" to it. The interval between the seventh and the tonic is a semitone, and semitones produce tension (the theme from the movie "Jaws" is built around the semitone). On the other hand, the interval between the fifth and the tonic produces resolution; if you alternate playing the tonic and the fifth, you get a "happy" sound. In general, the intervals in the major scale sound happy. A large portion of popular music is constructed from major scales, from "Hey Jude" by the Beatles to "Shake It Off" by Taylor Swift.

The minor scale, on the other hand, is often described as producing "sad" music. It is constructed by the following formula: whole step, half step, whole step, whole step, half step, whole step, whole step. If we use this formula to construct the C minor scale, we get the notes C, D, E♭, F, G, A♭, B♭, C. Note that this is the C major scale, with the third, sixth, and seventh lowered by a half step. One simple interpretation, then, is that the C minor scale is a "several semitones off" the C major scale; and since semitones produce tension, we end up with a more dissonant set of notes. (This is not entirely true, because where the semitone differences are located is crucial.)

[2] However, there is a style known as atonal music that uses the chromatic scale in an organized fashion; notable composers in this school include Schoenberg [3].

When we construct the A minor scale using the above formula, we obtain A, B, C, D, E, F, G, A. As noted earlier, we obtain the same notes as the C major scale, but in a different order. Because the notes in these two scales are the same, A minor is referred to as the *relative minor* of C major, and C major is referred to as the *relative major* of A minor. By simply changing the distance between the notes and the tonic, we have an entirely different mood produced by the same notes.

3 Markov Processes

Earlier, we mentioned that a naïve approach to random music generation might be to pick notes from the chromatic scale uniformly at random. It is clear that this would sound rather unmotivated and chaotic, so we now lay the foundation for a more sophisticated model.

Definition *A* Markov process *on a random variable X_t is a stochastic process where knowledge of X_u for $u < t$ does not change the expected behavior of X_s, where $s > t$. In other words, the future state of the system depends only on the current state of the system.*

The particular type of Markov process used in the current study is a *discrete-time Markov chain.* Applied to this class of Markov processes, i.e. for a discrete random variable X_t, over discrete time steps $t \in [0, T]$ the above definition becomes:

$$\Pr[X_{n+1} = k | X_n = a_n, X_{n-1} = a_{n-1}, ..., X_0 = a_0] = \Pr[X_{n+1} = k | X_n = a_n] \tag{1}$$

Equation (1) is known as the Markov property. The Markov property enables a discrete-time Markov process to be conveniently modeled by a matrix, where the row i denotes the current state X_n and the column j denotes the future state X_{n+1}, for $i, j = 0, 1, 2, ...$.

Definition *The* transition probability P_{ij} *of a discrete-time Markov chain is the probability that the system, currently in state i at time n, will transition to state j at time $n + 1$.*

Using the above definition, a matrix $\mathbf{P}$ of transition probabilities can be constructed:

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & \cdots \\ P_{10} & P_{11} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Because the ith row is a probability distribution of the state X_{n+1}, the sum of all its entries must be equal to 1. We are interested in the future behavior of the system, i.e. the transition probabilities for all n. We would like a general result so we can find the probability of being in any state at any time step. In the second time step, the probability of being in state 0 given that we started in state 0 is the dot product $P_{0k} \cdot P_{k0}$, or the first row of $\mathbf{P}$ times the first column of $\mathbf{P}$. (We hop from state 0 to state k, and then back to 0 from state k). In general, the probability of being in state m given that we started in state r is the dot product $P_{rk} \cdot P_{km}$. If we iterate over all initial states r and final states m, we recognize the result as matrix multiplication. Therefore, the transition probabilities at state n are given by $\mathbf{P}^n$ [5].

Another useful operation that can be performed on $\mathbf{P}$ is left-multiplication by a row vector. Since rows represent probability distributions, the ith element of the row, $i = 0, 1, 2, ...$ gives the probability of being in the ith state. We can use this to specify an initial condition, or any other distribution after time 1. We can also find a convergent state of the Markov chain.

