FiveThirtyEight Riddler 05-15-2020

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The below is a response to the FiveThirtyEight Riddler Classic for the week of 05-15-2020:

Let R_i denote the ith dice roll.

First, we consider the advantage of disadvantage roll. Let $M_{ij} = \min(R_i, R_j)$. Then let $X = \max(M_{12}, M_{34})$. X is the result of an advantage of disadvantage roll. First, we find the distribution of M_{12}. Observe that the probability of M_{12} being greater than some k is the same as the probabilty that both R_1 and R_2 are greater than k. So,

$$\begin{aligned}
P(M_{12} > k) &= P(R_1 > k, R_2 > k) \\
&= \sum_{i=k+1}^{20} \sum_{j=k+1}^{20} P(R_1 = i, R_2 = j) \\
&= \sum_{i=k+1}^{20} \sum_{j=k+1}^{20} P(R_1 = i)P(R_2 = j) \text{ dice rolls are independent} \\
&= \sum_{i=k+1}^{20} \sum_{j=k+1}^{20} \left(\frac{1}{20}\right)^2 \\
&= 1 - \frac{1}{10}k + \frac{k^2}{400}
\end{aligned}$$

Taking the complement yields

$$P(M_{12} \leq k) = 1 - P(M_{12} > k) = \frac{1}{10}k - \frac{k^2}{400}$$

Then we can compute $P(X \leq k)$, which is the same as both M_{12} and M_{34} being less than k. This yields,

$$\begin{aligned}
P(X \leq k) &= P(M_{12} \leq k, M_{34} \leq k) \\
&= P(M_{12} \leq k)P(M_{34} \leq k) \\
&= \left(\frac{1}{10}k - \frac{1}{400}k^2\right)^2 \\
&= \frac{1}{100}k^2 - \frac{1}{2000}k^3 + \frac{1}{160000}k^4
\end{aligned}$$

We can then compute $E[X]$ with the tail sum expectation formula, so,

$$\begin{aligned}
E[X] &= \sum_{k=0}^{20} P(X > k) \\
&= \sum_{k=0}^{20} 1 - P(X \leq k) \\
&= \sum_{k=0}^{20} 1 - \frac{1}{100}k^2 + \frac{1}{2000}k^3 - \frac{1}{160000}k^4 \\
&= \frac{786667}{80000} \\
&\approx 9.8333
\end{aligned}$$

Next, we consider the disadvantage of advantage roll. Let $\tilde{M}_{ij} = \max(R_i, R_j)$. Then let $Y = \min(\tilde{M}_{12}, \tilde{M}_{34})$. Y is the result of an advantage of disadvantage roll. Once again, we find the distribution of $\tilde{M}_{12}$. The probability that $\tilde{M}_{12} \leq k$ is the same as the probabilty that R_1 and R_2 are greater than k. So,

$$\begin{aligned}
P(\tilde{M}_{12} \leq k) &= P(R_1 \leq k, R_2 \leq k) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} P(R_1 = i)P(R_2 = j) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{k} \left(\frac{1}{20}\right)^2 \\
&= \frac{k^2}{400}
\end{aligned}$$

Then the probability that $Y > k$ is the same as the probability that both $\tilde{M}_{12}$

and $\tilde{M}_{34}$ are greater than k. So,

$$\begin{aligned}
P(Y > k) &= P(\tilde{M}_{12} > k, \tilde{M}_{34} > k) \\
&= P(\tilde{M}_{12} > k)P(\tilde{M}_{34} > k) \\
&= \left(1 - \frac{k^2}{400}\right)^2 \\
&= 1 - \frac{1}{400}k^2 + \frac{1}{160000}k^4
\end{aligned}$$

Once again using the tail sum formula for expectations yields,

$$\begin{aligned}
E[Y] &= \sum_{k=0}^{20} P(Y > k) \\
&= \sum_{k=0}^{20} 1 - \frac{1}{400}k^2 + \frac{1}{160000}k^4 \\
&= \frac{893333}{80000} \\
&\approx 11.1666
\end{aligned}$$

Finally, the expectation of a single roll, R, is given by $E[R] = \frac{1+20}{2} = 10.5$. So, the expectations are $E[Y] > E[R] > E[X]$.

Bonus question: We are comparing $P(X \geq k), P(Y \geq k)$, and $P(R \geq k)$. Observe that $P(X \geq k) = P(X > k - 1)$. Observe that for $k = 1$, these are all probability 1. Using the previous computations for these, we can compare and obtain the following,

$$\begin{aligned}
&P(Y \geq k) > P(X \geq k) \,\forall\, k = 2, \ldots, 20 \\
&P(Y \geq k) > P(R \geq k) \, k = 2, \ldots, 13 \text{ (less for } k > 13) \\
&P(X \geq k) > P(R \geq k) \, k = 2, \ldots, 8 \text{ (less for } k > 9)
\end{aligned}$$