Definition *The* stationary-state *of a discrete-time Markov chain is the vector of probailities π such that $\pi\boldsymbol{P}$=π.*

Proof Any matrix $\mathbf{P}$ admitting a diagonalization can be written as $\mathbf{P} = VDV^{-1}$, where D is diagonal matrix of eigenvalues. It is well known that $\mathbf{P}^n = VD^nV^{-1}$. Therefore, we can see that as $n \longrightarrow \infty$, $\mathbf{P}$ can only settle down to a stationary state if it has at least one eigenvalue equal to one (if all eigenvalues are less than one, D^n, and consequently P^n, go to zero). By definition, the stationary distribution $\pi = \begin{bmatrix}\pi_0 & ... & \pi_N\end{bmatrix}$ is the vector of probabilities π_k that the system will be in state k in the long run. Taking the limit as $n \longrightarrow \infty$ of $\mathbf{P}^n_{ij} = \mathbf{P}^{n-1}\mathbf{P}$, we obtain π on the left, by definition, and π on the right from $\mathbf{P}^{n-1}$ being approximately equal to $\mathbf{P}^n$ for large values of n. So $\pi = \pi\mathbf{P}$, as claimed. $\square$

The next theorem will be useful when we solve for the stationary states.

Theorem *The left eigenvectors of a matrix are contained in the inverse transpose of the matrix of right eigenvectors.*

Proof The right eigenvectors are the v such that $Av = \lambda v$. Taking transposes, we obtain $v^TA^T = \lambda v^T$; since we are assuming A has eigenvectors, it can be diagonalized as $A = VDV^{-1}$, so the above equation can be written as $v^T(V^{-1})^TDV^T = \lambda v^T$. The left eigenvectors are the v' such that $v'A = \lambda' v'$, where v' is a row vector. Note that the diagonal matrix of A equals the diagonal matrix of A^T, so $\lambda = \lambda'$. We have obtained an equation of the form $v'V'DV^{-1\prime} = \lambda v^T$; comparing terms, we see that V', the matrix of left eigenvectors, equals $(V^{-1})^T$, or the inverse tranpose of the matrix containing the right eigenvectors. $\square$

4 A Music-Theory Derived Markov Model

We investigated whether a musical composition obeying the Markov property can produce a pleasant-sounding melody. Recall from music theory that a good melody makes use of the intervals between notes, in particular the intervals between the tonic and each other note in the scale. The Markov property will force each note to be generated only as a function of the note before it, without any consideration of the tonic (unless the tonic is the preceding note).

First, we generated a Markov matrix for a major and minor scale such that transition has equal frequency (i.e. $P_{ij} = 1/8$ for all i, j). This is our "control", to see if common composition practices (translating to non-uniform probabilities) generate more familiar-sound music.

Next, we will used a major scale and a minor scale to generate melodies based on the rules of music theory; since each scale contains eight notes, the transition matrix will have dimension 8 x 8. Because this matrix is music-theory derived, certain intuitions from music theory were used to generate the transition probabilities. Enumerating all 64 is tedious, so a few examples of the reasoning will be given for each scale. Note that if the transition probability is greater than 1/8, the transition is desirable from a music theory standpoint; if the transition probability is less than this, then the transition is less desirable from a music theory standpoint. The numerals I, II, III, ..., VIII refer to the first, second, etc note (also known as "degrees") of the scale. We use the scale degrees because the note names themselves (A, B, C, etc) are not important; only the intervals between notes are important. Thus, these transition probabilities apply equally well to the G major or C major or any other major scale, and similarily for any minor scale.

Finally, we went ahead and decided to make a Markov matrix that will produce the most dissonant music possible. Dissonance is a type of tension that results when dissonant notes are played together or in sequence. For example, the interval between B and F is colloquially known as the "devil's interval" because it sounds so unsettling; it is so disturbing (both from its sound and from more technical music theory considerations) that it was forbidden by some Medieval composers [6]. Semitones are also a major source of dissonance. If we wanted to produce maximum dissonance, we would draw notes at random from the chromatic scale; there is no true tonic in the chromatic scale, so the piece would be all over the place. However, we restrict ourselves to the major scale.

Transition Probabilities — Major Scale

Transition	Probability	Reasoning
I - V	1/5	This is the *major fifth* interval, which is very commonly used.
V - I	1/5	See above.
VII - VIII	1/2	The transition between the VII and the I resolves tension.
I - VII	1/30	This transition produces tension.
IV - V	1/5	The IV commonly transitions to the V or I.
IV - I	1/4	See above.

The full matrix is located in the appendix.

Transition Probabilities — Minor Scale

Transition	Probability	Reasoning
I - III	1/4	This is the *minor third* interval, one of the intervals that gives the minor scale its melancholy flavor.
V - I	1/5	Even though we're in the minor scale, this is still the *major fifth* interval (seven semitones).
I-VI	1/4	Minor sixth interval — another interval that gives the minor scale its distinct character.
V - VI	1/4	Now that we're in the minor scale, producing tension via semitone transitions is desirable.
VII - VIII	1/10	Going up on tone is now less desirable than going up one semitone.
V - VII	1/5	Minor third interval.

The full matrix is located in the appendix.

Transition Probabilities — Chaotic Major Scale

Transition	Probability	Reasoning
I-V	1/20	Our goal is minimize harmonious sounds, and this is perhaps the most harmonious interval.
IV - VII	3/5	The infamous tritone.
I-VII	3/5	Major sevenths produce dissonance.
VII - I	1/20	Resolving to the tonic removes tension, and we don't want that.
II - IV	3/5	Minor third interval.
I - VIII	1/20	Playing an octave is a common motif used to make the music sound more upbeat.

The full matrix is located in the appendix.

5 A Data-Driven Model

The music-theory derived model is a useful one. In the next section, we will discuss the efficacy of this model in generating new musical melodies. Before we do that, however, we will develop a different model.

There are at least three flaws of the music-theory derived model.

- It lacks specificity. It uses music theory to merely make guesses, and thus lacks the specificity to emulate a particular genre or style.
- It is difficult to generalize. If we wanted to create a similar transition model for a different musical feature, there might not exist agreed-upon music theory to inform our decisions about transition probabilities for that feature. One such feature we may be interested in is the duration of musical notes, which is not taken into account in the music-theory model (since, unlike the sequences of notes, no general structure exists for rhythm).
- It is hard to expand the number of states. There are only so many transition probabilities that a human is capable of considering. Even encoding a Markov model with just 8 states forces us to consider 64 different transition probabilities. If we wished to consider more, this number would grow astoundingly large (as the square of the number of states).

We propose an adaptation of our previous model in an attempt to solve these issues. From this point onward, we refer to this model as the data-driven model.

For the data-driven model, we require a particular sequence of notes. In theory, this can obtained from a musical score or even simply listening to the song. In practice, it is easier to have a computer gather this information from a MIDI file. MIDI, which stands for Musical Instrument Digital Interface, is a standardized protocol for communicating and storing sequences of musical notes.

To build our model, we consider the pitches of consecutive notes in a MIDI file. Let X_i be the random variable representing the sequence of consecutive notes in a particular song. From the sequence of notes, we can compute the empirical Markov probability matrix,

$$P_{ij} = \frac{1}{T} \sum_{t=1}^{T} \mathbf{1} \left\{ X_t = i \cap X_{t+1} = j \right\},$$

where T is the total number of notes in our sequence, and $\mathbf{1}$ denotes an indicator random variable.

By relying on data obtained from other musical compositions, we can develop a transition matrix with no prior assumptions from music theory.

This new model solves the specificity problem. Rather than being an approximation of the transition probabilities of an entire system (all of Western music), it approximates the transition probabilities of a single song, written by a particular artist. This means that sequences generated using this transition matrix are likely to sound musically similar to the source.

Furthermore, it allows for easy generalization. When extracting data from MIDI, we considered the set of states to be all 128 possible midi pitches, ranging from C_{-1} to G_9, rather than just a single octave.

But why model pitch alone? This model is easily extendable to any musical feature that can be represented as a state. The next most obvious feature is note-duration, which can also be posed as a Markov problem.

In the context of note-duration, the Markov matrix P_{ij} describes the probability of going from a note of duration i quarter notes to one with j quarter notes.

The following process details how we used two features to generate a musical melody:

1. Using a MIDI file containing an existing piece of music, obtain two transition matrices: one for note pitch and one for note duration.
2. Generate a sequence X_i of note pitches from the first transition matrix.
3. Generate a sequence Y_i of note durations (in quarter notes) from the second transition matrix.
4. The pitch-duration pairs (X_i, Y_i) describe the melody. Encode them into a MIDI file for listening.

We ran this model on several musical pieces. The results are described below.

6 Results

6.1 Music-Theory Derived Model

We begin with the uniform major scale case, which is named **uniform_major_random_music.mp3**. As expected, the note sequence itself sounds pleasant, since all the notes are drawn from the C major scale. However, the piece overall sounds meandering, the musical equivalent of a run-on sentence. The uniform case illustrates the importance of musical compositions resolving to the tonic every once in a while, to establish a "home" for the piece. A similar story can be told for the uniform minor scale case (named **uniform_minor_random_music.mp3**. In fact, it does not sound all that different from the major scale case (perhaps slightly more mysterious).

Next, we consider the music-theory derived major scale case, which is named **major_random_music.mp3**. Here, the transition probabilities are not uniform, but depend on the location of the note relative to the tonic. Since the tonic has a much higher probability of appearing, this sample is generally more satisfying than the random case. The tension in the composition is resolved somewhat frequently, but randomly; it still contains a lot of tension. It should be noted that an initial condition was applied to make the composition start on the note C.

How often does the tonic appear in the long run? We use the theorem above about the stationary states of the Markov matrix:

$$\pi \mathbf{P} = \pi$$

With immense computational power at our disposal, we don't solve this by hand. Instead we diagonalize $\mathbf{P}$:

$$\mathbf{P} = VDV^{-1}$$

We now invoke the earlier theorem about left eigenvectors. The left eigenvectors are contained as columns in $(V^{-1})^T$. We are interested in only one vector, since π must be unique. How do we know which vector? The general left eigenvector problem is $vA = \lambda v$; clearly $\lambda = 1$ for us, so the column j of $(V^{-1})^T$ that contains π is the column j of D that contains the eigenvalue 1. Computation reveals

$$\pi = \begin{bmatrix} 0.199 & 0.116 & 0.107 & 0.121 & 0.192 & 0.084 & 0.061 & 0.120 \end{bmatrix}$$

The tonic appears about one in five notes, and, along with the fifth, appears most often. Our finding accords with experience: the tonic and the major fifth generally appear frequently in major scale compositions, because this interval gives the major scale much of its character. The major seventh, on the other hand, comes up in about 1/20 notes, which also helps establish the character of the major scale because the main function of the major seventh in this scale is to be to briefly build tension and resolve it quickly.

We move on to the music-theory derived minor scale case (named **minor_random_music.mp3**). It sounds more melancholy than the uniform minor scale case, but not to the extent that most minor scale compositions achieve. Just like the major scale case, the piece sounds wandering. The stationary distribution is

$$\pi = \begin{bmatrix} 0.161 & 0.111 & 0.146 & 0.131 & 0.146 & 0.149 & 0.104 & 0.053 \end{bmatrix}$$

We note the tonic appears more frequently than the uniform case. The other notes appearing frequently are the minor third, major fifth, and minor sixth. This is exactly what we want in a minor scale, so the stationary distribution computation confirms our intuition.

Finally, the dissonant major scale case (named **chaotic_major_music.mp3**) doesn't really sounds like anything; it lacks character. That's because the tonic doesn't show up often. Applying the above stationary distribution computation, we obtain

$$\pi = \begin{bmatrix} 0.086 & 0.021 & 0.183 & 0.275 & 0.085 & 0.026 & 0.297 & 0.026 \end{bmatrix}$$

The most frequently occurring notes are B and F — the tritone interval. By comparison, the tonic shows up less than one-tenth of the time. The home for this composition, then, is the tritone interval, but the reason why the tritone interval is so dissonant is that it lacks a harmonic center; the B wants to be a C, and the F wants to be an F$\sharp$. Thus, this composition is centered around the most unstable interval in music, causing it to be devoid of musical character.

6.2 Data-Driven Model

We first trained this model on a piece of classical music known as "Fantasia," by the composer Telemann. This piece features prominent major-scale melodies and is meant to be the analogue to the first music-theory derived model that we included.

Sequences generated using this model were still in some sense wandering, as were the sequences from the music-theory derived model. Despite this, the generated sequences of notes were often pleasing melodies which shared a strong resemblance with the original Telemann piece. One of these generated sequences, called **telemann_generated.mp3**), has been included for listening.

We also trained the model on Chopin's "Prelude in E Minor Op. 28 No. 4" and Coltrane's "Giant Steps".

Although analyzing the stationary distribution on all the states is less useful in this case than in the previous (since the number of states is now much larger), we can still approximate something much like the stationary distribution.

After allowing each of these models to generate sequences of 10,000 notes, we obtained the following distribution of pitch classes:

Here we note several things. The Chopin piece remains in E minor, never using notes such as C# or F because they do not fall in the key of E minor (and they were never used in the original piece). Even if we did not know what key the Telemann piece was in, we would likely guess D major, since D is the most

Pitch Class	**Frequency in Telemann Transition Model**	**Frequency in Chopin Transition Model**	**Frequency in Coltraine Transition Model**
C	0.0135	0.1316	0.0755
C#	0.0912	0	0.0435
D	0.1749	0.0160	0.1119
D#	0.0107	0.0099	0.0998
E	0.1243	0.0909	0.0389
F	0.0018	0	0.0736
F#	0.1401	0.1584	0.0749
G	0.1046	0.0320	0.1233
G#	0.0246	0.0168	0.0847
A	0.1578	0.1960	0.0489
A#	0.0193	0.0747	0.1261
B	0.1373	0.2738	0.0987

played pitch class. In addition, notes such as F, which do not belong in D major, are rarely ever played in the generated sequences.

Finally, looking at the Coltrane song, we see that although there are notes that are played more frequently, there is no clear key center as in the previous two examples. In this case, the distribution is much closer to a uniform distribution than the other cases, sometimes giving an atonal feel in the generated melodies.

7 Appendix

7.1 The Major Scale Markov Matrix

The P_{ij} entry of the matrix corresponds to the probability of transition from scale degree $i+1$ to scale degree $j+1$. (We follow the convention that i denotes row number and j denotes column number, $i, j = 0, 1, 2, ..., 7$) For instance, if we are in C major, the P_{04} entry corresponds to the probability of going to G given the current note is C. The matrix is generated by first filling in the probabilities for the most common transitions, and then using the fact that all rows must sum to 1 to fill in the rest.

$$\begin{bmatrix} 1/5 & 1/10 & 1/10 & 1/5 & 1/5 & 1/10 & 1/30 & 1/15 \\ 1/4 & 1/8 & 11/128 & 11/128 & 1/4 & 1/8 & 1/64 & 1/16 \\ 1/6 & 1/16 & 1/16 & 1/5 & 1/5 & 37/480 & 37/240 & 37/480 \\ 1/4 & 1/8 & 1/8 & 1/16 & 1/5 & 23/160 & 1/64 & 5/64 \\ 1/5 & 1/6 & 1/8 & 1/16 & 1/6 & 61/1680 & 1/7 & 1/10 \\ 1/7 & 1/6 & 1/7 & 1/10 & 1/6 & 1/7 & 4/105 & 1/10 \\ 1/8 & 1/32 & 1/8 & 1/32 & 1/8 & 1/32 & 1/32 & 1/2 \\ 1/5 & 1/10 & 1/10 & 1/6 & 1/5 & 1/45 & 1/90 & 1/5 \end{bmatrix}$$

7.2 The Minor Scale Markov Matrix

The notation here is the same as above, except with the minor scale. So, if we are in A minor, the P_{04} entry corresponds to the probability of going to a D given the current note is an A.

$$\begin{bmatrix} 1/10 & 1/20 & 1/4 & 1/10 & 1/10 & 1/4 & 1/20 & 1/10 \\ 1/6 & 1/10 & 1/4 & 1/5 & 1/6 & 1/50 & 7/100 & 2/75 \\ 1/5 & 1/4 & 1/10 & 1/10 & 1/10 & 1/8 & 1/10 & 1/40 \\ 1/8 & 1/5 & 1/8 & 1/10 & 1/8 & 1/5 & 1/10 & 1/40 \\ 1/5 & 1/14 & 1/14 & 1/12 & 1/10 & 1/4 & 1/5 & 1/42 \\ 1/10 & 1/50 & 1/8 & 1/4 & 1/4 & 1/10 & 21/200 & 1/20 \\ 1/10 & 1/8 & 1/8 & 1/8 & 1/4 & 3/40 & 1/10 & 1/10 \\ 1/2 & 1/20 & 1/10 & 1/20 & 1/20 & 1/20 & 1/10 & 1/10 \end{bmatrix}$$

7.3 The Chaotic Major Scale Markov Matrix

The notation here is the same as the notation for the major scale matrix above.

$$\begin{bmatrix} 1/20 & 1/20 & 1/20 & 1/20 & 1/20 & 1/10 & 3/5 & 1/20 \\ 0 & 1/20 & 1/20 & 3/5 & 1/20 & 1/20 & 1/5 & 0 \\ 0 & 1/20 & 1/20 & 2/5 & 2/5 & 1/20 & 1/20 & 0 \\ 0 & 0 & 2/5 & 0 & 0 & 0 & 3/5 & 0 \\ 1/20 & 1/20 & 13/20 & 1/20 & 1/20 & 1/20 & 1/20 & 1/20 \\ 13/20 & 1/20 & 1/20 & 1/20 & 1/20 & 1/20 & 1/20 & 1/20 \\ 1/5 & 0 & 0 & 3/5 & 0 & 0 & 3/20 & 1/20 \\ 1/20 & 1/20 & 1/20 & 1/20 & 1/20 & 1/20 & 13/20 & 1/20 \end{bmatrix}$$

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